

Recurrence spike spectra

K. Hauke Kraemer · Frank Hellmann · Jürgen Kurths · Norbert Marwan

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Abstract A novel kind of power spectrum is constructed, the *inter spike spectrum*, which transforms spike-train-like signals into their frequency domain. This method clearly shows the apparent cycles in the data and overcomes the problems when using the obvious idea of Fourier-transforming it. We invent this instructive approach with the idea of transforming the τ -recurrence rate of a recurrence plot (RP), which often has a spiky appearance. The τ -recurrence rate is the density of recurrence points along diagonals of the RP, which are parallel to the main diagonal with a distance of τ . In this context the inter spike spectrum can be interpreted as a nonlinear power spectrum of a potentially high dimensional system which constitutes the RP. The proposed measure is simple to compute, robust to noise and is able to detect bifurcations inducing regular-regular, regular-chaos as well as chaos-chaos transitions.

Keywords Bifurcations · Recurrence Analysis · Decomposition · Frequency Analysis

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1 Introduction

Recurrence Plots (RPs) provide a vivid representation of complex dynamics stemming from potentially high dimensional systems, Eq. (1).

$$R_{i,j}(\varepsilon) = \Theta(\varepsilon - D_{i,j}) = \Theta(\varepsilon - \|\vec{x}_i - \vec{x}_j\|), \quad \vec{x} \in \mathbb{R}^d, \quad i, j \in [1, \dots, N]. \quad (1)$$

K. Hauke Kraemer · Jürgen Kurths · Norbert Marwan · Frank Hellmann
Potsdam Institute for Climate Impact Research, Member of the Leibniz Association, 14473 Potsdam, Germany
E-mail: hkraemer@pik-potsdam.de

K. Hauke Kraemer · Jürgen Kurths
Institute of Physics and Astronomy, University of Potsdam, 14476 Potsdam, Germany

Norbert Marwan
Institute of Geosciences, University of Potsdam, 14476 Potsdam, Germany

Jürgen Kurths
Institute of Physics, Humboldt Universität zu Berlin, 12489 Berlin, Germany

The simple idea to track recurring states of the d -dimensional trajectory \vec{x}_i of the system under study not only allows for a beneficial visualization of the dynamics, but also for its quantification, using certain structures in the RP, such as diagonal or vertical lines [10].

Some of these recurrence quantification measures, the entropy of diagonal lines and the entropy of recurrence times, can be related to the Kolmogorov-Sinai entropy [8, 1]. However, these quantifiers have a free parameter, the minimal considered line length, and are usually biased, due to the finite size of the RP and thickened diagonal lines, which needs to be corrected [5]. Moreover, the mentioned statistics cannot account for changing regular (non-chaotic) dynamics, such as period-doubling bifurcations. A rather simple idea is to look at the τ -recurrence rate of the RP (τ -RR, Eq. 2) [9, 17]. This is the density of recurrence points along the diagonals of the recurrence matrix, as a function of the distance τ (sampling units) to the main diagonal:

$$\tau\text{-RR}(\varepsilon) = RR(\tau, \varepsilon) = \frac{1}{N - \tau} \sum_{i=1}^{N-\tau} R_{i,i+\tau}. \quad (2)$$

τ -RR serves as an estimator for the probability that the system recurs after time $\tau\Delta t$, with Δt being the sampling time of the trajectory $\vec{x}_i = \vec{x}(\Delta t \cdot i), i = 1, \dots, N$. Zbilut and Marwan [17] pointed out that τ -RR could be used as a plugin value for the auto-correlation function $C(\tau)$ and, hence, via the Wiener-Kinchim theorem a “generalized” powerspectrum can be obtained. This is reasonable, since the average distances for a given lag τ

$$\bar{D}(\tau) = \frac{1}{N - \tau} \sum_{i=1}^{N-\tau} D_{i,i+\tau} \quad (3)$$

can be directly read from the distance matrix \mathbf{D} and is also preserved in its thresholded version τ -RR. There are clear advantages for a recurrence-derived powerspectrum, i.e., Fourier transforming (FT) τ -RR (Fig. 1D), instead of $C(\tau)$ (Fig. 1C): There are no assumptions for stationarity or sampling, when constructing a RP. Furthermore, the correlation structures of higher dimensional spaces can be resolved in the recurrence-derived Fourier-spectrum.

However, there is a drawback to this approach. Whenever τ -RR is a spike-train-like signal, which it is in most cases (see Fig. 1) especially for map-data (low-resolution data), a FT of such a signal leads to a spike-train-like image in the frequency domain (e.g. [13, 3], see Fig. 1D). Thus, it is not intuitive how to extract meaningful information about dominant frequencies of the systems’ state space trajectory.

For clarification, consider the signal we would like to analyze (e.g. the τ -RR of a system) to be a Dirac comb (DC) with inter-spike period T_{is} :

$$DC_{is}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_{is}), \quad (4)$$

i.e., a series of Dirac delta functions for a period T_{is} . There is only one single period – T_{is} – in this signal (Fig. 2A, D), so in principle we would strive for a single peak in the frequency domain of this signal at a frequency $f = 1/T_{is}$. Surprisingly the Fourier spectrum does not meet this expectation and instead of a single frequency, there are exceptionally many frequencies excited (Fig. 2B, E). This is, because the Fourier components add constructively for every frequency $1/T_{is}$ and therefore $DC_{is}(t)$ coincides with its own Fourier transform up to a factor $1/T_{is}$.

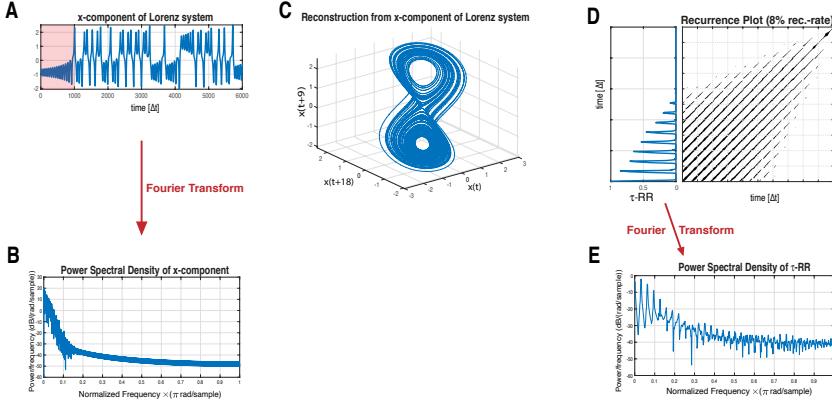


Fig. 1 Schematic illustration of a τ -recurrence rate based spectrum. **A** x -component time series of the Lorenz63-System (Eq. (8)) and **B** its corresponding Fourier power spectrum. **C** Reconstruction from x -component of Lorenz system. **D** Recurrence Plot (8% rec.-rate) and the corresponding τ -recurrence rate obtained from the state space trajectory in **C**. The shaded interval in the time series in **A** corresponds to the shown subset. **E** Fourier Power spectrum obtained from the τ -recurrence (subset shown in panel **D**) [17].

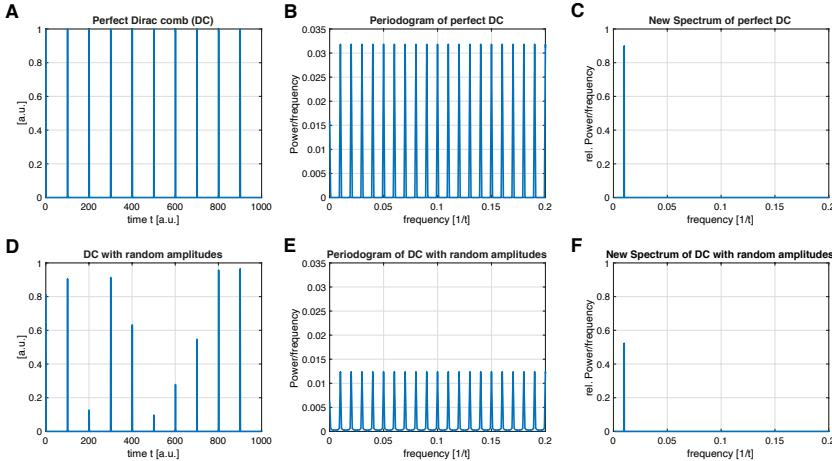


Fig. 2 The transformation of a Dirac comb (series of Dirac delta functions) with a single inter-spike period $T_{is} = 100$ ($\hat{f} = 0.01$) into the frequency domain. **A** Dirac Comb (DC) with equal amplitudes and **B** its FFT-based powerspectrum. **C** Proposed inter spike spectrum of the signal in **A** showing a single frequency, which corresponds to the inter-spike period T_{is} ($f = 0.01$). **D** DC with randomly chosen amplitudes and same T_{is} as in **A**, and **E** its FFT-based powerspectrum. **F** Proposed inter spike spectrum of the signal in **D** showing a single frequency, which corresponds to the expected inter-spike period T_{is} ($f = 0.01$).

In this article we propose a new way of transforming a spike-train-like signal into its frequency domain. This novel *inter spike spectrum* does not show resonance behavior of the signal's inherent inter-spike frequencies (Fig. 2C, F). Section 2 explains the idea, which can be used to decompose any arbitrary signal and is not restricted to the τ -RR, which we exemplify here. However, the more spiky the signal is, the more outstanding our new approach is compared to a FT. In Section 3 we exemplify its use when transforming the τ -

RR of a system under study. In this case, the inter spike spectrum can unravel characteristic time scales of high dimensional systems, which is not possible when using a FT.

2 Method

The idea is as simple as it is obvious. The signal, which we would like to transform – we focus on τ -RR in this article, but this can be applied to any sort of signal – is decomposed into a set of appropriate basis functions. Instead of using trigonometric functions, as it is the idea in the Fourier decomposition, we use Dirac combs (DC) with different inter-spike periods as basis functions, Eq. (4). Let $s(t_i)$ be the normalized signal we want to transform with length N and $t_i = i \cdot \Delta t$, $i = 1, \dots, N$, where Δt denotes the sampling time and $s(t_i) \in [0, 1] \forall i$. In the following we label this time series as a $(1 \times N)$ -dimensional vector \mathbf{s} . First, N different DC's of length N are constructed with inter-spike periods $T_{is} \in [1, \dots, N]$. Second, in order to account for possible phase shifts of these basis functions occurring in \mathbf{s} , each of these N different DC's also need to be shifted one step further $T_{is} - 1$ times. This leaves us with a total number of $M = \sum_{i=1}^N i$ basis functions which we can arrange as rows of a $(M \times N)$ -sized matrix \mathbf{X} (Fig. 3 illustrates the described procedure)

$$\mathbf{X}_{i,j} = \sum_{k=0}^N \delta(j - 1 - kT(i) - i + T(i)), \quad (5)$$

$$T(i) = n \quad , \forall n : \frac{n(n-1)}{2} + 1 \leq i < \frac{n(n+1)}{2} + 1, \quad n \in \mathbb{N}_+. \quad (6)$$

Due to the shifting of each of the basis functions of inter-spike period T_{is} , \mathbf{X} is not linear independent anymore. Thus, an under-determined linear system

$$\mathbf{X}^T \beta = \mathbf{s} \quad (7)$$

has to be solved for β , a $(M \times 1)$ -sized vector carrying the loadings we are interested in. Along a variety of algorithms, which can solve this problem, we are particularly interested in those solutions, which promote sparsity in β , since our goal is to decompose the signal \mathbf{s} into a minimal number of basis functions (for an excellent overview of the topic we refer to Brunton and Kutz [2]). Therefore, we here choose the *least absolute shrinkage and selection operator* (LASSO regression) [16]. Finally, we group loadings which correspond to basis functions having the same period T_{is} into β_f and obtain the *inter spike spectrum* by simply plotting β_f as a function of the frequency $f = T_{is}^{-1}$ (Fig. 2C, F).

3 Application

We exemplify the use of the inter spike spectrum in combination with the τ -RR as outlined in Section 1 on several interesting research questions. The procedure is the following:

- (1) Compute a RP of the trajectory of the system, Eq. (1).
- (2) Compute the τ -RR of that RP, Eq. (2).
- (3) Transform the τ -RR into the proposed inter spike spectrum, see Section 2.

X	
1 1 1 1 1	$T_u = 1$
1 0 1 0 1	$T_u = 2$
0 1 0 1 0	shift by 1
1 0 0 1 0	$T_u = 3$
0 1 0 0 1	shift by 1
0 0 1 0 0	shift by 2
1 0 0 0 1	$T_u = 4$
0 1 0 0 0	shift by 1
0 0 1 0 0	⋮
0 0 0 1 0	shift by 3
1 0 0 0 0	$T_u = 5$
0 1 0 0 0	shift by 1
0 0 1 0 0	⋮
0 0 0 1 0	shift by 4
0 0 0 0 1	

Fig. 3 Example for a set of basis functions for an input signal of length $N = 5$, aligned in the matrix \mathbf{X} .

3.1 Period estimation for different dynamics in the Rössler system

First, we consider the Rössler system (Eq. (9) in Appendix B.2) in three different dynamical setups. We use the proposed inter spike spectrum to identify the type of dynamics. We set the parameters $b = 2$, $c = 4$ and analyze period-2 limit cycle dynamics ($a = 0.36$, Fig. 4A, D, G, J), period-3 limit cycle dynamics ($a = 0.41$, Fig. 4B, E, H, K) and chaotic dynamics ($a = 0.428$, Fig. 4C, F, I, L).

The inter spike spectra unravel the specific dynamics, which are also apparent in the state space portraits (Fig. 4A, B, C) and in the τ -RRs (Fig. 4G, H, I). The proposed idea is also robust to noise (see Fig. 6 in the Appendix). This is because the peaks of the τ -RR are insensitive to noise. While the peak shape does change in the presence of noise, its position does not, and this is what the inter spike spectrum encrypts after all.

3.2 Bifurcations in the Logistic map

We consider the Logistic map $x_{n+1} = r \cdot x_n (1 - x_n)$ for changing control parameter r . We vary r from $r = 3.4$ to $r = 4$ in steps of 0.001. For each setting of r

- (1) a time series of length $N = 201$ is computed with a random initial condition $u_0 \in [0, 1]$, neglecting the first 1,000 samples as transients,
- (2) 100 iterative Amplitude Adjusted Fourier Transform (iAAFT) surrogates [14, 15] are computed,
- (3) the time series and its iAAFT surrogates are embedded in a 2-dimensional state space using a time delay of unity,
- (4) from the 2-dimensional trajectories RPs, Eq. (1), are computed under a threshold $\varepsilon = 0.05$,
- (5) τ -RR, Eq. (2), is computed from the RP of the signal and from the RPs of the surrogates,
- (6) inter spike spectra are obtained from τ -RR of the signal and from the τ -RRs of the surrogates, see Section 2, and finally,
- (7) from the distribution of the surrogate inter spike spectra the 95th percentile is computed. The peaks of the inter spike spectrum of the signal which exceed this percentile are counted.

In this example, the according null hypothesis is that the data stems from a process which yields the same auto-correlation, hence the same Fourier powerspectrum, and the same am-

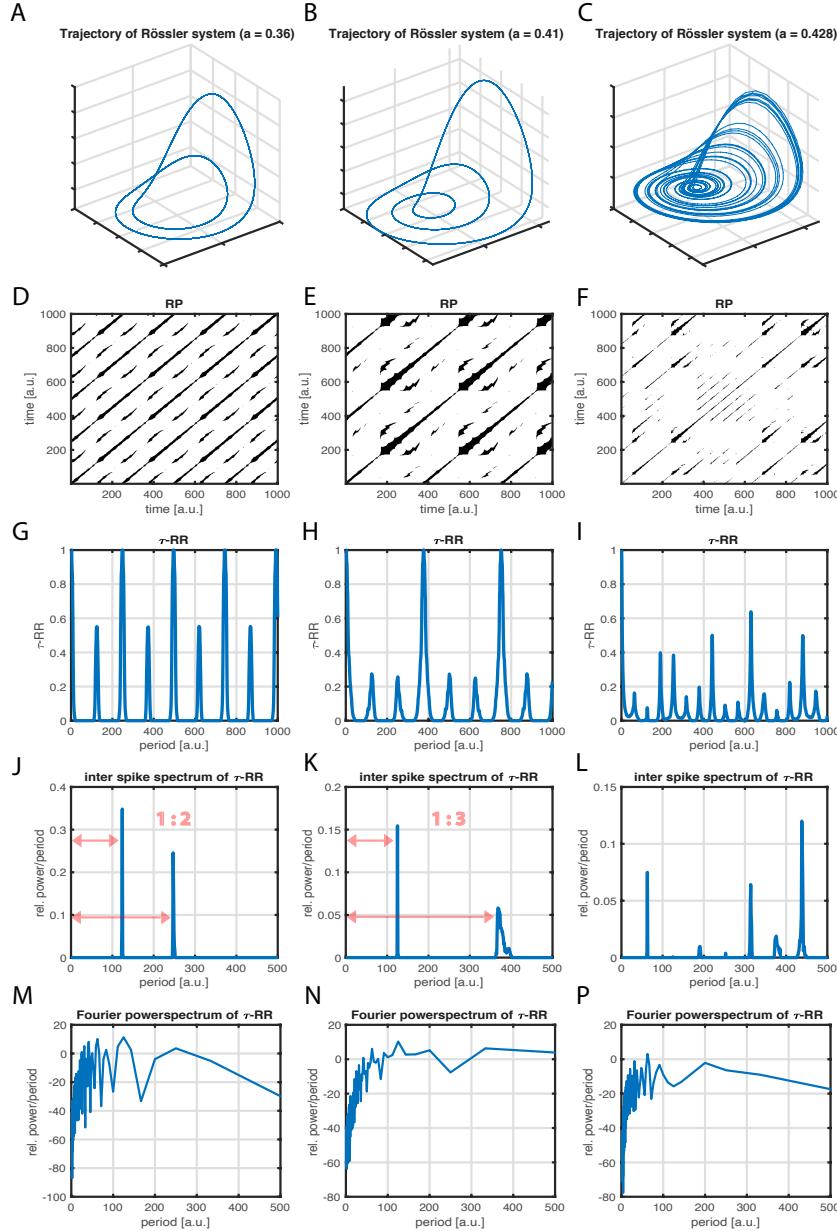


Fig. 4 Inter spike spectra of the τ -RR of the Rössler system in three different dynamical regimes with parameters $b = 2$, $c = 4$. Trajectory of the system in **A** period-2 (parameter $a = 0.36$), **B** in a period-3 (parameter $a = 0.41$) and **C** in a chaotic regime (parameter $a = 0.428$). **D, E, F** The corresponding RPs, obtained by using a recurrence threshold corresponding to a 10% global recurrence rate for D & E and 5% for F. **G, H, I** τ -RR's of the shown RPs. **J, K, L** The proposed inter spike spectra of the τ -RR's shown in panels G, H, I. Spectra were obtained with a regularization threshold corresponding to $\rho = 0.95$ accordance of τ -RR's and re-composed signals. The distance ratio of the peaks reflect the limit cycle dynamic. **M, N, P** Fourier power spectra of the τ -RR's shown in panels G, H, I.

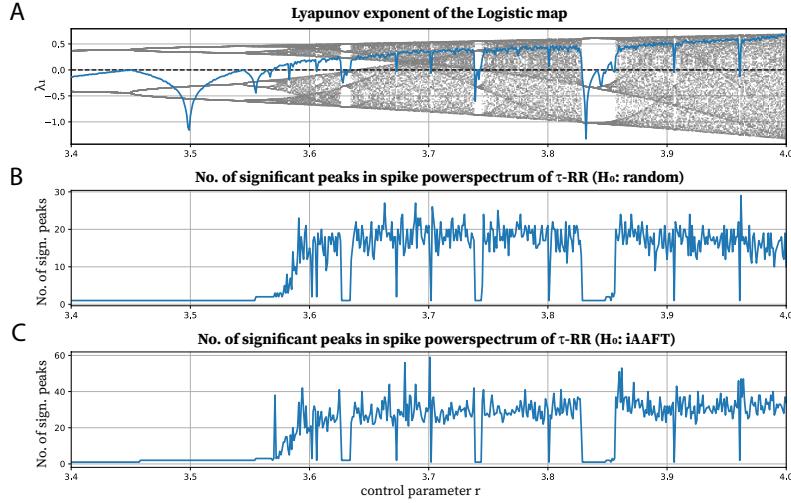


Fig. 5 **A** Lyapunov exponent of the Logistic map as a function of the control parameter r . **B** Number of significant peaks ($\alpha = 0.05$) in the inter spike spectrum of the τ -RR, random surrogates. **C** Same as **B**, but for iterative Amplitude Adjusted Fourier Transform (iAAFT) surrogates [14, 15].

plitude distribution. We consider the number of significant peaks in the inter spike spectrum with respect to the control parameter in order to distinguish the corresponding dynamics (Fig. 5C). A correlation with the positive Lyapunov exponent (Fig. 5A) is discernible ($\rho_{\text{Pearson}} = 0.88$). Moreover, this analysis can tackle period-doubling, since it “measures” the dominant cycles via the inter spike spectrum.

A less computationally intensive approach is to compute surrogates for the τ -RR analytically, rather than computing a RP and its τ -RR for each iAAFT surrogate of the time series. This translates into a null hypothesis that the τ -RR and its corresponding inter spike spectrum stems from a RP of a random signal. In this case the probability of finding a black point in the RP can be obtained from a binomial distribution with probability parameter p set to the recurrence rate of the RP of the signal. This way 100 surrogate τ -RRs are computed in step (5). The results are similar to the ones obtained from the iAAFT surrogates and nearly as good (Fig. 5B). The first period doubling at $r \approx 3.458$ cannot be detected, while the correlation to the positive Lyapunov exponent is only slightly worse ($\rho_{\text{Pearson}} = 0.84$).

The described procedure does work well for map data, because most often the τ -RR for those kind of data reveals a “spiky enough” nature. On the contrary, highly sampled (flow-) data often yield not as spiky τ -RR’s and, thus, the number of significant peaks in the inter spike spectrum may not be sensitive enough to detect period-doubling bifurcations. Nevertheless the according inter spike spectra is still revealing important information (Fig. 4) and practitioners can design appropriate quantifying statistics based on these spectra, which suit the research task.

3.3 Evolutionary spike spectrogram of CENOGRID isotope time series

4 Conclusion

A novel type of powerspectrum, the *inter spike spectrum*, has been proposed. The method decomposes any arbitrary signal into basis functions which consist of (lagged) Dirac combs (DC) of different inter-spike period. The loading for each period is obtained by a regularized regression, which promotes sparsity in its solution (we chose LASSO). Since there are $M = \sum_{i=1}^N i$ basis functions for a signal of length N the regression can get computationally intensive for $N > 1,000$. When plotting the computed loadings as a function of the frequency (or period) the inter spike spectrum is obtained. This novel powerspectrum is superior to an ordinary FFT-based powerspectrum, when the signal has a spike-train-like appearance. The invention of the proposed method has been motivated by the idea of transforming τ -recurrence rate signals (τ -RR's) into their frequency domain. This general idea [17] allows for a frequency analysis of high dimensional systems, because the RP is a representation of the system's state space trajectory. The τ -RR of a recurrence plot (RP) usually has a spiky shape, especially for map-like data, and the inter spike spectrum can reliably reveal the system's dominant frequencies, which is not possible when Fourier transforming the τ -RR or the underlying signal itself. Since the position of the peaks in the τ -RR are not sensitive to noise, the corresponding inter spike spectrum also yields robust results in the presence of noise. We have successfully used the idea of transforming the τ -RR for the detection of bifurcations in the Logistic map. By constructing appropriate surrogates of the inter spike spectra, and thus a null model, the number of significant peaks in the inter spike spectrum correlated well with the positive Lyapunov exponent. This measure was also able to resolve period-doubling bifurcations.

We could think of a broad range of applications of the proposed idea. The inter spike spectrum itself can serve as a valuable tool for the analysis of any sort of spike-train-like data. On the other hand, the inter spike spectrum of the τ -RR of a signal can serve as a generalized, nonlinear frequency analysis tool for complex systems. When there is only a subset of state variables available, the state space has to be reconstructed as a pre-processing step. Recent findings [6] show that this reconstruction process can be reliably automated and applied to multivariate data as well. This would allow for a “running window” approach, in order to detect transitions. Due to the mentioned computational constraints of our proposed method, a window size $w \leq 1,000$ would possibly suffice for most data, especially when it is map-like, i.e., not highly sampled.

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Conflict of interest

The authors declare that they have no conflict of interest.

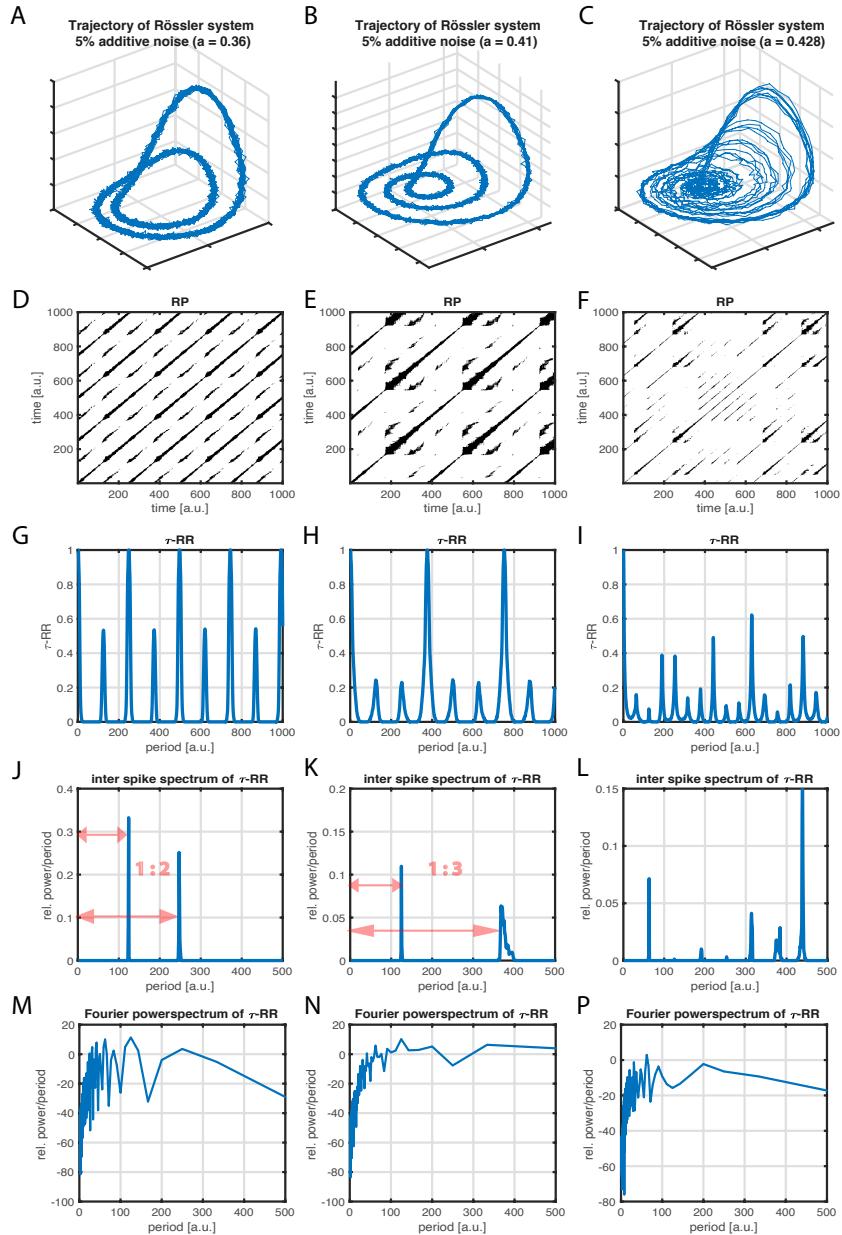
A Inter spike spectra for noisy Rössler system

Fig. 6 Same as in Fig. 4, but here with 5% additive Gaussian white noise. The appearance of the inter spike spectra in **J**, **K**, **L** and the Fourier spectra in **M**, **N**, **P** are unaffected by the additive noise noise.

B Exemplary models

B.1 Lorenz system

The classical Lorenz-63 system [7] is defined as

$$\begin{aligned}\dot{x} &= \sigma(y - x) \\ \dot{y} &= x(r - z) - y \\ \dot{z} &= xy - \beta z.\end{aligned}\tag{8}$$

For producing Fig. 1 we set the initial condition to $u_0 = [0.0, 10.0, 0.0]$, used a sampling time of $\Delta t = 0.01$ and discarded the first 2,000 points of the integration as transients. The parameters have been set to $\sigma = 10, \beta = 8/3, r = 28$ and we used a time series consisting of 6,000 samples.

B.2 Rössler system

The Rössler system [12] is defined as

$$\begin{aligned}\dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c).\end{aligned}\tag{9}$$

For producing Figs. 6, 4 the initial condition for producing panels **A** & **B** was set to $u_0 = [.7, -1, 0.4]$ with a sampling time of $dt = 0.05$ and in case of panel **C** $u_0 = [-0.1242, -2.5415, 0.2772]$ with a sampling time of $dt = 0.1$. The first 5,000 samples were discarded as transients and trajectories of length $N = 5,000$ were obtained from which we computed the RPs and the corresponding τ -RR's. For the inter spike spectra only the first 1,000 values of the τ -RR's were considered.

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