Question Say whether the following is true or false and support your answer by a proof: For any integer n, the number $n^2 + n + 1$ is odd.

Claim $(\forall n \in \mathcal{Z})(n^2 + n + 1)$ is odd.

Proof By algebra and definition of divisibility.

Let arbitrary $n \in \mathcal{Z}$ be given. $(n^2 + n + 1)$ being odd is equivalent to:

$$(\exists m \in \mathcal{Z})(2m + 1 = n^2 + n + 1)$$

$$(\exists m \in \mathcal{Z})(2m = n^2 + n)$$

$$(\exists m \in \mathcal{Z})[2m = n(n + 1)] \tag{1}$$

We shall prove that (1) is true by showing $n(n + 1)$ is even when n is either even and odd.

Case 1 Assume n is even. Therefore, there exists $a \in \mathcal{Z}$ such that $n = 2a$.

$$n(n + 1) = 2a(2a + 1)$$

Let $m = a(2a + 1)$. Clearly m is an integer and satisfies (1). In other words, $n(n + 1)$ is even because it is product of even and any integers.

Case 2 Assume n is odd. Therefore, there exists $b \in \mathcal{Z}$ such that $n = 2b + 1$.

$$n(n + 1) = (2b + 1)(2b + 2) = 2(2b + 1)(b + 1)$$

Let $m = (2b+1)(b+1)$. Clearly, m is an integer and satisfies (1). In other words, $n(n+1)$ is also even because one of its factor: $n+1$ is even.

Since any integer: n is either even or odd, both cases above show that: $(\forall n \in \mathcal{Z})(n^2+n)$ is even; and $(\forall n \in \mathcal{Z})(n^2+n+1)$ is odd.