Question/Claim A classic unsolved problem in number theory asks if there are infinitely many pairs of "twin primes", pairs of primes separated by 2, such as 3 and 5, 11 and 13, or 71 and 73. Prove that the only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7.

1 Lemma

For any integer n, at least one of the integers $n, n+2, n+4$ is divisible by 3.

Proof See proof in Test Flight, Question 5.

2 Proof of Claim

Proof By contradiction.

Let S be the set of all prime triples. Since $3, 5, 7$ are valid prime triple, S has at least one members: $|S| \geq 1$.

Let S' be the set of all prime triples excluding $3, 5, 7$:

$$S' = \{(p, p+2, p+4) | p \in \mathcal{P}, (p+2) \in \mathcal{P}, (p+4) \in \mathcal{P}, p \geq 5\}$$

where $\mathcal{P}$ is set of all primes.

Assume there are more than one prime triples, $|S| > 1$. This implies $|S'| \geq 1$. Since prime numbers are also integers, the Lemma above implies that for every prime triple in S', at least one of the primes $p, p+2, p+4$ is divisible by 3. In other words, there exists at least one primes which is at least 5 and divisible by 3. This contradicts the definition of prime number. Therefore, S' must be empty set. This implies that $|S| = 1$, and the only prime triple is 3, 5, 7.