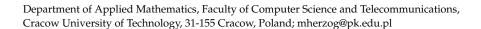


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# **Semi-Exponential Operators**

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**Abstract:** In this paper we study approximation properties of exponential-type operators for functions from exponential weighted spaces. We focus on some modifications of these operators and we derive a new example of such operators. A probabilistic approach for these modifications is also demonstrated.

Keywords: exponential operators; exponential weighted spaces; borel measures; Laplace transform

#### 1. Introduction

This paper is devoted to determine a new operator, which we call a semi-exponential operator. The idea comes mainly from papers [1,2], which concern exponential-type operators and semi-exponential operators. In general, the exponential-type operators, introduced in [1], are considered an interesting subject for many authors. The authors examine approximation properties of these operators for functions from different function spaces. In literature we can also find many modifications of these operators which are focused on examining the rate of convergence of these modifications compared to initial exponential-type operators (for example [3–8]). In particular, the truncated version of exponential-type operators and its modifications are also investigated (for example [9–12]). As it turns out the truncated operators shall be applied to Computer Aided Geometric Design. Moreover, the exponential-type operators can be seen in papers connected with differential equations, also considered in abstract spaces, for example in papers [2,3,13,14].

At the beginning of this paper we recall some basic definitions and expressions, which we are going to use in the main part of the paper. Later on, we specify the well-known exponential operators and operators from paper [2] which we call semi-exponential operators. We bring up the main differences between them. Following papers [1,2] we derive a new example of the semi-exponential operator. At the end of this paper we interpret the operator from the probabilistic point of view, similarly to paper [3].

## 2. Preliminaries

Let  $-\infty \le a < b \le +\infty$  and let us denote the space of all real-valued continuous functions on (a,b) by C((a,b)), and the Banach space of all continuous, bounded functions on (a,b) endowed with the sup-norm  $\|\cdot\|_{\infty}$  by CB((a,b)).

We shall also consider the following closed subspaces of CB((a, b)):

$$C_0((a,b)) = \{ f \in C((a,b)) \mid \lim_{x \to g} f(x) = 0 \},$$

where  $g \in \{a, b\}$  and

$$C_*((a,b)) = \{ f \in C((a,b)) \mid \lim_{x \to \sigma} f(x) = G \},$$

where  $G \in \mathbb{R}$ . For any positive integer q

$$v_q(x) = e^{-q|x|}$$
 for  $x \in (a, b)$ 



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is the exponential weight function. The space

$$E_q = \{ f \in C((a,b)) \mid v_q f \in CB((a,b)) \}$$

is a Banach space with the norm

$$||f||_q = ||v_q f||_{\infty}$$
 for  $f \in E_q$ 

and

$$E_a^0 = \{ f \in C((a,b)) \mid v_q f \in C_0((a,b)) \}$$

is a closed subspace of  $E_q$ . Moreover, for every  $q \ge 1$  we have

$$E_0 := CB((a,b)) \subset E_q^0 \subset E_q \subset E_{q+1}^0$$

and

$$\|\cdot\|_q \le \|\cdot\|_{\infty}$$
 on  $CB((a,b))$ ,  
 $\|\cdot\|_{q+1} \le \|\cdot\|_q$  on  $E_q$ .

We shall denote by  $E_{\infty} = \bigcup_{a=0}^{\infty} E_a$ .

## 3. Exponential and Semi-Exponential Operators

In [1] Ismail and May proposed some generalizations of integral operators as follows

$$S_{\lambda}(f;t) = \int_{a}^{b} W(\lambda,t,u)f(u)du, \tag{1}$$

with the normalization condition

$$\int_{a}^{b} W(\lambda, t, u) du = 1 \tag{2}$$

where  $W(\lambda, t, u)$  – the kernel of  $S_{\lambda}$  is a positive function satisfying the following homogenous partial differential equation

$$\frac{\partial}{\partial t}W(\lambda, t, u) = \frac{\lambda}{p(t)}W(\lambda, t, u)(u - t) \tag{3}$$

for  $\lambda > 0$ , p an analytic and positive function on (a, b).

The meaning of the generalization we can detect by investigating papers concerning approximation process for the well-known operators

• the Gauss–Weierstrass operator for  $p(t) \equiv 1$ 

$$S_{\lambda}(f;t) = \sqrt{\frac{\lambda}{2\pi}} \int_{\mathbb{R}} \exp(-\lambda(u-t)^2/2) f(u) du, \quad t \in \mathbb{R},$$

• the Szász–Mirakjan operator for p(t) = t

$$S_{\lambda}(f;t) = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} f\left(\frac{k}{\lambda}\right), \quad t \in (0,+\infty),$$

• the Bernstein polynomial operator for p(t) = t(1-t)

$$S_n(f;t) = \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} f\left(\frac{k}{n}\right), \quad t \in [0,1],$$

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• the Baskakov operator for p(t) = t(1+t)

$$S_{\lambda}(f;t) = (1+t)^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!} \left(\frac{t}{1+t}\right)^k f\left(\frac{k}{n}\right), \quad t \in (0,\infty),$$

• the Post–Widder operator for  $p(t) = t^2$ 

$$S_{\lambda}(f;t) = \frac{(\lambda/t)^{\lambda}}{\Gamma(\lambda)} \int_0^{\infty} e^{-(\lambda u/t)} u^{\lambda-1} f(u), \quad t \in (0,\infty),$$

• the Ismail–May operators for  $p(t) = t^2 + 1$ 

$$S_{\lambda}(f;t) = \frac{\lambda 2^{\lambda-2}}{\pi \Gamma(\lambda)} (1+t^2)^{-\lambda/2} \int_{-\infty}^{\infty} e^{\lambda u \arctan t} \left| \Gamma\left(\frac{\lambda}{2} (1+iu)\right) \right|^2 f(u) du, \quad t \in \mathbb{R},$$

• and for  $p(t) = 2t^{3/2}$ 

$$S_{\lambda}(f;t) = e^{-\lambda\sqrt{t}} \left( f(0) + \lambda \int_0^{\infty} e^{-\lambda u/\sqrt{t}} u^{-1/2} I_1(2\lambda\sqrt{u}) f(u) du \right), \quad t \in (0,+\infty).$$

These examples are exponential-type operators, which means they fulfill the differential Equation (3) and the normalization condition (2). Moreover, the authors prove that the two conditions and the generating function p determine uniquely the approximation operator  $S_{\lambda}$ . Let us recall some particular properties of the exponential operators which are important from the standpoint of this paper.

**Lemma 1** (Proposition 2.1 [1]). Let  $S_{\lambda}(\cdot;t)$  be an exponential operator. Then for  $t \in (a,b)$  we have

- (a)  $S_{\lambda}(e_0;t) = e_0(t)$ ,
- (*b*)  $S_{\lambda}(e_1;t) = e_1(t),$
- (c)  $S_{\lambda}(e_2;t) = e_2(t) + \frac{p(t)}{\lambda}$

where  $e_i(t) = t^i$  for  $i \in \{0, 1, 2\}$ .

We immediately conclude that operators  $S_{\lambda}$  must preserve linear functions.

In [2] Tyliba and Wachnicki studied a very similar problem we have just mentioned. They investigated a differential equation with minor changes of the right hand-side of (3), which means

$$\frac{\partial}{\partial t}K(\lambda, t, u) = \frac{\lambda}{p(t)}K(\lambda, t, u)(u - t) - \beta K(\lambda, t, u) \tag{4}$$

for  $\beta > 0$  and with the same normalization condition

$$\int_{a}^{b} K(\lambda, t, u) du = 1 \tag{5}$$

and with the previous requirements relating to  $\lambda$ , a, b and p = p(t).

From now on we shall call the operators

$$W_{\lambda}(f;t) = \int_{a}^{b} K(\lambda,t,u) f(u) du$$

a semi-exponential operators with the kernel  $K(\lambda, t, u)$ .

The authors prove lemmas and theorems using similar methods as in [1]. In the paper it is proved that  $W_{\lambda}$  are approximation operators for functions  $f \in E_q$ , there are also stated some estimates with the same rate of convergence as in [1]. As we can see the

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> minor changes in the Equation (3) are the reason why the operators are not retaining linear functions. In this case we have following

> **Lemma 2** (Lemma 2.1 [2]). Let  $W_{\lambda}(\cdot;t)$  be a semi-exponential operator. Then for  $t \in (a,b)$  we

(a)  $W_{\lambda}(e_0;t)=e_0(t),$ 

(b)  $W_{\lambda}(e_1;t) = e_1(t) + \frac{\beta p(t)}{\lambda},$ (c)  $W_{\lambda}(e_2;t) = e_2(t) + \frac{p(t) + 2\beta t p(t)}{\lambda} + \frac{\beta p(t) p'(t) + \beta^2 p(t)}{\lambda^2},$ 

where  $e_i(t) = t^i$  for  $i \in \{0, 1, 2\}$ .

**Theorem 1** ([1,2] respectively). Let  $x \in (a,b)$ . If  $f \in E_q$  and f''(x) exists, then

(a) ref. [1]  $\lim_{\lambda \to \infty} \lambda(S_{\lambda}(f;x) - f(x)) = \frac{p(x)f''(x)}{2}$ ,

(b) ref. [2]  $\lim_{\lambda \to \infty} \lambda(W_{\lambda}(f;x) - f(x)) = \beta p(x) f'(x) + \frac{p(x)f''(x)}{2}$ .

If  $f \in C^2([c,d])$ , then the above convergence is uniform in any interior interval  $[a_1, b_1] \subset (c, d).$ 

In general the exact formula for the exponential and semi-exponential operators is not too obvious. In the case of exponential-type operators we can see the examples at the beginning of this section. In paper [2] the authors give two examples of semi-exponential operators. For  $p(t) \equiv 1$  we have the semi-exponential Gauss–Weierstrass operator

$$W_{\lambda}(f;t) = \sqrt{\frac{\lambda}{2\pi}} \int_{\mathbb{R}} \exp\left(-\frac{\lambda(u-t-\frac{\beta}{\lambda})}{2}\right) f(u) du,$$

and for p(t) = t we have the semi-exponential Szász–Mirakjan operator

$$W_{\lambda}(f;t) = e^{-(\beta+\lambda)} \sum_{k=0}^{\infty} \frac{(\beta+\lambda)^k t^k}{k!} f\left(\frac{k}{\lambda}\right).$$

The purpose of this paper is to find more examples of semi-exponential operators and to look at them from the probabilistic point of view.

## 4. A New Example of the Semi-Exponential Operator

In this section we shall derive a new example of the semi-exponential operator. The crucial tool for motivation of our thesis is based on Theorem 3.10 of [2] and its proof. We shall recall the theorem

**Theorem 2.** The kernel  $K(\lambda,t,u)$  of the semi-exponential operator  $W_{\lambda}$  can be obtained by the partial differential Equation (4) and the normalization condition (5).

Let  $K(\lambda, t, u)$  – the kernel of the semi-exponential opererator  $W_{\lambda}$  satisfy (4) and (5) and  $p(t) = t^2$  for  $t \in (0, +\infty)$ . Notice that the solution of the differential Equation (4) shall be given in the following form

$$K(\lambda, t, u) = C(\lambda, u) \exp\left(\lambda \int_{c}^{t} \frac{u - \theta}{\theta^{2}} d\theta - \beta t\right), \tag{6}$$

for some  $c \in (0, +\infty)$ . We define an auxiliary function  $\xi$  as follows

$$\xi(\lambda, t, u) = \exp\left(-\lambda \int_{0}^{t} \frac{u - \theta}{\theta^{2}} d\theta + \beta t\right) K(\lambda, t, u).$$

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and we calculate the partial derivative of the function  $\xi$  with respect to t

$$\begin{split} \frac{\partial \xi}{\partial t}(\lambda, t, u) &= K(\lambda, t, u) \left( \frac{-\lambda(u - t)}{t^2} + \beta \right) \exp\left( -\lambda \int_c^t \frac{u - \theta}{\theta^2} d\theta + \beta t \right) \\ &+ \frac{\partial}{\partial t} K(\lambda, t, u) \exp\left( -\lambda \int_c^t \frac{u - \theta}{\theta^2} d\theta + \beta t \right). \end{split}$$

Now applying the Equation (4) we have

$$\frac{\partial \xi}{\partial t}(\lambda, t, u) = 0,$$

hence the function  $\xi$  depends only on  $\lambda$  and u.

The normalization condition (5) yields

$$\exp\left(\lambda \int_{c}^{t} \frac{d\theta}{\theta} + \beta t\right) = \int_{0}^{+\infty} \exp\left(\lambda u \int_{c}^{t} \frac{d\theta}{\theta^{2}}\right) C(\lambda, u) du,$$

which is equivalent for c = 1 to the following connection

$$t^{\lambda} \exp(\beta t) = \int_0^{+\infty} \exp\left(-\frac{\lambda u}{t}\right) C(\lambda, u) du.$$

Substituting  $\lambda = \nu + 1$ ,  $\beta = \frac{a}{\lambda} = \frac{a}{\nu + 1}$ ,  $t = \frac{\lambda}{p} = \frac{\nu + 1}{p}$  to the relation above, we get

$$\frac{e^{\frac{a}{p}}}{p^{\nu+1}} = (\nu+1)^{-\nu-1} \int_0^{+\infty} \exp(-pu)C(\nu,u)du. \tag{7}$$

The left-hand side of the equation above can also be expressed using the Laplace transform. It means

$$\frac{e^{\frac{a}{p}}}{p^{\nu+1}} = \int_0^{+\infty} \exp(-pu) \left(\frac{u}{a}\right)^{\frac{\nu}{2}} I_{\nu}(2\sqrt{(\nu+1)\beta u}) du \tag{8}$$

for p > 0 and v > -1, where  $I_v$  is the modified Bessel function of the first kind. When we compare (7) and (8) we immediately have the expression for C(v, u)

$$C(\nu, u) = (\nu + 1) \left(\frac{(\nu + 1)u}{\beta}\right)^{\frac{\nu}{2}} I_{\nu}(2\sqrt{(\nu + 1)\beta u}). \tag{9}$$

Now we are able to write the explicit formula for the kernel of the operator  $W_{\lambda}$  as well as the formula of the operator itself. Using the connections (6) and (9) we obtain

$$K(\lambda, t, u) = \frac{\lambda(\frac{\lambda u}{\beta})^{\frac{\lambda - 1}{2}} I_{\lambda - 1}(2\sqrt{\lambda \beta u})}{t^{\lambda} \exp(\beta t) \exp(\frac{\lambda u}{t})}$$

By the definition of the operator  $W_{\lambda}$  we have

$$W_{\lambda}(f;t) = \int_{0}^{+\infty} K(\lambda,t,u) f(u) du = \frac{\lambda}{t^{\lambda} \exp(\beta t)} \int_{0}^{+\infty} \frac{\left(\frac{\lambda u}{\beta}\right)^{\frac{\lambda-1}{2}} I_{\lambda-1}(2\sqrt{\lambda \beta u})}{\exp\left(\frac{\lambda u}{t}\right)} f(u) du.$$
 (10)

Using the substitution  $2\sqrt{\lambda\beta u} = k$  we can get an equivalent formula to the statement above,

$$W_{\lambda}(f;t) = \frac{1}{(2\beta t)^{\lambda} \exp(\beta t)} \int_{0}^{+\infty} \exp\left(-\frac{k^{2}}{4\beta t}\right) k^{\lambda} I_{\lambda-1}(k) f\left(\frac{k^{2}}{4\beta \lambda}\right) dk.$$

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### 5. A Probabilistic Approach to the Semi-Exponential Operators

This section is devoted to demonstrate how we can interpret the semi-exponential operators in the probabilistic view. In the paper [3] Altomare and Raşa investigate the case of exponential-type operators. The authors consider the operators for functions from the following polynomial weighted spaces

$$P_m = \{ f \in C([0, +\infty)) : w_m f \in CB([0, +\infty)) \}$$

and

$$P_m^0 = \{ f \in C([0, +\infty)) : w_m f \in C_0([0, +\infty)) \}$$

where  $m \ge 1$  and

$$w_m(x) = \frac{1}{1 + x^m}$$

for  $x \ge 0$  is a polynomial weight function. It is easy to observe that  $P_q \subset E_q$  for every  $q \ge 1$ . Assume that p is an analytic function on  $[0, +\infty)$  and fulfilling the following condition

$$p(x) > 0$$
 for  $x > 0$  and  $p(0) = 0$ . (11)

Furthermore we shall assume that there exists a family  $(m_{n,x})_{n\geq 1,x\geq 0}$  of probability Borel measures on  $[0,+\infty)$  such that

$$E_{\infty} \subset \bigcap_{n \ge 1, x \ge 0} L^{1}(m_{n,x}) \tag{12}$$

and the differential equation

$$\frac{d}{dx} \int_{0}^{+\infty} f(u) dm_{n,x}(u) = \frac{n}{p(x)} \int_{0}^{+\infty} (u - x) f(u) dm_{n,x}(u) - \beta \int_{0}^{+\infty} f(u) dm_{n,x}(u)$$
 (13)

is fulfilled for  $f \in E_{\infty}$ ,  $n \ge 1$ ,  $x \ge 0$  and  $\beta$  a non-negative real number, as it was assumed in (4). We consider the Equation (13) in the sense of the theory of generalized functions [15]. If the assumption (12) is fulfilled, then we can define a sequence of positive linear operators on  $E_{\infty}$  by the following formula

$$A_n(f;x) = \int_0^{+\infty} f(u) dm_{n,x}(u).$$
 (14)

It was shown in [2] that if we take p(x) = x for  $x \ge 0$ , then we have the modified exponential operator of Szász–Mirakjan type

$$A_n(f;x) = e^{-(n+\beta)} \sum_{k=0}^{\infty} \frac{(n+\beta)^k x^k}{k!} f\left(\frac{k}{n}\right).$$

In this case

$$m_{n,x} = \sum_{k=0}^{\infty} e^{-(n+\beta)} \frac{(n+\beta)^k x^k}{k!} \varepsilon_{k/n},$$

where  $\varepsilon_{k/n}$  denotes the unit mass concentrated at  $\frac{k}{n}$ .

An interesting example is the operator which we have just introduced in the previous section. We take  $p(x) = x^2$  for  $x \ge 0$  and we have a semi-exponential operator defined by (10), it means that

$$A_n(f;x) = \frac{n}{x^n \exp{(\beta x)}} \int_0^{+\infty} \frac{\left(\frac{nu}{\beta}\right)^{\frac{n-1}{2}} I_{n-1}(2\sqrt{n\beta u})}{\exp{(\frac{nu}{t})}} f(u) du.$$

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Now the measure  $m_{n,x}$  is defined by

$$m_{n,x} = \begin{cases} \varepsilon_0, & x = 0; \\ \mu_{n,x} \lambda_1, & x > 0, \end{cases}$$

where  $\lambda_1$  is the Lebesgue measure on  $[0, +\infty)$  and

$$\mu_{n,x}(u) = \left(\frac{n}{x}\right)^n \exp\left(-\beta x - \frac{nu}{x}\right) \left(\frac{u}{n\beta}\right)^{\frac{n-1}{2}} I_{n-1}(2\sqrt{n\beta u}),$$

 $I_{n-1}$  stands for the modified Bessel function of the first kind. By Definition (14) we have the following

**Remark 1.** *If*  $f \in CB([0,+\infty))$  *then*  $A_n(f) \in CB([0,+\infty))$  *and* 

$$||A_n(f)||_{\infty} \leq ||f||_{\infty}.$$

Moreover, if we take the auxiliary function  $\psi_x(t) = t - x$  for  $t \in [0, +\infty)$  then the differential Equation (13) and the definition (14) yields

$$A_n(e_1f;x) = \left(e_1(x) + \frac{\beta p(x)}{n}\right) A_n(f;x) + \frac{p(x)}{n} \frac{d}{dx} A_n(f;x)$$
 (15)

for  $f \in E_{\infty}$ , where  $e_i(x) = x^i$ . Using the connection above we can achieve the relations of Lemma 2 again, for example  $f \in \{e_0, e_1, e_2\}$ 

$$A_n(e_1; x) = e_1(x) + \frac{\beta p(x)}{n},$$

$$A_n(e_2; x) = \left(e_1(x) + \frac{\beta p(x)}{n}\right) A_n(e_1; x) + \frac{p(x)}{n} \frac{d}{dx} A_n(e_1; x)$$

$$= e_2(x) + \frac{p(x) + 2\beta x p(x)}{n} + \frac{\beta^2 p^2(x) + \beta p(x) p'(x)}{n^2}$$

and additionally

$$A_n(e_3; x) = \left(e_1(x) + \frac{\beta p(x)}{n}\right) A_n(e_2; x) + \frac{p(x)}{n} \frac{d}{dx} A_n(e_2; x)$$

$$= e_3(x) + \frac{3xp(x) + 3\beta x^2 p(x)}{n} + \frac{3\beta p^2(x) + 3x\beta^2 p^2(x) + 3xp(x)p'(x) + p(x)p'(x)}{n^2}$$

$$+ \frac{\beta^3 p^3(x) + 2\beta^2 p^2(x)p'(x) + \beta p(p'(x))^2 + \beta^2 p^2(x)p''(x) + \beta p^2 p''(x)}{n^3}.$$

Notice that the semi-exponential operators  $A_n$  are well-defined and continuous on the exponential weight spaces  $E_q$ . Moreover,  $||L_n|| \le 1 + \frac{K(q)}{n}$  where K(q) > 0 is a constant depending only on q.

**Lemma 3.** Let  $A_n$  be a semi-exponential operator,  $\varphi_m \in C^{\infty}([0, +\infty))$  and for all  $r \geq 0$   $\varphi_m^{(r)} = O(e_{m+1-r})$  as  $x \to +\infty$  then

$$A_n(e_m; x) = e_m(x) + \frac{\varphi_m(x)}{n} \tag{16}$$

for every  $m \geq 1$ .

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**Proof.** Let  $A_n$  be a semi-exponential operator. We use induction on  $m \ge 1$  to prove the lemma. By using (15) for m = 1 and m = 2 we have our statement. Now let us suppose that (16) is true for some  $m \ge 2$ . By using (15) we calculate

$$A_n(e_{m+1};x) = \left(e_1(x) + \frac{\beta p(x)}{n}\right) A_n(e_m;x) + \frac{p(x)}{n} \frac{d}{dx} A_n(e_m;x).$$

By the induction assumption we derive

$$A_n(e_{m+1};x) = e_{m+1}(x) + \frac{\varphi_{m+1}(x)}{n}$$

where

$$\varphi_{m+1}(x) = e_1(x)\varphi_m(x) + mp(x)e_{m-1}(x) + \frac{p(x)\varphi'_m(x)}{n} + \beta e_m(x)p(x) + \frac{\beta p(x)\varphi_m(x)}{n},$$

and  $\varphi_{m+1}^{(r)} = O(e_{m+2-r})$  as  $x \to +\infty$ . This ends the proof of Lemma 3.  $\ \Box$ 

Now we can claim

**Theorem 3.** Let  $A_n$  be a semi-exponential operator and for every  $r \ge 0$   $p^{(r)} = O(e_{1+\varepsilon-r})$  as  $x \to +\infty$  and  $\varepsilon < \frac{1}{2}$  then

- (i)  $A_n(E_q) \subset E_q$  and  $A_n(E_q^0) \subset E_q^0$  for every  $n \ge 1$ .
- (ii) Each  $A_n$  is continuous from  $E_q$  into itself and  $||A_n|| \le 1 + \frac{K(q)}{n}$ .

**Proof.** Let  $n \ge 1$ , q > 0 and  $\varepsilon < \frac{1}{2}$  be fixed. By definition of  $E_q$  for any  $f \in E_q$  we have

$$|f(t)| \le ||f||_q e^{qt} \quad t \ge 0.$$

By linearity and positivity of  $A_n$  and the definition of the norm in  $E_q$  we can write

$$|A_n(f;x)| \le A_n(|f|;x) \le ||f||_q A_n(e^{qt};x) \le ||f||_q \sum_{m=0}^{\infty} \frac{q^m}{m!} A_n(e_m;x)$$

Lemma 3 yields

$$|A_n(f;x)| \le ||f||_q \sum_{m=0}^{\infty} \frac{q^m}{m!} \left( e_m(x) + \frac{\varphi_m(x)}{n} \right).$$

Hence

$$\frac{|A_n(f;x)|}{e^{qx}} \le \|f\|_q \left(1 + \frac{1}{n} \sum_{m=0}^{\infty} \frac{q^m}{m!} \frac{\varphi_m(x)}{e^{qx}}\right) \le \|f\|_q \left(1 + \frac{M}{n} \sum_{m=0}^{\infty} \frac{q^m}{m!} \left(\frac{m+\varepsilon}{qe}\right)^{m+\varepsilon}\right)$$

for some constant M > 0 because  $\varphi_m = O(e_{m+\varepsilon})$  as  $x \to +\infty$ . If we use d'Alembert's ratio test we get the convergence of the series and we can write

$$||A_n(f;\cdot)||_q \le ||f||_q \left(1 + \frac{K(q)}{n}\right)$$

According to the estimation above we conclude that operators  $A_n$  are continuous from  $E_q$  into itself and  $||A_n|| \le 1 + \frac{K(q)}{n}$ .

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To prove the last statement in Theorem 3 we take a function  $f \in E_m^0$  and  $\epsilon > 0$ . From the definition of the space  $E_m^0$  we have

$$|f(t)| < \epsilon e^{qt}$$

for  $t \ge a_1$ . Let  $M = \sup\{|f(t)| : t \in [0, a_1]\}$ . There exists  $b_1 \ge a_1$  such that

$$M \le \epsilon e^{qx}$$

for  $x \ge b_1$ . Now we are prepared to write the following estimation for  $x \ge b_1$ 

$$|A_n(f;x)| \le \int_0^{a_1} |f(t)| dm_{n,x} + \int_{a_1}^{\infty} |f(t)| dm_{n,x} \le M + \epsilon A_n(e^{qt};x).$$

By the normalization condition and the previous estimation for  $A_n(e^{qt};x)$  we get

$$||A_n(f;\cdot)||_q \le \epsilon \left(2 + \frac{K(q)}{n}\right)$$

which proves our assertion in the case  $f \in E_q^0$ .  $\square$ 

**Remark 2.** If we try to use the theorem for the semi-exponential Post–Widder operator it will fail. It is easy to see that in this case the function  $p(x) = x^2$  and it does not fulfill the crucial assumption  $p = O(e_{1+\varepsilon})$  as  $x \to +\infty$ . On the other hand we recall the semi-exponential Szász–Mirakjan operator for p(x) = x and in this case the assumption is fulfilled.

#### 6. Conclusions

Someone could ask how we apply the new semi-exponential operator we have just derived. According to the applications of exponential-type operators, it seems to have lots of them. Now we are going to focus our attention on applying it in the theory of differential equations. Notice that in this paper we investigate the semi-exponential operators in spaces  $E_q$  which are larger then the spaces  $P_n$  considered in [3].

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