Question/Claim Prove (from the definition of a limit of a sequence) that if the sequence $\{a_n\}_{n=1}^{\infty}$ tends to limit L as $n \to \infty$, then for any fixed number $M > 0$, the sequence $\{Ma_n\}_{n=1}^{\infty}$ tends to the limit ML.

Proof By definition of limit of sequence.

Given that the sequence $\{a_n\}_{n=1}^{\infty}$ tends to limit L as $n \to \infty$, the following statement is true by the definition of limit of sequence:

$$(\forall\epsilon > 0)(\exists n \in \mathcal{N})(\forall m \geq n)[|a_m - L| < \epsilon] \tag{1}$$

We shall prove that:

$$(\forall\epsilon > 0)(\exists n \in \mathcal{N})(\forall m \geq n)[|Ma_m - ML| < \epsilon] \tag{2}$$

Let $\epsilon > 0$ be given. Since (1) is true for any arbitrary $\epsilon > 0$, it is also true for $\frac{\epsilon}{M}$ where $0 < \frac{\epsilon}{M} < \epsilon$. In other words by replacing $\frac{\epsilon}{M}$ with ϵ in (1), the following statement is true:

$$(\exists n \in \mathcal{N})(\forall m \geq n)[|a_m - L| < \frac{\epsilon}{M}] \tag{3}$$

Pick minimum natural number: n_0 that satisfies (3). In other words,

$$(\forall m \geq n_0)[|a_m - L| < \frac{\epsilon}{M}] \tag{4}$$

By multiplying M to both sides of inequality in (4):

$$(\forall m \geq n_0)[M|a_m - L| < \epsilon] \tag{5}$$

Since $M > 0$, (5) is equivalent to:

$$\begin{aligned} &(\forall m \geq n_0)[|M(a_m - L)| < \epsilon] \\ &(\forall m \geq n_0)[|Ma_m - ML| < \epsilon] \qquad (6) \end{aligned}$$

For any arbitrary $\epsilon > 0$, (6) implies (2). By the definition of limit of sequence, if the sequence $\{a_n\}_{n=1}^{\infty}$ tends to limit L as $n \to \infty$, then for any fixed number $M > 0$, the sequence $\{Ma_n\}_{n=1}^{\infty}$ tends to the limit ML.