Question Give an example of a family of intervals $A_n, n = 1, 2, ...,$ such that $A_{n+1} \subset A_n$ for all n and $\cap_{n=1}^{\infty} A_n$ consists of a single real number. Prove that your example has the stated property.

Let $A_n = \{x | x \in \mathcal{R}, 0 \leq x \leq \frac{1}{n}\}, n \in \mathcal{N}$. We shall prove the following claims:

1. $A_{n+1} \subset A_n$.
2. $\cap_{n=1}^{\infty} A_n = \{0\}$.

1 Proof of Claim 1

Proof by definition of proper subset.

Clearly, $\frac{1}{n+1} < \frac{1}{n}$. For all $x \in A_{n+1}$, the following inequality holds:

$$0 \leq x \leq \frac{1}{n+1} < \frac{1}{n}$$

This shows that all members of A_{n+1} is contained in A_n by the definition of A_n and A_{n+1}. At the same time, due to the strict inequality, not all members of A_n is contained in A_{n+1}. In particular, for all y that satisfies:

$$\frac{1}{n+1} < y \leq \frac{1}{n}$$

$y \in A_n$ and $y \notin A_{n+1}$. Hence by definition of proper subset, $A_{n+1} \subset A_n$.

2 Lemma 1

For any natural number n,

$$\cap_{i=1}^{n} A_i = A_n \tag{1}$$

Proof By mathematical induction.

Initial Step For $n = 1$, identity (1) reduces to:

$$A_1 = \cap_{i=1}^{1} A_i = A_1$$

which is true since both sides are equal to A_1.

Inductive Step Assume identity (1) is true for n:

$$\cap_{i=1}^{n} A_i = A_n \tag{2}$$

Take intersection of A_{n+1} on both sides of (2),

$$A_{n+1} \cap (\cap_{i=1}^{n} A_i) = A_{n+1} \cap A_n$$

$$\cap_{i=1}^{n+1} A_i = A_{n+1} \cap A_n \tag{3}$$

From the proof of Claim 1, $A_{n+1} \subset A_n$. Since all members of A_{n+1} are contained in A_n, (3) reduces to:

$$\cap_{i=1}^{n+1} A_i = A_{n+1} \cap A_n = A_{n+1}$$

which is identity (1) with $n + 1$ in place of n. Hence by the principle of mathematical induction, the identity holds for all $n \in \mathcal{N}$.

3 Lemma 2

$$| \cap_{i=1}^{n+1} A_i | < | \cap_{i=1}^{n} A_i | \tag{4}$$

Proof By Claim 1 and Lemma 1.

From (1),

$$\cap_{i=1}^{n} A_i = A_n, \cap_{i=1}^{n+1} A_i = A_{n+1}$$

From Claim 1, $|A_{n+1}| < |A_n|$ since A_{n+1} is a proper subset of A_n. Hence,

$$| \cap_{i=1}^{n+1} A_i| = |A_{n+1}| < |A_n| = | \cap_{i=1}^{n} A_i|$$

4 Lemma 3

$$\lim_{n\to\infty} A_n = \{0\} \tag{5}$$

Proof By definition of limit.

By the definition of A_n, its greatest lower bound is 0 for any n. For any given n, the lowest upper bound of A_n is $\frac{1}{n}$. Clearly, $\lim_{n\to\infty} \frac{1}{n} = 0$. Hence,

$$\lim_{n\to\infty} A_n = \{x|0 \le x \le \lim_{n\to\infty} \tfrac{1}{n}\} = \{x|0 \le x \le 0\} = \{x|x = 0\} = \{0\}$$

5 Proof of Claim 2

Proof by definition of limit.

From Lemma 2, since $\cap_{i=1}^{n} A_i$ is decreasing in sizes and bound as n increases, the limit for $\cap_{n=1}^{\infty} A_n$ exists. By definition of limit,

$$\cap_{n=1}^{\infty} A_n = \lim_{n\to\infty} \cap_{i=1}^{n} A_i \tag{6}$$

By Lemmas 1 and 3, (6) reduces to:

$$\cap_{n=1}^{\infty} A_n = \lim_{n\to\infty} \cap_{i=1}^{n} A_i = \lim_{n\to\infty} A_n = \{0\} \tag{7}$$

In conclusion, $A_n = \{x | x \in \mathcal{R}, 0 \leq x \leq \frac{1}{n}\}, \forall n \in \mathcal{N}$ have the following properties:

1. $A_{n+1} \subset A_n$.
2. $\cap_{n=1}^{\infty} A_n = \{0\}$.