Vijay Gupta · Ravi P. Agarwal

Convergence Estimates in Approximation Theory

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Vijay Gupta
School of Applied Sciences
Netaji Subhas Institute of Technology
New Delhi, India

Ravi P. Agarwal
Department of Mathematics
Texas A&M University - Kingsville
Kingsville, TX, USA

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Preface

The aim of the general approximation methods concerning linear positive operators is to deal with convergence behavior. The accuracy can be ascertained to a desired degree by applying different methods. We are also concerned with the amount of computation required to achieve this accuracy. A direct theorem provides the order of approximation for functions of specified smoothness. The converse of the direct result, that is, the inverse theorem, infers the nature of smoothness of the function from its order of approximation. Asymptotic analysis is a key tool for exploring the ordinary and partial differential equations that arise in the mathematical modeling of real-world phenomena. The rate of convergence infers the speed at which a convergent sequence approaches its limit.

This work treats the convergence results mainly for linear positive operators. After the well-known theorem due to Weierstrass and the important convergence theorem of Korovkin, many new operators were proposed and constructed by several researchers. The theory of these operators has been an important area of research in the last few decades. The basic results and direct estimates in both local and global approximation are also presented. We also discuss the asymptotic expansion of some of the linear positive operators, which is important for convergence estimates. We know that to improve the order of approximation, we can consider the combinations. By considering the linear combinations, we have to slacken the positivity conditions of the operators. Here we discuss linear and iterative combinations and present some results. Some operators reproduce constant and linear functions. Two decades ago it was observed that if we modify the original operators, we can have a better approximation. Later some researchers studied this direction and observed that some operators that do not even preserve linear functions can give a better approximation if they are modified to preserve linear functions. The overconvergence phenomena for certain operators, which were not discussed in the book by Gal, [77] are also discussed here. We present some results of overconvergencies to larger sets in the complex plane.

In this book, the crucial role of the rate of convergence for functions of bounded variation and for functions having derivatives of bounded variation is emphasized. New and efficient methods that are applicable to general operators are also discussed. The advantages of these methods consist of obtaining improved and even optimal estimates, as well as of broadening the applicability of the results. Several results have been established for different exponential-type operators and integral operators. Still, this type of study can be extended to other generalizations of the known operators. Also, the rate of convergence for functions of bounded variation on mixed summation–integral-type operators having different basis functions has not been studied to date, including Szász–Baskakov, Baskakov–Szász, and Beta–Szász operators, among others. It can be considered an open problem. We mention some results without proofs, while in other cases, proofs and outlines are given. The book is useful for beginners and for those who are working on analysis and related areas.

This monograph is divided into 11 chapters:

In Chap. 1, which is introductory in nature, we present some basic definitions and the standard theorems, which are important for the convergence point of view.

In Chap. 2, we discuss some important results on linear positive operators related to convergence. Some direct results, which include pointwise convergence, asymptotic formulas, and error estimations, are presented here. We also deal with some important results, including discretely defined operators, Kantorovich- and Durrmeyer-type operators, mixed summation–integral-type operators, Phillips operators, and other integral-type operators.

In Chap. 3, we discuss the asymptotic behavior of some of the linear positive operators. We discuss the complete asymptotic expansion of the Baskakov–Kantorovich, Szász–Baskakov, Meyer–König–Zeller–Durrmeyer, and Beta operators.

In Chap. 4, we mention approximation for certain combinations. The linear combinations are no longer positive operators. Here we study some results for linear and iterative combinations. Also, we consider a different form of the linear combinations and present direct estimates for combinations of Szász–Baskakov operators.

In Chap. 5, we present the techniques for getting a better approximation. Many well-known approximating operators L_n, preserve the constant as well as linear functions. In 1983, King considered the modification of the classical Bernstein polynomials so that the modified form preserves the test function e_2. A better approximation can be achieved for the modified form. In this chapter, we present some results for different operators discussed in recent years.

In Chap. 6, we study the overconvergence phenomenon for certain operators that were not discussed in the book by Gal [77]. We present some results of overconvergencies to larger sets in the complex plane than in the real domain. We discuss some very recent results on complex Baskakov–Stancu operators, complex Favard–Szász–Mirakjan–Stancu operators, complex Beta operators of the second kind, genuine Durrmeyer–Stancu polynomials, and certain complex Durrmeyer-type operators.

In Chap. 7, the pointwise approximation properties of some approximation operators of the probabilistic type are established and studied. The rates of convergence for Legendre–Fourier series, Hermite–Fejér polynomials, exponential-type operators (which include Bernstein, Szász, Baskakov, Gamma, and Weierstrass operators) are also discussed. The rates of convergence of such operators for a bounded variation function f are given at those points x, where $f(x+)$ and $f(x-)$ exist. Here we also present some results on the Durrmeyer variants of such operators. We also present the related results for a general class of summation–integral-type operators and the Meyer–König and Zeller operators.

In Chap. 8, we discuss various Bézier variants of the approximation operators. They have close relationships with the fields of geometry modeling and design. In this chapter, we present some important results on the approximation of Bézier variants of a series of approximation operators, which include Bézier variants of the Bernstein, Bleimann–Butzer–Hann, Balazs–Kantorovich, Szász–Kantorovich, Baskakov, Baskakov–Kantorovich, Baskakov–Durrmeyer, and MKZ operators, among others.

In Chap. 9, we study some other related results. We present the rate of approximation on nonlinear operators. Also, we discuss the rate of convergence in terms of Chanturiya's modulus of variation. We also present some results for bounded and absolutely continuous functions. Finally, we present the rate of convergence for functions having derivatives coinciding a.e. with the function of bounded variation.

In Chap. 10, we present results related to simultaneous approximation. Although lot of papers have appeared on such topic, in this chapter we mention the rate of simultaneous approximation for certain Durrmeyer–Bézier operators for functions of bounded variation. We also present the rate of convergence in simultaneous approximation for certain summation–integral-type operators having derivatives of bounded variation.

In the brief final chapter, Chap. 11, we present the future scope of related topics and mention some open problems.

New Delhi, India — Vijay Gupta
Kingsville, TX, USA — Ravi P. Agarwal

Acknowledgments

I dedicate this book in the memory of my father, the late P. D. Gupta (1935–1996), who inspired and encouraged me to opt for mathematics as my main subject at an early stage. I am indebted to my mother, Uma Gupta, for her constant blessings. I appreciate the constant support of my wife, Shalini, and my children, Arushi and Anshay, at each stage of this book's writing.

-Vijay Gupta

I dedicate this book in the memory of my father, the late Radhey Shiam Agarwal.

-Ravi P. Agarwal

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Chapter 1
Preliminaries

Approximation theory has been an established field of mathematics in the past century. This chapter deals with the basic definitions and standard theorems of approximation theory, which are important for the convergence point of view. A polynomial function is a function $P(x)$ defined on a real line $\mathbb{R}$ by

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n; x \in \mathbb{R},$$

where n is a nonnegative integer and $a_0, a_1, \ldots, a_n$ are real numbers. The degree of a polynomial is n if $a_n \neq 0$.

Let us denote by H_n the set of all polynomials whose degree does not exceed n. Let $f \in C[a,b]$ and let $P(x)$ be an arbitrary polynomial. We set

$$\Delta(P) = \max_{a \leq x \leq b} |P(x) - f(x)|.$$

Also, if

$$E_n = E_n(f) = \inf_{P \in H_n} \{\Delta(P)\},$$

then the quantity E_n is said to be the best approximation to $f(x)$ by the polynomials belonging to H_n.

A linear positive operator is a function L that has the following properties:

1. The domain D of L is a nonempty set of real functions all having the same real domain T.
2. For all $f \in D$, $L(f)$ is again a real function with domain T.
3. If $f, g \in D$ and $a, b \in R$, then $af + bg \in D$ and

$$L(af + bg) = aL(f) + bL(g).$$

4. If $f \in D$ and $f(x) \geq 0$, for all $x \in T$, then $L(f(x)) \geq 0$, for all $x \in T$.

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In order to decide the uniform convergence of a sequence of linear positive operators to the continuous functions, Bhoman [42] and Korovkin [174] found the answer by giving a very simple criterion.

1.1 Korovkin's Theorem

For a linear positive operator L_n, Korovkin [174] established the following important theorem:

Theorem 1.1 ([174]). *If the three conditions*

$$L_n(1,x) = 1 + \alpha_n(x),$$

$$L_n(t,x) = x + \beta_n(x),$$

$$L_n(t^2,x) = x^2 + \gamma_n(x)$$

are satisfied for the sequence of linear positive operators $L_n(f,x)$, where $\alpha_n(x), \beta_n(x)$, and $\gamma_n(x)$ converge uniformly to zero in the interval $[a,b]$, then the sequence $L_n(f,x)$ converges uniformly to the function $f(x)$ in this interval if $f(t)$ is bounded, continuous in the interval $[a,b]$, continuous on the right at point b, and continuous on the left at point a.

Proof. Since the function $f(t)$ is bounded [i.e., $|f(t)| < M$], then for all x and t, the inequality

$$-2M < f(t) - f(x) < 2M \tag{1.1}$$

holds. Now, if the function $f(x)$ is continuous on the interval $[a,b]$, continuous on the right at point b, and continuous on the left at point a, then we can find a $\delta > 0$ for $\epsilon > 0$ such that the inequality

$$-\epsilon < f(t) - f(x) < \epsilon \tag{1.2}$$

holds if $|t-x| < \delta, x \in [a,b]$. If we set $\psi(t) = (t-x)^2$, where $x \in [a,b]$ is fixed, inequalities (1.1) and (1.2) imply

$$-\epsilon - \frac{2M}{\delta^2}\psi(t) < f(t) - f(x) < \epsilon + \frac{2M}{\delta^2}\psi(t).$$

In view of the monotonicity and linearity of $L_n(f,x)$, these inequalities imply

$$\begin{aligned} -\epsilon L_n(1,x) - \frac{2M}{\delta^2}L_n(\psi,x) &\le L_n(f,x) - f(x)L_n(1,x) \\ &\le \epsilon L_n(1,x) + \frac{2M}{\delta^2}L_n(\psi,x). \end{aligned} \tag{1.3}$$

Next, by the linearity of $L_n(f, x)$, we have

$$\begin{aligned} L_n(\psi, x) &= L_n(t^2 - 2tx + x^2, x) = L_n(t^2, x) - 2xL_n(t, x) + x^2 L_n(1, x) \\ &= x^2 + \gamma_n(x) - 2x\,(x + \beta_n(x)) + x^2\,(1 + \alpha_n(x)) \\ &= \gamma_n(x) - 2x\beta_n(x) + x^2\alpha_n(x) = \delta_n(x), \end{aligned} \tag{1.4}$$

where $\delta_n(x)$ converges uniformly to zero in the interval $[a, b]$.

From the above equation and the first condition of Korovkin's theorem, we see that the right-hand side of inequality (1.3) converges uniformly to ϵ and the left-hand side to $-\epsilon$ in the interval $[a, b]$. Thus, there exists a number $N(\epsilon)$ such that the inequality

$$-2\epsilon < L_n(f, x) - f(x)L_n(1, x) < 2\epsilon$$

will hold if $n > N(\epsilon)$, $x \in [a, b]$. Finally, since $\epsilon > 0$ is arbitrary, the sequence

$$L_n(f, x) - f(x)L_n(1, x)$$

will converge uniformly to zero in the interval $[a, b]$, and hence we conclude that the sequence $L_n(f, x)$ converges uniformly to $f(x)$ in the interval $[a, b]$. ■

Definition 1.1. The function

$$T_n(x) = a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

is called a trigonometric polynomial of order n if $a_n^2 + b_n^2 \neq 0$, and the series

$$\frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \cdots + (a_n \cos nx + b_n \sin nx) + \cdots$$

is called a trigonometric series.

Obviously, the product of two trigonometric polynomials of degree m and n, respectively, is a polynomial of degree $m + n$.

Another important result given by Korovkin [174] for periodic functions is the following:

Theorem 1.2 ([174]). *If the three conditions*

$$L_n(1, x) = 1 + \alpha_n(x),$$

$$L_n(t, x) = x + \beta_n(x),$$

$$L_n(t^2, x) = x^2 + \gamma_n(x)$$

are satisfied for the sequence of linear positive operators $L_n(f, x)$*, where* $\alpha_n(x), \beta_n(x)$*, and* $\gamma_n(x)$ *converge uniformly to zero in the interval* $[a, b]$*, then the sequence* $L_n(f, x)$ *converges uniformly to the function* $f(x)$ *in this interval if* $f(t)$ *is bounded, has period* 2π*, is continuous in the interval* $[a, b]$*, is continuous on the right at point* b*, and is continuous on the left at point* a*.*

Shisha and Mond [212] obtained Korovkin's result in the quantitative form and established the following theorem:

Theorem 1.3 ([212]). *Let* $\infty < a < b < \infty$*, and let* $L_1, L_2, \ldots$ *be positive operators all having the same domain* D*, which contains the restrictions of* $1, t, t^2$ *to* $[a, b]$*. For* $n = 1, 2, \ldots$*, suppose* $L_n(1)$ *is bounded. Let* $f \in D$ *be continuous in* $[a, b]$*, with modulus of continuity* ω*. Then, for* $n = 1, 2, \ldots$*,*

$$||f - L_n(f)|| \leq ||f||.||L_n(1) - 1|| + ||L_n(1) + 1||\omega(\mu_n),$$

where $\mu_n = ||(L_n([t - x]^2))(x)||^{1/2}$*, and* $||.||$ *stands for the sup-norm over* $[a, b]$*. In particular, if* $L_n(1) = 1$*, then the conclusion becomes*

$$||f - L_n(f)|| \leq 2\omega(\mu_n).$$

In another paper, Shisha and Mond [211] gave the full proof of the following theorem:

Theorem 1.4 ([212]). *Let* $L_1, L_2, \ldots$ *be positive operators whose common domain* D *consists of real functions with domain* $(-\infty, \infty)$*. Suppose* $1, \cos x, \sin x, f$ *belong to* D*, where* f *is an everywhere continuous,* 2π*-periodic function, with modulus of continuity* ω*. Let* $-\infty < a < b < \infty$*, and suppose that for* $n = 1, 2, \ldots, L_n(1)$ *is bounded in* $[a, b]$*. Then for* $n = 1, 2, \ldots$*,*

$$||f - L_n(f)|| \leq ||f||.||L_n(1) - 1|| + ||L_n(1) + 1||\omega(\mu_n),$$

where $\mu_n = \pi|| \left(L_n \sin^2 \frac{t-x}{2}\right)(x)||^{1/2}$*, and* $||.||$ *stands for the sup-norm over* $[a, b]$*. In particular, if* $L_n(1) = 1$*, the conclusion becomes*

$$||f - L_n(f)|| \leq 2\omega(\mu_n).$$

1.2 Weierstrass Approximation Theorems

The theory of approximation on linear positive operators $P(x)$ concerns how the functions can be best approximated with simpler functions and quantitatively characterizes the errors induced in so doing. The goal is to minimize the maximum value of $|f(x) - P(x)|$.

On the convergence of linear positive operators, the most important basic result is due to the German mathematician Karl Weierstrass (1815–1897), who established an important theorem, namely, the *Weierstrass approximation theorem*. There are many proofs available of this important theorem. The early ones are due to Runge (1885), Picard (1891), Lerch (1892 and 1903), Volterra (1897), Lebesgue (1898), Mittag-Leffler (1900), Fejér (1900 and 1916), Landau (1908), De la Vallée Poussin (1908), Jackson (1911), Sierpinski (1911), Bernstein (1912), and Montel (1918) (see e.g. [LNT] and the references therein). We mention in this chapter the proof by Korovkin [174], which is based on the Bernstein polynomials.

Theorem 1.5 ([174] Weierstrass' first approximation theorem). *If f is a continuous function defined on $[a, b]$, then for a given $\epsilon > 0$, there is a polynomial $P(x)$ on $[a, b]$ such that $|f(x) - P(x)| < \epsilon$, for all $x \in [a, b]$.*

Proof. For a function f defined on $[0, 1]$, consider the nth-degree Bernstein polynomial defined by

$$(B_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right).$$

The first three moments are calculated as

$$B_n(1, x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = [x + (1-x)]^n = 1.$$

Next, the first-order moment is given by

$$\begin{aligned} B_n(t, x) &= \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \frac{k}{n} \\ &= \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} x^k (1-x)^{n-k} \\ &= \sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-k-1)!} x^{k+1} (1-x)^{n-k-1} \\ &= x B_{n-1}(1, x) = x. \end{aligned}$$

The second-order moment of the Bernstein polynomial is given by

$$B_n(t^2, x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \frac{k^2}{n^2}$$

$$
\begin{aligned}
&= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} \left(\frac{k(k-1)+k}{n^2} \right) \\
&= \frac{1}{n^2} \left[\sum_{k=2}^{n} \frac{n!}{(k-2)!(n-k)!} x^k (1-x)^{n-k} \right. \\
&\quad \left. + \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} x^k (1-x)^{n-k} \right] \\
&= \frac{1}{n^2} \left[\sum_{k=0}^{n-2} \frac{n!}{k!(n-k-2)!} x^{k+2} (1-x)^{n-k-2} \right. \\
&\quad \left. + \sum_{k=0}^{n-1} \frac{n!}{k!(n-k-1)!} x^{k+1} (1-x)^{n-k-1} \right] \\
&= \frac{1}{n^2} \left[n(n-1)x^2 B_{n-2}(1,x) + nxB_{n-1}(1,x) \right] \\
&= \frac{1}{n^2} \left[n(n-1)x^2 + nx \right] = x^2 + \frac{x(1-x)}{n}.
\end{aligned}
$$

Thus, $B_n(e_i(t), x)$ converges uniformly to $e_i(x)$ in the interval $[0, 1]$, where $e_i(t) = t^i, i = 0, 1, 2$. Therefore, by Theorem 1.1, $B_n(f, x)$ converges uniformly to the function $f(x)$ in the interval $[0, 1]$. Now we shall generalize the result for the interval $[a, b]$. In fact, let the function $f(x)$ be continuous in the interval $[a, b]$; then we introduce the function

$$\phi(y) = f[a + y(b-a)].$$

Since the function $\phi(y)$ is continuous in the interval $[0, 1]$, there is a polynomial $P(y)$ such that

$$|P(y) - \phi(y)| < \epsilon, y \in [0, 1].$$

Substituting

$$y = \frac{x-a}{b-a}, Q(x) = P\left(\frac{x-a}{b-a}\right),$$

and because

$$\phi\left(\frac{x-a}{b-a}\right) = f\left(a + (b-a)\frac{x-a}{b-a}\right) = f(x),$$

we obtain

$$\epsilon > \left| P\left(\frac{x-a}{b-a}\right) - \phi\left(\frac{x-a}{b-a}\right) \right| = |Q(x) - f(x)|, x \in [a, b].$$

This completes the proof of the theorem. ■

Subsequently, M. H. Stone generalized the above theorem to compact subsets of $\mathbb{R}^n$ known as the *Stone–Weierstrass theorem*. In the case of the trigonometric polynomials, Weierstrass gave the following theorem.

Theorem 1.6 ([174] Weierstrass' second approximation theorem). *If a function $f(x)$ has period 2π and is continuous on the real axis, then we can find a trigonometric polynomial $T(x)$ for $\epsilon > 0$ such that the inequality*

$$|T(x) - f(x)| < \epsilon, -\pi < x < \pi$$

holds.

The simple proof of this second theorem was established by De La Vallée-Poussin [236].

Definition 1.2. For $f \in C_{2\pi}$, the De La Vallée-Poussin singular integral is given by

$$V_n(x) = \frac{(2n)!!}{(2n-1)!!}\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^{2n}\frac{t-x}{2} f(t)dt,$$

where

$$n!! = \begin{cases} n(n-2)\cdots 2, & n \text{ even}, \\ n(n-2)\cdots 1, & n \text{ odd}. \end{cases}$$

Theorem 1.7. *If $V_n(x)$ denotes the De La Vallée-Poussin singular integral (Definition 1.2), then for real values,*

$$\lim_{n\to\infty} V_n(x) = f(x)$$

holds uniformly.

Definition 1.3. A function defined by

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x),$$

where $0 < a < 1$ and b denotes an odd positive integer, is called the Weierstrass function, which is continuous everywhere but differentiable nowhere.

1.3 Order of Approximation

For practical purposes, not only is the fact of the uniform convergence of a sequence of polynomials important, but the speed of convergence is more important; that is, for the polynomial $P_n(x)$ of degree n, it is important how fast e_n approaches zero, where

$$e_n = \max_{a \le x \le b} |P_n(x) - f(x)|.$$

Definition 1.4. Let $L_n(f,x) = \int_a^b K_n(x,t)f(t)dt$, where $K_n(x,t) \ge 0$ is the kernel of distribution, be a positive operator on $C(a,b)$ (a,b may be $\pm\infty$) into C^∞; $L_n(f,x)$ is said to be an exponential-type operator if

- $$\int_a^b K_n(x,t)dt = 1$$

- $$\frac{\partial}{\partial x}K_n(x,t) = \frac{n}{p(x)}K_n(x,t)(t-x),$$

  where $p(x)$ is a polynomial of degree ≤ 2, $p(x) > 0$ on (a,b).
  It is said to be regular if it further satisfies

- $$\int_a^b K_n(x,t)dt = a(n),$$

  where $a(n)$ is a rational function of $n, a(n) \to 1$ as $n \to \infty$.

Definition 1.5. We say $f(n) = O(g(n))$, defined as Big-O, if and only if there exist n_0 and a positive constant C, such that $|f(n)| \le C\,|g(n)|, n > n_0$.

We say $f(n) = o(g(n))$, defined as small-o, if $\lim\limits_{n\to\infty} \dfrac{f(n)}{g(n)} = 0$.

Definition 1.6. The modulus of continuity of f is defined as

$$\omega_f(\delta) \equiv \omega(f,\delta) = \sup_{\substack{x-\delta \le t \le x+\delta \\ t\in[a,b]}} |f(t) - f(x)|\,.$$

Let $f \in C[a,b], k \ge 1$, and $\delta > 0$. Then the kth-order modulus of continuity of f is defined by

$$\omega_k(f,\delta,a,b) = sup\{|\triangle_h^k f(x)| : |h| \le \delta, x, x+kh \in [a,b]\},$$

where $\triangle_h^k f(x)$ is the kth-order forward difference of $f(x)$ with step length h.

- If n is a natural number, then

$$\omega_f(n\delta) \leq n\omega_f(\delta).$$

- If $\lambda > 0$, where λ is any positive real number, then

$$\omega_f(\lambda\delta) \leq (\lambda + 1)\omega_f(\delta).$$

- If $\omega_f(\delta)$ is an increasing function of δ and if we let $0 < \delta_1 < \delta_2$, then we have

$$\omega_f(\delta_1) \leq \omega_f(\delta_2).$$

- If $f(x)$ is uniformly continuous on (a, b), then it is necessary and sufficient that

$$\lim_{\delta\to 0} \omega_f(\delta) = 0.$$

Definition 1.7. A function f defined on an interval $[a, b]$ is said to satisfy the Lipschitz condition of exponent α and the coefficient M if it satisfies the condition

$$|f(x) - f(y)| \leq M|x - y|^\alpha, \alpha > 0.$$

In this case, we say that $f \in Lip_M\alpha, \alpha > 0$. In other words, if the inequality $\omega(f, \delta) \leq C\delta^\alpha$ holds in the interval $[a, b]$, then we say f belongs to the class $Lip_M\alpha, \alpha > 0$.

Obviously, a function satisfying a Lipschitz condition is uniformly continuous. If $f \in Lip\ \ \alpha$ and $\alpha > 1$, then f is a constant function.

Definition 1.8. A function f is said to belong to the Dini–Lipschitz class if the equality

$$\lim_{\delta\to 0} \omega(f, \delta) \ln \frac{1}{\delta} = 0$$

holds.

Definition 1.9. Let us assume that $0 < a_1 < a_2 < b_2 < b_1 < \infty$. For sufficiently small $\eta > 0$, we define the linear approximating function, namely, the Steklov mean $f_{\eta,m}$ of mth-order corresponding to $f \in C_\alpha[0, \infty) \equiv \{f \in C[0, \infty) : |f(t)| \leq C.[g(t)]^\alpha, C > 0\}$ [where $g(t)$ may be some growth function with norm-$||.||_\alpha$ on $C_\alpha[0, \infty)$ that is defined as $||f||_\alpha = \sum_{t\in(0,\infty)} |f(t)|(g(t))^{-\alpha}$] by

$$f_{\eta,m}(x) = \eta^{-m} \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} (f(x) - \Delta_t^m f(x))dt_1 \cdots dt_m,$$

where $t = m^{-1}\sum_{i=1}^{m} t_i, t \in [a_1, b_1]$ and $\Delta_t^m f(x)$ is the mth forward difference of f with step length t. It is easily observed that the Steklov mean satisfies the following properties (see [153]):

(i) $f_{\eta,m}$ has continuous derivatives up to order m on $[a_1, b_1]$.
(ii) $||f_{\eta,m}^{(r)}||_{C[a_2,b_2]} \leq C_1\eta^{-r}\omega_r(f, \eta, a_2, b_2), r = 1, 2, \cdots, m.$
(iii) $||f - f_{\eta,m}||_{C[a_2,b_2]} \leq C_2\omega_m(f, \eta, a_1, b_1).$
(iv) $||f_{\eta,m}||_{C[a_2,b_2]} \leq C_3||f||_\alpha,$

where $C_i, i = 1, 2, 3$ are certain constants that depend on a_1, a_2, b_1, b_2 but are independent of f and n, and $\omega_m(f, \eta, a_1, b_1)$ is the modulus of continuity of order m corresponding to f.

Definition 1.10. Let $C_B[0, \infty)$ be the space of all real-valued continuous bounded functions f on $[0, \infty)$, endowed with the norm $||f|| = \sup_{x\in[0,\infty)} |f(x)|$. The Peetre K-functional is defined by

$$K_2(f, \delta) = \sum_{k=0}^{\infty} \inf\{||f - g|| + \delta||g'|| : g \in W_\infty^2\},$$

where $W_\infty^2 = \{g \in C_B[0, \infty) : g', g'' \in C_B[0, \infty)\}$. By the DeVore–Lorentz property, Theorem 2.4 of [60], there is an absolute constant $M > 0$ such that

$$K_2(f, \delta) \leq M\omega_2(f, \sqrt{\delta}),$$

where $\omega_2(f, \sqrt{\delta})$ is the modulus of continuity of order 2.

Theorem 1.8 ([174] First Jackson theorem). *For any continuous periodic function f, the inequality*

$$E_n(f) = ||T_n(f, .) - f|| \leq 6\omega(f, n^{-1})$$

holds, where $T_n(f, x)$ is a trigonometric polynomial of degree $m \leq n$, which deviates the least from the function $f(x)$.

In the following theorem, we denote by $T_{n,0}(f, x)$ a trigonometric polynomial of degree not greater than n and without an absolute term, which deviates the least from the function f. Also, its deviation from the function is $E_{n,0}(f)$.

Theorem 1.9 ([174] Second Jackson theorem). *If the derivative of a periodic function f is continuous and*

$$E_{n,0}(f') = ||T_{n,0}(f', .) - f'|| = \max_{-\pi\leq x\leq\pi} |T_{n,0}(f', x) - f'(x)|,$$

then

$$E_n(f) \leq \frac{6E_{n,0}(f')}{n}.$$

For algebraic polynomials, Jackson gave the following theorems:

Theorem 1.10 ([174] Third Jackson theorem). *If the function f is continuous in the interval* $[-1, 1]$*, then*

$$E_n(f) = ||P_n(f, .) - f|| \leq 6\omega(f, n^{-1}),$$

where $\omega(f, \delta)$ is the modulus of continuity of the function f in this interval, and $P_n(f, x)$ is an algebraic polynomial of degree not greater than n, which deviates the least from the function f.

Theorem 1.11 ([174] Fourth Jackson theorem). *If the function f is m times differentiable in the interval $[-1, 1], m \geq 1$, and $f^{(m)}(x)$ is continuous, then the inequality*

$$E_n(f) \leq \frac{6^m E_{n-m}(f^{(m)})}{n(n-1)\cdots(n-m+1)}$$

holds, where $n > m$.

1.4 Differential Properties of Function

In this section, we mention certain results (without proof) in the form of the definitions; for details, we refer the reader to [174].

Definition 1.11 (Bernstein's first inequality). Let

$$T_n(x) = A + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$$

be an nth-order trigonometric polynomial. Then for $|T_n(x)| \leq M$, the inequality

$$|T_n^{(s)}(x)| \leq 2^s n^s M$$

holds.

Definition 1.12 (Bernstein's second inequality). Let $P_n(x)$ be an algebraic polynomial of degree n, $|P_n(x)| \leq M, a \leq x \leq b$; then the inequality

$$|P_n^{(m)}(x)| \leq Dn^m M, a' \leq x \leq b',$$

holds, where the constant D does not depend on n.

Definition 1.13 (Bernstein's inequality). For the Bernstein-type operators $L_n(f, x)$, Ditzian–Totik [62] provided the following Bernstein inequality:

$$||\varphi^{2r} L_n^{(2r)} f||_p \leq C n^r ||f||_p,$$

where $\varphi(x)$ denotes the step-weight function.

Definition 1.14 (First Bernstein theorem). If the inequality

$$E_n(f) = |T_n(f,.) - f|| \leq \frac{A}{n^\alpha}, n = 1, 2, \ldots, 0 < \alpha < 1,$$

holds for the continuous and periodic function $f(x)$, then the function f belongs to the class $Lip_s \alpha$.

Definition 1.15 (Second Bernstein theorem). If the inequality

$$E_n(f) = |T_n(f,.) - f|| \leq \frac{A}{n^{k+\alpha}}, n = 1, 2, \ldots, 0 < \alpha < 1,$$

holds for the periodic function $f(x), k$ is a natural number, and then the function f is differentiable k times and $f^{(k)}$ belongs to the class $Lip_s \alpha$.

Definition 1.16 (Third Bernstein theorem). If the condition

$$E_n(f) \leq \frac{A}{n^{k+\alpha}}, n = 1, 2, \ldots, 0 < \alpha < 1,$$

where k is a natural number or 0 is satisfied for a function $f(x)$ given in the interval $[a, b]$, then the function $f(x)$ is k times differentiable in the interval $[a', b'], a < a' < b' < b$, and $f^{(k)}(x) \in Lip\alpha$.

Definition 1.17. By Zygmund class Z, we mean the class of functions f that satisfy the condition

$$|f(x+h) + f(x-h) - 2f(x)| \leq Mh, h > 0.$$

Definition 1.18 (Zygmund's theorem). In order for the inequality

$$E_n(f) \leq \frac{A}{n}, n \in \mathbb{N},$$

to hold, it is necessary and sufficient for the periodic function f to belong to the class Z.

Definition 1.19. Let $\omega_k(f, \delta, a, b)$ be the kth-order modulus of continuity. Then the generalized Zygmund class $\mathrm{Liz}(\alpha, k, a, b)$ is the class of functions f on $[a, b]$ for which

$$\omega_{2k}(f, \delta, a, b) \leq C\delta^{\alpha k},$$

where C is the constant.

1.5 Notations and Inequalities

Definition 1.20. A function f is said to be convex on $[a,b]$ if, for any $x_1, x_2 \in [a,b]$,

$$\lambda f(x_1) + (1-\lambda) f(x_2) \geq f(\lambda x_1 + (1-\lambda) f(x_2)),$$

for any $\lambda \in [0,1]$.

Definition 1.21. For $1 \leq p < \infty$, the space $L_p[a,b]$ is defined as the class of all complex-valued functions for which $\int_a^b |f(x)|^p dx < \infty$, where the integration is taken in the Lebesgue sense. The norm in $L_p[a,b]$ is defined by

$$\|f\|_{L_p[a,b]} = \left(\int_a^b |f(x)|^p dx \right)^{1/p},$$

and the functions equal almost everywhere (a.e.) are identified. The space $L_\infty[a,b]$ consists of the complex-valued measurable functions that are essentially bounded and is normed by

$$\|f\|_{L_\infty[a,b]} = \inf\{M : |f(x)| \leq M \text{ a.e. on } [a,b]\}.$$

Definition 1.22. The following are the basic inequalities:

(i) Hölder's inequality for summation:

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \left(|x_i|^p \right)^{1/p} \left(|y_i|^q \right)^{1/q},$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

(ii) Hölder's inequality for integration:

$$\int |x_i y_i| \leq \left(\int |x_i|^p \right)^{1/p} \left(\int |y_i|^q \right)^{1/q},$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

(iii) Cauchy–Schwarz's inequality for summation:

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \left(|x_i|^2 \right)^{1/2} \left(|y_i|^2 \right)^{1/2}.$$

(iv) Cauchy–Schwarz's inequality for integration:

$$\int |x_i y_i| \leq \left(\int |x_i|^2\right)^{1/2} \left(\int |y_i|^2\right)^{1/2}.$$

(v) Minkowski's inequality for summation:

$$\sum_{i=1}^{\infty} \left(|x_i + y_i|^p\right)^{1/p} \leq \left(|x_i|^p\right)^{1/p} + \left(|y_i|^p\right)^{1/p}.$$

(vi) Minkowski's inequality for integration:

$$\int \left(|x_i + y_i|^p\right)^{1/p} \leq \left(\int |x_i|^p\right)^{1/p} + \left(\int |y_i|^p\right)^{1/p}.$$

(vii) Jensen's inequality for integration: If ϕ is a convex function and f is integrable on $[a, b]$, then

$$\int \phi(f(t))dt \geq \phi \int (f(t))dt.$$

(viii) Bunyakovaki's inequality for integration is given by

$$\int_a^b f(x)\phi(x)dx \leq \left(\int_a^b f^2(x)dx \int_a^b \phi^2(x)dx\right)^{1/2}.$$

Definition 1.23. The hypergeometric function is given by

$${}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k,$$

and the Pochhammer–Berens confluent hypergeometric function is defined by

$${}_1F_1(a; b; x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{x^k}{k!},$$

with the Pochhammer symbol $(n)_k$ given as

$$(n)_k = n(n+1)(n+2)\cdots(n+k-1).$$

We can write $(1)_k = k!$.

Definition 1.24. We denote by $x^{\underline{i}}$ the falling factorial, which is defined as

$$x^{\underline{i}} = x(x-1)\cdots(x-j+1), x^{\underline{0}} = 1.$$

The rising factorial is denoted by $x^{\bar{i}}$ and is defined as

$$x^{\bar{i}} = x(x+1)\cdots(x+j-1), x^{\bar{0}} = 1.$$

The following theorem is the well-known Berry–Esseen bound for the central limit theorem of probability theory:

Theorem 1.12 ([179]). *Let $\{\xi_k\}_{k=1}^{\infty}$ be a sequence of independent and identically distributed random variables with the expectation $E(\xi_1\} = a_1$ and the variance $E(\xi_1 - a_1)^2 = \sigma^2 > 0$, $E|\xi_1 - a_1|^3 = \rho < \infty$, and let F_n stand for the distribution function of $\sum_{k=1}^{n}(\xi_1 - a_1)/\sigma\sqrt{n}$. Then there is an absolute constant C, $1/\sqrt{2\pi} \leq C < 0.8$ such that for all t and n,*

$$\left| F_n(t) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} du \right| < \frac{C\rho}{\sigma^3\sqrt{n}}.$$

1.6 Bounded Variation

Definition 1.25. Let f be a finite real-valued function defined over a closed interval $[a, b]$. For a given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\left| \sum_{r=1}^{n} [f(b_r) - f(a_r)] \right| < \epsilon$$

whenever $\left| \sum_{r=1}^{n} [(b_r) - (a_r)] \right| < \delta$, where

$$a_1 < b_1 \leq a_2 < b_2 \leq a_3 < b_3 \cdots \leq a_n < b_n.$$

Without altering the meaning of the definition, we replace the condition $\left| \sum_{r=1}^{n} [f(b_r) - f(a_r)] \right| < \epsilon$ with the stronger condition

$$\sum_{r=1}^{n} |f(b_r) - f(a_r)| < \epsilon.$$

Then f is said to be absolutely continuous on the interval $[a, b]$. The collection of all absolutely continuous functions on $[a, b]$ is denoted by $AC([a, b])$. The sum and difference of two absolutely continuous functions are also absolutely continuous. If the two functions are defined on a closed and bounded interval, then their product is also absolutely continuous.

Definition 1.26. Suppose a real-valued function f is defined on a closed interval $[a, b]$. As the interval $[a, b]$ is divided by points $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$, then a partition P of $[a, b]$ is a finite collection

$$P = \{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\}.$$

Suppose $f : [a, b] \to \mathbb{R}$. If P is any partition of $[a, b]$, define

$$V(P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|.$$

We define the total variation of f on $[a, b]$ by

$$V_a^b(f) = \sup_P V(P),$$

where the supremum is taken over all possible partitions of $[a, b]$. In the case $V_a^b(f) < \infty$, we say that f is of bounded variation on $[a, b]$.

The following are some basic properties on functions of bounded variation:

- A monotonic function defined on a closed interval $[a, b]$ is a function of bounded variation.
- The sum, difference, or product of functions of bounded variation is also a function of bounded variations.
- An absolutely continuous function is a function of bounded variation.
- An integral is a function of bounded variation.
- A continuous function may or may not be a function of bounded variation.
- A function of bounded variation is necessarily bounded.
- If a function f is a function of bounded variation over $[a, b]$ and c is any point in $[a, b]$, then f is also of bounded variation over $[a, c]$ and $[c, b]$, and vice versa.
- If f' exists and is bounded on $[a, b]$, then f is of bounded variation on $[a, b]$.

Chapter 2
Approximation by Certain Operators

In the theory of approximation following the well-known Weierstrass theorem, the study on direct results was initiated by Jackson's classical work [160] on algebraic and trigonometric polynomials of best approximation. However, because no general constructional guidelines are available for producing fast analytical approximation methods of a given type, it seems best to start with an appropriate sequence of linear positive operators and then to modify it to suit the desired requirements. Apart from the earlier known examples of linear positive operators, several new sequences and classes of operators were constructed and studied following the preceding idea. In this context, in the past five decades, integral modifications of the well-known operators of the Kantorovich and Durrmeyer types have been introduced and studied. Also, several mixed summation–integral-type operators have been introduced and their approximation behaviors studied. Twenty-five years ago, the q-Bernstein polynomials were introduced; next, the q-analogs of many well-known operators were constructed and their approximation properties were discussed. Some of the results on q-operators were compiled by the present authors in their 2013 book [31]. We will not discuss the q-operators here. In the present chapter, we study the phenomena of ordinary and simultaneous approximation (approximation of derivatives of the functions by the corresponding order derivatives of the operators). We present some of the direct approximation theorems in ordinary and simultaneous approximation.

2.1 Discretely Defined Operators

In this section, we mention some of the discretely defined operators and their approximation results. For $f \in C[0,1]$, the n-degree Bernstein polynomials are defined as

V. Gupta and R.P. Agarwal, *Convergence Estimates in Approximation Theory*,
DOI 10.1007/978-3-319-02765-4_2,

$$B_n(f,x) := \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), x \in [0,1], \tag{2.1}$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

It is well known that the Bernstein polynomials satisfy the following properties:

1. For all $n \geq 1$,

   $$B_n(f,0) = f(0), \quad B_n(f,1) = f(1),$$

   so that a Bernstein polynomial for f interpolates f at both endpoints of the interval $[0,1]$.
2. The Bernstein polynomial is obviously linear since it follows that

   $$B_n(af + bg, x) = aB_n(f,x) + bB_n(g,x),$$

   for all functions f and g defined on $[0,1]$ and a, b real.
3. The Bernstein polynomials can be expressed in the form of forward differences as

   $$B_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} \Delta_h^k f(0) x^k,$$

   where Δ is the forward difference operator with step size $h = 1/n$.
4. If $f(x)$ is convex on $[0,1]$, then for all $0 \leq x \leq 1$, we have

   $$B_n(f;x) \geq f(x), n \geq 1$$

   and

   $$B_{n-1}(f;x) \geq B_n(f,x), n \geq 2.$$

In pointwise convergence, the error term incorporates a function at the point under consideration and indicates bias of the method toward certain points, usually at the endpoints of the interval. The errors in the approximation could be with respect to a certain functional norm, seminorm, pointwise local or global. The common characteristic of a global result is the involvement with the whole domain of the functions, while with a local result it concerns limited subsets of the domain. It is obvious that the Weierstrass approximation theorem is just a corollary of the following theorem:

Theorem 2.1 ([37] Pointwise convergence). *For a function $f(x)$ bounded on* $[0,1]$*, the relation*

$$\lim_{n\to\infty} B_n(f,x) = f(x)$$

holds at each point of continuity x of f, and the relation holds uniformly on $[0,1]$ *if $f(x)$ is continuous on this interval* $[0,1]$.

Theorem 2.2 ([181, 239] Voronovskaja's asymptotic formula). *Let $f(x)$ be bounded in* $[0,1]$*, and suppose that the second derivative $f''(x)$ exists at a certain point x of* $[0,1]$*. Then*

$$\lim_{n\to\infty} n[B_n(f,x) - f(x)] = \frac{x(1-x)}{2} f''(x).$$

Theorem 2.3 ([38]). *If $q \in \mathbb{N}$ is even and $f \in C^q[0,1]$, then*

$$\lim_{n\to\infty} n^{q/2}\left[B_n(f,x) - f(x) - \sum_{r=1}^{q} B_n((e_1-x)^r, x)\frac{f^{(r)}(x)}{r!}\right] = 0,$$

uniformly on $[0,1]$.

In simultaneous approximation, Lorentz, in his book [181], proved the following pointwise convergence theorem:

Theorem 2.4 ([181] Simultaneous approximation). *Let $f(x)$ be bounded in* $[0,1]$*, and let $f^{(k)}(x_0)$ exist at a point $0 \le x_0 \le 1$; then $B_n^{(k)}(x_0) \to f^{(k)}(x_0)$.*

Many researchers have written papers on convergence behaviors of the Bernstein polynomials and for its generalizations. Ditzian [61] established the estimate

$$|f(x) - B_n(f,x)| \le C\omega^2_{\varphi^\lambda}(f, n^{-1/2}\varphi(x)^{1-\lambda}), x \in [0,1], \tag{2.2}$$

where $\lambda \in [0,1], \varphi(x) = \sqrt{x(1-x)}$ and the Ditzian–Totik modulus of smoothness of second order is given by

$$\omega^2_\phi(f,\delta) := \sup_{|h|\le\delta}\ \sup_{x\pm h\phi(x)\in[0,1]} |f(x-\phi(x)h) - 2f(x) + f(x+\phi(x)h)|,$$

in which $\phi : [0,1] \to R$ is the step-weight function.

The case $\lambda = 0$ in (2.2) gives the classical local estimate, and $\lambda = 1$ gives the global estimate developed by Ditzian–Totik [62]. The inequality was further extended by Felten [69] as

$$|f(x) - B_n(f,x)| \le C\omega_2\left(f, n^{-1/2}\frac{\varphi(x)}{\phi(x)}\right), x \in [0,1]. \tag{2.3}$$

In this continuation, Floater [74] derived an error bound and an asymptotic formula for derivatives of the Bernstein polynomials by differentiating a remainder formula of Stancu.

Theorem 2.5 ([74] Error bound). *Let* $f \in C^{k+2}[0,1]$ *for some* $k \geq 0$*; then*

$$\left|(B_n f)^{(k)} - f^{(k)}\right| \leq \frac{1}{2n}\Big(k(k-1)||f^{(k)}|| + k|1-2x|.||f^{(k+1)}|| \\ +x(1-x)||f^{(k+2)}||\Big),$$

where $||.||$ *is the max norm on* $[0,1]$.

Theorem 2.6 ([74] Asymptotic formula). *Let* $f \in C^{k+2}[0,1]$ *for some* $k \geq 0$*; then*

$$\lim_{n\to\infty}[(B_n f)^{(k)} - f^{(k)}(x)] = \frac{1}{2}\frac{d^k}{dx^k}[x(1-x)f''(x)],$$

uniformly for $x \in [0,1]$.

The well-known result on the Bernstein polynomials of equivalence is due to Knoop and Zhou [173] and Totik [232]: It states that for $f \in C[0,1]$ and for all $n \in \mathbb{N}$,

$$C_1\omega_2^{\varphi}\left(f, \frac{1}{\sqrt{n}}\right) \leq ||B_n(f,.) - f|| \leq C_2\omega_2^{\varphi}\left(f, \frac{1}{\sqrt{n}}\right),$$

where $C_1, C_2 > 0$ are independent of n and f and ω_2^{φ} is the Ditzian–Totik modulus of second order with $\varphi = \sqrt{x(1-x)}$. Gal [76] established the results for the lower bound of the Bernstein polynomials.

Theorem 2.7 ([76] Lower bound). *If* $f \in C^2[0,1]$ *satisfies the condition* $f''(x) \geq m > 0$ *for all* $x \in [0,1]$*, then*

$$|(B_n f)(x) - f(x)| \geq c_1\omega_2\left(f, \sqrt{\frac{x(1-x)}{n}}\right),$$

with $C_1 = \frac{m}{2||f''||} < \frac{1}{2}$.

Theorem 2.8 ([76] Lower bounds). *Let us denote* $e_s(t) = t^s, t \in [0,1], s \in \mathbb{N}, s \geq 3$*:*

(i) There do not exist $k \in \{1, 2, \ldots, s-1\}$ *and* $c_{s,k} > 0$ *(independent of* n *and* x*) such that*

$$B_n(e_s)(x) - x^s \geq c_{s,k}\omega_k\left(e_s, \sqrt{\frac{x(1-x)}{n}}\right), \forall x \in [0,1], n \in \mathbb{N}.$$

(ii) There exists $C_s > 0$ (independent of n and x) such that

$$B_n(e_s)(x) - x^s \geq C_s\omega_s\left(e_s, \sqrt{\frac{x(1-x)}{n}}\right), \forall x \in [0,1], n \in \mathbb{N}.$$

(iii) For any polynomial $P_s(x)$ of degree s with positive coefficients [i.e., $P_s(x) = a_0 + a_1 + \cdots + a_s x^s, a_s \geq 0$ for all $k = 0, 1, \ldots, s$], there exists $C_s > 0$ (independent of n and x) such that

$$B_n(P_s)(x) - P_s(x) \geq C_s\omega_s\left(P_s, \sqrt{\frac{x(1-x)}{n}}\right), \forall x \in [0,1], n \in \mathbb{N}.$$

Szász [227], in 1950, and Mirakyan [194] (also spelled Mirakian or Mirakjan), in 1941, generalized the Bernstein polynomials to an infinite interval as

$$S_n(f,x) := \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right), x \in [0,\infty), \tag{2.4}$$

where

$$s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}.$$

Stancu [221] obtained the following result on a uniform norm, using probabilistic methods:

Theorem 2.9 ([221]). *Let $f \in C^1[0,a], a > 0$; then for $n \in \mathbb{N}$, we have*

$$||S_n(f,.) - f|| \leq (a + \sqrt{a}).\frac{1}{\sqrt{n}}\omega(f', 1/\sqrt{n}).$$

Singh [213] obtained the following sharp estimate in the simultaneous approximation:

Theorem 2.10 ([213]). *Let $f \in C^{r+1}[0,a], a > 0$; then for $n \in \mathbb{N}$, we have*

$$||S_n^{(r)}(f,.) - f^{(r)}|| \leq \frac{r}{n}||f^{(r+1)}|| + K_{n,r}.\frac{1}{\sqrt{n}}\omega(f^{(r+1)}, 1/\sqrt{n}),$$

where $K_{n,r} = \left[(a/2) + (r/2\sqrt{n}) + (r^2/4n)\left((r^2/4n) + a\right)^{1/2}.\left(1 + (r/2\sqrt{n})\right)\right].$

Totik [231] represented the Szász operators in the form of a difference function as

$$S_n(f,x) := \sum_{k=0}^{\infty}\frac{(-nx)^k}{k!}\sum_{k=0}^{\infty} = \sum_{k=0}^{\infty}\Delta_{1/n}^k(f;0)\frac{(nx)^k}{k!},$$

where

$$\Delta_h^k(f;x) = \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i} f(x+ih).$$

Totik [231] introduced the modified modulus of smoothness as

$$\omega(\delta) = \sup_{\substack{0\le x<\infty \\ 0<h\le\delta}} |\Delta^2_{h\sqrt{x}}(f;x)|, \delta > 0,$$

where for an absolute constant K, $\omega(\lambda\delta) \le K\lambda^2\omega(\delta), \lambda \ge 1$. He obtained the following equivalence result for the Szász operators:

Theorem 2.11 ([231]). *Let $f \in C_B[0,\infty)$. Then the following are equivalent:*

(i) $S_n(f,x) - f(x) = o(1), n \to \infty$.
(ii) $\omega(\delta) = o(1), \delta \to 0$.
(iii) $f(x+h\sqrt{x}) - f(x) = o(1)$, *as* $h \to 0$ *uniformly in* x.
(iv) The function $f(x^2)$ *is uniformly continuous.*

Totik showed that equivalence of $(ii) \Leftrightarrow (i)$ holds even if $f \in C_B[0,\infty)$ is replaced by the weaker assumption on $f \in C[0,\infty), \omega(1) < \infty$.

Theorem 2.12 ([231]). *Let* $0 < \alpha \le 1$. *For* $f \in C_B[0,\infty)$, *the following are equivalent:*

(i) $S_n(f,x) - f(x) = o(n^{-\alpha})$.
(ii) $\omega(\delta) = O(\delta^{2\alpha})$.

Sun [225] gave an estimate for derivatives of these operators on functions of bounded variation with growth of order $O(t^{\alpha t})$ and remarked that, unfortunately for the continuous derivatives, his estimate does not include the case $f' \in Lip\ 1$, on every finite subinterval of $[0,\infty)$. For such a case, he obtained $S_n^{(r)}(f,x) - f^{(r)}(x) = O(\log n/n), r = 0,1,2,\ldots$. This degree is worse than the usual degree $1/n$. He also raised the question of whether a unified approach can be developed, which may improve the estimate for the class $f' \in Lip\ 1$ on every finite subinterval of $[0,\infty)$. Kasana and Agrawal [169] extended the studies and estimated a result for linear combinations of the Szász operators.

The modified Szász–Mirakjan operators discussed in [240] are defined as

$$S_n(f;m,x) := \frac{1}{g((nx+1)^2;m)} \sum_{k=0}^{\infty} \frac{(nx+1)^{2k}}{(k+m)!} f\left(\frac{k+m}{n(nx+1)}\right), \tag{2.5}$$

where $x \in [0,\infty)$ and $g(t;m) = \sum_{k=0}^{\infty} \frac{t^k}{(k+m)!}, t \in [0,\infty)$.

Walczak [240] considered the space $C_p, p \in \mathbb{N}_0$, associated with the weight function $w_0(x) := 1, w_p(x) := (1 + x^p)^{-1}, p \geq 1$ and consisting of all real-valued functions on $[0,\infty)$, for which $w_p(x) f(x)$ is uniformly continuous and bounded on $[0,\infty)$. The norm on C_p is defined as $||f||_p := \sup_{x\in[0,\infty)} w_p(x)|f(x)|$. In [240], it was proved that if $f \in C_p$,, then for the operators (2.5), one has the following estimate:

$$||S_n(f;m,.) - f||_p \leq M_0\omega\left(f;C_p;\frac{1}{n}\right), m,n \in \mathbb{N},$$

where M_0 is an absolute constant and the modulus of continuity $\omega(f;C_p;t) := \sup_{0\leq h\leq t} ||f(x+h) - f(x)||_p, t \in [0,\infty)$. In particular, if $f \in C_p^1 := \{f \in C_p : f' \in C_p\}, p \in \mathbb{N}_0$,, then

$$||S_n(f;m,.) - f||_p \leq \frac{M_1}{n},$$

where M_1 is an absolute constant.

From [241], we know that the Szász–Mirakjan operators are defined in terms of a sample of the given function f on the points k/n, called *knots*. For the operators $S_n(f;m,x)$, the knots are the numbers $(k+m)/(n(nx+1))$ for fixed m. Thus, the question arises of whether the knots $(k+m)/(n(nx+1))$ cannot be replaced by a given subset of points that are independent of x, provided this will not change the degree of convergence. In connection with this question, Walczak and Gupta [241] introduced the operators $L_n(f;p;r;s,x)$ for $f \in B_p, p \in \mathbb{N}$, which is a class of all real-valued continuous functions f on $[0,\infty)$, for which $w_p(x)x^k f^{(k)}(x), k = 0,1,2,\ldots,p$, is continuous and bounded on $[0,\infty)$ and $f^{(p)}(x)$ is uniformly continuous on $[0,\infty)$:

$$L_n(f;p;r;s,x) := \begin{cases} \frac{1}{I_r(n^s x)} \sum_{k=0}^{\infty} \frac{(n^s x)^{2k+r}}{2^{2k+r} k!\Gamma(r+k+1)} \sum_{j=0}^{p} \frac{f^{(j)}\left(\frac{2k}{n^s}\right)\left(x - \frac{2k}{n^s}\right)^j}{j!}, & x > 0, \\ f(0), & x=0, \end{cases} \tag{2.6}$$

where I_r is the modified Bessel function

$$I_r := \sum_{k=0}^{\infty} \frac{t^{2k+r}}{2^{2k+r} k!\Gamma(r+k+1)}.$$

Walczak and Gupta [241] estimated the rate of convergence of the operators $L_n(f;p;r;s,x)$.

Theorem 2.13. *Fix $p \in \mathbb{N}_0, r \in [0,\infty)$ and $s > 0$. Then there exists a positive constant $M \equiv M(p,r,s)$ such that for $f \in B_{2p+1}$, we have*

$$||L_n(f;2p+1;r;s,.) - f||_{2p+1} \leq M\omega\left(f^{(2p+1)};C_0;n^{-s}\right).$$

Theorem 2.14. *Fix $p \in \mathbb{N}_0, r \in [0,\infty)$ and $s > 0$. Then there exists a positive constant $M \equiv M(p,r,s)$ such that for $f \in B_{2p+2}$, we have*

$$||L_n(f;2p+2;r;s,.) - f||_{2p+2} \leq \frac{M(p,r,s)}{n^s}||f^{(2p+2)}||_0.$$

The Baskakov operator $V_n(f,x)$ (see [34]) is defined as

$$V_n(f,x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f(k/n). \tag{2.7}$$

Several papers on this operator in a local character are available in the literature. Concerning global approximation, Becker [36] established an equivalence theorem for the Baskakov operator in polynomial-weight spaces, where the order of approximation takes the form $(x(1+x)/n)^\alpha$. Totik [230] considered the modified modulus of smoothness as

$$\omega(\delta) = \sup_{\substack{0\leq x<\infty \\ 0<h\leq\delta}} |\Delta^2_{h\sqrt{x(1+x)}}(f;x)|, \delta > 0,$$

and he obtained the following equivalence results for the Baskakov operators:

Theorem 2.15 ([230]). *Let $f \in C_B[0,\infty)$; then the following are equivalent:*

(i) $V_n(f,x) - f(x) = o(1), n \to \infty.$
(ii) $\omega(\delta) = o(1), \delta \to 0.$
(iii) $f(x) - f(x+hx) = o(1)$, *as* $h \to 0$ *uniformly in* $x \in [0,\infty)$.
(iv) The function $f(e^x)$ *is uniformly continuous on* $[0,\infty)$.

Theorem 2.16 ([230]). *Let $0 < \alpha \leq 1$. For $f \in C_B[0,\infty)$, the following are equivalent:*

(i) $V_n(f,x) - f(x) = o(n^{-\alpha}).$
(ii) $\omega(\delta) = O(\delta^{2\alpha}).$

It is well known that the weighted approximation is not a simple extension, because the Baskakov operators are unbounded for the usual weighted norm $||f||_w = ||wf||_\infty$. For $f \in C_B[0,\infty)$, Xun and Zhou [244] introduced the norm

$$||f||_w = ||wf||_\infty + |f(0)|.$$

They discussed the rate of convergence for the Baskakov operators with the Jacobi weights and obtained

$$w(x)|V_n(f,x) - f(x)| = O(n^{-\alpha}) \Leftrightarrow K(f,t)_w = O(t^\alpha),$$

where $w(x) = x^a(1+x)^{-b}, 0 < a < 1, b.0, 0 < \alpha < 1$, and $C_B[0,\infty)$ is the set of bounded continuous functions on $[0,\infty)$. In 2011, Feng [70] introduced a new norm and a new K-functional. Using the K-functional, he established direct and inverse theorems for the Baskakov operators with the Jacobi-type weight. Let

$$C_{a,b,\lambda} = \left\{f : f \in C_B[0,\infty), \varphi^{2(2-\lambda)} w f \in C_B[0,\infty)\right\}$$

$$C^0_{a,b,\lambda} = \{f : f \in C_{a,b,\lambda}, f(0) = 0\},$$

where $\varphi(x) = \sqrt{x(1+x)}\; w(x) = x^a(1+x)^{-b}, x \in [0,\infty), 0 \le a < \lambda \le 1$, and $b \ge 0$. The K-functional is defined as

$$K_{\varphi^\lambda}(f,t)_{w,\lambda} = \inf_{g\in D}\left\{||\varphi^{2(1-\lambda)}(f-g)||_w + t||\varphi^{2(2-\lambda)}g''||_w\right\},$$

where $D = \{g : g \in C^0_{a,b,\lambda}, g' \in A.C._{loc}[0,\infty), ||\varphi^{2(2-\lambda)}g''||_w < \infty$.

Theorem 2.17 ([70]). *If $f \in C^0_{a,b,\lambda}$, then we have*

$$||\varphi^{2(1-\lambda)}(V_n(f,.) - f)||_w \le MK_{\varphi^\lambda}(f, n^{-1})_{w,\lambda}.$$

Theorem 2.18 ([70]). *Suppose $f \in C^0_{a,b,\lambda}, 0 < \alpha < 1$. Then the following statements are equivalent:*

(i) $\varphi^{2(1-\lambda)}(x)w(x)|V_n(f,x) - f(x)| = O(n^{-\alpha}), n \ge 2.$
(ii) $K_{\varphi^\lambda}(f,t)_{w,\lambda} = O(t^\alpha), 0 < t < 1.$

The Meyer–König–Zeller operators [192] are defined as

$$M_n(f,x) = \sum_{k=0}^{\infty}\binom{n+k}{k}x^k(1-x)^{n+1}f\left(\frac{k}{n+k}\right), 0 \le x < 1. \tag{2.8}$$

Usually, M_n is defined also for $x = 1$ by $M_n(f,1) = f(1)$. Totik [230] also obtained equivalence results for the M_n operators defined by (2.8).

Theorem 2.19 ([230]). *For a bounded and continuous function $f : [0,1) \to \mathbb{R}$, the following are equivalent:*

(i) $M_n(f,x) - f(x) = o(1), n \to \infty.$
(ii) $\Delta^2_{h(1-x)}(f;x) = o(1)$ *as* $h \to 0$ *uniformly in* $x \in [0,1)$.
(iii) $f(x) - f(x + h(1-x)) = o(1)$, *as* $h \to 0$ *uniformly in* $x \in [0,1)$.
(iv) The function $f\left(\frac{e^x}{1+e^x}\right)$ *is uniformly continuous on* $[0,\infty)$.

Theorem 2.20 ([230]). *Let $0 < \alpha \le 1$. If $f : [0,1) \to \mathbb{R}$ is continuous and bounded, then the following are equivalent:*

(i) $M_n(f, x) - f(x) = o(n^{-\alpha})$.
(ii) $x^{\alpha}(1-x)^{2\alpha}|\Delta_h^2(f; x)| \leq Kh^{2\alpha}\left(0 < h \leq \frac{1}{4}\sqrt{x}(1-x), 0 \leq x < 1\right)$.

Mastroianni [188] introduce and study a sequence L_n of discrete linear positive operators to approximate unbounded functions on the interval $[0,\infty) := \mathbb{R}_+$. Let $(\phi_n)_{n\geq 1}$ be a sequence of real-valued functions defined on $\mathbb{R}_+$ that are infinitely differentiable on $\mathbb{R}_+$ and that satisfy the following conditions:
$\phi_n(0) = 1$ for every $n \in \mathbb{N}$.
$(-1)^k \phi_n^{(k)} \geq 0$, for every $n \in \mathbb{N}, x \in \mathbb{R}_+, k \in \mathbb{N}\bigcup\{0\}$.
For each $(n, k) \in \mathbb{N}\times\mathbb{N}_0$, there are a number $p(n, k) \in \mathbb{N}$ and a function $\alpha_{n,k} \in \mathbb{R}_+$, such that $\phi_n^{(i+k)} = (-1)^k \phi_{p(n,k)}^{(i)}(x)\alpha_{n,k}(x), i \in \mathbb{N}_0, x \in \mathbb{R}_+$ and

$$\lim_{n\to\infty} \frac{n}{p(n,k)} = \lim_{n\to\infty} \frac{\alpha_{n,k}(x)}{n^k} = 1.$$

Agratini and Vecchia [18] consider that $E_2(\mathbb{R}_+) := \{f \in C(\mathbb{R}_+) : \frac{f(x)}{1+x^2}$ is convergent as $x \to \infty\}$. The Mastroianni operators L_n map $E_2(\mathbb{R}_+)$ into $C(\mathbb{R}_+)$ and are defined as

$$L_n(f, x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k \phi_n^{(k)}(x) f\left(\frac{k}{n}\right). \tag{2.9}$$

The special cases of Mastroianni's [188] operators are the well-known Szász–Favard–Mirakyan and the Baskakov operators:

1. $\phi_n(x) = e^{-nx}, p(n, k) = n$ and $\alpha_{n,k}(x) = n^k$ (constant function on $\mathbb{R}_+$): We obtain the Szász–Favard–Mirakyan operators.
2. $\phi_n(x) = (1 + x)^{-n}, p(n, k) = n + k$ and for $\alpha_{n,k}(x) = n(n + 1)\cdots(n + k - 1)(1 + x)^{-k}$: We obtain the Baskakov operators.

Agratini and Vecchia [18] obtained some direct results for the operators L_n:

Theorem 2.21 ([18]). *If f is locally $Lip\alpha$ on $E \subset \mathbb{R}_+, \alpha \in (0, 1]$, one has*

$$|L_n(f, x) - f(x)| \leq C_f\left(v_n^{\alpha/2}(x^2 + x)^{\alpha/2} + 2d^{\alpha}(x, E)\right), x \geq 0,$$

where $d(x, E)$ represents the distance between x and E, and v_n is defined in

$$L_n((t - x)^2, x) \leq \max\{u_n, |\tau_{n,1}|a_n^{-1}\}(x^2 + x) := v_n(x^2 + x),$$

with $\tau_{n,j} := \phi_n^{(j)}(0)/a_n^j$ and $u_n := \tau_{n,2} + 2\tau n, 1 + 1$.

Along the lines of Ditzian–Totik, Agratini and Vecchia [18] consider $\varphi \in \mathbb{R}^{\mathbb{R}_+}$ an admissible weight function. The second-order K-functional for $f \in C_B[0,\infty)$ is defined as

$$K_{2,\varphi}(f,t) = \inf_g \left\{||f-g|| + t||\varphi^2 g''|| : g' \in A.C._{loc}(\mathbb{R}_+)\right\}, t > 0,$$

where $||.||$ stands for sup-norm, and $g' \in A.C._{loc}(\mathbb{R}_+)$ means that g is differentiable and g' is absolutely continuous on every compact subset of $\mathbb{R}_+$.

Theorem 2.22 ([18]). *If $a_n = -\phi_n'(0), n \in \mathbb{N}$, and φ is an admissible weight function such that φ^2 is concave, then*

$$L_n(f,x) \leq 2K_{2,\varphi}\left(f, \frac{\upsilon_n x(1+x)}{2\varphi^2(x)}\right)$$

holds for every $x > 0$, where υ_n is as defined in Theorem 2.21.

2.2 Kantorovich Operators

The Bernstein polynomials are not suitable for approximation to general discontinuous functions. However, by replacing $f(k/n)$ in the definition of Bernstein polynomial by an integral mean of $f(x)$ over a small interval around k/n, we may obtain better results. To approximate Lebesgue integrable functions on the interval [0, 1], Kantorovich [164] introduced the modified Bernstein polynomials as

$$P_n(f,x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt, \tag{2.10}$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, 0 \leq x \leq 1.$$

For $f \in L_1[0,1]$, Lorentz [180] proved in his dissertation that

$$\int_0^1 |P_n(f,x) - f(x)|dx \to 0 (n \to \infty).$$

He also established the following result:

Theorem 2.23 ([180]). *$f \in AC[0,1]$ if and only if*

$$\lim_{n\to\infty} \mathrm{var}_{[0,1]}[P_n(f,x) - f(x)] = 0.$$

Decades later, in 1971, Hoeffding [155] also studied Bernstein–Kantorovich polynomials and obtained a quantitative version of Lorentz's result:

Theorem 2.24 ([155]). *If F is the difference of two convex absolutely continuous functions on* $[0,1]$ *and* $J(F') = \int_0^1 \sqrt{x(1-x)}|df(x)|$ *is finite, then*

$$\text{var}_{[0,1]}[P_n(F,x) - F(x)] = O(n^{-1/2}).$$

Theorem 2.25 ([155]). *Let f be a Lebesgue integrable function on bounded variation inside* $(0,1)$. *Then*

$$\int_0^1 |P_n(f,x) - f(x)|dx \leq (2/e)^{1/2} J(f) n^{-1/2},$$

where $J(f) = J(F')$, as given in Theorem 2.24.

Bojanic and Sisha [46] proved the following theorem:

Theorem 2.26. *Let f be a Lebesgue integrable function on* $[0,1]$. *Then, for $n \geq 2$, we have*

$$\int_0^1 \sqrt{x(1-x)}|P_{n-1}(f,x) - f(x)|dx \leq \frac{2\pi^2}{3}\omega(f, n^{-1/2})_{L_1}.$$

In 1985, Agrawal and Prasad [25] improved the estimate of Theorem 2.23 and established the following theorem:

Theorem 2.27. *Let f be a Lebesgue integrable function on* $[0,1]$. *Then, for $n \geq 2$,*

$$\int_0^1 \sqrt{x(1-x)}|P_{n-1}(f,x) - f(x)|dx \leq 6.128384\omega(f, n^{-1/2})_{L_1}.$$

The Kantorovich variant of Szász–Mirakjan operators is defined as

$$S_n(f,x) = n\sum_{k=0}^{\infty} s_{n,k}(x) \int_{k/n}^{(k+1)/n} f(t)dt, \tag{2.11}$$

where $s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}$. In the L_p-norm, Totik [229] obtained the following estimate:

Theorem 2.28. *Let $0 < \alpha < 1$ and $1 < p < \infty$. For an $f \in L^p[0,\infty)$, the following are equivalent:*

(i) $||S_n f - f||_{L^p[0,\infty)} \leq K n^{-\alpha}, n = 1, 2, \ldots.$
(ii) $||\Delta^*_{h\sqrt{x}}(f,x)||_{L^p(h^2,\infty)} + h^\alpha ||\Delta^1_h(f)||_{L^p[0,\infty)} \leq K h^{2\alpha} (h \geq 0),$

where $\Delta^*_h(f,x) = f(x-h) - 2f(x) + f(x+h)$.

Baskakov–Kantorovich operators are defined as

$$V_n^*(f,x) = n\sum_{k=0}^{\infty} b_{n,k}(x) \int_{k/n}^{(k+1)/n} f(t)dt, \quad n \in \mathbf{N}, \tag{2.12}$$

where $b_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}$. Abel and Gupta [4] introduced the Baskakov–Kantorovich–Bézier operators $V^*_{n,\alpha}$ defined by

$$V^*_{n,\alpha}(f,x) = n\sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k}(x)\int_{k/n}^{(k+1)/n} f(t)dt, \quad (n\in\mathbf{N},\ \ \alpha\geq 1,\ \text{ or }\ 0<\alpha<1),$$

where $Q^{(\alpha)}_{n,k}(x) = J^{\alpha}_{n,k}(x) - J^{\alpha}_{n,k+1}(x)$, $J_{n,k}(x) = \sum_{j=k}^{\infty} b_{n,j}(x)$. Abel and Gupta [4] estimated the rate of convergence for the operators $V^*_{n,\alpha}$, which is discussed here in Chap. 7. Additionally, Gupta and Radu [122] proposed the q-analog of the Baskakov–Kantorovich operators as

$$V^*_{n,q}(f,x) = [n]_q\sum_{k=0}^{\infty} b_{n,k}(q;x)\int_{q[k]_q/[n]_q}^{[k+1]_q/[n]_q} f\left(q^{-k+1}t\right)d_qt, \quad n\in\mathbf{N},$$

where $b_{n,k}(q;x) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{k(k-1)/2}\frac{x^k}{(1+x)_q^{n+k}}$. In [122], some direct results and the statistical convergence of these operators are discussed. For details on q-operators, we refer the reader to the 2013 book by Aral, Gupta, and Agrawal [31].

2.3 Durrmeyer-Type Operators

In 1967, J. L. Durrmeyer modified the Bernstein polynomials in order to approximate Lebesgue integrable functions on [0, 1]. The Bernstein–Durrmeyer operators introduced in [65] are defined as

$$D_n(f,x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)f(t)dt, x\in[0,1], \tag{2.13}$$

where the Bernstein basis function is defined as

$$p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}.$$

Derriennic [58] first studied these operators in detail, and she obtained some direct results in ordinary and simultaneous approximation. The Bernstein–Durrmeyer operators preserve only the constant functions. Also, these operators satisfy the commutativity property, that is, $D_n(D_m(f,x)) = D_m(D_n(f,x))$.

We can write the binomial coefficient as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{(-1)^k(-n)_k}{k!}.$$

In terms of hypergeometric functions, the Bernstein–Durrmeyer operators defined by (2.13) can be expressed as

$$\begin{aligned} D_n(f,x) &= (n+1)\int_0^1 f(t)\,[(1-x)(1-t)]^n \sum_{k=0}^{n} \frac{(-n)_k(-n)_k}{(k!)^2}\left[\frac{xt}{(1-x)(1-t)}\right]^k dt \\ &= (n-1)\int_0^1 f(t)\,[(1-x)(1-t)]^n \sum_{k=0}^{n} \frac{(-n)_k(-n)_k}{(1)_k k!}\left[\frac{xt}{(1-x)(1-t)}\right]^k dt \\ &= (n+1)\int_0^1 f(t)\,[(1-x)(1-t)]^n \;{}_2F_1\left(-n,-n;1;\frac{xt}{(1-x)(1-t)}\right) dt. \end{aligned}$$

For $n > 0$ and $r > -1$, by simple computation, we have

$$D_n(t^r,x) = \frac{\Gamma(n+2)\Gamma(r+1)}{\Gamma(n+r+2)}\,{}_2F_1\left(-n,-r;1;x\right). \tag{2.14}$$

Theorem 2.29 ([58]). *For any continuous function f on $[0,1]$, the sequence D_n converges uniformly and*

$$\sup_{x\in[0,1]} |D_n(f,x) - f(x)| \le 2\omega(f, n^{-1/2}), \textit{for } n \ge 3.$$

Theorem 2.30 ([58] Asymptotic formula). *If f is integrable in $[0,1]$ and admits a derivative of second order on $[0,1]$, then*

$$\lim_{n\to\infty} n[D_n(f,x) - f(x)] = (1-2x)f'(x) + x(1-x)f''(x).$$

Theorem 2.31 ([58]). *If f is integrable and admits a derivative of order r at $x \in [0,1]$, then*

$$\lim_{n\to\infty} \frac{d^r}{dx^r} D_n(f,x) = \frac{d^r}{dx^r} f(x).$$

Theorem 2.32 ([58]). *If f admits a derivative of order r that is continuous on $[0,1]$, then $\frac{d^r}{dx^r}D_n(f,x)$ converges uniformly to $f^{(r)}$ and*

$$\sup_{x\in[0,1]}\left|\frac{(n+r+1)!(n-r)!}{(n+1)!n!}\frac{d^r}{dx^r}D_n(f,x) - \frac{d^r}{dx^r}f(x)\right| \le K_r\omega(f^{(r)}, n^{-1/2}),$$

where K_r is a constant independent of f and n.

Agrawal and Kasana [23] extended the studies and established direct results in simultaneous approximation for the Bernstein–Durrmeyer operators.

Theorem 2.33 ([23]). *Let $f \in C^{r+1}[0,1]$. Then for all $n \geq 2(r+1).(r+2)-1$,*

$$\left\|\frac{(n+r+1)!(n-r)!}{(n+1)!n!}D_n^{(r)}(f,x)-f^{(r)}\right\| \leq \frac{(r+1)}{n+r+2)}||f^{(r+1)}||$$
$$+\alpha_{n,r}\left(1+\frac{\alpha_{n,r}\sqrt{n}}{2}\right)\omega(f^{(r)},n^{-1/2}),$$

where

$$\alpha_{n,r}=\left\{\frac{(n+1)}{2(n+r+2)(n+r+3)}\right\}^{1/2}.$$

Theorem 2.34 ([23]). *Let f be a bounded and integrable function on $[0,1]$ admitting a derivative of order $(r+2)$ at a point $x \in [0,1]$. Then*

$$\lim_{n\to\infty} n\left[\frac{(n+r+1)!(n-r)!}{(n+1)!n!}D_n^{(r)}(f,x)-f^{(r)}(x)\right] \leq (r+1)(1-2x)f^{(r+1)}(x)$$
$$+x(1-x)f^{(r+2)}(x).$$

Further, this limit holds uniformly if $f^{(r+2)}$ is continuous on $[0,1]$.

To approximate Lebesgue integrable functions on the interval $[0,1]$, the modified Bernstein polynomials (see [113]) are defined as

$$P_{n,\alpha,\beta}(f,x)=(n-\alpha+1)\sum_{k=\beta}^{n-\alpha+\beta}p_{n,k}(x)\int_0^1 p_{n-\alpha,k-\beta}(t)f(t)dt$$
$$+\sum_{k\in I_n}p_{n,k}(x)f\left(\frac{k}{n}\right), \tag{2.15}$$

where $p_{n,k}(x)$ is as defined in (2.13), for $n \geq \alpha$, where α,β are integers satisfying $\alpha \geq \beta \geq 0$, and $I_n \subseteq \{0,1,2,\ldots,n\}$ is a certain index set. For $\alpha=\beta=0$, $I_n=\{0\}$, this definition reduces to the Bernstein–Durrmeyer operators (2.13). When $\alpha=\beta=1$, $I_n=\{0\}$, we obtain the different form of the operators introduced and studied by Gupta and Maheshwari [118]. Later we denote $P_n(f,x) \equiv P_{n,1,1}(f,x)$ as

$$P_n(f,x)=n\sum_{k=0}^{n}p_{n,k}(x)\int_0^1 p_{n-1,k-1}(t)f(t)dt+(1-x)^n f(0). \tag{2.16}$$

Pointwise convergence and an asymptotic formula in simultaneous approximation were studied by Gupta and Ispir [113].

Theorem 2.35 ([113]). *Let $f \in C[0,1]$, and let $f^{(r)}$ exist at a point $x \in (0,1)$. Then $P_n^{(r)}(f,x) = f^{(r)}(x) + o(1)$ as $n \to \infty$.*

Theorem 2.36 ([113]). *Let $f \in C[0,1]$. If $f^{(r+2)}$ exists at a point $x \in (0,1)$, then*

$$\lim_{n\to\infty} n\left[P_n^{(r)}(f,x) - f^{(r)}(x)\right] = x(1-x)f^{(r+2)}(x)$$
$$+[r - x(1+2r)]f^{(r+1)}(x) - r^2 f^{(r)}(x).$$

In 1985, Kasana et al. [168] and Mazhar and Totik [191] independently introduced the Durrmeyer-type modification of the well-known Szász–Mirakyan operators as

$$M_n(f,x) = n\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^{\infty} s_{n,k}(t)f(t)dt, \tag{2.17}$$

where the Szász–Mirakyan basis function is defined by

$$s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}.$$

Also, Mazhar and Totik [191] proposed another modification of Szász–Mirakyan operators as

$$(L_n f)(x) \equiv L_n(f,x) = n\sum_{k=1}^{\infty} s_{n,k}(x)\int_0^{\infty} s_{n,k-1}(v)f(v)dv + e^{-nx}f(0), \tag{2.18}$$

which was introduced earlier by Phillips [199]. Obviously, M_n is a perfect Durrmeyer-type [65] analog of the Szász–Mirakyan operators, but L_n has much nicer properties than M_n. The main difference is that L_n preserves constant as well as linear functions, while M_n preserves only the constant ones. Mazhar and Totik [191] obtained the following results:

Theorem 2.37 ([191]). *For every continuous function $f \in C[0,\infty)$, we have*

$$|L_n(f,x) - f(x)| \le 11\omega_2(f, \sqrt{x/n}).$$

Theorem 2.38 ([191]). *Let $f \in C[0,\infty)$, $f(x) = O(e^{Ax})(A > 0)$. If $L_n(f,x) - f(x) = o_x(x/n)$ $x \ge 0, n \to \infty$, then f is linear function. Furthermore,*

$$|L_n(f,x) - f(x)| \le \frac{Kx}{n}, x \ge 0, n = 1,2,\ldots,$$

holds if and only if f has a derivative belonging to $Lip\ 1$, where

$$Lip\ 1 = \left\{f : |f(x+h) - f(x)| \le K_f h, x \ge 0, h > 0\right\}.$$

Theorem 2.39 ([191]). *Let $f \in C[0,\infty)$ be bounded. Then with $0 < \alpha < 1$,*

$$|L_n(f,x) - f(x)| \le \frac{Kx}{n}, x \ge 0, n = 1, 2, \ldots,$$

holds if and only if f has a derivative belonging to $Lip\ 1$, where

$$Lip\ 1 = \left\{f : |f(x+h) - f(x)| \le K\left(\frac{x}{n}\right)^{\alpha}, x \ge 0, n = 1, 2, \ldots\right\}$$

also holds if and only if $f \in Lip^2\ 2\alpha := \left\{f \in C[0,\infty) : \omega_2(f,\delta) \le K_f\delta^{\alpha}, \delta > 0\right\}$.

From the result of Shisha and Mond [212], Mazhar and Totik [191] also proved

$$(1 + (1 + \sqrt{x}))|M_n(f,x) - f(x)| \le K\omega(f, n^{-1/2}).$$

Theorem 2.40 ([191]). *Let $f \in C_B[0,\infty)$. Then $L_n(f,x) - f(x) = o(1) (n \to \infty)$ is satisfied uniformly on $[0,\infty)$ iff $f(x^2)$ is uniformly continuous on $[0,\infty)$.*

Theorem 2.41 ([191]). *Let $f \in C_B[0,\infty)$. Then $M_n(f,x) - f(x) = o(1) (n \to \infty)$ is satisfied uniformly on $[0,\infty)$ iff $f(x^2)$ is uniformly continuous on $[0,\infty)$.*

Gupta [95] studied the Szász–Durrmeyer operators M_n and obtained an asymptotic formula and error estimation in simultaneous approximation. He considered the class Ł$[0,\infty)$ of all measurable functions defined on $[0,\infty)$ such that

$$\text{Ł}[0,\infty) := \left\{f : \int_0^\infty e^{-nt} f(t)dt < \infty \ \text{ for some positive integer } n\right\}.$$

This class is certainly bigger than the class of all Lebesgue integrable functions on $[0,\infty)$. Furthermore, he defined

$$\text{Ł}_\alpha[0,\infty) := \left\{f \in \text{Ł}[0,\infty) : f(t) = O(e^{\alpha t}), t \to \infty, \alpha > 0\right\}.$$

Theorem 2.42 ([95] Asymptotic formula). *Let $f \in$ Ł$_\alpha[0,\infty)$, and let it be bounded on every finite subinterval of $[0,\infty)$ admitting a derivative of order $(r+2)$ at a fixed point $x \in (0,\infty)$. Then*

$$\lim_{n\to\infty} n[M_n^{(r)}(f,x) - f^{(r)}(x)] = (r+1) f^{(r+1)}(x) + x f^{(r+2)}(x).$$

Theorem 2.43 ([95] Error estimation). *Let $f \in$ Ł$_\alpha[0,\infty)$ be bounded on every finite subinterval of $[0,\infty)$ if $f^{(r+1)}$ exists and is continuous on $(a-\eta, b+\eta), \eta > 0$. Then for n sufficiently large,*

$$||M_n^{(r)}(f,.) - f^{(r)}|| \leq \frac{(r+1)}{n}||f^{(r+1)}|| + C_1 n^{-1/2}\omega(f^{(r+1)}, n^{-1/2}) + O(n^{-s}),$$

where the constant C_1 *is independent of* f *and* n*,* $\omega(f,\delta)$ *is the modulus of continuity of* f *on* $(a-\eta, b+\eta)$*, and* $||.||$ *denotes the sup-norm on* $[a,b]$.

The modified Lupas operators [208] are defined as

$$V_n(f,x) = \int_0^\infty W(n,x,t)f(t)dt = (n-1)\sum_{v=0}^{\infty} b_{n,v}(x)\int_0^\infty b_{n,v}(t)f(t)dt, \tag{2.19}$$

where

$$b_{n,v}(x) = \binom{n+v-1}{v}\frac{x^v}{(1+x)^{n+v}}.$$

In 1991, Sinha et al. [215] improved the results obtained in [208] and called the operators V_n “modified Baskakov operators." In reality, the operators (2.19) are Durrmeyer-type modifications of the Baskakov operators. Sinha et al. [215] considered the class Ł of all Lebesgue measurable functions f on $[0,\infty)$ as

$$Ł = \left\{ f : \int_0^\infty \frac{|f(t)|}{(1+t)^n}dt < \infty \text{ for some positive integer } n \right\}.$$

The class Ł is bigger than the class of all Lebesgue integrable functions on $[0,\infty)$.

Theorem 2.44 ([215] Asymptotic formula). *Let* $f \in Ł$ *be bounded on every finite subinterval of* $[0,\infty)$ *admitting a derivative of order* $(r+2)$ *at a fixed point* $x \in (0,\infty)$*. Let* $f(t) = O(t^\alpha)$ *as* $t \to \infty$ *for some* $\alpha > 0$*. Then we have*

$$\lim_{n\to\infty} n[V_n^{(r)}(f,x) - f^{(r)}(x)] = r(1+r)f^{(r)}(x) \\ +(r+1)(1+2x)f^{(r+1)}(x)+x(1+x)f^{(r+2)}(x).$$

Theorem 2.45 ([215] Error estimation). *Let* $f \in Ł$ *be bounded on every finite subinterval of* $[0,\infty)$ *and* $f(t) = O(t^\alpha)$ *as* $t \to \infty$ *for some* $\alpha > 0$*. If* $f^{(r+1)}$ *exists and is continuous on* $< a,b > \subset (0,\infty)$*, where* $< a,b >$ *denotes an open interval containing the closed interval* $[a,b]$*, then for* n *sufficiently large,*

$$||V_n^{(r)}(f,.) - f^{(r)}|| \leq C_1\left(||f^{(r)}|| + ||f^{(r+1)}||\right)n^{-1} \\ +C_2 n^{-1/2}\omega(f^{(r+1)}, n^{-1/2}) + O(n^{-s}),$$

for any $s > 0$*, where the constants* C_1 *and* C_2 *are independent of* f *and* n*,* $\omega(f,\delta)$ *is the modulus of continuity of* f *on* $< a,b >$*, and norm-*$||.||$ *denotes the sup-norm on* $[a,b]$.

Let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence of real functions having the following properties on an interval $[0,b], b>0$, for every $n\in\mathbb{N}, k\in\mathbb{N}_0$:
$\phi_n\in C^\infty[0,b], \phi_n(0)=1$.
ϕ_n is completely monotone; that is, $(-1)^k\phi_n^{(k)}\geq 0$.
There exists a $c\in\mathbb{Z}: \phi_n^{(k+1)}=-n\phi_{n+c}^{(k)}, n>\max\{0,-c\}$. By Chap. 4 of [209], suitable sequences $(\phi_n)_{n\in\mathbb{N}}$ are given by

1. $\phi_n(x)=(1-x)^n$ for the interval $[0,1]$ with $c=-1$,
2. $\phi_n(x)=e^{-nx}$ for the interval $[0,\infty)$ with $c=0$,
3. $\phi_n(x)=(1+cx)^{-n/c}$ for the interval $[0,\infty)$ with $c>0$.

With $I=[0,\infty)$, when $c\geq 0$ and $I=[0,1]$, when $c=-1$, Heilmann [150] proposed the general Durrmeyer-type operators as

$$H_n(f,x)=(n-c)\sum_{v=0}^{\infty}p_{n,v}(x)\int_I p_{n,v}(t)f(t)dt,\quad x\in I,\tag{2.20}$$

where $n\in\mathbb{N}, n>\max\{0,-c\}$, and $p_{n,v}(x)=(-1)^v\frac{x^v}{v!}\phi_n^{(v)}(x)$.

Theorem 2.46 ([150] Global direct result). *Let* $\varphi^{2s}f^{(2s)}\in L_p(I), 1\leq p<\infty$, $\varphi(x)=\sqrt{x(1+cx)}$. *Then*

$$||\varphi^{2s}(H_nf-f)^{(2s)}||_p\leq C\left\{\omega_\varphi^2(f^{(2s)},n^{-1/2})_{\varphi^{2s},p}+n^{-1}||\varphi^{2s}f^{(2s)}||_p\right\}$$

holds with a constant C *independent of* n.

Theorem 2.47 ([150] Converse result). *Let* $f,\varphi^{2s}f^{(2s)}\in L_p(I), 1\leq p<\infty$, $\varphi(x)=\sqrt{x(1+cx)}$, $s\in\mathbb{N}, s<\alpha<s+1$. *Then* $||\varphi^{2s}(H_nf-f)^{(2s)}||_p=O(n^{s-\alpha})$ *implies*

$$\omega_\varphi^{2(s+1)}(f,t)_p=O(t^{2\alpha}).$$

In the other paper, Heilmann [149] derived

(a) A global direct result for the rate of approximation of the rth derivative of a function $f\in L_p^r(R_+):=\{g: g^{(r)}\in L_p(R_+), 1\leq p\leq\infty$ by a sequence $H_n^{(r)}$ in the L_p-metric.
(b) A local convergence $H_n^{(r)}\to f^{(r)}$ and a Voronovskaja theorem for functions $f\in L_1^r(R_+):=\{g: g^{(r)}\in L_1[0,R]$. For every $0<R<\infty$, there exist positive constants M,m depending on g such that $|g^{(r)}(t)|\leq M(1+t^m), t\in R_+\}$. It was noticed that $L_p^r(R_+)$ is not contained in $L_1^r(R_+)$ [e.g., a function $g^{(r)}(t)=t^{-1/2}, 0<t\leq 1$, and $g^{(r)}(t)=t^{-2}, t\geq 1$, belongs to $L_1(R_+)$, but not to $L_1^r(R_+)$]. For $r=0$, $L_p^0(R_+)=L_p(R_+)$ and $L_1^0(R_+)=L_1(R_+)$. If $f\in L_p^r(R_+)\cup L_1^r(R_+), r\in\mathbb{N}_0, 1\leq p<\infty, n>(c+m)r, x\in R_+$, then

$$(H_n f)^{(r)}(x) = (n-c)\prod_{l=0}^{r-1}\frac{n+cl}{n-c(l+1)}\sum_{v=0}^{\infty}p_{n+cr,v}(x)\int_0^\infty p_{n-cr,v+r}(t)f^{(r)}(t)dt.$$

The operators $(H_n f)^{(r)}$ are not positive. Therefore, Heilmann [149] introduced

$$H_{n,r}f = D^r H_n I^r f, f \in L_p(R_+)\cup L_1(R_+),$$

where D and I denote the differentiation integration operators, respectively.

Theorem 2.48 ([149] Global direct result). *Let* $\varphi(x) = \sqrt{x(1+cx)}, n \in \mathbb{N}, n > c(r+6)$. *If* $f \in L_p(R_+), 1 \le p < \infty$, *then*

$$||(H_{n,r}f - f||_p \le C\left\{\omega_\varphi^2(f, n^{-1/2})_p + n^{-1}||f||_p\right\},$$

with a constant C *independent of* n.

2.4 Szász–Beta-Type Operators

In [112] and [126] some approximation properties of mixed Szász-Baskakov operators were discussed. To approximate integrable functions on $[0,\infty)$, Gupta et al. [140] introduced a mixed sequence of operators by modifying Szász operators with the weights of Beta basis functions as

$$S_n(f,x) = \sum_{v=1}^{\infty}s_{n,v}(x)\int_0^\infty b_{n,v-1}(t)f(t)dt + e^{-nx}f(0), \quad x \in [0,\infty), \tag{2.21}$$

where

$$s_{n,v}(x) = e^{-nx}\frac{(nx)^v}{v!}, b_{n,v}(t) = \frac{1}{B(n,v+1)}\frac{t^v}{(1+t)^{n+v+1}}.$$

In the kernel form, (2.21) may be written as

$$S_n(f,x) = \int_0^\infty W_n(x,t)f(t)dt,$$

where $W_n(x,t) = \sum_{v=1}^\infty s_{n,v}(x)b_{n,v-1}(t) + e^{-nx}\delta(t)$, and $\delta(t)$ is the Dirac delta function. They obtained some direct results. To prove the main results of [140], the following basic lemmas are required:

Lemma 2.1 ([137]). *For* $m \in \mathbb{N}\bigcup\{0\}$, *if the mth-order moment is defined as*

$$U_{n,m}(x) = \sum_{v=0}^{\infty}s_{n,v}(x)\left(\frac{v}{n}-x\right)^m,$$

then $U_{n,0}(x) = 1, \ \ U_{n,1}(x) = 0$, *and*

$$nU_{n,m+1}(x) = x\left[U_{n,m}^{(1)}(x) + mU_{n,m-1}(x)\right].$$

Consequently, $U_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$, *where* $[\alpha]$ *denotes the integral part of* α.

Lemma 2.2 ([140]). *Let the function* $\mu_{n,m}(x), m \in N^0$, *be defined as*

$$\mu_{n,m}(x) = \sum_{v=1}^{\infty} s_{n,v}(x) \int_0^{\infty} b_{n,v-1}(t-x)^m dt + (-x)^m e^{-nx}.$$

Then

$$\mu_{n,0}(x) = 1, \quad \mu_{n,1}(x) = \frac{x}{n-1}, \quad \mu_{n,2}(x) = \frac{x^2(n+2) + 2nx}{(n-1)(n-2)}.$$

Also, we have the recurrence relation

$$\begin{aligned}(n-m-1)\mu_{n,m+1}(x) = x\left[\mu_{n,m}^{(1)}(x) + m(x+2)\mu_{n,m-1}(x)\right]\\ + [m + x(2m+1)]\,\mu_{n,m}(x), \quad n > m+1.\end{aligned}$$

Consequently, for each $x \in [0, \infty)$ *and from this recurrence relation, we have* $\mu_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$.

Lemma 2.3. *Let* δ *be a positive number. Then, for every* $\gamma > 0, \ x \in (0, \infty)$, *there exists a constant* $M(s,x)$ *independent of n and depending on s and x, such that*

$$\int_{|t-x|>\delta} W_n(x,t)t^{\gamma} dt \leq M(s,x)n^{-s}, \quad s = 1, 2, 3, \ldots,$$

hold locally.

Lemma 2.4 ([137]). *The polynomials* $Q_{i,j,r}(x)$ *exist independently of n and v, such that*

$$x^r D^r[s_{n,v}(x)] = \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i(v-nx)^j Q_{i,j,r}(x)s_{n,v}(x), \quad D \equiv \frac{d}{dx}.$$

Lemma 2.5 ([140]). *Let* $n > r \geq 1$ *and* $f^{(i)} \in C_B[0,\infty)$ *for* $i \in \{0, 1, 2, \ldots, r\}$. *Then*

$$S_n^{(r)}(f,x) = \frac{n^r}{(n-1)(n-2)\cdots(n-r)} \sum_{v=0}^{\infty} s_{n,v}(x) \int_0^{\infty} b_{n-r,v+r-1}(t) f^{(r)}(t) dt.$$

Proof. By simple computation, we have the following identities:

$$s'_{n,v}(x) = n[s_{n,v-1}(x) - s_{n,v}(x)], \tag{2.22}$$

$$b'_{n,v}(t) = n[b_{n+1,v-1}(t) - b_{n+1,v}(t)], \tag{2.23}$$

where $x, t \in [0, \infty)$. Furthermore, the lemma is proved by mathematical induction. Using the above identities (2.22) and (2.23), we have

$$\begin{aligned}
S'_n(f,x) &= \sum_{v=1}^{\infty} s'_{n,v}(x) \int_0^{\infty} b_{n,v-1}(t) f(t) dt - ne^{-nx} f(0) \\
&= \sum_{v=1}^{\infty} n[s_{n,v-1}(x) - s_{n,v}(x)] \int_0^{\infty} b_{n,v-1}(t) f(t) dt - ne^{-nx} f(0) \\
&= ns_{n,0}(x) \int_0^{\infty} b_{n,0}(t) f(t) dt - ne^{-nx} f(0) \\
&\quad + n \sum_{v=1}^{\infty} s_{n,v}(x) \int_0^{\infty} [b_{n,v}(t) - b_{n,v-1}(t)] f(t) dt \\
&= ne^{-nx} \int_0^{\infty} n(1+t)^{-n-1 f(t) dt} \\
&\quad + n \sum_{v=1}^{\infty} s_{n,v}(x) \int_0^{\infty} \left(\frac{-1}{n-1}\right) b'_{n,v-1}(t) f(t) dt - ne^{-nx} f(0).
\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}
S'_n(f,x) &= ne^{-nx} f(0) + ne^{-nx} \int_0^{\infty} (1+t)^{-n} f'(t) dt \\
&\quad + \sum_{v=1}^{\infty} \frac{n}{n-1} s_{n,v}(x) \int_0^{\infty} b_{n,v-1}(t) f'(t) dt - ne^{-nx} f(0) \\
&= \frac{n}{n-1} \sum_{v=0}^{\infty} s_{n,v}(x) \int_0^{\infty} b_{n,v-1}(t) f'(t) dt,
\end{aligned}$$

which was to be proved. If we suppose that

$$S_n^{(i)}(f,x) = \frac{n^i}{(n-1)(n-2)\cdots(n-i)} \sum_{v=0}^{\infty} s_{n,v}(x) \int_0^{\infty} b_{n-i,v+i-1}(t) f^{(i)}(t) dt,$$

then by (2.22) and (2.23), and integration by parts, we obtain

$$
\begin{aligned}
S_n^{(i+1)}(f,x) &= \frac{n^i}{(n-1)(n-2)\cdots(n-i)}\sum_{v=0}^{\infty}[s_{n,v-1}(x)-s_{n,v}(x)]\\
&\quad \int_0^{\infty} b_{n-i,v+i-1}(t)f^{(i)}(t)dt\\
&\quad +\frac{n^i}{(n-1)(n-2)\cdots(n-i)}(-ne^{-nx})\int_0^{\infty} b_{n-i,i-1}(t)f^{(i)}(t)dt\\
&= \frac{n^i}{(n-1)(n-2)\cdots(n-i)}ns_{n,0}(x)\int_0^{\infty} b_{n-i,i}(t)f^{(i)}(t)dt\\
&\quad +\frac{n^i}{(n-1)(n-2)\cdots(n-i)}\sum_{v=1}^{\infty}ns_{n,v}(x)\int_0^{\infty} b_{n-i,v+i}(t)f^{(i)}(t)dt\\
&\quad -\frac{n^i}{(n-1)(n-2)\cdots(n-i)}ne^{-nx}\int_0^{\infty} b_{n-i,i-1}(t)f^{(i)}(t)dt\\
&= \frac{n^{i+1}}{(n-1)(n-2)\cdots(n-i)}s_{n,0}(x)\int_0^{\infty}[b_{n-i,i}(t)-b_{n-i,i-1}(t)]f^{(i)}(t)dt\\
&\quad +\frac{n^{i+1}}{(n-1)(n-2)\cdots(n-i)}\sum_{v=1}^{\infty}s_{n,v}(x)\\
&\quad \int_0^{\infty}[b_{n-i,v+i}(t)-b_{n-i,v+i-1}(t)]f^{(i)}(t)dt\\
&= \frac{n^{i+1}}{(n-1)(n-2)\cdots(n-i)}s_{n,0}(x)\int_0^{\infty}\left(\frac{-1}{n-i-1}\right)b'_{n-i-1,i}(t)f^{(i)}(t)dt\\
&\quad +\frac{n^{i+1}}{(n-1)(n-2)\cdots(n-i)}\sum_{v=1}^{\infty}s_{n,v}(x)\\
&\quad \times\int_0^{\infty}\left(\frac{-1}{n-i-1}\right)b'_{n-i-1,v+i}(t)f^{(i)}(t)dt\\
&= \frac{n^{i+1}}{(n-1)(n-2)\cdots(n-i-1)}\sum_{v=0}^{\infty}s_{n,v}(x)\int_0^{\infty} b_{n-i-1,v+i}(t)f^{(i+1)}(t)dt,
\end{aligned}
$$

which completes the proof. ∎

Let us denote

$$C_\gamma[0,\infty)\equiv\{f\in C[0,\infty): f(t)\le Mt^{\gamma},\ \text{for some}\ M>0,\ \gamma>0\}.$$

Theorem 2.49 ([140] Point-wise Convergence). *Let* $f\in C_\gamma[0,\infty)$, $\gamma>0$, *and* $f^{(r)}$ *exists at a point* $x\in(0,\infty)$, *then*

$$\left[\frac{d^r}{dw^r}S_n(f,w)\right]_{w=x} = f^{(r)}(x)+o(1),\quad \text{as } n\to\infty.$$

Proof. By Taylor's expansion of f, we have

$$f(t)=\sum_{i=0}^{r-1}\frac{f^{(i)}(x)}{i!}(t-x)^i+\varepsilon(t,x)(t-x)^r,$$

where $\varepsilon(t,x)\to 0$ as $t\to x$. Hence,

$$\begin{aligned}\left[\frac{d^r}{dw^r}S_n(f,w)\right]_{w=x} &= \left[\int_0^\infty \frac{d^r}{dw^r}W_n(t,w)f(t)dt\right]_{w=x}\\ &= \left[\sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\int_0^\infty \frac{d^r}{dw^r}W_n(t,w)(t-x)^i dt\right]_{w=x}\\ &\quad+\left[\int_0^\infty \frac{d^r}{dw^r}W_n(t,w)\varepsilon(t,x)(t-x)^r dt\right]_{w=x}\\ &:= R_1+R_2.\end{aligned}$$

First, to estimate R_1, using binomial expansion of $(t-x)^i$ and Lemma 2.2, we have

$$\begin{aligned}R_1 &= \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\sum_{v=0}^{i}\binom{i}{v}(-x)^{i-v}\left[\frac{\partial^r}{\partial w^r}\int_0^\infty W_n(t,w)t^v dt\right]_{w=x}\\ &= \left[\frac{f^{(r)}(x)}{i!}\frac{\partial^r}{\partial w^r}\int_0^\infty W_n(t,w)t^r dt\right]_{w=x}=f^{(r)}(x)+o(1),\quad n\to\infty.\end{aligned}$$

Next, using Lemma 2.4, we have

$$\begin{aligned}|R_2| &\le \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i\frac{|Q_{i,j,r}(x)|}{x^r}\sum_{v=1}^{\infty}|v-nx|^j s_{n,v}(x)\int_0^\infty b_{n,v-1}(t)|\varepsilon(t,x)|.|t-x|^r dt\\ &\quad+(-n)^r e^{-nx}|\varepsilon(0,x)|(-x)^r\\ &:= R_3+R_4.\end{aligned}$$

Since $\varepsilon(t,x)\to 0$ as $t\to x$ for a given $\varepsilon>0$ there exists a $\delta>0$, such that $|\varepsilon(t,x)|<\varepsilon$ whenever $0<|t-x|<\delta$. Thus, for some $M_1>0$, we can write

$$\begin{aligned}R_3 &\le M_1\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i\sum_{v=1}^{\infty}s_{n,v}(x)|v-nx|^j\Bigg\{\varepsilon\int_{|t-x|<\delta}b_{n,v-1}(t)|t-x|^r\\ &\quad+\int_{|t-x|\ge\delta}b_{n,v-1}(t)M_2|t-x|^\gamma dt\Bigg\}\\ &:= R_5+R_6,\end{aligned}$$

where $M_1=\sup\limits_{\substack{2i+j\le r\\ i,j\ge 0}}\dfrac{|Q_{i,j,r}(x)|}{x^r}$ and M_2 is independent of t.

Applying Schwarz inequality for integration and summation respectively, we obtain

$$R_5 \leq \varepsilon M_1 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \sum_{v=1}^{\infty} s_{n,v}(x)|v-nx|^j \left(\int_0^{\infty} b_{n,v-1}(t)dt\right)^{1/2}$$
$$\times \left(\int_0^{\infty} b_{n,v-1}(t-x)^{2r}dt\right)^{1/2}$$
$$\leq \varepsilon M_1 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \sum_{v=1}^{\infty} s_{n,v}(x) \left(\sum_{v=1}^{\infty} s_{n,v}(x)(v-nx)^{2j}\right)^{1/2}$$
$$\times \left(\sum_{v=1}^{\infty} s_{n,v}(x) \int_0^{\infty} b_{n,v-1}(t-x)^{2r}dt\right)^{1/2}.$$

Using Lemmas 2.1 and 2.2, we get

$$R_5 \leq \varepsilon M_1 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i\, O\left(n^{j/2}\right) O\left(n^{-r/2}\right) = O(1).$$

$$R_6 \leq M_2 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \sum_{v=1}^{\infty} s_{n,v}(x)|v-nx|^j \int_{|t-x|\geq\delta} b_{n,v-1}(t)|t-x|^{\gamma}dt$$
$$\leq M_2 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \sum_{v=1}^{\infty} s_{n,v}(x)|v-nx|^j \left(\int_{|t-x|\geq\delta} b_{n,v-1}(t)dt\right)^{1/2}$$
$$\times \left(\int_{|t-x|\geq\delta} b_{n,v-1}(t-x)^{2\gamma}dt\right)^{1/2}$$
$$\leq M_2 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \left(\sum_{v=1}^{\infty} s_{n,v}(x)(v-nx)^{2j}\right)^{1/2}$$
$$\times \left(\sum_{v=1}^{\infty} s_{n,v}(x) \int_0^{\infty} b_{n,v-1}(t)(t-x)^{2\gamma}dt\right)^{1/2}$$
$$= \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i\, O\left(n^{j/2}\right) O\left(n^{-\gamma/2}\right) = O\left(n^{(r-\gamma)/2}\right) = o(1).$$

Thus, due to arbitrariness of $\varepsilon > 0$ it follows that $R_3 = o(1)$. Also, $R_4 \to 0$ as $n \to \infty$ and hence, $R_2 = o(1)$. Collecting the estimates of R_1 and R_2, we get the required result. ∎

Theorem 2.50 ([140] Asymptotic Formula). *Let $f \in C_\gamma[0,\infty)$, $\gamma > 0$. If $f^{(r+2)}$ exists at a point $x \in (0,\infty)$, then*

$$\lim_{n\to\infty} n\left[\left[\frac{d^r}{dw^r}S_n(f,w)\right]_{w=x} - f^{(r)}(x)\right] = \frac{r(r+1)}{2}f^{(r)}(x) + [x(1+r)+r]f^{(r+1)}(x) + \frac{x}{2}(2+x)f^{(r+2)}(x).$$

Proof. Using Taylor's expansion of f, we have

$$f(t) = \sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^{r+2},$$

where $\varepsilon(t,x) \to 0$ as $t \to x$ and $\varepsilon(t,x) = O((t-x)^\gamma)$, $t \to \infty$ for some $\gamma > 0$. Applying Lemma 2.2, we have

$$\begin{aligned}
& n\left[\left[\frac{d^r}{dw^r}S_n(f,w)\right]_{w=x} - f^{(r)}(x)\right] \\
&= n\left[\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}\int_0^\infty \frac{d^r}{dw^r}W_n(t,w)(t-x)^i\,dt - f^{(r)}(x)\right]_{w=x} \\
&\quad + \left[n\int_0^\infty \frac{d^r}{dw^r}W_n(t,w)\varepsilon(t,x)(t,x)^{r+2}dt\right]_{w=x} \\
&:= E_1 + E_2.
\end{aligned}$$

$$\begin{aligned}
E_1 &= \left[n\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}\int_0^\infty \frac{d^r}{dw^r}W_n(t,w)t^j\,dt - nf^{(r)}(x)\right]_{w=x} \\
&= \frac{f^{(r)}(x)}{r!}n\left[S_n^{(r)}(t^r,x) - (r!)\right] \\
&\quad + \frac{f^{(r+1)}(x)}{(r+1)!}n\left[(r+1)(-x)S_n^{(r)}(t^r,x) + S_n^{(r)}(t^{r+1},x)\right] \\
&\quad + \frac{f^{(r+2)}(x)}{(r+2)!}n\left[\frac{(r+2)(r+1)}{2}x^2S_n^{(r)}(t^r,x)\right. \\
&\quad \left. + (r+2)(-x)S_n^{(r)}(t^{r+1},x) + S_n^{(r)}(t^{r+2},x)\right].
\end{aligned}$$

It is easily verified from Lemma 2.2 that for each $x \in (0, \infty)$,

$$S_n(t^i, x) = \frac{(n-i-1)!}{(n-1)!}(nx)^i + i(i-1)\frac{(n-i-1)!}{(n-1)!}(nx)^{i-1} + O(n^{-2}).$$

Therefore,

$$\begin{aligned}
E_1 &= nf^{(r)}(x)\left[\frac{n^r(n-r-1)!}{(n-1)!} - 1\right] \\
&\quad + n\frac{f^{(r+1)}(x)}{(r+1)!}\left[(r+1)(-x)(r!)\left\{\frac{n^r(n-r-1)!}{(n-1)!}\right\}\right. \\
&\quad \left. + \left\{\frac{n^{r+1}(n-r-2)!}{(n-1)!}(r+1)!x + r(r+1)\frac{n^r(n-r-2)!}{(n-1)!}(r!)\right\}\right] \\
&\quad + n\frac{f^{(r+2)}(x)}{(r+2)!} \times \left[\frac{(r+1)(r+2)x^2}{2}\frac{n^r(n-r-1)!}{(n-1)!}(r!)\right. \\
&\quad + (r+2)(-x)\left\{\frac{n^{r+1}(n-r-2)!}{(n-1)!}(r+1)!x + r(r+1)\frac{n^r(n-r-2)!}{(n-1)!}(r!)\right\} \\
&\quad + \left\{\frac{n^{r+2}(n-r-3)!}{(n-1)!}\frac{(r+2)!}{2}x^2\right\} \\
&\quad \left. + (r+1)(r+2)\frac{n^{r+1}(n-r-3)!}{(n-1)!}(r+1)!x + O(n^{-2})\right].
\end{aligned}$$

In order to complete the proof of the theorem it is sufficient to show that $E_2 \to 0$ as $n \to \infty$, which can be easily proved along the lines of the Theorem 2.49 and by using Lemmas 2.1, 2.2 and 2.4. ■

Let $C_B[0, \infty)$ be as defined in Definition 1.10. By [60], there exists $C_1 > 0$ such that

$$K_2(f, \delta) \le C_1\omega_2(f, \sqrt{\delta}). \tag{2.24}$$

Theorem 2.51 ([140]). *Let $f \in C_B[0, \infty)$. Then there exists an absolute constant $C_2 > 0$ such that*

$$|S_n(f, x) - f(x)| = C_2\omega_2\left(f, \sqrt{\frac{x(x+1)}{n-1}}\right) + \omega\left(f, \frac{x}{n-1}\right),$$

for every $x \in [0, \infty)$ and $n = 3, 4, \ldots$.

Proof. We define a new operator $\hat{S}_n : C_B[0, \infty) \to C_B[0, \infty)$ as follows:

$$\hat{S}_n(f, x) = S_n(f, x) + f(x) - f\left(\frac{nx}{n-1}\right). \tag{2.25}$$

Then, by Lemma 2.2, we obtain $\hat{S}_n(t-x,x) = 0$. Now, let $x \in [0,\infty)$ and $g \in W_\infty^2$. From Taylor's formula,

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du, \quad t \in [0,\infty),$$

we get

$$\begin{aligned}
\hat{S}_n(g,x) - g(x) &= \hat{S}_n\left(\int_0^t (t-u)g''(u)du, x\right)\\
&= S_n\left(\int_0^t (t-u)g''(u)du, x\right)\\
&\quad + \int_x^{nx/(n-1)} \left(\frac{n}{n-1}x - u\right) g''(u)du. \qquad (2.26)
\end{aligned}$$

On the other hand,

$$\left|\int_0^t (t-u)g''(u)du\right| \leq (t-x)^2 ||g''|| \qquad (2.27)$$

and

$$\begin{aligned}
&\left|\int_x^{nx/(n-1)} \left(\frac{n}{n-1}x - u\right) g''(u)du\right|\\
&\leq \left(\frac{nx}{n-1} - x\right)^2 ||g''||\\
&\leq \frac{x^2}{(n-1)^2}.||g''|| \leq \frac{x(1+x)}{(n-1)^2}||g''||. \qquad (2.28)
\end{aligned}$$

Thus, by (2.26)–(2.28) and by the positivity of S_n, we have

$$|\hat{S}_n(g,x) - g(x)| \leq S_n\left((t-x)^2, x\right).||g''|| + \frac{x(1+x)}{(n-1)^2}.||g''||.$$

Hence, in view of Lemma 2.2,

$$\begin{aligned}
|\hat{S}_n(g,x) - g(x)| &\leq \left(\frac{2nx + (n+2)x^2}{(n-1)(n-2)} + \frac{x(1+x)}{(n-1)^2}\right).||g''||\\
&\leq \left(\frac{2n}{(n-2)} + \frac{1}{(n-1)}\right).\frac{x(1+x)}{n-1}||g''||\\
&\leq \frac{7}{(n-1)}.x(1+x).||g''||. \qquad (2.29)
\end{aligned}$$

Again, applying Lemma 2.2, we find

$$|S_n(f,x)| \leq \sum_{v=1}^{\infty} s_{n,v}(x)\int_0^{\infty} b_{n,v-1}(t)f(t)dt + e^{-nx}|f(0)| \leq ||f||.$$

This means that S_n is a contraction; that is, $||S_n f|| \leq ||f||,\ \ f \in C_B[0,\infty)$. Thus, by (2.25),

$$||\hat{S}_n f|| \leq ||S_n f|| + 2||f|| \leq 3||f||, \quad f \in C_B[0,\infty). \tag{2.30}$$

Using (2.25), (2.29), and (2.30), we obtain

$$\begin{aligned}
|S_n(f,x) - f(x)| &\leq |\hat{S}_n(f,x) - f(x)| + \left|f(x) - f\left(\frac{nx}{(n-1)}\right)\right| \\
&\leq |\hat{S}_n(f-g,x) - (f-g)(x)| + |\hat{S}_n(g,x) - g(x)| \\
&\quad + \left|f(x) - f\left(\frac{nx}{(n-1)}\right)\right| \\
&\leq 4||f-g|| + \frac{7}{(n-1)}.x(1+x).||g''|| + \left|f(x) - f\left(\frac{nx}{(n-1)}\right)\right| \\
&\leq 7\left\{||f-g|| + \frac{x(1+x)}{(n-1)}.||g''||\right\} \\
&\quad + \sup_{t,t-(x/(n-1))\in[0,\infty)}\left|f\left(t - \frac{x}{(n-1)}\right) - f(t)\right| \\
&\leq 7\left\{||f-g|| + \frac{x(1+x)}{(n-1)}.||g''||\right\} + \omega\left(f, \frac{x}{(n-1)}\right).
\end{aligned}$$

Now, taking the infimum on the right-hand side over all $g \in W_\infty^2$ and using (2.24), we get the assertion of the theorem. ■

Theorem 2.52 ([140]). *Let $n > r+2 \geq 3$ and $f^{(i)} \in C_B[0,\infty)$ for $i \in \{0,1,2,\dots,r\}$. Then for $x \in [0,\infty)$, we have*

$$\begin{aligned}
&\left|S_n^{(r)}(f,x) - f^{(r)}(x)\right| \\
&\leq \left(\frac{n^r}{(n-1)(n-2)\cdots(n-r)} - 1\right)||f^{(r)}|| + \frac{n^r}{(n-1)(n-2)\cdots(n-r)} \\
&\quad \times \left[1 + \left(\frac{(n+(r+1)(r+2))x^2 + 2(n+r(r+2))x + r(r+1)}{n-r-2}\right)^{1/2}\right] \\
&\quad .\omega(f^{(r)}, (n-r-1)^{-1/2}).
\end{aligned}$$

Proof. Obviously,

$$\int_0^\infty b_{n-r,v+r-1}(t)dt = 1 \quad \text{and} \quad \sum_{v=0}^\infty s_{n,v}(x) = 1.$$

We can use Lemma 2.5 to obtain

$$\begin{aligned}S_n^{(r)}(f,x) - f^{(r)}(x) &= \frac{n^r}{(n-1)(n-2)\cdots(n-r)} \sum_{v=0}^\infty s_{n,v}(x) \\ &\quad \times \int_0^\infty b_{n-r,v+r-1} \left[f^{(r)}(t) - f^{(r)}(x) \right] dt \\ &\quad + \left[\frac{n^r}{(n-1)(n-2)\cdots(n-r)} - 1 \right] f^{(r)}(x). \quad (2.31)\end{aligned}$$

Taking into account the well-known property $\omega(f^{(r)},\lambda\delta) \leq (1+\lambda)\omega(f^{(r)},\delta)$, where $\lambda \geq 0$, we get

$$\begin{aligned}|S_n^{(r)}(f,x) - f^{(r)}(x)| &\leq \frac{n^r}{(n-1)(n-2)\cdots(n-r)} \sum_{v=0}^\infty s_{n,v}(x) \\ &\quad \times \int_0^\infty b_{n-r,v+r-1} \left| f^{(r)}(t) - f^{(r)}(x) \right| dt \\ &\quad + \left[\frac{n^r}{(n-1)(n-2)\cdots(n-r)} \right] ||f^{(r)}|| \\ &\leq \frac{n^r}{(n-1)(n-2)\cdots(n-r)} \sum_{v=0}^\infty s_{n,v}(x) \\ &\quad \times \int_0^\infty b_{n-r,v+r-1}(t)(1+\delta^{-1}|t-x|)\omega\left(f^{(r)},\delta\right) dt \\ &\quad + \left[\frac{n^r}{(n-1)(n-2)\cdots(n-r)} - 1 \right] ||f^{(r)}||. \quad (2.32)\end{aligned}$$

Further, using Cauchy's inequality, we have

$$\begin{aligned}&\sum_{v=0}^\infty s_{n,v}(x) \int_0^\infty b_{n-r,v+r-1}(t)|t-x|dt \\ &\quad \leq \left\{ \sum_{v=0}^\infty s_{n,v}(x) \int_0^\infty b_{n-r,v+r-1}(t)(t-x)^2 dt \right\}^{1/2}. \quad (2.33)\end{aligned}$$

Also, by direct computations, we have

$$\sum_{v=0}^{\infty} s_{n,v}(x)\int_0^{\infty} b_{n-r,v+r-1}(t)(t-x)^2dt$$
$$= \frac{n+(r+1)(r+2)}{(n-r-1)(n-r-2)}x^2 + \frac{2n+2r(r+2)}{(n-r-1)(n-r-2)}x + \frac{r(r+1)}{(n-r-1)(n-r-2)}.$$

Thus, by (2.32) and (2.33), we obtain

$$|S_n^{(r)}(f,x) - f^{(r)}(x)|$$
$$\leq \frac{n^r}{(n-1)(n-2)\ldots(n-r)}\left[1+\delta^{-1}\left(\frac{n+(r+1)(r+2)}{(n-r-1)(n-r-2)}x^2 + \frac{2n+2r(r+2)}{(n-r-1)(n-r-2)}x + \frac{r(r+1)}{(n-r-1)(n-r-2)}\right)^{1/2}\right]\omega(f^{(r)},\delta)$$
$$+\left(\frac{n^r}{(n-1)(n-2)\cdots(n-r)}-1\right)||f^{(r)}||.$$

Choosing $\delta^{-1} = \sqrt{n-r-1}$, we obtain the desired result. ■

A slightly different form of summation–integral-type Szász–Beta operators is discussed in [119], which, for $x \in [0,\infty)$, is defined as

$$\mathbf{S}_n(f,x) = \int_0^{\infty} K_n(x,t)f(t)dt = \sum_{v=1}^{\infty} s_{n,v}(x)\int_0^{\infty} b_{n,v}(t)dt + e^{-nx}f(0), \quad (2.34)$$

where

$$s_{n,v}(x)=e^{-nx}\frac{(nx)^v}{v!}, b_{n,v}(t)=\frac{1}{B(n+1,v)}\frac{t^{v-1}}{(1+t)^{n+v+1}} := \frac{(n+1)_v}{(v-1)!}\frac{t^{v-1}}{(1+t)^{n+v+1}},$$

and $(n)_k$ represents the Pochhammer symbol given by

$$(n)_v = n(n+1)(n+2)(n+3)\cdots(n+v-1).$$

An alternate form in terms of the Pochhammer–Berens confluent hypergeometric function of these operators is

$$\mathbf{S}(f,x) = (n+1)\int_0^{\infty}\frac{e^{-nx}nxf(t)}{(1+t)^{n+2}}\,{}_1F_1\left(n+2;2;\frac{nxt}{1+t}\right)dt + s_{n,0}(x)f(0).$$

Remark 2.1. For $n > 0$ and $r > 0$, the rth moments of these operators are given by

$$\mathbf{S}_n(t^r, x) = \frac{x\Gamma(n-r+1)\Gamma(r)}{\Gamma(n)} \ L^1_{r-1}(-nx),$$

where $L^1_{r-1}(-nx)$ is the generalized Laguerre polynomials, given by the relation

$$L^m_n(x) = \frac{(m+n)!}{m!n!} \ {}_1F_1\left(-n; m+1; x\right).$$

Remark 2.2. One can easily observe that

$$\mathbf{S}_n(t^i, x) = \frac{n^{i-1}(n-i)!}{(n-1)!}x^i + i(i-1)\frac{n^{i-2}(n-i)!}{(n-1)!}x^{i-1} + O(n^{-2}).$$

The main advantage of studying these operators is that they reproduce constant and linear functions. Some direct estimates in simultaneous approximation were established in [119].

Theorem 2.53 ([119] Asymptotic formula). *Let $f \in C_\gamma[0,\infty),\ \gamma > 0$. If $f^{(r+2)}$ exists at a point $x \in (0,\infty)$, then*

$$\lim_{n\to\infty} n\left[S_n^{(r)}(f,x) - f^{(r)}(x)\right] = \frac{r(r-1)}{2} f^{(r)}(x) + (x+1)rf^{(r+1)}(x) \\ +(x^2+x)f^{(r+2)}(x).$$

Theorem 2.54 ([119] Error estimation). *Let $f \in C_\gamma[0,\infty),\ \gamma > 0$ and $r \leq m \leq (r+2)$. If $f^{(m)}$ exists and is continuous on $(a-\eta, b+\eta), \eta > 0$, then for n sufficiently large, we have*

$$||S_n^{(r)}(f,.) - f^{(r)}|| \leq C_1 n^{-1}\sum_{j=r}^{m} ||f^{(i)}|| + C_2\omega(f^{(r+1)}, n^{-1/2}) + O(n^{-2}),$$

where the constants C_1 and C_2 are independent of f and n, $\omega(f,\delta)$ is the modulus of continuity of f on $(a-\eta, b+\eta)$, and $||.||$ denotes the sup-norm on the interval $[a,b]$.

Proof. By a Taylor expansion of f, we have

$$f(t) = \sum_{i=0}^{m} \frac{f^{(i)}(x)}{i!}(t-x)^i + \frac{f^{(m)}(\xi) - f^{(m)}(x)}{m!}(t-x)^m\chi(t) + h(t,x)(1-\chi(t)),$$

where ξ lies between t and x and $\chi(t)$ is the characteristic function on the interval $(a-\eta, b+\eta)$. For $t \in (a-\eta, b+\eta), x \in [a,b]$, we have

$$f(t)=\sum_{i=0}^{m}(t-x)^i\frac{f^{(i)}(x)}{i!}+(t-x)^i\frac{f^{(m)}(\xi)-f^{(m)}(x)}{m!}.$$

For $t\in[0,\infty)\setminus(a-\eta,b+\eta)$, we define

$$h(t,x)=f(t)-\sum_{i=0}^{m}(t-x)^i\frac{f^{(i)}(x)}{i!}.$$

$$\begin{aligned}\mathbf{S}_n^{(r)}(f,x)-f^{(r)}(x)&=\left\{\sum_{i=0}^{m}\frac{f^{(i)}(x)}{i!}\mathbf{S}_n^{(r)}((t-x)^i,x)-f^{(r)}(x)\right\}\\&\quad+\mathbf{S}_n^{(r)}\left(\frac{f^{(m)}(\xi)-f^{(m)}(x)}{m!}(t-x)^m\chi(t),x\right)\\&\quad+\mathbf{S}_n^{(r)}\left(h(t,x)(1-\chi(t)),x\right)\\&:=E_1+E_2+E_3.\end{aligned}$$

Using Remark 2.2, we obtain

$$\begin{aligned}E_1&=\sum_{i=0}^{m}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}\frac{d^r}{dx^r}\\&\quad\times\left[\frac{n^{j-1}(n-j)!}{(n-1)!}x^j+j(j-1)\frac{n^{j-2}(n-j)!}{(n-1)!}x^{j-1}+O(n^{-2})\right]-f^{(r)}(x).\end{aligned}$$

Consequently,

$$\|E_1\|_{C[a,b]}\le C_1n^{-1}\sum_{i=r}^{m}\|f^{(i)}\|_{C[a,b]}+O(n^{-2}),\text{uniformly on }[a,b].$$

Next, we estimate E_2 as follows:

$$\begin{aligned}|E_2|&\le\int_0^\infty K_n^{(r)}(x,t)\left|\frac{f^{(m)}(\xi)-f^{(m)}(x)}{m!}\right||t-x|^m\chi(t)dt\\&\le\frac{\omega(f^{(m)},\delta)}{m!}\int_0^\infty|K_n^{(r)}(x,t)|\left(1+\frac{|t-x|}{\delta}\right)|t-x|^m\,dt.\end{aligned}$$

Next, by Remark 2.1, it follows that

$$\sum_{v=1}^{\infty}s_{n,v}(x)|v-nx|^j\int_0^\infty b_{n,v}(t)\,|t-x|^q\,dt=O((n^{(j=q)/2}).$$

Therefore, by Lemma 2.4, we get

$$\sum_{v=1}^{\infty}|s_{n,v}^{(r)}(x)|\int_0^{\infty}b_{n,v}(t)\,|t-x|^q\,dt$$
$$\leq C_2\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i\Big[\sum_{v=1}^{\infty}s_{n,v}(x)|v-nx|^j\int_0^{\infty}b_{n,v}(t)|t-x|^q dt\Big]$$
$$=O(n^{(r-q)/2}),$$

uniformly in x, where $C_2=\sup\limits_{\substack{2i+j\leq r\\ i,j\geq 0}}\sup\limits_{x\in[a,b]}\dfrac{|Q_{i,j,r}(x)|}{x^r}$. Choosing $\delta=n^{-1/2}$, we obtain, for $s>0$,

$$||E_2||_{C[a,b]}\leq\frac{\omega(f^{(m)},n^{-1/2})}{m!}\big[O(n^{(r-m)/2})+n^{1/2}O(n^{(r-m-1)/2})+O(n^{-s})\big]$$
$$\leq C_2 n^{-(m-r)/2}\omega(f^{(m)},n^{-1/2}).$$

Since $t\in[0,\infty)\setminus(a-\eta,b+\eta)$, we can choose δ such that $|t-x|\geq\delta$ for all $x\in[a,b]$. Thus, by Lemma 2.4, we get

$$||E_2||_{C[a,b]}\leq\sum_{v-1}^{\infty}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}n^i s_{n,v}(x)|v-nx|^j\int_{|t-x|\geq\delta}b_{n,v}(t)|h(t,x)|dt$$
$$+(-n)^r e^{-nx}|h(0,x)|.$$

If β is any integer greater than or equal to $\{\gamma,m\}$, then we can find a constant C_3 such that $|h(t,x)|\leq C_3|t-x|^{\beta}$ for $|t-x|\geq\delta$. Hence, using the Schwarz inequality for both integration and summation, Lemma 2.1 and Remark 2.1, we easily see that $E_3=O(n^{-q})$ for any $q>0$, uniformly on $[a,b]$. Combining the estimates of $E_1,\ E_2,\ E_3$, we see that the required result is immediate. ■

Theorem 2.55 ([119] Direct result). *Let $f\in C_B[0,\infty)$. Then for all $x\in[0,\infty)$ and $n=2,3,4,\ldots$, there exists an absolute positive constant $C>0$ such that*

$$|\boldsymbol{S}_n(f,x)-f(x)|\leq C\omega_2\left(f,\sqrt{\frac{x(2+x)}{n-1}}\right),$$

where $C_B[0,\infty)$ is given in Definition 1.10.

Proof. Let $g\in W_{\infty}^2$ and $x,t\in[0,\infty)$. Applying the Taylor expansion

$$g(t)=g(x)+g'(x)(t-x)+\int_x^t(t-u)g''(u)du,$$

we have by Remark 2.1 that

$$\mathbf{S}_n(g,x) - g(x) = \mathbf{S}_n\left(\int_x^t (t-u)g''(u)du, x\right).$$

On the other hand, $|\int_x^t (t-u)g''(u)du| \leq (t-x)^2||g''||$. Therefore,

$$|\mathbf{S}_n(g,x) - g(x)| \leq \mathbf{S}_n\left((t-x)^2, x\right)||g''|| = \frac{x(2+x)}{n-1}||g''||.$$

Also, we have

$$|\mathbf{S}_n(f,x)| \leq \sum_{v=1}^{\infty} s_{n,v}(x)\int_0^{\infty} b_{n,v}(t)|f(t)|dt + e^{-nx}|f(0)| \leq ||f||.$$

Thus,

$$\begin{aligned}|\mathbf{S}_n(f,x) - f(x)| &\leq |\mathbf{S}_n(f-g,x) - (f-g)(x)| \\ &\quad + |\mathbf{S}_n(g,x) - g(x)| \\ &\leq 2||f-g|| + \frac{x(2+x)}{n-1}||g''||.\end{aligned}$$

Now taking the infimum over all $g \in W_\infty^2$ and using the inequality $K_2(f,\delta) \leq C\omega_2(f,\sqrt{\delta}), \delta > 0$, we get the required result. ■

For $0 \leq \lambda \leq 1$ and $C_B[0,\infty)$ as given in Definition 1.10, the weighted modulus of continuity and K-functional are defined as

$$\omega_{\varphi^\lambda}^2(f,t) = \sup_{0<h\leq t}\ \sup_{x \pm h\varphi^\lambda(x)\in[0,\infty)} |\Delta_{h\varphi^\lambda}^2 f|.$$

$$D_\lambda^2 = \{f \in C_B[0,\infty) : f' \in A.C._{loc(0,\infty)}, ||\varphi^{2\lambda} f''|| < \infty\},$$

$$K_{\varphi^\lambda}^2(f,t^2) = \inf\{||f-g|| + t^2||\varphi^{2\lambda}g''|| : g \in D_\lambda^2\},$$

$$\overline{D}_\lambda^2 = \{f \in D_\lambda^2 : ||f''|| < \infty\},$$

$$\overline{K}_{\varphi^\lambda}^2(f,t^2) = \inf\{||f-g|| + t^2||\varphi^{2\lambda}g''|| + t^{4/(2-\lambda)}||g''|| : g \in \overline{D}_\lambda^2\}.$$

It is well known that

$$\omega_{\varphi^\lambda}^2(f,t) \sim K_{\varphi^\lambda}^2(f,t^2) \sim \overline{K}_{\varphi^\lambda}^2(f,t^2).$$

Qi and Zhang [204] obtained the pointwise approximation equivalence theorems using the above unified modulus of smoothness.

Theorem 2.56 ([204] Direct result). *Let* $f \in C_B[0,\infty), 0 \leq \lambda \leq 1$, $\delta_n^2(x) = \varphi^2(x) + \frac{1}{n}$, *and* $\varphi(x) = \sqrt{x(2+x)}, n \geq 4$. *Then*

$$|S_n(f,x) - f(x)| \leq C\omega_{\varphi^\lambda}^2 \left(f, n^{-1/2}\delta_n^{1-\lambda}(x)\right).$$

Theorem 2.57 ([204] Inverse result). *Let* $f \in C_B[0,\infty), 0 < \alpha < 2, 0 \leq \lambda \leq 1$, $\delta_n^2(x) = \varphi^2(x) + \frac{1}{n}$, *and* $\varphi(x) = \sqrt{x(2+x)}, n \geq 3$. *Then*

$$|S_n(f,x) - f(x)| = O((n^{-1/2}\delta_n^{1-\lambda}(x))^\alpha)$$

if and only if $\omega_{\varphi^\lambda}^2(f,t) = O(t^\alpha)$.

2.5 Phillips Operators

Let $C_B[0,\infty)$ be as defined in Definition 1.10 equipped with the norm $||f|| = \sup_{x\geq 0}|f(x)|$. The Phillips operators (see [190, 199]) are defined by

$$(L_n f)(x) \equiv L_n(f,x) = n\sum_{k=1}^{\infty} p_{n,k}(x)\int_0^\infty p_{n,k-1}(v)f(v)dv + e^{-nx}f(0), \quad (2.35)$$

where

$$p_{n,k}(x) = e^{-nx}.\frac{(nx)^k}{k!},$$

$k = 0, 1, \ldots$, and $x \geq 0$.

Taking into account the construction given in [71], we define the following Phillips-type operator:

$$L_n^t(f,x) \equiv (L_n^t f)(x) = \frac{1}{G(x,t)}.\int_0^\infty g(x,t,\theta).L_n(f,\theta)d\theta \quad (2.36)$$

and $L_n^t(f,0) = f(0)$, where $g : [0,\infty) \times (t_0,\infty) \times [0,\infty) \to [0,\infty)$ is a function such that $g(x,t,.) \in C[0,\infty) \bigcap L^1[0,\infty)$ for all $(x,t) \in [0,\infty) \times (t_0,\infty)$ and $t_0 \geq 0$, $\quad G : (0,\infty) \times (t_0,\infty) \to (0,\infty)$,

$$G(x,t) = \int_0^\infty g(x,t,\theta)d\theta, \quad (2.37)$$

with the assumptions

$$\frac{1}{G(x,t)} \cdot \int_0^\infty g(x,t,\theta).(\theta - x)d\theta = 0 \tag{2.38}$$

and

$$\frac{1}{G(x,t)} \cdot \int_0^\infty g(x,t,\theta).(\theta - x)^2 d\theta \leq \beta(t)x \tag{2.39}$$

for every $(x,t) \in (0,\infty) \times (t_0,\infty)$ and $\beta : (t_0,\infty) \to (0,\infty)$ a given function. The parameter t may depend only on the natural number n. For example, an operator of type (2.36) is the following:

$$S_n^t(f,x) = \left(1+\frac{n}{t}\right)^{-tx} \cdot \left\{ n \sum_{k=1}^{\infty} \left(1+\frac{t}{n}\right)^{-k} \cdot \frac{tx(tx+1)\cdots(tx+k-1)}{k!} \right.$$
$$\left. \cdot \int_0^\infty p_{n,k-1}(v)f(v)dv + f(0) \right\},$$

where $f \in C_B[0,\infty)$, $x \geq 0$ and $t > 0$ (see [71]). The Phillips operators are closely related to the Szász operators [71] defined by

$$S_n(f,x) \equiv (S_n f)(x) = \sum_{k=0}^{\infty} p_{n,k}(x) f\left(\frac{k}{n}\right). \tag{2.40}$$

Indeed, if we replace the discrete values $f(k/n)$ in (2.40) by the integral terms

$$n \int_0^\infty p_{n,k-1}(v)f(v)dv, \quad k = 1, 2, \ldots,$$

then we arrive at (2.35). Another generalization of (2.35) can be found in [99].

In [20, 21, 123], the local approximation properties of the operators defined by (2.35) were studied. Let us consider $C_N[0,\infty) := \{f \in C[0,\infty) : |f(t)| \leq Ce^{Nt}, C > 0\}$. The norm $||.||_{C_N}$ for functions belonging to the class $C_N[0,\infty)$ is defined as

$$||f||_{C_N} = \sup_{x \in [0,\infty)} |f(t)|e^{-Nt}.$$

Theorem 2.58 ([21] Pointwise convergence). *If $p = 1, 2, 3, \ldots$, $f \in C_N[0,\infty)$ for some $N > 0$ and $f^{(p)}$ exist at a point $x \in [0,\infty)$, then*

$$L_n^{(p)}(f,x) = f^{(p)}(x) + o(1), as \;\; n \to \infty. \tag{2.41}$$

Furthermore, if $f^{(p)}$ exists and is continuous on $(a-\eta, b+\eta) \subset (0,\infty), \eta > 0$, then (2.41) holds uniformly in $x \in [a,b]$.

Theorem 2.59 ([21] Asymptotic formula). *Let $f \in C_N[0,\infty)$ for some $n > 0$. If $f^{(p+2)}$ exists at a point $x \in (0,\infty)$, then*

$$\lim_{\lambda\to\infty} \lambda[L_n^{(p)}(f,x) - f^{(p)}(x)] = pf^{(p+1)}(x) + xf^{(p+2)}(x). \tag{2.42}$$

Additionally, if $f^{(p+2)}$ exists and is continuous on $(a-\eta, b+\eta) \subset (0,\infty), \eta > 0$, then (2.42) holds uniformly in $x \in [a,b]$.

Theorem 2.60 ([21] Error estimation). *Let $f \in C_N[0,\infty)$ for some $n > 0$ and $p \le q \le p+2$. If $f^{(q)}$ exists and is continuous on $(a-\eta, b+\eta) \subset (0,\infty), \eta > 0$, then for λ sufficiently large,*

$$||L_n^{(p)}(f,.) - f^{(p)}|| \le \max\{Cn^{(-(q-p)/2}\omega(f^{(q)}, n^{-1/2}), C'n^{-1}\},$$

where $C = C(p), C' = C'(p,f)$, $\omega(f^{(q)},\delta)$ is the modulus of continuity of $f^{(q)}$ on $(a-\eta, b+\eta)$, and $||.||$ denotes the sup-norm on $[a,b]$.

In [72], direct results are formulated by the second-order Ditzian–Totik modulus of smoothness given by

$$\omega_\varphi^2(f,\delta) = \sup_{0<h\le\delta} \sup_{x\pm h\varphi(x)\in[0,\infty)} |f(x+h\varphi(x)) - 2f(x) + f(x-h\varphi(x))|,$$

$\varphi(x) = \sqrt{x}, x \ge 0$. The corresponding K-functional is

$$K_{2,\varphi}(f,\delta^2) = \inf_{h\in W_\infty^2(\varphi)} \{||f-h|| + \delta^2||\varphi^2 h''||\},$$

where

$$W_\infty^2(\varphi) = \{h \in C_B[0,\infty) : h' \in AC_{loc}[0,\infty),\ \varphi^2 h'' \in C_B[0,\infty)\}.$$

It is known (see [62]) that $K_{2,\varphi}(f,\delta^2)$ and $\omega_\varphi^2(f,\delta)$ are equivalent; that is, there are an absolute positive constant and $C > 0$ such that

$$C^{-1}\omega_\varphi^2(f,\delta) \le K_{2,\varphi}(f,\delta^2) \le C\omega_\varphi^2(f,\delta). \tag{2.43}$$

Finta and Gupta [72] estimated a global direct result for Phillips-type operators. To prove the direct result, they used the following lemmas:

Lemma 2.6 ([72]). *The operators L_n and L_n^t are linear positive and*

$$L_n(u-x,x) = L_n^t(u-x,x) \equiv 0$$

and

$$L_n((u-x)^2,x) = \frac{2}{n}.\varphi(x)^2, \qquad x \ge 0.$$

Moreover, $||L_n f|| \le ||f||$ and $||L_n^t f|| \le ||f||, \quad f \in C_B[0,\infty)$.

Proof. These properties can be derived by simple computations using (2.35)–(2.38):

$$\sum_{k=1}^{\infty} p_{n,k}(x) = 1 \tag{2.44}$$

and

$$n\int_0^{\infty} p_{n,k-1}(v)dv = 1, \quad k = 1,2,3,\ldots. \tag{2.45}$$

■

Lemma 2.7 ([72]). *For* $h \in W_{\infty}^2(\varphi)$*, we have*

$$n\int_0^{\infty} p_{n,k-1}(v)\left|\int_{k/n}^{v}(v-w)h''(w)dw\right|dv \le \frac{1}{n}||\varphi^2 h''||,$$

where $k = 1,2,\ldots$.

Proof. By [62], we obtain

$$\frac{|v-w|}{w} \le \frac{|v-(k/n)|}{k/n}$$

for w between k/n and v. Hence,

$$\begin{aligned}\left|\int_{k/n}^{v}(v-w)h''(w)dw\right| &\le \left|\int_{k/n}^{v}\frac{|v-w|}{w}.w|h''(w)|dw\right| \\ &\le \left|\int_{k/n}^{v}\frac{|v-w|}{w}dw\right|.||\varphi^2 h''|| \\ &\le \frac{n}{k}.\left(v-\frac{k}{n}\right)^2.||\varphi^2 h''||.\end{aligned}$$

Thus, by (2.45), after simple computation, we have

$$\begin{aligned}n\int_0^{\infty} p_{n,k-1}(v)\left|\int_{k/n}^{v}(v-w)h''(w)dw\right|dv &\le \frac{n^2}{k}||\varphi^2 h''||\int_0^{\infty} p_{n,k-1}(v)\left(v-\frac{k}{n}\right)^2 dv \\ &= \frac{n^2}{k}||\varphi^2 h''||.\frac{k}{n^3}=\frac{1}{n}.||\varphi^2 h''||,\end{aligned}$$

which was to be proved. ■

Lemma 2.8. *If*

$$r_{n,k}(x)=\left(\frac{k}{n}-x\right)^2-\frac{k}{n^2},$$

$k=0,1,2,\ldots$, *and* $x\geq 0$, *then for* $x>0$, *we have*

$$(L_nf)''(x)=\frac{n^2}{x^2}.\sum_{k=1}^{\infty}p_{n,k}(x).n\int_0^{\infty}p_{n,k-1}(v)f(v)dv.r_{n,k}(x)+n^2e^{-nx}f(0)$$

and

$$(S_nf)''(x)=\frac{n^2}{x^2}.\sum_{k=0}^{\infty}p_{n,k}(x)f\left(\frac{k}{n}\right).r_{n,k}(x).$$

Proof. Because

$$xp'_{n,k}(x)=(k-nx).p_{n,k}(x), \tag{2.46}$$

$$p''_{n,k}(x)=\frac{1}{x^2}\left[(k-nx)^2-k\right].p_{n,k}(x). \tag{2.47}$$

Hence,

$$\begin{aligned}(L_nf)''(x)&=n\sum_{k=1}^{\infty}p_{n,k}(x)''.n\int_0^{\infty}p_{n,k-1}(v)f(v)dv+n^2e^{-nx}f(0)\\&=\frac{n^2}{x^2}.\sum_{k=1}^{\infty}p_{n,k}(x).n\int_0^{\infty}p_{n,k-1}(v)f(v)dv.r_{n,k}(x)+n^2e^{-nx}f(0).\end{aligned}$$

The expression of $(S_nf)''$ is known, and it can be obtained in a similar way from (2.40). ∎

Lemma 2.9. *We have*

$$||\varphi^2(L_nh)''||\leq 8||\varphi^2(h)''||, \qquad h\in W^2_{\infty}(\varphi).$$

Proof. If we use Lemma 2.8, (2.51), (2.52), (2.45), and Lemma 2.7, then

$$\begin{aligned}&\varphi(x)^2\left|(L_nh)''(x)-(S_nh)''(x)\right|\\&\leq\frac{n}{x}.||\varphi^2h''||.\sum_{k=1}^{\infty}p_{n,k}(x).\left|r_{n,k}(x)\right|\end{aligned}$$

$$\leq \frac{n}{x}.||\varphi^2 h''||.\sum_{k=1}^{\infty} p_{n,k}(x).\left[\left(\frac{k}{n}-x\right)^2+\frac{k}{n^2}\right]$$

$$\leq \frac{n}{x}.||\varphi^2 h''||.\left[S_n\left((u-x)^2,x\right)+\frac{1}{n}S_n(u,x)\right]$$

$$= \frac{n}{x}.||\varphi^2 h''||.\frac{2x}{n} = 2||\varphi^2 h''||. \tag{2.48}$$

Following [62], we have

$$||\varphi^2(S_n h)''|| \leq 6||\varphi^2(h)''||, \quad h \in W_\infty^2(\varphi).$$

Hence, by (2.48), we have

$$||\varphi^2(L_n h)''|| \leq ||\varphi^2(L_n h)'' - \varphi^2(S_n h)''|| + ||\varphi^2(S_n h)''|| \leq 8||\varphi^2(h)''||.$$

■

Theorem 2.61 ([72] Direct estimate). *Let $(L_n)_{n\geq 1}$ and $(L_n^t)_{n\geq 1}$ be defined as in (2.35) and (2.36). Then there exists $C > 0$ such that*

$$||L_n f - f|| \leq C\omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right) \tag{2.49}$$

and

$$||L_n^t f - f|| \leq C\omega_\varphi^2\left(f, (n^{-1}+\beta(t))^{1/2}\right) \tag{2.50}$$

for all $f \in C_B[0,\infty)$ and $n = 1, 2, \ldots$.

Proof. Let $h \in W_\infty^2(\varphi)$ and $x \geq 0$. By a Taylor expansion,

$$h(v) = h\left(\frac{k}{n}\right) + h'\left(\frac{k}{n}\right)\left(v-\frac{k}{n}\right) + \int_{k/n}^{v}(v-w)h''(w)dw \tag{2.51}$$

and by (2.45), we obtain

$$|L_n(h,x) - S_n(h,x)|$$
$$\leq n\sum_{k=1}^{\infty} p_{n,k}(x)\left|\int_0^\infty p_{n,k-1}(v)\left[h(v)-h\left(\frac{k}{n}\right)\right]dv\right|$$
$$= n\sum_{k=1}^{\infty} p_{n,k}(x)\left|\int_0^\infty p_{n,k-1}(v)\left[h'\left(\frac{k}{n}\right)\left(v-\frac{k}{n}\right)+\int_{k/n}^{v}(v-w)h''(w)dw\right]dv\right|$$

$$= n\sum_{k=1}^{\infty} p_{n,k}(x)\left|\int_0^{\infty} p_{n,k-1}(v)\left[\int_{k/n}^{v}(v-w)h''(w)dw\right]dv\right|$$
$$\leq n\sum_{k=1}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k-1}(v)\left|\int_{k/n}^{v}(v-w)h''(w)dw\right|dv,$$

because

$$\int_0^{\infty} p_{n,k-1}(v)\left(v-\frac{k}{n}\right)dv = 0, \tag{2.52}$$

$k = 1, 2, \ldots$. Using Lemma 2.7 and (2.44), we get

$$|L_n(h,x) - S_n(h,x)| \leq \frac{1}{n}||\varphi^2 g''||.\sum_{k=1}^{\infty} p_{n,k}(x) \leq \frac{1}{n}||\varphi^2 h''||. \tag{2.53}$$

Now, let $f \in C_B[0,\infty)$. Then, in view of Lemma 2.6, (2.53), and [62], we have

$$\begin{aligned}
&|L_n(h,x) - S_n(h,x)| \\
&\quad\leq |L_n(f-h,x) - (f-h)x| + |L_n(h,x) - S_n(h,x)| + |S_n(h,x) - h(x)| \\
&\quad\leq 2||f-h|| + \frac{2}{n}||\varphi^2 h''||.
\end{aligned}$$

Hence,

$$||L_n f - S_n f|| \leq 2\inf_{h\in W_\infty^2(\varphi)}\left\{||f-h|| + \frac{1}{n}||\varphi^2 h''||\right\} = 2K_{2,\varphi}\left(f,\frac{1}{n}\right),$$

which completes the proof of (2.49).

Let $f \in C_B[0,\infty)$ and $x > 0$. By a Taylor expansion,

$$(L_n f)(\theta) = (L_n f)(x) + (L_n f)'(x).(\theta - x) + \int_x^{\theta}(\theta - w)(L_n f)''(w)dw$$

andby (2.37), (2.38), [62], we have

$$\begin{aligned}
&|(L_n^t f)(x) - (L_n f)(x)| \\
&\quad= \left|\frac{1}{G(x,t)}.\int_0^{\infty} g(x,t,\theta)\left\{\int_x^{\theta}(\theta-w)(L_n f)''(w)dw\right\}d\theta\right| \\
&\quad\leq \frac{1}{G(x,t)}.\int_0^{\infty} g(x,t,\theta)\left|\int_x^{\theta}\frac{(\theta-w)}{\varphi(w)^2 dw}\right|d\theta.||\varphi^2(L_n f)''||
\end{aligned}$$

$$\leq \frac{1}{G(x,t)}.\int_0^\infty g(x,t,\theta)(\theta-x)^2 d\theta.\frac{1}{\varphi(x)^2}||\varphi^2(L_n f)''||.$$

Hence, in view of (2.39), we get

$$||L_n^t f - L_n f|| \leq \beta(t)||\varphi^2(L_n f)''||. \tag{2.54}$$

Next, using (2.54) and Lemma 2.9, we get

$$||L_n^t h - L_n h|| \leq 8\beta(t)||\varphi^2 h''||.$$

Then, by Lemma 2.6, we have

$$\begin{aligned}||L_n^t f - L_n f|| &\leq ||L_n^t(f-h) - L_n(f-h)|| + ||L_n^t h - L_n h||\\ &\leq 2||f-h|| + 8\beta(t)||\varphi^2 h''||\\ &\leq 8\left\{||f-h|| + \beta(t)||\varphi^2 h''||\right\}.\end{aligned} \tag{2.55}$$

By (2.49), (2.43), and the definition of $K_{2,\varphi}(f, n^{-1})$, we obtain

$$||L_n f - f|| \leq C\left\{||f-h|| + \frac{1}{n}||\varphi^2 h''||\right\}. \tag{2.56}$$

Then (2.55) and (2.56) imply

$$||L_n^t f - f|| \leq ||L_n^t f - L_n f|| + ||L_n f - f|| \leq C\left\{||f-h|| + \left(\beta(t) + \frac{1}{n}\right)||\varphi^2 h''||\right\}$$

or

$$||L_n^t f - f|| \leq C.K_{2,\varphi}\left(f, \frac{1}{n}\right) + \beta(t).$$

Again using (2.43), we arrive at (2.50), which completes the proof of the theorem. ■

In 2011, Heilmann and Tachev [151] proved the commutativity of the Phillips operators as well as their commutativity with an appropriate differential operator and established a strong converse inequality of type A for the approximation of real-valued continuous bounded functions f on the interval $[0,\infty)$. In the following, the norm is defined as $||f|| = \sup_{x\geq 0}|f(x)|$. They also obtained a strong Voronovskaja-type theorem.

Theorem 2.62 ([151] Voronovskaja-type theorem). *Let* $g \in C_B[0,\infty), \varphi^2 g''',$ $\varphi^3 g''' \in C_B[0,\infty)$, *and* $n > 0$. *Then*

$$\left\|L_n g - g - \frac{1}{n}\varphi^2 g''\right\| \leq \frac{\sqrt{6}}{2}\frac{1}{n}\max\left\{\frac{4}{3}\sqrt{1+2c}\frac{1}{\sqrt{n}}||\varphi^3 g'''||, \sqrt{\frac{1+2c}{c}}\frac{1}{n}||\varphi^2 g'''||\right\},$$

where c *denotes an arbitrary positive constant.*

Theorem 2.63 ([151]). *For every* $f \in C_B[0,\infty)$ *and* $n > 0$,

$$||L_n f - f|| \leq 2K_\varphi^2\left(f, \frac{1}{n}\right)$$

holds.

Theorem 2.64 ([151]). *For every* $f \in C_B[0,\infty)$ *and* $n > 0$,

$$K_\varphi^2\left(f, \frac{1}{n}\right) \leq 92.16\,||L_n f - f||$$

holds.

2.6 Integral Modification of Jain Operators

In 1972, Jain [161] introduced the following operators:

$$B_n^\beta(f,x) = \sum_{k=0}^{\infty} l_{n,k}^{(\beta)}(x) f(k/n), \tag{2.57}$$

where $0 \leq \beta < 1$ and the basis function is defined as

$$l_{n,k}^{(\beta)}(x) = \frac{nx(nx+k\beta)^{k-1}}{k!}e^{-(nx+k\beta)}.$$

In [161], $\sum_{k=0}^{\infty} l_{n,k}^{(\beta)}(x) = 1$. As a special case when $\beta = 0$, the operators (2.57) reduce to the well-known Szász–Mirakyan operators. In 1985, Umar and Razi [235] extended the studies, proposing the Kantorovich-type integral modification of the operators (2.57).

To approximate Lebesgue integrable functions on the interval $[0,\infty)$, Gupta et al. [146] introduced a new sequence of summation–integral-type operators, defined as

$$D_{n,c}^{(\beta)}(f,x) = n\sum_{k=1}^{\infty} l_{n,k}^{(\beta)}(x)\int_0^{\infty} p_{n+c,k-1}(t,c)f(t)\,dt + l_{n,0}^{(\beta)}(x)f(0), \quad (2.58)$$

where

$$l_{n,k}^{(\beta)}(x) = \frac{nx(nx+k\beta)^{k-1}}{k!}e^{-(nx+k\beta)},\ p_{n,k}(t,c) = \frac{(-t)^k}{k!}\phi_{n,c}^{(k)}(t),$$

with $c \geq 0$, and two special cases:

1. $\phi_{n,0}(t) = e^{-nt}$,
2. $\phi_{n,1}(t) = (1+t)^{-n}$.

For the case $\beta = 0,\ c = 0$, our operators $D_{n,c}^{(\beta)}$ reduce to Phillips operators [199]. For $\beta = 0,\ c = 1$, we get the Szász–Beta-type operators introduced by Gupta et al. [140]. One can easily observe by simple computation that

$$\int_0^{\infty} p_{n,k}(t,c)t^r\,dt = \frac{(k+r)!}{k!}\frac{1}{\prod_{i=1}^{r+1}(n-ic)}. \quad (2.59)$$

The present section deals with the approximation properties of the operators $D_{n,c}^{(\beta)}(f,x)$. Here we present some direct theorems, which include pointwise convergence, asymptotic formula, rate of approximation in terms of modulus of continuity, and weighted approximation.

Lemma 2.10 ([161]). *For $\beta < 1$, we have*

$$B_n^{\beta}(1,x) = 1,\ B_n^{\beta}(t,x) = \frac{x}{1-\beta}$$

and

$$B_n^{\beta}(t^2,x) = \frac{x^2}{(1-\beta)^2} + \frac{x}{n(1-\beta)^3}.$$

Lemma 2.11 ([146]). *For $\beta < 1$, we have*

$$D_{n,c}^{(\beta)}(1,x) = 1,\quad D_{n,c}^{(\beta)}(t,x) = \frac{nx}{(n-c)}\cdot\frac{1}{(1-\beta)},$$

$$D_{n,c}^{(\beta)}(t^2,x) = \frac{nx}{(n-c)(n-2c)}\left[\frac{nx}{(1-\beta)^2} + \frac{1+(1-\beta)^2}{(1-\beta)^3}\right].$$

Remark 2.3. From Lemma 2.11, it follows that

$$D_{n,c}^{(\beta)}(t-x,x) = \frac{\beta x}{1-\beta} + \frac{cx}{(n-c)(1-\beta)},$$

$$D_{n,c}^{(\beta)}((t-x)^2,x) = \frac{x^2}{(1-\beta)^2}\left[\beta^2+\frac{nc(1+4\beta)+2c^2(1-2\beta)}{(n-c)(n-2c)}\right]+\frac{nx[1+(1-\beta)^2]}{(n-c)(n-2c)(1-\beta)^3}.$$

Furthermore, $D_{n,c}^{(\beta)}((t-x)^m,x)$ is a polynomial in x of degree m, and for every $x \in [0,\infty)$,

$$D_{n,c}^{(\beta)}((t-x)^m,x) = O\left(\frac{1}{n^{[(m+1)/2]}}\right), \quad \text{as } n \to \infty.$$

Theorem 2.65 ([146]). *Let $f \in C[0,\infty)$ and $\beta \to 0$ as $n \to \infty$. Then the sequence $\{D_{n,c}^{(\beta)}(f,x)\}$ converges uniformly to $f(x)$ in $[a,b]$, where $0 \le a < b < \infty$.*

Proof. In view of Lemma 2.11, $D_{n,c}^{(\beta)}(1,x) = 1$ for every $n \in \mathbf{N}$, $D_{n,c}^{(\beta)}(t,x) = \frac{nx}{(n-c)(1-\beta)}$ tends to x, and $D_{n,c}^{(\beta)}(t,x) = \frac{nx}{(n-c)(n-2c)}\left[\frac{nx}{(1-\beta)^2} + \frac{1+(1-\beta)^2}{(1-\beta)^3}\right]$ tends to x^2 as $n \to \infty$, uniformly on every compact subset of $[0,\infty)$. Hence, by the Bohman–Korovkin theorem, the required result follows. ∎

Theorem 2.66 ([146]). *Let f be bounded and integrable on $[0,\infty)$ and have a second derivative at a point $x \in [0,\infty)$ with $\beta \to 0$ as $n \to \infty$. Then*

$$\lim_{n\to\infty} n[D_{n,c}^{(\beta)}(f,x) - f(x)] = cxf'(x) + \frac{x(cx+2)}{2}f''(x).$$

Remark 2.4. For $0 \le \beta \le \beta'/n < 1$, from Remark 2.3, we have

$$\alpha_1 := D_{n,c}^{(\beta)}(t-x,x) \le \frac{\beta' x}{n(1-\beta)} + \frac{cx}{(n-c)(1-\beta)},$$

$$\alpha_2 := D_{n,c}^{(\beta)}((t-x)^2,x) \le \frac{x^2}{(1-\beta)^2}\left[\frac{\beta\beta'}{n}+\frac{nc(1+4\beta)+2c^2(1-2\beta)}{(n-c)(n-2c)}\right] + \frac{nx[1+(1-\beta)^2]}{(n-c)(n-2c)(1-\beta)^3}.$$

Theorem 2.67 ([146]). *Let $f \in C_B[0,\infty)$ and $0 \le \beta \le \beta'/n < 1$. Then*

$$|D_{n,c}^{(\beta)}(f,x) - f(x)| \le C\omega_2(f,\sqrt{\delta_n}) + \omega\left(f, \left|\frac{\beta' x}{n(1-\beta)} + \frac{cx}{(n-c)(1-\beta)}\right|\right),$$

where C is a positive constant and $\delta_n = \alpha_1^2 + \alpha_2$, which are given in Remark 2.4.

The next result is the weighted approximation theorem, where the approximation formula holds on the interval $[0,\infty)$.

Let $B_{x^2}[0,\infty) = \{f : \text{for every } x \in [0,\infty), |f(x)| \leq M_f(1+x^2),\ M_f \text{ being a constant depending on } f\}$. By $C^*_{x^2}[0,\infty)$, we denote the subspace of all continuous functions belonging to $B_{x^2}[0,\infty)$ and satisfying the condition $\lim_{x\to\infty} \frac{f(x)}{1+x^2}$ is finite. The norm on $C^*_{x^2}[0,\infty)$ is $\|f\|_{x^2} = \sup_{x\in[0,\infty)} \frac{f(x)}{1+x^2}$.

Theorem 2.68 ([146]). *Let $\beta \to 0$ as $n \to \infty$. Then for each $f \in C^*_{x^2}[0,\infty)$ and $n > 2c$, we have*

$$\lim_{n\to\infty} \|D^{(\beta)}_{n,c}(f) - f\|_{x^2} = 0.$$

Proof. Using [75], we see that it is sufficient to verify the following conditions:

$$\lim_{n\to\infty} \|D^{(\beta)}_{n,c}(t^\nu, x) - x^\nu\|_{x^2} = 0, \nu = 0, 1, 2. \tag{2.60}$$

Since $D^{(\beta)}_{n,c}(1,x) = 1$, therefore for $\nu = 0$ (2.60) holds. By Lemma 2.11 for $n > c$, we have

$$\begin{aligned}
\|D^{(\beta)}_{n,c}(t,x) - x\|_{x^2} &= \sup_{x\in[0,\infty)} \frac{|D^{(\beta)}_{n,c}(t,x) - x|}{1+x^2}, \\
&\leq \left(\frac{n}{(n-c)(1-\beta)} - 1\right) \sup_{x\in[0,\infty)} \frac{x}{1+x^2} \\
&\leq \left(\frac{n}{(n-c)(1-\beta)} - 1\right),
\end{aligned}$$

and the condition (2.60) holds for $\nu = 1$ as $n \to \infty$. Again by Lemma 2.11, for $n > 2c$, we have

$$\begin{aligned}
\|D^{(\beta)}_{n,c}(t^2,x) - x^2\|_{x^2} &= \sup_{x\in[0,\infty)} \frac{|D^{(\beta)}_{n,c}(t^2,x) - x^2|}{1+x^2}, \\
&\leq \left[\frac{n^2}{(n-c)(n-2c)(1-\beta)^2} - 1\right] \sup_{x\in[0,\infty)} \frac{x^2}{1+x^2} \\
&\quad + \left(\frac{n[1+(1-\beta)^2]}{(n-c)(n-2c)(1-\beta)^3}\right) \sup_{x\in[0,\infty)} \frac{x}{1+x^2},
\end{aligned}$$

and so the condition (2.60) holds for $\nu = 2$ as $n \to \infty$.

This completes the proof of the theorem. ■

2.7 Generalized Bernstein–Durrmeyer Operators

Stancu [220] introduced a sequence of positive linear operators depending on the parameters α and β, satisfying the condition $0 \leq \alpha \leq \beta$, the so-called Bernstein–Stancu operators, as

$$B_{n,\alpha,\beta}\left(f;x\right) = \sum_{k=0}^{n} f\left(\frac{k+\alpha}{n+\beta}\right) p_{n,k}\left(x\right), \tag{2.61}$$

where $p_{n,k}\left(x\right) = \binom{n}{k} x^k \left(1-x\right)^{n-k}$, and showed that as classical Bernstein operators $B_n\left(f;x\right)$, $B_{n,\alpha,\beta}\left(f;x\right)$ also converges to $f\left(x\right)$ uniformly on $[0,1]$ for $f \in C\left[0,1\right]$. A new type of Bernstein–Durrmeyer operators, with the purpose of approximating Lebesgue integrable functions on the mobile subinterval of $[0,1]$, was introduced in [14] as

$$\bar{D}_{n,\alpha,\beta}\left(f;x\right) = \left(n+1\right)\left(\frac{n+\beta}{n}\right)^{2n+1} \sum_{k=0}^{n} \bar{p}_{n,k}\left(x\right) \int_{\frac{\alpha}{n+\beta}}^{\frac{n+\alpha}{n+\beta}} \bar{p}_{n,k}\left(t\right) f\left(t\right) dt, \tag{2.62}$$

where

$$\bar{p}_{n,k}\left(x\right) = \binom{n}{k}\left(x - \frac{\alpha}{n+\beta}\right)^{k}\left(\frac{n+\alpha}{n+\beta} - x\right)^{n-k}.$$

If $\alpha = \beta = 0$, these operators reduce to the usual Bernstein–Durrmeyer operators. The operators (2.62) can be written in terms of a hypergeometric function as

$$\bar{D}_{n,\alpha,\beta}\left(f;x\right) = (n+1)\left(\frac{n+\beta}{n}\right)^{2n+1} \int_{\frac{\alpha}{n+\beta}}^{\frac{n+\alpha}{n+\beta}} f(t)\left[\left(\frac{n+\alpha}{n+\beta}-x\right)\left(\frac{n+\alpha}{n+\beta}-t\right)\right]^{n}$$

$${}_2F_1\left(-n,-n;1;\frac{\left(x-\frac{\alpha}{n+\beta}\right)\left(t-\frac{\alpha}{n+\beta}\right)}{\left(\frac{n+\alpha}{n+\beta}-x\right)\left(\frac{n+\alpha}{n+\beta}-t\right)}\right) dt.$$

Lemma 2.12 ([14]). *For $n > 0$ and $r > -1$, we have*

$$\bar{D}_{n,\alpha,\beta}(t^r;x) = \sum_{i=0}^{r}\binom{r}{i}\frac{n^i\alpha^{r-i}}{(n+\beta)^r}\frac{\Gamma(n+2)\Gamma(i+1)}{\Gamma(n+i+2)}\,{}_2F_1\left(-n,-i;1;\frac{(n+\beta)x-\alpha}{n}\right).$$

Lemma 2.13 ([14]). *For the operators $D_{n,\alpha,\beta}\left(f;x\right)$, the following equalities hold:*

$$\bar{D}_{n,\alpha,\beta}\left(1;x\right)=1,$$
$$\bar{D}_{n,\alpha,\beta}\left(t;x\right)=\frac{n}{(n+2)}x+\frac{n+2\alpha}{(n+2)\left(n+\beta\right)},$$

$$\begin{aligned}\bar{D}_{n,\alpha,\beta}\left(t^2;x\right)=&\left(x-\frac{\alpha}{n+\beta}\right)^2\frac{n\left(n-1\right)}{(n+2)\left(n+3\right)}\\&+\left(\frac{n}{n+\beta}\right)\left(x-\frac{\alpha}{n+\beta}\right)\frac{4n}{(n+2)\left(n+3\right)}\\&+\left(\frac{n}{n+\beta}\right)^2\frac{2}{(n+2)\left(n+3\right)}+\frac{2n\alpha}{(n+\beta)\left(n+2\right)}\left(x-\frac{\alpha}{n+\beta}\right)\\&+\frac{2n\alpha}{(n+\beta)^2}\frac{1}{(n+2)}+\left(\frac{\alpha}{n+\beta}\right)^2.\end{aligned}$$

Lemma 2.14 ([14]). *For* $f\in C\,[0,1]$*, we have* $\left\|D_{n,\alpha,\beta}f\right\|\leq\|f\|$.

Lemma 2.15 ([14]). *For* $n\in\mathbb{N}$*, we have*

$$\bar{D}_{n,\alpha,\beta}\left((t-x)^2;x\right)\leq\frac{2}{n+2}\delta_n^2\left(x\right),$$

where $\delta_n^2\left(x\right)=\varphi^2\left(x\right)+\frac{1}{n+2}$,

$$\varphi^2\left(x\right)=\begin{cases}\left(x-\frac{\alpha}{n+\beta}\right)\left(\frac{n+\alpha}{n+\beta}-x\right), & x\in\left[\frac{\alpha}{n+\beta},\frac{n+\alpha}{n+\beta}\right]\\ 0 & , x\in\left[0,\frac{\alpha}{n+\beta}\right]\cup x\in\left[\frac{n+\alpha}{n+\beta},1\right]\end{cases}.$$

Recall the following K-functional:

$$K_2\left(f,\delta\right)=\inf\left\{\|f-g\|+\delta\left\|g^{''}\right\|:g\in W^2\right\}\ \ (\delta>0),$$

where $W^2=\left\{g\in C\,[0,1]:g^{'},g^{''}\in C\,[0,1]\right\}$ and $\|.\|$ is the uniform on $C\,[0,1]$. By [59], there is a positive constant $C>0$ such that

$$K_2\left(f,\delta\right)\leq C\omega_2\left(f,\sqrt{\delta}\right),\tag{2.63}$$

where the second-order modulus of smoothness for $f\in C\,[0,1]$ is defined as

$$\omega_2\left(f,\sqrt{\delta}\right)=\sup_{0<h\leq\sqrt{\delta}}\ \sup_{x,x+2h\in[0,1]}\left|f\left(x+2h\right)-2f\left(x+h\right)+f\left(x\right)\right|.$$

We define the usual modulus of continuity for $f \in C[0,1]$ as

$$\omega(f,\delta) = \sup_{0<h\leq\delta} \sup_{x,x+h\in[0,1]} |f(x+h) - f(x)|.$$

Theorem 2.69 ([14] Local approximation). *For the operators $\bar{D}_{n,\alpha,\beta}$, there is a constant $C > 0$ such that*

$$\left|\bar{D}_{n,\alpha,\beta}(f;x) - f(x)\right| \leq C\omega_2\left(f, (n+2)^{-1}\delta_n(x)\right) + \omega\left(f, \frac{1}{n+2}\right),$$

where $f \in C[0,1]$, $\delta_n(x) = \left[\varphi^2(x) + \frac{1}{n+2}\right]^{1/2}$, and $x \in \left[\frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta}\right]$.

Proof. We introduce the auxiliary operators as follows:

$$\mathcal{D}_{n,\alpha,\beta}(f;x) = \bar{D}_{n,\alpha,\beta}(f;x) + f(x) - f\left(\frac{n}{(n+2)}x + \frac{n+2\alpha}{(n+2)(n+\beta)}\right).$$

It is clear from Lemma 2.13 that

$$\mathcal{D}_{n,\alpha,\beta}(1;x) = \bar{D}_{n,\alpha,\beta}(1;x) = 1$$

and

$$\mathcal{D}_{n,\alpha,\beta}(t;x) = \bar{D}_{n,\alpha,\beta}(t;x) + x - \frac{n}{(n+2)}x - \frac{n+2\alpha}{(n+2)(n+\beta)} = x.$$

Let $g \in W^2$ and $t \in [0,1]$. By a Taylor expansion, we have

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)\,du.$$

Applying the operator $\mathcal{D}_{n,\alpha,\beta}$ to both sides of the above equation, we get

$$\begin{aligned}\mathcal{D}_{n,\alpha,\beta}(g;x) &= g(x) + \mathcal{D}_{n,\alpha,\beta}\left(\int_x^t (t-u)g''(u)\,du\right)\\ &= g(x) + \bar{D}_{n,\alpha,\beta}\left(\int_x^t (t-u)g''(u)\,du; x\right)\\ &\quad - \int_x^{\frac{n}{(n+2)}x+\frac{n+2\alpha}{(n+2)(n+\beta)}} \left(\frac{n}{(n+2)}x + \frac{n+2\alpha}{(n+2)(n+\beta)} - u\right)g''(u)\,du.\end{aligned}$$

Hence,

$$
\begin{aligned}
&\left|\mathcal{D}_{n,\alpha,\beta}\left(g;x\right)-g\left(x\right)\right| \\
&\leq \mathcal{D}_{n,\alpha,\beta}\left(\int_x^t \left|t-u\right|\left|g^{''}\left(u\right)\right|du;x\right) \\
&\quad+\int_x^{\frac{n}{(n+2)}x+\frac{n+2\alpha}{(n+2)(n+\beta)}}\left|\frac{n}{(n+2)}x+\frac{n+2\alpha}{(n+2)\left(n+\beta\right)}-u\right|\left|g^{''}\left(u\right)\right|du \\
&\leq \mathcal{D}_{n,\alpha,\beta}\left(\left(t-x\right)^2,x\right)\left\|g^{''}\right\| \\
&\quad+\left(\frac{n}{(n+2)}x+\frac{n+2\alpha}{(n+2)\left(n+\beta\right)}-x\right)^2\left\|g^{''}\right\|. \qquad (2.64)
\end{aligned}
$$

On the other hand, from Lemma 2.15, we have

$$
\begin{aligned}
&\bar{D}_{n,\alpha,\beta}\left(\left(t-x\right)^2,x\right)+\left(\frac{n}{(n+2)}x+\frac{n+2\alpha}{(n+2)\left(n+\beta\right)}-x\right)^2 \\
&\leq \frac{2}{n+2}\delta_n^2\left(x\right)+\left(\frac{n+2\alpha}{(n+2)\left(n+\beta\right)}-\frac{2x}{(n+2)}\right)^2 \\
&\leq \frac{2}{n+2}\delta_n^2\left(x\right)+\left(\frac{n}{(n+2)\left(n+\beta\right)}\right)^2 \\
&\leq \frac{2}{n+2}\delta_n^2\left(x\right)+\frac{1}{(n+2)^2}\leq\frac{4}{n+2}\delta_n^2\left(x\right). \qquad (2.65)
\end{aligned}
$$

Thus, by (2.64),

$$
\left|\mathcal{D}_{n,\alpha,\beta}\left(g;x\right)-g\left(x\right)\right|\leq\frac{4}{n+2}\delta_n^2\left(x\right)\left\|g^{''}\right\|, \qquad (2.66)
$$

where $x\in\left[\frac{\alpha}{n+\beta},\frac{n+\alpha}{n+\beta}\right]$. Furthermore, by Lemma 2.14, we get

$$
\begin{aligned}
\left|\mathcal{D}_{n,\alpha,\beta}\left(f;x\right)\right| &\leq \left|\bar{D}_{n,\alpha,\beta}\left(f;x\right)\right|+\left|f\left(x\right)\right|+\left|f\left(\frac{n}{(n+2)}x+\frac{n+2\alpha}{(n+2)\left(n+\beta\right)}\right)\right| \\
&\leq 3\left\|f\right\|, \qquad (2.67)
\end{aligned}
$$

for all $f\in C\left[0,1\right]$.

For $f \in C[0,1]$ and $g \in W^2$, using (2.66) and (2.67), we obtain

$$\begin{aligned}
&\left|\bar{D}_{n,\alpha,\beta}(f;x) - f(x)\right| \\
&= \left|\mathcal{D}_{n,\alpha,\beta}(f;x) - f(x) + f\left(\frac{n}{(n+2)}x + \frac{n+2\alpha}{(n+2)(n+\beta)}\right) - f(x)\right| \\
&\leq \left|\mathcal{D}_{n,\alpha,\beta}(f-g;x)\right| + \left|\mathcal{D}_{n,\alpha,\beta}(g;x) - g(x)\right| \\
&\quad + |g(x) - f(x)| + \left|f\left(\frac{n}{(n+2)}x + \frac{n+2\alpha}{(n+2)(n+\beta)}\right) - f(x)\right| \\
&\leq 4\|f-g\| + \frac{4}{n+2}\delta_n^2(x)\left\|g''\right\| + \omega\left(f, \left|\frac{n+2\alpha}{(n+2)(n+\beta)} - \frac{2x}{(n+2)}\right|\right).
\end{aligned}$$

Taking the infimum over all $g \in W^2$, we obtain

$$\begin{aligned}
\left|\bar{D}_{n,\alpha,\beta}(f;x) - f(x)\right| &\leq 4K_2\left(f, \frac{1}{n+2}\delta_n^2(x)\right) \\
&\quad + \omega\left(f, \left|\frac{n+2\alpha}{(n+2)(n+\beta)} - \frac{2x}{(n+2)}\right|\right),
\end{aligned}$$

and by inequality (2.63), we get

$$\left|\bar{D}_{n,\alpha,\beta}(f;x) - f(x)\right| \leq C\omega_2\left(f, (n+2)^{-1}\delta_n(x)\right) + \omega\left(f, (n+2)^{-1}\right),$$

and so the proof is completed. ■

Before we give the global approximation results, we recall some notations. Let $f \in C[0,1]$ and $\varphi(x) = \sqrt{\left(x - \frac{\alpha}{n+\beta}\right)\left(\frac{n+\alpha}{n+\beta} - x\right)}$, $x \in \left[\frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta}\right]$. The second-order Ditzian–Totik modulus of smoothness and corresponding K-functional are given by, respectively,

$$\omega_2^{\varphi}\left(f, \sqrt{\delta}\right) = \sup_{0<h\leq\sqrt{\delta}} \sup_{x \pm h\varphi(x) \in \left[\frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta}\right]} |f(x+h\varphi(x)) - 2f(x) + f(x-h\varphi(x))|,$$

$$\bar{K}_{2,\varphi}(f,\delta) = \inf\left\{\|f-g\| + \delta\left\|\varphi^2 g''\right\| + \delta^2\left\|g''\right\| : g \in W^2(\varphi)\right\} \quad (\delta > 0),$$

where $W^2(\varphi) = \left\{g \in C[0,1] : g' \in AC_{loc}[0,1],\ \varphi^2 g'' \in C[0,1]\right\}$ and $g' \in AC_{loc}[0,1]$ means that g is differentiable and g' is absolutely continuous on every closed interval $[a,b] \subset [0,1]$. There is a positive constant $C > 0$ such that

$$\bar{K}_{2,\varphi}(f,\delta) \leq C\omega_2^{\varphi}\left(f, \sqrt{\delta}\right). \tag{2.68}$$

Also, the Ditzian–Totik modulus of first order is given by

$$\overrightarrow{\omega}_{\psi}(f,\delta)=\sup_{0<h\le\delta}\ \sup_{x\pm h\varphi(x)\in\left[\frac{\alpha}{n+\beta},\frac{n+\alpha}{n+\beta}\right]}\left|f\left(x+h\psi(x)\right)-f(x)\right|,$$

where ψ is an admissible step-weight function on $[0,1]$.

Theorem 2.70 ([14] Global approximation). *Let $f\in C[0,1]$. Then*

$$\left\|\bar{D}_{n,\alpha,\beta}f-f\right\|\le C\omega_2^{\varphi}\left(f,(n+2)^{-1/2}\right)+\overrightarrow{\omega}_{\psi}\left(f,(n+2)^{-1}\right),$$

where $C>0$ is an absolute constant, $\varphi(x)=\sqrt{\left(x-\frac{\alpha}{n+\beta}\right)\left(\frac{n+\alpha}{n+\beta}-x\right)}$, and $x\in\left[\frac{\alpha}{n+\beta},\frac{n+\alpha}{n+\beta}\right]$.

Proof. Again, we consider the auxiliary operators

$$\mathcal{D}_{n,\alpha,\beta}(f;x)=\bar{D}_{n,\alpha,\beta}(f;x)+f(x)-f\left(\frac{n}{(n+2)}x+\frac{n+2\alpha}{(n+2)(n+\beta)}\right).$$

Using the definition of the operator $\tilde{D}_{n,\alpha,\beta}$ and Lemma 2.13, we obtain as in the proof of Theorem 2.69 that

$$\begin{aligned}&\left|\mathcal{D}_{n,\alpha,\beta}(g;x)-g(x)\right|\\&\le\bar{D}_{n,\alpha,\beta}\left(\int_x^t|t-u|\left|g''(u)\right|du;x\right)\\&\quad+\int_x^{\frac{n}{(n+2)}x+\frac{n+2\alpha}{(n+2)(n+\beta)}}\left|\frac{n}{(n+2)}x+\frac{n+2\alpha}{(n+2)(n+\beta)}-u\right|\left|g''(u)\right|du.\end{aligned}\tag{2.69}$$

Moreover, δ_n^2 is a concave function on $x\in\left[\frac{\alpha}{n+\beta},\frac{n+\alpha}{n+\beta}\right]$. For $u=\lambda x+(1-\lambda)t$, $\lambda\in[0,1]$, we get

$$\frac{|t-u|}{\delta_n^2(u)}=\frac{\lambda|t-x|}{\delta_n^2(\lambda x+(1-\lambda)t)}\le\frac{\lambda|t-x|}{\lambda\delta_n^2(x)+(1-\lambda)\delta_n^2(t)}\le\frac{|t-x|}{\delta_n^2(x)}.$$

Thus, if we use this inequality in (2.69), we have

$$\left|\mathcal{D}_{n,\alpha,\beta}(g;x)-g(x)\right|\le\bar{D}_{n,\alpha,\beta}\left(\int_x^t\frac{|t-u|}{\delta_n^2(u)}du;x\right)\left\|\delta_n^2g''\right\|$$

$$+\int_{x}^{\frac{n}{(n+2)}x+\frac{n+2\alpha}{(n+2)(n+\beta)}} \frac{\left|\frac{n}{(n+2)}x+\frac{n+2\alpha}{(n+2)(n+\beta)}-u\right|}{\delta_n^2(u)}du \left\|\delta_n^2 g''\right\|$$

$$\leq \frac{1}{\delta_n^2(x)}\left\|\delta_n^2 g''\right\| \Bigg[\bar{D}_{n,\alpha,\beta}\left((t-x)^2;x\right)$$

$$+\left(\frac{n+2\alpha}{(n+2)(n+\beta)}-\frac{2x}{(n+2)}\right)^2\Bigg]. \tag{2.70}$$

By inequality (2.65), we have

$$\begin{aligned}\left|\mathcal{D}_{n,\alpha,\beta}(g;x)-g(x)\right| &\leq \frac{4}{n+2}\left\|\delta_n^2 g''\right\| \\ &\leq \frac{4}{n+2}\left(\left\|\varphi^2 g''\right\|+\frac{1}{n+2}\left\|g''\right\|\right).\end{aligned}$$

Using (2.67) and (2.70), we have for $f \in C[0,1]$,

$$\begin{aligned}\left|\bar{D}_{n,\alpha,\beta}(f;x)-f(x)\right| &\leq \left|\mathcal{D}_{n,\alpha,\beta}(f-g;x)\right| \\ &\quad+\left|\mathcal{D}_{n,\alpha,\beta}(g;x)-g(x)\right|+|g(x)-f(x)| \\ &\quad+\left|f\left(\frac{n}{(n+2)}x+\frac{n+2\alpha}{(n+2)(n+\beta)}\right)-f(x)\right| \\ &\leq 4\|f-g\|+\frac{4}{n+2}\left\|\varphi^2 g''\right\|+\frac{4}{(n+2)^2}\left\|g''\right\| \\ &\quad+\left|f\left(\frac{n}{(n+2)}x+\frac{n+2\alpha}{(n+2)(n+\beta)}\right)-f(x)\right|.\end{aligned}$$

Taking the infimum over all $g \in W^2$, we obtain

$$\left|\bar{D}_{n,\alpha,\beta}(f;x)-f(x)\right| \leq 4\bar{K}_{2,\varphi}\left(f,\frac{1}{n+2}\right)+\left|f\left(\frac{n}{(n+2)}x+\frac{n+2\alpha}{(n+2)(n+\beta)}\right)-f(x)\right|. \tag{2.71}$$

On the other hand,

$$\begin{aligned}&\left|f\left(\frac{n}{(n+2)}x+\frac{n+2\alpha}{(n+2)(n+\beta)}\right)-f(x)\right| \\ &\quad=\left|f\left(x+\psi(x)\frac{[(n+2\alpha)-2x(n+\beta)]}{(n+2)(n+\beta)\psi(x)}\right)-f(x)\right|\end{aligned}$$

$$\leq \sup_{t, t+\psi(t)\frac{[(n+2\alpha)-2x(n+\beta)]}{(n+2)(n+\beta)\psi(x)}} \left| f\left(t + \psi(t) \frac{[(n+2\alpha)-2x(n+\beta)]}{(n+2)(n+\beta)\psi(x)}\right) - f(t) \right|$$

$$\leq \overrightarrow{\omega}_{\psi}\left(f, \frac{|(n+2\alpha)-2x(n+\beta)|}{(n+2)(n+\beta)\psi(x)}\right) \leq \overrightarrow{\omega}_{\psi}\left(f, \frac{2\left(x - \frac{\alpha}{n+\beta}\right)}{(n+2)\psi(x)}\right)$$

$$= \overrightarrow{\omega}_{\psi}\left(f, \frac{1}{(n+2)}\right).$$

So, from (2.68) and (2.71), we obtain

$$\left\| \bar{D}_{n,\alpha,\beta} f - f \right\| \leq C\omega_2^{\varphi}\left(f, (n+2)^{-1/2}\right) + \overrightarrow{\omega}_{\psi}\left(f, (n+2)^{-1}\right),$$

which is the desired result. ∎

Theorem 2.71 ([14] Voronovskaja's asymptotic formula). *Let* $f \in C[0,1]$. *If* $f^{''}$ *exists at a point* $x \in \left[\frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta}\right]$, *then*

$$\lim_{n\to\infty} n\left[\bar{D}_{n,\alpha,\beta}(f;x) - f(x)\right] = (1-2x) f^{'}(x) + 2x(1-x) f^{''}(x).$$

Proof. Using a Taylor expansion of f, we have

$$f(t) = f(x) + f^{'}(x)(t-x) + \frac{1}{2} f^{''}(x)(t-x)^2 + \varepsilon(t,x)(t-x)^2,$$

where $\varepsilon(t,x) \to 0$ as $t \to x$. Applying $\bar{D}_{n,\alpha,\beta}$, we get

$$\begin{aligned}
\left[\bar{D}_{n,\alpha,\beta}(f;x) - f(x)\right] = {} & f^{'}(x)\bar{D}_{n,\alpha,\beta}((t-x),x) \\
& + \frac{1}{2} f^{''}(x)\bar{D}_{n,\alpha,\beta}\left((t-x)^2, x\right) \\
& + \bar{D}_{n,\alpha,\beta}\left(\varepsilon(t,x)(t-x)^2, x\right),
\end{aligned}$$

$$\begin{aligned}
& \lim_{n\to\infty} n\left[\bar{D}_{n,\alpha,\beta}(f;x) - f(x)\right] \\
& = \lim_{n\to\infty} n f^{'}(x)\bar{D}_{n,\alpha,\beta}((t-x),x) + \lim_{n\to\infty} n\frac{1}{2} f^{''}(x)\bar{D}_{n,\alpha,\beta}\left((t-x)^2, x\right) \\
& \quad + \lim_{n\to\infty} n\bar{D}_{n,\alpha,\beta}\left(\varepsilon(t,x)(t-x)^2, x\right)
\end{aligned}$$

$$\lim_{n\to\infty} n\left[\bar{D}_{n,\alpha,\beta}(f;x) - f(x)\right]$$

$$= (1-2x) f^{'}(x) + 2x(1-x) f^{''}(x) + \lim_{n\to\infty} n\bar{D}_{n,\alpha,\beta}\left(\varepsilon(t,x)(t-x)^2, x\right)$$
$$= (1-2x) f^{'}(x) + 2x(1-x) f^{''}(x) + F.$$

In order for us to complete the proof, it is sufficient to show that $F = 0$. By the Cauchy–Schwarz inequality, we have

$$F = \lim_{n\to\infty} n\bar{D}_{n,\alpha,\beta}\left(\varepsilon^2(t,x), x\right)^{1/2} \bar{D}_{n,\alpha,\beta}\left((t-x)^4, x\right)^{1/2}. \tag{2.72}$$

Furthermore, since $\varepsilon^2(x,x) = 0$ and $\varepsilon^2(.,x) \in C[0,1]$, it follows that

$$\lim_{n\to\infty} n\bar{D}_{n,\alpha,\beta}\left(\varepsilon^2(t,x), x\right) = 0, \tag{2.73}$$

uniformly with respect to $x \in \left[\frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta}\right]$. So from (2.72) and (2.73), we get

$$\lim_{n\to\infty} n\bar{D}_{n,\alpha,\beta}\left(\varepsilon^2(t,x), x\right)^{1/2} \bar{D}_{n,\alpha,\beta}\left((t-x)^4, x\right)^{1/2} = 0.$$

Thus, we have

$$\lim_{n\to\infty} n\left[\bar{D}_{n,\alpha,\beta}(f;x) - f(x)\right] = (1-2x) f^{'}(x) + 2x(1-x) f^{''}(x),$$

which completes the proof. ■

2.8 Generalizations of Baskakov Operators

The usual Baskakov–Durrmeyer operators were first introduced in [208]. Several other generalizations of Baskakov operators have been proposed and studied in recent years. In this direction, Gupta [92] introduced a different integral modification of the Baskakov operators by considering the weight functions of Beta basis functions as

$$L_n(f,x) = \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} b_{n,k}(t) f(t) dt, x \in [0,\infty), \tag{2.74}$$

where $p_{n,k}(x) = \dfrac{(n)_k}{k!} \dfrac{x^k}{(1+x)^{n+k}}$ and $b_{n,k}(t) = \dfrac{n(n+1)_k}{k!} \dfrac{t^k}{(1+t)^{n+k+1}}$.

Gupta [92] showed that by considering the modification of the Baskakov operators in the form (2.74), one can have a better approximation over the usual Baskakov–Durrmeyer operators. Gupta [92] denoted by $H[0,\infty)$ the class of all measurable functions defined on $[0,\infty)$ satisfying

$$\int_0^\infty \frac{|f(t)|}{(1+t)^{n+1}}dt < \infty \quad \text{for some positive integer } n.$$

Obviously, this class is bigger than the class of all Lebesgue integrable functions on $[0,\infty)$. Gupta [92] obtained an asymptotic formula and error estimation in simultaneous approximation for the operators (2.74). We mention here the asymptotic formula:

Theorem 2.72 ([92]). *Let $H[0,\infty)$ be bounded on every finite subinterval of $[0,\infty)$, and let $f^{(r+2)}$ exist at a fixed point $x \in (0,\infty)$. Let $f(t) = O(t^\alpha)$, as $t \to \infty$ for some $\alpha > 0$. Then we have*

$$\lim_{n\to\infty}[L_n^{(r)}(f,x) - f^{(r)}(x)] = r^r f^{(r)}(x) + [(1+r) + x(1+2r)]f^{(r+1)}(x) \\ +x(1+x)f^{(r+2)}(x).$$

Motivated by the recent studies on certain Beta-type operators by Ismail and Simeonov [157] in the hypergeometric form, Gupta and Yadav [133] represented the operators (2.74) as

$$\begin{aligned} L_n(f,x) &= n\sum_{k=0}^\infty \frac{(n)_k}{k!}\frac{x^k}{(1+x)^{n+k}}\int_0^\infty \frac{(n+1)_k}{k!}\frac{t^k}{(1+t)^{n+k+1}}f(t)dt \\ &= n\int_0^\infty \frac{f(t)(1+x)}{[(1+x)(1+t)]^{n+1}}\sum_{k=0}^\infty \frac{(n)_k(n+1)_k}{(k!)^2}\frac{(xt)^k}{[(1+x)(1+t)]^k}dt. \end{aligned}$$

By hypergeometric series ${}_2F_1(a,b;c;x) = \sum_{k=0}^\infty \frac{(a)_k(b)_k}{(c)_k k!}x^k$ and using the equality $(1)_k = k!$, we can write

$$L_n(f,x) = n\int_0^\infty \frac{f(t)(1+x)}{[(1+x)(1+t)]^{n+1}}\ {}_2F_1\left(n, n+1; 1; \frac{xt}{(1+x)(1+t)}\right)dt.$$

Now, using ${}_2F_1(a,b;c;x) = {}_2F_1(b,a;c;x)$ and applying the Pfaff–Kummer transformation

$${}_2F_1(a,b;c;x) = (1-x)^{-a}\ {}_2F_1\left(a, c-b; c; \frac{x}{x-1}\right),$$

we have

$$L_n(f,x) = n\int_0^\infty f(t)\frac{1+x}{(1+x+t)^{n+1}}\ {}_2F_1\left(n+1, 1-n; 1; \frac{-xt}{1+x+t}\right)dt, \tag{2.75}$$

which is the alternate form of the operators (2.74) in terms of hypergeometric functions.

Based on the two parameters α, β satisfying the conditions $0 \le \alpha \le \beta$, in 1983, Stancu gave the Stancu-type generalization of Bernstein operators [222]. The Stancu-type generalization of the Baskakov–Beta operators (namely, BBS operators), for $0 \le \alpha \le \beta$, was introduced in [133] as

$$L_{n,\alpha,\beta}(f,x) = n\int_0^\infty f\left(\frac{nt+\alpha}{n+\beta}\right)\frac{1+x}{(1+x+t)^{n+1}}\ {}_2F_1\left(n+1, 1-n; 1; \frac{-xt}{1+x+t}\right)dt. \tag{2.76}$$

As a special case, if $\alpha = \beta = 0$, the operators (2.76) reduce to Baskakov–Beta operators (2.74). We consider

$$C_\gamma[0,\infty) = \{f \in C[0,\infty) : f(t) = O(t^\gamma),\ \gamma > 0\}.$$

The operators $L_{n,\alpha,\beta}(f,x)$ are well defined for $f \in C_\gamma[0,\infty)$. In the present section, we present the direct theorems, which include a Voronovskaja-type asymptotic formula and an estimation of error in simultaneous approximation for the BBS operators.

Lemma 2.16 ([133]). *For $0 \le \alpha \le \beta$, we have*

$$\begin{aligned} L_{n,\alpha,\beta}(t^r, x) = {} & x^r \frac{n^r}{(n+\beta)^r}\frac{(n+r-1)!(n-r-1)!}{((n-1)!)^2} \\ & + x^{r-1}\left\{r^2\frac{n^r}{(n+\beta)^r}\frac{(n+r-2)!(n-r-1)!}{((n-1)!)^2}\right. \\ & \left. + r\alpha\frac{n^{r-1}}{(n+\beta)^r}\frac{(n+r-2)!(n-r)!}{((n-1)!)^2}\right\} \\ & + x^{r-2}\left\{r(r-1)^2\alpha\frac{n^{r-1}}{(n+\beta)^r}\frac{(n+r-3)!(n-r)!}{((n-1)!)^2}\right. \\ & \left. + \frac{r(r-1)\alpha^2}{2}\frac{n^{r-2}}{(n+\beta)^r}\frac{(n+r-3)!(n-r+1)!}{((n-1)!)^2}\right\} + O(n^{-2}). \end{aligned}$$

Lemma 2.17 ([92]). *For $m \in \mathbb{N}\bigcup\{0\}$, if*

$$U_{n,m}(x) = \sum_{k=0}^\infty p_{n,k}(x)\left(\frac{k}{n} - x\right)^m,$$

then $U_{n,0}(x) = 1, U_{n,1}(x) = 0$, *and we have the recurrence relation*

$$nU_{n,m+1}(x) = x(1+x)\left[U'_{n,m}(x) + mU_{n,m-1}(x)\right].$$

Consequently, $U_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$, *where* $[m]$ *is an integral part of* m.

Lemma 2.18 ([133]). *If we define the central moments as*

$$\begin{aligned}\mu_{n,m}(x) &= L_{n,\alpha,\beta}((t-x)^m, x),\\ &= \sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} b_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta} - x\right)^m dt,\ \ m \in \mathbb{N}\end{aligned}$$

then the following recurrence relation holds:

$$\begin{aligned}&(n-m-1)\left(\frac{n+\beta}{n}\right)\mu_{n,m+1}(x)\\ &\ = x(1+x)\left[\mu'_{n,m}(x) + m\mu_{n,m-1}(x)\right]\\ &\quad + \left[(m+nx+1) + \left(\frac{n+\beta}{n}\right)\left(\frac{\alpha}{n+\beta} - x\right)(n-2m-1)\right]\mu_{n,m}(x)\\ &\quad - \left(\frac{\alpha}{n+\beta} - x\right)\left[\left(\frac{\alpha}{n+\beta} - x\right)\left(\frac{n+\beta}{n}\right) - 1\right]m\mu_{n,m-1}(x),\ \ n > m+1.\end{aligned}$$

Furthermore, one can obtain the first three moments as

$$\mu_{n,0}(x) = 1,\ \ \mu_{n,1}(x) = \frac{x\left(n+\beta(1-n)\right) + n + \alpha(n-1)}{(n+\beta)(n-1)},$$

$$\begin{aligned}\mu_{n,2}(x) &= \frac{x^2\left[2n^2(n+1) + n\beta(n\beta - 4 + 10n + 3\beta) + 2\beta^2\right]}{(n+\beta)^2(n-1)(n-2)}\\ &\quad + \frac{x\left[2n^2(n+2) + 2n^2\alpha - 2n^2\beta(1+\alpha) + 2n(2\beta - 2\alpha + 3\alpha\beta) - 4\alpha\beta\right]}{(n+\beta)^2(n-1)(n-2)}\\ &\quad + \frac{2n^2 + n^2\alpha(\alpha+2) - n\alpha(3\alpha+4) + 2\alpha^2}{(n+\beta)^2(n-1)(n-2)}.\end{aligned}$$

From the recurrence relation, we can easily verify that for all $x \in [0,\infty)$, *we have*

$$\mu_{n,m}(x) = O(n^{-[(m+1)/2]}).$$

Lemma 2.19 ([92]). *The polynomials* $q_{i,j,r}(x)$ *on* $[0,\infty)$, *independent of* n *and* k, *are such that*

$$x^r(1+x)^r \frac{d^r}{dx^r} p_{n,k}(x) = \sum_{\substack{2i+j \le r \\ i,j \ge 0}} n^i (k-nx)^j q_{i,j,r}(x) p_{n,k}(x).$$

Proof. Suppose that the result is true for r. Then by using the induction hypothesis and $x(1+x)p'_{n,k}(x) = (k-nx)p_{n,k}(x)$, we have

$$\begin{aligned}
&\frac{d^r}{dx^r}\left(x^k(1+x)^{-(n+k)}\right)\\
&= \frac{d}{dx}\left(\sum_{\substack{2i+j \le r \\ i,j \ge 0}} n^i (k-nx)^j q_{i,j,r}(x) x^{k-r}(1+x)^{-(n+k+r)}\right)\\
&= \sum_{\substack{2i+j \le r \\ i,j \ge 0}} n^{i+1}(k-nx)^{j-1}(-j q_{i,j,r}(x)) x^{k-r}(1+x)^{-(n+k+r)}\\
&\quad + \sum_{\substack{2i+j \le r \\ i,j \ge 0}} n^i (k-nx)^{j+1} q_{i,j,r}(x)) x^{k-r-1}(1+x)^{-(n+k+r+1)}\\
&\quad - \sum_{\substack{2i+j \le r \\ i,j \ge 0}} n^i (k-nx)^j (r(1+2x) q_{i,j,r}(x)) x^{k-r-1}(1+x)^{-(n+k+r+1)}\\
&\quad + \sum_{\substack{2i+j \le r \\ i,j \ge 0}} n^i (k-nx)^j q'_{i,j,r}(x) x^{k-r}(1+x)^{-(n+k+r)}\\
&= \sum_{\substack{2(i-1)+(j+1) \le r \\ i \ge 1, j \ge 0}} n^i (k-nx)^j (-(j+1) q_{i-1,j+1,r}(x)) x^{k-r}(1+x)^{-(n+k+r)}\\
&\quad + \sum_{\substack{2i+(j-1) \le r \\ i \ge 0, j \ge 1}} n^i (k-nx)^j q_{i,j-1,r}(x) x^{k-r-1}(1+x)^{-(n+k+r+1)}\\
&\quad - \sum_{\substack{2i+j \le r \\ i,j \ge 0}} n^i (k-nx)^j (r(1+2x) q_{i,j,r}(x)) x^{k-r-1}(1+x)^{-(n+k+r+1)}\\
&\quad + \sum_{\substack{2i+j \le r \\ i,j \ge 0}} n^i (k-nx)^j (q'_{i,j,r}(x)) x^{k-r}(1+x)^{-(n+k+r+1)}\\
&= \sum_{\substack{2i+j \le r+1 \\ i,j \ge 0}} n^i (k-nx)^j q_{i,j,r+1}(x) x^{k-r-1}(1+x)^{-(n+k+r+1)},
\end{aligned}$$

where

$$\begin{aligned}
q_{i,j,r+1}(x) = {}& q_{i,j-1,r}(x) - (j+1)x(1+x)q_{i-1,j+1,r}(x)\\
&-r(1+2x)q_{i,j,r}(x) + x(1+x)q'_{i,j,r}(x)
\end{aligned}$$

and $2i + j \leq r + 1, i, j \geq 0$, with the convention that $q_{i,j,r}(x) \equiv 0$ if any one of the constraints $2i + j \leq r$ and $i, j \geq 0$ is violated. Hence, the result is true for $r + 1$. For $r = 0$, the result is trivial. Therefore, by induction, the result holds for all r. ■

Using Lemmas 2.16–2.19, Gupta and Yadav [133] established the following asymptotic formula and error estimation.

Theorem 2.73 ([133] Asymptotic formula). *Let $f \in C_\gamma[0,\infty)$ be bounded on every finite subinterval of $[0,\infty)$ admitting the derivative of order $(r+2)$ at a fixed $x \in (0,\infty)$. Let $f(t) = O(t^\gamma)$ as $t \to \infty$ for some $\gamma > 0$. Then we have*

$$\begin{aligned}\lim_{n\to\infty} n\left(L^{(r)}_{n,\alpha,\beta}(f,x) - f^{(r)}(x)\right) = r(r-\beta)f^{(r)}(x) + [(1+r+\alpha)\\ +x(1+2r-\beta)]f^{(r+1)}(x)\\ +x(1+x)f^{(r+2)}(x).\end{aligned}$$

Theorem 2.74 ([133] Error estimation). *Let $f \in C_\gamma[0,\infty)$ for some $\gamma > 0$ and $r \leq m \leq r+2$. If $f^{(m)}$ exists and is continuous on $(a-\eta, b+\eta) \subset (0,\infty)$, $\eta > 0$, then for n sufficiently large,*

$$\begin{aligned}\|L^{(r)}_{n,\alpha,\beta}(f,x) - f^{(r)}(x)\|_{C[a,b]} \leq C_1 n^{-1}\sum_{i=r}^{m}\|f^{(i)}\|_{C[a,b]}\\ +C_2 n^{-1/2}\omega(f^{(m)}, n^{-1/2}) + O(n^{-2}),\end{aligned}$$

where C_1, C_2 are constants independent of f and n, $\omega(f,\delta)$ is the modulus of continuity of f on $(a-\eta, b+\eta)$, and $\|.\|_{C[a,b]}$ denotes the sup $-norm$ *on* $[a,b]$.

For $f \in C[0,\infty)$, a new form of the modification of the well-known Baskakov operator was introduced and studied by Finta in [71]. It is defined as

$$D_n(f,x) = \sum_{k=1}^{\infty} p_{n,k}(x)\int_0^\infty b_{n,k}(t)f(t)\,dt + p_{n,0}(x)f(0), \tag{2.77}$$

where

$$p_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}, \quad b_{n,k}(t) = \frac{1}{B(k,n+1)}\frac{t^{k-1}}{(1+t)^{n+k+1}}.$$

It is observed that $D_n(f,x)$ reproduces constant as well as linear functions. Verma et al. [238] considered Baskakov–Durrmeyer–Stancu (abbr. BDS) operators as follows:

$$D_n^{(\alpha,\beta)}(f,x) = \sum_{k=1}^{\infty} p_{n,k}(x) \int_0^{\infty} b_{n,k}(t) f\left(\frac{nt+\alpha}{n+\beta}\right) dt$$
$$+ p_{n,0}(x) f\left(\frac{\alpha}{n+\beta}\right), \tag{2.78}$$

where the basis functions are given in (2.77).

Lemma 2.20 ([238]). *If we define the central moments as*

$$\begin{aligned}\mu_{n,m}(x) &= D_n^{(\alpha,\beta)}((t-x)^m, x)\\ &= \sum_{k=1}^{\infty} p_{n,k}(x) \int_0^{\infty} b_{n,k}(t) \left(\frac{nt+\alpha}{n+\beta} - x\right)^m dt\\ &\quad + p_{n,0}(x)\left(\frac{\alpha}{n+\beta} - x\right)^m,\ m \in \mathbb{N},\end{aligned}$$

then $\mu_{n,0}(x) = 1$ and $\mu_{n,1}(x) = \frac{\alpha-\beta x}{n+\beta}$, and for $n > m$, we have the following recurrence relation:

$$\begin{aligned}&(n-m)(n+\beta)\mu_{n,m+1}(x)\\ &\quad = nx(1+x)[\mu'_{n,m}(x) + m\mu_{n,m-1}(x)]\\ &\qquad + [n(\alpha-\beta x) - 2m(\alpha-(n+\beta)x) + mn]\mu_{n,m}(x)\\ &\qquad + \left[(n+\beta)m\left(\frac{\alpha}{n+\beta} - x\right)^2 - mn\left(\frac{\alpha}{n+\beta} - x\right)\right]\mu_{n,m-1}(x).\end{aligned}$$

From the recurrence relation, we can easily verify that for all $x \in [0,\infty)$, we have

$$\mu_{n,m}(x) = O(n^{-[(m+1)/2]}).$$

Remark 2.5. From Lemma 2.20, we find that $D_n^{(\alpha,\beta)}(t^m, x)$ is a polynomial in x of degree exactly m, for all $m \in \mathbb{N}^0$. Furthermore, $D_n^{(\alpha,\beta)}(t^m, x) = \sum_{j=0}^{m} \binom{m}{j} \frac{\alpha^{n-j} n^j}{(n+\beta)^m} D_n(t^j, x)$, and we can write it as

$$\begin{aligned}D_n^{(\alpha,\beta)}(t^m, x) &= \frac{n^m (n+m-1)!(n-m)!}{(n+\beta)^m\, n!(n-1)!} x^m\\ &\quad + \frac{mn^{m-1}(n+m-2)!(n-m)!}{(n+\beta)^m\, n!(n-1)!}[n(m-1) + \alpha(n-m+1)]x^{m-1}\end{aligned}$$

$$+\frac{m(m-1)n^{m-2}\alpha(n+m-3)!(n-m+1)!}{(n+\beta)^m\, n!(n-1)!}$$
$$\times\left[n(m-2)+\frac{\alpha(n-m+2)}{2}\right]x^{m-2}+O(n^{-2}).$$

Lemma 2.21 ([145]). *Let f be $r-$times differentiable on $[0,\infty)$ such that $f^{(r-1)}(t)=O(t^\gamma)$, $\gamma>0$ as $t\to\infty$. Then for $r=1,2,\ldots,$ we have*

$$[D_n^{(\alpha,\beta)}]^{(r)}(f,x)=\frac{n^r(n+r-1)!(n-r)!}{(n+\beta)^r\, n!(n-1)!}\sum_{k=0}^{\infty}p_{n+r,k}(x)\int_0^\infty b_{n-r,k+r}f^{(r)}\left(\frac{nt+\alpha}{n+\beta}\right)dt.$$

Lemma 2.22 ([84]). *Let $f\in C[a,b]$. Then*

$$\|f_{\eta,2k}^{(i)}\|_{C[a,b]}\le C_i\{\|f_{\eta,2k}\|_{C[a,b]}+\|f_{\eta,2k}^{(2k)}\|_{C[a,b]}\},\ i=1,2,\ldots,2k-1,$$

where the $C_i^{\prime}s$ are certain constants independent of f.

Here we present the pointwise approximation, asymptotic formula, and error estimation in terms of a higher-order modulus of continuity in simultaneous approximation.

We denote $C_\gamma[0,\infty)=\{f\in C[0,\infty): f(t)=O(t^\gamma),\gamma>0\}$. It can be easily verified that the operators $D_n^{(\alpha,\beta)}(f,x)$ are well defined for $f\in C_\gamma[0,\infty)$.

Theorem 2.75 ([145] Pointwise convergence). *Let α, β be two parameters satisfying the conditions $0\le\alpha\le\beta$. If $r\in\mathbb{N}$, $f\in C_\gamma[0,\infty)$ for some $\gamma>0$ and $f^{(r)}$ exists at a point $x\in(0,\infty)$, then*

$$\lim_{n\to\infty}[D_n^{(\alpha,\beta)}]^{(r)}(f,x)=f^{(r)}(x). \tag{2.79}$$

Furthermore, if $f^{(r)}$ exists and is continuous on $(a-\eta,b+\eta)\subset(0,\infty)$, $\eta>0$, then (2.79) holds uniformly in $[a,b]$.

The proof of the above theorem follows by using Remark 2.5, Lemmas 2.19, and 2.20; we omit the details here.

Theorem 2.76 ([145] Asymptotic formula). *Let $f\in C_\gamma[0,\infty)$ be bounded on every finite subinterval of $[0,\infty)$ admitting the derivative of order $(r+2)$ at a fixed $x\in(0,\infty)$. Let $f(t)=O(t^\gamma)$ as $t\to\infty$ for some $\gamma>0$. Then we have*

$$\lim_{n\to\infty}n\left([D_n^{(\alpha,\beta)}]^{(r)}(f,x)-f^{(r)}(x)\right)=r(r-1-\beta)f^{(r)}(x)$$
$$+[r(1+2x)+\alpha-\beta x]f^{(r+1)}(x)$$
$$+x(1+x)f^{(r+2)}(x).$$

Using Lemmas 2.21, 2.20, and Remark 2.5, we find that the proof of Theorem 2.76 follows using standard techniques; we omit the details here.

Theorem 2.77 ([145] Error estimation). *Let $f \in C_\gamma[0,\infty)$ for some $\gamma > 0$ and $0 < a < a_1 < b_1 < b < \infty$. Then for n sufficiently large, we have*

$$\|[D_n^{(\alpha,\beta)}]^{(r)}(f,.) - f^{(r)}\|_{C[a_1,b_1]} \leq C_1\omega_2(f^{(r)}, n^{-1/2}, [a_1,b_1]) + C_2 n^{-k}\|f\|_\gamma,$$

where $C_1 = C_1(r)$ and $C_2 = C_2(r, f)$.

Proof. We can write

$$\begin{aligned}\|[D_n^{(\alpha,\beta)}]^{(r)}(f,.) - f^{(r)}\|_{C[a_1,b_1]} \;&\leq\; \|[D_n^{(\alpha,\beta)}]^{(r)}((f - f_{\eta,2}),.)\|_{C[a_1,b_1]}\\ &\quad + \|[D_n^{(\alpha,\beta)}]^{(r)}(f_{\eta,2},.) - f_{\eta,2}^{(r)}\|_{C[a_1,b_1]}\\ &\quad + \|f^{(r)} - f_{\eta,2}^{(r)}\|_{C[a_1,b_1]}\\ &:= S_1 + S_2 + S_3.\end{aligned}$$

Since $f_{\eta,2}^{(r)} = (f^{(r)})_{\eta,2}$, by property (iii) of the Steklov mean (see Definition 1.9), we get

$$S_3 \leq C_1\omega_2(f^{(r)}, \eta, [a,b]).$$

Next, using Theorem 2.76 and Lemma 2.22, we get

$$\begin{aligned}S_2 &\leq C_2 n^{-1}\sum_{i=r}^{2+r}\|f_{\eta,2}^{(i)}\|_{C[a,b]}\\ &\leq C_4 n^{-1}\{\|f_{\eta,2}\|_{C[a,b]} + \|f_{\eta,2}^{(2+r)}\|_{C[a,b]}\}.\end{aligned}$$

By applying properties (ii) and (iv) of the Steklov mean, we obtain

$$S_2 \leq C_4 n^{-1}\{\|f\|_\gamma + \eta^{-2}\omega_2(f^{(r)}, \eta, [a,b])\}.$$

Finally, we estimate S_1 by choosing a^*, b^* satisfying the condition $0 < a < a^* < a_1 < b_1 < b^* < b < \infty$. Also, let $\chi(t)$ denote the characteristic function on the interval $[a^*, b^*]$. Then

$$\begin{aligned}S_1 \;&\leq\; \|[D_n^{(\alpha,\beta)}]^{(r)}\big(\chi(t)(f(t) - f_{\eta,2}(t)),.\big)\|_{C[a_1,b_1]}\\ &\quad + \|[D_n^{(\alpha,\beta)}]^{(r)}\big((1-\chi(t))\big(f(t) - f_{\eta,2}(t)\big),.\big)\|_{C[a_1,b_1]}\\ &:= S_4 + S_5.\end{aligned}$$

By Lemma 2.21, we have

$$
\begin{aligned}
&[D_n^{(\alpha,\beta)}]^{(r)}\big(\chi(t)(f(t)-f_{\eta,2}(t)),x\big)\\
&\quad=\frac{n^r(n+r-1)!(n-r)!}{(n+\beta)^r\, n!(n-1)!}\sum_{k=0}^{\infty}p_{n+r,k}(x)\\
&\qquad\times\int_0^{\infty}b_{n-r,k+r}(t)\chi(t)\left[f^{(r)}\left(\frac{nt+\alpha}{n+\beta}\right)-f_{\eta,2}^{(r)}\left(\frac{nt+\alpha}{n+\beta}\right)\right]dt.
\end{aligned}
$$

Hence,

$$
\big\|[D_n^{(\alpha,\beta)}]^{(r)}\big(\chi(t)(f(t)-f_{\eta,2}(t)),.\big)\big\|_{C[a_1,b_1]}\le C_5\|f^{(r)}-f_{\eta,2}^{(r)}\|_{C[a^*,b^*]}.
$$

Now for $x\in[a_1,b_1]$ and $t\in[0,\infty)\backslash[a^*,b^*]$, we choose a $\delta>0$ satisfying $\left|\frac{nt+\alpha}{n+\beta}-x\right|\ge\delta$. By Lemma 2.19 and the Schwarz inequality, we have

$$
\begin{aligned}
I&=|[D_n^{(\alpha,\beta)}]^{(r)}\big((1-\chi(t))\big(f(t)-f_{\eta,2}(t)\big),x)|\\
&\le\sum_{\substack{2i+j\le r\\ i,j\ge 0}}n^i\frac{|q_{i,j,r}(x)|}{x^r(1+x)^r}\sum_{k=1}^{\infty}p_{n,k}(x)|k-nx|^j\\
&\quad\times\int_0^{\infty}b_{n,k}(t)\big((1-\chi(t))\left|f\left(\frac{nt+\alpha}{n+\beta}\right)-f_{\eta,2}\left(\frac{nt+\alpha}{n+\beta}\right)\right|dt\\
&\quad+\frac{(n+r-1)!}{(n-1)!}(1+x)^{-n-r}\big((1-\chi(t))\left|f\left(\frac{\alpha}{n+\beta}\right)-f_{\eta,2}\left(\frac{\alpha}{n+\beta}\right)\right|\\
&\le C_6\|f\|_\gamma\Bigg\{\sum_{\substack{2i+j\le r\\ i,j\ge 0}}n^i\sum_{k=1}^{\infty}p_{n,k}(x)|k-nx|^j\\
&\quad\int_{|t-x|<\delta}b_{n,k}(t)dt+\frac{(n+r-1)!}{(n-1)!}(1+x)^{-n-r}\Bigg\}\\
&\le C_6\|f\|_\gamma\Bigg\{\delta^{-2s}\sum_{\substack{2i+j\le r\\ i,j\ge 0}}n^i\sum_{k=1}^{\infty}p_{n,k}(x)|k-nx|^j\left(\int_0^{\infty}b_{n,k}(t)dt\right)^{1/2}\\
&\quad\times\left(\int_0^{\infty}b_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{4s}dt\right)^{1/2}+\frac{(n+r-1)!}{(n-1)!}(1+x)^{-n-r}\Bigg\}\\
&\le C_6\|f\|_\gamma\delta^{-2s}\sum_{\substack{2i+j\le r\\ i,j\ge 0}}n^i\left\{\sum_{k=1}^{\infty}p_{n,k}(x)(k-nx)^{2j}-(1+x)^{-n-r}(-nx)^{2j}\right\}^{1/2}\\
&\quad\times\left(\int_0^{\infty}b_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{4s}dt\right)^{1/2}+\|f\|_\gamma\frac{(n+r-1)!}{(n-1)!}(1+x)^{-n-r}.
\end{aligned}
$$

Hence, by making the use of Lemmas 2.17 and 2.20, we get

$$I \leq C_7\|f\|_\gamma \leq \delta^{-2m}O(n^{(i+j/2-s)}) \leq C_7 n^{-q}\|f\|_\gamma,\ q = s - r/2,$$

where the last term vanishes as $n \to \infty$. Now choosing $m > 0$ satisfying $q \geq k$, we have

$$I \leq C_7 n^{-1}\|f\|_\gamma.$$

Therefore, by property (iii) of the Steklov mean, we obtain

$$S_1 \leq C_9\omega_2(f^{(r)}, \eta, [a,b]) + C_7 n^{-1}\|f\|_\gamma.$$

If we choose $\eta = n^{-1/2}$, the theorem follows. ■

Another modification of the operators (2.77) was proposed by Gupta [103] (see also [124]). For $\alpha > 0$, a new type of Baskakov–Durrmeyer operator is defined as

$$B_{n,\alpha}(f,x) = \sum_{v=1}^{\infty} p_{n,v,\alpha}(x)\int_0^\infty b_{n,v,\alpha}(t)f(t)dt + (1+\alpha x)^{-n/\alpha}f(0), \tag{2.80}$$

where

$$p_{n,v,\alpha}(x) = \frac{\Gamma(n/\alpha + v)}{\Gamma(v+1)\Gamma(n/\alpha)}\frac{(\alpha x)^v}{(1+\alpha x)^{v+n/\alpha}}$$

and

$$b_{n,v,\alpha}(t) = \frac{\alpha\Gamma(n/\alpha + v + 1)}{\Gamma(v)\Gamma(n/\alpha + 1)}\frac{(\alpha t)^{v-1}}{(1+\alpha t)^{1+v+n/\alpha}}.$$

Some local and global direct results for the operators (2.80) were established in [103, 124]. Before presenting the main results, we will mention some lemmas without proofs.

Lemma 2.23 ([103]). *If we define the central moments as*

$$\begin{aligned} T_{n,m,\alpha}(x) &= B_{n,\alpha}((t-x)^m, x) \\ &= \sum_{k=1}^{\infty} p_{n,k,\alpha}(x)\int_0^\infty b_{n,k,\alpha}(t)\,(t-x)^m\,dt + (1+\alpha x)^{-n/\alpha}(-x)^m,\ m \in \mathbb{N}, \end{aligned}$$

then $T_{n,0,\alpha}(x) = 1$ and $T_{n,1,\alpha}(x) = 0$, and for $n > \alpha m$, we have the following recurrence relation:

$$(n-\alpha m)T_{n,m+1,\alpha}(x) = x(1+\alpha x)[T'_{n,m,\alpha}(x)+2mT_{n,m-1,\alpha}(x)] \\ +[n(\alpha-\beta x)+m(1+2\alpha x)T_{n,m,\alpha}(x).$$

From the recurrence relation, we can easily verify that for all $x \in [0,\infty)$, *we have*

$$\mu_{n,m}(x) = O(n^{-[(m+1)/2]}).$$

Remark 2.6. From Lemma 2.23, for each $x \in (0,\infty)$, we have

$$B_{n,\alpha}(t^i,x) = \frac{\Gamma(n/\alpha+i)\Gamma(n/\alpha-i+1)}{\Gamma(n/\alpha+1)\Gamma(n/\alpha)}x^i \\ +i(i-1)\frac{\Gamma(n/\alpha+i-1)\Gamma(n/\alpha-i+1)}{\Gamma(n/\alpha+1)\Gamma(n/\alpha)}x^{i-1}+O(n^{-2}).$$

Lemma 2.24 ([103]). *The polynomials* $Q_{i,j,r,\alpha}(x)$ *on* $[0,\infty)$, *independent of n and k, exist such that*

$$[x(1+\alpha x)]^r\frac{d^r}{dx^r}p_{n,k,\alpha}(x) = \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i(k-nx)^j Q_{i,j,r,\alpha}(x)p_{n,k,\alpha}(x).$$

Lemma 2.25 ([103]). *Let* f *be* $r-$*times differentiable on* $[0,\infty)$ *such that* $f^{(r-1)}(t) = O(t^\gamma)$, $\gamma > 0$ *as* $t \to \infty$. *Then for* $r = 1,2,\ldots$ *and* $n > \gamma + r$, *we have*

$$B^{(r)}_{n,\alpha}(f,x)=\frac{\Gamma(n/\alpha+r)\Gamma(n/\alpha-r+1)}{\Gamma(n/\alpha+1)\Gamma(n/\alpha)}\sum_{k=0}^{\infty}p_{n+\alpha r,k,\alpha}(x)\int_0^\infty b_{n-\alpha r,k+r,\alpha}(t)f^{(r)}(t)dt.$$

Let $C_B[0,\infty)$ be as defined in Definition 1.10 endowed with the norm $||f|| = \sup_{x\ge 0}|f(x)|$. The Peetre K-functional is defined by

$$K_2(f,\delta) = \inf\{||f-g||+\delta||g''|| : g \in W^2_\infty\},$$

where $W^2_\infty = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. By the DeVore–Lorentz property (Theorem 2.4 of [60]), there exists an absolute constant $M > 0$ such that

$$K_2(f,\delta) \le M\omega_2(f,\sqrt{\delta}),$$

where $\omega_2(f,\sqrt{\delta})$ is the usual modulus of continuity of second order.

Theorem 2.78 (Local result [103]). *Let* $f \in C_B[0,\infty)$. *Then for all* $x \in [0,\infty)$ *and* $n \in \mathbb{N}, n > \alpha$, *there exists an absolute positive constant* $M > 0$ *such that*

$$|B_{n,\alpha}(f,x)-f(x)| \le M\omega_2\left(f,\sqrt{\frac{2x(1+\alpha x)}{n-\alpha}}\right).$$

Proof. Let $g \in W_\infty^2$ and $x, t \in [0,\infty)$. By a Taylor expansion, we have

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du.$$

Using Lemma 2.23, we obtain

$$B_{n,\alpha}(g,x) - g(x) = B_{n,\alpha}\left(\int_x^t (t-u)g''(u)du, x\right).$$

Also, as $|\int_x^t (t-u)g''(u)du| \leq (t-x)^2||g''||$, we have

$$\begin{aligned}|B_{n,\alpha}(g,x) - g(x)| &\leq B_{n,\alpha}\left((t-x)^2, x\right)||g''|| \\ &= \frac{2x(1+\alpha x)}{n-\alpha}||g''||.\end{aligned}$$

Again, by Lemma 2.23, we have

$$\begin{aligned}|B_{n,\alpha}(f,x)| \leq &\sum_{k=1}^{\infty} p_{n,k,\alpha}(x) \int_0^\infty b_{n,k,\alpha}(t)|f(t)|dt \\ &+(1+\alpha x)^{-n/\alpha}|f(0)| \leq ||f||.\end{aligned}$$

Thus,

$$\begin{aligned}|B_{n,\alpha}(f,x) - f(x)| &\leq |B_{n,\alpha}(f-g,x) - (f-g)(x)| \\ &\quad + |B_{n,\alpha}(g,x) - g(x)| \\ &\leq 2||f-g|| + \frac{2x(1+\alpha x)}{n-\alpha}||g''||.\end{aligned}$$

Finally, taking the infimum over all $g \in W_\infty^2$ and using the inequality $K_2(f,\delta) \leq M\omega_2(f,\sqrt{\delta}), \delta > 0$, we get the required result. ∎

Theorem 2.79 (Simultaneous approximation [124]). *Let $n > r+1 \geq 2$, $x \in [0,\infty)$, and $f^{(i)} \in C_B[0,\infty)$ for $i \in \{0,1,2,\ldots,r\}$. Then*

$$\begin{aligned}|B_{n,\alpha}^{(r)}(f,x) - f^{(r)}(x)| \leq &\left(\frac{\Gamma(n/\alpha+r)\Gamma(n/\alpha-r+1)}{\Gamma(n/\alpha+1)\Gamma(n/\alpha)} - 1\right)\|f^{(r)}\| \\ &+2\frac{\Gamma(n/\alpha+r)\Gamma(n/\alpha-r+1)}{\Gamma(n/\alpha+1)\Gamma(n/\alpha)}\omega\big(f^{(r)}, \delta(n,r,x,\alpha)\big),\end{aligned}$$

where

$$\delta(n,r,x,\alpha) = \left\{ \frac{2n\alpha + 4\alpha^2 r(1+r)}{(n-\alpha r)(n-\alpha(r+1))} x^2 + \frac{2n + 4\alpha r(1+r)}{(n-\alpha r)(n-\alpha(r+1))} x \right.$$
$$\left. + \frac{r(1+r)}{(n-\alpha r)(n-\alpha(r+1))} \right\}^{1/2}.$$

Proof. Applying Lemma 2.25, we get

$$\begin{aligned} B_{n,\alpha}^{(r)}(f,x) - f^{(r)}(x) &= \left[\frac{\Gamma(n/\alpha + r)\Gamma(n/\alpha - r + 1)}{\Gamma(n/\alpha + 1)\Gamma(n/\alpha)} - 1 \right] f^{(r)}(x) \\ &\quad + \frac{\Gamma(n/\alpha + r)\Gamma(n/\alpha - r + 1)}{\Gamma(n/\alpha + 1)\Gamma(n/\alpha)} \\ &\quad \times \sum_{v=0}^{\infty} p_{n+\alpha r,v,\alpha}(x) \int_0^{\infty} b_{n-\alpha r,v+r,\alpha}(t) \left[f^{(r)}(t) - f^{(r)}(x) \right] dt, \end{aligned}$$

because $\int_0^\infty b_{n-\alpha r,v+r,\alpha}(t)dt = 1$ and $\sum_{v=0}^{\infty} p_{n+\alpha r,v,\alpha}(x) = 1$. Using the inequality $\omega\left(f^{(r)}, \lambda\delta\right) \leq (1+\lambda)\omega\left(f^{(r)}, \delta\right), \lambda \geq 0$, we get

$$\begin{aligned} |B_{n,\alpha}^{(r)}(f,x) - f^{(r)}(x)| &\leq \left[\frac{\Gamma(n/\alpha + r)\Gamma(n/\alpha - r + 1)}{\Gamma(n/\alpha + 1)\Gamma(n/\alpha)} - 1 \right] \|f^{(r)}\| \\ &\quad + \frac{\Gamma(n/\alpha + r)\Gamma(n/\alpha - r + 1)}{\Gamma(n/\alpha + 1)\Gamma(n/\alpha)} \\ &\quad \times \sum_{v=0}^{\infty} p_{n+\alpha r,v,\alpha}(x) \int_0^{\infty} b_{n-\alpha r,v+r,\alpha}(t) |f^{(r)}(t) - f^{(r)}(x)| dt \\ &\leq \frac{\Gamma(n/\alpha + r)\Gamma(n/\alpha - r + 1)}{\Gamma(n/\alpha + 1)\Gamma(n/\alpha)} \\ &\quad \times \sum_{v=0}^{\infty} p_{n+\alpha r,v,\alpha}(x) \int_0^{\infty} b_{n-\alpha r,v+r,\alpha}(t) \left(1+\delta^{-1}|t-x|\right) \omega\left(f^{(r)}, \delta\right) dt \\ &\quad + \left[\frac{\Gamma(n/\alpha + r)\Gamma(n/\alpha - r + 1)}{\Gamma(n/\alpha + 1)\Gamma(n/\alpha)} - 1 \right] \|f^{(r)}\|. \end{aligned} \tag{2.81}$$

Using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} &\sum_{v=0}^{\infty} p_{n+\alpha r,v,\alpha}(x) \int_0^{\infty} b_{n-\alpha r,v+r,\alpha}(t) |t-x| dt \\ &\quad \leq \left(\sum_{v=0}^{\infty} p_{n+\alpha r,v,\alpha}(x) \int_0^{\infty} b_{n-\alpha r,v+r,\alpha}(t)(t-x)^2 dt \right)^{\frac{1}{2}}. \end{aligned} \tag{2.82}$$

Also, by easy computation, we have

$$\sum_{v=0}^{\infty} p_{n+\alpha r,v,\alpha}(x)\int_0^{\infty} b_{n-\alpha r,v+r,\alpha}(t)(t-x)^2dt$$
$$=\sum_{v=0}^{\infty} p_{n+\alpha r,v,\alpha}(x)\left[\frac{(v+r+1)(v+r)}{(n-\alpha r)[n-\alpha(r+1)]}-\frac{2x(v+r)}{n-\alpha r}+x^2\right]$$
$$=\frac{[2n\alpha+4r\alpha^2(1+r)]x^2+[2n+4\alpha r(1+r)]x+r(1+r)}{(n-\alpha r)(n-\alpha(r+1))}. \quad (2.83)$$

After combining (2.81)–(2.83), we get

$$\left|B_{n,\alpha}^{(r)}(f,x)-f^{(r)}(x)\right|$$
$$\leq \frac{\Gamma(n/\alpha+r)\Gamma(n/\alpha-r+1)}{\Gamma(n/\alpha+1)\Gamma(n/\alpha)}\left(1+\delta^{-1}\left[\frac{2n\alpha+4\alpha^2r(1+r)}{(n-\alpha r)(n-\alpha(r+1))}x^2\right.\right.$$
$$\left.\left.+\frac{2n+4\alpha r(1+r)}{(n-\alpha r)(n-\alpha(r+1))}x+\frac{r(r+1)}{(n-\alpha r)(n-\alpha(r+1))}\right]^{\frac{1}{2}}\right)\omega(f^{(r)},\delta)$$
$$+\left[\frac{\Gamma(n/\alpha+r)\Gamma(n/\alpha-r+1)}{\Gamma(n/\alpha+1)\Gamma(n/\alpha)}-1\right]\|f^{(r)}\|.$$

Finally, if we choose $\delta=\delta(n,r,x,\alpha)$, we obtain the assertion of the theorem. ■

Theorem 2.80 (Global result [103]). *Let $f\in C_B[0,\infty)$ and $n\geq\alpha+1$. Then*

$$||B_{n,\alpha}(f,.)-f||\leq C\omega_\varphi^2(f,n^{-1/2}),$$

where $C>0$ is an absolute constant and $\varphi(x)=\sqrt{2x(1+\alpha x)}, x\in[0,\infty)$;

$$\omega_\varphi^2(f,\delta)=\sup_{0<h\leq\delta}\ \sup_{x\pm h\varphi(x)\in[0,\infty)}|f(x+h\varphi(x)-2f(x)+f9x-h\varphi(x))|$$

is the Ditzian–Totik modulus of smoothness of second order.

Proof. Using a Taylor expansion, we have

$$g(t)=g\left(\frac{k}{n}\right)+g'\left(\frac{k}{n}\right)\left(t-\frac{k}{n}\right)+\int_{k/n}^{t}(t-u)g''(u)du.$$

Also, by Lemma 2.23, we have

$$B_{n,\alpha}(g,x)-\hat{B}_{n,\alpha}(g,x)=\sum_{k=1}^{\infty} p_{n,k,\alpha}\int_0^{\infty}\left[g(t)-g\left(\frac{k}{n}\right)\right]b_{n,k,\alpha}(t)dt$$

$$= \sum_{k=1}^{\infty} p_{n,k,\alpha} \int_0^{\infty} b_{n,k,\alpha}(t) \left[\int_{k/n}^{t} (t-u) g''(u) du \right] dt,$$

where $\hat{B}_{n,\alpha}(f,x)$ denotes the modified Baskakov operators given by

$$\hat{B}_{n,\alpha}(f,x) = \sum_{k=0}^{\infty} \frac{\Gamma(n/\alpha + k)}{\Gamma(k+1)\Gamma(n/\alpha)} \frac{(\alpha x)^k}{(1+\alpha x)^{n/\alpha+k}} f(k/n), x \in [0,\infty).$$

Therefore, by using p. 140 of [62], we have

$$\begin{aligned}
&|B_{n,\alpha}(g,x) - \hat{B}_{n,\alpha}(g,x)| \\
&\le \sum_{k=1}^{\infty} p_{n,k,\alpha} \int_0^{\infty} b_{n,k,\alpha}(t) \left| \int_{k/n}^{t} \frac{|t-u| du}{2u(1+\alpha u)} \right| dt ||\varphi^2 g''|| \\
&\le \frac{1}{2} \sum_{k=1}^{\infty} p_{n,k,\alpha}(x) \int_0^{\infty} b_{n,k,\alpha}(t) \frac{(t-k/n)^2}{k/n} \\
&\quad \times \left(\frac{1}{1+\alpha k/n} + \frac{1}{1+\alpha t} \right) dt ||\varphi^2 g''|| \\
&\le \frac{1}{2} \sum_{k=1}^{\infty} p_{n,k,\alpha}(x) \frac{n}{k} \frac{n}{n+\alpha k} \int_0^{\infty} b_{n,k,\alpha}(t) \left(t - \frac{k}{n} \right)^2 dt ||\varphi^2 g''|| \\
&\quad + \frac{1}{2} \sum_{k=1}^{\infty} p_{n,k,\alpha}(x) \frac{n}{k} \int_0^{\infty} b_{n,k,\alpha}(t) \frac{(t-k/n)^2}{1+\alpha t} dt ||\varphi^2 g''||.
\end{aligned} \tag{2.84}$$

Also, by easy computation, we have

$$\int_0^{\infty} b_{n,k,\alpha}(t)(t-k/n)^2 dt = \frac{k}{n} \frac{n+\alpha k}{n} \frac{1}{n-\alpha} \tag{2.85}$$

and

$$\int_0^{\infty} b_{n,k,\alpha}(t) \frac{(t-k/n)^2}{1+\alpha t} dt = \frac{k}{n} \frac{1}{n} \frac{n+\alpha k}{n+\alpha(k+1)}. \tag{2.86}$$

Combining (2.84)–(2.86), we get

$$\begin{aligned}
&|B_{n,\alpha}(g,x) - \hat{B}_{n,\alpha}(g,x)| \\
&\le \frac{1}{2} \left\{ \sum_{k=1}^{\infty} p_{n,k,\alpha}(x) \frac{1}{n-\alpha} + \sum_{k=1}^{\infty} p_{n,k,\alpha}(x) \frac{1}{n} \frac{n+\alpha k}{n+\alpha(k+1)} \right\} ||\varphi^2 g''||
\end{aligned}$$

$$\leq \frac{1}{2}\left\{\frac{1}{n-\alpha}+\sum_{k=1}^{\infty}p_{n,k,\alpha}\frac{1}{n}\right\}||\varphi^2 g''|| \leq \frac{1}{n-\alpha}||\varphi^2 g''||. \tag{2.87}$$

Applying Lemma 2.23, for all $f \in C_B[0,\infty)$, we have

$$||B_{n,\alpha}(f,.)|| \leq ||f||. \tag{2.88}$$

Next, applying a Taylor expansion of g and the Schwarz inequality, and using the facts $\hat{B}_{n,\alpha}(1,x) = 1$, $\hat{B}_{n,\alpha}(t-x,x) = 0$, and $\hat{B}_{n,\alpha}((t-x)^2,x) = (x(1+\alpha x))/n$, we have

$$\begin{aligned}
|\hat{B}_{n,\alpha}(g,x)-g(x)| &\leq \left|\hat{B}_{n,\alpha}\left(\int_{k/n}^{t}(t-u)g''(u)du, x\right)\right| \\
&\leq \hat{B}_{n,\alpha}\left(\left|\int_{k/n}^{t}(t-u)g''(u)du\right|, x\right) \\
&\leq \hat{B}_{n,\alpha}((t-x)^2,x)||g''(u)|| \leq \frac{x(1+\alpha x)}{n}g''(u) \\
&= \frac{1}{n}||\hat{\varphi}^2 g''||,
\end{aligned} \tag{2.89}$$

where $\hat{\varphi}(x) = \sqrt{x(1+\alpha x)}$, $x \in [0,\infty)$. Combining the estimates of (2.87)–(2.89), we obtain

$$\begin{aligned}
|B_{n,\alpha}(f,x)-f(x)| \leq\ & |B_{n,\alpha}(f-g,x)-(f-g)(x)| \\
&+\left|B_{n,\alpha}(g,x)-\hat{B}_{n,\alpha}(g,x)\right|+\left|\hat{B}_{n,\alpha}(g,x)-g(x)\right| \\
\leq\ & 2||f-g||+\frac{1}{n-\alpha}||\varphi^2 g''||+\frac{1}{n}||\hat{\varphi}^2 g''|| \\
\leq\ & 2\left\{||f-g||+\frac{1}{n}\frac{1}{n-\alpha}||\varphi^2 g''||\right\} \\
&+\left\{||f-g||+\frac{1}{n}||\hat{\varphi}^2 g''||\right\}.
\end{aligned} \tag{2.90}$$

Considering the K-functionals

$$K_{\varphi}^2(f,\delta) = \inf\{||f-g||+\delta||\varphi^2 g''|| : g \in W_{\infty}^2\}, \delta > 0,$$

and

$$K_{\hat{\varphi}}^2(f,\delta) = \inf\{||f-g||+\delta||\hat{\varphi}^2 g''|| : g \in W_{\infty}^2\}, \delta > 0,$$

by (2.90) for all $f \in C_B[0,\infty)$ and $n \geq \alpha + 1$, we obtain

$$||B_{n,\alpha} f - f|| \leq 2K_{\varphi}^2(f,(n-\alpha)^{-1}) + K_{\hat{\varphi}}^2(f,n^{-1}).$$

Using pp. 11 and 37 of [62], the inequality $\hat{\varphi}(x) \leq \varphi(x)$, we get the desired result. ■

Using Lemmas 2.23–2.25, we also obtained in [103]—for the class $C_\gamma[0,\infty) = \{f \in C[0,\infty) : f(t) = O(t^\gamma), \gamma > 0\}$—the pointwise convergence, error estimation, and asymptotic formula for $B_{n,\alpha}$.

Theorem 2.81 ([103] Asymptotic formula). *Let $f \in C_\gamma[0,\infty)$. If $f^{(r+2)}$ exists at a point $x \in (0,\infty)$, then*

$$\lim_{n\to\infty} n\left(B_{n,\alpha}^{(r)}(f,x) - f^{(r)}(x)\right) = \alpha r(r-1)f^{(r)}(x) + r(1+2\alpha x)f^{(r+1)}(x) + x(1+\alpha x)f^{(r+2)}(x).$$

2.9 Mixed Summation–Integral Operators

While many mixed summation–integral operators are available in the literature, in this section we mention only the operators that we believe will be of interest to the reader.

Let us consider the following basis functions:

$$\begin{aligned}
b_{n,v}(x) &= \frac{1}{B(n,v+1)} \frac{x^v}{(1+x)^{n+v+1}}, \\
s_{n,v}(x) &= e^{-nx}\frac{(nx)^v}{v!}, \\
p_{n,v}(x) &= \binom{n+v-1}{v}\frac{x^v}{(1+x)^{n+v}}, \\
l_{n,k}^{(\beta)}(x) &= \frac{nx(nx+k\beta)^{k-1}}{k!}e^{-(nx+k\beta)}, \\
p_{n,v}^{[c]}(x) &= \left(\frac{c}{1+c}\right)^{ncx}\binom{ncx+v-1}{v}(1+c)^{-v} (v=0,1,\ldots), \\
m_{n,v}(x) &= \binom{n+v-1}{v}x^v(1-x)^n,
\end{aligned}$$

$$\hat{m}_{n,v}(x) = n\binom{n+v}{v} x^v (1-x)^{n-1}.$$

1. Beta–Szász operators (see [116]) are defined as

$$A_n(f,x) = \sum_{v=1}^{\infty} b_{n,v}(x) \int_0^{\infty} s_{n,v-1}(t) f(t)dt + (1+x)^{-n-1} f(0). \tag{2.91}$$

   The operators $A_n(f,x)$ preserve only constant functions; the conclusion of the asymptotic formula has the following form:

$$\lim_{n\to\infty} n\left(A_n^{(r)}(f,x) - f^{(r)}(x)\right) = \frac{r(r+1)}{2} f^{(r)}(x) + [r + x(1+r)] f^{(r+1)}(x) + \frac{x(1+x)}{2} f^{(r+2)}(x).$$

2. The discretely defined Baskakov–Szász operators [24] are given as

$$B_n(f,x) = n\sum_{v=1}^{\infty} p_{n,v}(x) \int_0^{\infty} s_{n,v-1}(t) f(t)dt + (1+x)^{-n} f(0). \tag{2.92}$$

   The operators $B_n(f,x)$ preserve constant and linear functions; the conclusion of the asymptotic formula is as follows:

$$\lim_{n\to\infty} n\left(B_n(f,x) - f(x)\right) = \frac{x(2+x)}{2} f''(x).$$

3. The Szász–Mirakyan–Baskakov-type operators (see [125]) are defined by

$$C_n(f,x) = (n-1)\sum_{v=0}^{\infty} s_{n,v}(x) \int_0^{\infty} p_{n,v}(t) f(t)dt. \tag{2.93}$$

   The operators $C_n(f,x)$ preserve only constant functions and have the following form of asymptotic formula:

$$\lim_{n\to\infty} n\left(C_n^{(r)}(f,x) - f^{(r)}(x)\right) = \frac{r(r+3)}{2} f^{(r)}(x) + [(r+1) + x(r+2)] f^{(r+1)}(x) + \frac{x(2+x)}{2} f^{(r+2)}(x).$$

4. The discretely defined Szász–Mirakyan–Baskakov-type operators (see [125]) are given as

$$D_n(f,x) = (n-1)\sum_{v=1}^{\infty} s_{n,v}(x)\int_0^{\infty} p_{n,v-1}(t)f(t)dt + e^{-nx}f(0). \tag{2.94}$$

The operators $D_n(f,x)$ preserve only constant functions; the conclusion of the asymptotic formula is as follows:

$$\lim_{n\to\infty} n\Big(D_n(f,x) - f(x)\Big) = 2xf'(x) + \frac{x(2+x)}{2}f''(x).$$

5. In 2013, Gupta [105] proposed the mixed Durrmeyer-type integral modification of the operators studied in [6] as

$$(D_{n,c,d}f)(x) = (n-d)\sum_{v=0}^{\infty} p_{n,v}^{[c]}(x)\int_0^{\infty} b_{n,v}^{[d]}(t)f(t)dt, x \geq 0, \tag{2.95}$$

where $b_{n,v}^{[d]}(t) = (-1)^v \frac{t^v}{v!}\phi_{n,d}^{(v)}(t)$, with two special cases:

(a) $\phi_{n,0}(t) = e^{-nt}$.
(b) $\phi_{n,1}(t) = (1+t)^{-n}$.

The operators $(D_{n,c,d}f)(x)$ reproduce the constant functions only, and the asymptotic formula has the form

$$\lim_{n\to\infty} n[(D_{n,c,d}f)(x) - f(x)] = (1+2xd)f'(x) + \frac{x(xcd+1+2c)}{2c}f''(x).$$

6. The discretely defined Jain–Beta operators considered in [228] are as follows:

$$E_n(f,x) = \sum_{v=1}^{\infty} l_{n,v}^{(\beta)}(x)\int_0^{\infty} \frac{1}{B(n+1,v)}\frac{t^{v-1}}{(1+t)^{n+v+1}}f(t)dt + e^{-nx}f(0). \tag{2.96}$$

The operators $E_n(f,x)$ reproduce only constant functions. The Korovkin-type convergence theorem and statistical convergence for the operators $E_n(f,x)$ were discussed in [228].

7. For the function $f \in L_1[0,1]$, the Durrmeyer variant of mixed MKZ operators is defined in [107] as

$$M_n(f,x) = \sum_{v=0}^{\infty} m_{n,v}(x)\int_0^1 \hat{m}_{n,v}(t)f(t)dt. \tag{2.97}$$

Abel et al. [7] obtained sharp estimates of the first and second central moments and established the rate of convergence by the first modulus of continuity. They also obtained the following Voronovskaja-type asymptotic formula:

Let $x \in [0, 1]$. For each function $f \in L_\infty[0, 1]$ that admits a second derivative at x, the operators satisfy

$$\lim_{n\to\infty} n\Big(M_n(f, x) - f(x)\Big) = (1-x)(1-2x)f'(x) + x(1-x)^2 f''(x).$$

For the function $f \in L_1[0, 1]$ and $\alpha \geq 1$, the Bézier variant of (2.97) considered in [7] is

$$M_{n,\alpha}(f, x) = \sum_{v=0}^{\infty} Q_{n,v}^{(\alpha)}(x) \int_0^1 \hat{m}_{n,v}(t) f(t)dt, \tag{2.98}$$

where

$$Q_{n,v}^{(\alpha)}(x) = \left(\sum_{j=v}^{\infty} m_{n,j}(x)\right)^{\alpha} - \left(\sum_{j=v+1}^{\infty} m_{n,j}(x)\right)^{\alpha}.$$

The following direct result was also established by Abel et al. [7]:

For every $f \in C[0, 1], \alpha \geq 1$, and $\delta > 0$, we have

$$|M_{n,\alpha}(f, x) - f(x)| \leq \left(1 + \frac{1-x}{\delta}\sqrt{\alpha\frac{2x+53/(n-4)}{n+1}}\right)\omega(f, \delta), x \in [0, 1], n \geq 5.$$

Moreover, if f is differentiable on $[0, 1]$ with f' bounded on $[0, 1]$, we have

$$|M_{n,\alpha}(f, x) - f(x)| \leq (1-x)\sqrt{\alpha\frac{2x + 53/(n-4)}{n+1}}$$
$$\times\left(|f'(x)| + \left(1 + \frac{1-x}{\delta}\sqrt{\alpha\frac{2x+53/(n-4)}{n+1}}\right)\omega(f', \delta)\right).$$

Chapter 3
Complete Asymptotic Expansion

Another topic of interest on linear positive operators is the asymptotic expansion. The commendable work on complete asymptotic expansion for different operators was done in the last two decades by Abel and collaborators (see, e.g., [1–3, 11–13]). In this chapter, we discuss the asymptotic expansion of some of the operators.

Throughout this chapter, we denote by $K[q,x], q \in \mathbb{N}$ and $x \in I$ the class of all functions $f \in L_\infty(I)$ that are q-times differentiable at x.

3.1 Baskakov–Kantorovich Operators

Baskakov–Kantorovich operators for $x \in I \equiv [0,\infty)$ are defined as

$$V_n^*(f,x) = n\sum_{k=0}^{\infty} b_{n,k}(x)\int_{k/n}^{(k+1)/n} f(t)dt, \quad n \in \mathbf{N},$$

where $b_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}$.

Throughout this section, for $0 \le j \le k \le s$, we will denote the numbers $Z(s,k,j)$ by

$$Z(s,k,j) = (-1)^{k-j}\sum_{r=k}^{s}(-1)^{s-r}\binom{s}{r}S_{r-j}^{r-k}\sigma_{r+1}^{r+1-j}\left(1-\frac{j}{r+1}\right), \tag{3.1}$$

where S_j^i and σ_j^i denote the Stirling numbers of the first and second kind, respectively, which are defined as

$$x^{\underline{j}} = \sum_{i=0}^{j} S_j^i x^i,\; x^j = \sum_{i=0}^{j}\sigma_j^i x^{\underline{i}},\; j = 0,1,\ldots. \tag{3.2}$$

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Abel and Gupta [4] established the complete asymptotic expansion for Baskakov–Kantorovich operators. To find the asymptotic expansion, we need the following lemmas:

Lemma 3.1 ([4]). *For $e_r = t^r, r = 0, 1, \ldots$, we have the following representation:*

$$V_n^*(e_r, x) = \sum_{k=0}^{r} n^{-k} \sum_{j=0}^{k} (-1)^{k+j} S_{r-j}^{r-k} \sigma_{r+1}^{r+1-j} \left(1 - \frac{j}{r+1}\right) x^{r-j}.$$

Proof. By direct computation and using the second identity in (3.2), we have

$$\begin{aligned} V_n^*(e_r, x) &= \frac{n^{-r}}{r+1} \sum_{k=0}^{\infty} b_{n,k}(x) \left((k+1)^{r+1} - k^{r+1}\right) \\ &= \frac{n^{-r}}{r+1} \sum_{k=0}^{\infty} b_{n,k}(x) \sum_{j=0}^{r+1} \sigma_{r+1}^{j} \left((k+1)^{\underline{j}} - k^{\underline{j}}\right) \\ &= \frac{n^{-r}}{r+1} \sum_{j=0}^{r} (j+1) \sigma_{r+1}^{j+1} \sum_{k=0}^{\infty} b_{n,k}(x) k^{\underline{j}} \\ &= \frac{n^{-r}}{r+1} \sum_{j=0}^{r} (j+1) \sigma_{r+1}^{j+1} (n+j-1)^{\underline{j}} x^j. \end{aligned}$$

Next, using the identity

$$(n+j-1)^{\underline{j}} = (-1)^j (-n)^{\underline{j}} = (-1)^j \sum_{k=0}^{j} S_j^k (-n)^k,$$

we conclude that

$$\begin{aligned} V_n^*(e_r, x) &= \frac{1}{r+1} \sum_{k=0}^{r} n^{k-r} \sum_{j=k}^{r} (-1)^{k+j} (j+1) S_j^k \sigma_{r+1}^{j+1} x^j \\ &= \frac{1}{r+1} \sum_{k=0}^{r} n^{-k} \sum_{j=0}^{k} (-1)^{k+j} (r-j+1) S_{r-j}^{r-k} \sigma_{r+1}^{r-j+1} x^{r-j}. \end{aligned}$$

■

Lemma 3.2 ([4]). *For $\psi_x^s = (t-x)^s, s = 0, 1, 2, \ldots$, the central moments possess the representation*

$$V_n^*(\psi_x^s, x) = \sum_{k=[(s+1)/2]}^{s} n^{-k} \sum_{j=0}^{k} x^{s-j} Z(s, k, j),$$

where $Z(s, k, j)$ is defined in (3.1).

Proof. By the binomial theorem and Lemma 3.1, we have

$$\begin{aligned}
V_n^*(\psi_x^s, x) &= \sum_{r=0}^{s} \binom{s}{r} (-x)^{s-r} V_n^*(e_r, x) \\
&= \sum_{k=0}^{s} n^{-k} \sum_{j=0}^{k} (-1)^{k+j} x^{s-j} \sum_{r=k}^{s} (-1)^{s-r} \binom{s}{r} S_{r-j}^{r-k} \sigma_{r+1}^{r+1-j} \left(1 - \frac{j}{r+1}\right) \\
&= \sum_{k=0}^{s} n^{-k} \sum_{j=0}^{k} x^{s-j} Z(s,k,j).
\end{aligned}$$

It remains for us to show that $V_n^*(\psi_x^s, x) = O(n^{-[(s+1)/2]})$. It suffices to show that for $0 \le j \le k$, $Z(s,k,j) = 0$ holds if $2k < s$. Following [162], we have

$$S_r^{r-k} = \sum_{\mu=0}^{k} C_{k,k-\mu} \binom{r}{k+\mu}, \sigma_r^{r-k} \sum_{\nu=0}^{k} \overline{C}_{k,k-\nu} \binom{r}{k+\nu}, \tag{3.3}$$

where $C_{k,k} = \overline{C}_{k.k} = 0$ for $k \ge 1$. First, consider the case $j \ge 1$, using (3.3). For $1 \le j \le k$, we have

$$\begin{aligned}
S_{r-j}^{r-k} \sigma_{r+1}^{r+1-j} &= \sum_{\mu=0}^{k-j} C_{k-j,k-j-\mu} \binom{r-j}{k-j+\mu} \sum_{\nu=1}^{j} \overline{C}_{j,j-\nu} \binom{r+1}{j+\nu} \\
&= \sum_{\nu=0}^{k-j} \sum_{\nu=1}^{j} (r+1)^{\underline{k+1}} P(k,j,\mu,\nu,r),
\end{aligned}$$

where

$$P(k,j,\mu,\nu,r) = \frac{C_{k-j,k-j-\mu}}{(k-j+\mu)!} \frac{\overline{C}_{j,j-\nu}}{(j+\nu)!} (r-j)^{\underline{\nu-1}} (r-k)^{\underline{\mu}}$$

is a polynomial in the variable r of degree $\le \mu + \nu - 1$. Thus,

$$\begin{aligned}
Z(s,k,j) &= (-1)^{k-j} \sum_{\mu=0}^{k-j} \sum_{\nu=1}^{j} \sum_{r=k}^{s} (-1)^{s-r} \binom{s}{r} r^{\underline{k}} (r+1-j) P(j,k,\mu,\nu,r) \\
&= (-1)^j s^{\underline{k}} \sum_{\mu=0}^{k-j} \sum_{\nu=1}^{j} \sum_{r=0}^{s-k} (-1)^{s-r} \binom{s-k}{r} (r+k+1-j) P(k,j,\mu,\nu,r+k).
\end{aligned}$$

Since $(r+k+1-j)P(k,j,\mu,\nu,r+k)$ is a polynomial in variable r of degree $\leq \mu+\nu \leq k$, the inner sum vanishes if $k < s-k$,. In the case $j=0$, we have

$$
\begin{aligned}
Z(s,k,0) &= (-1)^k \sum_{r=k}^{s} (-1)^{s-r} \binom{s}{r} S_r^{r-k} \\
&= (-1)^k \sum_{r=k}^{s} (-1)^{s-r} \binom{s}{r} \sum_{\mu=0}^{k} C_{k,k-\mu} \binom{r}{k+\mu} \\
&= (-1)^k s^{\underline{k}} \sum_{\mu=0}^{k} \frac{C_{k,k-\mu}}{(k+\mu)!} \sum_{r=0}^{s-k} (-1)^{s-k-r} \binom{s-k}{r} r^{\underline{\mu}},
\end{aligned}
$$

which vanishes if $k < s-k$. This completes the proof of the lemma. ∎

Remark 3.1. For $s = 0, 1, 2, \ldots,$ as a consequence of Lemma 3.2, we have

$$V_n^*(\psi_x^s, x) = O(n^{-[(s+1)/2]}), n \to \infty.$$

Theorem 3.1 ([4] Asymptotic expansion). *Let $q \in \mathbb{N}$ and $x \in (0,\infty)$. For each function $f \in K[2q,x]$, the Baskakov–Kantorovich operators possess the asymptotic expansion*

$$V_n^*(f,x) = f(x) + \sum_{k=1}^{q} n^{-k} \sum_{s=k}^{2k} \frac{f^{(s)}(x)}{s!} \sum_{j=0}^{k} x^{s-j} Z(s,k,j) + o(n^{-q}), \quad n \to \infty,$$

where the numbers $Z(s,k,j)$ are as defined in (3.1).

If we use Remark 3.1 and Lemma 3.2, the theorem follows after simple computation.

Corollary 3.1 ([4]). *For each $f \in K[2,x], x \in (0,\infty)$, we have*

$$\lim_{n\to\infty} n[V_n^*(f,x) - f(x)] = \frac{1}{2}\left[f'(x) + x(1+x)f''(x)\right].$$

Example 3.1. For $q = 3$, we have

$$
\begin{aligned}
V_n^*(f,x) &= f(x) + \frac{f'(x) + x(1+x)f''(x)}{2n} \\
&\quad + \frac{4f^{(2)}(x) + 2x(1+x)(5+4x)f^{(3)}(x) + 3x^2(1+x)^2 f^{(4)}(x)}{24n^2} \\
&\quad + \frac{1}{48n^3}\Big(2f^{(3)}(x) + 2x(1+x)(5+10x+6x^2)f^{(4)}(x) \\
&\quad + x^2(1+x)^2(7+8x)f^{(5)}(x) + x^3(1+x)^3 f^{(6)}(x)\Big).
\end{aligned}
$$

3.2 Baskakov–Szász–Durrmeyer Operators

Let $W_\gamma(0,\infty), \gamma > 0$ denote the space of all locally integrable functions f on $(0,\infty)$ satisfying $f(t) = O(e^{\gamma t})$ as $t \to \infty$. Gupta and Srivastava [125] proposed the Durrmeyer-type Baskakov–Szász operators as

$$V_n(f,x) = n\sum_{v=0}^{\infty} b_{n,v}(x)\int_0^\infty s_{n,v}(t)f(t)dt, \quad n \in \mathbf{N},$$

where

$$b_{n,v}(x) = \binom{n+v-1}{v}\frac{x^v}{(1+x)^{n+v}}, s_{n,v}(t) = e^{-nt}\frac{(nt)^v}{v!}.$$

For $f \in W_\gamma(0,\infty)$, the operators $V_n(f,x)$ are well defined for each integer $n > \gamma$. Abel et al. [8] established the complete asymptotic expansion for the Baskakov–Szász operators. Throughout this section, we denote the number $Z(s,k,j)$ for $0 \leq j \leq k \leq s$ as

$$Z(s,k,j) = \binom{s}{j}\sum_{m=k}^{s}(-1)^{s-m}\binom{s-j}{m-j}\begin{bmatrix} m-j \\ m-k \end{bmatrix} m^{\underline{j}}. \tag{3.4}$$

The quantity $\begin{bmatrix} j \\ i \end{bmatrix}$ in (3.4) denotes the Stirling numbers of the first kind defined as

$$x^{\overline{j}} = \sum_{i=0}^{j}\begin{bmatrix} j \\ i \end{bmatrix} x^i, j = 0,1,\ldots.$$

In the traditional notation of Stirling numbers of the first kind, we have $S_j^i = (-1)^{j-i}\begin{bmatrix} j \\ i \end{bmatrix}$. The following results are required for asymptotic expansion:

Lemma 3.3 ([8]). *For all $m = 0,1,2,\ldots$ and $x \in (0,\infty)$, we have*

$$V_n(e_m,x) = \sum_{k=0}^{m} n^{-k}\sum_{j=0}^{k}\binom{m}{j}\begin{bmatrix} m-j \\ m-k \end{bmatrix} m^{\underline{j}} x^{m-j}, \quad n \to \infty.$$

Proof. Obviously, with $n\int_0^\infty s_{n,v}(t)t^m dt = n^{-m}(v+m)^{\underline{m}}$, we have

$$V_n(e_m,x) = n^{-m}\sum_{v=0}^{\infty} b_{n,v}(x)(v+m)^{\underline{m}}$$

$$= n^{-m}(1+x)^{-n} \sum_{v=0}^{\infty} \binom{n+v-1}{v} \left(\frac{\partial}{\partial y}\right)^m y^{k+m} \Big|_{y=x/(1+x)}$$

$$= n^{-m}(1+x)^{-n} \left(\frac{\partial}{\partial y}\right)^m \frac{y^m}{(1-y)^n} \Big|_{y=x/(1+x)}.$$

Using the Leibniz formula, we have

$$V_n(e_m, x) = n^{-m}(1+x)^{-n} \sum_{j=0}^{m} \binom{m}{j} m^{\underline{m-j}} y^j \frac{(-1)^j (-n)^{\underline{j}}}{(1-y)^{n+j}}$$

$$= \frac{1}{n^m} \sum_{j=0}^{m} \binom{m}{j} m^{\underline{m-j}} n^{\overline{j}} x^j$$

$$= \sum_{j=0}^{m} \binom{m}{j} m^{\underline{m-j}} x^j \sum_{i=0}^{j} \begin{bmatrix} j \\ i \end{bmatrix} n^{i-m}$$

$$= \sum_{k=0}^{m} n^{-k} \sum_{j=0}^{k} \binom{m}{j} \begin{bmatrix} m-j \\ m-k \end{bmatrix} m^{\underline{j}} x^{m-j}.$$

This completes the proof of the lemma. ∎

Lemma 3.4 ([8]). *For each integer $q > 0$ and for each $x \in (0, \infty)$, we have*

$$V_n(\psi_x^s, x) = \sum_{k=[(s+1)/2]}^{q} n^{-k} \sum_{s=k}^{2k} \frac{f^{(s)}(x)}{s!} \sum_{j-=0}^{k} Z(s,k,j) x^{s-j} + o(n^{-q}), n \to \infty.$$

Remark 3.2. For $s = 0, 1, 2, \ldots,$ as a consequence of Lemma 3.4, we have

$$V_n(\psi_x^s, x) = O(n^{-[(s+1)/2]}), \quad n \to \infty.$$

Lemma 3.5 ([8]). *Let I be an interval. For $q \in \mathbb{N}$ and fixed $x \in I$, let $A_n : L_\infty(I) \to C(I)$ be a sequence of linear positive operators with the property*

$$A_n(\psi_x^s, x) = O(n^{-[(s+1)/2]}), \quad n \to \infty.$$

Then, for each $f \in L_\infty(I)$ that is $2q$-times differentiable at x, we have the asymptotic relation

$$A_n(f, x) = \sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} A_n(\psi_x^s, x) + o(n^{-q}), \quad n \to \infty.$$

If, in addition, $f^{(2q+2)}(x)$ exists, the term $o(n^{-q})$ above can be replaced by $O(n^{-(q+1)})$.

Lemma 3.6 ([8]). *Let $x > 0$ be given. Assume that $f \in W_\gamma(0,\infty)$ vanishes in a neighborhood of x. Then for each positive constant q, $V_n(f,x) = O(n^{-q})$ as $n \to \infty$.*

Theorem 3.2 ([8] Asymptotic expansion). *Let $q \in \mathbb{N}$ and $x \in (0,\infty)$. For each function $f \in K[2q,x]$, the operators $V_n(f,x)$ possess the asymptotic expansion*

$$V_n(f,x) = f(x) + \sum_{k=1}^{q} n^{-k} \sum_{s=k}^{2k} \frac{f^{(s)}(x)}{s!} \sum_{j=0}^{k} x^{s-j} Z(s,k,j) + o(n^{-q}), \quad n \to \infty,$$

where the numbers $Z(s,k,j)$ are as defined in (3.4).

Proof. In view of Lemma 3.6, we can assume without loss of generality that the function $f \in K[2q,x]$ is bounded on $(0,\infty)$. By Lemma 3.4 and applications of Lemma 3.5, we have

$$\begin{aligned} V_n(f,x) &= \sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} V_n(\psi_x^s, x) + o(n^{-q}) \\ &= \sum_{k=0}^{q} n^{-k} \sum_{s=k}^{2k} \frac{f^{(s)}(x)}{s!} \sum_{j=0}^{k} Z(s,k,j) x^{s-j} + o(n^{-q}), \quad n \to \infty, \end{aligned}$$

which completes the proof of the theorem. ■

For the parameters $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{Z}$, Abel et al. [10] considered general Baskakov–Szász–Durrmeyer operators $V_{n,\alpha,\beta}$ as

$$V_{n,\alpha,\beta}(f,x) = n \sum_{v=\max\{0,-\beta\}}^{\infty} b_{n+\alpha,v}(x) \int_0^\infty s_{n,v+\beta}(t) f(t)\,dt, \quad n \in \mathbf{N}.$$

The following main result concerning the asymptotic formula was established later.

Theorem 3.3 ([10] Asymptotic expansion). *Let $q \in \mathbb{N}$ and $x \in (0,\infty)$. For each function $f \in K[2q,x]$, the operators $V_{n,\alpha,\beta}(f,x)$ possess the asymptotic expansion*

$$V_{n,\alpha,\beta}(f,x) = f(x) + \sum_{k=1}^{q} n^{-k} c_{k,\alpha,\beta}(f;x) + o(n^{-q}), \quad n \to \infty,$$

where the coefficients $c_{k,\alpha,\beta}(f;x), k = 1,2,\ldots$ are given by

$$c_{k,\alpha,\beta}(f;x) = \sum_{s=k}^{2k} \frac{f^{(s)}(x)}{s!} \sum_{j=0}^{k} x^{s-j} Z_{\alpha,\beta}(s,k,j)$$

and

$$Z_{\alpha,\beta}(s,k,j)$$
$$=\binom{s}{j}\sum_{m=k}^{s}(-1)^{s-m}\binom{s-j}{m-j}(\beta+m)^{\underline{j}}\sum_{i=0}^{k-j}\binom{m-j-i}{m-k}\left[\begin{matrix}m-j\\m-j-i\end{matrix}\right]\alpha^{k-j-i}.$$

Moreover, if $f\in K[2(q+r),x]$, *for* $r\in\mathbb{N}_0$, *the differential operators possess the asymptotic expansion*

$$V^{(r)}_{n,\alpha,\beta}(f,x)=f^{(r)}(x)+\sum_{k=1}^{q}n^{-k}c^{(r)}_{k,\alpha,\beta}(f;x)+o(n^{-q}),\quad n\to\infty.$$

Corollary 3.2 ([10]). *Let* $x\in(0,\infty)$. *For each* $f\in K[2,x]$, *the following asymptotic formula holds:*

$$\lim_{n\to\infty}[V_{n,\alpha,\beta}(f,x)-f(x)]=(1+\beta+\alpha x)f'(x)+\frac{1}{2}x(2+x)f''(x).$$

Moreover, if $f\in K[2r+2,x]$, *for* $r\in\mathbb{N}_0$, *we have*

$$\begin{aligned}\lim_{n\to\infty}[V^{(r)}_{n,\alpha,\beta}(f,x)-f^{(r)}(x)]&=\frac{1}{2}r(r-1+2\alpha)f^{(r)}(x)\\&+[(r+\alpha)x+1+\beta+r]f^{(r+1)}(x)\\&+\frac{1}{2}x(2+x)f^{(r+2)}(x).\end{aligned}$$

3.3 Meyer–König–Zeller–Durrmeyer Operators

For the parameters $\alpha\in\mathbb{R}$ and $\beta\in\mathbb{Z}$, the general Meyer–König–Zeller operators for $n+\alpha>0$ considered by Abel et al. [9] are defined as

$$M_{n,\alpha,\beta}(f,x)=(n+\alpha)\sum_{k=\max\{0,-\beta\}}^{\infty}m_{n,k}(x)\int_0^1 m_{n+\alpha,k+\beta}(t)(1-t)^{-2}f(t)dt,\tag{3.5}$$

where $m_{n,k}(x)=\binom{n+k}{k}x^k(1-x)^{n+1}$. One can obtain the special cases of these operators as

1. $M_{n,0,0}\equiv M_n$ is the natural MKZ–Durrmeyer operators.
2. $M_{n,2,0}\equiv M_n^{[\text{Chen}]}$ [53].

3. $M_{n-1,0,-2} \equiv M_n^{[\mathrm{Guo}]}$ [89].
4. $M_{n-1,1,0} \equiv M_n^{[\mathrm{Gupta}]}$ [107].

Abel et al. [9] obtained a complete asymptotic expansion for the operators (3.5).

Theorem 3.4 ([9] Asymptotic expansion). *Let $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{Z}$. Assume that $q \in \mathbb{N}$ and $x \in I \equiv (0,1)$. For each function $f \in K[2q,x]$, the operators $M_{n,\alpha,\beta}$ possess the asymptotic expansion*

$$M_{n,\alpha,\beta}(f,x) = f(x) + \sum_{k=1}^{q} \frac{c_{k,\alpha,\beta}(f;x)}{(n+\alpha+\beta+1)^{\overline{k}}} + o(n^{-q}) \quad (n \to \infty),$$

where the coefficients $c_{k,\alpha,\beta}(f;x)$ $(k = 1,2,\ldots)$ are given by

$$c_{k,\alpha,\beta}(f;x) = \sum_{s=0}^{2k} \frac{f^{(s)}(x)}{s!}(1-x)^s a_{s,k,\alpha,\beta}(x),$$

with

$$a_{s,k,\alpha,\beta}(x) = (k-1)! \sum_{j=0}^{k} (-1)^{k-j} \binom{k+\beta}{k-j} x^j \times \sum_{r=0}^{s} (-1)^r \binom{s}{r} \binom{r+\alpha+\beta+j-1}{j} \binom{r+j-1}{k-1} r.$$

Remark 3.3. For functions $f \in \bigcap_{q=1}^{\infty} K[q,x]$, we have the complete asymptotic expansion

$$M_{n,\alpha,\beta}(f,x) = f(x) + \sum_{k=1}^{\infty} \frac{c_{k,\alpha,\beta}(f;x)}{(n+\alpha+\beta+1)^{\overline{k}}} \quad (n \to \infty).$$

Remark 3.4. For the reader's convenience, we list the initial coefficients explicitly:

$$\begin{aligned} c_{1,\alpha,\beta}(f;x) &= (1-x)(\beta+1-(\alpha+\beta+1)x)f'(x) + x(1-x)^2 f^{(2)}(x), \\ c_{2,\alpha,\beta}(f;x) &= (\alpha+\beta+1)x(1-x)(\beta+2-(\alpha+\beta+2)x)f'(x) \\ &\quad + \frac{1}{2}(1-x)^2\Big((\beta+1)(\beta+2) - 2(\beta+2)(\alpha+\beta+3)x \\ &\quad + (\alpha+\beta+2)(\alpha+\beta+7)x^2\Big) f^{(2)}(x) \\ &\quad + x(1-x)^3\Big(2+(\beta+1)x-2(\alpha+4)x\Big) f^{(3)}(x) \frac{1}{2} x^2 (1-x)^4 f^{(4)}(x). \end{aligned}$$

An immediate consequence of Theorem 3.4 is the following Voronovskaja-type formula:

Corollary 3.3 ([9]). *Let $x \in I$. For each function $f \in K[2,x]$, the operators $M_{n,\alpha,\beta}$ satisfy*

$$\lim_{n\to\infty} n\left(M_{n,\alpha,\beta}(f,x)-f(x)\right) = (1-x)(\beta+1-(\alpha+\beta+1)x)f'(x)+x(1-x)^2 f^{(2)}(x).$$

In the above-mentioned special cases, we obtain the formulas

$$\lim_{n\to\infty} n\left(M_{n,0,0}(f,x)-f(x)\right) = (1-x)^2\left(xf'(x)\right)',$$

$$\lim_{n\to\infty} n\left(M_n^{[\mathrm{Chen}]}(f,x)-f(x)\right) = \left(x(1-x)^2 f'(x)\right)',$$

$$\lim_{n\to\infty} n\left(M_n^{[\mathrm{Guo}]}(f,x)-f(x)\right) = (1-x)^2\left(-f'(x)+xf''(x)\right),$$

$$\lim_{n\to\infty} n\left(M_n^{[\mathrm{Gupta}]}(f,x)-f(x)\right) = (1-x)\left(x(1-x)f'(x)\right)'.$$

For the operators $M_{n-1,1,0}$ by Gupta and Abel [107], we obtain a concise representation (3.6), which can be useful for the calculation of the coefficients $c_{k,1,0}(f;x)$ by computer algebra software. To this end, for $m = 0,1,2,\ldots$, we denote by $T_m(f;z_0)$ the Taylor polynomial of degree m, given by

$$T_m(f;z_0)(z) = \sum_{s=0}^{m} \frac{f^{(s)}(z_0)}{s!}(z-z_0)^s,$$

provided f admits a derivative of order m in z_0.

Corollary 3.4 ([9]). *Under the assumption of Theorem 3.4, the operators $M_n^{[\mathrm{Gupta}]} \equiv M_{n-1,1,0}$ possess the asymptotic expansion*

$$M_n^{[\mathrm{Gupta}]}(f,x) = f(x) + \sum_{k=1}^{q} \frac{c_{k,1,0}(f;x)}{(n+1)^{\overline{k}}} + o(n^{-q}) \quad (n\to\infty),$$

where the coefficients $c_{k,1,0}(f;x)$ $(k=1,2,\ldots)$ are given by

$$c_{k,1,0}(f;x) = \frac{(-1)^k}{k!}\left\{\frac{\partial^{2k}}{\partial z^k \partial y^k}\left[y^k T_{2k}(f;x)\left(\frac{1-z(1-y)}{1-z(x-y)}\right)\right]\right\}\Bigg|_{\substack{z=1\\y=x}}. \quad (3.6)$$

Remark 3.5 ([9]). For the reader's convenience, we list the initial coefficients (3.6) explicitly:

$$c_{1,1,0}(f;x) = (1-x)(1-2x)f'(x) + x(1-x)^2 f^{(2)}(x),$$

$$
\begin{aligned}
c_{2,1,0}(f;x) &= 2x(1-x)(2-3x)f'(x) \\
&\quad +(1-x)^2(1-2x)(1-6x)f^{(2)}(x) \\
&\quad +x(1-x)^3(2-5x)f^{(3)}(x)+\frac{1}{2}x^2(1-x)^4f^{(4)}(x), \\
c_{3,1,0}(f;x) &= 6x^2(1-x)(3-4x)f'(x) \\
&\quad +6x(1-x)^2(3-15x+16x^2)f^{(2)}(x) \\
&\quad +(1-x)^3(1-18x+81x^2-92x^3)f^{(3)}(x) \\
&\quad +x(1-x)^4(3-21x+31x^2)f^{(4)}(x) \\
&\quad +\frac{1}{2}x^2(1-x)^5(3-8x)f^{(5)}(x)+\frac{1}{6}x^3(1-x)^6f^{(6)}(x).
\end{aligned}
$$

3.4 Beta Operators of the First Kind

The Beta operators $L_n(n \in N)$ for Lebesgue integrable functions f on the interval $I = (0,1)$ are defined by

$$
L_n(f,x) = \frac{1}{B(nx, n(1-x))} \int_0^1 t^{nx-1}(1-t)^{n(1-x)-1} f(t)dt, \tag{3.7}
$$

where $B(m,n)$ is the Beta function. These operators were introduced by Lupas [185] in a slightly different form. Equation (3.7) was given by Khan [171].

Throughout this section, let the numbers $Z(s,k,j)$ for $0 \le j \le k \le s$ be given by

$$
\begin{aligned}
Z(s,k,j) = \sum_{r=k}^{s} (-1)^{s-r} \binom{s}{r} \left[\begin{matrix} r \\ r-j \end{matrix} \right] \sum_{i=0}^{k-j} \left\{ \begin{matrix} r-j \\ r-j-i \end{matrix} \right\} \\
\times \binom{r-j-i}{k-j-i} (1-r)^{\underline{k-j-i}}.
\end{aligned} \tag{3.8}
$$

The quantities $\left[\begin{matrix} j \\ i \end{matrix} \right]$ and $\left\{ \begin{matrix} j \\ i \end{matrix} \right\}$ above denote the Stirling numbers of the first and second kind, respectively, defined by

$$
x^{\underline{j}} = \sum_{i=0}^{j} (-1)^{j-i} \left[\begin{matrix} j \\ i \end{matrix} \right] x^i, \; x^j = \sum_{i=0}^{j} \left\{ \begin{matrix} j \\ i \end{matrix} \right\} x^{\underline{i}}, \; j = 0, 1, \ldots,
$$

where $x^{\underline{i}} = x(x-1)\cdots(x-i+1)$, $x^{\underline{0}} = 1$ is the falling factorial.

To obtain the asymptotic expansion, we need the following results:

Lemma 3.7 ([11]). *For $r = 0, 1, 2, \ldots,$ the moments $L_n(e^r, x)$ of the Beta operators possess the representation*

$$L_n(e^r,x)=\sum_{k=0}^{r}\frac{1}{n^{\overline{k}}}\sum_{j=0}^{k}\left[\begin{matrix} r \\ r-j \end{matrix}\right]x^{r-j}\sum_{i=0}^{k-j}\left\{\begin{matrix} r-j \\ r-j-i \end{matrix}\right\}\binom{r-j-i}{k-j-i}(1-r)^{\underline{k-j-i}}.$$

Proof. Direct computation of the moments yields

$$L_n(e^r,x)=\frac{B(nx+r,n(1-x))}{B(nx,n(1-x))}=\frac{(nx)^{\overline{r}}}{n^{\overline{r}}}.$$

We take advantage of the identities

$$(nx)^{\overline{r}}=\sum_{j=0}^{r}\left[\begin{matrix} r \\ j \end{matrix}\right](nx)^j,\quad n^j=\sum_{i=0}^{j}\left\{\begin{matrix} j \\ i \end{matrix}\right\}(n)^{\underline{i}},$$

and

$$(n)^{\underline{i}}=\sum_{k=0}^{i}\binom{i}{k}(n+r-1)^{\underline{k}}(1-r)^{\underline{i-k}},$$

which is the Vandermonde convolution. Combining these formulas, we obtain

$$\begin{aligned} L_n(e^r,x) &= \frac{(nx)^{\overline{r}}}{(n)^{\overline{r}}}=\frac{1}{(n)^{\overline{r}}}\sum_{j=0}^{r}\left[\begin{matrix} r \\ j \end{matrix}\right]x^j\sum_{i=0}^{j}\left\{\begin{matrix} j \\ i \end{matrix}\right\}\sum_{k=0}^{i}\binom{i}{k}(n+r-1)^{\underline{k}}(1-r)^{\underline{i-k}} \\ &= \sum_{k=0}^{r}\frac{(n+r-1)^{\underline{k}}}{(n)^{\overline{r}}}\sum_{j=k}^{r}\left[\begin{matrix} r \\ j \end{matrix}\right]x^j\sum_{i=k}^{j}\left\{\begin{matrix} j \\ i \end{matrix}\right\}\binom{i}{k}(1-r)^{\underline{i-k}}. \end{aligned}$$

Observing that

$$\frac{(n+r-1)^{\underline{i}}}{(n)^{\overline{r}}}=\frac{1}{n^{\overline{r-k}}}$$

completes the proof after some computation. ∎

Lemma 3.8 ([11]). *For $s = 0, 1, 2, \ldots,$ the central moments of the Beta operators possess the representation*

$$L_n(\psi_x^s,x)=\sum_{k=[(s+1)/2]}\frac{1}{n^{\overline{k}}}x^{s-j}Z(s,k,j),$$

where the numbers $Z(s,k,j)$ are as defined in (3.8).

Proof. Application of the binomial formula yields for the central moments

$$
\begin{aligned}
L_n(\psi_x^s, x) &= \sum_{r=0}^{s} \binom{s}{r} (-x)^{s-r} L_n(e^r, x) \\
&= \sum_{k=0}^{s} \frac{1}{n^{\overline{k}}} \sum_{j=0}^{k} x^{s-j} \sum_{r=k}^{s} (-1)^{s-r} \binom{s}{r} \begin{bmatrix} r \\ r-j \end{bmatrix} \\
&\quad \times \sum_{i=0}^{k-j} \begin{Bmatrix} r-j \\ r-j-i \end{Bmatrix} \binom{r-j-i}{k-j-i} (1-r)^{\underline{k-j-i}} \\
&= \sum_{k=0}^{s} \frac{1}{n^{\overline{k}}} \sum_{j=0}^{k} x^{s-j} Z(s,k,j).
\end{aligned}
$$

It remains for us to prove that $L_n(\psi_x^s, x) = O\left(n^{-[(s+1)/2]}\right)$. It is sufficient to show that, for $0 \le j \le k$, $Z(s,k,j) = 0$ if $2k < s$. To this end, we recall some known facts about Stirling numbers. The Stirling numbers of the first and second kinds possess, respectively, the representations

$$
\begin{bmatrix} r \\ r-k \end{bmatrix} = (-1)^k \sum_{\mu=0}^{k} C_{k,k-\mu} \binom{r}{k+\mu}, \quad \begin{Bmatrix} r \\ r-k \end{Bmatrix} = \sum_{\nu=0}^{k} \overline{C}_{k,k-\nu} \binom{r}{k+\nu} \tag{3.9}
$$

($k = 0, \ldots, r$), where $C_{k,k} = \overline{C}_{k,k} = 0$, for $k \ge 1$ (see [162]). The coefficients $C_{k,i}$ and $\overline{C}_{k,i}$ are independent of r and satisfy certain partial difference equations [162]. Taking advantage of the representation (3.9), we obtain, for $0 \le i + j \le k$,

$$
\begin{aligned}
&\begin{bmatrix} r \\ r-j \end{bmatrix} \begin{Bmatrix} r-j \\ r-j-i \end{Bmatrix} \binom{r-j-i}{k-j-i} \\
&= (-1)^j \sum_{\mu=0}^{j} C_{j,j-\mu} \binom{r}{j+\mu} \sum_{\nu=0}^{i} \overline{C}_{i,i-\nu} \binom{r-j}{i+\nu} \binom{r-j-i}{k-j-i} \\
&= \sum_{\mu=0}^{j} \sum_{\nu=0}^{i} r^{\underline{k}} P(k,j,i,\mu,\nu;r),
\end{aligned}
$$

where

$$
P(k,j,i,\mu,\nu;r) = \frac{(-1)^j}{(k-j-i)!} \frac{C_{j,j-\mu}}{(j+\mu)!} \frac{\overline{C}_{i,i-\nu}}{(i+\nu)!} (r-j)^{\underline{\mu}} (r-j-i)^{\underline{\nu}}
$$

is a polynomial in the variable r of degree $\leq \mu + \nu$. Thus, we conclude that

$$\begin{aligned} Z(s,k,j) &= \sum_{i=0}^{k-j} \sum_{\mu=0}^{j} \sum_{\nu=0}^{i} \sum_{r=k}^{s} (-1)^{s-r} \binom{s}{r} r^{\underline{k}} P(k,j,i,\mu,\nu;r)(1-r)^{\underline{k-j-i}} \\ &= s^{\underline{k}} \sum_{i=0}^{k-j} \sum_{\mu=0}^{j} \sum_{\nu=0}^{i} \sum_{r=0}^{s-k} (-1)^{s-k-r} \binom{s-k}{r} \\ &\quad \times P(k,j,i,\mu,\nu;r+k)(1-r-k)^{\underline{k-j-i}}. \end{aligned}$$

Since $P(k,j,i,\mu,\nu;r+k)(1-r-k)^{\underline{k-j-i}}$ is a polynomial in the variable r of degree $\leq \mu + \nu + k - j - i \leq k$, the inner sum vanishes if $k < s - k$, that is, if $2k < s$. This completes the proof of Lemma 3.8. ∎

Remark 3.6. An immediate consequence of Lemma 3.8 is that, for $s = 0, 1, 2 \ldots,$

$$L_n(\psi_x^s, x) = O\left(n^{-[(s+1)/2]}\right) \quad (n \to \infty).$$

Theorem 3.5 ([11] Asymptotic expansion). *Let $q \in \mathbb{N}$ and $x \in I$. For each function $f \in K[2q, x]$, the Beta operators possess an asymptotic expansion into a reciprocal factorial series:*

$$L_n(f,x) = f(x) + \sum_{k=1}^{q} \frac{1}{n^{\overline{k}}} \sum_{s=k}^{2k} \frac{f^{(s)}(x)}{s!} \sum_{j=0}^{k} x^{s-j} Z(s,k,j) + o(n^{-q}) \quad (n \to \infty),$$

where the numbers $Z(s,k,j)$ are as defined in (3.8).

Proof. By Remark 3.6, the first conclusion in Lemma 3.5 is valid for the operators $L_n(f,x)$. Therefore, we can apply Lemma 3.5, and the assertion of Theorem 3.5 follows after some calculations by Lemma 3.8. ∎

Remark 3.7. For the reader's convenience, we give the series explicitly, for $q = 3$:

$$\begin{aligned} L_n(f,x) &= f(x) + \frac{x(1-x)f''(x)}{2n} \\ &\quad + \frac{-12x(1-x)f^{(2)}(x)+8x(1-x)(1-2x)f^{(3)}(x)+3x^2(1-x)^2 f^{(4)}(x)}{24n^{\overline{2}}} \\ &\quad + \frac{1}{48n^{\overline{3}}}\Big(-32x(1-x)(1-2x)f^{(3)}(x)+6x(1-x)(2-11x+11x^2)f^{(4)}(x) \\ &\quad +8x^2(1-x)^2(1-2x)f^{(5)}(x)+x^3(1-x)^3 f^{(6)}(x)\Big)+O(n^{-4}) \quad (n \to \infty). \end{aligned}$$

An immediate consequence of Theorem 3.5 is the following Voronovskaja-type formula.

Corollary 3.5 ([11]). *Let* $x \in I$. *For each function* $f \in K[2, x]$, *the operators* $L_n(f, x)$ *satisfy*

$$\lim_{n \to \infty} n\,(L_n(f, x) - f(x)) = \frac{1}{2} x(1 - x) f''(x).$$

Chapter 4
Linear and Iterative Combinations

The linear positive operators are conceptually simpler, and easier to construct and study, but they lack rapidity of convergence for sufficiently smooth functions. In the last century, Bernstein [37] defined the combinations $\hat{B}_n$ of the Bernstein polynomials B_n as

$$\begin{aligned}\hat{B}_n(f,x) &= B_n(f,x) - \frac{x(1-x)}{2n} B_n(f'',x) \\ &= \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \left(f\left(\frac{k}{n}\right) - \frac{x(1-x)}{2n} f''\left(\frac{k}{n}\right)\right).\end{aligned}$$

He subtracted the leading term $\frac{x(1-x)}{2n} B_n(f'',x)$ to achieve the order of approximation $O(n^{-2})$.

Also, Kantorovich's well-known result states that the optimum rate of convergence for any sequence of positive linear operators is at most $O(n^{-2})$. To improve the order of approximation, one can apply the technique of linear combinations and iterative combinations.

4.1 Linear Combinations

Thus, if we want to have a better order of approximation, we have to slacken the positivity condition. For $k \in N$, Butzer [48] considered the linear combinations B_n^{2k} of Bernstein polynomials B_n as

$$(2^k - 1) B_n^{2k} = 2^k B_{2n}^{2k-2} - B_n^{2k-2}, \quad B_n^0 = B_n.$$

Butzer [48] proved that if $f^{(2k)}$ exists at a point $x \in [0,1]$, then

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$$(B_n^{2k-2} - f)(x) = O(n^{-k}), \quad k \in N.$$

The limit $\lim_{n\to\infty} n^k (B_n^{2k-2} - f)(x)$ can be calculated by a recursion formula. In general, the linear combinations Butzer considered for Bernstein polynomials are defined as

$$B_n(f, k, x) = \sum_{j=0}^{k} C(j, k) L_{2^j n}(f, x),$$

where

$$C(j, k) = \prod_{\substack{i=0 \\ i \neq j}}^{k} \frac{2^j}{2^j - 2^i}.$$

Later, May [190] considered these combinations to improve the order of approximation of the Phillips operators and established some direct, inverse, and saturation results.

Independently, Rathore [205], in 1973, and May [189], in 1976, considered the more general combinations for the sequence of linear positive operators of the exponential type. The kth-order linear combinations $L_n(f, k, x)$ of the operators $L_{d_j n}(f, x)$, discussed in [189], are given by

$$L_n(f, k, x) = \sum_{j=0}^{k} C(j, k) L_{d_j n}(f, x),$$

where

$$C(j, k) = \prod_{\substack{i=0 \\ i \neq j}}^{k} \frac{d_j}{d_j - d_i}, \quad k \neq 0; C(0, 0) = 1,$$

and $d_0, d_1, \ldots, d_k$ are arbitrary but fixed distinct positive integers.

In an alternate form, the linear combinations $L_n(f, k, x)$ can be defined as

$$L_n(f, k, x) = \frac{1}{\triangle} \begin{vmatrix} L_{d_0 n}(f, x) & d_0^{-1} & d_0^{-2} & \ldots & d_0^{-k} \\ L_{d_1 n}(f, x) & d_1^{-1} & d_1^{-2} & \ldots & d_1^{-k} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ L_{d_k n}(f, x) & d_k^{-1} & d_k^{-2} & \ldots & d_k^{-k} \end{vmatrix},$$

where $\triangle$ is the Vandermonde determinant obtained by replacing the operator column of the above determinant by the entries 1.

The Lupas–Durrmeyer operators (also called Baskakov operators) [208] are defined as

$$V_n(f,x) = \int_0^\infty W(n,x,t)f(t)dt = (n-1)\sum_{v=0}^{\infty} b_{n,v}(x)\int_0^\infty b_{n,v}(t)f(t)dt,$$

where

$$b_{n,v}(x) = \binom{n+v-1}{v}\frac{x^v}{(1+x)^{n+v}}.$$

Kasana [170] estimated the direct, inverse, and saturation theorems for May's linear combinations [189]. We present the direct results with proofs and the inverse and saturation results (without proofs) on Baskakov–Durrmeyer operators V_n defined on a class $C_\gamma[0,\infty)$. We denote

$$C_\gamma[0,\infty) \equiv \{f \in C[0,\infty) : |f(t)| \le Mt^\gamma, \gamma > 0, M > 0\}.$$

The norm-$||.||_\gamma$ on $C_\gamma[0,\infty)$ is defined as $||f||_\gamma = \sup_{t\in(0,\infty)} |f(t)|t^{-\gamma}$.

Lemma 4.1. *Let $T_{n,m}(x) = V_n\big((t-x)^m, x\big)$. Then $T_{n,0}(x) = 1$, $T_{n,1}(x) = \frac{1+2x}{n-2}$, $n > 2$, and for $n > m+2$, we have*

$$(n-m-2)T_{n,m+1}(x) = x(1+x)[T'_{n,m}(x)+2mT_{n,m-1}(x)]+(m+1)(1+2x)T_{n,m}(x).$$

Corollary 4.1. *Let γ and δ be two positive numbers and $[a,b] \subset [0,\infty)$. Then for any $m > 0$, there exists a constant M_m such that*

$$\left|\left|\int_{|t-x|\ge\delta} W(n,x,t)t^\gamma dt\right|\right|_{C[a,b]} \le M_m n^{-m}.$$

Lemma 4.2. *For $p \in N$ and n sufficiently large,*

$$V_n((t-x)^p, k, x) = n^{-(k+1)}\{Q(p,k,x) + o(1)\}$$

holds, where $Q(p,k,x)$ is a certain polynomial in x of degree p and $t \in [0,\infty)$ is arbitrary but fixed.

Theorem 4.1 ([170] Asymptotic formula). *Let $f \in C_\gamma[0,\infty)$ for some $\gamma > 0$. If $f^{(2k+2)}$ exists at a point $x \in (0,\infty)$, then*

$$\lim_{n\to\infty} n^{k+1}[V_n(f,k,x) - f(x)] = \sum_{j=k+1}^{2k+2} \frac{f^{(j)}(x)}{j!}Q(j,k,x), \tag{4.1}$$

where $Q(j,k,x)$ are certain polynomials in x of degree j. Moreover,

$$Q(2k+2,k,x) = \frac{(-1)^k\{x(1+x)\}^{k+1}}{\prod_{j=0}^k d_j}\frac{(2k+2)!}{(k+1)!}$$

and

$$Q(2k+1,k,x) = \frac{(-1)^k(1+2x)\{x(1+x)\}^k}{\prod_{j=0}^k d_j}\frac{(2k+2)!}{2(k!)}.$$

Furthermore, if $f^{(2k+1)}$ exists and is absolutely continuous over $[a,b]$ and $f^{(2k+2)} \in L_\infty[a,b]$, then for any $[c,d] \subset (a,b)$,

$$||V_n(f,k,.) - f|| \leq Mn^{-k+1}\left\{||f||_\gamma + ||f^{(2k+2)}||_{L_\infty[a,b]}\right\} \tag{4.2}$$

holds.

Proof. By using a Taylor expansion, we have

$$\begin{aligned} V_n(f,k,x) - f(x) &= \sum_{j=0}^{k} C(j,k)V_{d_jn}(f(t)-f(x),x) \\ &\quad + \sum_{i=1}^{2k+2}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{k} C(j,k)V_{d_jn}((t-x)^{2k+2}\varepsilon(t,x),x) \\ &:= I_1 + I_2, \end{aligned}$$

where $\varepsilon(t,x) \to 0$ as $t \to x$. Using Lemma 4.2, we see that

$$I_1 = \sum_{i=k+1}^{2k+2}\frac{f^{(i)}(x)}{i!}Q(i,k,x) + o(1).$$

Since $\varepsilon(t,x) \to 0$ as $t \to x$, it follows that for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|\varepsilon(t,x)| < \varepsilon$ whenever $|t-x| < \delta$. For $|t-x| \geq \delta$, we observe that $|\varepsilon(t,x)| \leq K|t-x|^\gamma$.

$$\begin{aligned} I_2 &= \sum_{j=0}^{k} C(j,k)V_{d_jn}(\varepsilon(t,x)(t-x)^{2k+2},x) \\ &= \sum_{j=0}^{k} C(j,k)(d_j-1)\sum_{v=0}^{\infty} b_{d_jn,v}(x)\int_0^\infty b_{d_jn,v}(t)\varepsilon(t,x)(t-x)^{2k+2}dt \end{aligned}$$

$$= \sum_{j=0}^{k} C(j,k)(d_j - 1) \sum_{v=0}^{\infty} b_{d_j n,v}(x) \left(\int_{|t-x|<\delta} + \int_{|t-x|\geq\delta} \right)$$
$$\times b_{d_j n,v}(t).\varepsilon(t,x)(t-x)^{2k+2} dt.$$

Using Lemma 4.1, we have

$$\begin{aligned} I_3 &= \sum_{j=0}^{k} C(j,k)(d_j - 1) \sum_{v=0}^{\infty} b_{d_j n,v}(x) \\ &\quad \times \int_{|t-x|<\delta} b_{d_j n,v}(t).\varepsilon(t,x)(t-x)^{2k+2} dt \\ &= \varepsilon.O(n^{-(k+1)}). \end{aligned}$$

Next, applying Schwarz's inequality and Lemma 4.1, we have

$$\begin{aligned} I_4 &= \sum_{j=0}^{k} C(j,k)(d_j - 1) \sum_{v=0}^{\infty} b_{d_j n,v}(x) \int_{|t-x|\geq\delta} b_{d_j n,v}(t).\varepsilon(t,x)(t-x)^{2k+2} dt \\ &\leq K \sum_{j=0}^{k} C(j,k)(d_j - 1) \sum_{v=0}^{\infty} b_{d_j n,v}(x) \int_{|t-x|\geq\delta} b_{d_j n,v}(t)|t-x|^{2k+2+\gamma} dt \\ &\leq K \sum_{j=0}^{k} C(j,k) \left(\int_0^{\infty} W(d_j n, x, t) \frac{(t-x)^{2s}}{\delta^{2s-4k-2\gamma-4}} dt \right)^{1/2} \\ &= O(^{-s/2}) = o(n^{-(k+1)}). \end{aligned}$$

By combining I_3, I_4, we get $n^{k+1} I_2 \to 0$ as $n \to \infty$. This completes the proof of (4.1). The values of $Q(2k+2,k,x)$ and $Q(2k+1,k,x)$ can be obtained from Lemma 4.2. Next, we have

$$\begin{aligned} V_n(f,k,x) - f(x) &= V_n(\psi(t)(f(t)f(x)),k,x) \\ &= V_n((1-\psi(t))(f(t)-f(x)),k,x) \\ &:= I_5 + I_6. \end{aligned}$$

Proceeding along the lines of the estimate of I_4, we have, for all $t \in [c,d]$,

$$I_6 \leq K_1 n^{k+1} ||f||_{\gamma}.$$

Next, by the given hypothesis on f for $t \in [a,b]$ and $x \in [c,d]$, we have

$$f(t) - f(x) = \sum_{i=1}^{2k+1} \frac{f^{(i)}(x)}{i!}(t-x)^i + \frac{1}{(2k+1)!}\int_x^t (t-w)^{2k+1} f^{(2k+2)}(w)dw.$$

Thus,

$$\begin{aligned} I_5 &= \sum_{i=1}^{2k+1} \frac{f^{(i)}(x)}{i!}\left[V_n((t-x)^i,k,x) + V_n((\psi(t)-1)(t-x)^i,k,x)\right] \\ &= +\frac{1}{(2k+1)!}V_n\left(\psi(t)\int_x^t (t-w)^{2k+1} f^{(2k+2)}(w)dw,k,x\right) \\ &:= \sum_{i=1}^{2k+1} \frac{f^{(i)}(x)}{i!}[I_7+I_8]+I_9. \end{aligned}$$

By Lemma 4.2, we get $I_7 = O(n^{-(k+1)})$, uniformly for $x \in [c,d]$. Next, proceeding along the lines of I_4, for $x \in [c,d]$, we have $I_8 = O(n^{-(k+1)})$. Also, by Lemma 4.1, we get

$$I_9 = ||f^{(2k+2)}||_{L_\infty[a,b]}O(n^{(-k+1)}).$$

Combining the estimates of I_7, I_8, and I_9, we obtain

$$||I_5||_{C[c,d]} \le K_1 n^{-(k+1)}\left\{\sum_{i=1}^{2k+1}||f^{(i)}||_{C[c,d]} + ||f^{(2k+2)}||_{L_\infty[a,b]}\right\}.$$

Next, using the Goldberg and Meir property [84], we obtain

$$||I_5||_{C[c,d]} \le K_2 n^{-(k+1)}\left\{||f||_{C[c,d]} + ||f^{(2k+2)}||_{L_\infty[a,b]}\right\}.$$

Finally, combining the estimates of I_5 and I_6, the assertion (4.2) follows. ∎

Theorem 4.2 ([170] Error estimation). *Let $f \in C_\gamma[0,\infty)$ and $0 < a_1 < a_2 < b_2 < b_1 < \infty$. Then for n sufficiently large, there exists a constant M_k such that*

$$||V_n(f,k,.) - f||_{C[a_2,b_2]} \le M_k\left\{\omega_{2k+2}(f,n^{-1/2},a_1,b_1) + n^{(k+1)}||f||_\gamma\right\}.$$

Proof. By the linearity property, we have

$$\begin{aligned} V_n(f,k,x) - f(x) &= V_n(f - f_{\eta,2k+2},k,x) \\ &\quad + f_{\eta,2k+2}(x) - f(x) \\ &\quad + V_n(f_{\eta,2k+2},k,x) - f_{\eta,2k+2}(x) \\ &:= J_1 + J_2 + J_3. \end{aligned}$$

Clearly,

$$|J_1| \leq \sum_{j=0} K|C(j,k)| \int_0^\infty W(d_j n, x, t) \left| f_{\eta,2k+2}(t) - f(t) \right| dt.$$

Now, choose a^*, b^* such that $a_1 < a^* < a_2 < b_2 < b^* < b_1$. For all $x \in [a_2, b_2]$, there exists $\delta > 0$ such that

$$\int_0^\infty W(d_j n, x, t) \left| f_{\eta,2k+2}(t) - f(t) \right| dt$$
$$\leq ||f_{\eta,2k+2} - f||_{C[a^*,b^*]} + M_m n^{-m} ||f||_\gamma.$$

Hence, in view of the property of the Steklov mean $f_{\eta,2k+2}$, we have

$$||J_1||_{C[a_2,b_2]} \leq M_1 \omega_{2k+2}(f, n^{-1/2}, a_1, b_1) + M_m n^{-m} ||f||_\gamma.$$

Similarly,

$$||J_2||_{C[a_2,b_2]} \leq M_2 \omega_{2k+2}(f, n^{-1/2}, a_1, b_1).$$

Next,

$$f_{\eta,2k+2} = \sum_{i=0}^{2k+1} \frac{f_{\eta,2k+2}^{(i)}(x)}{i!}(t-x)^i + \frac{f_{\eta,2k+2}^{(2k+2)}(\xi)}{(2k+2)!},$$

where ξ lies between t and x. We separate the integral into two parts as in the estimate of J_1. Then, using Lemma 4.1, we obtain

$$||V_n(f_{\eta,2k+2}, k, .) - f_{\eta,2k+2}||_{C[a_2,b_2]}$$
$$\leq M_3 n^{-(k+1)} \sum_{i=k+1}^{2k+2} ||f_{\eta,2k+2}||_{C[a_2,b_2]} + M_m n^{-m} ||f_{\eta,2k+2}||_\gamma.$$

Again, using the Goldberg and Meir property [84], we obtain

$$||f_{\eta,2k+2}||_{C[a_2,b_2]}$$
$$\leq M_4 \left\{ ||f_{\eta,2k+2}||_{C[a_2,b_2]} + ||f_{\eta,2k+2}||_{C[a_2,b_2]}^{(2k+2)} \right\}, i = 1, 2, \ldots, 2k+2.$$

Choosing $m \geq k+1$, we have

$$\begin{aligned}&||V_n(f_{\eta,2k+2},k,.) - f_{\eta,2k+2}||_{C[a_2,b_2]}\\&\leq M_5 n^{-(k+1)}\left\{||f^{(2k+2)}_{\eta,2k+2}||_{C[a_2,b_2]} + ||f_{\eta,2k+2}||_{C[a_2,b_2]}\right\}.\end{aligned}$$

Again, by applying the Steklov mean property, choosing $\eta = n^{-1/2}$, and combining the estimates of J_1, J_2 and J_3, we get the required result. ■

Theorem 4.3 ([170] Inverse result). *Let $f \in C_\gamma[0,\infty)$ and $0 < \alpha < 2$. Then, in the following statements, $(i) \Rightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv)$ hold:*

(i) $||V_n(f,k,.) - f||_{C[a_1,b_1]} = O(n^{-(\alpha(k+1)/2)})$.
(ii) $f \in Liz(\alpha, k+1, a_2, b_2)$.
(iii) (a) For $m < \alpha(k+1) < m+1, m = 0,1,2,\ldots 2k+1$, $f^{(m)}$ exists and belongs to the class Lip$(\alpha(k+1)-m, a_2, b_2)$.
(b) For $\alpha(k+1) = m+1, m = 0,1,2,\ldots 2k$, $f^{(m)}$ exists and belongs to the class Lip$^*(1, a_2, b_2)$.
(iv) $||V_n(f,k,.) - f||_{C[a_3,b_3]} = O(n^{-(\alpha(k+1)/2)})$,
where $Liz(\alpha,k,a,b)$ denotes the generalized Zygmund class of functions for which $\omega_{2k}(f,h,a,b) \leq Mh^{\alpha k}$ when $k = 1$ $Liz(\alpha,1)$ reduces to the Zygmund class $Lip^*\alpha$.

Theorem 4.4 ([170] Saturation result). *Let $f \in C_\gamma[0,\infty)$ and $0 < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < \infty$. Then in the following statements, the implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ and $(iv) \Rightarrow (v) \Rightarrow (vi)$ hold:*

(i) $||V_n(f,k,.) - f)||_{C[a_1,b_1]} = O(n^{-(k+1)})$.
(ii) $f^{(2k+1)} \in A.C.[a_2,b_2]$ *and* $f^{(2k+2)} \in L_\infty[a_2,b_2]$.
(iii) $||V_n(f,k,.) - f||_{C[a_3,b_3]} = O(n^{-(k+1))})$.
(iv) $||V_n(f,k,.) - f||_{C[a_1,b_1]} = o(n^{-(k+1)})$.
(v) $f \in C^{2k+2}[a_2,b_2]$ *and* $\sum_{j=k+1}^{2k+2} Q(j,k,x)f^{(j)}(x) = 0, x \in [a_2,b_2]$,
where the polynomials $Q(j,k,x)$ are defined in Theorem 4.1.
(vi) $||V_n(f,k,.) - f||_{C[a_3,b_3]} = o(n^{-(k+1)})$.

In a simultaneous approximation for Baskakov (Lupas)–Durrmeyer operators, Agrawal et al. [27] obtained the following asymptotic formula and an estimation of error in simultaneous approximation for the linear combinations.

Theorem 4.5 ([27] Asymptotic formula). *Let f be integrable on $[0,\infty)$, admitting a $(2k+r+2)$th derivative at a point $x \in [0,\infty)$ with $f^{(r)}(x) = O(x^\alpha)$, where α is a positive integer not less than $2k+2$, as $x \to \infty$. Then*

$$\lim_{n\to\infty} n^{k+1}[V_n^{(r)}(f,k,x) - f^{(r)}(x)] = \sum_{i=1}^{2k+2} Q(i,k,r,x)f^{(i+r)}(x)$$

and

$$\lim_{n\to\infty} n^{k+1}[V_n^{(r)}(f,k+1,x) - f^{(r)}(x)] = 0,$$

where $Q(i,k,r,x)$ are certain polynomials in x of degree at most i. Furthermore, if $f^{(2k+r+2)}$ exists and is continuous on $< a,b >$, then the above limits hold uniformly on $[a,b]$. Here $< a,b >\subset [0,\infty)$ denotes an open interval containing the closed interval $[a,b]$.

Theorem 4.6 ([27] Error estimation). *Let $1 \leq p \leq 2k+2$ and f be integrable on $[0,\infty)$. If $f^{(p+r)}$ exists and is continuous on $< a,b >$, having the modulus of continuity $\omega(f^{(p+r)},\delta)$ on $< a,b >$, and $f^{(r)}(x) = O(x^{\alpha})$, where α is a positive integer $\geq p$, then for n sufficiently large, we have*

$$||V_n^{(r)}(f,k,.) - f^{(r)}|| \leq \max\{C_1 n^{-p/2}\omega(f^{(p+r)}, n^{-1/2}), C_2 n^{-(k+1)}\},$$

where $C_1 = C_1(k,p,r)$ and $C_2 = C_2(k,p,r,f)$.

In this continuation, Sinha et al. [214] obtained some local direct estimates in the L_p-norm for the linear combinations of Baskakov–Durrmeyer operators $V_n(f,x)$ using the linear approximating technique of Steklov means. In the following three theorems, $0 < a_1 < a_2 < b_2 < b_1 < \infty$ and $I_i = [a_i,b_i], i = 1,2$.

Theorem 4.7 ([214] Error estimation). *Let $f \in L_p[0,\infty), p > 1$. If f has $2k+2$ derivatives on I_1 with $f^{(2k+1)} \in A.C.(I_1)$ and $f^{(2k+2)} \in L_p(I_1)$, then for n sufficiently large,*

$$||V_n(f,k,.) - f||_{L_p(I_2)} \leq M_k n^{-(k+1)}\{||f^{(2k+2)}||_{L_p(I_1)} + ||f||_{L_p[0,\infty)}\},$$

where M_k is a constant independent of f and n.

Theorem 4.8 ([214] Error estimation). *Let $f \in L_1[0,\infty)$. If f has $2k+1$ derivatives on I_1 with $f^{(2k)} \in A.C.(I_1)$ and $f^{(2k+1)} \in B.V.(I_1)$, then for all n sufficiently large,*

$$\begin{aligned}||V_n(f,k,.) - f||_{L_1(I_2)} \leq M_k n^{-(k+1)}\Big\{&||f^{(2k+1)}||_{B.V.(I_1)} \\ &+||f^{(2k+1)}||_{L_1(I_2)} + ||f||_{L_1[0,\infty)}\Big\},\end{aligned}$$

where M_k is a constant independent of f and n.

Theorem 4.9 ([214] Error estimation). *Let $f \in L_p[0,\infty), p \geq 1$. Then for n sufficiently large,*

$$||V_n(f,k,.) - f||_{L_p(I_2)} \leq M_k \left(\omega_{2k+2}(f, n^{-1/2}, p, I_1), n^{-(k+1)}||f||_{L_p[0,\infty)}\right),$$

where M_k is a constant independent of f and n.

4.2 Iterative Combinations

Another approach to improve the order of approximation is to take the iterates of operators. Such an approach was considered first by Micchelli [193], who considered the iterative combinations of the well-known Bernstein polynomials $B_n(f, x)$ and was able to achieve a better order of approximation. The kth iterative combinations of the Bernstein polynomial are defined as

$$T_{n,k}(f, x) = (I - (I - B_n)^k)(f(t), x) = \sum_{p=1}^{k}(-1)^{p+1}\binom{k}{p} B_n^p(f, x),$$

where $B_n^p(f, x)$, $p \in N$ denotes the pth iterate, and $B_n^0(f, x) = I$. For $f \in C[0, 1]$, Micchelli obtained the following result:

$$||T_{n,k}(f) - f|| \leq \frac{3k}{2}(2^k - 1)\omega\left(f, n^{-1/2}\right), \quad n = 1, 2, \ldots.$$

Here $\omega(f, .)$ denotes the modulus of continuity of f and $||.||$ is the sup-norm on the interval $[0, 1]$.

Agrawal [22] sharpened the above result and also obtained an asymptotic formula for the iterative combinations of Bernstein polynomials.

Theorem 4.10 ([22]). *Let* $f \in C[0, 1]$. *Then for all n, we have*

$$||T_{n,k}(f) - f|| \leq \left[(2^k - 1) + \frac{n}{4}\left\{2^k - \left(2 - \frac{1}{n}\right)^k\right\}\right]\omega(f, n^{-1/2}).$$

Theorem 4.11 ([22]). *Let* $f \in C[0, 1]$. *If* $f^{(2k)}$ *exists at a point* $x \in [0, 1]$, *then*

$$\lim_{n\to\infty} n^k[T_{n,k}(f, x) - f(x)] = \sum_{j=2}^{2k}\frac{f^{(j)}(x)}{j!}Q(j, k, x)$$

and

$$\lim_{n\to\infty} n^k[T_{n,k+1}(f, x) - f(x)] = 0.$$

Furthermore, if $f^{(2k)}$ *exists and is continuous on* $[0, 1]$, *then the above limits hold uniformly in* $[0, 1]$.

Later, Agrawal [19] also investigated some problems concerning the degree of approximation in simultaneous approximation by Micchelli combinations of Bernstein polynomials. He obtained the following pointwise convergence, asymptotic formula, and error estimation:

Theorem 4.12 ([19] Pointwise convergence). *Let $f \in C[0,1]$ and $k, p \in N$. If $f^{(p)}$ exists at some point $x \in (0,1)$, then*

$$\lim_{n\to\infty} T_{n,k}^{(p)}(f,x) = f(x).$$

Furthermore, if $f^{(p)}$ exists and is continuous on $< a, b >$, where $< a, b >$ denotes an open interval containing the closed interval $[a, b]$, then the above limits hold uniformly in $[a, b]$.

Theorem 4.13 ([19] Asymptotic formula). *Let $f \in C[0,1]$ and $k, p \in N$. If $f^{(2k+p)}$ exists at some point $x \in (0,1)$, then*

$$\lim_{n\to\infty} n^k [T_{n,k}^{(p)}(f,x) - f^{(p)}(x)] = \sum_{j=p}^{2k+p} \frac{f^{(j)}(x)}{j!} Q(j,k,p,x),$$

where $Q(j,k,p,x)$ are certain polynomials in x. Furthermore, if $f^{(2k+p)}$ exists and is continuous on $< a, b >$, then the above limits hold uniformly in $[a, b]$.

Theorem 4.14 ([19] Error estimation). *Let $p \le q \le 2k+p$, $f \in C[0,1]$. If $f^{(q)}$ exists and is continuous on $< a, b >$, then*

$$||T_{n,k}^{(p)}(f,.) - f^{(p)}||_{C[a,b]} \le \max\left\{C_1 n^{-(q-p)/2}\omega(f^{(q)}, n^{-1/2}), C_2 n^{-k}\right\},$$

where $C_1 = C_1(k,p)$ and $C_2 = C_2(k,p,f)$.

For $f \in L_p[0,1], 1 \leqslant p < \infty$, the Bernstein–Durrmeyer operators D_n defined in (2.13) can be expressed as

$$D_n(f,x) = \int_0^1 W_n(t,x) f(t)\, dt,$$

where $W_n(t,x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x) p_{n,k}(t)$ is the kernel of the operators.

The iterative combination $\hat{T}_{n,k} : L_p[0,1] \to C^\infty[0,1]$ of the operators is defined as

$$\hat{T}_{n,k}(f;t) = \left(I - (I - D_n)^k\right)(f;t) = \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} D_n^r(f;t),\ k \in N,$$

where $D_n^0 \equiv I$ and $D_n^r \equiv D_n(D_n^{r-1})$ for $r \in N$. In the following two results, we denote $I_j = [a_j, b_j]\ j = 1, 2, 3, 0 < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < 1$.

Theorem 4.15 ([217] Direct theorem). *If $p \geqslant 1$, $f \in L_p[0,1]$, then for all n sufficiently large,*

$$\|\hat{T}_{n,k}(f;.) - f\|_{L_p(I_2)} \leqslant C_k \left(\omega_{2k}\Big(f, \frac{1}{\sqrt{n}}, p, I_1.\Big) + n^{-k}\|f\|_{L_p[0,1]}\right),$$

holds, where C_k is a constant independent of f and n.

Theorem 4.16 ([216] Local inverse theorem). *Let $f \in L_p[0,1], 1 \leqslant p < \infty$, $0 < \alpha < 2k$, and $\|\hat{T}_{n,k}(f,.) - f\|_{L_p(I_1)} = O(n^{-\alpha/2})$ as $n \to \infty$. Then $\omega_{2k}(f, \tau, p, I_2) = O(\tau^{\alpha})$ as $\tau \to 0$.*

Gupta and Noor [119] proposed the mixed sequence of summation–integral-type operators and introduced the integral modification of the Szász operators having weights of Beta basis functions as

$$\begin{aligned} S_n(f,x) &= \sum_{v=1}^{\infty} s_{n,v}(x) \int_0^{\infty} b_{n,v}(t) f(t)dt + e^{-nx} f(0) \\ &= \int_0^{\infty} K_n(x,t) f(t)dt, \quad x \in [0,\infty), \end{aligned} \tag{4.3}$$

where

$$s_{n,v}(x) = e^{-nx}\frac{(nx)^k}{k!}, b_{n,v}(t) = \frac{1}{B(n+1,v)}\frac{t^{v-1}}{(1+t)^{n+v+1}}$$

and

$$K_n(x,t) = \sum_{v=1}^{\infty} s_{n,v}(x) b_{n,v}(t) + e^{-nx}\delta(t),$$

$\delta(t)$ being the Dirac delta function. For these operators, they [119] obtained direct results in simultaneous approximation. These operators are a modification of the well-known Szász operators [227] defined by

$$\hat{S}_n(f,x) = \sum_{v=0}^{\infty} s_{n,v}(x) f\left(\frac{v}{n}\right).$$

Finta et al. [73] considered iterative combinations $S_{n,k}$ of the operators S_n as

$$S_{n,k}(f,x) = (I - (I - S_n)^k)(f(t),x) = \sum_{p=1}^{k} (-1)^{p+1} \binom{k}{p} S_n^p(f,x),$$

where $S_n^p(f,x)$, $p \in N$ denotes the pth iterate, and $S_n^0(f,x) = f$, $f \in C_\gamma[0,\infty) \equiv \{f \in C[0,\infty) : |f(t)| \leq Mt^\gamma, \gamma > 0, M > 0\}$. The norm-$||.||_\gamma$ on $C_\gamma[0,\infty)$ is defined as $||f||_\gamma = \sup_{t\in(0,\infty)} |f(t)|t^{-\gamma}$. Finta et al. [73] estimated the asymptotic formula, error estimation, and a global direct result for the iterative combinations of the operators (4.3). To prove the results, we need the following lemmas:

Lemma 4.3 ([119]). *For the function $U_{n,m}(x), m \in \mathbb{N}^0$ defined as*

$$U_{n,m}(x) = \sum_{v=0}^{\infty} s_{n,v}(x) \left(\frac{v}{n} - x\right)^m,$$

we have $U_{n,0}(x) = 1$ and $U_{n,1}(x) = 0$. Furthermore, the recurrence relation

$$nU_{n,m+1}(x) = x\left[U'_{n,m}(x) + mU_{n,m-1}(x)\right], \quad m = 1, 2, 3, \ldots$$

holds. Consequently, for each fixed $x \in [0, \infty), U_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$.

Lemma 4.4 ([119]). *For $m \in \mathbb{N}^0$ (the set of nonnegative integers), the mth-order moment for the operators S_n is defined as*

$$\begin{aligned} T_{n,m}(x) &= S_n((t-x)^m, x) \\ &= \sum_{v=1}^{\infty} s_{n,v}(x) \int_0^{\infty} b_{n,v}(t)(t-x)^m \, dt + e^{-nx}(-x)^m. \end{aligned}$$

Then $\mu_{n,0}(x) = 1, T_{n,1}(x) = 0, T_{n,2}(x) = x(2+x)/(n-1)$, and for $n \geq m$, the recurrence relation

$$(n-m)T_{n,m+1}(x) = x\left[T'_{n,m}(x) + m(2+x)T_{n,m-1}(x)\right]$$

holds.

Corollary 4.2. *Let δ be a positive number. Then for every $\gamma > 0, x \in (0, \infty)$, there exists a constant $M(s, x)$ independent of n and depending on s and x such that*

$$\sum_{v=1}^{\infty} s_{n,v}(x) \int_{|t-x|>\delta} b_{n,v}(t) t^{\gamma} dt \leq M(s, x) n^{-s}, \quad s = 1, 2, 3, \ldots,$$

locally on every bounded interval.

Lemma 4.5 ([73]). *The recurrence relation*

$$T^{[p+1]}_{n,m}(x) = \sum_{j=0}^{m} \binom{m}{j} \sum_{i=0}^{m-j} \frac{1}{i!} D^i \left(T^{[p]}_{n,m-j}(x)\right) T_{n,i+j}(x)$$

holds, where D denotes $\frac{d}{dx}$.

Lemma 4.6 ([73]). *For $p \in \mathbb{N}, m \in \mathbb{N}^0$ and $x \in [0, \infty)$, we have*

$$T^{[p]}_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right).$$

Lemma 4.7 ([73]). *For the lth moment ($l \in \mathbb{N}$) of $S_{n,k}$, we find that*

$$S_{n,k}((t-x)^l : x) = O(n^{-k}).$$

Lemma 4.8 ([73]). *For the function $s_{n,v}(x)$,*

$$x^r \frac{d^r}{dx^r}(s_{n,v}(x)) = \sum_{\substack{2i+j \le r,\\ i,j \ge 0}} n^i (v-nx)^j Q_{i,j,r}(x)\, s_{n,v}(x)$$

holds, where $Q_{i,j,r}(x)$ are certain polynomials in x independent of n and v.

Lemma 4.9 ([73]). *If f is r-times differentiable on $[0,\infty)$ such that $f^{(r-1)}(t) = O(t^{\alpha}), \alpha > 0$ as $t \to \infty$, then for $r = 1, 2, \ldots$ and $n > \alpha + r$, we have*

$$S^{(r)}_{n,k}(f,x) = \frac{n^{r-1}(n-r)!}{(n-1)!} \sum_{v=0}^{\infty} s_{n,v}(x) \int_0^{\infty} b_{n-r,v+r}(t) f^{(r)}(t)dt.$$

In the simultaneous approximation, the following are the main results for iterative combinations:

Theorem 4.17 ([73]). *Let $f \in C_{\gamma}[0,\infty)$ such that $f^{(2k+r)}$ exists at a point $x \in (0,\infty)$. Then*

$$\lim_{n\to\infty} n^k \left[S^{(r)}_{n,k}(f,x) - f^{(r)}(x)\right] = \sum_{j=r}^{2k+r} P(j,k,r,x) f^{(j)}(x),$$

where $P(j,k,r,x)$ are certain polynomials in x.

Proof. By a Taylor expansion of f, we have

$$f(t) = \sum_{i=0}^{2k+r} \frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^{2k+r},$$

where $\varepsilon \to 0$ as $t \to x$. Hence,

$$\begin{aligned} S^{(r)}_{n,k}(f,x) &= \sum_{p=1}^{k} (-1)^{p+1} \binom{k}{p} \frac{d^r}{dx^r} S^p_n(f,x) \\ &= \sum_{p=1}^{k} (-1)^{p+1} \binom{k}{p} \int_0^{\infty} K^{(r)}_n(x,y) S^{p-1}_n(f,y)dy \end{aligned}$$

$$= \sum_{p=1}^{k}(-1)^{p+1}\binom{k}{p}\int_0^\infty K_n^{(r)}(x,y)$$

$$\times \left\{ \sum_{j=r}^{2k+r} \frac{f^{(j)}(x)}{j!} S_n^{p-1}((t-x)^j, y) + S_n^{p-1}(\varepsilon(t,x)(t-x)^{2k+r}, y) \right\} dy$$

$$= \sum_{p=1}^{k}(-1)^{p+1}\binom{k}{p}\int_0^\infty K_n^{(r)}(x,y)[E_1 + E_2]dy.$$

Using the Binomial expansion of $(t-x)^j$ and Lemma 4.4, we have

$$E_1 = \sum_{j=r}^{2k+r} \frac{f^{(j)}(x)}{j!} \sum_{i=0}^{j} \binom{j}{i} (-x)^{j-i} S_{n,k}^{(r)}(t^i, x)$$

$$= f^{(r)}(x) + n^{-k} \sum_{j=r}^{2k+r} P(j,k,r,x) f^{(j)}(x) + o(n^{-k}),$$

where we have used the identity

$$\sum_{i=0}^{j}(-1)^i \binom{j}{i}\binom{i}{r} = \begin{cases} 0, & j > r \\ (-1)^r, & j = r. \end{cases}$$

Next we estimate E_2 as follows: If

$$I = \int_0^\infty K_n^{(r)}(x,y) S_n^{p-1}(\varepsilon(t,x)(t-x)^{2k+r}, y) dy,$$

then by applying Lemma 4.8, we have

$$|I| = \sum_{\substack{2i+j \le r \\ i,j \ge 0}} n^i \frac{|Q_{i,j,r}(x)|}{x^r} \sum_{v=1}^{\infty} |v - nx|^j s_{n,v}(x)$$

$$\times \int_0^\infty b_{n,v}(y) S_n^{p-1}(|\varepsilon(t,x)||t-x|^{2k+r}, y)dy + n^r e^{-nx}|\varepsilon(0,x)|x^{2k+r}.$$

The second term on the right-hand side of above expression tends to zero as $n \to \infty$. Since $\varepsilon(t,x) \to 0$ as $t \to x$, for a given $\varepsilon > 0$ such that $|\varepsilon(t,x)| < \varepsilon$ whenever $0 < |\varepsilon(t,x)| < \delta$, for $|t-x| \ge \delta$, we have $|\varepsilon(t,x)(t-x)^{2k+r}| \le Mt^\gamma$ for some $M > 0$. Hence,

$$|I| = \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \frac{|Q_{i,j,r}(x)|}{x^r} \sum_{v=1}^{\infty} |v-nx|^j s_{n,v}(x)$$

$$\times \left\{ \varepsilon \int_{|t-x|<\delta} b_{n,v}(y) S_n^{p-1}(|t-x|^{2k+r}, y)dy + \int_{|t-x|\ge\delta} b_{n,v}(y) S_n^{p-1}(Mt^{\gamma}, y)dy \right\}.$$

Applying Schwarz's inequality, and Lemmas 4.3 and 4.6, we see that $I_1 = \varepsilon O(1)$. Proceeding in a similar way by applying Schwarz's inequality and Corollary 4.2, we obtain $I_2 = o(1)$. Since $\varepsilon > 0$ is arbitrary, we have $I = o(n^{-k})$. This completes the proof of the theorem. ■

Theorem 4.18 ([73]). *Let $f \in C_\gamma[0,\infty), \gamma > 0$ and $0 < a_1 < a_2 < b_2 < \infty$. Then for all n sufficiently large, we have*

$$\left\| S_{n,k}^{(r)}(f,) - f^{(r)} \right\|_{C[a_2,b_2]} \le M \left\{ n^{-k} \|f\|_\gamma + \omega_{2k}\left(f^{(r)}, n^{-1/2}, a_1, b_1\right) \right\},$$

where M is independent of f and n.

Proof. Let $f_{\eta,2k}(t)$ be the Steklov mean of the $(2k)$th order corresponding to $f(t)$ over $[a_, b_1]$. Then we have

$$\begin{aligned} &\left\| S_{n,k}^{(r)}(f,.) - f^{(r)} \right\|_{C[a_2,b_2]} \\ &\leqslant \left\| S_{n,k}^{(r)}\left(f - f_{\eta,2k},.\right) \right\|_{C[a_2,b_2]} + \left\| S_{n,k}^{(r)}\left(f_{\eta,2k},.\right) - f_{\eta,2k}^{(r)} \right\|_{C[a_2,b_2]} \\ &\quad + \left\| f^{(r)} - f_{\eta,2k}^{(r)} \right\|_{C[a_2,b_2]} \\ &:= J_1 + J_2 + J_3, \ . \end{aligned}$$

Since $f_{\eta,2k}^{(r)} = \left(f^{(r)}\right)_{\eta,2k}$, by property (iii) of the Steklov mean (see Definition 1.9 with growth of order $O(t^\gamma)$, we have

$$J_3 \le M_1 \omega_{2k}(f^{(r)}, \eta, a_1, b_1).$$

Next, applying Theorem 4.17 and the interpolation property due to Goldberg and Meier [84], we get

$$\begin{aligned} J_2 &\le M_2 n^{-k} \sum_{j=r}^{2k+r} \left\| f_{\eta,2k}^{(j)} \right\|_{C[a_2,b_2]} \\ &\le M_3 n^{-k} \left(\left\| f_{\eta,2k} \right\|_{C[a_2,b_2]} + \left\| \left(f_{\eta,2k}^{(r)}\right)^{(2k)} \right\|_{C[a_2,b_2]} \right). \end{aligned}$$

Hence, by properties (ii) and (iv) of the Steklov mean, we have

$$J_2 \leq M_4 n^{-k} \left(\|f\|_\gamma + \eta^{-2k}\omega_{2k}\left(f^{(r)}, \eta, a_1, b_1\right)\right).$$

Let a^* and b^* be such that $0 < a_1 < a^* < a_2 < b_2 < b^* < b_1 < \infty$. Also, let $\psi(t)$ denote the characteristic function of the interval $[a^*, b^*]$. Then

$$\begin{aligned}
J_1 &= ||S_{n,k}^{(r)}(\psi(t)(f(t) - f_{\eta,2k}(t), .)||_{C[a_2,b_2]} \\
&\quad + ||S_{n,k}^{(r)}((1-\psi(t))(f(t) - f_{\eta,2k}(t), .)||_{C[a_2,b_2]} \\
&:= J_4 + J_5.
\end{aligned}$$

We may note here that to estimate J_4 and J_5, it is enough to consider their expressions without the iterative combinations. By Lemma 4.9, it is clear that

$$\begin{aligned}
&S_{n,k}^{(r)}(\psi(t)(f(t) - f_{\eta,2k}(t), x) \\
&= \frac{n^{r-1}(n-r)!}{(n-1)!}\sum_{v=0}^{\infty} s_{n,v}(x)\int_0^\infty b_{n-r,v+r}(t)\psi(t)(f^{(r)}(t) - f_{\eta,2k}^{(r)}(t))dt.
\end{aligned}$$

Hence,

$$||S_{n,k}^{(r)}(\psi(f - f_{\eta,2k}, .)||_{C[a_2,b_2]} \leq M_5||f^{(r)} - f_{\eta,2k}^{(r)}||_{C[a^*,b^*]}.$$

Next, for$x \in [a_2, b_2]$ and $t \in [0,\infty) \setminus [a^*, b^*]$, we can choose a $\delta_1 > 0$ satisfying $|t - x| \geq \delta_1$. Therefore, by Lemma 4.8, we have for $I \equiv S_n^{(r)}((1-\psi(t))(f(t) - f_{\eta,2k}(t), x)$,

$$\begin{aligned}
|I| \leq &\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i (v - nx)^j \frac{|Q_{i,j,r}(x)|}{x^r}\sum_{v=1}^{\infty} s_{n,v}(x)|v-nx|^j \\
&\times \int_0^\infty b_{n,v}(t)(1-\psi(t))|f(t) - f_{\eta,2k}(t)|dt \\
&+ n^r e^{-nx}(1-\psi(0))|f(0) - f_{\eta,2k}(0)|.
\end{aligned}$$

For sufficiently large n, the second term is zero. Thus, by Schwarz's inequality,

$$\begin{aligned}
|I| &\leq M_6||f||_\gamma \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \sum_{v=1}^{\infty} s_{n,v}(x)|v-nx|^j \int_{|t-x|\geq\delta_1} b_{n,v}(t)dt \\
&\leq M_6||f||_\gamma \delta_1^{-2s} \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \sum_{v=1}^{\infty} s_{n,v}(x)|v-nx|^j
\end{aligned}$$

$$\times\left(\int_0^\infty b_{n,v}(t)dt\right)^{1/2}\left(\int_0^\infty b_{n,v}(t)(t-x)^{4s}dt\right)^{1/2}$$

$$\leq M_6||f||_\gamma \delta_1^{-2s}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i\left(\sum_{v=1}^\infty (v-nx)^{2j}\right)^{1/2}$$

$$\times\left(\sum_{v=1}^\infty s_{n,v}(x)\int_0^\infty b_{n,v}(t)(t-x)^{4s}dt\right)^{1/2}.$$

Using Lemmas 4.3 and 4.4, we get

$$|I|\leq M_7||f||_\gamma \delta_1^{-2s}O(n^{i+j/2-s})\leq M_7||f||_\gamma,$$

where $q=s-(r/2)$. Now, choosing $q>0$ satisfying $q\geq k$, we obtain

$$|I|\leq M_7 n^{-k}||f||_\gamma.$$

Therefore, by property (iii) of the Steklov mean $f_{\eta,2k}$, we get

$$\begin{aligned}J_1 &\leq M_8||f^{(r)}-f^{(r)}_{\eta,2k}||_{C[a^*,b^*]}+M_7 n^{-k}||f||_\gamma\\ &\leq M_9\omega_{2k}(f^{(r)},\eta,a_1,b_1)+M_7 n^{-k}||f||_\gamma.\end{aligned}$$

If we choose $\eta=n^{-1/2}$, the theorem follows. ■

Next, we present the global approximation theorem, which can be used in the next theorem to prove the global result for the iterative combinations.

Theorem 4.19 ([73]). *Let $f\in C_B[0,\infty)$ and $n\geq 2$. Then*

$$\left\|S_n(f,)-f\right\|\leq C\omega_\varphi^2(f,n^{-1/2}),$$

where $C>0$ is an absolute constant, $\varphi(x)=\sqrt{x(2+x)}, x\in[0,\infty)$, and

$$\omega_\varphi^2(f,\delta)=\sup_{0<h\leq\delta}\ \sup_{x\pm h\varphi(x)\in(0,\infty)}|f(x+h\varphi(x))-2f(x)+f(x-h\varphi(x))|$$

is the Ditzian–Totik [62] second-order modulus of smoothness.

Proof. Let $g\in W_\infty^2(\varphi)$ (see Definition 1.10). By a Taylor expansion, we have

$$g(t)=g\left(\frac{v}{n}\right)+g'\left(\frac{v}{n}\right)\left(t-\frac{v}{n}\right)+\int_{v/n}^t (t-u)g''(u)du.$$

Next,

$$S_n(g,x) - \hat{S}_n(g,x) = \sum_{v=1}^{\infty} s_{n,v}(x) \int_0^{\infty} \left[g(t) - g\left(\frac{v}{n}\right)\right] b_{n,v}(t)dt$$
$$= \sum_{v=1}^{\infty} s_{n,v}(x) \int_0^{\infty} b_{n,v}(t) \left[\int_{v/n}^{t} (t-u)g''(u)du\right] dt.$$

Hence, by Lemma 9.6.1 of [62], we have

$$\begin{aligned}
&|S_n(g,x) - \hat{S}_n(g,x)| \\
&\leq \sum_{v=1}^{\infty} s_{n,v}(x) \int_0^{\infty} b_{n,v}(t) \left|\int_{v/n}^{t} \frac{|t-u|}{u(2+u)} du\right| dt.||\varphi^2 g''|| \\
&\leq \sum_{v=1}^{\infty} s_{n,v}(x) \int_0^{\infty} b_{n,v}(t) \left|\int_{v/n}^{t} \frac{|t-u|}{u(1+u)} du\right| dt.||\varphi^2 g''|| \\
&\leq \sum_{v=1}^{\infty} s_{n,v}(x) \int_0^{\infty} b_{n,v}(t) \frac{(t-v/n)^2}{v/n} \left(\frac{1}{1+v/n} + \frac{1}{1+t}\right) dt.||\varphi^2 g''|| \\
&\leq \left\{\sum_{v=1}^{\infty} s_{n,v}(x) \frac{1}{n-1} + \sum_{v=1}^{\infty} s_{n,v}(x) \frac{1}{n}.\frac{n+v}{n+v+1}\right\}.||\varphi^2 g''|| \\
&\leq \left\{\frac{1}{n-1} + \sum_{v=1}^{\infty} s_{n,v}(x) \frac{1}{n}\right\}.||\varphi^2 g''|| \leq \frac{2}{n-1}.||\varphi^2 g''||.
\end{aligned}$$

On the other hand, by Lemma 4.3, we have $||S_n f|| \leq ||f||$. Thus, by Lemma 9.6.1 of [62], we have

$$\begin{aligned}
&|S_n(g,x) - \hat{S}_n(g,x)| \\
&\leq |S_n(f-g,x) - (f-g)(x)| + |S_n(g,x) - \hat{S}_n(g,x)| + |\hat{S}_n(g,x) - g(x)| \\
&\leq 2||f-g|| + \frac{2}{n-1}||\varphi^2 g''|| + \frac{1}{n}||\hat{\varphi}^2 g''|| \\
&\leq 2\left\{||f-g|| = \frac{1}{n-1}||\varphi^2 g''||\right\} + \left\{||f-g|| + \frac{1}{n}||\hat{\varphi}^2 g''||\right\},
\end{aligned}$$

where $\hat{\varphi} = \sqrt{x}, x \in [0,\infty)$. If we consider the following K-functional:

$$K_{\varphi}^2(f,\delta) = \inf\{||f-g|| + \delta||\varphi^2 g''|| : g \in W_{\infty}^2\}, \delta > 0,$$

and

$$K^2_{\hat{\varphi}}(f,\delta) = \inf\{||f-g|| + \delta||\hat{\varphi}^2 g''|| : g \in W^2_\infty\}, \delta > 0,$$

respectively, then

$$\begin{aligned} ||S_n f - f|| &\le ||S_n f - \hat{S}_n f|| + ||\hat{S}_n f - f|| \\ &\le 2K^2_\varphi(f,(n-1)^{-1}) + K^2_{\hat{\varphi}}(f,n^{-1}), \end{aligned}$$

for all $f \in C_B[0,\infty)$ and $n \ge 2$. Using Theorem 2.1.1 of [62], inequality $\hat{\varphi}(x) \le \varphi(x), x \in [0,\infty)$, and Theorem 4.1.1 of [62], we get the assertion of the theorem. ■

Theorem 4.20 ([73]). *Let $f \in C_B[0,\infty)$ and $n \ge 2$. Then there exists a $C(k) > 0$ such that*

$$||S_{n,k} f - f|| \le c(k)\omega^2_\varphi(f, n^{-1/2}).$$

Proof. We have

$$\begin{aligned} S_{n,k}(f,x) - f(x) &= \sum_{p=1}^{k}(-1)^{p+1}\binom{k}{p}[S_n^p(f,x) - f(x)] - \sum_{p=0}^{k}(-1)^{p+1}\binom{k}{p} f(x) \\ &= \sum_{p=1}^{k}(-1)^{p+1}\binom{k}{p}[S_n^p(f,x) - f(x)]. \end{aligned}$$

Using $||S_n f|| \le ||f||$, we obtain

$$||S_n^p f|| = ||S_n(S_n^{p-1} f)|| \le ||S_n^{p-1} f|| \le \cdots \le ||f||.$$

Hence,

$$\begin{aligned} ||S_n^p f - f|| &= ||S_n^p f - S_n^{p-1} f|| + \cdots + ||S_n f - f|| \\ &\le ||S_n^{p-1}(S_n f - f)|| + \cdots + ||S_n f - f|| \end{aligned}$$

$$p||S_n f - f||.$$

In conclusion,

$$\begin{aligned} ||S_{n,k} f - f|| &\le \sum_{p=1}^{k}\binom{k}{p}||S_n^p f - f|| \le \sum_{p=1}^{k}\binom{k}{p} p||S_n f - f|| \\ &= ||S_n f - f|| \sum_{p=1}^{k}\binom{k-1}{p-1} = 2^{k-1}k||S_n f - f||. \end{aligned}$$

In view of Theorem 4.19, there exists a $C > 0$ absolute constant such that

$$||S_n f - f|| \leq C\omega_\varphi^2(f, n^{-1/2}).$$

Thus,

$$||S_{n,k} f - f|| \leq 2^{k-1} k C\omega_\varphi^2(f, n^{-1/2}),$$

and the constant $c(k)$ can be chosen as $2^{k-1}kC$, which completes the proof. ■

4.3 Another Form of Linear Combinations

In 1993, Bingzheng [40] considered the following linear combinations for the general class of Durrmeyer-type operators as

$$M_{n,r}(f, x) = \sum_{i=0}^{r-1} \alpha_i(n) M_{n_i}(f, x),$$

where, with an absolute constant $A, n_i, \alpha_i(n)$ satisfy

(a) $n = n_0 < n_1 \cdots < n_{r-1} \leq An$,
(b) $\sum_{i=0}^{r-1} |\alpha_i(n)| \leq A$,
(c) $\sum_{i=0}^{r-1} \alpha_i(n) = 1$,
(d) $\sum_{i=0}^{r-1} \alpha_i(n) n_i^{-k} = 0$, for $k = 1, 2, \ldots, r-1$.

Obviously, $M_{n,1}(f, x) = M_n(f, x)$, and these operators are defined as

$$M_n(f, x) = (n-c) \sum_{v=0}^{\infty} p_{n,v,c}(c) \int_I p_{n,v,c}(t) f(t) dt,$$

where

$$p_{n,v,c}(x) = (-1)^k \phi_{n,c}^{(k)}(x) \frac{x^k}{k!}, x \in I,$$

with $n \in N, n > \max\{0, -c\}$ and special cases

1. $\phi_{n,c}(x) = (1-x)^n$ for the interval $I = [0, 1]$, with $c = -1$,
2. $\phi_{n,c}(x) = e^{-nx}$ for the interval $I = [0, \infty)$, with $c = 0$,
3. $\phi_{n,c}(x) = (1 + cx)^{-n/c}$ for the interval $I = [0, \infty)$, with $c > 0$.

Let $C[I] \bigcap L_\infty[I]$ be the set of continuous bounded functions on I. $||.||_\infty$ denotes the supremum norm. The rth modulus of smoothness is defined in $C[I] \bigcap L_\infty[I]$ as $\omega_r(f, t) = \sup_{0 \leq h < t} ||\Delta_h^r f||_\infty$, where the forward differences $\Delta_h^r f(x) =$

$\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}f(x+(k-r/2)h)$, if $x \pm (r/2)h \in I$, $\Delta_h^r f(x) = 0$, otherwise. The Peetre K-functionals are defined as

$$K_r(f,t^r) = \inf g \in D_r\{||f-g||_\infty + t^r||g||_{D_r}\},$$

where the Sobolev space D_r and its norm are defined as $D_r = \{g \in C[I] : g^{(r-1)} \in A.C_{loc}, g^{(r)} \in L_\infty\}$, and $||g||_{D_r} = ||g||_\infty + ||g^{(r)}||_\infty$. It is well known that for $f \in C[I]$,

$$M_0^{-1}\omega_r(f,t) \leq K_r(f,t^r) \leq M_0\omega_r(f,t), \tag{4.4}$$

with the constant M_0 independent of f and $t > 0$. Following Heilmann's methods [148], Bingzheng [40] used the following representations for the derivatives of M_n :

Let $f \in C[I]\bigcap L_\infty[I], \phi(x) = \sqrt{x(1+cx)}, s \in \mathbb{N}, x \in I$. If $f^{(s)} \in C[I]\bigcap L_\infty[I]$, then

$$(M_n f)^{(s)}(x) = (-1)^s(n-c)\alpha(n,s,c)\sum_{k=0}^{\infty} p_{n+cs,k}(x)\int_I p_{n-cs,k+s}(t)f^{(s)}(t)dt,$$

with $0 < c^{-1} \leq \alpha(n,s,c) = \prod_{j=0}^{s-1}(n+cl)/(n-cj-c) \leq C, C$ is independent of n. In some situations, it is convenient to work with the operators $M_{n,s}$ given by

$$(M_{n,s}f)(x) = (-1)^s(n-c)\alpha(n,s,c)\sum_{k=0}^{\infty} p_{n+cs,k}(x)\int_I p_{n-cs,k+s}(t)f(t)dt,$$

where $f \in C[I]\bigcap L_\infty[I]$. It was observed that $(M_n f)^{(s)} = M_{n,s}f^{(s)}$ if $f^{(s)} \in C[I]\bigcap L_\infty[I]$. For these operators, the linear combinations take the form

$$M_{n,r,s}(f,x) = \sum_{i=0}^{r-1}\alpha_i(n)M_{n_i,s}(f,x),\ f \in C[I]\bigcap L_\infty[I]. \tag{4.5}$$

To prove the direct theorem, the following lemma must be used:

Lemma 4.10 ([40]). *For every $m \in \mathbb{N}, n > c(s+2m+2)$ and $x \in I$, we have*

$$|(M_{n,s}((t-x)^{2m})(x)| \leq C_1(\phi(x)^2 + n + 1/n^2)^m$$

$$|(M_{n,s}((t-x)^{2m+1})(x)| \leq C_2(\phi(x)^2 + n + 1/n^2)^m(1+2cx)/n,$$

where C_1, C_2 are independent of n.

Theorem 4.21 ([40] Direct theorem). *Let $f^{(s)} \in C[I]\bigcap L_\infty[I], s \in \mathbb{N}_0, r \in \mathbb{N}$. Then*

$$\begin{aligned}|(M_{n,r}(f,x)-f(x))^{(s)}| &\le MK_r(f^{(s)},(\phi(x)^2/n+1/n^2)^{r/2})\\ &\le M'K_r(f^{(s)},(\phi(x)^2/n+1/n^2)^{1/2}),\end{aligned}$$

where M, M' are constants independent of $f, n, x \in I$ and $\phi(x) = \sqrt{x(1+cx)}$.

Proof. From the definition of combinations and Lemma 4.10, we have

$$M_{n,r,s}((t-x)^k,x) = 0, k = 0,1,2,\dots,r-1.$$

Let $g \in D_r$. By Lemma 4.22 and Holder's inequality, we have

$$\begin{aligned}|M_{n,r,s}(g,x)-g(x)| &\le M_{n,r,s}\left(\int_x^t (t-u)^{r-1}g^{(r)}(u)/(r-1)!du,x\right)\\ &\le \sum_{i=0}^{r-1}|\alpha_i(n)|M_{n_i,s}(|t-x|^r,x)||g^{(r)}||_\infty\\ &\le \sum_{i=0}^{r-1}|\alpha_i(n)|M_{n_i,s}((t-x)^{2r},x)^{1/2}||g^{(r)}||_\infty\\ &\le A(C_1(\phi(x)^2/n+1/n^2)^{r/2}||g^{(r)}||_\infty\\ &= AC_1(\phi(x)^2/n+1/n^2)^{r/2}||g^{(r)}||_\infty.\end{aligned}$$

Thus, we have

$$\begin{aligned}|(M_{n,r,s}(f,x)-f(x))^{(s)}| &\le |M_{n,r,s}(f^{(s)}-g,x)|+|f^{(s)}-g|+|M_{n,r,s}(g,x)-g(x)|\\ &\le \sum_{i=0}^{r-1}|\alpha_i(n)|.|M_{n_i,s}(f^{(s)}-g,x)|\\ &\quad +AC_1(\phi(x)^2+n+1/n^2)^{r/2}||g^{(r)}||_\infty+|f^{(s)}-g|\\ &\le (A+1)(C_0+1)(C_1+1)\Big(||f^{(s)}-g||_\infty\\ &\quad +(\phi(x)^2+n+1/n^2)^{r/2}||g^{(r)}||_\infty\Big),\end{aligned}$$

where C_0 satisfies $||M_{n_i,s}(f,x)|| \le C_0||f||_\infty$. By taking the infimum over $g \in D_r$, we obtain

$$|(M_{n,r}(f,x)-f(x))^{(s)}| \le MK_r(f^{(s)},(\phi(x)^2/n+1/n^2)^{r/2}),$$

where M is a constant independent of f, x, and using (4.4), we obtain the required result. ■

Also, the following inverse result was discussed and proved in [40].

Theorem 4.22 ([40] Inverse theorem). *Let* $f^{(s)} \in C[I] \bigcap L_\infty[I], \phi(x) = \sqrt{x(1+cx)}, r \in \mathbb{N}$, *and* $s < \alpha < r+s, n > s+2r+2$. *Then*

$$|(M_{n,r}(f,x) - f(x))^{(s)}| \leq C(\phi(x)^2/n + 1/n^2)^{(\alpha-s)/2}),$$

with constant C *independent of* n, x, *if and only if* $\omega(f^{(s)}, h) = O(h^{\alpha-s})$.

Zhou [258] considered this type of combination for the Szász–Durrmeyer operators $L_n(f,x) = M_n(f,x)$ defined in (2.17) and obtained the following results:

Theorem 4.23 ([258] Direct theorem). *Let* $f \in C[0,\infty), r \in \mathbb{N}$. *Then we have*

$$\begin{aligned}|L_{n,r}(f,x) - f(x)| &\leq CK_r(f, (x/n + n^{-2})^{r/2}) \\ &\leq C'\omega_r(f, \sqrt{x/n + n^{-2}}),\end{aligned}$$

where C *and* C' *are constants independent of* $f, n \in \mathbb{N}$ *and* $x \geq 0$.

Theorem 4.24 ([258] Inverse theorem). *Let* $f \in C[0,\infty), r \in \mathbb{N}, 0 < \alpha < r$. *Then we have*

$$|L_{n,r}(f,x) - f(x)| \leq C(x/n + n^{-2})^{\alpha/2},$$

with a constant C *independent of* x *and* n, *if and only if* $\omega_r(f,h) = O(h^\alpha)$.

4.4 Combinations of Szász–Baskakov Operators

In this section, we consider the combinations of the Szász–Baskakov operators. This section is based on the results studied in [129]. Prasad et al. [203] introduced the Szász–Mirakyan–Baskakov operator as

$$S_n(f,x) = (n-1)\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^\infty b_{n,k}(t)f(t)dt, x \in \mathbb{R}_+ \equiv [0,\infty), \tag{4.6}$$

where $s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}$ and $b_{n,k}(t) = \binom{n+k-1}{k}\frac{t^k}{(1+t)^{n+k}} = \frac{(n)_k}{k!}\frac{t^k}{(1+t)^{n+k}}$, and $(n)_k$ represents the Pochhammer symbol.

Alternatively, the operators (4.6) can be represented as

$$S_n(f(t),x) = (n-1)\int_0^\infty \frac{e^{-nx}f(t)}{(1+t)^n}\,{}_1F_1\left(n;1;\frac{nxt}{1+t}\right)dt,$$

where the function ${}_1F_1$ is known as the Pochhammer–Berens confluent hypergeometric function.

Lemma 4.11 ([129]). *For $n > 0$ and $r \geq 0$, we have*

$$S_n(t^r, x) = \frac{\Gamma(n-r-1)\Gamma(r+1)}{\Gamma(n-1)}\ {}_1F_1(-r; 1; -nx). \tag{4.7}$$

Furthermore, we have

$$S_n(t^r, x) = \frac{\Gamma(n-r+1)\Gamma(r+1)}{\Gamma(n-1)}\ L_r(-nx),$$

where $L_r(-nx)$ are the Laguerre polynomials.

Proof. Substituting $f(t) = t^r$ in (4.6) and using the definition of the Beta integral, we have

$$\begin{aligned} S_n(t^r, x) &= (n-1)\sum_{k=0}^{\infty} \frac{e^{-nx}(nx)^k}{k!} \frac{(n)_k}{k!} \int_0^{\infty} \frac{t^{k+r}}{(1+t)^{n+k}} dt \\ &= (n-1)\sum_{k=0}^{\infty} \frac{e^{-nx}(nx)^k}{k!} \frac{(n)_k}{k!} \frac{\Gamma(k+r+1)\Gamma(n-r-1)}{\Gamma(n+k)}. \end{aligned}$$

Using $k! = (1)_k$ and $\Gamma(k+r+1) = \Gamma(r+1)(r+1)_k$, we have

$$\begin{aligned} S_n(t^r, x) &= (n-1)e^{-nx}\Gamma(n-r-1)\sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \frac{(n)_k}{(1)_k} \frac{\Gamma(r+1)(r+1)_k}{(n)_k \Gamma(n)} \\ &= \frac{e^{-nx}\Gamma(n-r-1)\Gamma(r+1)}{\Gamma(n-1)} \sum_{k=0}^{\infty} \frac{(r+1)_k}{(1)_k} \frac{(nx)^k}{k!} \\ &= \frac{e^{-nx}\Gamma(n-r-1)\Gamma(r+1)}{\Gamma(n-1)}\ {}_1F_1(r+1; 1; nx), \end{aligned}$$

which, on using ${}_1F_1(a; b; x) = e^x\ {}_1F_1(b-a; b; -x)$, leads us to (4.7).

It is obvious that the confluent hypergeometric functions can be related to the generalized Laguerre polynomials $L_n^m(x)$ with the relation

$$L_n^m(x) = \frac{(m+n)!}{m!n!}\ {}_1F_1\left(-n; m+1; x\right).$$

Thus,

$$S_n(t^r, x) = \frac{\Gamma(n-r-1)\Gamma(r+1)}{\Gamma(n-1)}\ L_r(-nx),$$

where $L_r(-nx) = L_r^0(-nx)$ is the simple Laguerre polynomials. ∎

Remark 4.1. For fixed $x \in I \equiv [0, \infty)$, define the function ψ_x by $\psi_x(t) = t - x$. The central moments for the operators S_n are given by

$$(S_n\psi_x^0)(x) = 1, (S_n\psi_x^1)(x) = \frac{1+2x}{n-2}, (S_n\psi_x^2)(x) = \frac{(n+6)x^2 + 2(n+3)x + 2}{(n-2)(n-3)}.$$

Moreover, let $x \in I$ be fixed. For $r = 0, 1, 2, ...$ and $n \in N$, the central moments for the operators S_n satisfy

$$(S_n\psi_x^r)(x) = O(n^{-[(r+1)/2]}).$$

In view of the above, an application of the Schwarz inequality, for $r = 0, 1, 2, ...$, yields

$$(S_n|\psi_x^r|)(x) \le \sqrt{(S_n\psi_x^{2r})(x)} = O(n^{-r/2}). \tag{4.8}$$

It is known that the simple generalized Laguerre polynomials $L_k(x)$ have the following representation:

$$L_k(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{k-j} \frac{x^j}{j!}. \tag{4.9}$$

Therefore, the coefficient of the leading term in (4.9) is $(-1)^k \cdot \frac{1}{k!}$. Hence,

$$\begin{aligned} S_n(t^k, x) &= \frac{\Gamma(n-k-1)k!}{\Gamma(n-1)} \cdot L_k(-nx) \\ &= \frac{\Gamma(n-k-1)k!}{\Gamma(n-1)} \cdot \sum_{j=0}^{k} \binom{k}{k-j} \frac{n^j x^j}{j!} \\ &= \frac{\Gamma(n-k-1)}{\Gamma(n-1)} \cdot x^k + \frac{\Gamma(n-k-1)k!}{\Gamma(n-1)} \sum_{j=0}^{k-1} \binom{k}{k-j} \frac{n^j x^j}{j!}. \end{aligned} \tag{4.10}$$

We consider the following linear combinations:

$$S_{n,r} = \sum_{i=0}^{r} \alpha_i(n) \cdot S_{n_i}, \tag{4.11}$$

where n_i, $i = 0, 1, \ldots, r$ are distinct positive numbers. Determine $\alpha_i(n)$ such that $S_{n,r}p = p$ for all $p \in \mathbf{P}_r$. This seems to be natural, as the operators S_n do not preserve linear functions. The requirement that each polynomial of degree at most r should be reproduced leads to a linear system of equations:

$$S_{n,r}(t^k, x) = x^k,\ 0 \le k \le r. \tag{4.12}$$

Therefore, (4.10) and (4.11) imply the system

$$\begin{aligned} &\alpha_0(n) + \alpha_1(n) + \cdots + \alpha_r(n) = 1 \\ &\sum_{i=0}^{r} \alpha_i(n) \cdot \frac{\Gamma(n_i - k - 1)}{\Gamma(n_i - 1)} \cdot n_i^k = 1, \quad 1 \le k \le r. \end{aligned} \tag{4.13}$$

The unique solution of this system is

$$\alpha_i(n) = \frac{\Gamma(n_i - 1)}{\Gamma(n_i - r - 1)} \cdot \prod_{\substack{j=0 \\ j \ne i}}^{r} \frac{1}{(n_i - n_j)}, \quad 0 \le i \le r. \tag{4.14}$$

To verify this, let us first set $k = r$ in the second equation in (4.13). By using (4.14), we set the left side of the second equation of (4.13) equal to

$$\sum_{i=0}^{r} n_i^r \prod_{\substack{j=0 \\ j \ne i}}^{r} \frac{1}{(n_i - n_j)} = f[n_0, n_1, \ldots, n_r], \quad f(t) = t^r, \tag{4.15}$$

where in (4.15) we have used the well-known formula for the representation of the divided difference $f[n_0, n_1, \ldots, n_r]$ over the knots $n_0, \ldots, n_r$ for $f(t) = t^r$. But the latter is equal to the leading coefficient in the Lagrange interpolation polynomial of degree r over the knots $n_0, \ldots, n_r$, which obviously is equal to 1. Furthermore, if $1 \le k < r$ in (4.13), then we should verify that

$$\sum_{i=0}^{r} \frac{\Gamma(n_i - k - 1)}{\Gamma(n_i - r - 1)} \cdot n_i^k \cdot \prod_{\substack{j=0 \\ j \ne i}}^{r} \frac{1}{(n_i - n_j)} = 1.$$

Consequently,

$$(n_i - r - 1)(n_i - r) \cdot (n_i - k - 2) \cdot n_i^k = h(n_i) \in \mathbf{P}_r,$$

with leading coefficient 1. So in a similar way as earlier, we see that the second equation in (4.13) holds for all $1 \le k \le r$. To verify the first equation in (4.13), we only need to observe that $\alpha_0(n) + \cdots + \alpha_r(n)$ equals the divided difference $f[n_0, \ldots, n_r]$ for

$$f(t) = (t - r - 1)(t - r) \cdots (t - 2) = t^r + \cdots,$$

and the latter is equal to 1. According to (4.10), by the same method we observe that

$$\sum_{i=0}^{r} \alpha_i(n) \frac{\Gamma(n_i - k - 1)k!}{\Gamma(n_i - 1)} \binom{k}{k-j} \frac{n_i^j x^j}{j!} = 0,$$

for $0 \leq j \leq k-1$ and $1 \leq k \leq r$. To obtain a direct estimate for approximation by linear combinations $S_{n,r}$, one needs two additional assumptions:

$$n = n_0 < n_1 < \cdots < n_r \leq A \cdot n (A = A(r)), \tag{4.16}$$

$$\sum_{i=0}^{r} |\alpha_i(n)| \leq C. \tag{4.17}$$

The first of these conditions guarantees that

$$\left(S_{n,r} |\psi_x^{r+1}|\right)(x) = O\left(n^{-\frac{r+1}{2}}\right), \quad n \to \infty, \tag{4.18}$$

which follows from (4.8). The second condition is that the sum of the absolute values of the coefficients should be bounded independent of n. This is due to the fact that the linear combinations are no longer positive operators. We end this section with the following example:

Example 4.1. Let $n_0 = n, n_1 = 2n, n_2 = 3n$. Then, by simple calculations, we verify that

$$\begin{aligned}
\alpha_0 &= \frac{(n-2)(n-3)}{(-n)(-2n)} = \frac{1}{2} - \frac{5}{2} \cdot \frac{1}{n} + \frac{3}{n^2}, \\
\alpha_1 &= -4 + 10\frac{1}{n} - 6 \cdot \frac{1}{n^2}, \\
\alpha_2 &= \frac{9}{2} - \frac{15}{2} \cdot \frac{1}{n} + \frac{3}{n^2}.
\end{aligned}$$

The two assumptions on α_i are fulfilled.

But if we choose $\alpha_0 = n, \alpha_1 = n+1, \alpha_2 = n+2$, it is easy to verify that the first condition in (4.13) is not satisfied. So we should be careful with the choice of the coefficients of the linear combination.

The main result in this section is

Theorem 4.25 ([129]). *Let* $f \in C_B[0, \infty)$. *Then for every* $x \in [0, \infty)$ *and for* $C > 0, n > r$, *we have*

$$|(S_{n,r} f)(x) - f(x)| \leq C \cdot \omega_{r+1}\left(f, \frac{1}{\sqrt{n}}\right),$$

where $C_B[0,\infty)$ *is the space of all real-valued continuous bounded functions* f *defined on* $[0,\infty)$.

Proof. The classical Peetre K_r-functional for $f \in C_B[0,\infty)$ is defined by

$$K_r(f,\delta^r) = \inf\{\|f-g\| + \delta^r \cdot \|g^{(r)}\| : g \in W_\infty^r\}, \quad \delta > 0, \tag{4.19}$$

where $W_\infty^r = \{g \in C_B[0,\infty), g^{(r)} \in C_B[0,\infty)\}$. From the classical book of DeVore–Lorentz, [60] there exists a positive constant C such that

$$K_r(f,\delta^r) \le C\omega_r(f,\delta). \tag{4.20}$$

Let $g \in W_\infty^r$. By a Taylor expansion of g, we get

$$g(t) = g(x) + \sum_{i=1}^{r} \frac{(t-x)^i}{i!} g^{(i)}(x) + \frac{(t-x)^{r+1}}{(r+1)!} \cdot g^{(r+1)}(\xi_{t,x}). \tag{4.21}$$

We apply the operator to both sides of (4.21). Now (4.12) implies

$$S_{n,r}(g,x) - g(x) = S_{n,r}\left(\frac{(t-x)^{r+1}}{(r+1)!} \cdot g^{(r+1)}(\xi_{t,x}); x\right).$$

Therefore,

$$|S_{n,r}(g,x) - g(x)| \le \sum_{i=0}^{r} |\alpha_i| \cdot S_{n_i}\left(\frac{|t-x|^{r+1}}{(r+1)!}; x\right) \cdot \|g^{(r+1)}\|.$$

From (4.17) and (4.18), it follows that

$$|S_{n,r}(g,x) - g(x)| \le C(r) \cdot n^{-(r+1)/2} \cdot \|g^{(r+1)}\|.$$

Consequently,

$$\begin{aligned} |S_{n,r}(f,x) - f(x)| &\le |S_{n,r}(f-g,x) - (f-g)(x)| + |S_{n,r}(g,x) - g(x)| \\ &\le 2\|f-g\| + C(r) \cdot n^{-(r+1)/2} \cdot \|g^{(r+1)}\|. \end{aligned}$$

Taking the infimum on the right side over all $g \in W_\infty^{r+1}$ and using (4.19) and (4.20), we get the required result. ∎

Corollary 4.3 ([129]). *If* $f^{(r+1)} \in C_B[0,\infty)$*, then*

$$|(S_{n,r}f)(x) - f(x)| \le C \cdot \left(\frac{1}{\sqrt{n}}\right)^{r+1} \cdot \|f^{(r+1)}\|_{C_B[0,\infty)}.$$

Theorem 4.26 ([129]). *Let* $f, f', \ldots, f^{(r+2)} \in C_B[0,\infty)$. *Then, if* $r = 2k+1, k = 0, 1, 2, \ldots$ *for* $x \in [0,\infty)$*, it follows that*

$$\lim_{n\to\infty} n^{k+1} \cdot [S_{n,2k+1}(f,x) - f(x)] = P_{2k+2}(x) \cdot f^{(2k+2)}(x),$$

where $P_{2k+2}(x) = \lim_{n\to\infty} \left(S_{n,2k+1}(\psi_x^{2k+2}(t),x)n^{k+1}\right)$.

Proof. By the Taylor expansion of f, we have

$$f(t) - f(x) = \sum_{i=1}^{2k+2} \frac{(t-x)^i}{i!} \cdot f^{(i)}(x) + \frac{(t-x)^{2k+2}}{(2k+2)!} \cdot R(t,x), \tag{4.22}$$

where $R(t,x)$ is a bounded function for all $t, x \in [0,\infty)$ and $\lim_{t\to x} R(t,x) = 0$. We apply $S_{n,2k+1}$ to both sides of (4.22) to get

$$S_{n,2k+1}(f,x) - f(x) = \frac{f^{(2k+2)}(x)}{(2k+2)!} \cdot S_{n,2k+1}(\psi_x^{2k+2},x) + I, \tag{4.23}$$

where

$$I = \frac{1}{(2k+2)!} \cdot S_{n,2k+1}\left((t-x)^{2k+2} \cdot R(t,x);x\right).$$

From (4.8), (4.17), and (4.18), we get

$$|S_{n,2k+1}(\psi_x^{2k+2},x)| = O\left(n^{-[(2k+3)/2]}\right) = O\left(n^{-(k+1)}\right). \tag{4.24}$$

Let $\varepsilon > 0$ be given. Since $\xi(t,x) \to 0$ as $t \to x$, then there exists a $\delta > 0$ such that when $|t-x| < \delta$, we have $|\xi(t,x)| < \varepsilon$, and when $|t-x| \geq \delta$, we write

$$|\xi(t,x)| \leq C \cdot \frac{(t-x)^2}{\delta^2}.$$

Thus, for all $t, x \in [0,\infty)$, we can write

$$|\xi(t,x)| \leq \varepsilon + C \cdot \frac{(t-x)^2}{\delta^2}$$

and

$$\begin{aligned} |I| &\leq C\varepsilon \cdot n^{-(k+1)} + \frac{C}{\delta^2} \cdot \left|S_{n,2k+1}((t-x)^{2k+4},x)\right| \\ &\leq C\varepsilon \cdot n^{-(k+1)} + \frac{C}{\delta^2} \cdot n^{-(k+2)}. \end{aligned}$$

Hence,

$$n^{k+1} \cdot |I| \leq C\varepsilon + \frac{C}{\delta^2}\frac{1}{n}.$$

So

$$\lim_{n\to\infty} n^{k+1} \cdot |I| = 0.$$

Combining the estimates (4.22)–(4.24), we get the desired result. This completes the proof of the theorem. ■

Chapter 5
Better Approximation

Many well-known approximating operators L_n, preserve the constant as well as the linear functions, that is, $L_n(e_0, x) = e_0(x)$ and $L_n(e_1, x) = e_1(x)$ for $e_i(x) = x^i$ $(i = 0, 1)$. These conditions hold specifically for the Bernstein polynomials, Szász–Mirakjan operators, Baskakov operators, Phillips operators, genuine Durrmeyer-type operators, and so on. For each of these operators, $L_n(e_2, x) \neq e_2(x)$, but instead there is a more complicated expression that tends to $e_2(x)$ for sufficiently large n. King [172] modified the well-known Bernstein polynomials $B_n(f, x)$ in order to preserve e_0 and e_2. He considered $r_n^*(x)$ as

$$r_n^*(x) = -\frac{1}{2(n-2)} + \sqrt{\left(\frac{n}{n-1}\right)x^2 + \frac{1}{4(n-1)^2}}, n = 2, 3, \ldots,$$

and $r_1^*(x) = x^2$. The modified form of the Bernstein polynomial becomes

$$V_n(f, x) = \sum_{k=0}^{n} \binom{n}{k} (r_n^*(x))^k (1 - r_n^*(x))^{n-k} f\left(\frac{k}{n}\right),$$

with $0 \leq r_n^*(x) \leq 1, n = 1, 2, \ldots, 0 \leq x \leq 1$. Obviously, $\lim_{n\to\infty} r_n^*(x) = x$. In [172],

$$V_n(e_0, x) = 1, V_n(e_1, x) = r_n^*(x), V_n(e_2, x) = x^2.$$

King [172] established quantitative estimates and compared them with estimates of approximation by Bernstein polynomials. The order of approximation of $V_n(f, x)$ to $f(x)$ is at least as good as the order of approximation to $f(x)$ by the Bernstein polynomial whenever $0 \leq x < 1/3$.

Some other properties for the modified operators V_n have been discussed by Gonska and Pitul [85]:

V. Gupta and R.P. Agarwal, *Convergence Estimates in Approximation Theory*,
DOI 10.1007/978-3-319-02765-4_5,

Since $V_n e_1 = r_n^*$, it is clear that V_n is not a polynomial operator. V_n interpolates at the endpoints 0 and 1.

5.1 Bernstein–Durrmeyer-Type Operators

Gupta [96] introduced the following positive linear operators, which modify the Bernstein–Durrmeyer operators:

$$P_n(f,x) = \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 b_{n,k}(t) f(t)dt, \tag{5.1}$$

where $x \in [0,1]$, $n \in \mathbb{N}$ and, for $\phi_n(x) = (1-x)^n$,

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \text{ and } b_{n,k}(t) = (-1)^{k+1} \frac{x^k}{k!} \phi_n^{(k+1)}(t).$$

Throughout the section, we use the test functions e_i as $e_i(x) = x^i$, $i = 0,1,2$, and the moment function $\varphi_x = t - x$.

Lemma 5.1. *For $x \in [0,1]$ and $n \in \mathbb{N}$, we have*

(i) $P_n(e_0,x) = 1$, $P_n(e_1,x) = \dfrac{nx+1}{n+1}$, $P_n(e_2,x) = \dfrac{n^2x^2 - n(x-4) + 2}{(n+1)(n+2)}$,

(ii) $P_n(\varphi_x,x) = \dfrac{1-x}{n+1}$, $P_n(\varphi_x^2,x) = \dfrac{-2x^2(n-1) + 2x(n-2)x + 2}{(n+1)(n+2)}$.

Also, Gupta and Maheshwari [118] introduced another modification of Bernstein–Durrmeyer operators and investigated their approximation properties for functions of bounded variation:

$$R_n(f,x) = n \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n-1,k-1}(t) f(t)dt + (1-x)^n f(0), \tag{5.2}$$

where $x \in [0,1]$, $n \in \mathbb{N}$, and the term $p_{n,k}(x)$ is given above.

Lemma 5.2. *For $x \in [0,1]$ and $n \in \mathbb{N}$, we have*

(i) $R_n(e_0,x) = 1$, $R_n(e_1,x) = \dfrac{nx}{n+1}$, $R_n(e_2,x) = \dfrac{nx(x(n-1)+2)}{(n+1)(n+2)}$,

(ii) $R_n(\varphi_x,x) = \dfrac{-x}{n+1}$, $R_n(\varphi_x^2,x) = \dfrac{x(1-x)(2n+1) - (1-3x)x}{(n+1)(n+2)}$.

In [111], Gupta and Duman modified the operators given by (5.1) and (5.2) such that the linear functions are preserved. They were able to achieve a better approximation over some compact interval.

We start with the operator P_n. Then, by defining

$$r_n(x) = \frac{(n+1)x - 1}{n},$$

we replace x in the definition of $P_n(f, x)$ by $r_n(x)$. So, to get $r_n(x) \in [0, 1]$ for any $n \in \mathbb{N}$, we have to use the restriction $x \in \left[\frac{1}{2}, 1\right]$. Then the following modification of the operators $P_n(f, x)$ is defined by (5.1)

$$\hat{P}_n(f, x) = \sum_{k=0}^{n} d_{n,k}(x) \int_0^1 b_{n,k}(t) f(t) dt, \tag{5.3}$$

where $x \in \left[\frac{1}{2}, 1\right]$, $n \in \mathbb{N}$, the term $b_{n,k}(t)$ is given in (5.1), and

$$d_{n,k}(x) = \binom{n}{k} \frac{(n+1)^{n-k}((n+1)x - 1)^k (1-x)^{n-k}}{n^n}.$$

In a similar manner, defining

$$q_n(x) = \frac{(n+1)x}{n},$$

from the definition of $R_n(f, x)$ given by (5.2) and using the restriction $x \in \left[0, \frac{1}{2}\right]$, we have the following positive linear operators at once:

$$\hat{R}_n(f, x) = n \sum_{k=0}^{n} t_{n,k}(x) \int_0^1 p_{n-1,k-1}(t) f(t) dt + \frac{(n-(n+1)x)^n}{n^n} f(0), \tag{5.4}$$

where $x \in \left[0, \frac{1}{2}\right]$, $n \in \mathbb{N}$, the term $p_{n-1,k-1}(t)$ is given above, and

$$t_{n,k}(x) = \binom{n}{k} \frac{(n+1)^k x^k (n-(n+1)x)^{n-k}}{n^n}.$$

The next results follow from Lemmas 5.1 and 5.2.

Lemma 5.3 ([111]). *For $x \in \left[\frac{1}{2}, 1\right]$ and $n \in \mathbb{N}$, we have*

(i) $\hat{P}_n(e_0, x) = 1$, $\hat{P}_n(e_1, x) = x$, $\hat{P}_n(e_2, x) = \dfrac{(n^2-1)x^2 + 2(n+1)x - 1}{n(n+2)}$,

(ii) $\hat{P}_n(\varphi_x, x) = 0$, $\hat{P}_n(\varphi_x^2, x) = \dfrac{(1-x)(2nx + x - 1)}{n(n+2)}$.

Lemma 5.4 ([111]). *For* $x \in \left[0, \frac{1}{2}\right]$ *and* $n \in \mathbb{N}$, *we have*

(i) $\hat{R}_n(e_0, x) = 1$, $\hat{R}_n(e_1, x) = x$, $\hat{R}_n(e_2, x) = \dfrac{x((n^2-1)x + 2n)}{n(n+2)}$,

(ii) $\hat{R}_n(\varphi_x, x) = 0$, $\hat{R}_n(\varphi_x^2, x) = \dfrac{x(2n(1-x) - x)}{n(n+2)}$.

Lemmas 5.3 and 5.4 easily show that the modified operators $\hat{P}_n$ and $\hat{R}_n$ preserve the linear functions; that is, for $h(t) = at + b$, where a, b are any real constants, we obtain

$$\hat{P}_n(h, x) = \hat{R}_n(h, x) = h(x).$$

On the other hand, the above lemmas guarantee that the following Korovkin-type approximation results hold.

Theorem 5.1 ([111]). *For all* $f \in C[0, 1]$, *the sequence* $\{\hat{P}_n(f, x)\}_{n \in \mathbb{N}}$ *is uniformly convergent to* $f(x)$ *with respect to* $x \in [1/2, 1]$.

Theorem 5.2 ([111]). *For all* $f \in C[0, 1]$, *the sequence* $\{\hat{R}_n(f, x)\}_{n \in \mathbb{N}}$ *is uniformly convergent to* $f(x)$ *with respect to* $x \in [0, 1/2]$.

In [111], the rates of convergence of the operators $\hat{P}_n$ and $\hat{R}_n$ are given by (5.3) and (5.4), respectively.

Theorem 5.3 ([111]). *For every* $f \in C[0, 1]$, $x \in \left[\frac{1}{2}, 1\right]$, *and* $n \in \mathbb{N}$, *we have*

$$\left| \hat{P}_n(f, x) - f(x) \right| \leq 2\omega(f, \delta_x),$$

where

$$\delta_x := \sqrt{\frac{(1-x)(2nx + x - 1)}{n(n+2)}}.$$

Remark 5.1 ([111]). For the operator P_n given by (5.1), we may write that, for every $f \in C[0, 1]$, $x \in [0, 1]$, and $n \in \mathbb{N}$,

$$|P_n(f, x) - f(x)| \leq 2\omega(f, \alpha_x), \tag{5.5}$$

where

$$\alpha_x := \sqrt{\frac{-2x^2(n-1) + 2x(n-2)x + 2}{(n+1)(n+2)}}.$$

Now we claim that the error estimation in Theorem 5.3 is better than that of (5.13) provided $f \in C[0, 1]$ and $x \in \left[\frac{1}{2}, \frac{3}{5}\right]$. Indeed, in order to get this better estimation, we must show that $\delta_x \leq \alpha_x$ for appropriate x's. Using also the restriction $x \in \left[\frac{1}{2}, 1\right]$, one can obtain that

$$\begin{aligned}\delta_x \le \alpha_x &\Leftrightarrow \frac{(1-x)(2nx+x-1)}{n(n+2)} \le \frac{-2x^2(n-1)+2x(n-2)x+2}{(n+1)(n+2)}\\ &\Leftrightarrow \frac{(1-x)((5n+1)x-(3n+1))}{n(n+1)(n+2)} \le 0\\ &\Leftrightarrow x \le \frac{3n+1}{5n+1}.\end{aligned}$$

Observe now that

$$\frac{3n+1}{5n+1} > \frac{3}{5} \quad \text{for any } n \in \mathbb{N}.$$

Thus, considering the above inequalities, we can say that if $x \in \left[\frac{1}{2}, \frac{3}{5}\right]$, then we have

$$\delta_x \le \alpha_x,$$

which corrects the claim.

A similar idea as in Theorem 5.3 immediately leads us to the following result.

Theorem 5.4 ([111]). *For every $f \in C[0,1]$, $x \in \left[0, \frac{1}{2}\right]$, and $n \in \mathbb{N}$, we have*

$$\left|\hat{R}_n(f,x) - f(x)\right| \le 2\omega(f, u_x),$$

where

$$u_x := \sqrt{\frac{x(2n(1-x)-x)}{n(n+2)}}.$$

Remark 5.2 ([111]). Furthermore, for the operator R_n given by (5.2), we get, for every $f \in C[0,1]$, $x \in [0,1]$ and $n \in \mathbb{N}$, that

$$|R_n(f,x) - f(x)| \le 2\omega(f, v_x),$$

where

$$v_x := \sqrt{\frac{x(1-x)(2n+1)-(1-3x)x}{(n+1)(n+2)}}.$$

Now considering Remark 5.2, we see that a similar claim is valid for the operators $\hat{R}_n$ on the interval $\left[\frac{2}{5}, \frac{1}{2}\right]$. Indeed, in order to get a better estimation, we must show that $u_x \le v_x$ for appropriate x's. So we may write that

$$\begin{aligned}u_x \le v_x &\Leftrightarrow \frac{x(2n(1-x)-x)}{n(n+2)} \le \frac{x(1-x)(2n+1)-(1-3x)x}{(n+1)(n+2)}\\ &\Leftrightarrow -\frac{x(x+(5x-2)n)}{n(n+1)(n+2)} \le 0\\ &\Leftrightarrow x \ge \frac{2n}{5n+1}.\end{aligned}$$

However, since

$$\frac{2n}{5n+1} < \frac{2}{5} \quad \text{for any } n \in \mathbb{N},$$

the above inequalities yield that if $x \in \left[\frac{2}{5}, \frac{1}{2}\right]$, then we have

$$u_x \leq v_x,$$

which corrects the claim again.

5.2 Phillips Operators

In 2010, Gupta [104] modified the well-known Phillips operators such that the modified form preserves the test functions e_0 and e_2. He was able to achieve a better approximation over the original Phillips operators.

The Phillips operators are defined as

$$P_n(f,x) = n\sum_{v=1}^{\infty} s_{n,v}(x)\int_0^{\infty} s_{n,v-1}(t)f(t)dt + e^{-nx}f(0), \quad x \in [0,\infty), \quad (5.6)$$

where $s_{n,v}(x) = \exp(-nx)\frac{(nx)^v}{v!}$. The operators (5.6) were introduced in [199].

Remark 5.3. It is obvious that for all $x \in [0,\infty)$, we have

(i) $P_n(e_0,x) = 1$,
(ii) $P_n(e_1,x) = x$,
(iii) $P_n(e_2,x) = \frac{nx^2+2x}{n}$.

Let $\{r_n^*(x)\}$ be the sequence of real-valued continuous functions defined on the interval $[0,\infty)$, with $0 \leq r_n^*(x) < \infty$. If we replace x by $r_n^*(x)$, which is defined by

$$r_n^*(x) = \frac{-1+\sqrt{1+n^2x^2}}{n},$$

we get the following modification of the Phillips operators defined as

$$P_n^*(f,x) = ne^{-nr_n^*(x)}\sum_{v=1}^{\infty}\frac{(nr_n^*(x))^v}{v!}\int_0^{\infty} s_{n,v-1}(t)f(t)dt + e^{-nr_n^*(x)}f(0). \quad (5.7)$$

One can see that P_n^* maps $C_B[0,\infty)$, the space of all bounded and continuous functions on $[0,\infty)$, into itself.

Remark 5.4. For each $x \in [0, \infty)$, we have

(i) $P_n^*(e_0, x) = 1$,
(ii) $P_n^*(e_1, x) = \frac{-1+\sqrt{1+n^2x^2}}{n}$,
(iii) $P_n^*(e_2, x) = x^2$.

Theorem 5.5 ([104]). *For each $f \in C_B[0, \infty)$ and $x \geq 0$, we have*

$$|P_n^*(f, x) - f(x)| \leq 2\omega(f, \delta_{n,x}),$$

where $\delta_{n,x} = \sqrt{2x(x - r_n^*(x)}$ and $\omega(f, \delta)$ is the modulus of continuity defined as $\omega(f, \delta) = \sup_{|t-x|\leq\delta; x,t\in[0,\infty)} |f(t) - f(x)|$.

Remark 5.5. For the Phillips operators $P_n(f, x)$ for each $f \in C_B[0, \infty)$ and $x \geq 0$, we have

$$|P_n(f, x) - f(x)| \leq 2\omega(f, \alpha_{n,x}),$$

where $\alpha_{n,x} = \sqrt{\frac{2x}{n}}$.

Now by Theorem 5.5 and Remark 5.5, one can easily observe that

$$2x(x - r_n^*(x)) \leq \frac{2x}{n},$$

for all $x \geq 0$. This implies that $\delta_{n,x} \leq \alpha_{n,x}$. Thus, by constructing the operators in the form (5.7), one can say that the error estimation for the modified Phillips operators (5.7) is better than the original Phillips operators defined by (5.6).

5.3 Szász–Mirakjan–Beta Operators

Gupta and Noor [119] proposed a sequence of mixed summation–integral-type operators, the so-called Szász–Mirakjan–Beta operators, as

$$U_n(f, x) = e^{-nx} \sum_{k=1}^{\infty} \frac{(nx)^k}{k!B(n + 1, k)} \int_0^{\infty} \frac{t^{k-1}}{(1 + t)^{n+k+1}} f(t)dt + e^{-nx} f(0), \quad (5.8)$$

where $f \in C[0, \infty)$ such that $|f(t)| \leq M(1 + t)^{\gamma}$ for some $M > 0, \gamma > 0$.

Remark 5.6. From [119], we know that

$$U_n(e_0, x) = 1, U_n(e_1, x) = x, U_n(e_2, x) = \frac{nx^2 + 2x}{n - 1}.$$

Duman et al. [64] modified the operators (5.8) along the lines of King [172] and observed that their modified operators have a better error estimation on the interval $[0, 2]$.

Let $\{r_n(x)\}$ be a sequence of real-valued continuous functions defined on $[0, \infty)$ with $0 \leq r_n(x) < \infty$. Then the operator U_n takes the form

$$U_n(f, r_n(x)) = e^{-nr_n(x)} \sum_{k=1}^{\infty} \frac{(nr_n(x))^k}{k!B(n+1,k)} \int_0^{\infty} \frac{t^{k-1}}{(1+t)^{n+k+1}} f(t)dt + e^{-nr_n(x)} f(0),$$

where $f \in C[0, \infty)$. Duman et al. [64] replaced $r_n(x)$ by $r_n^*(x)$, defined as

$$r_n^*(x) = \frac{1}{n}\left(-1 + \sqrt{1 + n(n-1)x^2}\right), \quad x \geq 0, n \in \mathbb{N}.$$

Then they got the following positive linear operators:

$$U_n^*(f, r_n(x)) = e^{-nr_n^*(x)} \sum_{k=1}^{\infty} \frac{(nr_n^*(x))^k}{k!B(n+1,k)} \int_0^{\infty} \frac{t^{k-1}}{(1+t)^{n+k+1}} f(t)dt + e^{-nr_n^*(x)} f(0). \tag{5.9}$$

The modified form of the operators reproduces the test function e_0 and e_2:

Lemma 5.5. *For every $x \geq 0$ and $\varphi_x = t - x$, we have*

$$U_n^*(\varphi_x, x) = -x + \frac{1}{n}\left(-1 + \sqrt{1 + n(n-1)x^2}\right)$$

and

$$U_n^*(\varphi_x^2, x) = 2x\left(x + \frac{1}{n} - \frac{\sqrt{1 + n(n-1)x^2}}{n}\right).$$

Theorem 5.6 ([64]). *For each $f \in C_B[0, \infty)$ and $x \geq 0$ and $n > 1$, we have*

$$|U_n^*(f, x) - f(x)| \leq 2\omega(f, \delta_{n,x}),$$

where $\delta_{n,x} = \sqrt{2x(x - r_n^*(x)}$ and $\omega(f, \delta)$ is the modulus of continuity defined as $\omega(f, \delta) = \sup_{|t-x| \leq \delta; x,t \in [0,\infty)} |f(t) - f(x)|$.

Remark 5.7. For the Szász–Mirakjan–Beta operators $U_n(f, x)$ for each $f \in C_B[0, \infty)$ and $x \geq 0$, we have

$$|U_n(f, x) - f(x)| \leq 2\omega(f, \alpha_{n,x}),$$

where $\alpha_{n,x} = \sqrt{\frac{x(2+x)}{n-1}}$.

According to Duman et al. [64], the error estimation in Theorem 5.6 is better than that in Remark 5.7 provided $f \in C_B[0,\infty)$ and $x \in [0,2]$. Indeed, for $0 \le x \le 2$, we have $x^2/2 \le 1$. Also, since $\left(n-\frac{1}{2}\right)^2 - n(n-1) = \frac{1}{4}$, we have

$$x^2\left[\left(n-\frac{1}{2}\right)^2 - n(n-1)\right] \le 1.$$

Thus,

$$-\frac{1}{n} + \frac{1}{n}\sqrt{1+n(n-1)x^2} \ge -\frac{1}{n} + \left(\frac{2n-1}{2n}\right)x.$$

Using the above inequality, one has

$$x - r_n^*(x) \le \frac{2+x}{2n}$$

or

$$2x(x - r_n^*(x)) \le \frac{x(2+x)}{n} \le \frac{x(2+x)}{n-1},$$

for $x \in [0,2]$ and $n > 1$. This guarantees that $\delta_{n,x} \le \alpha_{n,x}$ for $x \in [0,2]$ and $n > 1$.

5.4 Integrated Szász–Mirakjan Operators

Another integral modification of Szász–Mirakjan operators was proposed in [63] with a parameter c in the definition of [137] and for $c > 0$. They proposed the modified operators as

$$S_{n,c}(f,x) = \sum_{v=0}^{\infty} s_{n,v}(x) \int_0^{\infty} b_{n,v,c}(t) f(t)\,dt, \quad x \in [0,\infty), \tag{5.10}$$

where $s_{n,v}(x) = e^{-nx}\frac{(nx)^v}{v!}$, $b_{n,v,c}(t) = c\frac{\Gamma(\frac{n}{c}+v+1)}{\Gamma(v+1)\Gamma(\frac{n}{c})}\frac{(ct)^v}{(1+ct)^{(\frac{n}{c}+v+1)}}$. In case $c = 1$, the above operators reduce to the Szász–Beta operators [137]. These operators are linear positive operators and reproduce only constant function. The approximation properties of the operators $S_{n,c}(f,x)$ are different from the operators studied by Duman et al. [64].

Remark 5.8. It can be easily verified that

$$S_{n,c}(e_0,x) = 1, S_{n,c}(e_1,x) = \frac{(1+nx)}{n-c}$$

and

$$S_{n,c}(e_2, x) = \frac{n^2x^2 + 4nx + 2}{(n-c)(n-2c)}.$$

For $n > c$, set

$$r_n(x) = \frac{(n-c)x - 1}{n}. \tag{5.11}$$

Gupta and Deo [110] replaced x in the definition of $S_{n,c}(f, x)$ by $r_n(x)$ with $r_n(x) \in [0, \infty)$ for any $n \in \mathbb{N}$. Then the modified operators take the form

$$V_{n,c}(f, x) = \sum_{v=0}^{\infty} s_{n,v}(r_n(x)) \int_0^{\infty} b_{n,v,c}(t) f(t) dt, \tag{5.12}$$

where

$$s_{n,v}(r_n(x)) = e^{-nr_n(x)} \frac{(nr_n(x))^v}{v!}$$

and $x \in [0, \infty)$, $n \in \mathbb{N}$. The operators $V_{n,c}(f, x)$ preserve the constant and linear functions.

Lemma 5.6 ([110]). *For $x \in [0, \infty)$, $n \in \mathbb{N}$, and with $\varphi_x = t - x$, we have*

(i) $V_{n,c}(\varphi_x; x) = 0$,

(ii) $V_{n,c}(\varphi_x^2; x) = \dfrac{c(n-c)x^2 + 2(n-c)x - 1}{(n-c)(n-2c)}$.

Theorem 5.7 ([110]). *Let $f \in C_B[0, \infty)$. Then for every $x \in [0, \infty)$ and for $n > 2c, C > 0$, we have*

$$|V_{n,c}(f, x) - f(x)| \leq C\omega_2 \left(f, \sqrt{\frac{(n-c)^2x^2 + 2(n-c)x - 1}{(n-c)(n-2c)}} \right).$$

Theorem 5.8 ([110]). *For every $f \in C[0, \infty)$, $x \in [0, \infty)$, and $n \in \mathbb{N}$, we have*

$$|V_{n,c}(f; x) - f(x)| \leq 2\omega(f, \delta_x),$$

where

$$\delta_x := \sqrt{\frac{c(n-c)x^2 + 2(n-c)x - 1}{(n-c)(n-2c)}}.$$

Remark 5.9. For the operator $S_{n,c}$ given by (5.10), we may write that, for every $f \in C[0,\infty)$, $x \in [0,\infty)$, and $n \in \mathbb{N}$,

$$|S_{n,c}(f;x) - f(x)| \leq 2\omega(f, c_x), \tag{5.13}$$

where

$$c_x := \sqrt{\frac{(4cx + 2c^2x^2 + 2) + nx(cx + 2)}{(n-c)(n-2c)}}.$$

According to [110], the error estimation in Theorem 5.8 is better than that of Remark 5.9 provided $f \in C[0,\infty)$ and $x \in [0,\infty)$. Indeed, in order to get this better estimation, one must show that $\delta_x \leq c_x$. One can obtain

$$\begin{aligned}
\delta_x \leq c_x \Leftrightarrow\ & \frac{c(n-c)x^2 + 2(n-c)x - 1}{(n-c)(n-2c)} \\
\leq\ & \frac{(4cx + 2c^2x^2 + 2) + nx(cx + 2)}{(n-c)(n-2c)} \\
\Leftrightarrow\ & 3c^2x^2 + 6cx + 3 \geq 0 \\
\Leftrightarrow\ & (cx + 1)^2 \geq 0,
\end{aligned}$$

which holds. Thus, we have $\delta_x \leq c_x$, showing that modified operators give a better approximation than the original operators.

5.5 Beta Operators of the Second Kind

Stancu [223] introduced Beta operators L_n of the second kind in order to approximate the Lebesgue integrable functions on the interval $(0,\infty)$ as

$$(L_n f)(x) = \frac{1}{B(nx, n+1)} \int_0^\infty \frac{t^{nx-1}}{(1+t)^{nx+n+1}} f(t)dt. \tag{5.14}$$

Then Abel and Gupta in [5] obtained the rate of convergence via a decomposition technique.

Lemma 5.7. *For all $e_i(x) = x^i, i \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$, $x > 0$, with $n > i$, we have $(L_n e_0)(x) = 1$, $(L_n e_1)(x) = e_1(x)$, and $(L_n e_2)(x) = x^2 + \frac{x(1+x)}{n-1}$. Also, we have the recurrence relation*

$$(L_n e_{i+1})(x) = \frac{nx + i}{n - i}(L_n e_i)(x), \text{ for all } n > i.$$

For a fixed $x \in (0, \infty)$*, if we define the function* φ_x *by* $\varphi_x(t) = t - x$*, then*

(i) $(L_n\varphi_x^0)(x) = 1$,
(ii) $(L_n\varphi_x^1)(x) = 0$,
(iii) $(L_n\varphi_x^2)(x) = \frac{x(1+x)}{n-1}$.

The present section deals with the study of Stancu–Beta operators, which preserve the constant as well as linear functions but not the quadratic ones. In 2011, Gupta and Yadav [132] applied King's approach to propose the modified form of these operators, so that they preserve the quadratic functions, which results in a better approximation for the modified operators in the compact interval $(0, 1)$ for these operators.

Theorem 5.9 ([132]). *For the Stancu–Beta operators, we can write that, for every* $f \in C_B(0, \infty), x > 0$*, and* $n > 1$,

$$|(L_n f)(x) - f(x)| \leq 2\omega(f, \alpha_x),$$

where $\alpha_x = \sqrt{\frac{x(1+x)}{n-1}}$.

Let $r_n(x)$ be a sequence of real-valued continuous functions defined on $(0, \infty)$ with $0 < r_n(x) < \infty)$ and defined by

$$r_n(x) = \frac{-1 + \sqrt{1 + 4n(n-1)x^2}}{2n}.$$

Then we can define the modified form of the operators (5.14) as

$$(L_n^* f)(x) = \frac{1}{B(nr_n(x), n+1)} \int_0^\infty \frac{t^{nr_n(x)-1}}{(1+t)^{nr_n(x)+n+1}} f(t)dt. \tag{5.15}$$

Lemma 5.8. *For each* $x > 0$*, we have*

(i) $(L_n^* 1)(x) = 1$,
(ii) $(L_n^* t)(x) = \frac{-1+\sqrt{1+4n(n-1)x^2}}{2n}$,
(iii) $(L_n^* t^2)(x) = x^2$.

For a fixed $x \in (0, \infty)$*, define the function* φ_x *by* $\varphi_x(t) = t - x$*. The first central moments for the operators* L_n^* *are given by*

(i) $(L_n^*\varphi_x^0)(x) = 1$,
(ii) $(L_n^*\varphi_x^1)(x) = -x + \frac{-1+\sqrt{1+4n(n-1)x^2}}{2n}$,
(iii) $(L_n^*\varphi_x^2)(x) = 2x\left[x + \frac{1}{2n} - \frac{\sqrt{1+4n(n-1)x^2}}{2n}\right]$.

Theorem 5.10 ([132]). *For every* $f \in C_B(0, \infty)$*,* $x > 0$*, and* $n > 1$*, we have*

$$|(L_n^* f)(x) - f(x)| \leq 2\omega(f, \delta_x),$$

where

$$\delta_x = 2x\left[x + \frac{1}{2n} - \frac{1}{2n}\sqrt{1 + 4n(n-1)x^2}\right].$$

Remark 5.10 ([132]). In order to get a better estimation, we must show that $\delta_x < \alpha_x$ for appropriate x's. Indeed, for $0 < x < 1$, we have $x^2 < 1$. Also, since

$$x^2\left[(2n-1)^2 - 4n(n-1)\right] < 1,$$

or

$$1 + 4n(n-1)x^2 > (2n-1)^2x^2,$$

which gives

$$\sqrt{1 + 4n(n-1x^2)} > (2n-1)x,$$

we then obtain

$$-\frac{1}{2n} + \frac{1}{2n}\sqrt{1 + 4n(n-1)x^2} > -\frac{1}{2n} + \frac{2n-1}{2n}x,$$

and thus we have $x - r_n(x) < \frac{1+x}{2n}$,
that is,

$$2x(x - r_n(x)) < \frac{x(1+x)}{n} < \frac{x(1+x)}{n-1},$$

for $x \in (0,1)$ and $n > 1$. This guarantees that $\delta_x < \alpha_x$ for $x \in (0,1)$ and $n > 1$, which corrects our claim.

Chapter 6
Complex Operators in Compact Disks

If $f : G \to \mathbb{C}$ is an analytic function in the open set $G \subset \mathbb{C}$, with $\overline{D_1} \subset G$ (where $D_1 = \{z \in \mathbb{C} : |z| < 1\}$), then S. N. Bernstein proved that the complex Bernstein polynomials converge uniformly to f in $\overline{D_1}$ (see, e.g., Lorentz [182], p. 88). The main contributions for the complex operators are due to S. G. Gal; in fact, several important results have been compiled in his 2009 monograph [77]. Since that publication, several important results on Durrmeyer-type operators and other extensions of the known operators have been discussed. In this chapter, we present some of the important results on certain complex operators that were not discussed in [77].

6.1 Complex Baskakov–Stancu Operators

In this section, we present a Stancu-type generalization of the complex Baskakov operator studied in [82]. The Baskakov–Stancu operator of the real variable $x \in [0, \infty)$ is defined by

$$S_n^{\alpha,\beta}(f, x) = \sum_{\nu=0}^{\infty} p_{n,\nu}(x) f\left(\frac{\nu + \alpha}{n + \beta}\right),$$

where $p_{n,\nu}(x) = \binom{n+\nu-1}{\nu}\frac{x^\nu}{(1+x)^{n+\nu}}$ and α, β are two given parameters satisfying the conditions $0 \leq \alpha \leq \beta$. For $\alpha = \beta = 0$, we recapture the classical Baskakov operator.

Denoting

$$V_n^{\alpha,\beta}(f, x) = \sum_{\nu=0}^{\infty} \frac{n(n+1)\ldots.(n+\nu-1)}{(n+\beta)^\nu}\left[\frac{\alpha}{n+\beta}, \frac{\alpha+1}{n+\beta}, \ldots, \frac{\alpha+\nu}{n+\beta}; f\right] x^\nu,$$

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[where for $j = 0$, we take $n(n+1)\dots(n+j-1) = 1$], according to the method in the proof of Theorem 2 in Lupas [184], we have $S_n^{\alpha,\beta}(f,x) = V_n^{\alpha,\beta}(f,x)$ for all $x \geq 0$ [under the hypothesis on f that $S_n^{\alpha,\beta}(f,x)$ is well defined]. But if x is not positive (e.g., if $x = -1/2$), then reasoning exactly as in the case $\alpha = \beta = 0$ in Gal [77], p. 124, it is easy to see that $S_n^{\alpha,\beta}(f,x)$ and $V_n^{\alpha,\beta}(f,x)$ do not necessarily coincide.

Consequently, the corresponding complex versions (namely, replacing $x \geq 0$ by $z \in \mathbb{C}$), $S_n^{\alpha,\beta}(f,z)$ and $V_n^{\alpha,\beta}(f,z)$, do not necessarily represent the same operator.

In this section, we deal with the following complex form for the Baskakov–Stancu operator:

$$V_n^{\alpha,\beta}(f,z) = \sum_{\nu=0}^{\infty} \frac{n(n+1)\dots.(n+\nu-1)}{(n+\beta)^\nu} \left[\frac{\alpha}{n+\beta}, \frac{\alpha+1}{n+\beta}, \dots, \frac{\alpha+\nu}{n+\beta}; f\right] z^\nu,$$

which, for $\alpha = 0 = \beta$, was studied in Gal [77], pp. 124–134. Here $[x_0, x_1, \dots, x_m; f]$ denotes the divided difference of the function f on the distinct points $x_0, x_1, \dots, x_m$.

Let $H(\mathbb{D}_R)$ denote the class of the functions $f : [R, +\infty) \cup \overline{\mathbb{D}}_R \to \mathbb{C}$ (where $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}$) with all its derivatives bounded in $[0,\infty)$ by the same constant, analytic in $\mathbb{D}_R$, satisfying an exponential growth condition, namely, $|f(z)| \leq Me^{A|z|}$ for all $z \in \mathbb{D}_R$.

Following exactly the steps in Remark 1, pp. 124–125 in Gal [77], Gal et al. [82] found that for the functions $f \in H(\mathbb{D}_R)$, the operator $V_n^{\alpha,\beta}(f,z)$ is well defined for all $z \in \mathbb{C}$.

The following lemmas are required to prove the main theorems of this section:

Lemma 6.1 ([82]). *For all $n, k \in \mathbb{N} \cup \{0\}$, $0 \leq \alpha \leq \beta$, $z \in \mathbb{C}$, let us define*

$$V_n^{\alpha,\beta}(e_k,z) = \sum_{\nu=0}^{\infty} \frac{n(n+1)\dots.(n+\nu-1)}{(n+\beta)^\nu} \left[\frac{\alpha}{n+\beta}, \frac{\alpha+1}{n+\beta}, \dots, \frac{\alpha+\nu}{n+\beta}; e_k\right] z^\nu,$$

where $e_k(z) = z^k$. Then $V_n^{\alpha,\beta}(e_0,z) = 1$, and we have the following recurrence relation:

$$V_n^{\alpha,\beta}(e_{k+1},z) = \frac{z(1+z)}{n+\beta}(V_n^{\alpha,\beta}(e_k,z))' + \frac{nz+\alpha}{n+\beta}V_n^{\alpha,\beta}(e_k,z). \tag{6.1}$$

Consequently,

$$V_n^{\alpha,\beta}(e_1,z) = \frac{nz+\alpha}{n+\beta}, \quad V_n^{\alpha,\beta}(e_2,z) = \frac{n(n+1)z^2}{(n+\beta)^2} + \frac{nz(1+2\alpha)}{(n+\beta)^2} + \frac{\alpha^2}{(n+\beta)^2}.$$

Lemma 6.2 ([82]). *Let α, β satisfy $0 \leq \alpha \leq \beta$. Denoting $e_j(z) = z^j$ and $V_n^{0,0}(e_j)$ by $V_n(e_j)$, for all $n, k \in \mathbb{N} \cup \{0\}$, $0 \leq \alpha \leq \beta$, we have the following recursive* *relation for the images of the monomials e_k under $V_n^{\alpha,\beta}$ in terms of $V_n(e_j)$, $j = 0, 1, 2 \ldots, k$:*

$$V_n^{\alpha,\beta}(e_k, z) = \sum_{j=0}^{k} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} V_n(e_j, z).$$

Lemma 6.3 ([82]). *For all $n, k \in \mathbb{N} \cup \{0\}$, $0 \le \alpha \le \beta$ and $|z| \le r$, $r \ge 1$, we have*

$$|V_n^{\alpha,\beta}(e_k, z)| \le (k+1)!\, r^k.$$

Theorem 6.1 ([82] Upper estimate). *For $n_0 \in \mathbb{N}$ and $R > 0$ with $3 \le n_0 < 2R < +\infty$, let $f : [R, +\infty) \to \mathbb{C}$ and all its derivatives be bounded in $[0, \infty)$ by the same positive constant, analytic in $\mathbb{D}_R$,, that is, $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$,. Additionally, suppose that there exist $M > 0$ and $A \in (\frac{1}{R}, 1)$, with the property that $|c_k| \le M \frac{A^k}{k!}$, for all $k = 0, 1, \ldots$.*

Suppose that $0 \le \alpha \le \beta$ and $1 \le r < \min\{\frac{n_0}{2}, \frac{1}{A}\}$. Then, for all $|z| \le r$ and $n > n_0$, we have

$$|V_n^{\alpha,\beta}(f, z) - f(z)| \le \frac{\alpha + \beta r}{n+\beta} \sum_{k=1}^{\infty} |c_k| r^{k-1} + \frac{A_r(f)}{n+\beta} + \frac{\alpha B_r(f)}{n+\beta} + \frac{\beta C_r(f)}{n+\beta},$$

where $\sum_{k=1}^{\infty} |c_k| r^{k-1} < +\infty$, $B_r(f) = \sum_{k=1}^{\infty} |c_k| k r^{k-1} < +\infty$, and $C_r(f) = \sum_{k=1}^{\infty} |c_k| k r^k < +\infty$ follow from the analyticity of f and $A_r(f) = \sum_{k=1}^{\infty} |c_k|(1+r)k\,(k+1)! r^{k-1} < +\infty$ follows from $|c_k| \le M \frac{A^k}{k!}$.

Proof. By using the recurrence (6.1) in Lemma 6.1, we get

$$V_n^{\alpha,\beta}(e_k, z) - z^k = \frac{z(1+z)}{n+\beta} (V_n^{\alpha,\beta}(e_{k-1}, z))' + \frac{nz+\alpha}{n+\beta} \left(V_n^{\alpha,\beta}(e_{k-1}, z) - z^{k-1}\right) + \frac{nz+\alpha}{n+\beta} z^{k-1} - z^k$$

and

$$|V_n^{\alpha,\beta}(e_k, z) - z^k| \le \frac{|z|(1+|z|)}{n+\beta} |(V_n^{\alpha,\beta}(e_{k-1}, z))'| + \frac{n|z|+\alpha}{n+\beta} \left|V_n^{\alpha,\beta}(e_{k-1}, z) - z^{k-1}\right| + \frac{\alpha}{n+\beta} |z|^{k-1} + \frac{\beta}{n+\beta} |z|^k.$$

Clearly, $V_n^{\alpha,\beta}(e_0, z) - e_0 = 0$, and

$$|V_n^{\alpha,\beta}(e_1, z) - e_1(z)| = \left|\frac{nz+\beta}{n+\beta} - z\right| = \left|\frac{\alpha - \beta z}{n+\beta}\right| \le \frac{\alpha+\beta r}{n+\beta}.$$

Using Lemma 6.3 and Bernstein's inequality for the polynomial $V_n^{\alpha,\beta}(e_{k-1}, z)$ of degree $\le k-1$, we have

$$\begin{aligned}|(V_n^{\alpha,\beta}(e_{k-1}, z))'| &\le \frac{k-1}{r}\max\{|V_n^{\alpha,\beta}(e_{k-1}, z)| : |z| \le r\}\\ &\le \frac{k-1}{r}k!\,r^{k-1} \le (k+1)!\,r^{k-2}.\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}&|V_n^{\alpha,\beta}(e_k, z) - z^k|\\ &\le \frac{r(1+r)}{n+\beta}(k+1)!\,r^{k-2} + \frac{nr+\alpha}{n+\beta}\left|V_n^{\alpha,\beta}(e_{k-1}, z) - z^{k-1}\right|\\ &\quad + \frac{\alpha}{n+\beta}r^{k-1} + \frac{\beta}{n+\beta}r^k\\ &\le r\left|V_n^{\alpha,\beta}(e_{k-1}, z) - z^{k-1}\right| + \frac{r(1+r)}{n+\beta}(k+1)!\,r^{k-2} + \frac{\alpha}{n+\beta}r^{k-1} + \frac{\beta}{n+\beta}r^k.\end{aligned}$$

Taking $k = 2$ above, we obtain

$$|V_n^{\alpha,\beta}(e_2, z) - e_2(z)| \le r \cdot \frac{\alpha+\beta r}{n+\beta} + \frac{r(r+1)}{n+\beta}(2+1)!r^0 + \frac{\alpha}{n+\beta}r^1 + \frac{\beta}{n+\beta}r^2.$$

Then, for $k = 3$, it follows that

$$\begin{aligned}|V_n^{\alpha,\beta}(e_3, z) - e_3(z)| &\le r^2 \cdot \frac{\alpha+\beta r}{n+\beta} + \frac{r(r+1)}{n+\beta}(2+1)!r^1 + \frac{\alpha}{n+\beta}r^2 + \frac{\beta}{n+\beta}r^3\\ &\quad + \frac{r(r+1)}{n+\beta}(3+1)!r^1 + \frac{\alpha}{n+\beta}r^2 + \frac{\beta}{n+\beta}r^3\\ &= r^2 \cdot \frac{\alpha+\beta r}{n+\beta} + \frac{r(r+1)}{n+\beta}[(2+1)!+(3+1)!]r^1 + \frac{2\alpha}{n+\beta}r^2\\ &\quad + \frac{2\beta}{n+\beta}r^3.\end{aligned}$$

Reasoning by recurrence, for any $k \geq 2$, we finally get

$$|V_n^{\alpha,\beta}(e_k, z) - e_k(z)| \leq r^{k-1} \cdot \frac{\alpha + \beta r}{n + \beta} + \frac{r(r+1)}{n+\beta} \cdot \left[\sum_{j=2}^{k}(j+1)!\right] \cdot r^{k-2}$$

$$+ \frac{(k-1)\alpha}{n+\beta} r^{k-1} + \frac{(k-1)\beta}{n+\beta} r^k$$

$$\leq r^{k-1} \cdot \frac{\alpha + \beta r}{n+\beta} + \frac{r(1+r)}{n+\beta} k \cdot (k+1)!\, r^{k-2}$$

$$+ \frac{k\alpha}{n+\beta} r^{k-1} + \frac{k\beta}{n+\beta} r^k.$$

Clearly, this inequality is valid for $k = 1$, too.

Now, reasoning exactly as in the case of complex Baskakov operators in Remark 1 in Gal [77], pp. 128–130, we can write (note that here all the hypotheses on n_0 and r are involved in the statement)

$$V_n^{\alpha,\beta}(f, z) = \sum_{k=0}^{\infty} c_k V_n^{\alpha,\beta}(e_k, z),$$

which implies

$$|V_n^{\alpha,\beta}(f, z) - f(z)| \leq \sum_{k=1}^{\infty} |c_k| \cdot |V_n^{\alpha,\beta}(e_k, z) - z^k|$$

$$\leq \frac{\alpha + \beta r}{n+\beta} \sum_{k=1}^{\infty} |c_k| \cdot r^{k-1} + \sum_{k=1}^{\infty} |c_k| \frac{r(1+r)}{n+\beta} k \cdot (k+1)!\, r^{k-2}$$

$$+ \frac{\alpha}{n+\beta} \sum_{k=1}^{\infty} |c_k| k r^{k-1} + \frac{\beta}{n+\beta} \sum_{k=1}^{\infty} |c_k| k r^k$$

$$= \frac{\alpha + \beta r}{n+\beta} \sum_{k=1}^{\infty} |c_k| \cdot r^{k-1} + \frac{A_r(f)}{n+\beta} + \frac{\alpha B_r(f)}{n+\beta} + \frac{\beta C_r(f)}{n+\beta}.$$

Note here that by the analyticity of f, we clearly get $\sum_{k=1}^{\infty} |c_k| r^{k-1} < +\infty$, $B_r(f) < +\infty$, and $C_r(f) < +\infty$, while using the hypothesis $|c_k| \leq M \frac{A^k}{k!}$, we easily get

$$A_r(f) = (1+r) \cdot \sum_{k=1}^{\infty} |c_k| \cdot k \cdot (k+1)! \cdot r^{k-2} \leq M(1+r) \sum_{k=1}^{\infty} k(k+1) \cdot (rA)^k < +\infty,$$

since $rA < 1$. This proves the theorem. ∎

Theorem 6.2 ([82] Voronovskaja-type asymptotic formula). *Let $0 \le \alpha \le \beta$. Suppose that the hypothesis on the function f and that on the constants n_0, R, M, A in the statement of Theorem 6.1 hold, and let $1 \le r < \min\{\frac{n_0}{2}, \frac{1}{A}\}$ be fixed. For all $n > n_0$, $|z| \le r$, we have the following Voronovskaja-type result:*

$$\left|V_n^{\alpha,\beta}(f,z) - f(z) - \frac{\alpha-\beta z}{n+\beta}f'(z) - \frac{z(1+z)}{2n}f''(z)\right| \le \frac{M_{1,r}(f)}{n^2} + \frac{\sum_{j=2}^{6} M_{j,r}(f)}{(n+\beta)^2},$$

where, from $|c_k| \le M\frac{A^k}{k!}$, we get

$$M_{1,r}(f) = 16\sum_{k=0}^{\infty}|c_k|(k-1)(k-2)^2\, k!r^k < +\infty,\ M_{2,r}(f)$$

$$= \alpha^2 \cdot \sum_{k=2}^{\infty}|c_k| \cdot \frac{(k-1)k!}{2} r^{k-2} < +\infty,$$

$$M_{3,r}(f) = 2\alpha \sum_{k=0}^{\infty}|c_k|k^2 k!r^{k-1} < +\infty,\ M_{4,r}(f)$$

$$= \left(\frac{\beta^2}{2} + 2\beta\right)\sum_{k=0}^{\infty}|c_k|k^2(k+1)!r^k < +\infty,$$

$$M_{5,r}(f) = \alpha\beta \sum_{k=0}^{\infty}|c_k|k(k-1)r^{k-1} < +\infty,\ M_{6,r}(f)$$

$$= \beta^2 \sum_{k=0}^{\infty}|c_k|k(k-1)r^k < +\infty.$$

Proof. For all $z \in \mathbb{D}_R$, let us consider

$$V_n^{\alpha,\beta}(f,z) - f(z) - \frac{\alpha-\beta z}{n+\beta}f'(z) - \frac{z(1+z)}{2n}f''(z)$$

$$= V_n(f,z) - f(z) - \frac{z(1+z)}{2n}f''(z) + V_n^{\alpha,\beta}(f,z) - V_n(f,z) - \frac{\alpha-\beta z}{n+\beta}f'(z).$$

Taking $f(z) = \sum_{k=0}^{\infty} c_k z^k$, we get

$$V_n^{\alpha,\beta}(f,z) - f(z) - \frac{\alpha-\beta z}{n+\beta}f'(z) - \frac{z(1+z)}{2n}f''(z)$$

$$= \sum_{k=2}^{\infty} c_k \left(V_n(e_k,z) - z^k - \frac{z(1+z)}{2n}k(k-1)z^{k-2}\right)$$

$$+\sum_{k=2}^{\infty} c_k \left(V_n^{\alpha,\beta}(e_k, z) - V_n(e_k, z) - \frac{\alpha - \beta z}{n+\beta} k z^{k-1} \right).$$

To estimate the first sum, we use the Voronovskaja-type result for the Baskakov operators obtained in [77], Theorem 1.9.3, p. 130:

$$\left| V_n(f, z) - f(z) - \frac{z(1+z)}{2n} f''(z) \right| \le \frac{16}{n^2} \sum_{k=0}^{\infty} |c_k| (k-1)(k-2)^2 \, k! r^k.$$

Next, to estimate the second sum, by using Lemma 6.2, we obtain

$$\begin{aligned}
& V_n^{\alpha,\beta}(e_k, z) - V_n(e_k, z) - \frac{\alpha - \beta z}{n+\beta} k z^{k-1} \\
&= \sum_{j=0}^{k-1} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} V_n(e_j, z) + \left(\frac{n^k}{(n+\beta)^k} - 1 \right) V_n(e_k, z) - \frac{\alpha - \beta z}{n+\beta} k z^{k-1} \\
&= \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} V_n(e_j, z) + \frac{k n^{k-1} \alpha}{(n+\beta)^k} V_n(e_{k-1}, z) \\
&\quad - \sum_{j=0}^{k-1} \binom{k}{j} \frac{n^j \beta^{k-j}}{(n+\beta)^k} V_n(e_k, z) - \frac{\alpha - \beta z}{n+\beta} k z^{k-1} \\
&= \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} V_n(e_j, z) + \frac{k n^{k-1} \alpha}{(n+\beta)^k} [V_n(e_{k-1}, z) - z^{k-1}] \\
&\quad - \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \beta^{k-j}}{(n+\beta)^k} V_n(e_k, z) \\
&\quad - \frac{k n^{k-1} \beta}{(n+\beta)^k} [V_n(e_k, z) - z^k] + \left(\frac{n^{k-1}}{(n+\beta)^{k-1}} - 1 \right) \frac{k\alpha}{n+\beta} z^{k-1} \\
&\quad + \left(1 - \frac{n^{k-1}}{(n+\beta)^{k-1}} \right) \frac{k\beta}{n+\beta} z^k.
\end{aligned}$$

Now, if we use the inequalities

$$1 - \frac{n^k}{(n+\beta)^k} \le \sum_{j=1}^{k} \left(1 - \frac{n}{n+\beta} \right) = \frac{k\beta}{n+\beta},$$

$$\sum_{j=0}^{k-2}\binom{k-2}{j}\frac{n^j\alpha^{k-2-j}}{(n+\beta)^{k-2}}=\sum_{j=0}^{k-2}\binom{k-2}{j}\frac{n^j}{(n+\beta)^j}\cdot\frac{\alpha^{k-2-j}}{(n+\beta)^{k-2-j}}$$

$$=\left(\frac{n+\alpha}{n+\beta}\right)^{k-2}\leq 1,$$

$$|V_n(e_j,z)|\leq r^j(j+1)! \text{ (see Lemma 6.3 for } \alpha=\beta=0),$$

$$|V_n(e_k,z)-z^k|\leq\frac{r(1+r)k(k+1)!}{n}\leq\frac{2r^kk(k+1)!}{n}$$

(for this last inequality, take $\alpha=\beta=0$ in the proof of Theorem 6.1)

and

$$\left|\sum_{j=0}^{k-2}\binom{k}{j}\frac{n^j\alpha^{k-j}}{(n+\beta)^k}V_n(e_j,z)\right|\leq\sum_{j=0}^{k-2}\binom{k}{j}\frac{n^j\alpha^{k-j}}{(n+\beta)^k}|V_n(e_j,z)|$$

$$=\sum_{j=0}^{k-2}\frac{k(k-1)}{(k-j)(k-j-1)}\binom{k-2}{j}\frac{n^j\alpha^{k-j}}{(n+\beta)^k}|V_n(e_j,z)|$$

$$\leq\frac{k(k-1)}{2}\cdot\frac{\alpha^2}{(n+\beta)^2}(k-1)!r^{k-2}\sum_{j=0}^{k-2}\binom{k-2}{j}\frac{n^j\alpha^{k-2-j}}{(n+\beta)^(k-2)}$$

$$\leq\frac{k(k-1)}{2}\cdot\frac{\alpha^2}{(n+\beta)^2}(k-1)!r^{k-2},$$

it follows that

$$\left|V_n^{\alpha,\beta}(e_k,z)-V_n(e_k,z)-\frac{\alpha-\beta z}{n+\beta}kz^{k-1}\right|$$

$$\leq\left|\sum_{j=0}^{k-2}\binom{k}{j}\frac{n^j\alpha^{k-j}}{(n+\beta)^k}V_n(e_j,z)\right|+\frac{kn^{k-1}\alpha}{(n+\beta)^k}|V_n(e_{k-1},z)-z^{k-1}|$$

$$+\left|\sum_{j=0}^{k-2}\binom{k}{j}\frac{n^j\beta^{k-j}}{(n+\beta)^k}V_n(e_k,z)\right|+\frac{kn^{k-1}\beta}{(n+\beta)^k}|V_n(e_k,z)-z^k|$$

$$+\left|\left(\frac{n^{k-1}}{(n+\beta)^{k-1}}-1\right)\right|\frac{k\alpha}{n+\beta}|z|^{k-1}+\left|\left(1-\frac{n^{k-1}}{(n+\beta)^{k-1}}\right)\right|\frac{k\beta}{n+\beta}|z|^k$$

$$\leq\frac{(k-1)k!\alpha^2}{2(n+\beta)^2}r^{k-2}+\frac{kn^{k-1}\alpha}{(n+\beta)^k}\cdot\frac{2r^{k-1}(k-1)k!}{n}$$

$$+r^k(k+1)! \cdot \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \beta^{k-j}}{(n+\beta)^k} + \frac{kn^{k-1}\beta}{(n+\beta)^k} \cdot \frac{2r^k k(k+1)!}{n}$$

$$+\frac{k(k-1)\alpha\beta}{(n+\beta)^2} r^{k-1} + \frac{k(k-1)\beta^2}{(n+\beta)^2} r^k,$$

$$\leq \frac{(k-1)k!\alpha^2}{2(n+\beta)^2} r^{k-2} + \frac{2\alpha k^2 k!}{(n+\beta)^2} r^{k-1} + \frac{\beta^2 k^2 (k+1)!}{2(n+\beta)^2} r^k + \frac{2k^2(k+1)!\beta}{(n+\beta)^2} r^k$$

$$+\frac{k(k-1)\alpha\beta}{(n+\beta)^2} r^{k-1} + \frac{k(k-1)\beta^2}{(n+\beta)^2} r^k$$

$$\leq \frac{(k-1)k!\alpha^2}{2(n+\beta)^2} r^{k-2} + \frac{2\alpha k^2 k!}{(n+\beta)^2} r^{k-1} + \frac{k^2(k+1)!}{(n+\beta)^2} \left(\frac{\beta^2}{2} + 2\beta \right) r^k$$

$$+\frac{k(k-1)\alpha\beta}{(n+\beta)^2} r^{k-1} + \frac{k(k-1)\beta^2}{(n+\beta)^2} r^k,$$

which immediately proves the theorem. ■

Theorem 6.3 ([82] Exact order of approximation). *Suppose that the hypothesis on the function f and that on the constants n_0, R, M, A in the statement of Theorem 6.1 hold, and let $1 \leq r < \min\{\frac{n_0}{2}, \frac{1}{A}\}$ be fixed. Let $0 \leq \alpha \leq \beta$, and suppose that f is not a polynomial of degree ≤ 0. Then, for all $n > n_0$ and $|z| \leq r$, we have*

$$|V_n^{\alpha,\beta}(f,z) - f(z)| \geq \frac{C_r(f)}{n},$$

where the constant $C_r(f)$ depends only on f, α, β, and r.

Proof. For all $|z| \leq r$ and $n \in \mathbb{N}$, we can write

$$V_n^{\alpha,\beta}(f,z) - f(z) = \frac{1}{n} \Bigg[\frac{n}{n+\beta}(\alpha - \beta z) f'(z) + \frac{z(1+z)}{2} f''(z)$$

$$+\frac{1}{n} \cdot n^2 \left(V_n^{\alpha,\beta}(f,z) - f(z) - \frac{\alpha - \beta z}{n+\beta} f'(z) - \frac{z(1+z)}{2n} f''(z) \right) \Bigg]$$

$$= \frac{1}{n} \Bigg[(\alpha - \beta z) f'(z) + \frac{z(1+z)}{2} f''(z)$$

$$+\frac{1}{n} \cdot n^2 \Bigg(V_n^{\alpha,\beta}(f)(z) - f(z) - \frac{\alpha - \beta z}{n+\beta} f'(z)$$

$$-\frac{z(1+z)}{2n} f''(z) - \frac{\beta(\alpha - \beta z) f'(z)}{n(n+\beta)} \Bigg) \Bigg].$$

Applying the inequality

$$\|F+G\| \geq |\,\|F\|-\|G\|\,| \geq \|F\|-\|G\|,$$

we obtain

$$\|V_n^{\alpha,\beta}(f)-f\|_r \geq \frac{1}{n}\left[\left\|(\alpha-\beta e_1)f'+\frac{e_1(1+e_1)}{2}f''\right\|_r\right.$$
$$\left.-\frac{1}{n}\cdot n^2\left\|V_n^{\alpha,\beta}(f)-f-\frac{\alpha-\beta e_1}{n+\beta}-\frac{e_1(1+e_1)}{2n}f''-\frac{\beta(\alpha-\beta e_1)f'}{n(n+\beta)}\right\|_r\right].$$

Since f is not a polynomial of degree ≤ 0 in $\mathbb{D}_R$, we get $\left\|(\alpha-\beta e_1)f'+\frac{e_1(1+e_1)}{2}f''\right\|_r > 0$. Indeed, supposing the contrary gives us

$$(\alpha-\beta z)f'(z)+\frac{z(1+z)}{2}f''(z)=0, \quad \text{for all } z\in\overline{\mathbb{D}}_r.$$

Denoting $y(z)=f'(z)$, seeking $y(z)$ in the form $y(z)=\sum_{k=0}^{\infty}b_kz^k$, and replacing in the above differential equation, we easily get $b_k=0$ for all $k=0,1,\ldots$ (we can make similar reasonings here with those in Gal [77], pp. 75–76). Thus, we get that $f(z)$ is a constant function, which is contradiction.

Now, since by Theorem 6.2 it follows that

$$n^2\left\|V_n^{\alpha,\beta}(f)-f-\frac{\alpha-\beta e_1}{n+\beta}f'-\frac{e_1(1+e_1)}{2n}f''-\frac{\beta(\alpha-\beta e_1)f'}{n(n+\beta)}\right\|_r$$
$$\leq n^2\left\|V_n^{\alpha,\beta}(f)-f-\frac{\alpha-\beta e_1}{n+\beta}f'-\frac{e_1(1+e_1)}{2n}f''\right\|_r+\|\beta(\alpha-\beta e_1)f'\|_r$$
$$\leq \sum_{j=1}^{6}M_{j,r}(f)+\beta(\alpha+\beta r)\|f'\|_r,$$

there is an $n_1>n_0$ (depending on f and r only) such that for all $n\geq n_1$, we have

$$\left\|(\alpha-\beta e_1)f'+\frac{e_1(1+e_1)}{2}f''\right\|_r$$
$$-\frac{1}{n}\cdot n^2\left\|V_n^{\alpha,\beta}(f)-f-\frac{\alpha-\beta e_1}{n+\beta}f'-\frac{e_1(1+e_1)}{2n}f''-\frac{\beta(\alpha-\beta e_1)f'}{n(n+\beta)}\right\|_r$$
$$\geq \frac{1}{2}\left\|(\alpha-\beta e_1)f'+\frac{e_1(1+e_1)}{2}f''\right\|_r,$$

which implies that

$$\|V_n^{\alpha,\beta}(f)-f\|_r \geq \frac{1}{2n}\left\|(\alpha-\beta e_1)f'+\frac{e_1(1+e_1)}{2}f''\right\|_r$$

for all $n\geq n_1$.

For $n \in \{n_0+1, \ldots, n_1\}$, we get $\|V_n^{\alpha,\beta}(f) - f\|_r \ge \frac{1}{n} A_r(f)$ with $A_r(f) = n \cdot \|V_n^{\alpha,\beta}(f) - f\|_r > 0$, which implies that $\|V_n^{\alpha,\beta}(f) - f\|_r \ge \frac{C_r(f)}{n}$ for all $n \ge n_0$, with

$$C_r(f) = \min\left\{A_{r,n_0+1}(f), \ldots, A_{r,n_1}(f), \frac{1}{2}\left\|(\alpha - \beta e_1)f' + \frac{e_1(1+e_1)}{2} f''\right\|_r\right\},$$

which proves the theorem. ■

Theorem 6.4 ([82] Simultaneous approximation). *Suppose that the hypotheses on the function f and on the constants n_0, R, M, A in the statement of Theorem 6.1 hold, and let $1 \le r < \min\{\frac{n_0}{2}, \frac{1}{A}\}$ and $p \in \mathbb{N}$ be fixed. Let $0 \le \alpha \le \beta$, and suppose that f is not a polynomial of degree $\le p-1$. Then, for all $n > n_0$ and $|z| \le r$, we have*

$$\|[V_n^{\alpha,\beta}(f)]^{(p)} - f^{(p)}\|_r \sim \frac{1}{n},$$

where the constants $C_r(f)$ in the equivalence depend only on f, p, and r.

Proof. Denote by γ the circle or radius r_1 with $r < r_1 < \min\{n_0/2, 1/A\}$, and center 0. Since for $|z| \le r$ and $v \in \gamma$, we have $|v - z| \ge r1 - r$, by Cauchy's formula, for all $|z| \le r$ and $n > n_0$, we obtain

$$\begin{aligned}|[V_n^{\alpha,\beta}(f,z)]^{(p)} - f^{(p)}(z)| &= \frac{p!}{2\pi} \cdot \left|\int_\gamma \frac{V_n^{\alpha,\beta}(f,v) - f(v)}{(v-z)^{p+1}} dv\right| \\ &\le \frac{C_{r,r_1,\alpha,\beta}}{n} \cdot \frac{p!}{2\pi} \cdot \frac{2\pi r_1}{(r_1 - r)^{p+1}},\end{aligned}$$

which proves one of the inequalities in the equivalence.

To prove the converse inequality in the equivalence, we start from the relationship for $V_n^{\alpha,\beta}(f,v) - f(v)$ (in the proof of Theorem 6.3, with v instead of z there), replaced in Cauchy's formula

$$[V_n^{\alpha,\beta}(f,z)]^{(p)} - f^{(p)}(z) = \frac{p!}{2\pi i} \cdot \int_\gamma \frac{V_n^{\alpha,\beta}(f,v) - f(v)}{(v-z)^{p+1}} dv.$$

By standard reasonings such as those for the case of a classical complex Baskakov operator (see the proof of Theorem 1.9.6, p. 134 [77]), combined with those for the Bernstein–Stancu polynomials (see Gal [77], pp. 76–78, the proof of Theorem 1.6.5), the present proof finally reduces to the proof of the fact that $\|[(\alpha - \beta z)f'(z) + \frac{z(1+z)}{2} f''(z)]^{(p)}\|_r > 0$. But this can be shown by precisely following the steps in Gal [77], pp. 77–78 [where it is proved that $\|[(\alpha - \beta z)f'(z) + \frac{z(1-z)}{2} f''(z)]^{(p)}\|_r > 0$]. Because the methods are standard, we omit the details here. ■

6.2 Complex Favard–Szász–Mirakjan–Stancu Operators

The Favard–Szász–Mirakjan operators are important and have been studied intensively in connection with different branches of analysis, such as numerical analysis, approximation theory, and statistics. For a real function f of real variable $f : [0,\infty) \to \mathbb{R}$, the Favard–Szász–Mirakjan operators are defined as follows:

$$S_n(f,x) = e^{-nx} \sum_{\nu=0}^{\infty} \frac{(nz)^\nu}{\nu!} f\left(\frac{\nu}{n}\right),\ \ x \in [0,\infty),$$

where the convergence of $S_n(f,x) \to f(x)$ under the exponential growth condition on f, that is, $|f(x)| \le Ce^{Bx}$ for all $x \in [0,+\infty)$, with $C,\ B > 0$, was established in [66]. The actual construction of the Szász–Mirakyan operators and its various modifications require estimations of infinite series, which, in a certain sense, restrict their usefulness from the computational point of view. We deal with the following complex form for the Favard–Szász–Mirakjan–Stancu operators studied in [130], which are defined as

$$S_n^{\alpha,\beta}(f,z) = \sum_{\nu=0}^{\infty} \left[\frac{\alpha}{n+\beta}, \frac{\alpha+1}{n+\beta}, \dots, \frac{\alpha+\nu}{n+\beta}; f\right] z^\nu,$$

which, for $\alpha = 0 = \beta$, was studied in [77]. Here, $[x_0, x_1, \dots, x_m; f]$ denotes the divided difference of the function f on the knots $x_0, x_1, \dots, x_m$. Such a formula was first established by Lupas [184] for the special case $\alpha = \beta = 0$ and for functions of real variables. This formula holds for a complex setting too, since only algebraic calculations were used in [184]. To prove the main theorems, we need the following lemmas:

Lemma 6.4 ([130]). *For all $n,k \in \mathbb{N} \cup \{0\},\ 0 \le \alpha \le \beta,\ z \in \mathbb{C}$, let us define*

$$S_n^{\alpha,\beta}(e_k,z) = \sum_{\nu=0}^{\infty} \left[\frac{\alpha}{n+\beta}, \frac{\alpha+1}{n+\beta}, \dots, \frac{\alpha+\nu}{n+\beta}; e_k\right] z^\nu,$$

where $e_k(z) = z^k$. Then $S_n^{\alpha,\beta}(e_0,z) = 1$, and we have the following recurrence relation:

$$S_n^{\alpha,\beta}(e_{k+1},z) = \frac{z}{n+\beta}(S_n^{\alpha,\beta}(e_k,z))' + \frac{nz+\alpha}{n+\beta} S_n^{\alpha,\beta}(e_k,z). \tag{6.2}$$

Consequently,

$$S_n^{\alpha,\beta}(e_1,z) = \frac{nz+\alpha}{n+\beta},\quad S_n^{\alpha,\beta}(e_2,z) = \frac{n^2z^2}{(n+\beta)^2} + \frac{nz(1+2\alpha)}{(n+\beta)^2} + \frac{\alpha^2}{(n+\beta)^2}.$$

Lemma 6.5 ([130]). *Let α, β satisfy $0 \le \alpha \le \beta$. Denoting $e_j(z) = z^j$ and $S_n^{0,0}(e_j)$ by $S_n(e_j)$ for all $n, k \in \mathbb{N} \cup \{0\}$, we have the following recursive relation for the images of the monomials e_k under $S_n^{\alpha,\beta}$ in terms of $S_n(e_j)$, $j = 0, 1, 2 \ldots, k$:*

$$S_n^{\alpha,\beta}(e_k, z) = \sum_{j=0}^{k} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} S_n(e_j, z).$$

Lemma 6.6 ([130]). *For all $n, k \in \mathbb{N} \cup \{0\}$, $0 \le \alpha \le \beta$ and $|z| \le r$, $r \ge 1$, we have*

$$|S_n^{\alpha,\beta}(e_k, z)| \le (2r)^k.$$

The main results discussed in [130] follow.

Theorem 6.5 ([130] Upper estimate). *For $2 < R < +\infty$, let $f : [R, +\infty) \cup \overline{\mathbb{D}}_R \to \mathbb{C}$ be bounded on $[0, +\infty)$ and analytic in $\mathbb{D}_R$; that is, $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$.*

(a) Suppose that $0 \le \alpha \le \beta$ and $1 \le r < \frac{R}{2}$ are arbitrarily fixed. Then, for all $|z| \le r$ and $n \in \mathbb{N}$, we have

$$|S_n^{\alpha,\beta}(f, z) - f(z)| \le \frac{\alpha + \beta r}{n+\beta} \sum_{k=1}^{\infty} |c_k| r^{k-1} + \frac{A_r(f)}{n+\beta} + \frac{\alpha B_r(f)}{n+\beta} + \frac{\beta C_r(f)}{n+\beta},$$

where $\sum_{k=1}^{\infty} |c_k| r^{k-1} < +\infty$, $B_r(f) = \sum_{k=1}^{\infty} |c_k| k r^{k-1} < +\infty$, $C_r(f) = \sum_{k=1}^{\infty} |c_k| k r^k < +\infty$ and $A_r(f) = 2\sum_{k=1}^{\infty} |c_k|(k-1)(2r)^{k-1} < +\infty$.

(b) Suppose that $0 \le \alpha \le \beta$ and $1 \le r < r_1 < \frac{R}{2}$. Then for all $|z| \le r$ and $n \in \mathbb{N}$, we have

$$|[S_n^{\alpha,\beta}(f, z)]^{(p)} - f^{(p)}(z)| \le \frac{p! r_1}{(r_1 - r)^{p+1}} \cdot \frac{M_{r_1}(f)}{n+\beta},$$

where $M_{r_1}(f) = (\alpha + \beta r_1) \sum_{k=1}^{\infty} |c_k| \cdot r_1^{k-1} + A_{r_1}(f) + B_{r_1}(f) + C_{r_1}(f)$.

Proof. (a) Using the recurrence (6.2) in Lemma 6.4, we get

$$\begin{aligned} S_n^{\alpha,\beta}(e_k, z) - z^k = \; & \frac{z}{n+\beta} (S_n^{\alpha,\beta}(e_{k-1}, z))' + \frac{nz + \alpha}{n+\beta} \left(S_n^{\alpha,\beta}(e_{k-1}, z) - z^{k-1} \right) \\ & + \frac{nz + \alpha}{n+\beta} z^{k-1} - z^k \end{aligned}$$

and

$$|S_n^{\alpha,\beta}(e_k,z)-z^k| \le \frac{|z|}{n+\beta}|(S_n^{\alpha,\beta}(e_{k-1},z))'| + \frac{n|z|+\alpha}{n+\beta}\left|S_n^{\alpha,\beta}(e_{k-1},z)-z^{k-1}\right| + \frac{\alpha}{n+\beta}|z|^{k-1} + \frac{\beta}{n+\beta}|z|^k.$$

Clearly, $S_n^{\alpha,\beta}(e_0,z)-e_0=0$ and

$$|S_n^{\alpha,\beta}(e_1,z)-e_1(z)| = \left|\frac{nz+\beta}{n+\beta}-z\right| = \left|\frac{\alpha-\beta z}{n+\beta}\right| \le \frac{\alpha+\beta r}{n+\beta}.$$

Using Lemma 6.6 and Bernstein's inequality for the polynomial $S_n^{\alpha,\beta}(e_{k-1},z)$ of degree $\le k-1$, we have

$$\begin{aligned}|(S_n^{\alpha,\beta}(e_{k-1},z))'| &\le \frac{k-1}{r}\max\{|S_n^{\alpha,\beta}(e_{k-1},z)| : |z|\le r\}\\ &\le \frac{k-1}{r}(2r)^{k-1} = 2(k-1)(2r)^{k-2}.\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}&|S_n^{\alpha,\beta}(e_k,z)-z^k|\\ &\le \frac{k-1}{n+\beta}(2r)^{k-1} + \frac{nr+\alpha}{n+\beta}\left|S_n^{\alpha,\beta}(e_{k-1},z)-z^{k-1}\right| + \frac{\alpha}{n+\beta}r^{k-1} + \frac{\beta}{n+\beta}r^k\\ &\le r\left|S_n^{\alpha,\beta}(e_{k-1},z)-z^{k-1}\right| + \frac{k}{n+\beta}(2r)^{k-1} + \frac{\alpha}{n+\beta}r^{k-1} + \frac{\beta}{n+\beta}r^k.\end{aligned}$$

Setting $k=2$ in the preceding equation, we obtain

$$|S_n^{\alpha,\beta}(e_2,z)-e_2(z)| \le r\cdot\frac{\alpha+\beta r}{n+\beta} + \frac{1}{n+\beta}(2r) + \frac{\alpha}{n+\beta}r^1 + \frac{\beta}{n+\beta}r^2.$$

Then, for $k=3$, it follows that

$$\begin{aligned}|S_n^{\alpha,\beta}(e_3,z)-e_3(z)| &\le r^2\cdot\frac{\alpha+\beta r}{n+\beta} + \frac{1\cdot 2^1}{n+\beta}r^2 + \frac{\alpha}{n+\beta}r^2 + \frac{\beta}{n+\beta}r^3\\ &\quad + \frac{2\cdot 2^2}{n+\beta}r^2 + \frac{\alpha}{n+\beta}r^2 + \frac{\beta}{n+\beta}r^3\\ &= r^2\cdot\frac{\alpha+\beta r}{n+\beta} + \frac{1}{n+\beta}[(1\cdot 2^1)+(2\cdot 2^2)]r^2\\ &\quad + \frac{2\alpha}{n+\beta}r^2 + \frac{2\beta}{n+\beta}r^3.\end{aligned}$$

Reasoning by recurrence for any $k \geq 2$, we finally get

$$|S_n^{\alpha,\beta}(e_k, z) - e_k(z)| \leq r^{k-1} \cdot \frac{\alpha + \beta r}{n+\beta} + \frac{1}{n+\beta} \cdot \left[\sum_{j=1}^{k-1} j \cdot 2^j\right] \cdot r^{k-1}$$
$$+ \frac{(k-1)\alpha}{n+\beta} r^{k-1} + \frac{(k-1)\beta}{n+\beta} r^k$$
$$\leq r^{k-1} \cdot \frac{\alpha + \beta r}{n+\beta} + \frac{2(k-1)}{n+\beta} \cdot (2r)^{k-1}$$
$$+ \frac{k\alpha}{n+\beta} r^{k-1} + \frac{k\beta}{n+\beta} r^k.$$

Since the formula $\sum_{j=1}^{k-1} j 2^j = (k-2)2^k + 2$ can easily be proved by mathematical induction, this inequality is clearly valid for $k = 1$ too.

Now, reasoning exactly as in the case of complex Favard–Szász–Mirakjan operators in Remark 2 in [77], we can write

$$S_n^{\alpha,\beta}(f, z) = \sum_{k=0}^{\infty} c_k S_n^{\alpha,\beta}(e_k, z),$$

which implies

$$|S_n^{\alpha,\beta}(f, z) - f(z)| \leq \sum_{k=1}^{\infty} |c_k| \cdot |S_n^{\alpha,\beta}(e_k, z) - z^k|$$
$$\leq \frac{\alpha + \beta r}{n+\beta} \sum_{k=1}^{\infty} |c_k| \cdot r^{k-1} + \frac{2}{n+\beta} \sum_{k=1}^{\infty} |c_k|(k-1) \cdot (2r)^{k-1}$$
$$+ \frac{\alpha}{n+\beta} \sum_{k=1}^{\infty} |c_k| k r^{k-1} + \frac{\beta}{n+\beta} \sum_{k=1}^{\infty} |c_k| k r^k$$
$$= \frac{\alpha + \beta r}{n+\beta} \sum_{k=1}^{\infty} |c_k| \cdot r^{k-1} + \frac{A_r(f)}{n+\beta} + \frac{\alpha B_r(f)}{n+\beta} + \frac{\beta C_r(f)}{n+\beta}.$$

Note here that by the analyticity of f, we clearly get $\sum_{k=1}^{\infty} |c_k| r^{k-1} < +\infty$, $B_r(f) < +\infty$, $C_r(f) < +\infty$, and $A_r(f) = 2\sum_{k=1}^{\infty} |c_k| \cdot (k-1) \cdot (2r)^{k-1} < +\infty$, which proves (a).

(b) Denote by γ the circle of radius $r_1 > r$ with center 0. For any $|z| \leq r$ and $v \in \gamma$, we have $|v - z| \geq r_1 - r$. By Cauchy's formula for all $|z| \leq r$, it follows that

$$|[S_n^{\alpha,\beta}(f,z)]^{(p)} - f^{(p)}(z)| = \frac{p!}{2\pi}\left|\int_\gamma \frac{S_n^{\alpha,\beta}(f,z) - f(z)}{(v-z)^{p+1}}\right|$$

$$\leq \frac{p!r_1}{(r_1-r)^{p+1}} \cdot \left[\frac{\alpha+\beta r_1}{n+\beta}\sum_{k=1}^{\infty}|c_k| \cdot r_1^{k-1} + \frac{A_{r_1}(f)}{n+\beta} + \frac{\alpha B_{r_1}(f)}{n+\beta} + \frac{\beta C_{r_1}(f)}{n+\beta}\right],$$

which proves (b) and the theorem. ■

Theorem 6.6 ([130] Asymptotic formula). *For* $2 < R < +\infty$, *let* $f : [R,+\infty) \cup \overline{\mathbb{D}}_R \to \mathbb{C}$ *be bounded on* $[0,+\infty)$ *and analytic in* $\mathbb{D}_R$*; that is,* $f(z) = \sum_{k=0}^{\infty} c_k z^k$, *for all* $z \in \mathbb{D}_R$. *Also, let* $1 \leq r < \frac{R}{2}$ *and* $0 \leq \alpha \leq \beta$. *Then, for all* $|z| \leq r$ *and* $n \in \mathbb{N}$, *we have the following Voronovskaja-type result:*

$$\left|S_n^{\alpha,\beta}(f,z) - f(z) - \frac{\alpha-\beta z}{n+\beta}f'(z) - \frac{z}{2n}f''(z)\right| \leq \frac{M_{1,r}(f)}{n^2} + \frac{\sum_{j=2}^{6} M_{j,r}(f)}{(n+\beta)^2},$$

where

$$M_{1,r}(f) = 26\sum_{k=3}^{\infty}|c_k|(k-1)^2(k-2)(2r)^{k-2} < +\infty,$$

$$M_{2,r}(f) = \left(\frac{\alpha^2}{2} + 2\alpha\right) \cdot \sum_{k=2}^{\infty}|c_k| \cdot k(k-1)(2r)^{k-2} < +\infty,$$

$$M_{3,r}(f) = \frac{\beta^2}{2}\sum_{k=2}^{\infty}|c_k|k(k-1)(2r)^{k} < +\infty,$$

$$M_{4,r}(f) = \beta\sum_{k=2}^{\infty}|c_k|k(k-1)(2r)^{k-1} < +\infty,$$

$$M_{5,r}(f) = \alpha\beta\sum_{k=0}^{\infty}|c_k|k(k-1)r^{k-1} < +\infty,$$

$$M_{6,r}(f) = \beta^2\sum_{k=0}^{\infty}|c_k|k(k-1)r^{k} < +\infty.$$

Proof. For all $z \in \mathbb{D}_R$, let us consider

$$S_n^{\alpha,\beta}(f,z) - f(z) - \frac{\alpha-\beta z}{n+\beta}f'(z) - \frac{z}{2n}f''(z)$$

$$= S_n(f,z) - f(z) - \frac{z}{2n}f''(z) + S_n^{\alpha,\beta}(f,z) - S_n(f,z) - \frac{\alpha-\beta z}{n+\beta}f'(z).$$

Taking $f(z) = \sum_{k=0}^{\infty} c_k z^k$, we get

$$\begin{aligned}
&S_n^{\alpha,\beta}(f,z) - f(z) - \frac{\alpha-\beta z}{n+\beta} f'(z) - \frac{z}{2n} f''(z) \\
&= \sum_{k=2}^{\infty} c_k \left(S_n(e_k,z) - z^k - \frac{z}{2n} k(k-1) z^{k-2}\right) \\
&\quad + \sum_{k=2}^{\infty} c_k \left(S_n^{\alpha,\beta}(e_k,z) - S_n(e_k,z) - \frac{\alpha-\beta z}{n+\beta} k z^{k-1}\right).
\end{aligned}$$

To estimate the first sum, we use the Voronovskaja-type result for the Favard–Szász–Mirakjan operators obtained in [77], Theorem 1.8.5:

$$\left|S_n(f,z) - f(z) - \frac{z}{2n} f''(z)\right| \le \frac{26}{n^2} \sum_{k=3}^{\infty} |c_k|(k-1)^2(k-2)(2r)^{k-2}.$$

Next, to estimate the second sum, using Lemma 6.5, we obtain

$$\begin{aligned}
&S_n^{\alpha,\beta}(e_k,z) - S_n(e_k,z) - \frac{\alpha-\beta z}{n+\beta} k z^{k-1} \\
&= \sum_{j=0}^{k-1} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} S_n(e_j,z) + \left(\frac{n^k}{(n+\beta)^k} - 1\right) S_n(e_k,z) - \frac{\alpha-\beta z}{n+\beta} k z^{k-1} \\
&= \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} S_n(e_j,z) + \frac{k n^{k-1}\alpha}{(n+\beta)^k} S_n(e_{k-1},z) \\
&\quad - \sum_{j=0}^{k-1} \binom{k}{j} \frac{n^j \beta^{k-j}}{(n+\beta)^k} S_n(e_k,z) - \frac{\alpha-\beta z}{n+\beta} k z^{k-1} \\
&= \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} S_n(e_j,z) + \frac{k n^{k-1}\alpha}{(n+\beta)^k} [S_n(e_{k-1},z) - z^{k-1}] \\
&\quad - \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \beta^{k-j}}{(n+\beta)^k} S_n(e_k,z) - \frac{k n^{k-1}\beta}{(n+\beta)^k} [S_n(e_k,z) - z^k] \\
&\quad + \left(\frac{n^{k-1}}{(n+\beta)^{k-1}} - 1\right) \frac{k\alpha}{n+\beta} z^{k-1} + \left(1 - \frac{n^{k-1}}{(n+\beta)^{k-1}}\right) \frac{k\beta}{n+\beta} z^k.
\end{aligned}$$

Now, using the inequalities

$$1 - \frac{n^k}{(n+\beta)^k} \le \sum_{j=1}^{k} \left(1 - \frac{n}{n+\beta}\right) = \frac{k\beta}{n+\beta},$$

$$\sum_{j=0}^{k-2} \binom{k-2}{j} \frac{n^j \alpha^{k-2-j}}{(n+\beta)^{k-2}} = \sum_{j=0}^{k-2} \binom{k-2}{j} \frac{n^j}{(n+\beta)^j} \cdot \frac{\alpha^{k-2-j}}{(n+\beta)^{k-2-j}}$$

$$= \left(\frac{n+\alpha}{n+\beta}\right)^{k-2} \le 1,$$

$|S_n(e_j, z)| \le (2r)^k$ (see [77] for $\alpha = \beta = 0$),

$$|S_n(e_k, z) - z^k| \le \frac{k-1}{n} \cdot (2r)^{k-1},$$

(for this last inequality, take $\alpha = \beta = 0$ in the proof of Theorem 6.5)

and

$$\left| \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} S_n(e_j, z) \right|$$

$$\le \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} |S_n(e_j, z)|$$

$$= \sum_{j=0}^{k-2} \frac{k(k-1)}{(k-j)(k-j-1)} \binom{k-2}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} |S_n(e_j, z)|$$

$$\le \frac{k(k-1)}{2} \cdot \frac{\alpha^2}{(n+\beta)^2} (2r)^{k-2} \sum_{j=0}^{k-2} \binom{k-2}{j} \frac{n^j \alpha^{k-2-j}}{(n+\beta)^{(k-2)}}$$

$$\le \frac{k(k-1)}{2} \cdot \frac{\alpha^2}{(n+\beta)^2} (2r)^{k-2},$$

it follows that

$$\left| S_n^{\alpha,\beta}(e_k, z) - S_n(e_k, z) - \frac{\alpha - \beta z}{n+\beta} k z^{k-1} \right|$$

$$\le \left| \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} S_n(e_j, z) \right| + \frac{k n^{k-1} \alpha}{(n+\beta)^k} |S_n(e_{k-1}, z) - z^{k-1}|$$

$$+\left|\sum_{j=0}^{k-2}\binom{k}{j}\frac{n^j\beta^{k-j}}{(n+\beta)^k}S_n(e_k,z)\right|$$
$$+\frac{kn^{k-1}\beta}{(n+\beta)^k}|S_n(e_k,z)-z^k|+\left|\left(\frac{n^{k-1}}{(n+\beta)^{k-1}}-1\right)\right|\frac{k\alpha}{n+\beta}|z|^{k-1}$$
$$+\left|\left(1-\frac{n^{k-1}}{(n+\beta)^{k-1}}\right)\right|\frac{k\beta}{n+\beta}|z|^k$$
$$\leq\frac{k(k-1)\alpha^2}{2(n+\beta)^2}(2r)^{k-2}+\frac{kn^{k-1}\alpha}{(n+\beta)^k}\cdot\frac{(2r)^{k-2}(k-2)}{n}$$
$$+(2r)^k\cdot\sum_{j=0}^{k-2}\binom{k}{j}\frac{n^j\beta^{k-j}}{(n+\beta)^k}$$
$$+\frac{kn^{k-1}\beta}{(n+\beta)^k}\cdot\frac{(2r)^{k-1}(k-1)}{n}+\frac{k(k-1)\alpha\beta}{(n+\beta)^2}r^{k-1}+\frac{k(k-1)\beta^2}{(n+\beta)^2}r^k,$$
$$\leq\frac{k(k-1)\alpha^2}{2(n+\beta)^2}(2r)^{k-2}+\frac{k(k-2)\alpha}{(n+\beta)^2}\cdot(2r)^{k-2}+\frac{\beta^2k(k-1)}{2(n+\beta)^2}(2r)^k$$
$$+\frac{k(k-1)\beta}{(n+\beta)^2}(2r)^{k-1}$$
$$+\frac{k(k-1)\alpha\beta}{(n+\beta)^2}r^{k-1}+\frac{k(k-1)\beta^2}{(n+\beta)^2}r^k$$
$$\leq\frac{k(k-1)}{(n+\beta)^2}\left(\frac{\alpha^2}{2}+2\alpha\right)(2r)^{k-2}+\frac{\beta^2k(k-1)}{2(n+\beta)^2}(2r)^k+\frac{k(k-1)\beta}{(n+\beta)^2}(2r)^{k-1}$$
$$+\frac{k(k-1)\alpha\beta}{(n+\beta)^2}r^{k-1}+\frac{k(k-1)\beta^2}{(n+\beta)^2}r^k,$$

which immediately proves the theorem. ■

Theorem 6.7 ([130] Exact order). *For $2 < R < +\infty$, $1 \leq r < \frac{R}{2}$, let $f : [R,+\infty)\cup\overline{\mathbb{D}}_R \to \mathbb{C}$ be bounded on $[0,+\infty)$ and analytic in $\mathbb{D}_R$; that is, $f(z)=\sum_{k=0}^{\infty}c_kz^k$, for all $z\in\mathbb{D}_R$.*

Let $0<\alpha\leq\beta$ and suppose that f is not a polynomial of degree ≤ 0. Then, for all $n\in\mathbb{N}$ and $|z|\leq r$, we have

$$|S_n^{\alpha,\beta}(f,z)-f(z)|\geq\frac{C_r(f)}{n},$$

where the constant $C_r(f)$ depends only on f, α, β, and r.

Proof. For all $|z| \le r$ and $n \in \mathbb{N}$, we can write

$$
\begin{aligned}
&S_n^{\alpha,\beta}(f,z) - f(z)\\
&= \frac{1}{n}\Bigg[\frac{n}{n+\beta}(\alpha - \beta z) f'(z) + \frac{z}{2} f''(z)\\
&\quad + \frac{1}{n} \cdot n^2 \left(S_n^{\alpha,\beta}(f,z) - f(z) - \frac{\alpha - \beta z}{n+\beta} f'(z) - \frac{z}{2n} f''(z) \right)\Bigg]\\
&= \frac{1}{n}\Bigg[(\alpha - \beta z) f'(z) + \frac{z}{2} f''(z)\\
&\quad + \frac{1}{n} \cdot n^2 \left(S_n^{\alpha,\beta}(f)(z) - f(z) - \frac{\alpha - \beta z}{n+\beta} f'(z) - \frac{z}{2n} f''(z) - \frac{\beta(\alpha - \beta z) f'(z)}{n(n+\beta)} \right)\Bigg].
\end{aligned}
$$

Applying the inequality

$$\|F + G\| \ge |\, \|F\| - \|G\| \,| \ge \|F\| - \|G\|,$$

we obtain

$$
\begin{aligned}
\|S_n^{\alpha,\beta}(f) - f\|_r \ge \frac{1}{n} \Bigg[\left\|(\alpha - \beta e_1) f' + \frac{e_1}{2} f''\right\|_r\\
- \frac{1}{n} \cdot n^2 \left\| S_n^{\alpha,\beta}(f) - f - \frac{\alpha - \beta e_1}{n+\beta} - \frac{e_1}{2n} f'' - \frac{\beta(\alpha - \beta e_1) f'}{n(n+\beta)} \right\|_r \Bigg].
\end{aligned}
$$

Since f is not a polynomial of degree ≤ 0 in $\mathbb{D}_R$, we get $\left\|(\alpha - \beta e_1) f' + \frac{e_1}{2} f''\right\|_r > 0$. Indeed, supposing the contrary, we see that

$$(\alpha - \beta z) f'(z) + \frac{z}{2} f''(z) = 0, \text{ for all } z \in \overline{\mathbb{D}}_r.$$

Denoting $y(z) = f'(z)$, seeking $y(z)$ in the form $y(z) = \sum_{k=0}^{\infty} b_k z^k$, and replacing in the above differential equation, we easily get $b_k = 0$ for all $k = 0, 1, \ldots$ (we can make similar reasonings here with those in [77]). Thus, we get that $f(z)$ is a constant function, which is a contradiction.

Now, since by Theorem 6.6 it follows that

$$
\begin{aligned}
&n^2 \left\| S_n^{\alpha,\beta}(f) - f - \frac{\alpha - \beta e_1}{n+\beta} f' - \frac{e_1}{2n} f'' - \frac{\beta(\alpha - \beta e_1) f'}{n(n+\beta)} \right\|_r\\
&\le n^2 \left\| S_n^{\alpha,\beta}(f) - f - \frac{\alpha - \beta e_1}{n+\beta} f' - \frac{e_1}{2n} f'' \right\|_r + \|\beta(\alpha - \beta e_1) f'\|_r\\
&\le \sum_{j=1}^{6} M_{j,r}(f) + \beta(\alpha + \beta r)\|f'\|_r,
\end{aligned}
$$

there is an $n_1 > n_0$ (depending on f, α, β, and r only) such that for all $n \geq n_1$, we have

$$\begin{aligned}&\left\|(\alpha-\beta e_1)f' + \frac{e_1}{2}f''\right\|_r \\ &-\frac{1}{n}\cdot n^2\left\|S_n^{\alpha,\beta}(f) - f - \frac{\alpha-\beta e_1}{n+\beta}f' - \frac{e_1}{2n}f'' - \frac{\beta(\alpha-\beta e_1)f'}{n(n+\beta)}\right\|_r \\ &\geq \frac{1}{2}\left\|(\alpha-\beta e_1)f' + \frac{e_1}{2}f''\right\|_r,\end{aligned}$$

which implies that

$$\|S_n^{\alpha,\beta}(f) - f\|_r \geq \frac{1}{2n}\left\|(\alpha-\beta e_1)f' + \frac{e_1}{2}f''\right\|_r$$

for all $n \geq n_1$.

For $n \in \{n_0+1, \ldots, n_1\}$, we get $\|S_n^{\alpha,\beta}(f) - f\|_r \geq \frac{1}{n}A_r(f)$ with $A_r(f) = n \cdot \|S_n^{\alpha,\beta}(f) - f\|_r > 0$, which implies that $\|S_n^{\alpha,\beta}(f) - f\|_r \geq \frac{C_r(f)}{n}$ for all $n \geq n_0$, with

$$C_r(f) = \min\left\{A_{r,n_0+1}(f), \ldots, A_{r,n_1}(f), \frac{1}{2}\left\|(\alpha-\beta e_1)f' + \frac{e_1}{2}f''\right\|_r\right\},$$

which proves the theorem. ■

6.3 Complex Beta Operators of the Second Kind

The complex Beta operator of the first kind was first introduced in the case of real variables in Mühlbach [196] and later studied by Lupas [185], Khan [171], and Abel–Gupta–Mohapatra [11], among others. The complex Beta operators of the first kind (see [80]) are defined for all $n \in \mathbb{N}$ and $0 < \mathrm{Re}(z) < 1$ by

$$F_n(f,z) = \frac{1}{B(nz, n(1-z))}\int_0^1 t^{nz-1}(1-t)^{n(1-z)-1}f(t)dt,\ z \in \mathbb{C}, 0 < \mathrm{Re}(z) < 1, \tag{6.3}$$

where $B(\alpha, \beta)$ is Euler's Beta function, defined by

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}, \alpha, \beta \in \mathbb{C},\ \mathrm{Re}(\alpha),\ \mathrm{Re}(\beta) > 0.$$

The results discussed in [80] provided evidence of the overconvergence phenomenon for this complex Beta operator of the first kind, that is, the extensions of approximation properties with upper and exact quantitative estimates, from the real interval (0, 1) to strips in compact disks of the complex plane of the forms

$$SD^r(0,1) = \{z \in \mathbb{C}; |z| \leq r,\ 0 < \mathrm{Re}(z) < 1\}$$

and

$$SD^r[a,b] = \{z \in \mathbb{C}; |z| \leq r,\ a \leq \mathrm{Re}(z) \leq b\},$$

with $r \geq 1$ and $0 < a < b < 1$. For the Beta operators of the first kind, the upper estimate, exact order, and simultaneous approximation were considered in [80]. Also, the following asymptotic formula was established.

Theorem 6.8 ([80] Voronovskaja-type result). *Let $R > 1$, and suppose that $f : D_R \to \mathbb{C}$ is analytic in $D_R = \{z \in \mathbb{C} : |z| < R\}$; that is, we can write $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in D_R$. For any fixed $r \in [1, R)$ and for all $|z| \leq r$ with $0 < \mathrm{Re}(z) < 1$ and $n \in \mathbb{N}$, we have*

$$\left| F_n(f,z) - f(z) - \frac{z(1-z)f''(z)}{2n} \right| \leq \frac{M_r(f)}{n^2},$$

where $M_r(f) = \sum_{k=2}^{\infty} |c_k|(1+r)k(k+1)(k-1)^2 r^{k-1} < \infty$.

Very recently, Gal and Gupta [81] studied the complex Beta operators of the second kind. The complex Stancu Beta operators of the second kind will be defined for all $n \in \mathbb{N}$ and $z \in \mathbb{C}$ satisfying $0 < \mathrm{Re}(z)$ by

$$K_n(f,z) = \frac{1}{B(nz, n+1)} \int_0^{\infty} \frac{t^{nz-1}}{(1+t)^{nz+n+1}} f(t)dt, \tag{6.4}$$

where $B(\alpha, \beta)$ is Euler's Beta function, defined by

$$B(\alpha,\beta) = \int_0^{\infty} \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}},\ \alpha, \beta \in \mathbb{C},\ \mathrm{Re}(\alpha),\ \mathrm{Re}(\beta) > 0,$$

and f is supposed to be locally integrable and of polynomial growth on $(0, +\infty)$ as $t \to \infty$. This last hypothesis on f ensures the existence of $K_n(f; z)$ for sufficiently large n; that is, there exists n_0 depending on f such that $K_n(f; z)$ exists for all $n \geq n_0$ and $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$.

Note that because of the well-known formulas $B(\alpha,\beta) = \frac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ and $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, where Γ denotes Euler's Gamma function, for all $z \in \mathbb{N}$ with $\mathrm{Re}(z) > 0$ and sufficiently large n, we can easily can deduce the form

$$K_n(f,z) = \frac{nz(nz+1)\dots(nz+n)}{n!} \cdot \int_0^{\infty} \frac{t^{nz-1}}{(1+t)^{nz+n+1}} f(t)dt,\ \mathrm{Re}(z) > 0. \tag{6.5}$$

The results in this section will show the overconvergence phenomenon for this complex Stancu Beta integral operator of the second kind, that is, the extensions of approximation properties with upper and exact quantitative estimates, from the real interval $(0, r]$ to semidisks of the right half-plane of the form

$$SD^r(0, r] = \{z \in \mathbb{C}; |z| \le r, \ 0 < \text{Re}(z) \le r\}$$

and to subsets of semidisks of the form

$$SD^r[a, r] = \{z \in \mathbb{C}; |z| \le r, \ a \le \text{Re}(z) \le r\},$$

with $r \ge 1$ and $0 < a < r$.

Lemma 6.7 ([81]). *For all $e_p = t^p$, $p \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$, $z \in \mathbb{C}$ with $0 < \text{Re}(z)$, we have $K_n(e_0, z) = 1$, $K_n(e_1)(z) = e_1(z)$, and*

$$K_n(e_{p+1}, z) = \frac{nz + p}{n - p} K_n(e_p, z), \ \textit{for all } n > p.$$

Here $e_k(z) = z^k$.

Theorem 6.9 ([81] Upper estimate). *Let $D_R = \{z \in \mathbb{C}; |z| < R\}$, with $1 < R < \infty$, and suppose that $f : [R, \infty) \bigcup \overline{D}_R \to C$ is continuous in $[R, \infty) \bigcup \overline{D}_R$, analytic in D_R, that is, $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in D_R$, and $f(t)$ is of polynomial growth on $(0, +\infty)$ as $t \to \infty$. In addition, suppose that there exist $M > 0$ and $A \in \left(\frac{1}{2R}, \frac{1}{2}\right)$ such that $|c_k| \le M \cdot \frac{A^k}{k!}$, for all $k = 0, 1, 2, \ldots$ [which implies $|f(z)| \le Me^{A|z|}$ for all $z \in D_R$].*

Let $1 \le r < \frac{1}{2A}$. There exists $n_0 \in \mathbb{N}$ (depending only on f) such that $K_n(f, z)$ is analytic in $SD^r(0, r]$ for all $n \ge n_0$, and

$$|K_n(f, z) - f(z)| \le \frac{C}{n}, \ \textit{for all } n \ge n_1 \textit{ and } z \in SD^r[a, r],$$

for any $a \in (0, r)$. Here $C > 0$ is independent of n and z but depends on f, r, and a, and n_1 depends on f, r, and a.

Proof. In the definition of $K_n(f, z)$ in (6.4), for $z = x + iy$ with $x > 0$, note that it follows that $t^{nz-1} = e^{(nz-1)ln(t)} = e^{(nx-1)ln(t)} \cdot e^{inyln(t)}$ and $|t^{nz-1}| = t^{nx-1}$, which implies

$$|K_n(f, z)| \le \frac{1}{|B(nz, n+1)|} \int_0^{+\infty} \left| \frac{t^{nz-1}}{(1+t)^{nz+n+1}} \right| \cdot |f(t)| dt$$
$$= \frac{1}{|B(nz, n+1)|} \int_0^{+\infty} \frac{t^{nx-1}}{(1+t)^{nx+n+1}} \cdot |f(t)| dt.$$

But it is well known that because $f(t)$ is of polynomial growth as $t \to +\infty$, the last integral exists finitely for sufficiently large n.

Therefore, there exists n_0 depending only on f such that $K_n(f, z)$ is well defined for sufficiently large n and for z with $\mathrm{Re}(z) > 0$.

It remains to prove that $K_n(f, z)$ is, in fact, analytic for $\mathrm{Re}(z) > 0$ and n sufficiently large. For this purpose, from a standard result in the theory of improper integrals depending on a parameter, it suffices to prove that for any $\delta > 0$, the improper integral

$$\int_0^\infty \left[\frac{t^{nz-1}}{(1+t)^{nz+n+1}}\right]_z' \cdot f(t)dt$$

is uniformly convergent for $\mathrm{Re}(z) \geq \delta > 0$ and n sufficiently large.

But by simple calculation, we obtain

$$\left[\frac{t^{nz-1}}{(1+t)^{nz+n+1}}\right]_z' = \left[\frac{e^{(nz-1)ln(t)}}{e^{(nz+n+1)ln(1+t)}}\right]_z' = n[ln(t) - ln(1+t)] \cdot \frac{t^{nz-1}}{(1+t)^{nz+n+1}},$$

and since $ln(1+t) \leq 1+t$ for all $t \geq 0$, it easily follows that it remains to prove that the integral $\int_0^\infty \frac{t^{nz-1}}{(1+t)^{nz+n+1}} \cdot ln(t)f(t)dt$ is uniformly convergent for $\mathrm{Re}(z) \geq \delta > 0$ and n sufficiently large.

By $ln(t) < t$ for all $t \geq 1$ and by

$$\int_0^\infty \frac{t^{nz-1}}{(1+t)^{nz+n+1}} \cdot ln(t)f(t)dt = \int_0^1 \frac{t^{nz-1}}{(1+t)^{nz+n+1}} \cdot ln(t)f(t)dt + \int_1^\infty \frac{t^{nz-1}}{(1+t)^{nz+n+1}} \cdot ln(t)f(t)dt,$$

we clearly need to prove the uniform convergence, for all $\mathrm{Re}(z) \geq \delta > 0$ and n sufficiently large, for the integral $\int_0^1 \frac{t^{nz-1}}{(1+t)^{nz+n+1}} \cdot ln(t)f(t)dt$. But this follows immediately from the estimate

$$\left|\frac{t^{nz-1}}{(1+t)^{nz+n+1}}\right| \cdot |ln(t)| \cdot |f(t)| \leq Ct^{n\delta-1}|ln(t)|,$$

(see, e.g., [195], p. 19, Exercise 1.51), where $|f(t)| \leq C$ for all $t \in [0, 1]$.

In what follows, we deal with the approximation property. For this purpose, first let us define $S_n(z) = \sum_{k=0}^n c_k z^k$ if $|z| \leq r$ and $S_n(t) = f(t)$ if $t \in (r, +\infty)$, where $1 \leq r < \frac{1}{2A}$. Evidently, for each $n \in \mathbb{N}$, S_n is piecewise continuous on $[0, +\infty)$ (more exactly, has a discontinuity point of the first kind at $x = r$) but locally integrable on $[0, +\infty)$ and of polynomial growth as $t \to \infty$.

Clearly, $f(z) - S_n(z) = \sum_{k=n+1}^\infty c_k z^k$ if $|z| \leq r$ and $f(t) - S_n(t) = 0$ if $t \in (r, \infty)$. Also, it is immediate that $K_n(S_n)(z)$ is well defined for all $n \in \mathbb{N}$.

Therefore, for sufficiently large n and for $z \in SD^r(0, r]$, we have

$$|K_n(f,z) - f(z)| \le |K_n(f,z) - K_n(S_n,z)| + |K_n(S_n,z) - S_n(z)| + |S_n(z) - f(z)|$$
$$\le |K_n(f - S_n,z)| + \sum_{k=0}^{n} |c_k| \cdot |K_n(e_k,z) - e_k(z)| + |S_n(z) - f(z)|,$$

where $e_k(z) = z^k$.

First, we will obtain an estimate for $|S_n(z) - f(z)|$. Let $1 \le r < \frac{1}{2A} < r_1 < R$. By the hypothesis, we can make such a choice for r_1.

Denoting $M_{r_1}(f) = \max\{|f(z)|; |z| \le r_1\}$ and $\rho = \frac{r}{r_1}$, by $0 < \rho = \frac{r}{r_1} < 2Ar < 1$ and by Cauchy's estimate (see, e.g., [224], p. 184, Lemma 10.5), we get $|c_k| = \frac{|f^{(k)}(0)|}{k!} \le \frac{1}{k!} \cdot \frac{M_{r_1(f)}k!}{r_1^k} = \frac{M_{r_1}(f)}{r_1^k}$, which implies

$$|S_n(z) - f(z)| \le \sum_{k=n+1}^{\infty} |c_k| \cdot |z|^k \le \sum_{k=n+1}^{\infty} \frac{M_{r_1}(f)}{r_1^k} \cdot |z|^k \le \sum_{k=n+1}^{\infty} M_{r_1}(f)\frac{r^k}{r_1^k}$$
$$= M_{r_1}(f)\rho^{n+1} \sum_{k=0}^{\infty} \rho^k = \frac{M_{r_1}(f)}{1-\rho} \cdot \rho^{n+1},$$

for all $|z| \le r$ and $n \in \mathbb{N}$.

By now using Lemma 6.7 and taking into account the inequalities

$$\frac{1}{n-p} \le \frac{2(p+1)}{n+p}, \ \frac{1}{n-p} \le \frac{p+1}{n}, \ n \ge p+1,$$

for all $z \in SD^r(0, r]$ and $n \ge p+1$, we get

$$\begin{aligned}
&|K_n(e_{p+1},z) - e_{p+1}(z)| \\
&\quad = \left| \frac{nz+p}{n-p} K_n(e_p,z) - \frac{nz+p}{n-p} e_p(z) + \frac{nz+p}{n-p} e_p(z) - e_{p+1}(z) \right| \\
&\quad \le \frac{|nz+p|}{n-p} \left| K_n(e_p,z) - e_p(z) \right| + |e_p(z)| \cdot \left| \frac{nz+p}{n-p} - z \right| \\
&\quad \le |nz+p| \cdot \frac{2(p+1)}{n+p} \cdot \left| K_n(e_p,z) - e_p(z) \right| + r^p \cdot \frac{|p(1+z)|}{n-p} \\
&\quad \le \frac{nr+p}{n+p} \cdot 2(p+1)|K_n(e_p,z) - e_p(z)| + \frac{r^p \cdot 2pr \cdot (p+1)}{n} \\
&\quad \le 2r(p+1)\left[|K_n(e_p,z) - e_p(z)| + \frac{pr^p}{n} \right],
\end{aligned}$$

for all $p = 0, 1, \dots, n-1$.

Therefore, denoting $E_{p,n}(z) = |K_n(e_p, z) - e_p(z)|$, we have obtained

$$E_{p+1,n}(z) \leq r(2p+2)\left[E_{p,n}(z) + p \cdot \frac{r^p}{n}\right],$$

for all $p = 0, 1, \ldots, n-1$.

Since $E_{0,n}(z) = E_{1,n}(z) = 0$, for $p = 1$ in the above inequality, we get

$$E_{2,n}(z) \leq r(2 \cdot 1 + 2)\left[E_{1,n}(z) + \frac{r}{n}\right] \leq \frac{1 \cdot r^2}{n} \cdot (2 \cdot 1 + 2).$$

In what follows, we will use the obvious inequality $p \leq 2(p-1) + 2$, valid for all $p \geq 1$.

For $p = 2$ in the above recurrence inequality, it follows that

$$\begin{aligned} E_{3,n}(z) &\leq r(2 \cdot 2 + 2)\left[E_{2,n}(z) + 2 \cdot \frac{r^2}{n}\right] \\ &\leq \frac{r^3}{n}\left[(2 \cdot 2 + 2)(2 \cdot 1 + 2) + 2 \cdot (2 \cdot 2 + 2)\right] \\ &\leq \frac{r^3}{n}\left[(2 \cdot 2 + 2)(2 \cdot 1 + 2) + (2 \cdot 1 + 2)(2 \cdot 2 + 2)\right] \\ &\leq \frac{2r^3}{n}(2 \cdot 2 + 2)(2 \cdot 1 + 2). \end{aligned}$$

For $p = 3$ in the above recurrence inequality, we get

$$\begin{aligned} E_{4,n}(z) &\leq r(2 \cdot 3 + 2)\left[E_{3,n}(z) + 3 \cdot \frac{r^3}{n}\right] \\ &\leq r(2 \cdot 3 + 2) \cdot \left[\frac{2r^3}{n}(2 \cdot 1 + 2)(2 \cdot 2 + 2) + (2 \cdot 2 + 2) \cdot \frac{r^3}{n}\right] \\ &\leq \frac{3r^4}{n}(2 \cdot 3 + 2)(2 \cdot 2 + 2)(2 \cdot 1 + 2). \end{aligned}$$

By mathematical induction, we easily obtain

$$E_{p,n}(z) \leq \frac{(p-1) \cdot r^p}{n} \Pi_{i=1}^{p-1} 2(i+1) = \frac{(p-1) \cdot 2^{p-1} r^p}{n} \cdot p! \leq \frac{p \cdot p!(2r)^p}{2n},$$

for all $n \geq p + 1$ and $z \in SD^r(0, r]$.

Therefore, we obtain

$$\sum_{k=0}^{n}|c_k|\cdot|K_n(e_k,z)-e_k(z)|\le\frac{M}{2n}\sum_{k=0}^{n}k(2Ar)^k\le\frac{M}{2n}\sum_{k=0}^{\infty}k(2Ar)^k,$$

where the hypothesis on f obviously implies that $\sum_{k=0}^{\infty}k\cdot(2Ar)^k<\infty$.

Now, let us estimate $|K_n(f-S_n,z)|$. By the definition of S_n and by (6.5), we easily get

$$K_n(f-S_n,z)=\frac{nz(nz+1)\dots(nz+n)}{n!}\cdot\int_0^r\frac{t^{nz-1}}{(1+t)^{nz+n+1}}(f(t)-S_n(t))dt,$$

for all $z\in SD^r[a,r]$, $z=x+iy$, and $n\in\mathbb{N}$. Passing to the modulus, we see that

$$\begin{aligned}|K_n(f-S_n,z)|&\le\frac{nr(nr+1)\dots(nr+n)}{n!}\cdot\int_0^r\left|\frac{t^{nz-1}}{(1+t)^{nz+n+1}}\right|\cdot|f(t)-S_n(t)|dt\\&\le\|f-S_n\|_{C[0,r]}\cdot\frac{nr(nr+1)\dots(nr+n)}{n!}\cdot\int_0^r\frac{t^{nx-1}}{(1+t)^{nx+n+1}}dt\\&\le C_{r,r_1,f}\cdot\rho^{n+1}\cdot\frac{nr(nr+1)\dots(nr+n)}{n!}\cdot\int_0^r\frac{t^{nx-1}}{(1+t)^{nx+n+1}}dt.\end{aligned}$$

Now, let us estimate the integral $\int_0^r\frac{t^{nx-1}}{(1+t)^{nx+n+1}}dt$. For sufficiently large n (such that $na-1\ge 1$), we have

$$\begin{aligned}&\int_0^r\frac{t^{nx-1}}{(1+t)^{nx+n+1}}dt\\&=\int_0^1\frac{t^{nx-1}}{(1+t)^{nx+n+1}}dt+\int_1^r\frac{t^{nx-1}}{(1+t)^{nx+n+1}}dt\\&\le\int_0^1\frac{t^{nx-1}}{(1+t)^{nx-1}}dt+\int_1^r\frac{t^{nx+n+1}}{(1+t)^{nx+n+1}}dt\\&\le\left(\frac{1}{2}\right)^{nx-1}+(r-1)\left(\frac{r}{r+1}\right)^{nx+n+1}\\&\le r\left(\frac{r}{r+1}\right)^{nx-1}\le r\left(\frac{r}{r+1}\right)^{na-1},\end{aligned}$$

which immediately implies the estimate for $n\ge n_0$ (with n_0 depending only on f and a) and $z\in SD^r[a,r]$:

$$|K_n(f-S_n,z)|\le C_{r,r_1,f}\cdot\rho^{n+1}\cdot\frac{nr(nr+1)\dots(nr+n)}{n!}\cdot r\left(\frac{r}{r+1}\right)^{na-1}.$$

Collecting all of the above estimates, for sufficiently large n and $z \in S^r[a, r]$, we get

$$|K_n(f, z) - f(z)| \leq C_{r,r_1,f}\rho^{n+1} + \frac{M}{n}\sum_{k=0}^{\infty} k(2Ar)^k$$

$$+ C_{r,r_1,f} \cdot \rho^{n+1} \cdot \frac{nr(nr+1)\dots(nr+n)}{n!} \cdot r\left(\frac{r}{r+1}\right)^{na-1}. \tag{6.6}$$

In (6.6), we need to choose $n \geq 2/a$.

Now, denote

$$a_n = \frac{1}{n^2} \cdot \frac{nr(nr+1)\dots(nr+n)}{n!} = \frac{r}{n} \cdot \frac{(nr+1)\dots(nr+n)}{n!}.$$

We can write

$$\rho^{n+1} \cdot \frac{nr(nr+1)\dots(nr+n)}{n!} \cdot \left(\frac{r}{r+1}\right)^{na-1} = (n \cdot \rho^{n+1}) \cdot a_n \cdot \left[n\left(\frac{r}{r+1}\right)^{na-1}\right].$$

Note that because $0 < \rho < 1$ and $0 < r/(r+1) < 1$, it is clear that for sufficiently large n, we have $n \cdot \rho^{n+1} \leq \frac{c_1}{n}$ and $n\left(\frac{r}{r+1}\right)^{na-1} \leq \frac{c_2}{n}$, where $c_1 > 0$ and $c_2 > 0$ are independent of n and z. On the other hand, by simple calculation, we get

$$\frac{a_{n+1}}{a_n} = \frac{n}{(n+1)^2} \cdot \left(1 + \frac{r}{nr+1}\right)\left(1 + \frac{r}{nr+2}\right)\dots\left(1 + \frac{r}{nr+n}\right)$$

$$< \frac{n}{(n+1)^2}\left(1 + \frac{r}{nr+1}\right)^n < \frac{n}{(n+1)^2}\left(1 + \frac{r}{nr+1}\right)^{(nr+1)/r} \leq \frac{3n}{(n+1)^2} \leq 1,$$

for all $n \in \mathbb{N}$. We used the inequality $e < 3$ here. Therefore, the sequence $(a_n)_n$ is nonincreasing, which implies that it is bounded.

In conclusion, for sufficiently large n, we have

$$\rho^{n+1} \cdot \frac{nr(nr+1)\dots(nr+n)}{n!} \cdot \left(\frac{r}{r+1}\right)^{na-1} \leq \frac{c_3}{n^2},$$

which, coupled with (6.6), immediately implies the order of approximation $O(1/n)$ in the statement of the theorem. ■

Theorem 6.10 ([81] Asymptotic formula). *Let $D_R = \{z \in \mathbb{C}; |z| < R\}$, with $1 < R < \infty$, and suppose that $f : [R, \infty) \bigcup \overline{D}_R \to C$ is continuous in $[R, \infty) \bigcup \overline{D}_R$, analytic in D_R, that is, $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in D_R$, and*

$f(t)$ is of polynomial growth on $(0,+\infty)$ as $t \to \infty$. In addition, suppose that there exist $M > 0$ and $A \in \left(\frac{1}{2R}, \frac{1}{2}\right)$ such that $|c_k| \leq M \cdot \frac{A^k}{k!}$, for all $k = 0, 1, 2, \ldots$, [which implies $|f(z)| \leq Me^{A|z|}$ for all $z \in D_R$].

Let $1 \leq r < \frac{1}{2A}$. There exists $n_1 \in \mathbb{N}$ (depending on f, r, and a) such that for all $n \geq n_1$, $z \in SD^r[a,r]$, and $a \in (0,r)$, we have

$$\left|K_n(f,z) - f(z) - \frac{z(1+z)f''(z)}{2(n-1)}\right| \leq \frac{C}{n^2},$$

where $C > 0$ is independent of n and z but depends on f, r, and a.

Proof. Keeping the notations in the proof of Theorem 6.9, we can write

$$\begin{aligned}
&\left|K_n(f,z) - f(z) - \frac{z(1+z)f''(z)}{2(n-1)}\right| \\
&\quad = \left|\left(K_n(f-S_n,z) - (f(z)-S_n(z)) - \frac{z(1+z)[f(z)-S_n(z)]''}{2(n-1)}\right)\right. \\
&\qquad \left. + \left(K_n(S_n,z) - S_n(z) - \frac{z(1+z)S_n''(z)}{2(n-1)}\right)\right| \\
&\quad \leq \left|K_n(f-S_n,z) - (f(z)-S_n(z)) - \frac{z(1+z)[f(z)-S_n(z)]''}{2(n-1)}\right| \\
&\qquad + \left|K_n(S_n,z) - S_n(z) - \frac{z(1+z)S_n''(z)}{2(n-1)}\right| := A + B.
\end{aligned}$$

We get

$$A \leq |K_n(f-S_n,z)| + |f(z)-S_n(z)| + \left|\tfrac{z(1+z)[f(z)-S_n(z)]''}{2(n-1)}\right|$$

$$\leq |K_n(f-S_n,z)| + |f(z)-S_n(z)| + \tfrac{r(1+r)\left|f''(z)-S_n''(z)\right|}{2(n-1)} := A_1 + A_2 + A_3.$$

From the proof of Theorem 6.9, for all $z \in SD^r[a,r]$ with $a \in (0,r)$ and for sufficiently large n, we have

$$A_1 \leq \frac{C_1}{n^2} \text{ and } A_2 \leq C_2\rho^{n+1},$$

where $C_1 > 0$, $C_2 > 0$ are independent of n and z but may depend on f, r, and a and $0 < \rho < 1$.

In order to estimate A_3, let $0 < a_1 < a < r$, $1 \leq r < r_1 < \frac{1}{2A}$, and denote by $\Gamma = \Gamma_{a_1,r_1} = S_1 \bigcup L_1$ the closed curve composed by the segment in $\mathbb{C}$:

$$S_1 = \left\{ z = x + iy \in \mathbb{C}; x = a_1 \text{ and } -\sqrt{r_1^2 - a_1^2} \le y \le \sqrt{r_1^2 - a_1^2} \right\},$$

and by the arc

$$L_1 = \{z \in \mathbb{C} : |z| = r_1, \mathrm{Re}(z) \ge a_1\}.$$

Clearly, Γ, together with its interior, is exactly $SD^{r_1}[a_1, r_1]$ and from $r < r_1$, we have $SD^r[a, r] \subset SD^{r_1}[a_1, r_1]$, the inclusion being strictly.

By Cauchy's integral formula for derivatives, we have for all $z \in SD^r[a, b]$ and $n \in \mathbb{N}$ sufficiently large,

$$f''(z) - S_n''(z) = \frac{2!}{2\pi i} \int_\Gamma \frac{f(u) - S_n(u)}{(u - z)^3} du,$$

which, by the estimate of $||f - S_n(\cdot)||_{SD^{r_1}[a_1,r_1]}$ in the proof of Theorem 6.9 and by the inequality $|u - z| \ge d = \min\{r_1 - r, a - a_1\}$ valid for all $z \in SD^r[a, r]$ and $u \in \Gamma$, implies

$$\begin{aligned} ||f''(z) - S_n''(\cdot)||_{SD^r[a,r]} &\le \frac{2!}{2\pi} \cdot \frac{l(\Gamma)}{d^3} ||f - S_n(\cdot)||_{SD^{r_1}[a_1,r_1]} \\ &\le \frac{M_{r_1}(f)}{1 - \rho} \cdot \rho^{n+1} \cdot \frac{C_{r_1}(f) 2! l(\Gamma)}{2\pi d^3}, \end{aligned}$$

with $\rho = \frac{r}{r_1}$.

Note that here, by simple geometrical reasoning, for the length $l(\Gamma)$ of Γ, we get

$$\begin{aligned} l(\Gamma) &= l(S_1) + l(L_1) \\ &= 2\sqrt{r_1^2 - a_1^2} + 2r_1 \cdot \arccos(a_1/r_1), \end{aligned}$$

where $\arccos(\alpha)$ is considered expressed in radians.

Therefore, collecting all of the above estimates, we easily get $A \le \frac{C}{n^2}$ for sufficiently large n, with $C > 0$ independent of n and z (but depending on f, r, and a).

In the last part of the proof, we will obtain an estimate of the order $O(1/n^2)$ for $B = \left| K_n(S_n, z) - S_n(z) - \frac{z(1+z)S_n''(z)}{2(n-1)} \right|$ too, which implies the estimate in the statement.

Denoting $\pi_{k,n}(z) = K_n(e_k)(z)$ and

$$E_{k,n}(z) = \pi_{k,n}(z) - e_k(z) - \frac{z^{k-1}(1 + z)k(k - 1)}{2(n - 1)},$$

we first clearly see that $E_{0,n}(z) = E_{1,n}(z) = 0$. Then we can write

$$\left|K_n(S_n, z) - S_n(z) - \frac{z(1+z)S_n''(z)}{2(n-1)}\right| \leq \sum_{k=2}^{n} |c_k| \cdot |E_{k,n}(z)|,$$

and so it remains to estimate $E_{k,n}(z)$ for $2 \leq k \leq n$, by using the recurrence in Lemma 2.1.

In this sense, a simple calculation based on Lemma 6.7 leads us to the formula

$$\begin{aligned}
E_{k,n}(z) &= \frac{nz+k-1}{n-k+1} \cdot \pi_{k-1,n}(z) - z^k - \frac{z^{k-1}(1+z)k(k-1)}{2(n-1)} \\
&= \frac{nz+k-1}{n-k+1}\left[E_{k-1,n}(z) + z^{k-1} + \frac{z^{k-2}(1+z)(k-1)(k-2)}{2(n-1)}\right] \\
&\qquad - z^k - \frac{z^{k-1}(1+z)k(k-1)}{2(n-1)} \\
&= \frac{nz+k-1}{n-k+1}E_{k-1,n}(z) + \frac{(k-1)(k-2)z^{k-2}(1+z)[(1+z)k+(z-1)]}{2(n-1)(n-k+1)}.
\end{aligned}$$

If we take into account the inequalities valid for all $2 \leq k \leq n$ and $r \geq 1$,

$$\frac{1}{n-k+1} \leq \frac{2k}{n+k-1} \leq \frac{2k}{n+k}, \quad \frac{nr+k-1}{n+k-1} \leq r,$$
$$2(n-1)(n+k) \geq n^2, \quad k(1+r)+(r-1) \leq (k+1)(1+r),$$

this immediately implies, for all $2 \leq k \leq n$ and $|z| \leq r$ with $a \leq \mathrm{Re}(z) \leq r$,

$$\begin{aligned}
&|E_{k,n}(z)| \leq \\
&\leq \left|\frac{nz+k-1}{n-k+1}\right| \cdot |E_{k-1,n}(z)| + \left|\frac{(k-1)(k-2)z^{k-2}(1+z)[(1+z)k+(z-1)]}{2(n-1)(n-k+1)}\right| \\
&\leq \frac{nr+k-1}{n-k+1} \cdot |E_{k-1,n}(z)| + \frac{(k-1)(k-2)r^{k-2}(1+r)[(1+r)k+(r-1)]}{2(n-1)(n-k+1)} \\
&\leq \frac{2k(nr+k-1)}{n+k-1} \cdot |E_{k-1,n}(z)| + \frac{2k(k-1)(k-2)r^{k-2}(1+r)[(1+r)k+(r-1)]}{2(n-1)(n+k)} \\
&\leq 2kr \cdot |E_{k-1,n}(z)| + \frac{2k(k-1)(k-2)r^{k-2}(1+r)[(1+r)k+(r-1)]}{2(n-1)(n+k)} \\
&\leq 2rk|E_{k-1,n}(z)| + \frac{r^{k-2}(1+r)2k(k-1)(k-2)}{n^2} \cdot [k(1+r)+(r-1)] \\
&\leq 2rk|E_{k-1,n}(z)| + \frac{r^{k-1}(1+r)^2 k(k-1)(k-2)(k+1)}{n^2}.
\end{aligned}$$

Denoting $A(k,r) = 2(1+r)^2(k+1)k(k-1)(k-2)$, we have obtained

$$|E_{k,n}(z)| \le 2rk|E_{k-1,n}(z)| + \frac{r^{k-1}}{n^2} \cdot A(k,r).$$

Obviously, $E_{0,n}(z) = E_{1,n}(z) = E_{2,n} = 0$. Take the last inequality, $k = 3, 4, \ldots, n$.

For $k = 3$, we obtain $|E_{3,n}(z)| \le \frac{r^2}{n^2} \cdot A(3,r)$.

For $k = 4$, it follows that

$$|E_{4,n}(z)| \le r(2\cdot 4) \cdot |E_{3,n}(z)| + \frac{r^3}{n^2}A(4,r) \le \frac{r^3}{n^2} \cdot (2\cdot 4)\left[A(3,r) + A(4,r)\right].$$

For $k = 5$, analogously, we get

$$\begin{aligned}
|E_{5,n}(z)| &\le r(2\cdot 5) \cdot |E_{4,n}(z)| + \frac{r^4}{n^2}A(5,r) \\
&\le r(2\cdot 5)\left[(2\cdot 4)\frac{r^3}{n^2}[A(3,r) + A(4,r)]\right] + \frac{r^4}{n^2}A(5,r) \\
&\le \frac{r^4}{n^2} \cdot (2\cdot 4)(2\cdot 5)[A(3,r) + A(4,r) + A(5,r)].
\end{aligned}$$

Reasoning by mathematical induction, we finally easily obtain

$$\begin{aligned}
|E_{k,n}(z)| &\le \frac{r^{k-1}}{n^2} \cdot (2\cdot 4)(2\cdot 5)\ldots(2\cdot k) \cdot \sum_{j=3}^{k} A(j,r) \\
&= \frac{r^{k-1}}{n^2} \cdot \frac{2^{k-1}}{2^2} \cdot \frac{k!}{3!} \cdot \sum_{j=3}^{k} A(j,r) \\
&= \frac{(2r)^{k-1}}{24n^2} \cdot k! \cdot 2(1+r)^2 \sum_{j=3}^{k} (j-2)(j-1)j(j+1) \\
&\le \frac{k!(2r)^{k-1}}{12n^2} \cdot (1+r)^2(k-2)^2(k-1)k(k+1).
\end{aligned}$$

We conclude that

$$\begin{aligned}
B := &\left|K_n(S_n,z) - S_n(z) - \frac{z(1+z)S_n''(z)}{2(n-1)}\right| \le \sum_{k=2}^{n} |c_k| \cdot |E_{k,n}| \\
&\le \frac{MA(1+r)^2}{12n^2} \cdot \sum_{k=2}^{n} (k-2)^2(k-1)k(k+1)(2rA)^{k-1}
\end{aligned}$$

$$\leq \frac{MA(1+r)^2}{12n^2} \cdot \sum_{k=2}^{\infty}(k-2)^2(k-1)k(k+1)(2rA)^{k-1},$$

where since $2Ar < 1$ by hypothesis, we get

$$\sum_{k=2}^{\infty}(k-2)^2(k-1)k(k+1)(2rA)^{k-1} < +\infty.$$

Indeed, the fact that the last series is convergent follows from the uniform convergence of the series $\sum_{k=0}^{\infty} z^k$ and its derivative of order 5, for $|z| < 1$. This finishes the proof of the theorem. ■

Theorem 6.11 ([81] Exact order). *Let $D_R = \{z \in \mathbb{C}; |z| < R\}$, with $1 < R < \infty$, and suppose that $f : [R, \infty) \bigcup \overline{D}_R \to C$ is continuous in $[R, \infty) \bigcup \overline{D}_R$, analytic in D_R, that is, $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in D_R$, and $f(t)$ is of polynomial growth on $(0, +\infty)$ as $t \to \infty$. In addition, suppose that there exist $M > 0$ and $A \in \left(\frac{1}{2R}, \frac{1}{2}\right)$ such that $|c_k| \leq M \cdot \frac{A^k}{k!}$, for all $k = 0, 1, 2, \ldots$ [which implies $|f(z)| \leq Me^{A|z|}$ for all $z \in D_R$].*

Let $1 \leq r < \frac{1}{2A}$. If f is not a polynomial of degree ≤ 1, then there exists $n_1 \in \mathbb{N}$ (depending on f, r, and a) such that for all $n \geq n_1$, $z \in SD^r[a, r]$, and $a \in (0, r)$, we have

$$||K_n(f, \cdot) - f||_{SD^r[a,r]} \geq \frac{C_{r,a}(f)}{n},$$

where $C_{r,a}(f)$ depends only on f, a, and r. Here $\| \cdot \|_{SD^r[a,r]}$ denotes the uniform norm on $SD^r[a, r]$.

Proof. For all $|z| \leq r$ and $n > n_0$ (with n_0 depending only on f), we have

$$K_n(f, z) - f(z) = \frac{1}{(n-1)} \left[\frac{z(1+z)f''(z)}{2} \right.$$
$$\left. + \frac{1}{(n-1)} \left\{ (n-1)^2 \left(K_n(f, z) - f(z) - \frac{z(1+z)f''(z)}{2(n-1)} \right) \right\} \right].$$

Also, we have

$$||F + G||_{SD^r[a,r]} \geq \Big| ||F||_{SD^r[a,r]} - ||G||_{SD^r[a,r]} \Big| \geq ||F||_{SD^r[a,r]} - ||G||_{SD^r[a,r]}.$$

It follows that

$$||K_n(f, \cdot) - f||_{SD^r[a,r]} \geq \frac{1}{(n-1)} \left[\left\| \frac{e_1(1+e_1)f''}{2} \right\|_{SD^r[a,r]} \right.$$

$$-\frac{1}{(n-1)}\left\{(n-1)^2\left\|K_n(f,\cdot)-f-\frac{e_1(1+e_1)f''}{2(n-1)}\right\|_{SD^r[a,r]}\right\}\Bigg].$$

Taking into account that, by hypothesis, f is not a polynomial of degree ≤ 1 in D_R, we get $\left\|\frac{e_1(1+e_1)}{2}f''\right\|_{SD^r[a,r]} > 0$.

Indeed, suppose, to the contrary, that it follows that $\frac{z(1+z)}{2}f''(z) = 0$ for all $z \in \overline{D}_R$, which implies that $f''(z) = 0$, for all $z \in \overline{D}_R\backslash\{0,-1\}$. Because f is analytic, by the uniqueness of analytic functions, we get $f''(z) = 0$, for all $z \in D_R$, that is, f is a polynomial of degree ≤ 1, which contradicts the hypothesis.

Now by Theorem 6.10, for sufficiently large n, we have

$$(n-1)^2\left\|K_n(f,\cdot)-f-\frac{e_1(1+e_1)f''}{2(n-1)}\right\|_{SD^r[a,r]} \leq \frac{C(n-1)^2}{n^2} \leq M_r(f).$$

Therefore, there exists an index $n_1 > n_0$ depending only on f, a, and r, such that for any $n \geq n_1$, we have

$$\begin{aligned}&\left\|\frac{e_1(1+e_1)f''}{2}\right\|_{SD^r[a,r]}\\ &\quad -\frac{1}{(n-1)}\left\{(n-1)^2\left\|K_n(f,\cdot)-f-\frac{e_1(1+e_1)f''}{2(n-1)}\right\|_{SD^r[a,r]}\right\}\\ &\geq \frac{1}{4}\|e_1(1+e_1)f''\|_{SD^r[a,r]},\end{aligned}$$

which immediately implies

$$||K_n(f,\cdot)-f||_{SD^r[a,r]} \geq \frac{1}{4n}||e_1(1+e_1)f''||_{SD^r[a,r]}, \forall n \geq n_1.$$

This completes the proof. ∎

In simultaneous approximation, the following result was stated and proved in [81], and thus we omit the details here.

Theorem 6.12 ([81] Simultaneous approximation). *Let $D_R = \{z \in \mathbb{C}; |z| < R\}$, with $1 < R < \infty$, and suppose that $f : [R,\infty)\bigcup\overline{D}_R \to C$ is continuous in $[R,\infty)\bigcup\overline{D}_R$, analytic in D_R, that is, $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in D_R$, and $f(t)$ is of polynomial growth on $(0,+\infty)$ as $t \to \infty$. In addition, suppose that there exist $M > 0$ and $A \in \left(\frac{1}{2R},\frac{1}{2}\right)$ such that $|c_k| \leq M \cdot \frac{A^k}{k!}$, for all $k = 0,1,2,\dots$ [which implies $|f(z)| \leq Me^{A|z|}$ for all $z \in D_R$].*

Let $1 \leq r < r_1 < \frac{1}{2A}$, $0 < a_1 < a < r$, and $p \in \mathbb{N}$. If f is not a polynomial of degree $\leq \max\{1, p-1\}$, then there exists $n_1 \in \mathbb{N}$ (depending on f, r, and a) such that for all $n \geq n_1$ and $z \in SD^r[a,r]$, we have

$$||K_n^{(p)}(f,\cdot) - f^{(p)}||_{SD^r[a,r]} \sim \frac{1}{n},$$

where the constants in the equivalence depend only on f, r, r_1, a, a_1*, and* p*.*

6.4 Genuine Durrmeyer–Stancu Polynomials

Let α and β be two given real parameters satisfying the conditions $0 \leq \alpha \leq \beta$. For $f : [0,1] \to \mathbb{C}$ continuous on $[0,1]$, Mahmudov and Gupta [187] introduced the complex genuine Durrmeyer–Stancu operators as follows:

$$\begin{aligned} U_n^{(\alpha,\beta)}(f;z) &= p_{n,0}(z) f\left(\frac{\alpha}{n+\beta}\right) + p_{n,n}(z) f\left(\frac{n+\alpha}{n+\beta}\right) \\ &\quad + (n-1)\sum_{k=1}^{n-1} p_{n,k}(z) \int_0^1 p_{n-2,k-1}(t) f\left(\frac{nt+\alpha}{n+\beta}\right) dt \end{aligned}$$

Here $p_{n,k}(z) = \binom{n}{k} z^k (1-z)^{n-k}$, $p_{-1,k} = 0$. For $\alpha = \beta = 0$, they become the genuine Bernstein–Durrmeyer operators $U_n(f;z) = U_n^{(0,0)}(f;z)$. Note that this case is investigated in [78]. Let $\mathbb{D}_R := \{z \in \mathbb{C} : |z| < R\}$, $H(\mathbb{D}_R)$ denote the set of all analytic functions on $\mathbb{D}_R$. For $f \in H(\mathbb{D}_R)$, we assume that $f(z) = \sum_{m=0}^{\infty} a_m z^m$.

First, we describe a method to recursively compute the images of the monomials e_m under $U_n^{(\alpha,\beta)}$ in terms of $U_n(e_j)$, $j = 0, \ldots, m$, where $e_m(z) = z^m$. Then we establish a recurrence formula for the moments $U_n^{(\alpha,\beta)}(e_{m+1})$.

Lemma 6.8 ([187]). *For all* $m, n \in \mathbb{N} \cup \{0\}$, $0 \leq \alpha \leq \beta$, *and* $z \in \mathbb{C}$, *we have*

$$U_n^{(\alpha,\beta)}(e_m;z) = \sum_{j=0}^{m} \binom{m}{j} \frac{n^j \alpha^{m-j}}{(n+\beta)^m} U_n(e_j;z).$$

Lemma 6.9 ([187]). *For all* $m, n \in \mathbb{N} \cup \{0\}$, $0 \leq \alpha \leq \beta$, *and* $|z| \leq r$, $r \geq 1$, *we have*

$$\left|U_n^{(\alpha,\beta)}(e_m;z)\right| \leq r^m.$$

Lemma 6.10 ([187]). *For all* $m, n \in \mathbb{N}$, $0 \leq \alpha \leq \beta$, *and* $z \in \mathbb{C}$, *we have*

$$U_n^{(\alpha,\beta)}(e_{m+1};z) = \frac{z(1-z)}{n+m}\frac{n}{n+\beta}\left(U_n^{(\alpha,\beta)}(e_m;z)\right)'$$

$$+\frac{(m+zn)\,n+\alpha\,(2m+n)}{(n+\beta)\,(n+m)}U_n^{(\alpha,\beta)}\,(e_m;z)$$
$$-\frac{1}{n+m}\frac{\alpha m\,(\alpha+n)}{(n+\beta)^2}U_n^{(\alpha,\beta)}\,(e_{m-1};z)\,.$$

Theorem 6.13 ([187]). *Let* $0 \leq \alpha \leq \beta$ *and* $1 \leq r < R$. *Then for all* $|z| \leq r$, *we have*

$$\left|U_n^{(\alpha,\beta)}\,(f;z)-f\,(z)\right| \leq \frac{A_{r,\beta}\,(f)}{n+\beta}+\frac{\alpha B_r\,(f)}{n\,(n+\beta)},$$

where

$$A_{r,\beta}\,(f)=2\,(1+r)\sum_{m=1}^{\infty}\left(\frac{m\,(m-1)}{2}+\beta m\right)r^{m-1}, \quad B_r\,(f)$$
$$=\alpha\,(1+r)\sum_{m=2}^{\infty}m\,(m-1)\,r^{m-2}.$$

Proof. Let us start with the recurrence formula:

$$\begin{aligned}U_n^{(\alpha,\beta)}\,(e_m;z)-z^m &= \frac{z\,(1-z)}{n+m-1}\frac{n}{n+\beta}\left(U_n^{(\alpha,\beta)}\,(e_{m-1};z)\right)' \\ &+\frac{(m-1+zn)\,n+\alpha\,(m-1+n)}{(n+\beta)\,(n+m-1)}\left[U_n^{(\alpha,\beta)}\,(e_{m-1};z)-z^{m-1}\right] \\ &+\frac{\alpha\,(m-1)}{(n+\beta)\,(n+m-1)}U_n^{(\alpha,\beta)}\,(e_{m-1};z) \\ &-\frac{\alpha\,(m-1)\,(\alpha+n)}{(n+\beta)^2\,(n+m-1)}U_n^{(\alpha,\beta)}\,(e_{m-2};z) \\ &+\frac{(m-1)\,n\,(1-z)+(\alpha-z\beta)\,(n+m-1)}{(n+\beta)\,(n+m-1)}z^{m-1}.\end{aligned} \tag{6.7}$$

Taking into account that $U_n^{(\alpha,\beta)}\,(e_{m-1};z)$ is a polynomial of degree $m-1$, by the well-known Bernstein inequality and Lemma 6.8, we obtain

$$\left|\left(U_n^{(\alpha,\beta)}\,(e_{m-1};z)\right)'\right| \leq \frac{m-1}{r}\max\left\{\left|U_n^{(\alpha,\beta)}\,(e_{m-1};z)\right| : |z| \leq r\right\} \leq (m-1)\,r^{m-2}. \tag{6.8}$$

Using (6.8) from the above recurrence formula (6.7), we get

$$\begin{aligned}&\left|U_n^{(\alpha,\beta)}\,(e_m;z)-z^m\right| \\ &\leq \frac{r\,(1+r)}{n+\beta}\,(m-1)\,r^{m-2}+\frac{nr+\alpha}{n+\beta}\left|U_n^{(\alpha,\beta)}\,(e_{m-1};z)-e_{m-1}\,(z)\right|\end{aligned}$$

$$
\begin{aligned}
&+\frac{\alpha(m-1)}{n(n+\beta)}(1+r)r^{m-2}+\frac{(m-1)(r+1)+\alpha+\beta r}{n+\beta}r^{m-1}\\
&\leq r\left|U_n^{(\alpha,\beta)}(e_{m-1};z)-e_{m-1}(z)\right|+\frac{m-1}{n+\beta}(1+r)r^{m-1}\\
&+\frac{\alpha(m-1)}{n(n+\beta)}(1+r)r^{m-2}+\frac{(m-1+\beta)}{n+\beta}(1+r)r^{m-1}\\
&\leq r\left|U_n^{(\alpha,\beta)}(e_{m-1};z)-e_{m-1}(z)\right|+2\frac{(m-1+\beta)}{n+\beta}(1+r)r^{m-1}\\
&+\frac{\alpha(m-1)}{n(n+\beta)}(1+r)r^{m-2}.
\end{aligned}
$$

By writing the last inequality for $m=1,2,\ldots,$ we easily obtain, step by step, the following:

$$
\begin{aligned}
&\left|U_n^{(\alpha,\beta)}(e_m;z)-e_m(z)\right|\\
&\leq r\left(r\left|U_n^{(\alpha,\beta)}(e_{m-2};z)-e_{m-2}(z)\right|+2\frac{(m-2+\beta)}{n+\beta}(1+r)r^{m-2}\right.\\
&\left.+\frac{\alpha(m-2)}{n(n+\beta)}(1+r)r^{m-3}\right)\\
&+2\frac{(m-1+\beta)}{n+\beta}(1+r)r^{m-1}+\frac{\alpha(m-1)}{n(n+\beta)}(1+r)r^{m-2}\\
&=r^2\left|U_n^{(\alpha,\beta)}(e_{m-2};z)-e_{m-2}(z)\right|+2\frac{1+r}{n+\beta}r^{m-1}(m-1+\beta+m-2+\beta)\\
&+\frac{\alpha}{n(n+\beta)}(1+r)r^{m-2}(m-1+m-2)\\
&\leq\ldots\leq\frac{2(1+r)}{n+\beta}\left(\frac{m(m-1)}{2}+\beta m\right)r^{m-1}+\frac{\alpha(1+r)}{n(n+\beta)}m(m-1)r^{m-2},
\end{aligned}
$$

which implies

$$
\begin{aligned}
\left|U_n^{(\alpha,\beta)}(f;z)-f(z)\right|&\leq\sum_{m=1}^{\infty}|a_m|\left|U_n^{(\alpha,\beta)}(e_m;z)-e_m(z)\right|\\
&\leq\frac{A_{r,\beta}(f)}{n+\beta}+\frac{\alpha B_r(f)}{n(n+\beta)}.
\end{aligned}
$$

■

Theorem 6.14 ([187]). *Suppose that $f \in H(\mathbb{D}_R)$, $0 \le \alpha \le \beta$, and $1 \le r < R$. Then, for all $|z| \le r$, we have*

$$\left|U_n^{(\alpha,\beta)}(f;z) - f(z) - \frac{\alpha - \beta z}{n+\beta} f'(z) - \frac{z(1-z)}{n+1} f''(z)\right|$$
$$\le \frac{M_r^1(f)}{n^2} + \frac{\alpha^2 M_r^2(f) + \left(\beta^2 + 2\beta\right) M_r^4(f) + \alpha\beta M_r^5(f)}{(n+\beta)^2}$$
$$+ \frac{\alpha M_r^2(f) + \beta M_r^6(f)}{n(n+\beta)},$$

where

$$M_r^1(f) = 16 \sum_{m=3}^{\infty} |a_m| m(m-1)(m-2)^2 r^m,$$

$$M_r^2(f) = \frac{1}{2} \sum_{m=2}^{\infty} |a_m| m(m-1) r^{m-2},$$

$$M_r^3(f) = 2 \sum_{m=2}^{\infty} |a_m| m(m-1)(m-2) r^{m-2},$$

$$M_r^4(f) = \frac{1}{2} \sum_{m=2}^{\infty} |a_m| m(m-1) r^m,$$

$$M_r^6(f) = \sum_{m=2}^{\infty} |a_m| m(m-1) r^{m-1}, \quad M_r^5(f) = 2 \sum_{m=2}^{\infty} |a_m| m^2(m-1) r^m.$$

Proof. For all $z \in \mathbb{D}_R$, we immediately obtain

$$U_n^{(\alpha,\beta)}(f;z) - f(z) - \frac{\alpha - \beta z}{n+\beta} f'(z) - \frac{z(1-z)}{n+1} f''(z)$$
$$= U_n(f;z) - f(z) - \frac{z(1-z)}{n+1} f''(z)$$
$$+ U_n^{(\alpha,\beta)}(f;z) - U_n(f;z) - \frac{\alpha - \beta z}{n+\beta} f'(z)$$
$$= \sum_{m=2}^{\infty} a_m \left(U_n(e_m;z) - e_m(z) - \frac{z(1-z)}{n+1} m(m-1) z^{m-2}\right)$$
$$+ \sum_{m=1}^{\infty} a_m \left(U_n^{(\alpha,\beta)}(e_m;z) - U_n(e_m;z) - \frac{\alpha - \beta z}{n+\beta} m z^{m-1}\right).$$

To estimate the first sum, we use a Voronovskaja-type result for the genuine Durrmeyer operators obtained in [78, Theorem 2.4], namely,

$$\left|U_n(f;z) - f(z) - \frac{z(1-z)}{n+1} f''(z)\right| \leq \frac{16}{n^2} \sum_{m=3}^{\infty} |a_m| \, m(m-1)(m-2)^2 r^m.$$

To estimate the second series, we rewrite it as follows:

$$\begin{aligned}
&U_n^{(\alpha,\beta)}(e_m;z) - U_n(e_m;z) - \frac{\alpha-\beta z}{n+\beta} m z^{m-1} \\
&= \sum_{j=0}^{m-1} \binom{m}{j} \frac{n^j \alpha^{m-j}}{(n+\beta)^m} U_n(e_j;z) + \left(\frac{n^m}{(n+\beta)^m} - 1\right) U_n(e_m;z) - \frac{\alpha-\beta z}{n+\beta} m z^{m-1} \\
&= \sum_{j=0}^{m-2} \binom{m}{j} \frac{n^j \alpha^{m-j}}{(n+\beta)^m} U_n(e_j;z) + \frac{m n^{m-1}\alpha}{(n+\beta)^m} U_n(e_{m-1};z) \\
&\quad - \sum_{j=0}^{m-1} \binom{m}{j} \frac{n^j \beta^{m-j}}{(n+\beta)^m} U_n(e_m;z) - \frac{\alpha-\beta z}{n+\beta} m z^{m-1} \\
&= \sum_{j=0}^{m-2} \binom{m}{j} \frac{n^j \alpha^{m-j}}{(n+\beta)^m} U_n(e_j;z) + \frac{m n^{m-1}\alpha}{(n+\beta)^m} \left(U_n(e_{m-1};z) - z^{m-1}\right) \\
&\quad - \sum_{j=0}^{m-2} \binom{m}{j} \frac{n^j \beta^{m-j}}{(n+\beta)^m} U_n(e_m;z) \\
&\quad + \left(\frac{n^{m-1}}{(n+\beta)^{m-1}} - 1\right) \frac{\alpha}{n+\beta} m z^{m-1} + \frac{m n^{m-1}\beta}{(n+\beta)^m} \left(z^m - U_n(e_m;z)\right) \\
&\quad + \left(1 - \frac{n^{m-1}}{(n+\beta)^{m-1}}\right) \frac{\beta z}{n+\beta} m z^{m-1}.
\end{aligned}$$

Using

$$1 - \frac{n^k}{(n+\beta)^k} \leq \sum_{j=1}^{k} \left(1 - \frac{n}{n+\beta}\right) = \frac{k\beta}{n+\beta}$$

$$\begin{aligned}
\left|\sum_{j=0}^{m-2} \binom{m}{j} \frac{n^j \alpha^{m-j}}{(n+\beta)^m} U_n(e_j;z)\right| &\leq \sum_{j=0}^{m-2} \binom{m}{j} \frac{n^j \alpha^{m-j}}{(n+\beta)^m} \left|U_n(e_j;z)\right| \\
&= \sum_{j=0}^{m-2} \frac{(m-1)m}{(m-j-1)(m-j)} \binom{m-2}{j}
\end{aligned}$$

$$\begin{aligned}
&\frac{n^j \alpha^{m-j}}{(n+\beta)^m}\left|U_n\left(e_j;z\right)\right| \\
&\leq \frac{m(m-1)}{2}\frac{\alpha^2}{(n+\beta)^2}r^{m-2}\sum_{j=0}^{m-2}\frac{n^j\alpha^{m-2-j}}{(n+\beta)^{m-2}} \\
&\leq \frac{m(m-1)}{2}\frac{\alpha^2}{(n+\beta)^2}r^{m-2},
\end{aligned}$$

we get

$$\begin{aligned}
&\left|U_n^{(\alpha,\beta)}(e_m;z)-U_n(e_m;z)-\frac{\alpha-\beta z}{n+\beta}mz^{m-1}\right| \\
&\leq \frac{m(m-1)\alpha^2}{2(n+\beta)^2}r^{m-2}+\frac{2m(m-1)(m-2)\alpha}{n(n+\beta)}r^{m-2}+\frac{m(m-1)\beta^2}{2(n+\beta)^2}r^m \\
&+\frac{m(m-1)\alpha\beta}{(n+\beta)^2}r^{m-1}+\frac{2m^2(m-1)\beta}{n(n+\beta)}r^m+\frac{m(m-1)\beta}{(n+\beta)^2}r^m.
\end{aligned}$$

■

Theorem 6.15 ([187]). *Let $R>1$, $f\in H(\mathbb{D}_R)$, and $0<\alpha\leq\beta$. If f is not a constant, then for any $r\in[1,R)$, we have*

$$\left\|U_n^{(\alpha,\beta)}(f)-f\right\|_r \geq \frac{1}{n+1}C_r(f), \quad n\in\mathbb{N},$$

where the constant $C_r(f)$ depends on f and r but is independent of n.

Proof. For all $z\in\mathbb{D}_R$ and $n\in\mathbb{N}$, we get

$$\begin{aligned}
&U_n^{(\alpha,\beta)}(f;z)-f(z) \\
&=\frac{1}{n+1}\left\{\frac{n+1}{n+\beta}(\alpha-\beta z)f'(z)+z(1-z)f''(z)\right. \\
&\quad\left.+(n+1)\left(U_n^{(\alpha,\beta)}(f;z)-f(z)-\frac{\alpha-\beta z}{n+\beta}f'(z)-\frac{z(1-z)}{n+1}f''(z)\right)\right\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left\|U_n^{(\alpha,\beta)}(f)-f\right\|_r \\
&\geq \frac{1}{n+1}\left\{\begin{array}{c}\left\|\frac{n+1}{n+\beta}(\alpha e_0-\beta e_1)f'+e_1(1-e_1)f''\right\|_r \\ -(n+1)\left\|U_n^{(\alpha,\beta)}(f)-f-\frac{\alpha e_0-\beta e_1}{n+\beta}f'-\frac{e_1(1-e_1)}{n+1}f''\right\|_r\end{array}\right\}.
\end{aligned}$$

Because, by hypothesis, f is not a constant function in $\mathbb{D}_R$, it follows that

$$\left\| \frac{n+1}{n+\beta} \left(\alpha e_0 - \beta e_1\right) f' + e_1 \left(1 - e_1\right) f'' \right\|_r > 0.$$

Indeed, if we assume the contrary, it follows that $\frac{n+1}{n+\beta} \left(\alpha - \beta z\right) f' \left(z\right) + z \left(1 - z\right) f'' \left(z\right) = 0$ for all $z \in \overline{\mathbb{D}}_r$. Writing the expansion of $f' \left(z\right)$ and $f'' \left(z\right)$ in the last equality, we can see that $a_m = 0$ for $m = 1, 2, \ldots$. Thus, f is constant, which contradicts the hypothesis.

Now, by Theorem 6.14, we have

$$(n+1) \left| U_n^{(\alpha,\beta)} \left(f; z\right) - f \left(z\right) - \frac{\alpha - \beta z}{n+\beta} f' \left(z\right) - \frac{z \left(1-z\right)}{n+1} f'' \left(z\right) \right| \to 0 \quad \text{as} \quad n \to \infty.$$

Moreover,

$$\left\| \frac{n+1}{n+\beta} \left(\alpha e_0 - \beta e_1\right) f' + e_1 \left(1 - e_1\right) f'' \right\|_r \to \left\| \left(\alpha e_0 - \beta e_1\right) f' + e_1 \left(1 - e_1\right) f'' \right\|_r$$

as $n \to \infty$. Consequently, there exists n_1 (depending only on f and r) such that for all $n \geq n_1$, we have

$$\begin{aligned} &\left\| \frac{n+1}{n+\beta} \left(\alpha e_0 - \beta e_1\right) f' + e_1 \left(1 - e_1\right) f'' \right\|_r \\ &- (n+1) \left\| U_n^{(\alpha,\beta)} \left(f\right) - f - \frac{\alpha e_0 - \beta e_1}{n+\beta} f' - \frac{e_1 \left(1 - e_1\right)}{n+1} f'' \right\|_r \\ &\geq \frac{1}{2} \left\| \frac{n+1}{n+\beta} \left(\alpha e_0 - \beta e_1\right) f' + e_1 \left(1 - e_1\right) f'' \right\|_r, \end{aligned}$$

which implies

$$\left\| U_n^{(\alpha,\beta)} \left(f\right) - f \right\|_r \geq \frac{1}{2\left(n+1\right)} \left\| \left(\alpha e_0 - \beta e_1\right) f' + e_1 \left(1 - e_1\right) f'' \right\|_r, \quad \text{for all } n \geq n_1.$$

For $1 \leq n \leq n_1 - 1$, we have

$$\left\| U_n^{(\alpha,\beta)} \left(f\right) - f \right\|_r \geq \frac{1}{n+1} \left((n+1) \left\| U_n^{(\alpha,\beta)} \left(f\right) - f \right\|_r \right) = \frac{1}{n+1} M_{r,n} \left(f\right) > 0,$$

which finally implies that

$$\left\| U_n^{(\alpha,\beta)} \left(f\right) - f \right\|_r \geq \frac{1}{n+1} C_r \left(f\right), \text{for all } n,$$

with $C_r \left(f\right) = \min \left\{ M_{r,1} \left(f\right), \ldots, M_{r,n_1-1} \left(f\right), \frac{1}{2} \left\| \left(\alpha e_0 - \beta e_1\right) f' + e_1 \left(1 - e_1\right) f'' \right\|_r \right\}$.

■

6.5 New Complex Durrmeyer Operators

In 2010, Anastassiou and Gal [28] studied the approximation properties of the complex Bernstein–Durrmeyer operator defined by

$$D_n(f,z) = (n+1)\sum_{k=0}^{n} p_{n,k}(z)\int_0^1 f(t)p_{n,k}(t)dt, z \in \mathbb{C}, \tag{6.9}$$

where

$$p_{n,k}(z) := \binom{n}{k} z^k(1-z)^{n-k}.$$

We have

$$D_n(1,z) = 1, D_n(t,z) = \frac{1+nz}{n+2}, D_n(t^2,z) = \frac{n(n-1)z^2+4nz+2}{(n+2)(n+3)},$$

and the following asymptotic formula holds (see [28]):

$$\lim_{n\to\infty} n[D_n(f,z) - f(z)] = [z(1-z)f'(z)]',$$

uniformly in compact disks $|z| \le r$ with $r \ge 1$. Now, if we change the origin and scale of the above operators and we define $r_n(x) = \frac{(n+2)x-1}{n}$, then we get $r_n(x) \in [0,1]$ for all $x \in \left[\frac{1}{3},\frac{2}{3}\right]$. Also, by replacing $z = x \in [0,1]$ in the definition of $D_n(f,z)$ by $r_n(x)$, we can define the new Durrmeyer-type operators as

$$T_n(f,x) = (n+1)\sum_{k=0}^{n} m_{n,k}(x)\int_0^1 f(t)p_{n,k}(t)dt,\ x \in \left[\frac{1}{3},\frac{2}{3}\right], \tag{6.10}$$

with

$$m_{n,k}(x) := \binom{n}{k}\frac{((n+1)-(n+2)x)^{n-k}((n+2)x-1)^k}{n^n},$$

where by simple computation, we now get

$$T_n(1,x) = 1, T_n(t,x) = x,$$

$$T_n(t^2,x) = \frac{(n-1)(n+2)^2x^2+2(n^2+3n+2)x-(n+1)}{n(n+2)(n+3)}$$

and the following new asymptotic formula:

$$\lim_{n\to\infty} n[T_n(f,x) - f(x)] = x(1-x)f''(x), \text{ uniformly in } \left[\frac{1}{3}, \frac{2}{3}\right].$$

Thus, by Korovkin's theory, the new real operators $T_n(f,x)$ keep good approximation properties on the interval $\left[\frac{1}{3}, \frac{2}{3}\right]$. The aim of this section is to present approximation results for the complex operator $T_n(f,z)$, $z \in \mathbb{C}$, $n \in \mathbb{N}$. Such results provide evidence of the overconvergence phenomenon for this new type of Durrmeyer-type operator, that is, the extensions of approximation properties with exact quantitative estimates, from the real interval $\left[\frac{1}{3}, \frac{2}{3}\right]$, to compact disks in the complex plane centered at the origin. To prove the main results, we need the following lemmas:

Lemma 6.11 ([79]). *For all $e_p = t^p$, $p \in \mathbb{N} \cup \{0\}$, and $z \in \mathbb{C}$, we have*

$$T_n\left(e_{p+1}, z\right) = \frac{[(n+2)z-1][(n+1)-(n+2)z]}{n(n+2)(n+p+2)} T_n'\left(e_p, z\right) + \frac{(n+2)z+p}{n+p+2} T_n\left(e_p, z\right). \tag{6.11}$$

Lemma 6.12 ([79]). *For all $n \in \mathbb{N}$ and $z \in \mathbb{C}$, we have $T_n(f,z) = D_n\left(f, r_n(z)\right)$.*

Theorem 6.16 ([79]). *Let $1 \le r < \frac{R}{4}$ and $f(z) = \sum_{k=0}^{\infty} c_k z^k$ for all $|z| < R$. For all $|z| \le r$ and $n \in \mathbb{N}$, we have*

$$|T_n(f,z) - f(z)| \le \frac{C_r(f)}{n},$$

where $C_r(f) = 2\sum_{p=1}^{\infty} |c_p| p(p+1)(4r)^p + \sum_{p=1}^{\infty} |c_p| \cdot p \cdot (3r)^p < 3\sum_{p=1}^{\infty} |c_p| p(p+1)(4r)^p < \infty$.

Proof. By Lemma 6.12, we have

$$\begin{aligned}|T_n(f,z) - f(z)| &= |D_n\left(f, r_n(z)\right) - f\left(r_n(z)\right) + f\left(r_n(z)\right) - f(z)| \\ &\le |D_n\left(f, r_n(z)\right) - f\left(r_n(z)\right)| + |f\left(r_n(z)\right) - f(z)|.\end{aligned}$$

On the other hand, since for $|z| \le r$ and $r \ge 1$ we have $|r_n(z)| \le 4r$ and $|f\left(r_n(z)\right)| = \left|\sum_{k=0}^{\infty} c_k \left(r_n(z)\right)^k\right| \le \sum_{k=0}^{\infty} |c_k|(4r)^k < \infty$, by making the substitution $u = \frac{(n+2)z-1}{n}$ in Corollary 2.2(ii) in [28], we immediately obtain

$$|D_n\left(f, r_n(z)\right) - f\left(r_n(z)\right)| \le \frac{M_{4r}(f)}{n},$$

where $M_{4r}(f) = 2\sum_{p=1}^{\infty} |c_p| p(p+1)(4r)^p$.

Also, by the mean value theorem in complex analysis, for all $|z| \leq r$, it follows that

$$|f(r_n(z)) - f(z)| \leq |r_n(z) - z| \|f'\|_{3r} \leq \frac{3r}{n} \|f'\|_{3r} \leq \frac{1}{n} \sum_{p=1}^{\infty} |c_p| p(3r)^p,$$

which proves the theorem. Note here the notation $\|f\|_r = \sup_{|z| \leq r} |f(z)|$. ■

Theorem 6.17 ([79] Voronovskaja-type result). *Let $R > 4$, and suppose that $f : D_R \to \mathbb{C}$ is analytic in $D_R = \{z \in \mathbb{C} : |z| < R\}$; that is, we can write $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in D_R$. For any fixed $r \in \left[1, \frac{R}{4}\right)$ and for all $n \in \mathbb{N}$, $|z| \leq r$, we have*

$$\left| T_n(f, z) - f(z) - \frac{z(1-z) f''(z)}{n} \right| \leq \frac{M_r(f)}{n^2},$$

where $M_r(f) = \sum_{k=1}^{\infty} |c_k| k B_{k,r} (4r)^k < \infty$ and

$$B_{k,r} = A_k + 5k^3(r+1),$$

with

$$\begin{aligned} A_k &= k^3 + 2k^2 + 3k + 2 + 2(k-1)^2(k+2) + (k-1) \\ &\quad [2k^2 + 4k + 4 + 2(k+2)(2k+3)] \\ &\quad +(k-1)[2(k-1)^2 + 1 + (k-1)(k+2)] \\ &\quad +(k-1)^2(k+2) + (k-1)(k-2)^2. \end{aligned}$$

Proof. We denote $e_k(z) = z^k, k = 0, 1, 2, \ldots$ and $\pi_{k,n}(z) = T_n(e_k, z)$. By the above Lemma 6.12 and by the proof of Corollary 2.2(ii) in [28], we can clearly write $T_n(f, z) = \sum_{k=0}^{\infty} c_k \pi_{k,n}(z)$. Also, since

$$\begin{aligned} \frac{z(1-z) f''(z)}{n} &= \frac{z(1-z)}{n} \sum_{k=2}^{\infty} c_k k(k-1) z^{k-2} \\ &= \frac{1}{n} \sum_{k=1}^{\infty} c_k [k(k-1) - k(k-1)z] z^{k-1}, \end{aligned}$$

it follows that

$$\left| T_n(f,z) - f(z) - \frac{z(1-z)f''(z)}{n} \right|$$
$$\leq \sum_{k=1}^{\infty} |c_k| \left| \pi_{k,n}(z) - e_k(z) - \frac{k(k-1)(1-z)z^{k-1}}{n} \right|,$$

for all $z \in D_R, n \in \mathbb{N}$.

By Lemma 6.11, for all $n \in \mathbb{N}, z \in \mathbb{C}$, and $k = 0, 1, 2, \ldots$, we have

$$\pi_{k+1,n}(z) = \frac{[(n+2)z-1][(n+1)-(n+2)z]}{n(n+2)(n+k+2)}\pi'_{k,n}(z) + \frac{(n+2)z+k}{n+k+2}\pi_{k,n}(z).$$

If we denote

$$E_{k,n}(z) = \pi_{k,n}(z) - e_k(z) - \frac{k(k-1)(1-z)z^{k-1}}{n},$$

then, because by Lemma 6.12 we have $\pi_{k,n}(z) = D_n(e_k, r_n(z))(z)$, it follows by (2.1) in [28] that $E_{k,n}(z)$ is a polynomial of degree less than or equal to k. By simple computation and by using the above recurrence relation, we are led to

$$E_{k,n}(z) = \frac{[(n+2)z-1][(n+1)-(n+2)z]}{n(n+2)(n+k+1)}E'_{k-1,n}(z)$$
$$+\frac{(n+2)z+k-1}{n+k+1}E_{k-1,n}(z) + X_{k,n}(z), \tag{6.12}$$

where

$$X_{k,n}(z) = \frac{[(n+2)z-1][(n+1)-(n+2)z]}{n(n+2)(n+k+1)}\cdot$$
$$\cdot\left[(k-1)z^{k-2} + \frac{(k-1)(k-2)^2}{n}z^{k-3} - \frac{(k-1)^2(k-2)}{n}z^{k-2}\right]$$
$$+\frac{(n+2)z+k-1}{n+k+1}\left[z^{k-1} + \frac{(k-1)(k-2)}{n}z^{k-2} - \frac{(k-1)(k-2)}{n}z^{k-1}\right]$$
$$-z^k - \frac{k(k-1)}{n}z^{k-1} + \frac{k(k-1)}{n}z^k = \frac{(n+2)^2z-(n+2)^2z^2-(n+1)}{n(n+2)(n+k+1)}\cdot$$
$$\cdot\left[(k-1)z^{k-2} + \frac{(k-1)(k-2)^2}{n}z^{k-3} - \frac{(k-1)^2(k-2)}{n}z^{k-2}\right]$$
$$+\frac{(n+2)z+k-1}{n+k+1}\left[z^{k-1} + \frac{(k-1)(k-2)}{n}z^{k-2} - \frac{(k-1)(k-2)}{n}z^{k-1}\right]$$

$$-z^k - \frac{k(k-1)}{n}z^{k-1} + \frac{k(k-1)}{n}z^k = \frac{z^{k-2}}{n(n+2)(n+k+1)}\cdot$$

$$\cdot\Bigg[(k-1)(n+2)^2z + \frac{(k-1)(k-2)^2}{n}(n+2)^2 - \frac{(k-1)^2(k-2)}{n}(n+2)^2z$$

$$-(n+2)^2(k-1)z^2 - \frac{(k-1)(k-2)^2}{n}(n+2)^2z + \frac{(k-1)^2(k-2)}{n}(n+2)^2z^2$$

$$-(n+1)(k-1) - \frac{(k-1)(k-2)^2}{nz}(n+1) + \frac{(k-1)^2(k-2)}{n}(n+1)$$

$$+n(n+2)^2z^2 + (k-1)(k-2)(n+2)^2z - (k-1)(k-2)(n+2)^2z^2$$

$$+(k-1)n(n+2)z + (k-1)^2(k-2)(n+2) - (k-1)^2(k-2)(n+2)z$$

$$-n(n+2)(n+k+1)z^2 - k(k-1)(n+2)(n+k+1)z$$

$$+k(k-1)(n+2)(n+k+1)z^2\Bigg]$$

$$= \frac{z^{k-2}}{n(n+k+1)}\Bigg[z\Bigg\{(k-1)(n+2) - (k-1)^2(k-2)\frac{(n+2)}{n}$$

$$-(k-1)(k-2)^2\frac{(n+2)}{n}$$

$$+(k-1)(k-2)(n+2) + (k-1)n - (k-1)^2(k-2) - k(k-1)(n+k+1)\Bigg\}$$

$$+z^2\Bigg\{-(n+2)(k-1) + (k-1)^2(k-2)\frac{(n+2)}{n} + n(n+2)$$

$$-(k-1)(k-2)(n+2) - n(n+k+1) + k(k-1)(n+k+1)\Bigg\}$$

$$+(k-1)\Bigg\{\frac{(k-2)^2}{n}(n+2) - \frac{n+1}{n+2} + (k-1)(k-2)\frac{(n+1)}{n(n+2)}$$

$$+(k-1)(k-2)\Bigg\}\Bigg]$$

$$-\frac{z^{k-3}}{n^2(n+2)(n+k+1)}(k-1)(k-2)^2(n+1)$$

$$= \frac{z^{k-2}}{n(n+k+1)}\Bigg[z(k-1)\Bigg\{-2k^2 + 4k - 4 - (k-2)(2k-3)\frac{(n+2)}{n}\Bigg\}$$

$$+z^2\Bigg\{k^3 - 2k^2 + 3k - 2 + (k-1)^2(k-2)\frac{(n+2)}{n}\Bigg\}$$

$$+(k-1)\left\{(k-2)^2\frac{(n+2)}{n}-\frac{(n+1)}{(n+2)}+(k-1)(k-2)\right\}\Bigg]$$
$$+\frac{z^{k-2}}{n^2(n+2)(n+k+1)}(k-1)^2(k-2)(n+1)$$
$$-\frac{z^{k-3}}{n^2(n+2)(n+k+1)}(k-1)(k-2)^2(n+1),$$

for all $k \geq 1, n \in \mathbb{N}$, and $|z| \leq r$.

Now, using the estimate in Theorem 6.16 for $f(z) = e_k(z)$, we get

$$|\pi_{k,n}(z) - e_k(z)| \leq \frac{3(k+1)^2(4r)^k}{n},$$

for all $k, n \in \mathbb{N}, |z| \leq r$, with $1 \leq r$.

Since for all $k, n \in \mathbb{N}, k \geq 1$, and $|z| \leq r$, we have

$$\left|\frac{[(n+2)z-1][(n+1)-(n+2)z]}{n(n+2)(n+k+1)}\right| \leq \frac{[(n+2)r+1][(n+1)+(n+2)r]}{n(n+2)(n+k+1)}$$
$$\leq \frac{(n+2)^2(r+1)^2}{n(n+2)(n+k+1)} \leq \frac{(r+1)^2}{n},$$

and

$$\left|\frac{(n+2)z+k-1}{n+k+1}\right| \leq \frac{(n+2)r+k-1}{n+k+1} \leq 3r,$$

from (6.12), it easily follows that

$$|E_{k,n}(z)| \leq \frac{(r+1)^2}{n}|E'_{k-1,n}(z)| + 3r|E_{k-1,n}(z)| + |X_{k,n}(z)|.$$

Now we shall find the estimation of $|E'_{k-1,n}(z)|$ for $k \geq 1$. Taking into account the fact that $E_{k-1,n}(z)$ is a polynomial of degree $\leq k-1$, we have

$$|E'_{k-1,n}(z)| \leq \frac{k-1}{r}||E_{k-1,n}||_r$$
$$\leq \frac{k-1}{r}\left[||\pi_{k-1,n} - e_{k-1}||_r + \left|\left|\frac{(k-1)(k-2)e_{k-2}[1-e_1]}{n}\right|\right|_r\right]$$
$$\leq \frac{k-1}{r}\left[\frac{3k^2(4r)^{k-1}}{n} + \frac{r^{k-2}k^2(1+r)}{n}\right]$$

$$\leq \frac{k^3}{n}\left[3(4r)^{k-1} + \frac{1+r}{r}r^{k-1}\right] \leq \frac{5k^3(4r)^{k-1}}{n}.$$

Thus,

$$\frac{(r+1)^2}{n}|E'_{k-1,n}(z)| \leq \frac{5k^3(r+1)(4r)^k}{n^2}$$

and

$$|E_{k,n}(z)| \leq \frac{5k^3(r+1)(4r)^k}{n^2} + 3r|E_{k-1,n}(z)| + |X_{k,n}(z)|,$$

where

$$\begin{aligned}|X_{k,n}(z)| \leq \frac{r^{k-2}}{n^2}\Big[& r^2(k^3+2k^2+3k+2+2(k-1)^2(k+2)) \\ & +r(k-1)(2k^2+4k+4+2(k+2)(2k+3)) \\ & +(k-1)(2(k-2)^2+1+(k-1)(k+2))\Big] \\ & +\frac{r^{k-2}(k-1)^2(k+2)}{n^2} + \frac{r^{k-3}(k-1)(k-2)^2}{n^2} \leq \frac{(4r)^k}{n^2}A_k,\end{aligned}$$

for all $|z| \leq r, k \geq 1, n \in \mathbb{N}$, with

$$\begin{aligned}A_k = & \, k^3+2k^2+3k+2+2(k-1)^2(k+2)+(k-1) \\ & [2k^2+4k+4+2(k+2)(2k+3)] \\ & +(k-1)[2(k-1)^2+1+(k-1)(k+2)] \\ & +(k-1)^2(k+2)+(k-1)(k-2)^2.\end{aligned}$$

Thus, for all $|z| \leq r, k \geq 1, n \in \mathbb{N}$,

$$|E_{k,n}(z)| \leq 3r|E_{k-1,n}(z)| + \frac{(4r)^k}{n^2}B_{k,r} \leq 4r|E_{k-1,n}(z)| + \frac{(4r)^k}{n^2}B_{k,r},$$

where $B_{k,r}$ is a polynomial of degree 3 in k defined as

$$B_{k,r} = A_k + 5k^3(r+1).$$

But $E_{0,n}(z) = 0$, for any $z \in C$, and therefore by writing the last inequality for $k = 1, 2, \ldots$, we easily obtain, step by step, the following:

$$|E_{k,n}(z)| \le \frac{(4r)^k}{n^2}\sum_{j=1}^{k} B_{j,r} \le \frac{k(4r)^k}{n^2} B_{k,r}.$$

We conclude that

$$\left|T_n(f,z) - f(z) - \frac{z(1-z)f''(z)}{n}\right| \le \sum_{k=1}^{\infty}|c_k||E_{k,n}| \le \frac{1}{n^2}\sum_{k=1}^{\infty}|c_k|kB_{k,r}(4r)^k.$$

As $f^{(4)}(z) = \sum_{k=4}^{\infty} c_k k(k-1)(k-2)(k-3)z^{k-4}$ and the series is absolutely convergent in $|z| \le r$, it easily follows that $\sum_{k=4}^{\infty}|c_k|k(k-1)(k-2)(k-3)r^{k-4} < \infty$, which implies that $\sum_{k=1}^{\infty}|c_k|kB_{k,r}r^k < \infty$. This completes the proof of the theorem. ∎

Remark 6.1. It is interesting to note that although by Lemma 6.12, the operator T_n is the M_n operator but taken on the value $\frac{(n+2)z-1}{n}$ instead of z, their asymptotic limits differ essentially. More exactly, while by Theorem 6.17 we have $\lim_{n\to\infty} n[T_n(f,z) - f(z)] = z(1-z)f''(z)$, uniformly for $|z| \le r$, by Theorem 2.4 in [28], we have $\lim_{n\to\infty} n[M_n(f,z) - f(z)] = [z(1-z)f'(z)]'$, uniformly for $|z| \le r$.

Theorem 6.18 ([79] Exact order of approximation). *Let $R > 4$, and suppose that $f : D_R \to \mathbb{C}$ is analytic in D_R; that is, we can write $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in D_R$. If f is not a polynomial of degree ≤ 1, then for any $r \in \left[1, \frac{R}{4}\right)$, we have*

$$\|T_n(f,\cdot) - f\|_r \ge \frac{C_r(f)}{n}, n \in \mathbb{N},$$

where $C_r(f)$ depends only on f and r.

Proof. For all $z \in D_r$ and $n \in \mathbb{N}$, we have

$$T_n(f,z) - f(z)$$
$$= \frac{1}{n}\left[z(1-z)f''(z) + \frac{1}{n}\left\{n^2\left(T_n(f,z) - f(z) - \frac{z(1-z)f''(z)}{n}\right)\right\}\right].$$

Also, we have

$$||F + G||_r \ge |||F||_r - ||G||_r| \ge ||F||_r - ||G||_r.$$

It follows that

$$||T_n(f,\cdot) - f||_r$$
$$\ge \frac{1}{n}\left[||e_1(1-e_1)f''||_r - \frac{1}{n}\left\{n^2\left\|T_n(f,\cdot) - f - \frac{e_1(1-e_1)f''}{n}\right\|_r\right\}\right].$$

Taking into account that, by hypothesis, f is not a polynomial of degree ≤ 1 in D_R, we get $||e_1(1-e_1)f''||_r > 0$.

Indeed, suppose to the contrary that it follows that $z(1-z)f''(z) = 0$ for all $|z| \leq r$; that is, $f(z) = az + b$ for all $|z| \leq r$ with $z \neq 0$ and $z \neq 1$. But since f is analytic in $\overline{D_R}$ and $r \geq 1$, by the identity theorem on analytic functions, it follows that $f(z) = az + b$ for all $|z| \leq r$, and therefore $f(z) = az + b$ for all $|z| < R$, which contradicts the hypothesis.

Now by Theorem 6.17, we have

$$n^2 \left|\left| T_n(f,\cdot) - f - \frac{e_1(1-e_1)f''}{n} \right|\right|_r \leq M_r(f).$$

Therefore, there exists an index n_0 depending only on f and r such that for all $n \geq n_0$, we have

$$||e_1(1-e_1)f''||_r - \frac{1}{n}\left\{ n^2 \left|\left| T_n(f,\cdot) - f - \frac{e_1(1-e_1)f''}{n} \right|\right|_r \right\} \geq \frac{1}{2}||e_1(1-e_1)f''||_r,$$

which immediately implies

$$||T_n(f,\cdot) - f||_r \geq \frac{1}{2n}||e_1(1-e_1)f''||_r, \forall n \geq n_0.$$

For $n \in \{1, 2, \ldots n_0 - 1\}$, we obviously have $||T_n(f,\cdot) - f||_r \geq \frac{M_{r,n}(f)}{n}$, with $M_{r,n}(f) = ||T_n(f,\cdot) - f||_r > 0$. Indeed, if we had $||T_n(f,\cdot) - f||_r = 0$, then it would follow that $T_n(f,z) = f(z)$ for all $|z| \leq r$, which is valid only for f, a linear function, contradicting the hypothesis on f. Therefore, we finally obtain $||T_n(f,\cdot) - f||_r \geq \frac{C_r(f)}{n}$ for all n, where

$$C_r(f) = \min\{M_{r,1}(f), M_{r,2}(f), \ldots, M_{r,n_0-1}(f), \frac{1}{2}||e_1(1-e_1)f''||_r\},$$

which completes the proof. ■

Theorem 6.19 ([79] Simultaneous approximation). *Let $R > 4$, and suppose that $f : D_R \to \mathbb{C}$ is analytic in D_R; that is, $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in D_R$. Also, let $1 \leq r < r_1 < \frac{R}{4}$ and $p \in \mathbb{N}$ be fixed. If f is not a polynomial of degree $\leq \max\{1, p-1\}$, then we have*

$$||T_n^{(p)}(f,\cdot) - f^{(p)}||_r \sim \frac{1}{n},$$

where the constants in the equivalence depend only on f, r, r_1, and p.

Proof. Denoting by Γ the circle of radius r_1 with its center at 0 (where $r_1 > r \geq 1$), by Cauchy's integral formula for derivatives, we have for all $|z| < r$ and $n \in \mathbb{N}$,

$$T_n^{(p)}(f,z) - f^{(p)}(z) = \frac{p!}{2\pi i}\int_\Gamma \frac{T_n(f,u)-f(u)}{(u-z)^{p+1}}du,$$

which, by Theorem 6.16 and by the inequality $|u-z| \ge r_1 - r$ valid for all $|z| \le r$ and $u \in \Gamma$, implies

$$||T_n^{(p)}(f,\cdot) - f^{(p)}||_r \le \frac{p!}{2\pi}\frac{2\pi r_1}{(r_1-r)^{p+1}}||T_n(f,\cdot)-f||_r \le C_{r_1}(f)\frac{p!r_1}{n(r_1-r)^{p+1}}.$$

It remains to prove the lower estimation for $||T_n^{(p)}(f,\cdot) - f^{(p)}||_r$.

First, evidently for all $u \in \Gamma$ and $n \in \mathbb{N}$, we can write the identity

$$T_n(f,u) - f(u)$$
$$= \frac{1}{n}\left[u(1-u)f''(u) + \frac{1}{n}\left\{n^2\left(T_n(f,u)-f(u)-\frac{u(1-u)f''(u)}{n}\right)\right\}\right].$$

Substituting it in the earlier Cauchy integral formula, we obtain

$$T_n^{(p)}(f,z) - f^{(p)}(z) = \frac{1}{n}\Bigg[[z(1-z)f''(z)]^{(p)}$$
$$+\frac{1}{n}\frac{p!}{2\pi i}\int_\Gamma \frac{n^2\left(T_n(f,u)-f(u)-\frac{u(1-u)f''(u)}{n}\right)}{(u-z)^{p+1}}du\Bigg].$$

This implies

$$\left|\left|T_n^{(p)}(f,\cdot) - f^{(p)}\right|\right|_r \ge \frac{1}{n}\Bigg[||[e_1(1-e_1)f'']^{(p)}||_r$$
$$-\frac{1}{n}\left|\left|\frac{p!}{2\pi i}\int_\Gamma \frac{n^2\left(T_n(f,u)-f(u)-\frac{u(1-u)f''(u)}{n}\right)}{(u-\cdot)^{p+1}}du\right|\right|_r\Bigg].$$

Now, applying Theorem 6.17, we get

$$\left|\left|\frac{p!}{2\pi i}\int_\Gamma \frac{n^2\left(T_n(f,u)-f(u)-\frac{u(1-u)f''(u)}{n}\right)}{(u-\cdot)^{p+1}}du\right|\right|_r$$
$$\le \frac{p!}{2\pi}\frac{2\pi r_1 n^2}{(r_1-r)^{p+1}}\left|\left|T_n(f,\cdot)-f-\frac{e_1(1-e_1)f''}{n}\right|\right|_r \le \frac{M_{r_1}(f)p!r_1}{(r_1-r)^{p+1}}.$$

But, by hypothesis on f, we have $||[e_1(1-e_1)f'']^{(p)}||_r > 0$.

Indeed, if we suppose the contrary, it would follow that $[z(1-z)f''(z)]^{(p)} = 0$, for all $|z| \leq r$. This would imply that $z(1-z)f''(z) = Q_{p-1}(z)$, for all $|z| \leq r$, where $Q_{p-1}(z)$ is a polynomial of degree $\leq p-1$.

Suppose first that $p = 1$. It would follow that $z(1-z)f''(z) = C$, where C is a constant, and because of the analyticity of f'' in $|z| \leq r$ with $r \geq 1$, it necessarily would follow that $C = 0$; that is, $f''(z) = 0$, for all $|z| \leq r$, which is equivalent to $f(z) = az + b$, for all $|z| \leq r$, contradicting the hypothesis in Theorem 6.19 for $p = 1$.

Suppose now that $p = 2$. It would follow that $z(1-z)f''(z) = az+b$ and because of the analyticity of f'' in $|z| \leq r$ with $r \geq 1$, it necessarily would follow that $a = b = 0$; that is, $f''(z) = 0$, for all $|z| \leq r$, which is equivalent to $f(z) = Az+B$, for all $|z| \leq r$, again contradicting the hypothesis in Theorem 6.18 for $p = 2$.

Now, if we suppose that $p \geq 3$, the analyticity of f'' would imply that $Q_{p-1}(0) = Q_{p-1}(1) = 0$ (because otherwise the analyticity of $f''(z)$ for $|z| \leq r$, with $r \geq 1$, would be contradicted). Therefore, after simplification with $z(1-z)$, we would obtain that f'' is a polynomial of degree $\leq p-3$, namely, that f would be a polynomial of degree $\leq p-1$, contradicting the hypothesis.

In conclusion, $||[e_1(1-e_1)f'']^{(p)}||_r > 0$, and in continuation, reasoning exactly as in the proof of Theorem 6.18, we immediately get the desired conclusion. ■

6.6 Complex q-Durrmeyer-Type Operators

In recent years, applications of q-calculus in the area of approximation theory and number theory have been an active area of research. In 2008, Gupta and Wang [131] introduced certain q-Durrmeyer-type operators of the real variable $x \in [0, 1]$ and studied some approximation results in the case of real variables. Agrawal and Gupta [15] extended this study in 2012 to the complex variable for analytic functions in compact disks. They established the quantitative Voronovskaja-type estimate. In this way, we provided evidence of the overconvergence phenomenon for these q-Durrmeyer polynomials, namely, the extensions of approximation properties (with quantitative estimates) from the real interval $[0, 1]$ to compact disks in the complex plane.

For ready reference, we begin with some notations and definitions of q-calculus: For each nonnegative integer k, the q-integer $[k]_q$ and the q-factorial $[k]_q!$ are defined by

$$[k]_q := \begin{cases} (1-q^k)/(1-q), & q \neq 1 \\ k, & q = 1 \end{cases},$$

and

$$[k]_q! := \begin{cases} [k]_q\,[k-1]_q \cdots [1]_q, & k \geq 1 \\ 1, & k = 0 \end{cases},$$

respectively. For the integers $n,\ k,\ n \geq k \geq 0$, the q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}\,.$$

In this section, we present approximation results for the complex q-Durrmeyer operators (introduced and studied in the case of a real variable by Gupta-Wang [131]), defined by

$$M_{n,q}(f;z) = [n+1]_q \sum_{k=1}^{n} q^{1-k} p_{n,k}(q;z) \int_0^1 f(t) p_{n,k-1}(q;qt) d_q t$$
$$+ f(0) p_{n,0}(q;z), \tag{6.13}$$

where $z \in \mathbb{C}, n = 1, 2, \ldots ; q \in (0,1)$ and $(a-b)_q^m = \Pi_{j=0}^{m-1}(a - q^j b)$, q-Bernstein basis functions, are defined as

$$p_{n,k}(q;z) := \begin{bmatrix} n \\ k \end{bmatrix}_q z^k (1-z)_q^{n-k};$$

also, here the q-Beta functions [163] are given as

$$B_q(m,n) = \int_0^1 t^{m-1}(1-qt)_q^{n-1} d_q t, \ \ m, n > 0.$$

Throughout this section, we use the notation $D_R = \{z \in \mathbb{C} : |z| < R\}$, and by $H(D_R)$, we mean the set of all analytic functions on $f : D_R \to \mathbb{C}$ with $f(z) = \sum_{k=0}^{\infty} a_k z^k$ for all $z \in D_R$. The norm $\|f\|_r = \max\{|f(z)| : |z| \leq r\}$. We denote $\pi_{p,n}(q;z) = M_{n,q}(e_p;z)$ for all $e_p = t^p$, $p \in \mathbb{N} \cup \{0\}$.

The following main theorems were studied for the operators (6.13).

Theorem 6.20 ([15] Upper bound). *Let $f(z) = \sum_{p=0}^{\infty} a_p z^p$ for all $|z| < R$, and let $1 \leq r \leq R$; then for all $|z| \leq r$, $q \in (0,1)$ and $n \in \mathbb{N}$,*

$$\left|M_{n,q}(f;z) - f(z)\right| \leq \frac{K_r(f)}{[n+2]_q},$$

where $K_r(f) = (1+r)\sum_{p=1}^{\infty} |a_p| p(p+1) r^{p-1} < \infty$.

Theorem 6.21 ([15] Voronovskaja-type asymptotic result). *Suppose that $f \in H(D_R)$, $R > 1$. Then, for any fixed $r \in [1, R]$ and for all $n \in \mathbb{N}, |z| \leq r$ and*

$q \in (0, 1)$, we have

$$\left| M_{n,q}(f; z) - f(z) - \frac{z(1-z)f''(z) - 2zf'(z)}{[n]_q} \right| \le \frac{M_r(f)}{[n]_q^2} + 2(1-q) \sum_{k=1}^{\infty} |a_k| k r^k,$$

where $M_r(f) = \sum_{k=1}^{\infty} |a_k| k B_{k,r} r^k < \infty$, and

$$B_{k,r} = (k-1)(k-2)(2k-3) + 8k(k-1)^2 + 6(k-1)k^2 + 4k(k-1)^2(1+r).$$

Remark 6.2 ([15]). For $q \in (0, 1)$ fixed, we have $\frac{1}{[n]_q} \to 1 - q$ as $n \to \infty$. Thus, Theorem 6.2 does not provide convergence. But this can be improved by choosing $1 - \frac{1}{n^2} \le q_n < 1$ with $q_n \nearrow 1$ as $n \to \infty$. Indeed, since in this case $\frac{1}{[n]_{q_n}} \to 0$ as $n \to \infty$ and $1 - q_n \le \frac{1}{n^2} \le \frac{1}{[n]_{q_n}^2}$ from Theorem 6.21, we get

$$\left| M_{n,q_n}(f; z) - f(z) - \frac{z(1-z)f''(z) - 2zf'(z)}{[n]_{q_n}} \right| \le \frac{M_r(f)}{[n]_{q_n}^2} + \frac{2}{[n]_{q_n}^2} \sum_{k=1}^{\infty} |a_k| k r^k.$$

Theorem 6.22 ([15] Exact order). *Let $1 - \frac{1}{n^2} \le q_n < 1$, $n \in \mathbb{N}$, $R > 1$, and let $f \in H(D_R)$, $R > 1$. If f is not a polynomial of degree 0, then for any $r \in [1, R)$, we have*

$$\| M_{n,q_n}(f; \cdot) - f \|_r \ge \frac{C_r(f)}{[n]_{q_n}}, \quad n \in \mathbb{N},$$

where the constant $C_r(f) > 0$ depends on f, r, and the sequence $(q_n)_{n \in \mathbb{N}}$, but it is independent of n.

Corollary 6.1 ([15]). *Let $1 - \frac{1}{n^2} < q_n < 1$ for all $n \in \mathbb{N}$, $R > 1$, and suppose that $f \in H(D_R)$. If f is not a polynomial of degree 0, then for any $r \in [1, R)$, we have*

$$\| M_{n,q_n}(f; \cdot) - f \|_r \sim \frac{1}{[n]_{q_n}}, \quad n \in \mathbb{N},$$

where the constants in the above equivalence depend on $f, r, and (q_n)_n$ but are independent of n.

In 2012, Ren and Zeng [206] extended the studies on such operators. The operators (6.13) preserve only constant functions. To make the convergence faster, Ren and Zeng [206] modified the operators (6.13) by changing the scale of reference by replacing x by $[n+2]_q x / [n]_q$, so that they reproduce constant as well as linear functions. The operators discussed in [206] take the following form:

$$R_{n,q}(f;x) = [n+1]_q \sum_{k=1}^{n} q^{1-k} t_{n,k}(q;x) \int_0^1 f(t) p_{n,k-1}(q;qt) d_q t \\ + f(0) t_{n,0}(q;x), \tag{6.14}$$

where $x \in [0, \frac{1}{[3]_q}], n \in \mathbb{N}, q \in (0,1)$, q-Bernstein basis functions are defined as in (6.13), and

$$t_{n,k}(q;x) := \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{[n+2]_q x}{[n]_q} \right)^k \left(1 - \frac{[n+2]_q x}{[n]_q} \right)_q^{n-k}.$$

Remark 6.3 ([206]). If $e_k(x) = x^k, k = 0,1,2$, for $0 < q < 1, x \in [0, \frac{1}{[3]_q}], n \in \mathbb{N}$, one has

$$R_{n,q}(e_0;x), R_{n,q}(e_1;x) = x, R_{n,q}(e_2;x) = \frac{[2]_q x}{[n+3]_q} + \frac{[n-1]_q [n+2]_q q^2 x^2}{[n+3]_q [n]_q}.$$

For the complex operators $R_{n,q}(f;z), z \in \mathbb{C}, n \in \mathbb{N}, 0 < q < 1$, and $t_{n,k}(q;z) := \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{[n+2]_q z}{[n]_q} \right)^k \left(1 - \frac{[n+2]_q z}{[n]_q} \right)_q^{n-k}$, Ren and Zeng [206] obtained the following main results:

Theorem 6.23 ([206] Upper bound). *Let $0 < q < 1, R > 3, D_R = \{z \in \mathbb{C} : |z| < R\}$. Suppose that $f : D_R \to \mathbb{C}$ is analytic in D_R; that is, $f(z) = \sum_{m=0}^{\infty} c_m z^m$ for all $z \in D_R$. Take $1 \le r \le R/3$.*

(i) For all $|z| \le r$ and $n \in \mathbb{N}$, we have

$$\left| R_{n,q}(f;z) - f(z) \right| \le \frac{K_r(f)}{[n]_q},$$

where $K_r(f) = (1+r) \sum_{m=2}^{\infty} |c_m| m(m-1)([3]_q r)^{m-1} < \infty$.

(ii) (Simultaneous approximation) If $1 \le r \le r_1 < R/3$ are arbitrary fixed, then for all $|z| \le r$ and $n, p \in \mathbb{N}$, we have

$$\left| R_{n,q}^{(p)}(f;z) - f^{(p)}(z) \right| \le \frac{K_{r_1}(f) p! r_1}{[n]_q (r_1 - r)^{p+1}},$$

where $K_{r_1}(f)$ is as defined in part (i).

Theorem 6.24 ([206] Voronovskaja-type asymptotic result). *Let $0 < q < 1, R > 3$, and suppose that $f : D_R \to \mathbb{C}$; that is, $f(z) = \sum_{k=0}^{\infty} c_k z^k$ for all $z \in D_R$. For any fixed $r \in [1, R/3]$ and for all $n \in \mathbb{N}, |z| \le r$, we have*

$$\left| R_{n,q}(f;z) - f(z) - \frac{z(1-z)f''(z)}{[n]_q} \right| \leq \frac{M_r(f)}{[n]_q^2},$$

where $M_r(f) = \sum_{k=2}^{\infty} |c_k|(k-1)F_{k,r}([3]_q r)^k < \infty$, *and*

$$F_{k,r} = (k-1)(k-2)\,(2k-3)+6k(k-1)^2+4\,(k-1)\,k^2+4(k-2)(k-1)^2(1+r).$$

Theorem 6.25 ([206] Exact order). *Let* $0 < q_n < 1$ *satisfy* $\lim_{n\to\infty} q_n = 1$, $R > 3$, $D_R = \{z \in \mathbb{C} : |z| < R\}$. *Suppose that* $f : D_R \to \mathbb{C}$ *is analytic in* D_R. *If* f *is not a polynomial of degree* ≤ 1, *then for any* $r \in [1, R/3)$, *we have*

$$\|R_{n,q_n}(f;\cdot) - f\|_r \geq \frac{C_r(f)}{[n]_{q_n}}, \quad n \in \mathbb{N},$$

where $||f||_r = \max\{|f(z)| : |z| \leq r\}$ *and the constant* $C_r(f) > 0$ *depends on* f, r *and on the sequence* $\{q_n\}_{n\in\mathbb{N}}$ *but is independent of* n.

6.7 Complex q-Bernstein–Schurer Operators

The approximation properties of a class of complex q-Bernstein–Schurer operators was studied by Ren and Zeng in 2013 [207]. They obtained the order of simultaneous approximation and a Voronovskaja-type result with quantitative estimate for these complex q-Bernstein–Schurer operators attached to analytic functions on compact disks. More importantly, their results show the overconvergence phenomenon for these complex operators. The complex q-Bernstein–Schurer operators defined for any fixed $p \in \{0, 1, 2, \ldots\}$ by

$$S_{n,p}(f;q;z) = \sum_{k=0}^{n+p} \begin{bmatrix} n+p \\ k \end{bmatrix}_q z^k \prod_{s=0}^{n+p-k-1} (1-q^s z) f([k]_q/[n]_q), \tag{6.15}$$

where $z \in \mathbb{C}, n \in \mathbb{N}, 0 < q < 1$, an empty product is taken to be equal to 1. Obviously, for $p = 0$, the operators defined by (6.15) reduce to q Bernstein polynomials. Denote by $f[x_0, x_1, \ldots, x_k]$ the usual divided difference; that is, $f[x_0] = f(x_0)$, $f[x_0, x_1] = \frac{f(x_1)-f(x_0)}{x_1-x_0}$, and $f[x_0, x_1, \ldots, x_k] = \frac{f(x_1,\ldots,x_k)-f(x_0,\ldots,x_{k-1})}{x_k-x_0}$. Denote by $\triangle_q^k$ the finite difference of order k, that is, $\triangle_q^0 f_j = f_j = f(\frac{[j]_q}{[n]_q})$, $\triangle_q^k f_j = \triangle_q^{k-1} f_{j+1} - q^{k-1}\triangle_q^{k-1} f_j$, where $x_j = \frac{[j]_q}{[n]_q}$, $f_j = f(x_j)$. The complex q-Bernstein–Schurer operators given by (6.15) may be expressed in the form

$$S_{n,p}(f;q;z) = \sum_{j=0}^{n+p} \begin{bmatrix} n+p \\ j \end{bmatrix}_q \triangle_q^j f_0 \cdot e_j(z),$$

where $e_j(z) = z^j$, $\triangle_q^j f_0$ is represented as

$$\triangle_q^j f_0 = \triangle_q^j f(x_0) = \frac{[j]_q! q^{j(j-1)/2}}{[n]_q^j} f[x_0, x_1, \ldots, x_j].$$

Theorem 6.26 ([207] Upper estimate). *Let $0 < q < 1$. For fixed $p \in \mathbb{N}\bigcup\{0\}$ and $R > p+1$, let us denote $D_R = \{z \in \mathbb{C} : |z| < R\}$ and let us suppose that $f : D_R \to \mathbb{C}$ is analytic in D_R; that is, $f(z) = \sum_{k=0}^{\infty} c_k z^k$ for all $z \in D_R$. Also, assume that $1 \leq r$ and $r(p+1) < R$*

(i) For all $|z| \leq r$ and $n \in \mathbb{N}$, we have

$$|S_{n,p}(f;q;z) - f(z)| \leq M_{r,n}^{(p,q)}(f),$$

where

$$0 < M_{r,n}^{(p,q)} = \sum_{k=0}^{\infty} |c_k| \left\{ \frac{2(k-1)[k-1]_q}{[n]_q} [([p]_q + 1)r]^k \right.$$
$$\left. + \frac{1}{[n]_q} [(p]_q + 1)r]^k - \frac{r^k}{[n]_q} \right\} < \infty. \tag{6.16}$$

(ii) If $1 \leq r < r_1 \leq r_1(p+1) < R$, then for all $|z| \leq r$ and $n, m \in \mathbb{N}$, we have

$$|S_{n,p}^{(m)}(f, q; z) - f^{(m)}(z)| \leq \frac{M_{r_1,n}^{(p,q)}(f) m! r_1}{(r_1 - r)^{m+1}}.$$

Theorem 6.27 ([207] Voronovskaja-type result). *Let $0 < q < 1$. For fixed $p \in \mathbb{N}\bigcup\{0\}$ and $R > p+1$, let us denote $D_R = \{z \in \mathbb{C} : |z| < R\}$ and let us suppose that $f : D_R \to \mathbb{C}$ is analytic in D_R; that is, $f(z) = \sum_{k=0}^{\infty} c_k z^k$ for all $z \in D_R$. For all $|z| \leq 1$ and $n \in \mathbb{N}$, we have*

$$\left| S_{n,p}(f;q;z) - f(z) - \frac{[p]_q z}{[n]_q} f'(z) - \frac{z(1-z)}{2[n]_q} f''(z) \right|$$
$$\leq \frac{M_{p,q}(f)}{[n]_q^2} + (1-q) \sum_{k=1}^{\infty} k|c_k|([p_q + 1)^k,$$

where $0 < M_{p,q}(f) = \sum_{k=1}^{\infty} k|c_k|(A_{k,q} + B_{k,p,q})([p]_q + 1)^{k-1} < \infty$, $A_{k,q} = 2(k-1)[2(k-1)[k-1]_q + 1]$, and $B_{k,q,p} = (k-1)\{[p]_q^2 + [p]_q(3k - 4 + 2[k-1]_q) + 2k^2 - 5k + (k-2)([k-2]_q + [k-1]_q)\}$.

Theorem 6.28 ([207] Exact order). *Let $1 - \frac{1}{n^2} \leq q_n < 1, n \in \mathbb{N}$. For fixed $p \in \mathbb{N}\bigcup\{0\}$ and $R > p+1$, let us denote $D_R = \{z \in \mathbb{C} : |z| < R\}$ and let us suppose that $f : D_R \to \mathbb{C}$ is analytic in D_R. Assume that we have $1 \leq r$ and $\tau(p+1) < R$.*

If f is not a function of the form $f(z) = \frac{c_1}{2[p]_{q_n}+1}.(z-1)^{2[p]_{q_n}+1} + c_2$, with arbitrary complex constants c_1 and c_2, then for any $r \in [1, R)$, we have

$$\left|\left|S_{n,p}(f;q;.) - f\right|\right|_r \geq \frac{C_{r,p,q_n}(f)}{[n]_{q_n}},$$

where $||f||_r = \max\{|f(z)| : |z| \leq r\}$ and the constant $C_{r,p,q_n}(f) > 0$ depends only on f, p, r and on the sequence $\{q_n\}_{n\in\mathbb{N}}$, but it is independent of n.

Theorem 6.29 ([207] Simultaneous approximation). *Let $1 - \frac{1}{n^2} \leq q_n < 1 \in \mathbb{N}$ and $R > p + 1$, let us denote $D_R = \{z \in \mathbb{C} : |z| < R\}$, and let us suppose that $f : D_R \to \mathbb{C}$ is analytic in D_R. Also, let $m \in \mathbb{N}, 1 \leq r < r_1 \leq r_1(p+1) < R$. If f is not a function of the form $f(z) = \frac{c_1}{2[p]_{q_n}+1}.(z-1)^{2[p]_{q_n}+1} + c_2 + Q_{m-1}(z)$, where c_1 and c_2 are arbitrary complex constants, $Q_{m-1}(z) = 0$ if $m = 1$ and $Q_{m-1}(z) = \sum_{k=1}^{m-1} a_k^* z^k$ if $m \geq 2$, then we have*

$$\left|\left|S_{n,p}^{(m)}(f;q_n;.) - f^{(m)}\right|\right|_r \asymp \frac{1}{[n]_{q_n}}, n \in \mathbb{N},$$

where $||f||_r = \max\{|f(z)| : |z| \leq r\}$ and the constants in the equivalence depend on f, r, r_1, p, m and on the sequence $\{q_n\}_{n\in\mathbb{N}}$ but are independent of n.

Chapter 7
Rate of Convergence for Functions of Bounded Variation

By Jordan's theorem , a function is with bounded variation (BV) if and only if it can be represented as the difference of two increasing (decreasing) functions. It is well known that univariate functions of bounded variation possess at most a countable number of discontinuities, all of the first kind. In particular, for $f \in BV(\mathbb{R})$ and $x \in \mathbb{R}$, the one-sided limits $\lim_{h\to 0} f(x+h)$ and $\lim_{h\to 0} f(x-h)$ exist, and for all $x \in \mathbb{R}$, except at most a countable number of points f satisfy the one-sided continuity condition

$$f(x) = \lim_{h\to 0+} f(x+h) = \lim_{h\to 0+} f(x-h),$$

respectively. Thus,

$$f(x) = \lim_{h\to 0+} \frac{1}{2}[f(x+h) + f(x-h)].$$

This chapter deals with the rate of convergence for certain sequences of operators.

7.1 Fourier and Fourier–Legendre Series

Let f be a 2π-periodic function of bounded variation on $[-\pi, \pi]$, and let $\hat{S}_n(f, x)$ be the nth partial sums of the Fourier series of f. If $g_x(t) = f(x+t) + f(x-t) - f(+) - f(x-)$, then

$$\hat{S}_n(f, x) - \frac{1}{2}[f(x+) + f(x-)] = \frac{1}{\pi}\int_0^n \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}dt. \tag{7.1}$$

Also, the well-known Dirichlet–Jordan theorem (see [259]) states that

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$$\lim_{n\to\infty}\left(\hat{S}_n(f,x)-\frac{1}{2}[f(x+)+f(x-)]\right)=0. \tag{7.2}$$

In 1979, Bojanic [43] estimated the quantitative version of the Dirichlet–Jordan theorem, by proving the inequality.

Theorem 7.1 ([43]). *Let g be a 2π-periodic function of bounded variation on $[0,\pi]$ with $g(0)=0$. Then*

$$\left|\frac{1}{\pi}\int_0^n \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}dt\right| \le \frac{3}{n}\sum_{k=1}^{n} V_0^{\pi/k}(g),$$

for $n=1,2,\ldots$.

By (7.1) and Theorem 7.1, Bojanic [43] concluded that for $n\ge 1$,

$$\left|\hat{S}_n(f,x)-\frac{1}{2}[f(x+)+f(x-)]\right| \le \frac{3}{n}\sum_{k=1}^{n} V_0^{\pi/k}(g_x), \tag{7.3}$$

where $V_0^y(g_x)$ is the total variation of g_x on $[0,y]$, $y\in[0,\pi]$.

Since $g_x(t)$ is a function of bounded variation on $[0,\pi]$, continuous at a point $t=0$, the total variation $V_0^y(g_x)$, $y\in[0,\pi]$ is a continuous function at $y=0$. Consequently, $V_0^{\pi/k}(g_x)\to 0$ as $k\to\infty$, which implies that the right-hand side of (7.3) converges to zero, and Dirichlet–Jordan's theorem is thus verified.

Another result related to (7.3) was established by Natanson [198], who proved that for a 2π-periodic continuous function of bounded variation on $[-\pi,\pi]$, one has

$$\left|\hat{S}_n(f,x)-f(x)\right| \le \frac{2}{\pi n}\int_{4/\pi n}^{\pi}\omega_{V(f)}(2t)t^{-2}dt+\frac{1}{\pi n}V_{-\pi}^{\pi}(f), \tag{7.4}$$

where $\omega_{V(f)}(\delta)$ is the modulus of continuity of the total variation $V_{-\pi}^{t}(f)$, $t\in[-\pi,\pi]$, of f.

As for continuous functions of bounded variation, we have $V_0^{\delta}(g_x)\le V_{x-\delta}^{x+\delta}(f)\le 2\omega_{V(f)}(\delta)$. Thus, (7.3) becomes

$$\left|\hat{S}_n(f,x)-f(x)\right| \le \frac{6}{n}\sum_{k=1}^{n}\omega_{V(f)}\left(\frac{\pi}{k}\right),$$

which is equivalent to Natanson's estimate (7.4).

In this continuation, Bojanic and Vuilleumier [47] established the rate of approximation by Fourier–Legendre series. After that, several researchers began to obtain the estimates on the rate of convergence for different operators.

Let P_n be the Legendre polynomial of degree n normalized so $P_n(1)=1$, and let f be a function of bounded variation on $[-1,1]$:

$$S_n(f,x) = \sum_{k=0}^{n} a_k(f)P_k(x), a_k = \left(k+\frac{1}{2}\right)\int_{-1}^{1} P_k(t)f(t)dt.$$

In the last century, E. W. Hobson [154] showed that the Fourier–Legendre series of a function of bounded variation on $[-1,1]$ converges at every point $x \in (-1,1)$ to $\frac{1}{2}[f(x+)+f(x-)]$.

In 1981, Bojanic and Vuilleumier [47] represented the polynomials $S_n(f,x)$ as

$$S_n(f,x) = \int_{-1}^{1} K_n(x,t)f(t)dt, K_n(x,t) = \sum_{k=0}^{n}\left(k+\frac{1}{2}\right)P_k(x)P_k(t),$$

and they established the quantitative estimate and general result on the rate of convergence for Legendre–Fourier series.

Theorem 7.2 ([47]). *Let f be a function of bounded variation on $[-1,1]$. Then for every $x \in (-1,1)$ and $n \geq 2$, one has*

$$\left|S_n(f,x) - \frac{1}{2}(f(x+)+f(x-))\right| \leq \frac{28(1-x)^{-3/2}}{n}\sum_{k=1}^{n} V_{x-(1+x)/k}^{x+(1-x)/k}(g_x)$$
$$+\frac{1}{\pi n(1-x^2)}|f(x+)-f(x-)|,$$

where

$$g_x(t) = \begin{cases} f(t)-f(x-), & , \quad -1 \leq t < x, \\ 0, & , \quad t = x, \\ f(t)-f(x+), & , \quad x < t \leq 1, \end{cases}$$

and $V_a^b(g_x)$ is the total variation of g_x on $[a,b]$.

As a special case, if f is a continuous function of bounded variation, Theorem 7.2 becomes

$$\left|S_n(f,x) - \frac{1}{2}(f(x+)+f(x-))\right| \leq \frac{28(1-x)^{-3/2}}{n}\sum_{k=1}^{n} V_{x-(1+x)/k}^{x+(1-x)/k}(g_x).$$

Also, the right-hand side of Theorem 7.2 converges to 0 as $n \to \infty$ since the continuity of $g_x(t)$ at $t = x$ implies $V_{x-\delta}^{x+\delta}(g_x) \to 0$, as $\delta \to 0$.

Theorem 7.3 ([47]). *Let $K_n(x,t)$ be a continuous function of two variables on $[a,b]\times[a,b]$, and let L_n be the operator that transforms a function f of bounded variation on $[a,b]$ into the function $\int_a^b K_n(x,t)f(t)dt, x \in [a,b]$. If, for a fixed $x \in (a,b)$ and $n \geq 1$, the kernel $K_n(x,t)$ satisfies the conditions* *(i)* $\left|\int_a^x K_n(x,t)dt - \frac{1}{2}\right| \le \frac{A(x)}{n}$ *and* $\left|\int_x^b K_n(x,t)dt - \frac{1}{2}\right| \le \frac{A(x)}{n}$,

(ii) $\int_{x-(x-a)/n}^{x+(b-x)/n} |K_n(x,t)|dt \le B(x)$,

(iii) $\left|\int_a^t K_n(x,t)dt\right| \frac{C(x)}{n(t-x)}, (a \le t < x < b)$,

$\left|\int_t^b K_n(x,t)dt\right| \frac{C(x)}{n(t-x)}, (a < x < t \le b)$,

where $A(x)$, $B(x)$, *and* $C(x)$ *are positive functions on* (a,b), *then there exists a positive number* $M(f,x)$ *depending only on* f *and* x *such that*

$$\left|L_n(f,x) - \frac{1}{2}(f(x+) + f(x-))\right| \le \frac{M(f,x)}{n} \sum_{k=1}^{n} V_{x-(x-a)/k}^{x+(b-x)/k}(g_x),$$

where

$$g_x(t) = \begin{cases} f(t) - f(x-), & a \le t < x, \\ 0, & t = x, \\ f(t) - f(x+), & x < t \le b. \end{cases}$$

7.2 Hermite–Fejér Polynomials

Let f be a function defined on $[-1,1]$. The Hermite–Fejér interpolation polynomial $H_n(f,x)$ of f based on the zeros $x_{kn} = \cos\left(\frac{(2k-1)\pi}{2n}\right), k = 1,2,\ldots,n$ of the Chebyshev polynomials $T_n(x) = \cos(n \cos^{-1} x)$ is defined by

$$H_n(f,x) = \sum_{k=1}^{n} f(x_{kn})(1 - x_{kn}x)\left(\frac{T_n(x)}{n(x - x_{kn})}\right)^2. \tag{7.5}$$

If $f(x)$ is a continuous function on $[-1,1]$, Fejér [67] proved that the polynomials $H_n(f,x)$ converge uniformly to $f(x)$. The convergence behavior of $H_n(f,x)$ of a bounded variation function on $[-1,1]$ was studied by Bojanic and Cheng [44].

Theorem 7.4 ([44]). *If* $f \in BV[-1,1]$ *and is continuous at* $x \in (-1,1)$, *then* $H_n(f,x)$ *converges to* $f(x)$ *when* n *tends to* ∞, *and the rate of convergence of* $H_n(f,x)$ *to* $f(x)$ *satisfies the inequality*

$$|H_n(f,x) - f(x)| \le \frac{64T_n^2(x)}{n} \sum_{k=1}^{n} V_{x-\pi/k}^{x+\pi/k}(f) + 2V_{x-\pi[T_n(x)]/2n}^{x+\pi[T_n(x)]/2n}(f),$$

where $V_a^b(f)$ *is the total variation of* f *on* $[a,b]$; *this estimate cannot be improved asymptotically.*

If x is a point of discontinuity of f, where $f(x+) \neq f(x-)$, the sequence $(H_n(f,x))$ is no longer convergent. It follows from the following observation (see [44]):

$$\lim_{n\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} H_n(f,x) = \frac{1}{2}[f(x+) + f(x-)] \pm \frac{1}{2}[f(x+) - f(x-)]\beta(x), \quad (7.6)$$

where $\beta(x) = 1$ if $x = \cos(\alpha\pi)$ and α is irrational, and

$$\beta(x) = \left(\frac{\sin(\pi/2q)}{\pi/2q}\right)^2 \left(1 - \sum_{k=1}^{\infty} \frac{8qk}{(4q^2k^2-1)^2}\right),$$

if $x = \cos(p\pi/q)$. The above equation (7.6) shows that, unlike Fourier series of 2π-periodic functions of bounded variation, which all converge to $(f(x+) + f(x-))/2$, the Hermite–Fejér interpolation polynomials of a function of bounded variation converge only if $f(x+) = f(x-)$.

Although smoother functions have many applications in various fields, such as computer vision, CAGD , graphics, and image processing, the asymptotic behavior of Hermite–Fejér interpolation polynomials for functions smoother than continuous functions has been studied only for functions with continuous derivatives.

7.3 Exponential-Type Operators

In 1976, May [189] studied some direct, inverse, and saturation results for the linear combinations of exponential-type operators. He was able to improve the order of approximation by considering linear combinations in local approximation. In fact, exponential-type operators include some well-known operators as special cases. In this section, we present the rate of convergence for BV functions for certain exponential-type operators.

Let f be a function defined on $[0,1]$. The Bernstein polynomial of degree n is defined as

$$B_n(f,x) = \sum_{k=0}^{n} p_{n,k}(x) f(k/n), \quad (7.7)$$

where the Bernstein basis function is defined by

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

Bernstein [37] showed that $B_n(f,x)$ converges to $f(x)$ uniformly on the interval $[0,1]$ if f is continuous. As for discontinuous functions, Herzog and Hill [152] proved that if f is bounded on $[0,1]$ and x is a point of discontinuity of the first kind, then

$$\lim_{n\to\infty} B_n(f,x) = \frac{1}{2}\left[f(x+) + f(x-)\right].$$

As a special case, if f is of bounded variation on $[0,1]$, then this convergence holds for every $x \in (0,1)$. Cheng [54] estimated the rate of convergence for Bernstein polynomials of functions of bounded variation.

Theorem 7.5 ([54]). *Let f be a function of bounded variation on* $[0,1]$. *Then for every $x \in (0,1)$ and $n \geq (3/x(1-x))^8$, one has*

$$\left|B_n(f,x) - \frac{1}{2}(f(x+) + f(x-))\right| \leq \frac{3}{nx(1-x)} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x) + \frac{18(x(1-x))^{-5/2}}{n^{1/6}} |f(x+) - f(x-)|,$$

where

$$g_x(t) = \begin{cases} f(t) - f(x-), & 0 \leq t < x, \\ 0, & t = x, \\ f(t) - f(x+), & x < t \leq 1, \end{cases}$$

and $V_a^b(g_x)$ is the total variation of g_x on $[a,b]$.

In 1989, Guo and Khan [91] observed that the preceding Theorem 7.5 can be improved considerably. They extended a modified form of this result in three ways: (i) hold for all n; (ii) be asymptotically sharp; (iii) hold for a general class of operators, including the classical operators such as Bernstein, Szász, Baskakov, Gamma , and Weierstrass operators. They considered the following operators due to Feller [68]. Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with a finite variance such that $E(X_1) = x \in I \subseteq R = (-\infty,\infty)$, $Var(X_1) = \sigma^2(x) > 0$. Set $S_n = X_1 + X_2 + \ldots + X_n$. For a function f, define the approximation operator as

$$L_n(f,x) = E\{f(S_n/n)\} = \int_{-\infty}^{\infty} f(t/n)dF_{n,x}(t), \tag{7.8}$$

where $F_{n,x}(t)$ is the distribution function (df) of S_n and $|f|$ is $F_{n,x}$-integrable.

Theorem 7.6 ([91]). *Let $f \in BV(-\infty,\infty)$. Then for every $x \in (-\infty,\infty)$ and all $n = 1,2,\ldots$, for the Feller operator (7.8), we have*

$$\left|L_n(f,x) - \frac{1}{2}(f(x+) + f(x-))\right| \leq \frac{P(x)}{n} \sum_{k=0}^{n} V_{I_k}(g_x) + \frac{Q(x)}{n^{1/2}} |f(x+) - f(x-)|, \tag{7.9}$$

where $I_k = [x - 1/\sqrt{k}, x + 1/\sqrt{k}], k = 1, 2, \ldots, n, I_0 = (-\infty, \infty)$, $P(x) = 2\sigma^2(x) + 1$, $Q(x) = 2E|X_1 - x|^3/\sigma^3(x)$, *and*

$$g_x(t) = \begin{cases} f(t) - f(x-), & t < x, \\ 0, & t = x, \\ f(t) - f(x+), & t > x. \end{cases}$$

Furthermore, (7.9) is asymptotically sharp when $f \in BV(-\infty, \infty)$, *and* $F_{1,x}(t)$ *is either absolutely continuous with respect to the Lebesgue measure or is a lattice point distribution, and x is a lattice point.*

Corollary 7.1. *Let* $M_n = \{t : P(S_n/n \le t\} = P(S_n/n \ge t)\}$. *Then by Theorem 7.6, if* $x \in M_n$, *we have*

$$|L_n(f, x) - f(x)| \le \frac{P(x)}{n} \sum_{k=0}^{n} V_{I_k}(g_x)$$

regardless of the size of the saltus of f at x.

The special cases of Theorem 7.6 are the following:

1. **Bernstein operators**

   Let $P(X_1 = 1) - x = 1 - P(X_1 = 0), x \in (0, 1)$. Then

$$L_n(f, x) = B_n(f, x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} f(k/n).$$

Now $\sigma^2(x) = x(1 - x)$ and $E|X_1 - x|^3 = \sigma^2(x)(2x^2 - 2x + 1)$. Therefore,

$$\left|B_n(f, x) - \frac{1}{2}(f(x+) + f(x-))\right| \le \frac{2x(1-x)+1}{n} \sum_{k=0}^{n} V_{I_k}(g_x) + \frac{2(2x^2-2x+1)}{\sqrt{nx(1-x)}} |f(x+) - f(x-)|. \quad (7.10)$$

In [91], the author remarked in passing that $I_k = [x - k^{-1/2}, x + k^{-1/2}]$ could be replaced by $I_k^* = [x - xk^{-1/2}, x + (1 - x)k^{-1/2}]$ and that (7.10) would be modified as

$$\left|B_n(f, x) - \frac{1}{2}(f(x+) + f(x-))\right| \le \frac{3}{nx(1-x)} \sum_{k=0}^{n} V_{I_k^*}(g_x) + \frac{2(2x^2-2x+1)}{\sqrt{nx(1-x)}} |f(x+) - f(x-)|. \quad (7.11)$$

2. **Szász operators**

   Let $P(X_1 = j) = e^{-x}x^j/(j!)$, $j = 0, 1, \ldots$. Then we obtain the Szász operator $S_n(f, x)$

$$L_n(f, x) = S_n(f, x) = \sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!}f(k/n),$$

   where $f(t)$ is defined over $[0, \infty)$. Now $\sigma^2(x) = x$ and

$$E|X_1 - x|^3 = x + 2e^{-x}\sum_{k=0}^{[x]}(x-k)^3\frac{x^k}{k!}.$$

   Therefore, if $f \in BV[0, \infty)$, then for all $x \in (0, \infty)$, we have

$$\left|S_n(f, x) - \frac{1}{2}(f(x+) + f(x-))\right| \le \frac{2x+1}{n}\sum_{k=0}^{n} V_{I_k}(g_x) + \frac{2E|X_1-x|^3}{x\sqrt{nx)}}|f(x+) - f(x-)|. \quad (7.12)$$

   One could use the simple bound for $E|X_1 - x|^3 \le 8x^3 + 6x^2 + x$ to have

$$\left|S_n(f, x) - \frac{1}{2}(f(x+) + f(x-))\right| \le \frac{2x+1}{n}\sum_{k=0}^{n} V_{I_k}(g_x) + \frac{2(8x^2+6x+1)}{\sqrt{nx}}|f(x+) - f(x-)|. \quad (7.13)$$

   In 1984, Cheng [55] also established the rate of convergence for functions of bounded variation.

Theorem 7.7 ([55]). *Let f be a function of bounded variation on every finite subinterval of $[0, \infty)$, and let $f(t) = O(t^{\alpha t})$ for some $\alpha > 0$ as $t \to \infty$. If $x \in (0, \infty)$ is irrational, then for n sufficiently large, we have*

$$\left|S_n(f, x) - \frac{1}{2}(f(x+) + f(x-))\right| \le \frac{(3+x)x^{-1}}{n}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x)/\sqrt{k}}(g_x) + \frac{O(x^{-1/2})}{n^{1/2}}|f(x+) - f(x-)| + O(1)(4x)^{4\alpha x}(nx)^{-1/2}\left(\frac{e}{4}\right)^{nx},$$

where

$$g_x(t) = \begin{cases} f(t) - f(x-), & 0 \le t < x, \\ 0, & t = x, \\ f(t) - f(x+), & x < t < \infty\,, \end{cases}$$

and $V_a^b(g_x)$ *is the total variation of* g_x *on* $[a, b]$.

3. **Baskakov operators**

   Let $P(X_1 = j) = 1/(1+x)(x/(1+x))^j$, $j = 0, 1, \ldots$. Then we obtain the Baskakov operator $V_n(f, x)$

$$L_n(f, x) = V_n(f, x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f(k/n).$$

   Now, $\sigma^2(x) = x(1+x)$ and

$$E|X_1 - x|^3 = 2x^3 + 3x^2 + x + \frac{2}{1+x} \sum_{k=0}^{[x]} (x-k)^3 \left(\frac{x}{1+x}\right)^k.$$

   Therefore, if $f \in BV[0, \infty)$, then for all $x \in (0, \infty)$, we have

$$\left| V_n(f, x) - \frac{1}{2}(f(x+) + f(x-)) \right| \le \frac{2x(1+x)+1}{n} \sum_{k=0}^{n} V_{I_k}(g_x) + \frac{2E|X_1 - x|^3}{(x(1+x))^{3/2}\sqrt{n}} |f(x+) - f(x-)|. \tag{7.14}$$

   Again, one could use the bound $E|X_1 - x|^3 \le 16x^3 + 9x^2 + x$ to obtain

$$\left| V_n(f, x) - \frac{1}{2}(f(x+) + f(x-)) \right| \le \frac{2x(1+x)+1}{n} \sum_{k=0}^{n} V_{I_k}(g_x) + \frac{2(16x^2 + 9x + 1)}{(1+x)^{3/2}\sqrt{nx}} |f(x+) - f(x-)|. \tag{7.15}$$

4. **Gamma operators**

   Let X_1 have probability density over $(0, \infty)$:

$$f_{X_1}(y) = \begin{cases} (1/x)e^{-y/x}, & y > 0, \\ 0, & y \le 0. \end{cases}$$

Then

$$L_n(f,x) = G_n(f,x) = \frac{x^{-n}}{(n-1)!}\int_0^\infty f(y/n)y^{n-1}e^{-y/x}dy.$$

Now, $\sigma(x) = x^2$ and $E|X_1-x|^3 = 2x^3(6/e-1)$. Hence, for any $f \in BV[0,\infty)$ and $x \in (0,\infty)$, we have

$$\left|G_n(f,x) - \frac{1}{2}(f(x+)+f(x-))\right| \le \frac{2x^2+1}{n}\sum_{k=0}^{n} V_{I_k}(g_x) + \frac{4(6/e-1)}{\sqrt{n}}|f(x+)-f(x-)|. \quad (7.16)$$

5. **Weierstrass operators**

Let X_1 have probability density defined over $(-\infty,\infty)$ by $g^*(y-x)$, where $g^*(y)$ is the derivative of $G^*(y) = (2\pi)^{-1/2}\int_{-\infty}^{y} e^{-u^2/2}du$. Then $L_n(f,x)$ reduces to Weierstrass operators

$$W_n(f,x) = \sqrt{\frac{n}{2\pi}}\int_{-\infty}^{\infty} f(x+u)e^{-nu^2/2}du.$$

Now, $\sigma^2(x) = 1$, and by Corollary 7.1, we have

$$\left|W_n(f,x) - \frac{1}{2}(f(x+)+f(x-))\right| = \frac{3}{n}\sum_{k=0}^{n} V_{I_k}(g_x),$$

for $f \in BV(-\infty,\infty)$ and $x \in (-\infty,\infty)$ regardless of the size of the saltus of f at x.

7.4 Bernstein–Durrmeyer-Type Polynomials

Durrmeyer [65] introduced the integral modification of the classical Bernstein polynomials, which was studied in detail by Derriennic [58], who established some approximation properties of these operators. If f is a function defined on [0, 1], the Durrmeyer operators are defined as

$$D_n(f,x) = \int_0^1 K_n(x,t)f(t)dt = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)f(t)dt, \quad (7.17)$$

where the kernel $K_n(x,t) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)p_{n,k}(t)$ and the Bernstein basis function $p_{n,k}(x)$ is given in (7.7).

Guo [88] estimated the rate of convergence for functions of bounded variation. He used the Berry–Esseen theorem to obtain the upper bound for Bernstein basis functions. Here are the basic lemmas and rate of convergence for Bernstein–Durrmeyer operators given in [88]:

Lemma 7.1. *For every $x \in (0,1)$ and $0 \le k \le n$, we have*

$$p_{n,k}(x) \le \frac{5}{2\sqrt{nx(1-x)}}.$$

Lemma 7.2. *For every $0 \le j \le n$) and n sufficiently large, we have*

$$\left|\sum_{k=0}^{j} p_{n,k}(x) - \sum_{k=0}^{j} p_{n+1,k}(x)\right| \le \frac{2}{\sqrt{nx(1-x)}}.$$

For n sufficiently large, Guo considered $\frac{x(1-x)}{n} \le D_n((t-x)^2, x) \le \frac{2x(1-x)}{n}$. Obviously,

$$\int_0^y K_n(x,t)dt \le \frac{2x(1-x)}{n(x-y)^2}, 0 \le y < x, \tag{7.18}$$

and

$$\int_z^1 K_n(x,t)dt \le \frac{2x(1-x)}{n(z-x)^2}, x < z \le 1. \tag{7.19}$$

Theorem 7.8 ([88]). *Let f be a function of bounded variation on every finite subinterval of $[0,1]$, and let $V_a^b(g_x)$ be the total variation of g_x on $[a,b]$. Then for n sufficiently large,*

$$\left|D_n(f,x) - \frac{1}{2}(f(x+) + f(x-))\right| \le \frac{5}{nx(1-x)} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x)$$
$$+ \frac{13(x(1-x))^{-1}}{4\sqrt{n}} |f(x+) - f(x-)|,$$

where the auxiliary function g_x is as defined in Theorem 7.2.

Proof. We can write

$$\left|D_n(f,x) - \frac{1}{2}(f(x+) + f(x-))\right| \le \frac{1}{2}|f(x+) - f(x-)|.|D_n(sign(t-x),x)|$$
$$+|D_n(g_x,x)|.$$

First,

$$\begin{aligned} D_n(sign(t-x),x) &= \int_x^1 K_n(x,t)dt - \int_0^x K_n(x,t)dt \\ &:= A_n(x) - B_n(x). \end{aligned}$$

Using $(n+1)\int_x^1 p_{n,j}(t)dt = \sum_{k=0}^{j} p_{n+1,k}(x)$, we have

$$\begin{aligned} A_n(x) &= \int_x^1 K_n(x,t)dt \\ &= (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_x^1 p_{n,k}(t)dt = \sum_{k=0}^{n} p_{n,k}(x)\sum_{j=0}^{k} p_{n+1,j}(x). \end{aligned}$$

Using Lemma 7.2, we have $|A_n(x) - S| \le \frac{2}{\sqrt{nx(1-x)}}$, where $S = \sum_{k=0}^{n} \left(p_{n,k}(x)\sum_{j=0}^{k} p_{n,j}(x)\right)$. If $I = (p_{n,0}(x) + p_{n,1}(x) + p_{n,2}(x) + \ldots. + p_{n,n}(x))(p_{n,0}(x)+p_{n,1}(x)+p_{n,2}(x)+\ldots.+p_{n,n}(x))$. Then with $A_n(x)+B_n(x)=1$, we have

$$\begin{aligned} |A_n(x) - B_n(x)| = |2A_n(x) - 1| &= \sum_{k=0}^{n} p_{n,k}^2(x) \\ &\le p_{n,k}(x)\sum_{k=0}^{n} p_{n,k}(x) \le \frac{13}{2\sqrt{nx(1-x)}}. \end{aligned}$$

Next,

$$\begin{aligned} D_n(g_x,x) &= \left(\int_{I_1} + \int_{I_2} + \int_{I_3}\right) K_n(x,t)g_x(t)dt \\ &:= E_1 + E_2 + E_3, \end{aligned}$$

where $I_1 = [0, x - x/\sqrt{n}]$, $I_2 = [x - x/\sqrt{n}, x + (1-x)/\sqrt{n}]$, $I_3 = [x + (1+x)/\sqrt{n}, 1]$.

First, we estimate E_2. For $t \in I_2$, we have $|g_x(t)| = |g_x(t) - g_x(x)| \le V_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}}(g_x)$. Therefore,

$$E_2 = V_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}}(g_x)\int_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}} d_t((\lambda_n(x,t)),$$

where $\lambda_n(x,t) = \int_0^t K_n(x,u)du$. Since $\int_a^b d_t(\lambda_n(x,t)) \le 1$ for all $[a,b] \subseteq [0,1]$, we have

$$E_2 \le V_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}}(g_x).$$

Next, let $y = x - x/\sqrt{n}$. Using Lebesgue–Stieltjes integration by parts, we find

$$\begin{aligned} E_1 &= \int_0^y g_x(t) d_t(\lambda_n(x,t)) \\ &\le V_{y+}^x(g_x)\lambda_n(x,y) + \int_0^y \lambda_n(x,t) d_t(-V_t^x(g_x)). \end{aligned}$$

Using (7.18), we have

$$\begin{aligned} E_1 &\le V_{y+}^x(g_x)\frac{2x(1-x)}{n(x-y)^2} + \frac{2x(1-x)}{n}\int_0^y \frac{1}{(x-t)^2} d_t(-V_t^x(g_x)) \\ &\le \frac{2(1-x)}{nx}\left(V_0^x(g_x) + \sum_{k=1}^n V_{x-x/\sqrt{k}}^x(g_x)\right) \le \frac{4}{nx(1-x)}\sum_{k=1}^n V_{x-x/\sqrt{k}}^x(g_x). \end{aligned}$$

Along similar lines, one has

$$E_3 \le \frac{4}{nx(1-x)}\sum_{k=1}^n V_x^{x+(1-x)/\sqrt{k}}(g_x).$$

Thus,

$$D_n(g_x, x) = \frac{5}{nx(1-x)}\sum_{k=1}^n V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x).$$

Combining the estimates of $D_n(g_x, x)$ and $D_n(sign(t-x), x)$, we get the desired result. ■

There may be several variants of Durrmeyer-type operators, such as

$$U_{n,\alpha,\beta}(f,x) = \sum_{k\in I_n} p_{n,k}(x) f\left(\frac{k}{n}\right) + (n-\alpha+1)\sum_{k=0}^{n-\alpha+\beta} p_{n,k}(x)\int_0^1 p_{n-\alpha,k-\beta}(t) f(t)dt,$$

for $n \ge \alpha$, where α, β are integers satisfying $\alpha \ge \beta \ge 0$ and $I_n \subseteq \{0, 1, 2, \ldots, n\}$ is a certain index set. For $\alpha{=}\beta{=}0$, $I_n = 0$. This definition reduces to Bernstein–Durrmeyer operators D_n. An important example is the special case $\alpha{=}2, \beta{=}1, I_n = \{0, n\}$, which was introduced by Goodman and Sharma [86]. In 2008, Abel et al. [13] considered the case $\alpha = 0, \beta = 1, I_n = \{0\}$, namely, the operators

$$U_{n,0,1}(f,x) = U_n(f,x) = (n+1)\sum_{k=1}^n p_{n,k}(x)\int_0^1 p_{n,k-1}(t) f(t)dt + (1-x)^n f(0). \tag{7.20}$$

Theorem 7.9 ([13]). *Let f be a function of bounded variation on the interval* $[0,1]$. *Furthermore, let (a_k) be a strictly decreasing sequence of positive numbers so that $a_1 = 1$. Then for each $x \in (0,1)$ and all $n \in N$, the following holds:*

$$\left|U_n(f,x) - \frac{1}{2}(f(x+)+f(x-))\right| \le V_{x-xa_n}^{x+(1-x)a_n}(g_x) + \frac{2+1/\sqrt{8e}}{\sqrt{nx(1-x)}}|f(x+)-f(x-)| + C_n(x)(1-a_2^2)^{-1}\sum_{k=1}^{n-1}(a_{k+1}^{-2}-a_k^{-2})V_{x-a_k}^{x+(1-x)a_k}(g_x),$$

where the auxiliary function g_x is as defined in Theorem 7.2, with

$$C_n(x) = \frac{2x(1-x)n + 6x^2}{(n+2)(n+3)}\max\{x^{-2},(1-x)^{-2}\}.$$

For the special case $a_k = 1/\sqrt{k}$, Abel et al. obtained the following corollary.

Corollary 7.2. *Let f be a function of bounded variation on the interval* $[0,1]$. *Then for each $x \in (0,1)$, any given number $\varepsilon > 0$ and all n sufficiently large, the following holds:*

$$\left|U_n(f,x) - \frac{1}{2}(f(x+)+f(x-))\right| \le \frac{2+1/\sqrt{8e}}{\sqrt{nx(1-x)}}|f(x+)-f(x-)| + \frac{4+\varepsilon}{nx(1-x)}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x).$$

7.5 Szász–Mirakyan–Durrmeyer-Type Operators

The Szász–Mirakyan–Durrmeyer operators are defined as

$$M_n(f,x) = n\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^{\infty} s_{n,k}(t)f(t)dt, \tag{7.21}$$

where $s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}$. The rate of convergence for functions of bounded variation for such operators was obtained in [108] and [208]. Later, in 1999, Gupta and Pant [120] improved the result by using the Berry–Esseen theorem with better bounds.

Lemma 7.3. *For every $x \in (0, \infty)$, we have*

$$s_{n,k}(x) \leq \frac{32x^2 + 24x + 5}{2\sqrt{nx}}.$$

For $n \geq 2$, they considered $2x/n \leq M_n((t-x)^2, x) \leq (2x+1)/n$. The kernel is defined by $K_n(x,t) = n\sum_{k=0}^{\infty} s_{n,k}(x)s_{n,k}(t)$. Obviously,

$$\int_0^y K_n(x,t)dt \leq \frac{2x+1}{n(x-y)^2}, 0 \leq y < x, \tag{7.22}$$

and

$$\int_z^{\infty} K_n(x,t)dt \leq \frac{2x+1}{n(z-x)^2}, x < z < \infty. \tag{7.23}$$

Theorem 7.10 ([120]). *Let f be a function of bounded variation on every finite subinterval of $[0,\infty)$, and let $f(t) = O(e^{\alpha t})$ for some $\alpha > 0$ as $t \to \infty$. If $x \in (0,\infty)$ and $n \geq 4\alpha$, then*

$$\begin{aligned}\left|M_n(f,x) - \frac{1}{2}(f(x+) + f(x-))\right| \leq & \frac{x^2+6x+3}{nx^2}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x)\\ &+\frac{32x^2+24x+5}{2\sqrt{nx}}|f(x+) - f(x-)|\\ &+\sqrt{\frac{2(2x+1)}{n}}\frac{e^{2\alpha x}}{x} + \frac{e^{\alpha x}(2x+1)}{nx^2},\end{aligned}$$

where the auxiliary function is as defined in Theorem 7.7, and $V_a^b(g_x)$ is the total variation of g_x on $[a,b]$.

Proof. We can write

$$\begin{aligned}\left|M_n(f,x) - \frac{1}{2}(f(x+) + f(x-))\right| \leq & \frac{1}{2}|f(x+) - f(x-)|.|M_n(sign(t-x),x)|\\ &+|M_n(g_x,x)|.\end{aligned}$$

Using the techniques as given in [208], we have

$$\begin{aligned}|M_n(sign(t-x),x)| &= \left|\int_x^{\infty} K_n(x,t)dt - \int_0^x K_n(x,t)dt\right|\\ &:= A_n(x) - B_n(x).\end{aligned}$$

Also, by applying the identity $n\int_x^\infty s_{n,k}(t)dt = \sum_{j=0}^{k} s_{n,j}(x)$, we have

$$\begin{aligned}
A_n(x) &= \int_x^\infty K_n(x,t)dt \\
&= n\sum_{k=0}^{\infty} s_{n,k}(x)\int_x^\infty s_{n,k}(t)dt = \sum_{k=0}^{\infty} s_{n,k}(x)\sum_{j=0}^{k} s_{n,j}(x) \\
&= s_{n,0}^2(x)+s_{n,1}^2(x)+s_{n,2}^2(x)+\ldots . \\
&\quad +s_{n,0}(x)(s_{n,1}(x)+s_{n,2}(x)+\ldots .)+s_{n,1}(x)(s_{n,2}(x)+s_{n,3}(x)+\ldots .)+\ldots .
\end{aligned}$$

Let $I = (s_{n,0}(x) + s_{n,1}(x) + s_{n,2}(x) + \ldots .)(s_{n,0}(x) + s_{n,1}(x) + s_{n,2}(x) + \ldots .)$. Then with $A_n(x) + B_n(x) = 1$, we have

$$\begin{aligned}
|A_n(x) - B_n(x)| = |2A_n(x) - 1| &= \sum_{k=0}^{\infty} s_{n,k}^2(x) \\
&\le s_{n,k}(x)\sum_{k=0}^{\infty} s_{n,k}(x) \le \frac{32x^2 + 24x + 5}{2\sqrt{nx}}.
\end{aligned}$$

Next,

$$\begin{aligned}
M_n(g_x, x) &= \left(\int_{I_1} + \int_{I_2} + \int_{I_3}\right) K_n(x,t)g_x(t)dt \\
&:= E_1 + E_2 + E_3,
\end{aligned}$$

where $I_1 = [0, x - x/\sqrt{n}]$, $I_2 = [x - x/\sqrt{n}, x + x/\sqrt{n}], and I_3 = [x + x/\sqrt{n}, \infty)$.

First, we estimate E_2. For $t \in I_2$, we have $|g_x(t)| = |g_x(t) - g_x(x)| \le V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x)$. Therefore,

$$E_2 = V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x)\int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} K_n(x,t)dt = V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x)\int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} d_t((\lambda_n(x,t)),$$

where $\lambda_n(x,t) = \int_0^t K_n(x,u)du$. Since $\int_a^b d_t(\lambda_n(x,t)) \le 1$ for all $[a,b] \subseteq [0,\infty)$, we have

$$E_2 \le V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x) \le \frac{1}{n}\sum_{k=0}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x).$$

Using (7.22) and proceeding as in [208], we have

$$E_1 \leq \frac{2(2x+1)}{nx^2} \sum_{k=0}^{n} V_{x-x/\sqrt{k}}^{x}(g_x).$$

Finally,

$$\begin{aligned} E_3 &= \int_x^{2x} g_x(t) d_t(Q_n(x,t)) - g_x(2x) \int_{2x}^{\infty} K_n(x,u) du \\ &\quad + \int_{2x}^{\infty} g_x(t) d_t(\lambda_n(x,t)) \\ &:= E_{31} + E_{32} + E_{33}. \end{aligned}$$

After simple computation, the estimates turn out to be as follows; details can be found in [208] and [120]:

$$|E_{31}| \leq \frac{2(2x+1)}{nx^2} \sum_{k=0}^{n} V_x^{x+x/\sqrt{k}}(g_x).$$

$$|E_{32}| \leq \frac{(2x+1)}{nx^2} \sum_{k=0}^{n} V_x^{x+x/\sqrt{k}}(g_x).$$

and

$$\begin{aligned} |E_{33}| &\equiv \left| n \sum_{k=0}^{\infty} s_{n,k}(x) \int_{2x}^{\infty} s_{n,k}(t) g_x(t) dt \right| \\ &\leq n \sum_{k=0}^{\infty} s_{n,k}(x) \int_{2x}^{\infty} s_{n,k}(t)(e^{\alpha t} + e^{\alpha x}) dt \\ &\leq \frac{n}{x} \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} s_{n,k}(t) |t-x| e^{\alpha t} dt \\ &\quad + \frac{e^{\alpha x}}{x^2} n \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} s_{n,k}(t)(t-x)^2 dt \\ &\leq \frac{1}{x} \left(M_n((t-x)^2, x) \right)^{1/2} \left(n \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} s_{n,k}(t) e^{2\alpha t} dt \right)^{1/2} \\ &\quad + \frac{e^{\alpha x}}{x^2} M_n((t-x)^2, x). \end{aligned}$$

Next, with the assumption that $n > 2\alpha$, we have

$$n\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^{\infty} s_{n,k}(t)e^{2\alpha t}\,dt$$
$$= n\sum_{k=0}^{\infty} s_{n,k}(x)\frac{n^k}{k!}\int_0^{\infty} t^k e^{-(n-2\alpha)t}\,dt = \frac{n}{n-2\alpha}e^{2\alpha x.n/(n-2\alpha)} \le 2e^{4\alpha x},$$

for $n \ge 4\alpha$. Therefore,

$$|E_{33}| \le \sqrt{\frac{2(2x+1)}{n}\frac{e^{2\alpha x}}{x}} + \frac{e^{\alpha x}}{x^2}.\frac{2x+1}{n}.$$

If we combine the estimates of $E_1, E_2, E_{31} - E_{32}$, the desired result follows. ■

The Szász–Durrmeyer operators reproduce only constant functions. There is another modification of Szász operators [227], namely, Phillips operators, which reproduce constant as well as linear functions. Thus, Phillips operators are sometimes called genuine Szász–Durrmeyer operators. For $f \in C[0,\infty)$, the Phillips operators (see [199]) are defined as

$$P_n(f,x) = n\sum_{k=1}^{\infty} s_{n,k}(x)\int_0^{\infty} s_{n,k-1}(t)f(t)\,dt + e^{-nx}f(0),$$

where $s_{n,k}(x)$ is the Szász basis function is given by

$$s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}.$$

In this direction, Gupta and Shrivastava in [128] estimated the rate of convergence of Phillips operators for functions of bounded variation, using a probabilistic approach.

Let $B_N[0,\infty)$, $N \ge 0$, be the class of all measurable functions f such that $|f(t)| \le Me^{Nt}$ for all $t \in [0,\infty)$ and some positive M for which the operators $P_n(f,t)$ are well defined.

Theorem 7.11 ([128]). *Let $f \in B_N[0,\infty)$ be a function of bounded variation on every finite subinterval of $[0,\infty)$, and let*

$$g_x(t) = \begin{cases} f(t) - f(x+) \; if \;\; x < t < \infty, \\ \qquad 0 \qquad\qquad if \;\; t = x, \\ f(t) - f(x-) \;\; if \;\; 0 \le t < x \end{cases}$$

We denote by $V_a^b(g_x)$ the total variation of g_x on $[a,b]$. Then for all $x > 0$ and $n > 4N$, we have

$$\left| P_n(f,x) - \frac{1}{2}\{f(x+) + f(x-)\} \right|$$
$$\leq \frac{7}{nx} \sum_{m=1}^{n} V_{x-x/\sqrt{m}}^{x+x/\sqrt{m}}(g_x) + \frac{32x^2 + 24x + 5}{4\sqrt{nx}} |f(x+) - f(x-)|$$
$$+ \sqrt{\frac{2}{nx}} \left(1 + \frac{1}{\sqrt{nx}}\right) e^{2Nx}.$$

7.6 Baskakov–Durrmeyer-Type Operators

In 1991, Wang and Guo [243] studied the rate of convergence behavior of Lupas–Durrmeyer (Baskakov–Durrmeyer) operators. Let f be defined on $(0, \infty)$ with bounded variation in each finite interval and $f(x) = O(x^r), x \to \infty$. Denote $\{f\}$ by $BV_{loc,r}(0, \infty)$. The Lupas–Durrmeyer operators are defined as

$$V_n(f, x) = (n-1) \sum_{k=0}^{\infty} b_{n,k}(x) \int_0^{\infty} b_{n,k}(t) f(t) dt,$$

where

$$b_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}.$$

If $\{\xi_i\}$ are independent random variables with the same geometric distribution $P(\xi_i = k) = x^k(1-x), i = 1, 2, 3, \ldots,$ then $E\xi_i = x/(1-x), D\xi_i = x/(1-x)^2, \eta_n = \sum_{i=1}^{n} \xi_i$, with

$$P(\eta_n = k) = \binom{n+k-1}{k} x^k (1-x)^n.$$

For the above estimate, we refer to [56]. Then in [243], the following upper bound for the Baskakov basis function was established:

$$\binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} \leq \frac{33}{\sqrt{n}} \left(\frac{1+x}{x}\right)^{3/2}. \tag{7.24}$$

The following estimate is based on the bound (7.24).

Theorem 7.12 ([243]). *Let f be a function belonging to $BV_{loc,r}(0, \infty)$. Then for every $x \in (0, \infty)$ and n sufficiently large,*

$$\left|V_n(f,x)-\frac{1}{2}(f(x+)+f(x-))\right| \le \frac{5(1+x)}{nx}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x)$$
$$+\frac{50}{\sqrt{n}}\left(\frac{1+x}{x}\right)^{3/2}|f(x+)-f(x-)|$$
$$+\frac{(1+x)^r}{x^4}O(n^{-2}),$$

where the auxiliary function is as defined in Theorem 7.7.

Gupta and Srivastava [127] also obtained the rate of convergence for BV functions of Baskakov–Durrmeyer operators.

Let $B_\alpha[0,\infty), \alpha > 0$, be the class of all measurable complex-valued functions satisfying the growth condition $|f(t)| \le M(1+t)^\alpha$ for all $y \in [0,\infty)$ and some $M > 0$. The another sequence of Baskakov–Durrmeyer-type positive linear operators M_n (see [26]) is defined by

$$M_n(f,x) = (n-1)\sum_{k=1}^{\infty} b_{n,k}(x)\int_0^\infty b_{n,k-1}(t)f(t)dt + (1+x)^{-n}f(0).$$

Lemma 7.4 ([99]). *For all $x \in (0,\infty)$ and $k \in N$, we have*

$$b_{n,k}(x) < \frac{\sqrt{(1+x)}}{\sqrt{2enx}}.$$

Proof. By Theorem 2 of [246], it follows that

$$\binom{n+k-1}{k} t^k(1-t)^n < \frac{1}{\sqrt{2enx}}, \quad t \in (0,1]. \tag{7.25}$$

Replacing the variable t by $x/(1+x)$ in (7.25), we get

$$\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}} = b_{n,k}(x) < \frac{\sqrt{(1+x)}}{\sqrt{2enx}}.$$

■

The mth-order moment has been defined as

$$M_n\left((t-x)^m,x\right) = T_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right),$$

where $[\alpha]$ is an integral part of α. In particular, given any number $\lambda > 2$ and any $x > 0$, there is an integer $N(\lambda,x) > 2$ such that

$$T_{n,2}(x) \leq \frac{\lambda x(1+x)}{n}, \quad \text{for all } n \in N(\lambda, x).$$

Lemma 7.5 ([99]). *If* $n \in N$, *then*

(i) *For* $0 \leq y < x$, *we have*

$$\int_0^y W_n(x,t)dt \leq \frac{\lambda x(1+x)}{n(x-y)^2}.$$

(ii) *For* $x \leq z < \infty$, *we have*

$$\int_z^\infty W_n(x,t)dt \leq \frac{\lambda x(1+x)}{n(z-x)^2}.$$

Lemma 7.6 ([99]). *Let* $\{\xi_i\}_{i=1}^\infty$ *be a sequence of independent random variables with the same geometric distribution:*

$$P(\xi_i = k) = \left(\frac{x}{1+x}\right)^k \frac{1}{1+x}, \quad k \in N, \quad x > 0.$$

Then $E(\xi_1) = x$, $\rho_2 = E(\xi_1 - E\xi_1)^2 = x^2 + x$ *and* $\rho_3 = E|\xi_1 - E\xi_1|^3 \leq 3x(1+x)^2$.

Proof. We have

$$E(\xi_1^r) = \sum_{k=0}^\infty k^r \left(\frac{x}{1+x}\right)^k \frac{1}{1+x}.$$

By direct computation, we can easily check that

$$\sum_{k=0}^\infty k^r \left(\frac{x}{1+x}\right)^k \frac{1}{1+x} = 1, \quad E\xi_1 = x, \quad E\xi_1^2 = 2x^2 + x,$$
$$E\xi_1^3 = 6x^3 + 6x^2 + x, \quad E\xi_1^4 = 24x^4 + 36x^3 + 14x^2 + x.$$

Hence,

$$E(\xi_1 - E\xi_1)^2 = x^2 + x \quad \text{and} \quad E(\xi_1 - E\xi_1)^4 = 9x^4 + 18x^3 + 10x^2 + x.$$

Using Holder's inequality, we obtain

$$\begin{aligned}\rho_3 &= E|\xi_1 - E\xi_1|^3 \leq \sqrt{E(\xi_1 - E\xi_1)^4 E(\xi_1 - E\xi_1)^2} \leq \sqrt{(x^2+x)(9x^4+18x^3+10x^2+x)} \\ &\leq 3x(1+x)^2.\end{aligned}$$

∎

Lemma 7.7 ((See [242].) (Berry–Esseen theorem.)). *Let $X_1, X_2, \ldots . X_n$ be n independent and identically distributed random variables with zero mean and a finite absolute third moment. Then*

$$\sup_{x \in R} |F_n(x) - \phi(x)| < (0.82) l_{3,n},$$

where F_n is the distribution function of $(X_1 + X_2 + \ldots . + X_n)(n\rho_2)^{-1}$, ϕ is the standard normal distribution function, and $l_{3,n}$ is the Lyapunov ratio given by

$$l_{3,n} = \frac{\rho_3}{\rho_2^{3/2}\sqrt{n}}, \quad \rho_3 = E|X_1|^3.$$

Lemma 7.8 ([99]). *For $x \in (0, \infty)$ and $n, k \in N$, we have*

$$\left| \sum_{j=0}^{k-1} b_{n,j}(x) - \sum_{j=0}^{k-1} b_{n-1,j}(x) \right| \leq \frac{12 + 13x}{2\sqrt{nx(1+x)}}.$$

Proof. First,

$$\begin{aligned} I \equiv \left| \sum_{j=0}^{k-1} b_{n,j}(x) - \sum_{j=0}^{k-1} b_{n-1,j}(x) \right| &= \left| \sum_{j=0}^{k-1} b_{n,j}(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(k-nx)/\sqrt{nx(1+x)}} e^{-t^2/2} dt \right. \\ &+ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(k-(n-1)x)/\sqrt{(n-1)x(1+x)}} e^{-t^2/2} dt \\ &\left. - \frac{1}{\sqrt{2\pi}} \int_{(k-nx)/\sqrt{nx(1+x)}}^{(k-(n-1)x)/\sqrt{(n-1)x(1+x)}} e^{-t^2/2} dt - \sum_{j=0}^{k-1} b_{n-1,j}(x) \right|. \end{aligned}$$

Now, using Lemma 7.7, we get

$$\begin{aligned} I &\leq \left| \sum_{j=0}^{k-1} b_{n,j}(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(k-nx)/\sqrt{nx(1+x)}} e^{-t^2/2} dt \right| \\ &+ \left| \sum_{j=0}^{k-1} b_{n,j}(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(k-(n-1)x)/\sqrt{(n-1)x(1+x)}} e^{-t^2/2} dt \right| \\ &+ \left| \frac{1}{\sqrt{2\pi}} \int_{(k-nx)/\sqrt{nx(1+x)}}^{(k-(n-1)x)/\sqrt{(n-1)x(1+x)}} e^{-t^2/2} dt \right| \\ &\leq \frac{2(0.82)\rho_3}{\rho_2^{3/2}\sqrt{n}} + \frac{x}{\sqrt{2\pi nx(1+x)}}. \end{aligned}$$

Now, using Lemma 7.6, we have

$$I \leq \frac{6x(1+x)^2}{[x(1+x)]^{3/2}\sqrt{n}} + \frac{x}{\sqrt{2\pi nx(1+x)}} \leq \frac{12+13x}{2\sqrt{nx(1+x)}}.$$

■

Theorem 7.13 ([99]). *Let $f \in B_\alpha[0,\infty)$ be a function of bounded variation on every finite subinterval of $[0,\infty)$. Then for a fixed point $x \in (0,\infty), \lambda > 2$, and $n \geq \max\{1+\alpha, N(\lambda,x)\}$, we have*

$$\begin{aligned}\left|M_n(f,x) - \frac{1}{2}(f(x+)+f(x-))\right| &\leq \frac{3\lambda+(3\lambda+1)x}{nx}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x)\\ &\quad+\frac{27x+25}{4\sqrt{nx(1+x)}}|f(x+)-f(x-)|\\ &\quad+M(2^\alpha-1)\frac{(1+x)^\alpha}{x^{2\alpha}}O(n^{-\alpha})+\frac{2M\lambda(1+x)^{\alpha+1}}{nx},\end{aligned}$$

where the auxiliary function is as defined in Theorem 7.7.

Proof. Clearly,

$$M_n(f,x)-\frac{f(x+)+f(x-)}{2} = M_n(sgn_x(t),x)\left\{\frac{f(x+)-f(x-)}{2}\right\}+M_n(g_x,x), \tag{7.26}$$

where

$$sgn_x(t) = \begin{cases} -1, & 0 \leq t < x,\\ 0, & t = x,\\ +1, & t > x.\end{cases}$$

First, we observe that

$$\begin{aligned}M_n(sgn_x(t),x) &= \int_0^\infty sgn_x(t)W_n(x,t)dt = \int_x^\infty W_n(x,t)dt - \int_0^x W_n(x,t)dt\\ &:= A_n(x) - B_n(x).\end{aligned}$$

Using $(n-1)\int_x^\infty b_{n,k}(t)dt = \sum_{j=0}^k b_{n-1,j}(x)$, we have

$$\begin{aligned}A_n(x) &= (n-1)\sum_{k=1}^\infty b_{n,k}(x)\int_x^\infty b_{n,k-1}(t)dt + \int_x^\infty (1+x)^{-n}\delta(t)dt\\ &= \sum_{k=1}^\infty b_{n,k}(x)\sum_{j=0}^{k-1} b_{n-1,j}(x), \quad \delta(t)=0, \quad \text{as} \quad x>0.\end{aligned}$$

Next, using Lemma 7.8, we get

$$\left|A_n(x)-\sum_{k=1}^{\infty}b_{n,k}(x)\sum_{j=0}^{k-1}b_{n,j}(x)\right| = \left|\sum_{k=1}^{\infty}b_{n,k}(x)\left(\sum_{j=0}^{k-1}b_{n-1,j}(x)-\sum_{j=0}^{k-1}b_{n,j}(x)\right)\right|$$
$$\leq \frac{12+13x}{2\sqrt{nx(1+x)}}. \tag{7.27}$$

Let

$$\begin{aligned}
S &= \sum_{k=1}^{\infty}b_{n,k}(x)\sum_{j=0}^{k-1}b_{n,j}(x)\\
&= b_{n,0}b_{n,1}+b_{n,2}(b_{n,0}+b_{n,1})+b_{n,3}(b_{n,0}+b_{n,1}+b_{n,2})+\ldots..\\
&= \left[b_{n,0}^2+b_{n,1}(b_{n,0}+b_{n,1})+b_{n,2}(b_{n,0}+b_{n,1}+b_{n,2})+\ldots..\right]-\left[b_{n,0}^2+b_{n,1}^2+\ldots\right]\\
&:= S_1-\left[b_{n,0}^2+b_{n,1}^2+\ldots\right].
\end{aligned}$$

We have

$$S_1 = b_{n,0}^2+b_{n,1}^2+b_{n,2}^2+\ldots+b_{n,0}\left[b_{n,1}+b_{n,2}+\ldots\right]+b_{n,1}\left[b_{n,2}+b_{n,3}+\ldots\right]+\ldots.$$

Next, $I=(b_{n,0}+b_{n,1}+b_{n,2}+\ldots)(b_{n,0}+b_{n,1}+b_{n,2}+\ldots)$. Hence, by Lemma 7.4, we have

$$2S_1-I = b_{n,0}^2+b_{n,1}^2+b_{n,2}^2+\ldots\leq\frac{\sqrt{1+x}}{\sqrt{2enx}}.$$

Thus, $S=S_1-(2S_1-I)=I-S_1$.

$$\left|S-\frac{1}{2}\right| = \left|S_1-\frac{1}{2}\right|\leq\frac{\sqrt{1+x}}{2\sqrt{2enx}}\leq\frac{1+x}{4\sqrt{nx(1+x)}}. \tag{7.28}$$

If we combine (7.27) and (7.28), it follows that

$$\left|A_n(x)-\frac{1}{2}\right|\leq\frac{1}{2\sqrt{nx(1+x)}}\left[\frac{1+x}{2}+12+13x\right]=\frac{27x+25}{4\sqrt{nx(1+x)}}.$$

Now, as $A_n(x)+B_n(x)=\int_0^{\infty}W_n(x,t)dt=1$, we have

$$|A_n(x)-B_n(x)| = |2A_n(x)-I|\leq\frac{27x+25}{4\sqrt{nx(1+x)}}. \tag{7.29}$$

Next, we estimate $M_n(g_x, x)$. By Lebesgue–Stieltjes integral representation, we have

$$M_n(g_x,x)=\int_0^\infty W_n(x,t)g_x(t)dt=\left(\int_{I_1}+\int_{I_2}+\int_{I_3}\right)W_n(x,t)g_x(t)dt :=E_1+E_2+E_3,$$

where $I_1 = [0, x-x/\sqrt{n}]$, $I_2 = [x-x/\sqrt{n}, x+x/\sqrt{n}]$, and $I_3 = [x+x/\sqrt{n}, \infty)$. Suppose $\beta_n(t,x) = \int_0^t W_n(x,t)dt$. We first estimate E_1. Writing $y = x - x/\sqrt{n}$ and using Lebesgue–Stieltjes integration by parts, we have

$$E_1 = \int_0^y g_x(t)d_t(\beta_n(t,x)) = g_x(y^+)\beta_n(y,x) - \int_0^y (\beta_n(t,x))d_t g_x(t).$$

Since $|g_x(y^+)| \le V_{y+}^x(g_x)$, it follows from Lemma 7.5 that

$$\begin{aligned}|E_1| &\le V_{y+}^x(g_x)\beta_n(y,x) + \int_0^y \beta_n(t,x)d_t\left(-V_t^x(g_x)\right)\\ &\le V_{y+}^x(g_x)\frac{\lambda x(1+x)}{n(x-y)^2} + \frac{\lambda x(1+x)}{n}\int_0^y \frac{1}{(x-t)^2}d_t(-V_{t+}^x(g_x)).\end{aligned}$$

Integrating the last term by parts, we have, after simple computation,

$$|E_1| \le \frac{\lambda x(1+x)}{n}\left[\frac{V_0^x(g_x)}{x^2} + 2\int_0^y \frac{V_t^x(g_x)}{(x-t)^3}dt\right].$$

Now, replacing the variable y in the last integral by $x - x/\sqrt{u}$, we obtain

$$|E_1| \le \frac{2\lambda(1+x)}{nx}\sum_{k=1}^n V_{x-x/\sqrt{k}}^x(g_x). \tag{7.30}$$

Next, we estimate E_2. For $t \in [x - x/\sqrt{n}, x + x/\sqrt{n}]$, we have

$$|g_x(t)| = |g_x(t) - g_x(x)| \le V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x),$$

and, therefore,

$$|E_2| \le V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x) \le \frac{1}{n}\int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} d_t(\beta_n(t,x)).$$

Since $\int_a^b d_t\beta_n(t,x) \le 1$, for $(a,b) \subset [0,\infty)$, we thus have

$$|E_2| \le V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x) \le \frac{1}{n}\sum_{k=1}^n V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x). \tag{7.31}$$

Finally, we estimate E_3. By setting $z = x + x/\sqrt{n}$, we have

$$E_3 = \int_z^\infty g_x(t) W_n(x,t)dt = \int_z^\infty g_x(t) d_t(\beta_n(t,x)).$$

We define $Q_n(x,t)$ on $[0, 2x]$ as

$$Q_n(x,t) = 1 - \beta_n(t,x), \quad 0 \le t < 2x, \quad \text{and} \quad 0, t = 2x.$$

Thus,

$$\begin{aligned}
E_3 &= \int_z^{2x} g_x(t) d_t(Q_n(x,t)) - g_x(2x) \int_{2x}^\infty W_n(x,t)dt \\
&\quad + \int_{2x}^\infty g_x(t) d_t(\beta_n(x,t)) \\
&:= E_{31} + E_{32} + E_{33}.
\end{aligned} \tag{7.32}$$

By partial integration, we get

$$E_{31} = g_x(z^-) Q_n(x, z^-) + \int_x^{2x} d_t(g_x(t)) \overline{Q}_n(x,t),$$

where $\overline{Q}_n(x,t)$ is the normalized form of $Q_n(x,t)$. Since $Q_n(x,z^-) = Q_n(x,z)$ and $|g + x(z^-)| \le V_x^{z^-}(g_x)$, we have

$$|E_{31}| = V_x^{z^-}(g_z) Q_n(x,z) + \int_z^{2x} d_t(V_x^t(g_x)) \overline{Q}_n(x,t).$$

Applying Lemma 7.5 and the fact that $\overline{Q}_n(x,t) \le Q_n(x,t)$ on $[0, 2x]$, we get

$$\begin{aligned}
|E_{31}| &\le \frac{V_x^{z^-}(g_x)\lambda x(1+x)}{n(z-x)^2} + \frac{\lambda x(1+x)}{n} \int_z^{2x} \frac{1}{(x-t)^2} d_t(V_x^t(g_x)) \\
&\quad + \frac{1}{2}\left[V_{2x-}^{2x}(g_x) \int_{2x}^\infty W_n(x,u)du \right] \\
&\le \frac{V_x^{z^-}(g_x)\lambda x(1+x)}{n(z-x)^2} \frac{\lambda x(1+x)}{n} \int_z^{2x} \frac{1}{(x-t)^2} d_t(V_x^t(g_x)) \\
&\quad + \frac{1}{2}\left[V_{2x-}^{2x}(g_x) \frac{\lambda x(1+x)}{nx^2} \right].
\end{aligned}$$

Thus, arguing similarly as in the estimate of E_1, we get

$$|E_{31}| \le \frac{2\lambda(1+x)}{nx} \sum_{k=1}^n V_x^{x+x/\sqrt{k}}(g_x). \tag{7.33}$$

Again by Lemma 7.5, we get

$$|E_{32}| \leq \frac{\lambda x(1+x)}{nx^2} g_x(2x) \leq \frac{\lambda(1+x)}{nx} \sum_{k=1}^{n} V_x^{x+x/\sqrt{k}}(g_x). \tag{7.34}$$

Finally, we estimate E_{33} as follows:

$$|E_{33}| \leq M \int_{2x}^{\infty} W_n(x,t)[(1+t)^\alpha + (1+x)^\alpha]dt, \quad \text{where} \quad n > \alpha.$$

For $2x \leq t$, we have the inequality

$$(1+t)^\alpha - (1+x)^\alpha \leq (2^\alpha - 1)\frac{(1+x)^\alpha}{x^\alpha}(t-x)^\alpha.$$

Thus,

$$\begin{aligned} |E_{33}| &\leq M(2^\alpha - 1)\frac{(1+x)^\alpha}{x^\alpha} \int_{2x}^{\infty} W_n(t,x)[(t-x)^\alpha]dt + 2M \int_{2x}^{\infty} W_n(t,x)dt \\ & M(2^\alpha - 1)\frac{(1+x)^\alpha}{x^\alpha} \int_{2x}^{\infty} W_n(t,x)\frac{(t-x)^{2\alpha}}{x^\alpha}dt + 2M \int_{2x}^{\infty} W_n(t,x)dt. \end{aligned}$$

Now, using Lemmas 7.5 and 7.6, we obtain

$$|E_{33}| \leq M(2^\alpha - 1)\frac{(1+x)^\alpha}{x^\alpha} O(n^{-\alpha}) + \frac{2M\lambda(1+x)^{\alpha+1}}{nx}. \tag{7.35}$$

Finally, collecting the estimates of (7.26), (7.29)–(7.35), we get the required result. This completes the proof of the theorem. ■

7.7 Baskakov–Beta Operators

In 1994, Gupta observed [92] that by taking the weights of Beta basis functions in the integral modification of Baskakov operators, we have a better approximation. Also, for the rate of convergence of functions of bounded variation, some approximation properties become simpler for Baskakov–Beta operators. The Baskakov–Beta operators [92, 94], for $x \in [0, \infty)$, are defined as

$$\begin{aligned} B_n(f,x) &= \int_0^\infty K_n(x,t) f(t)dt \\ &= \sum_{k=0}^{\infty} b_{n,k}(x) \int_0^\infty v_{n,k}(t) f(t)dt, \end{aligned}$$

where the kernel $K_n(x,t) = \sum_{k=0}^{\infty} b_{n,k}(x) v_{n,k}(t)$ and

$$b_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}, \quad v_{n,k}(t) = \frac{1}{B(k+1,n)} \frac{t^k}{(1+t)^{n+k+1}},$$

with $B(m,n) = \dfrac{(m-1)!(n-1)!}{(n+m-1)!}$. In 1998, Gupta and Arya [109] used the following upper estimate for the Baskakov basis function:

If y is a positive-valued random variable with nondegenerate probability distribution, then $E(y^3) \leq [E(y^2)E(y^4)]^{1/2}$, provided $E(y^2), E(y^3), E(y^4) < \infty$. Using this, they got the following upper bound:

$$\binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} \leq \frac{8\sqrt{1+9x(1+x)}+2}{5\sqrt{nx(1+x)}}. \tag{7.36}$$

Using the bound (7.36) and other basic results, Gupta and Arya [109] established the rate of convergence.

Theorem 7.14 ([109]). *Let f be a function of bounded variation on every finite subinterval of $[0,\infty)$. If $|f(t)| \leq M(1+t)^\alpha$ for $t \in [0,\infty)$, where $M > 0, \alpha \in N_0$, and we'll choose a number $\lambda > 2$. Then for $n > \max\{1+\alpha, N(\lambda,x)\}$, we have*

$$\left| B_n(f,x) - \frac{1}{2}(f(x+)+f(x-)) \right| \leq \frac{5\lambda+(5\lambda+1)x}{nx} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x)$$
$$+\frac{4[1+9x(1+x)]^{1/2}+1}{5\sqrt{nx(1+x)}}|f(x+)-f(x-)|$$
$$+M(2^\alpha-1)\frac{(1+x)^\alpha}{x^{2\alpha}}O(n^{-\alpha}) + \frac{2\lambda M(1+x)^{\alpha+1}}{nx},$$

where the auxiliary function is as defined in Theorem 7.7.

Proof. Clearly, we can write

$$\left| B_n(f,x) - \frac{f(x+)+f(x-)}{2} \right| \leq |B_n(sgn_x(t),x)|.\frac{1}{2}|f(x+)-f(x-)|$$
$$+|B_n(g_x,x)|, \tag{7.37}$$

where

$$sgn_x(t) = \begin{cases} -1, & 0 \leq t < x, \\ 0, & t = x, \\ +1, & t > x. \end{cases}$$

First, we observe that

$$\begin{aligned} B_n(sgn_x(t),x) &= \int_0^\infty sgn_x(t)K_n(x,t)dt = \int_x^\infty K_n(x,t)dt - \int_0^x K_n(x,t)dt \\ &:= A_n(x) - B_n(x). \end{aligned}$$

Using the fact $\int_x^\infty v_{n,k}(t)dt = \sum_{j=0}^k b_{n,j}(x)$, we have

$$\begin{aligned} A_n(x) &= \int_0^\infty K_n(x,t)dt = \sum_{k=0}^\infty b_{n,k}(x)\int_x^\infty v_{n,k}(t)dt \\ &= \sum_{k=0}^\infty b_{n,k}(x)\sum_{j=0}^k v_{n,j}(x). \end{aligned}$$

For simplicity, we shall use the notation b_k instead of $b_{n,k}$:

$$\begin{aligned} I &= \sum_{k=0}^\infty\sum_{j=0}^\infty b_j b_k = \sum_{k=0}^\infty\sum_{j=0}^\infty b_j b_k + \sum_{k=0}^\infty\sum_{j=k}^\infty b_j b_k - \sum_{k=0}^\infty b_k^2 \\ &= 2A_n(x) - \sum_{k=0}^\infty b_k^2, \end{aligned}$$

because

$$sum_{k=0}^\infty \sum_{j=k}^\infty b_j b_k = \sum_{j=0}^\infty\sum_{k=0}^j b_j b_k = \sum_{k=0}^\infty\sum_{j=0}^k b_k b_j = A_n.$$

Applying the bound (7.36), we get

$$|2A_n(x) - I| \le \frac{8\sqrt{1+9x(1+x)}+2}{5\sqrt{nx(1+x)}}\sum_{k=0}^\infty b_{n,k}(x) \le \frac{8\sqrt{1+9x(1+x)}+2}{5\sqrt{nx(1+x)}}.$$

Because $A_n(x) - B_n(x) = \int_0^\infty K_n(x,t)dt = 1$, therefore

$$|A_n(x) - B_n(x)| \le \frac{8\sqrt{1+9x(1+x)}+2}{5\sqrt{nx(1+x)}}.$$

The estimate of $B_n(g_x,x)$ follows along the lines of Theorem 7.13, by using the moment estimates of Baskakov–Beta operators; we omit the details. ■

7.8 General Summation–Integral-Type Operators

In 2003, Srivastava and Gupta [218] introduced a general sequence of linear positive operators $G_{n,c}(f, x)$, which, applied to f, are defined as

$$G_{n,c}(f; x) = n \sum_{k=1}^{\infty} p_{n,k}(x; c) \int_0^{\infty} p_{n+c,k-1}(t; c) f(t)\, dt + p_{n,0}(x; c) f(0), \tag{7.38}$$

where

$$p_{n,k}(x; c) = \frac{(-x)^k}{k!} \phi_{n,c}^{(k)}(x)$$

and

$$\phi_{n,c}(x) = \begin{cases} e^{-nx}, & c = 0, \\ (1 + cx)^{-n/c}, & c \in \mathbb{N} := \{1, 2, 3, \ldots,\}, \\ (1 - x)^{-n}, & c = -1, \end{cases}$$

where $\{\phi_{n,c}(x)\}_{n=1}^{\infty}$ is a sequence of functions, defined on the closed interval $[0, b]$ $(b > 0)$, that satisfies the following properties for every $n \in \mathbb{N}$ and $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$:

(i) $\phi_{n,c} \in C^{\infty}([a, b])$ $(b > a \geq 0)$.
(ii) $\phi_{n,c}(0) = 1$.
(iii) $\phi_{n,c}(x)$ is completely monotone, so that $(-1)^k \phi_{n,c}^{(k)}(x) \geq 0$ $(0 \leq x \leq b)$.
(iv) There is an integer c such that

$$\phi_{n,c}^{(k+1)}(x) = -n\phi_{n+c,c}^{(k)}(x) \; (n > \max\{0, -c\};\ x \in [0, b]).$$

The following are the special cases of the operators $G_{n,c}(f; x)$ in (7.38), which have the following forms:

1. If $c = 0$, then by simple computation, one has $p_{n,k}(x; 0) = e^{-nx} \frac{(nx)^k}{k!}$, and the operators become the Phillips operators $G_{n,0}(f; x)$, introduced by [199], which, for $x \in [0, \infty)$, are defined by

$$G_{n,0}(f; x) = n \sum_{k=1}^{\infty} p_{n,k}(x; 0) \int_0^{\infty} p_{n,k-1}(t; 0) f(t)\, dt + p_{n,0}(x; 0) f(0).$$

2. If $c = 1$, then by simple computation, one has $p_{n,k}(x;1) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}$, and the operators become the Durrmeyer-type Baskakov operators $G_{n,1}(f;x)$, which were introduced by [139] and, for $x \in [0,\infty)$, are defined as

$$G_{n,1}(f;x) = n\sum_{k=1}^{\infty} p_{n,k}(x;1)\int_0^{\infty} p_{n+1,k-1}(t;1)\, f(t)\,dt + p_{n,0}(x;1)\, f(0).$$

3. If $c = -1$, then by simple computation, one has $p_{n,k}(x;-1) = \binom{n}{k} x^k(1-x)^{n-k}$, and the operators become the Bernstein–Durrmeyer-type $G_{n,-1}(f;x)$ studied in [118] and its q analog that appeared in [87]. In this case, summation runs from 1 to n, integration from 0 to 1, and $x \in [0,1]$ is defined as

$$G_{n,-1}(f;x) = n\sum_{k=1}^{n} p_{n,k}(x;-1)\int_0^{1} p_{n-1,k-1}(t;-1)\, f(t)\,dt + p_{n,0}(x;-1)\, f(0).$$

Later Ispir and Yuksel [158] named the operators (7.38) Srivastava–Gupta operators. They considered the Bézier variant of these operators and estimated the rate of convergence, which we will discuss in the next chapter. Some other approximation properties of these operators were discussed in [57] and [237].

Let $H_\alpha(0,\infty)$ $(\alpha \geq 0)$ be the class of all locally integrable functions defined on $(0,\infty)$ and satisfying the growth condition $|f(t)| \leq Mt^\alpha$, $M > 0, t \to \infty$.

Theorem 7.15 ([218]). *Let $f \in H_\alpha(0,\infty)$ $(\alpha \geq 0)$, and suppose that the one-sided limits $f(x-)$ and $f(x+)$ exist for some fixed point $x \in (0,\infty)$. Then for $r \in N, c \in N^0$ and $\lambda > 2$, there is a positive constant $M > 0$ independent of n such that for n sufficiently large,*

$$\left|G_{n,c}(f;x) - \frac{1}{2}(f(x+)+f(x-))\right| \leq \frac{2\lambda(1+cx)+x}{nx}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x) + Mn^{-r}$$
$$+A_{n,c}(x)|f(x+)-f(x-)|,$$

where the auxiliary function is as defined in Theorem 7.7, $V_a^b(g_x)$ is the total variation of g_x on $[a,b]$, and

$$A_{n,c}(x) = \begin{cases} \frac{\sqrt{1+cx}}{\sqrt{8enx}}, & c \in N \\ \frac{1}{2\sqrt{\pi nx}}, & c = 0 \end{cases}$$

Theorem 7.16. *Let f be a function of bounded variation on every finite subinterval of $[0,1]$. Suppose also that the one-sided limits $f(x-)$ and $f(x+)$ exist for some fixed point $x \in (0,1)$. Then for $\lambda > 2$ and n sufficiently large,*

$$\left|G_{n,-1}(f;x) - \frac{1}{2}(f(x+) + f(x-))\right| \leq \frac{5\lambda}{nx(1-x)} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x)$$
$$+ \frac{1}{\sqrt{8enx(1-x)}}|f(x+) - f(x-)|,$$

where the auxiliary function g_x is as defined in Theorem 7.2.

In response to a question posed by Gupta in private communication to Rogalski, regarding the better bounds, Bastien and Rogalski [35] established a better bound for the Baskakov basis function, which can be used in the above Theorem 7.16; this was also remarked in [218].

7.9 Meyer–König–Zeller Operators

In 1988, Guo [89] introduced certain integral modifications of Meyer–König–Zeller operators defined on [0, 1] as

$$M_n(f,x) = \frac{(n+k-2)(n+k-3)}{n-2} \sum_{k=1}^{\infty} p_{n,k+1}(x) \int_0^1 p_{n-2,k-1}(t) f(t) dt, \tag{7.39}$$

where $p_{n,k}(x) = \binom{n+k-1}{k} x^k (1-x)^n$. He obtained the following rate of convergence by considering the bound $\binom{n+k-1}{k} x^k (1-x)^n \leq \frac{33}{\sqrt{n}x^{3/2}}$:

Theorem 7.17. *Let f be a function of bounded variation on* [0, 1], $x \in (0, 1)$. *Then for n sufficiently large,*

$$\left|M_n(f,x) - \frac{1}{2}(f(x+) + f(x-))\right| \leq \frac{7}{nx} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x)$$
$$+ \frac{50}{\sqrt{n}x^{3/2}}|f(x+) - f(x-)|,$$

where

$$g_x(t) = \begin{cases} f(t) - f(x-) & , \quad 0 \leq t < x \\ 0 & , \quad t = x \\ f(t) - f(x+) & , \quad x < t \leq 1 \end{cases}$$

Later, in 1995, Gupta [93] improved the above result by using a better bound for the MKZ basis function and obtained the following main result:

Theorem 7.18. *Let f be a function of bounded variation on* $[0,1]$, $x \in (0,1)$. *Then for n sufficiently large,*

$$\left|M_n(f,x) - \frac{1}{2}(f(x+) + f(x-))\right| \leq \frac{7}{nx}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x) + \frac{114}{15\sqrt{n}x^{3/2}}|f(x+) - f(x-)|,$$

where $g_x(t)$ is as defined in Theorem 7.17.

In 1998, Zeng [246] obtained the optimum bound for the MKZ basis function and obtained the better approximation. In the preceding Theorem 7.18, he obtained the coefficient of $|f(x+) - f(x-)|$ as $\frac{3}{\sqrt{8e}x^{3/2}}$, but while calculating the estimate, there was a mistake. Later Gupta and Kumar [114] gave the correct estimate of this theorem and found the coefficient of $|f(x+) - f(x-)|$ to be $\left(3 + \frac{1}{\sqrt{8e}}\right)\frac{1}{\sqrt{n}x^{3/2}}$.

In this continuation, Guo [90] considered another integrated MKZ operator on $[0,1]$ as

$$K_n(f,x) = \sum_{k=0}^{\infty} m_{n,k}(x)\int_{I_k} f(t)dt, \tag{7.40}$$

where $I_k = \left[\frac{k}{n+k}, \frac{k+1}{n+k+1}\right]$ and $m_{n,k}(x) = (n+1)\binom{n+k+1}{k}x^k(1-x)^n$. Guo [90] established the following main result:

Theorem 7.19. *Let f be a function of bounded variation on* $[0,1]$, $x \in (0,1)$. *Then for n sufficiently large,*

$$\left|K_n(f,x) - \frac{1}{2}(f(x+) + f(x-))\right| \leq \frac{5}{nx}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x) + \frac{49}{\sqrt{n}x^{3/2}}|f(x+) - f(x-)|,$$

where $g_x(t)$ is as defined in Theorem 7.17.

Love et al. [183] claimed to have improved the preceding theorem by considering the correct moment estimation, but the result was still not correct.

Theorem 7.20 ([183]). *Let f be a function of bounded variation on* $[0,1]$. *Then for* $0 < \delta \leq x \leq 1-\delta$ *and any integer* $n > 2$,

$$\left|K_n(f,x) - \frac{1}{2}(f(x+) + f(x-))\right| \leq \frac{5}{\sqrt{n}x^{3/2}}|f(x+) - f(x-)|$$
$$+\frac{1}{n-1}\left\{1 + \frac{8x(1-x)}{\delta^2}\left(1 + \frac{1-\delta}{12(n-2)\delta}\right)\right\}$$
$$\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x),$$

where $g_x(t)$ *is as defined in Theorem 7.17.*

As above, using the bound of Zeng [246] in the above Theorem 7.20, Gupta [97] correctly obtained the coefficient of $|f(x+) - f(x-)|$: It was $\left(\frac{8}{3} + \frac{1}{\sqrt{2e}}\right)\frac{1}{\sqrt{n}x^{3/2}}$.

In order to approximate Lebesgue integrable functions on [0, 1], Gupta and Abel [107] proposed the Durrmeyer variant of Meyer–König and Zeller as operators as

$$D_n(f,x) = \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^1 b_{n,k}(t) f(t) dt, x \in [0,1], n \in \mathbb{N},$$

where $p_{n,k}(x)$ is given in (7.39) and $b_{n,k}(t) = \frac{(n+k)!}{k!(n-1)!} t^k (1-t)^{n-1}$. Gupta and Abel [107] estimated the rate of convergence of these operators for functions of bounded variation.

Theorem 7.21 ([107]). *Let* f *be a function of bounded variation on* $[0,1]$. *Then for each* $x \in (0,1)$, $K > 12.25$, *there is a constant* N *independent of* f *and* n, *such that for all* $n > N$, *we have*

$$\left|D_n(f,x) - \frac{1}{2}(f(x+) + f(x-))\right| \leq \frac{1}{\sqrt{8enxn}}|f(x+) - f(x-)|$$
$$+\frac{K}{nx(1-x)} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x),$$

where $g_x(t)$ *is as defined in Theorem 7.17.*

For $f \in L^1[0,1]$, Gavrea and Trif [83] introduced another Durrmeyer-type modification of the Meyer–König and Zeller operators as

$$T_n(f,x) = \sum_{k=0}^{n} p_{n+1,k}(x) \int_0^1 b_{n,k}(t) f(t) dt$$
$$+ \sum_{k=0}^{n} p_{n+1,k}(1-x) \int_0^1 m_{n,k}(1-t) f(t) dt, \tag{7.41}$$

where $p_{n+1,k}(x)$ and $m_{n,k}(t)$ can be derived from (7.39) and (7.40), respectively. They [83] used the identity

$$\sum_{k=0}^{n} p_{n+1,k}(x) + \sum_{k=0}^{n} p_{n+1,k}(1-x) = 1, x \in R,$$

to define the operators (7.41). The probabilistic proof of this identity was given by Zeilberger [245]. In 2012, Trif [234] estimated the rate of convergence for functions of bounded variation of the operators (7.41).

Theorem 7.22 ([234]). *If $f : [0,1] \to \mathbb{R}$ is a function of bounded variation on* $[0,1]$, *then for each $n \geq 4$ and $x \in (0,1)$, one has*

$$\left|T_n(f,x) - \frac{1}{2}(f(x+) + f(x-))\right| \leq \frac{1}{2\sqrt{n\varphi(x)}}|f(x+) - f(x-)| + V_{a_{k,x}}^{b_{k,x}}(g_x)$$
$$+ \frac{84}{(n-1)\varphi^2(x)}\left[V_0^1(g_x) + \sum_{k=1}^{n-1} V_{a_{k,x}}^{b_{k,x}}(g_x)\right],$$

where $\varphi(x) : \min\{x, 1-x\}$,

$$a_{k,x} := x - \frac{x}{\sqrt{k}}, b_{k,x} := x + \frac{1-x}{\sqrt{k}},$$

where $g_x : [0,1] \to \mathbb{R}$ is the auxiliary function as defined in Theorem 7.17.

Chapter 8
Convergence for Bounded Functions on Bézier Variants

The various Bézier variants (BV) of the approximation operators are important research topics in approximation theory. They have close relationships with geometry modeling and design. Let $p_{n,k}(x) = \binom{n}{k} x^k(1-x)^{n-k}, (0 \le k \le n)$ be Bernstein basis functions. The Bézier basis functions, which were introduced in 1972 by Bézier [39], are defined as $J_{n,k}(x) = \sum_{j=k}^{n} p_{n,j}(x)$. For $\alpha \ge 1$ and a function f defined on $[0,1]$, the Bernstein–Bézier operators $B_{n,\alpha}(f,x)$, which were introduced by Chang, [51] are defined as

$$B_{n,\alpha}(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) Q_{n,k}^{(\alpha)}(x), \tag{8.1}$$

where $Q_{n,k}^{(\alpha)}(x) = (J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)), (J_{n,n+1}(x) \equiv 0)$.

Chang [51] studied the convergence of the operators (8.1) and showed that for $f \in C[0,1]$, $\lim_{n\to\infty} B_{n,\alpha}(f,x) = f(x)$ uniformly on $[0,1]$. In this continuation, Li and Gong [176] established the rate of convergence in terms of the modulus of continuity for Bernstein–Bézier operators.

Theorem 8.1 ([176]). *For $f \in C[0,1]$, we have*

$$||B_{n,\alpha}(f,x) - f||_{C[0,1]} \le \begin{cases} (1+\alpha/4)\omega(n^{-1/2}, f), & \alpha \ge 1, \\ M\omega(n^{-\alpha/2}, f)\ , & 0 < \alpha < 1, \end{cases}$$

where $\omega(\delta, f)$ is the modulus of continuity $f(x)$, and M is a constant depending only on α and f.

In 1986, Liu [177] established an inverse theorem for Bernstein–Bézier polynomials.

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Theorem 8.2 ([177]). *For $f \in C[0,1], \alpha \geq 1$ and $0 < \beta < 1$ such that*

$$|B_{n,\alpha}(f,x) - f| \leq M\left(\max\left\{n^{-1}, \sqrt{x(1-x)}/n^{1/2}\right\}\right)^{\beta},$$

where M is a constant $f \in Lip\beta$.

For $f \in L_1[0,1]$, the Bernstein–Kantorovich–Bézier operators are defined as

$$L_{n,\alpha}(f,x) = (n+1)\sum_{k=0}^{n} Q_{n,k}^{(\alpha)}(x)\int_{I_k} f(t)dt, \tag{8.2}$$

where $I_k = [k/(n+1), (k+1)/(n+1)]$, $0 \leq k \leq n$, and $Q_{n,k}^{(\alpha)}(x)$ is as defined in (8.1).

Also, in 1991, Liu [178] studied the operators (8.2) in the L_p-norm and estimated the direct result.

Theorem 8.3 ([178]). *Let $1 \leq p < \infty$. Then for any $f \in L_p(0,1)$, we have*

$$||L_{n,\alpha}(f,.) - f||_p \leq M\omega(f, n^{-1/2})_p,$$

where M is a constant depending only on α and p, and $\omega(f,t)_p$ is the modulus of continuity of f in $L_p[0,1]$.

Zeng and collaborators (see [247, 248, 253, 255]) have done commendable work on the Bézier variants of different operators. They established the rate of convergence for functions of bounded variation for several operators. In this chapter, we mention some results on the rate of convergence on certain Bézier-type operators for bounded functions and for functions of BV.

Throughout this chapter, we define a metric form:

$$\Omega_x(f,\xi) = \sup_{t\in[x-\xi,x+\xi]\bigcap[0,\infty)} |f(t) - f(x)|,$$

where $\xi \geq 0$. It satisfies each of the following properties:

1. $\Omega_x(f,h)$ is monotone nondecreasing with respect to h.
2. $\lim_{h\to\infty}\Omega_x(f,h) = 0$ if f is continuous at the point x.
3. If f is a function of bounded variation on $[a,b]$ and $V_a^b(f)$ denotes the total variation of f on $[a,b]$, then

$$\Omega_x(f,h) \leq V_{x-h}^{x+h}(f).$$

8.1 Bernstein–Bézier-Type Operators

The rate of convergence of the Bézier variant of classical Bernstein polynomials and Bernstein–Kantorovich operators for functions of bounded variation was studied first by Zeng and Piriou [255]. They observed [255] that the operators $B_{n,\alpha}(f,x)$,

$L_{n,\alpha}(f,x)$ defined by (8.1) and (8.2), respectively, are not Feller operators except for $\alpha = 1$, namely, $B_{n,\alpha}(t,x) \neq x$ and $L_{n,\alpha}(t,x) \neq x$. Zeng and Piriou established the rate of convergence by considering better bounds. For the auxiliary function g_x used in this section, we refer to Theorem 7.5.

Theorem 8.4 ([255]). *Let f be of bounded variation on* $[0,1]$. *Then for every* $x \in [0,1]$ *and* $n \geq 1$, *we have*

$$\left| B_{n,\alpha}(f,x) - \left[\frac{1}{2^\alpha} f(x+) + \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right] \right|$$
$$\leq \frac{3\alpha}{nx(1-x)+1} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x)$$
$$+ \frac{2\alpha}{\sqrt{nx(1-x)}+1} \left(|f(x+) - f(x-)| + e_n(x)|f(x) - f(x-)| \right),$$

where

$$e_n(x) = \begin{cases} 0, & x \neq k/n \ \text{for all} \ k \in N, \\ 1, & x = k/n \ \text{for a} \ k \in N. \end{cases}$$

When $x = 0$ *(resp.,* $x = 1$*), we set* $1/2^\alpha f(x+) + (1 - 1/2^\alpha) f(x-) = f(0)$ *(resp.,* $f(1)$*).*

Theorem 8.5 ([255]). *Let f be of bounded variation on* $[0,1]$. *Then for every* $x \in (0,1)$ *and* $n > 1/3x(1-x)$, *we have*

$$\left| L_{n,\alpha}(f,x) - \left[\frac{1}{2^\alpha} f(x+) + \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right] \right|$$
$$\leq \frac{5\alpha}{nx(1-x)+1} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x)$$
$$+ \frac{4\alpha}{\sqrt{nx(1-x)}+1} |f(x+) - f(x-)|.$$

In 2000, Zeng and Chen [251] extended the studies in this direction, introducing the Bézier variant of Bernstein–Durrmeyer operators. For a function defined on $[0,1]$ and $\alpha \geq 1$, the Durrmeyer–Bézier operators $D_{n,\alpha}(f,x)$ are defined by

$$D_{n,\alpha}(f,x) = (n+1) \sum_{k=0}^{n} Q_{n,k}^{(\alpha)}(x) \int_0^1 p_{n,k}(t) f(t) dt, \tag{8.3}$$

where $p_{n,k}(t)$ is the Bernstein basis function.

Theorem 8.6 ([251]). *Let f be of bounded variation on $[0,1]$ and $\alpha \geq 1$. Then for every $x \in (0,1)$ and $n > 1/(x(1-x))$, we have*

$$\left| D_{n,\alpha}(f,x) - \left[\frac{1}{\alpha+1} f(x+) + \frac{\alpha}{\alpha+1} f(x-) \right] \right| \leq \frac{8\alpha}{nx(1-x)} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x) + \frac{2\alpha}{\sqrt{nx(1-x)}} |f(x+) - f(x-)|.$$

8.2 Bleimann–Butzer–Hann–Bézier Operators

Bleimann et al. [41] introduced a sequence of positive linear operators defined on the space of real functions on the infinite interval $[0,\infty)$ by

$$L_n(f,x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n-k+1}\right), \quad x \in [0,\infty); n \in N := \{1,2,3,\ldots\}, \tag{8.4}$$

where

$$p_{n,k}(x) = \binom{n}{k} p_x^k q_x^{n-k}, \quad p_x = \frac{x}{1+x}; q_x = 1 - p_x = \frac{1}{1+x}.$$

In fact, we can write

$$Z_{n,x} = \frac{S_{n,x}}{n - S_{n,x} + 1}, \quad S_{n,x} = \xi_{1,x} + \cdots + \xi_{n,x},$$

where $\xi_{1,x}, \ldots, \xi_{n,x}$ are independent random variables having the same Bernoulli distribution with parameters

$$p_x = \frac{x}{1+x},$$

that is,

$\mathrm{Prob}\{\xi_{k,x} = 1\} = p_x$ and $\mathrm{Prob}\{\xi_{k,x} = 0\} = q_x = 1 - p_x$,

so that $S_{n,x}$ has a binomial distribution with parameters n, p_x. This probabilistic representation plays an important technical role. Srivastava and Gupta [219] considered the Bézier variant of the Bleimann–Butzer–Hann operators as follows:

$$L_{n,\alpha}(f,x) = \sum_{k=0}^{n} Q_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n-k+1}\right), \quad x \in [0,\infty); n \in N; \alpha \geq 1, \tag{8.5}$$

where

$$Q_{n,k}^{(\alpha)}(x) = (J_{n,k}(x))^{\alpha} - (J_{n,k+1}(x))^{\alpha} \quad J_{n,k+1}(x) = 0,$$

and

$$J_{n,k}(x) = \sum_{j=k}^{n} p_{n,j}(x)$$

One can easily verify that $L_{n,\alpha}(f, x)$ are positive linear operators and that

$$L_{n,1}(f, x) \equiv L_n(f, x)$$

in terms of the Bleimann–Butzer–Hann operators $L_n(f, x)$ defined by (8.4).

Some basic properties of $J_{n,k}(x)$ are given below:

1. $J_{n,k}(x) - J_{n,k+1}(x) = p_{n,k}(x) \quad k \in N_0 = N \cup \{0\}$.
2. $J_{n,0}(x) > J_{n,1}(x) > \cdots J_{n,k}(x) > J_{n,k+1}(x) > \cdots$.
3. For $k \in N, 0 < J_{n,k}(x) < 1$ and $J_{n,k}(x)$ increases strictly on $[0, \infty)$.

By the Lebesgue–Stieltjes integral representation, we have

$$L_{n,\alpha}(f, x) = \int_0^{\infty} f(t) d_t \left(K_{n,\alpha}(x, t)\right),$$

where the kernel $K_{n,\alpha}(x, t)$ is defined by

$$K_{n,\alpha}(x, t) = \begin{cases} \sum_{k<(n-k+1)t} Q_{n,k}^{(\alpha)}(x), & 0 < t < \infty, \\ 0, & t = 0. \end{cases}$$

Theorem 8.7 ([219]). *Let f be a function of bounded variation on every finite subinterval of the infinite interval $[0, \infty)$. Suppose also that, for some $r \in N$, $f(t) = O(t^r), \ t \to \infty$. Then, for $x \in (0, \infty), \ \alpha \geq 1$, and n sufficiently large,*

$$\begin{aligned} &\left| L_{n,\alpha}(f, x) - \frac{1}{2^{\alpha}} f(x+) - \left(1 - \frac{1}{2^{\alpha}}\right) f(x-) \right| \\ &\leq \frac{9\alpha(1 + x)}{\sqrt{nx}} (|f(x+) - f(x-)| + \varepsilon_n(x)|f(x) - f(x-)|) \\ &\quad + \frac{7\alpha(1 + x)^2}{(n + 2)x} \sum_{k=1}^{n} \Omega_x \left(f_x, \frac{x}{\sqrt{k}}\right) + O(n^{-1}), \quad n \to \infty, \end{aligned}$$

where

$$\varepsilon_n(x) = \begin{cases} 1, & (n + 1)p_x \in N, \\ 0, & otherwise, \end{cases}$$

and $f_x(t)$ is as defined in Theorem 7.7.

8.3 Balazs–Kantorovich–Bézier Operators

Agratini [17] defined the Kantorovich variant of the Bernstein-type rational operators [33] as

$$L_n(f,x) = na_n \sum_{k=0}^{n} p_{n,k}(x) \int_{I_{n,k}} f(t)dt, \quad n \in N, \quad x \geq 0, \tag{8.6}$$

where $I_{n,k} = [k/na_n, (k+1)/na_n]$. For $\alpha \geq 1$, Gupta and Zeng [135] defined the Bézier variant of these Balazs–Kantorovich operators (8.6) as

$$\begin{aligned} L_{n,\alpha}(f,x) &= na_n \sum_{k=0}^{n} Q_{n,k}^{\alpha}(x) \int_{I_{n,k}} f(t)dt \\ &= \int_0^{\infty} W_{n,\alpha}(x,t) f(t)dt, \end{aligned}$$

where $Q_{n,k}^{\alpha}(x) = J_{n,k}^{\alpha} - J_{n,k+1}^{\alpha}(x)$, $J_{n,k}(x) = \sum_{j=k}^{n} p_{n,j}(x)$ is the basis function, $W_{n,\alpha}(x,t) = na_n \sum_{k=0}^{\infty} Q_{n,k}^{\alpha}(x)\chi_{n,k}(t)$, and $\chi_{n,k}(t)$ is the characteristic function of the interval $I_{n,k}$ with respect to $[0,\infty)$.

Let

$$\begin{aligned} \Phi_{\text{loc},r} = \{f : &\text{ f is bounded in every finite subinterval of } [0,\infty), \\ &f(t) = O(t^r), r > 0 \text{ as, } t \to \infty\}. \end{aligned}$$

Lemma 8.1 ([135]). *For $x \in (0,\infty)$, we have*

$$\left| \sum_{k > na_n x/(1+a_n x)} p_{n,k}(x) - \frac{1}{2} \right| \leq \frac{[1 + (a_n x)^2 + 0.5(1+a_n x)^2]}{(1+a_n x)[1+\sqrt{na_n x}]}.$$

Lemma 8.2. *For all $x \geq 0$, we have*

$$\begin{aligned} L_n(1,x) &= 1, \\ L_n(t,x) &= \frac{x}{1+a_n x} + \frac{1}{2na_n}, \\ L_n(t^2,x) &= \frac{n-1}{n}\frac{x^2}{(1+a_n x)^2} + \frac{2x}{na_n(1+a_n x)} + \frac{1}{3n^2 {a_n}^2}. \end{aligned}$$

Thus,

$$L_n((t-x)^2,x) = \frac{1}{3n^2 {a_n}^2} + \frac{n{a_n}^3 x^4 - {a_n}^2 x^3 - a_n x^2 + x}{n^2 {a_n}^2}$$

$$\leq \frac{1 + na_n x + n^2 {a_n}^4 x^4}{n^2 {a_n}^2} = O((na_n)^{-1}).$$

Lemma 8.3 ([135]). *Let $x \in (0, \infty)$. Then, for sufficiently large n, we have*

$$\beta_{n,\alpha}(x, y) = \int_0^y W_{n,\alpha}(x, t)dt \leq \frac{\alpha}{(x - y)^2} \left(\frac{1 + na_n x + n^2 {a_n}^4 x^4}{n^2 {a_n}^2} \right), \; 0 \leq y < x, \tag{8.7}$$

and

$$1 - \beta_{n,\alpha}(x, z) = \int_z^\infty W_{n,\alpha}(x, t)dt \leq \frac{\alpha}{(z - x)^2} \left(\frac{1 + na_n x + n^2 {a_n}^4 x^4}{n^2 {a_n}^2} \right), \; x \leq z < \infty. \tag{8.8}$$

Theorem 8.8 ([135]). *Let $f \in \Phi_{\text{loc},r}$, $f(x+)$ and $f(x-)$ exist at a fixed point $x \in (0, \infty)$. Then, for $\alpha \geq 1$, $r > 1$, and for sufficiently large n, we have*

$$\begin{aligned}
&\left| L_{n,\alpha}(f, x) - \frac{1}{2^\alpha} f(x+) - \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right| \\
&\quad \leq \alpha \left[\frac{[1 + (a_n x)^2 + 0.5(1 + a_n x)^2]}{(1 + a_n x)[1 + \sqrt{na_n x}]} + \frac{1 + a_n x}{\sqrt{2ena_n x}} \right] |f(x+) - f(x-)| \\
&\quad + \frac{4\alpha + 4\alpha na_n x + n{a_n}^2 x^2 + 4\alpha n^2 {a_n}^4 x^4}{n^2 {a_n}^2 x^2} \sum_{k=1}^n \Omega_x(g_x, x/\sqrt{k}) + O((na_n)^{-r}),
\end{aligned}$$

where the auxiliary function $g_x(t)$ is defined in Theorem 7.7.

Proof. Since $L_{n,\alpha}(\delta_x, x) = 0$, we have

$$\begin{aligned}
&\left| L_{n,\alpha}(f, x) - \frac{1}{2^\alpha} f(x+) - \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right| \\
&\quad \leq |L_{n,\alpha}(g_x, x)| + \frac{|f(x+) - f(x-)|}{2^\alpha} L_{n,\alpha}(\text{sign}_x(t), x).
\end{aligned} \tag{8.9}$$

Since $p_{n,k} \leq \frac{1 + a_n x}{\sqrt{2ena_n x}}$, and using Lemma 8.1, we have

$$|L_{n,\alpha}(\text{sign}_x(t), x)| \leq \alpha 2^\alpha \left[\frac{[1 + (a_n x)^2 + 0.5(1 + a_n x)^2]}{(1 + a_n x)[1 + \sqrt{na_n x}]} + \frac{1 + a_n x}{\sqrt{2ena_n x}} \right]. \tag{8.10}$$

Next, we estimate $L_{n,\alpha}(g_x(t), x)$ by decomposing the integral into four parts as

$$\begin{aligned}
\int_0^\alpha g_x(t) W_{n,\alpha}(x, t) d_t &= \left(\int_0^{x - x/\sqrt{n}} + \int_{x - x/\sqrt{n}}^{x + x/\sqrt{n}} + \int_{x + x/\sqrt{n}}^{2x} + \int_{2x}^\infty \right) g_x(t) W_{n,\alpha}(x, t) d_t \\
&:= E_1 + E_2 + E_3 + E_4.
\end{aligned}$$

First, we estimate E_1. Since $\Omega_x(g_x, \lambda)$ is monotone nondecreasing with respect to λ, we integrate by parts with $y = x - x/\sqrt{n}$ to get

$$|E_1| = \left| \int_0^{x-x/\sqrt{n}} g_x(t) d_t(\beta_{n,\alpha}(x,t)) \right| \leq \int_0^{x-x/\sqrt{n}} \Omega_x(g_x, x-t) d_t(\beta_{n,\alpha}(x,t))$$
$$\leq \Omega_x(g_x, x-y)(\beta_{n,\alpha}(x,y)) + \int_0^y \hat{\beta}_{n,\alpha}(x,t) d(-\Omega_x, x-t),$$

where $\hat{\beta}_{n,\alpha}(x,t)$ is the normalized form of $\beta_{n,\alpha}(x,t)$. Since $\hat{\beta}_{n,\alpha}(x,t) \leq \beta_{n,\alpha}(x,t)$ on $(0,\infty)$, using (8.7) and rearranging the last integral, we have

$$|E_1| \leq \alpha \left(\frac{1 + na_n x + n^2 {a_n}^4 x^4}{n^2 {a_n}^2 x^2} \right) \Omega_x(g_x, x)$$
$$+ \alpha \left(\frac{1 + na_n x + n^2 {a_n}^4 x^4}{n^2 {a_n}^2} \right) \int_0^y \Omega_x(g_x, x-t) \frac{2}{(x-t)^3} dt.$$

Now, putting $t = x - x/\sqrt{u}$ in the last integral, we have

$$\int_0^y \Omega_x(g_x, x-t) \frac{2}{(x-t)^3} dt \leq \frac{1}{x^2} \sum_{k=1}^{n} \Omega_x \left(g_x, x/\sqrt{k} \right).$$

So we get

$$|E_1| \leq 2\alpha \left(\frac{1 + na_n x + n^2 {a_n}^4 x^4}{n^2 {a_n}^2 x^2} \right) \sum_{k=1}^{n} \Omega_x \left(g_x, x/\sqrt{k} \right). \tag{8.11}$$

Since $g_x(x) = 0$, we have

$$|E_2| \leq \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} |g_x(t) - g_x(x)| d_t(\beta_{n,\alpha}(x,t)) \frac{1}{n} \sum_{k=1}^{n} \Omega_x \left(g_x, x/\sqrt{k} \right). \tag{8.12}$$

Along similar lines as estimating E_1, we get

$$|E_3| \leq 2\alpha \left(\frac{1 + na_n x + n^2 {a_n}^4 x^4}{n^2 {a_n}^2 x^2} \right) \sum_{k=1}^{n} \Omega_x(g_x, x/\sqrt{k}). \tag{8.13}$$

Finally, to estimate E_4, let $r > 1$ be such that $f(t) = O(t^{2r}),\ \ t \to \infty$, and for a certain constant $C > 0$ depending on f, x, r, we have

$$|E_4| \leq Cna_n \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \int_{2x}^{\infty} \chi_{n,k}(t) t^{2r} dt \leq Cna_n\alpha \sum_{k=0}^{\infty} p_{n,k}(x) \int_{2x}^{\infty} \chi_{n,k}(t) t^{2r} dt.$$

By Lemma 8.2, we have

$$|E_4| = O((na_n)^{-r}), \quad n \to \infty. \tag{8.14}$$

Collecting (8.9)–(8.14), we complete the proof of the theorem. ■

One can observe from the following example that a class of bounded functions on the interval $[0, \infty)$ need not be of bounded variation.

Example 8.1. Consider the function

$$f(x) = \begin{cases} 0, & x = 0, \\ x \sin(\frac{1}{x^2}), & 0 < x \leq 1, \\ \sin 1, & x \geq 1. \end{cases}$$

8.4 Szász–Kantorovich–Bézier Operators

Gupta et al. [138] considered the Szász–Kantorovich–Bézier operators $S_{n,\alpha}$ defined on the space of real functions on the infinite interval $I = [0, \infty)$ by

$$S_{n,\alpha}(f;x) = n \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt \quad (n \in \mathbf{N}, \quad \alpha \geq 1, \ \ or \ \ 0 < \alpha < 1), \tag{8.15}$$

where

$$Q_{n,k}^{(\alpha)}(x) = (J_{n,k}(x))^{\alpha} - (J_{n,k+1}(x))^{\alpha},$$

and

$$J_{n,k}(x) = \sum_{j=k}^{\infty} s_{n,j}(x), s_{n,k}(x) = \exp(-nx)\frac{(nx)^k}{k!}.$$

Some basic properties of $J_{n,k}(x)$ are given in [138]. For $\alpha \geq 1$, we have

$$Q_{n,k}^{(\alpha)}(x) \leq \alpha(J_{n,k}(x) - J_{n,k+1}(x)) = \alpha s_{n,k}(x).$$

For $0 < \alpha < 1$, we have

$$\alpha s_{n,k}(x) \leq Q_{n,k}^{(\alpha)}(x) \leq s_{n,k}^{\alpha}(x).$$

It is easy to verify that $S_{n,\alpha}$ are positive linear operators and that $S_{n,1}$ is just the Szász–Kantorovich operator studied in [231]. Let the kernel function $K_{n,\alpha}(x,t)$ be defined by

$$K_{n,\alpha}(x,t) = n \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \varphi_{n,k}(t),$$

where $\varphi_{n,k}(t)$ denotes the characteristic function of the interval $I_k = [k/n, \ (k+1)/n]$ with respect to $I = [0, \infty)$. Then, by Lebesgue–Stieltjes integral representation, we have

$$S_{n,\alpha}(f, x) = \int_0^{\infty} f(t) K_{n,\alpha}(x,t) dt. \tag{8.16}$$

The case $\alpha \geq 1$ for the rate of convergence was considered in [138].

The case $0 < \alpha < 1$ was studied in [136]. We mention here the basic results and the main results on the rate of convergence that were studied in [136].

Let

$$\Phi_B = \{ f \,|\, f \text{ is measurable on } [0, \ \infty) \text{ and is bounded on every finite subinterval of } [0, \ \infty), \text{ for some } r > 0, \ f(t) = O((1+t)^r) \text{ as } t \to \infty \}.$$

For $f \in \Phi_B, \ x \in [0, \infty), \text{ and } \eta \geq 0.$

Lemma 8.4. *For each fixed $x \in (0, \infty)$, if $T_{n,m}(x) = S_n((t-x)^m, x)$, then*

$$S_n(1, x) = 1, S_n(t-x, x) = \frac{1}{2n} \ and \ S_n((t-x)^2, x) = \frac{1+3nx}{2n^2}.$$

We define

$$T_{n,\gamma}(x) = S_{n,1}(|x-t|^{\gamma}, x).$$

Lemma 8.5 ([136]). *Let the kernel function $K_{n,\alpha}(x,t), 0 < \alpha < 1$ be defined as in (8.16). Then*

(i) For $0 < y < x$, we have

$$\int_0^y K_{n,\alpha}(x,t) dt \leq \frac{3nx+1}{2n^2(x-y)^2}.$$

(ii) For $z < x < \infty$, we have

$$\int_z^{\infty} K_{n,\alpha}(x,t) dt \leq \frac{2^{1-\alpha}(T_{n,2/\alpha}(x))^{\alpha}}{(z-x)^2}.$$

Proof. Choose an integer $k' \in [0, \infty)$ such that $y \in [k'/n, (k'+1)/n)$. Then $y = \frac{k'}{n} + \frac{\varepsilon}{n}$ with some $\varepsilon \in [0, 1)$, and we have

$$
\begin{aligned}
\int_0^y K_{n,\alpha}(x,t)dt &= \sum_{k=0}^{k'-1} Q_{n,k}^{(\alpha)}(x) + \varepsilon Q_{n,k'}^{(\alpha)}(x) \\
&= 1-(1-\varepsilon)J_{n,k'}^{\alpha}(x) - \varepsilon J_{n,k'+1}^{\alpha}(x) \\
&\leq 1-(1-\varepsilon)J_{n,k'}(x) - \varepsilon J_{n,k'+1}(x) \\
&= \sum_{k=0}^{k'-1} Q_{n,k}(x) + \varepsilon Q_{n,k'}(x) = \int_0^y K_{n,1}(x,t)dt \\
&\leq \frac{1}{(x-y)^2} T_{n,2}(x) \leq \frac{3nx+1}{2n^2(x-y)^2}.
\end{aligned}
$$

This completes the proof of (i).

Next,

$$
\begin{aligned}
\int_z^\infty K_{n,\alpha}(x,t)dt &= nQ_{n,k'}^{(\alpha)}(x)\left(\frac{k'+1}{n} - z\right) + \sum_{k=k'+1}^{\infty} Q_{n,k}^{(\alpha)}(x) \\
&= nQ_{n,k'}^{(\alpha)}(x)\left(\frac{k'+1}{n} - z\right) + \left(\sum_{k=k'+1}^{\infty} s_{n,k}(x)\right)^{\alpha} \\
&\leq s_{n,k'}^{\alpha}(x) + \left(\sum_{k=k'+1}^{\infty} s_{n,k}(x)\right)^{\alpha} \leq 2^{1-\alpha}\left(\sum_{k=k'}^{\infty} s_{n,k}(x)\right)^{\alpha} \\
&\leq \frac{2^{1-\alpha}}{(z-x)^2}\left(\sum_{k=k'}^{\infty} n s_{n,k}(x)\int_{k/n}^{(k+1)/n} |t-x|^{2/\alpha} dt\right)^{\alpha} \\
&\leq \frac{2^{1-\alpha}}{(z-x)^2}\left(T_{n,2/\alpha}(x)\right)^{\alpha}.
\end{aligned}
$$

■

Lemma 8.6 ([136]). *For $x > 0$ and $0 < \alpha < 1$, all n sufficiently large,*

$$Q_{n,k}^{(\alpha)}(x) \leq \frac{\alpha 4^{1-\alpha}}{\sqrt{2enx}}$$

holds.

Proof. By using the bound given in [256], for sufficiently large n, we have

$$s_{n,k}(x) \leq \frac{1}{\sqrt{2enx}}.$$

Next, by the mean value theorem, we have

$$Q_{n,k'}^{(\alpha)}(x) = \alpha(\zeta_{n,k'}(x))^{\alpha-1}[J_{n.k'}(x) - J_{n,k'+1}(x)] = \alpha(\zeta_{n,k'}(x))^{\alpha-1} s_{n,k'}(x), \quad (8.17)$$

where $J_{n,k'+1}(x) < \zeta_{n,k'}(x) < J_{n,k'}(x)$. But in view of Lemma 4 of [138], we have

$$\zeta_{n,k'}(x) > J_{n,k'+1}(x) = \sum_{j=k'+1}^{\infty} s_{n,j}(x) > \frac{1}{4}, \tag{8.18}$$

for some $n > n_0(x)$. Hence, by (8.17) and (8.18), we have

$$Q_{n,k'}^{(\alpha)}(x) < \alpha 4^{1-\alpha} s_{n,j}(x) < \frac{\alpha 4^{1-\alpha}}{\sqrt{2enx}}.$$

■

Lemma 8.7 ([136]). *If x belongs to the interval $I_{k'}$ defined in (8.16) for some nonnegative integer k', then for $0 < \alpha < 1$, we have*

$$\left|\sum_{k=k'}^{\infty} s_{n,k}^{\alpha}(x) - \frac{1}{2^{\alpha}}\right| \leq \alpha 4^{1-\alpha}\frac{0.8\sqrt{1+3x}}{\sqrt{nx}}.$$

Proof. By the mean value theorem, we have

$$\left|\sum_{k=k'}^{\infty} s_{n,k}^{\alpha}(x) - \frac{1}{2^{\alpha}}\right| = \alpha(\xi_{n,k'}(x))^{\alpha-1}\left|\sum_{k=k'}^{\infty} s_{n,k}(x) - \frac{1}{2}\right|, \tag{8.19}$$

where $\xi_{n,k'}(x)$ lies between $\frac{1}{2}$ and $\sum_{k=k'}^{\infty} s_{n,k}(x)$. Using Lemma 2 of [247], we find that

$$\left|\sum_{k=k'}^{\infty} s_{n,k}(x) - \frac{1}{2}\right| \leq \frac{0.8\sqrt{1+3x}}{\sqrt{nx}} \tag{8.20}$$

As $\sum_{k=k'}^{\infty} s_{n,k} > \frac{1}{4}$, for all $n > n_0(x)$. Hence, $(\xi_{n,k'}(x))^{\alpha-1} \leq 4^{1-\alpha}, n > n_0(x)$. Therefore, by combining (8.19) and (8.20), we get the desired result. ■

Theorem 8.9 ([136]). *Let $0 < \alpha < 1$, and let $f \in \Phi_B$ and $f(x+)$, $f(x-)$ exist at a fixed point $x \in (0,\infty)$. Then, for $n > 11(1+3x)/x$, we have*

$$\left|S_{n,\alpha}(f,x) - \frac{f(x+) + (2^{\alpha}-1)f(x-)}{2^{\alpha}}\right| \leq B_{\alpha}\frac{3\alpha+5+x}{nx}\sum_{k=1}^{n}\Omega_x(g_x, x/\sqrt{k})$$
$$+ \frac{2\sqrt{1+3x}\,|f(x+)-f(x-)|}{\sqrt{nx}} + O(n^{-r}),$$

where B_{α} is a constant depending on α, $f(t) = O(t^r)$, and the auxiliary function $g_x(t)$ is defined in Theorem 7.7.

Proof. For any $f \in \Phi_B$, if $f(x+)$ and $f(x-)$ exist at x, then by decomposition (cf. [253], p. 1449),

$$f(t) = \frac{1}{2^\alpha} f(x+) + \left(1 - \frac{1}{2^\alpha}\right) f(x-) + g_x(t) + \frac{f(x+) - f(x-)}{2^\alpha} \mathrm{sgn}_{\alpha,x}(t)$$
$$+ \eta_x(t) \left[f(x) - \frac{1}{2^\alpha} f(x+) - \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right],$$

where

$$\mathrm{sgn}_{\alpha,x}(t) = \begin{cases} 2^\alpha - 1, & t > x, \\ 0, & t = x, \\ -1, & t < x; \end{cases} \qquad \eta_x(t) = \begin{cases} 1, & t = x, \\ 0, & t \neq x. \end{cases}$$

Obviously,

$$S_{n,\alpha}(\eta_x, x) = 0.$$

Hence, it follows that

$$\left| S_{n,\alpha}(f, x) - \frac{1}{2^\alpha} f(x+) - \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right| \leq |S_{n,\alpha}(g_x, x)|$$
$$+ \left| \frac{f(x+) - f(x-)}{2^\alpha} S_{n,\alpha}(sgn_{x,\alpha}, x) \right|. \tag{8.21}$$

We need to estimate $|S_{n,\alpha}(\mathrm{sgn}_{x,\alpha}, x)|$ and $|S_{n,\alpha}(g_x, x)|$. Let $x \in I_{k'}$ [see (8.16)] for some k'. Then

$$S_{n,\alpha}(\mathrm{sgn}_{x,\alpha}, x) = (2^\alpha - 1) \sum_{k=k'}^{\infty} Q_{n,k}^{(\alpha)}(x) - \sum_{k=0}^{k'} Q_{n,k}^{(\alpha)}(x)$$
$$+ n Q_{n,k'}^{(\alpha)}(x) \left((2^\alpha - 1) \left(\frac{k'+1}{n} - x \right) - \left(x - \frac{k'}{n} \right) \right)$$
$$2^\alpha \sum_{k=k'}^{\infty} Q_{n,k}^{(\alpha)}(x) - 1 + 2^\alpha \sum_{k=k'}^{\infty} n Q_{n,k'}^{(\alpha)}(x) \left(\frac{k'+1}{n} - x \right).$$

By Lemmas 8.6 and 8.7, we have

$$|S_{n,\alpha}(\mathrm{sgn}_{x,\alpha}, x)| \leq 2^\alpha \left| \left(\sum_{k=k'}^{\infty} s_{n,k}(x) \right)^\alpha - \frac{1}{2^\alpha} \right| + 2^\alpha Q_{n,k'}^{(\alpha)}(x).$$

$$\leq 2^{\alpha}\frac{\alpha 4^{1-\alpha}(0.8\sqrt{1+3x}+(2e)^{-1/2})}{\sqrt{nx}} \leq 2^{\alpha}\frac{2\sqrt{1+3x}}{\sqrt{nx}}. \tag{8.22}$$

Next, we estimate $|S_{n,\alpha}(g_x,x)|$. We write

$$S_{n,\alpha}(g_x,x) = \int_{[0,\infty)} g_x(t)K_{n,\alpha}(x,t)dt = \sum_{j=1}^{4}\int_{A_j} g_x(t)K_{n,\alpha}(x,t)dt, \tag{8.23}$$

where

$$A_1 := [0,\ x - x/\sqrt{n}],\quad A_2 := (x - x/\sqrt{n},\ x + x/\sqrt{n}],$$
$$A_3 := (x + x/\sqrt{n},\ 2x],\qquad A_4 := (2x,\ \infty).$$

First, note that $g_x(x) = 0$. Thus,

$$\left|\int_{A_2} g_x(t)K_{n,\alpha}(x,t)dt\right| \leq \Omega_x\left(g_x, x/\sqrt{n}\right) \leq \frac{1}{n}\sum_{k=1}^{n}\Omega_x\left(g_x, x/\sqrt{k}\right). \tag{8.24}$$

To estimate $|\int_{A_1} g_x(t)K_{n,\alpha}(x,t)dt|$, note that $\Omega_x(g_x,\eta)$ is monotone increasing with respect to η. Thus, it follows that

$$\left|\int_0^{x-x/\sqrt{n}} g_x(t)d_tK_{n,\alpha}(x,t)\right| \leq \int_0^{x-x/\sqrt{n}} \Omega_x(g_x,x-t)d_tK_{n,\alpha}(x,t).$$

Integrating by parts with $y = x - x/\sqrt{n}$, we have

$$\int_0^{x-x/\sqrt{n}} \Omega_x(g_x,x-t)d_tK_{n,\alpha}(x,t) \leq \Omega_x(g_x,x-y)K_{n,\alpha}(x,y)$$
$$+\int_0^{y}\hat{K}_{n,\alpha}(x,t)d(-\Omega_x,x-t)), \tag{8.25}$$

where $\hat{K}_{n,\alpha}(x,t)$ is the normalized form of $K_{n,\alpha}(x,t)$. Since $\hat{K}_{n,\alpha}(x,t) \leq K_{n,\alpha}(x,t)$ on $(0,\infty)$, from (8.25) and Lemma 8.5, it follows that

$$\left|\int_{A_1} g_x(t)d_tK_{n,\alpha}(x,t)\right| \leq \Omega_x(g_x,x-y)\frac{3nx+1}{2n^2(x-y)^2}$$

$$+\frac{3nx+1}{2n^2}\int_0^y \frac{1}{(x-t)^2}d(-\Omega_x(g_x,x-t)). \tag{8.26}$$

Since

$$\int_0^y \frac{1}{(x-t)^2}d(-\Omega_x(g_x,x-t)) = -\frac{\Omega_x(g_x,x-y)}{(x-y)^2}$$
$$+\frac{\Omega_x(g_x,x)}{x^2}+\int_0^y \Omega_x(g_x,x-t)\frac{2}{(x-t)^3}dt.$$

Thus, we have

$$\left|\int_0^{x-x/\sqrt{n}} g_x(t)d_tK_{n,\alpha}(x,t)\right| \leq \frac{3nx+1}{2n^2x^2}\Omega_x(g_x,x)$$
$$+\frac{3nx+1}{2n^2}\int_0^{x-x/\sqrt{n}} \Omega_x(g_x,x-t)\frac{2}{(x-t)^3}dt.$$

Putting $t = x - x/\sqrt{u}$ for the last integral, we get

$$\int_0^{x-x/\sqrt{n}} \Omega_x(g_x,x-t)\frac{2}{(x-t)^3}dt = \frac{1}{x^2}\int_1^n \Omega_x(g_x,x/x\sqrt{u})du$$
$$\leq \frac{1}{x^2}\sum_{k=1}^n \Omega_x(g_x,x/\sqrt{k}).$$

Consequently,

$$\left|\int_{A_1} g_x(t)K_{n,\alpha}(x,t)dt\right| \leq \frac{3nx+1}{2n^2x^2}\left(\Omega_x(g_x,x)+\sum_{k=1}^n \Omega_x(g_x,x/\sqrt{k})\right)$$
$$\leq \frac{3nx+1}{n^2x^2}\sum_{k=1}^n \Omega_x(g_x,x/\sqrt{k}). \tag{8.27}$$

Using a similar method to estimate $\left|\int_{A_3} g_x(t)K_{n,\alpha}(x,t)dt\right|$, we get

$$\left| \int_{A_3} g_x(t) K_{n,\alpha}(x,t)dt \right| \leq B_\alpha \frac{3\alpha+1}{nx} \sum_{k=1}^{n} \Omega_x(g_x, x/\sqrt{k}), \tag{8.28}$$

where B_α is a constant depending only on α.

Finally, we estimate

$$\left| \int_{A_4} g_x(t) K_{n,\alpha}(x,t)dt \right|$$

as $f(t) = O(t^r)$. Then, proceeding along the lines of [4], we have

$$\left| \int_{A_4} g_x(t) K_{n,\alpha}(x,t)dt \right| \leq Cn \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \int_{2x}^{\infty} \varphi_{n,k}(t) t^{2r} dt = O(n^{-r}). \tag{8.29}$$

The theorem follows by collecting the estimates of (8.21)–(8.24), (8.26)–(8.28), and (8.29). ∎

8.5 Baskakov–Bézier Operators

In 2002, Zeng and Gupta [253] introduced the Bézier variant of the well-known Baskakov operators $B_{n,\alpha}$ for $\alpha \geq 1$ as

$$B_{n,\alpha}(f,x) = \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) f(k/n) \quad (n \in \mathbf{N}, \quad \alpha \geq 1, \quad \text{or} \quad 0 < \alpha < 1), \tag{8.30}$$

where $Q_{n,k}^{(\alpha)}(x) = J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x),\ J_{n,k}(x) = \sum_{j=k}^{\infty} b_{n,j}(x),$

and

$$b_{n,j}(x) = \binom{n+j-1}{j} \frac{x^j}{(1+x)^{n+j}}$$

is the Baskakov basis function. When $\alpha = 1$, the operators (8.30) reduce to the classical Baskakov operators. Some basic properties of $J_{n,k}(x)$ mentioned in [253] are as follows:

1. $J_{n,k}(x) - J_{n,k+1}(x) = b_{n,k}(x), k = 0, 1, 2, \ldots.$
2. $J'_{n,k}(x) = n b_{n+1,k-1}(x), k = 1, 2, 3, \ldots.$
3. $J_{n,k}(x) = n \int_0^x b_{n+1,k-1}(t)dt, k = 1, 2, 3, \ldots.$
4. $J_{n,0}(x) > J_{n,1}(x) > \cdots > J_{n,k}(x) > J_{n,k+1}(x) > \cdots,$

and for every natural $k, 0 \leq J_{n,k}(x) < 1$ and $J_{n,k}(x)$, it increases strictly on $[0, \infty)$. By the Lebesgue–Stieltjes integral representation, one can write

$$B_{n,\alpha}(f, x) = \int_0^{\infty} f(t) d_t (K_{n,\alpha}(x, t) \quad (n \in \mathbf{N}, \quad \alpha \geq 1, \text{ or } \quad 0 < \alpha < 1), \tag{8.31}$$

where

$$K_{,\alpha}(x, t) = \begin{cases} \sum_{k \leq nt} Q_{n,k}^{(\alpha)}(x), & , \quad 0 < t < \infty, \\ 0, & , \quad t = 0. \end{cases}$$

Obviously, $\int_0^{\infty} d_t (K_{n,\alpha}(x, t)) = B_{n,\alpha}(1, x) = 1$, and $Q_{n,k}^{(\alpha)}(x) > 0$ for $x \in [0, \infty)$ and $k = 0, 1, 2, \ldots$. Thus, the operator $B_{n,\alpha}(f, x)$ is an operator of the probabilistic type. Consider the following class of function $I_{\text{loc}B}\Phi_B$:

$$I_{\text{loc}B} = \{ f \,|\, f \text{ is bounded on every finite subinterval of } [0, \ \infty)\}.$$

Lemma 8.8 ([253]). *For all $x \in [0, \infty)$ and $k \in N$,*

$$J_{n,k}^{\alpha}(x) b_{n,k}(x) \leq Q_{n,k}^{(\alpha)}(x) \leq \alpha b_{n,k}(x) < \frac{\alpha\sqrt{1+x}}{\sqrt{2enx}}.$$

holds.

Lemma 8.9 ([253]). *For $x \in [0, \infty)$,*

$$\left| \sum_{k > nx} b_{n,k}(x) - \frac{1}{2} \right| \leq \frac{3\sqrt{1+x}}{\sqrt{nx}}$$

holds, and for $0 \leq t < x$, we have

$$\sum_{k \leq nt} Q_{n,k}^{(\alpha)}(x) \leq \frac{\alpha x(1+x)}{n(t-x)^2}.$$

Lemma 8.10 ([253]). *For any fixed $\beta > 0$, $x \in (0, \infty)$, and for n sufficiently large,*

$$\sum_{k > 2nx} \left(\frac{k}{n} \right)^{\beta} b_{n,k}(x) = o_x(n^{-1/2})$$

holds.

Theorem 8.10 ([253]). *Let f be a function belonging to $I_{\text{loc}B}$, and let $f(t) = O(t^{\beta})$ for some $\beta > 0$ as $t \to \infty$. Then, for every $\alpha \geq 1$, $x \in (0, \infty)$ in which $f(x+)$ and $f(x-)$ exist and n is sufficiently large, we have*

$$\left| B_{n,\alpha}(f,x) - \frac{f(x+) + (2^\alpha - 1) f(x-)}{2^\alpha} \right| \le \frac{5\alpha\sqrt{x+1}}{\sqrt{nx}} \sum_{k=1}^{n} \Omega_x(g_x, x/\sqrt{k})$$

$$+ \frac{3\alpha\sqrt{1+x}}{\sqrt{nx}} |f(x+) - f(x-)| + \varepsilon_n(x)|f(x+) - f(x-)| + o_x(n^{-1/2}),$$

where the auxiliary function $g_x(t)$ is defined in Theorem 7.7,

$$\Omega_x(f,\lambda) = \sup_{t\in[x-\lambda,\ x+\lambda]} |f(t) - f(x)|$$

and

$$\varepsilon_n(x) = \begin{cases} 1, & x = \frac{k'}{n}, k' \in N, \\ 0, & x \ne \frac{k}{n}, k \in N. \end{cases}$$

Proof. Note that for all t, we can write

$$f(t) = \frac{1}{2^\alpha} f(x+) + \left(1 - \frac{1}{2^\alpha}\right) f(x-) + g_x(t) + \frac{f(x+) - f(x-)}{2^\alpha} \mathrm{sgn}_{\alpha,x}(t)$$
$$+ \eta_x(t) \left[f(x) - \frac{1}{2^\alpha} f(x+) - \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right],$$

where

$$\mathrm{sgn}_{\alpha,x}(t) = \begin{cases} 2^\alpha - 1, & t > x, \\ 0, & t = x, \\ -1, & t < x; \end{cases} \qquad \eta_x(t) = \begin{cases} 1, & t = x, \\ 0, & t \ne x. \end{cases}$$

Hence,

$$\left| B_{n,\alpha}(f,x) - \frac{1}{2^\alpha} f(x+) - \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right|$$
$$\le |B_{n,\alpha}(g_x, x)| + \left| \frac{f(x+) - f(x-)}{2^\alpha} B_{n,\alpha}(\mathrm{sgn}_{x,\alpha}, x) \right.$$
$$\left. + \left[f(x) - \frac{1}{2^\alpha} f(x+) - \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right] B_{n,\alpha}(\eta_x, x) \right|. \tag{8.32}$$

By direct calculation,

$$B_{n,\alpha}(\mathrm{sgn}_{x,\alpha}, x) = -\sum_{k<nx} Q_{n,k}^{(\alpha)}(x) + (2^\alpha - 1) \sum_{k>nx} Q_{n,k}^{(\alpha)}(x)$$

$$2^{\alpha} \sum_{k>nx} Q_{n,k}^{(\alpha)}(x) - 1 + \varepsilon_n(x) Q_{n,k'}^{(\alpha)}(x)$$

$$2^{\alpha} \left(\sum_{k>nx} b_{n,k}(x) \right)^{\alpha} - 1 + \varepsilon_n(x) Q_{n,k'}^{(\alpha)}(x)$$

and $B_{n,\alpha}(\eta_x, x) = \varepsilon_n(x) Q_{n,k'}^{(\alpha)}(x)$. Thus, by Lemmas 8.8 and 8.9, we have

$$\left| \frac{f(x+) - f(x-)}{2^{\alpha}} B_{n,\alpha}(\mathrm{sgn}_{x,\alpha}, x) \right.$$
$$\left. + \left[f(x) - \frac{1}{2^{\alpha}} f(x+) - \left(1 - \frac{1}{2^{\alpha}} \right) f(x-) \right] B_{n,\alpha}(\eta_x, x) \right|$$
$$\leq \frac{3\alpha\sqrt{1+x}}{\sqrt{nx}} \left(|f(x+) - f(x-)| + \varepsilon_n(x) f(x) - f(x-)| \right).$$

Next, by Lemma 8.10 and using the standard techniques, we have

$$|B_{n,\alpha}(g_x, x)| \leq \frac{5\alpha(1+x)}{nx} \sum_{k=1}^{n} \Omega_x \left(g_x, \frac{x}{\sqrt{k}} \right) + o_x(n^{-1/2}).$$

Combining the above estimates, one can establish the theorem immediately. ■

Gupta [101] also studied the other case of the rate of convergence of Baskakov–Bézier operators, that is, $0 < \alpha < 1$.

Theorem 8.11 ([101]). *Let f be a function of bounded variation on every finite subinterval of $[0, \infty)$. Let $f(t) = O(t^r)$ for some $r \in N$ as $t \to \infty$. Then, for every $0 < \alpha < 1$, $x \in (0, \infty)$, there exists a constant $M(f, \alpha, r, x)$ such that for n sufficiently large,*

$$\left| B_{n,\alpha}(f, x) - \frac{f(x+) + (2^{\alpha} - 1) f(x-)}{2^{\alpha}} \right| \leq \frac{5\sqrt{x+1}}{\sqrt{nx}} \sum_{k=1}^{n} \Omega_x(g_x, x/\sqrt{k})$$
$$+ \frac{3\sqrt{1+x}}{\sqrt{nx}} |f(x+) - f(x-)| + \varepsilon_n(x) |f(x+) - f(x-)| + \frac{M(f, \alpha, r, x)}{n^m},$$

where $\Omega_x(f, \lambda)$, $\varepsilon_n(x)$, and $g_x(t)$ is defined in Theorem 8.10.

8.6 Baskakov–Kantorovich–Bézier Operators

In 2003, Abel and Gupta [4] introduced a Bézier variant of the Baskakov–Kantorovich operators $V_{n,\alpha}^*$ defined by

$$V_{n,\alpha}^*(f,x) = n\sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x)\int_{I_k} f(t)dt \quad (n\in\mathbf{N},\ \ \alpha\geq 1,\ \text{ or }\ 0<\alpha<1), \tag{8.33}$$

where $I_k = [k/n,\ (k+1)/n]$, $Q_{n,k}^{(\alpha)}(x) = J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)$, $J_{n,k}(x) = \sum_{j=k}^{\infty} b_{n,j}(x)$, and

$$b_{n,j}(x) = \binom{n+j-1}{j}\frac{x^j}{(1+x)^{n+j}}$$

is the Baskakov basis function.

Abel and Gupta [4] studied the rate of convergence of the operators (8.33) for functions of bounded variation in the case $\alpha\geq 1$. They considered $W(0,\infty)$ to be the class of functions f that are locally integrable on $(0,\infty)$ and are of polynomial growth as $t\to\infty$; that is, for some positive r, $f(t) = O(t^r))$ as $t\to\infty$ holds.

Theorem 8.12 ([4]). *Assume that $f\in W(0,\infty)$ is a function of bounded variation on every finite subinterval of $(0,\infty)$. Furthermore, let $\alpha\geq 1, x\in(0,\infty)$ and $\lambda>1$ be given. Then for each $r\in N$, there exists a constant $M(f,\alpha,r,x)$ such that for sufficiently large n, the Bézier-type Baskakov–Kantorovich operators $V_{n,\alpha}^*(f,x)$ satisfy the estimate*

$$\left|V_{n,\alpha}^*(f,x) - \frac{f(x+)+(2^\alpha-1)f(x-)}{2^\alpha}\right| \leq \frac{4\alpha\lambda(1+x)+x}{nx}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x)$$
$$+\frac{7\alpha\sqrt{1+x}}{2\sqrt{nx}}|f(x+)-f(x-)| + \frac{M(f,\alpha,r,x)}{n^r},$$

where $V_a^b(g_x)$ is the total variation of g_x on $[a,b]$ and the auxiliary function $g_x(t)$ is defined in Theorem 7.7.

Since the other case is equally important, recently Zeng–Gupta–Agratini [257] established the rate of convergence of the Baskakov–Kantorovich–Bézier operators in the case $0<\alpha<1$. Consider the following class of functions Φ_B:

$$\Phi_B = \{\ f\,|\,f \text{ is integrable and is bounded on every finite subinterval}$$
$$\text{of } [0,\ \infty),\ \text{for some } r\in\mathbf{N},\ f(t) = O(t^r) \text{ as } t\to\infty\},$$

for $f\in\Phi_B,\ \ x\in[0,\infty),\ $ and $\eta\geq 0$.

Let the kernel function $K_{n,\alpha}(x,t)$ be defined by

$$K_{n,\alpha}(x,t) = n\sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x)\varphi_{n,k}(t), \tag{8.34}$$

where $\varphi_{n,k}(t)$ denotes the characteristic function of the interval $I_k = [k/n,\ (k+1)/n]$ with respect to $I = [0,\infty)$. Then, by Lebesgue–Stieltjes integral representation, we have

$$V_{n,\alpha}^*(f,x) = \int_0^{\infty} f(t)K_{n,\alpha}(x,t)dt. \tag{8.35}$$

Let f be defined on $[0,\infty)$, $f(t) = O(t^r)$, and let f be the bounded variation on every finite subinterval of $[0,\infty)$. Then the function f satisfies the conditions of Theorem 8.12. Therefore, Theorem 8.13 subsumes the approximation of functions of bounded variation as a special case.

Lemma 8.11 ([4, Lemma 4]). *For each fixed* $x \in (0,\infty)$, *let* $T_{n,m}(x) = V_{n,1}^*((t-x)^m, x)$. *Then*

$$V_{n,1}^*(1,x) = 1,\ V_{n,1}^*(t-x,x) = \frac{1}{2n},\ V_{n,1}^*\left((t-x)^2,x\right) = \frac{1+3nx(1+x)}{3n^2},$$

and

$$T_{n,m}(x) = O\left(n^{-\lfloor (m+1)/2\rfloor}\right)\ \ (n\to\infty).$$

Lemma 8.12 ([257]). *Let the kernel function* $K_{n,\alpha}(x,t), 0<\alpha<1$ *be defined as in (8.34). Then*

(i) For $0<y<x$, *we have*

$$\int_0^y K_{n,\alpha}(x,t)dt \le \frac{(1+x)^2}{n(x-y)^2}.$$

(ii) For $z<x<\infty$, *we have*

$$\int_z^{\infty} K_{n,\alpha}(x,t)dt \le \frac{C_\alpha}{n(z-x)^2},$$

where C_α *is a constant depending only on* α.

Proof. Choose an integer $k' \in [0,\infty)$ such that $y \in [k'/n,(k'+1)/n)$. Then $y = \frac{k'}{n} + \frac{\varepsilon}{n}$ with some $\varepsilon \in [0,1)$, and we have

$$
\begin{aligned}
\int_0^y K_{n,\alpha}(x,t)dt &= \sum_{k=0}^{k'-1} Q_{n,k}^{(\alpha)}(x) + \varepsilon Q_{n,k'}^{(\alpha)}(x) \\
&= 1-(1-\varepsilon)J_{n,k'}^{\alpha}(x) - \varepsilon J_{n,k'+1}^{\alpha}(x) \\
&\leq 1-(1-\varepsilon)J_{n,k'}(x) - \varepsilon J_{n,k'+1}(x) \\
&= \sum_{k=0}^{k'-1} Q_{n,k}(x) + \varepsilon Q_{n,k'}(x) = \int_0^y K_{n,1}(x,t)dt \\
&\leq \frac{1}{(x-y)^2} T_{n,2}(x) \leq \frac{(1+x)^2}{n(x-y)^2}.
\end{aligned}
$$

This completes the proof of (i).

Next, if $z \in [k''/n, (k''+1)/n)$, then

$$
\begin{aligned}
\int_z^{\infty} K_{n,\alpha}(x,t)dt &= nQ_{n,k''}^{(\alpha)}(x)\left(\frac{k''+1}{n} - z\right) + \sum_{k=k''+1}^{\infty} Q_{n,k}^{(\alpha)}(x) \\
&\leq \sum_{k=k''}^{\infty} Q_{n,k}^{(\alpha)}(x) \\
&= \left(\sum_{k=k''}^{\infty} b_{n,k}(x)\right)^{\alpha}.
\end{aligned}
$$

Now, by applying Lemma 8.11 and the method that was presented in Lemma 7 of [13], we obtain inequality (ii). ■

Lemma 8.13 ([257]). *Let $0 < \alpha \leq 1$. If x belongs to interval $I_{k'}$ for some nonnegative integer k', then for $n > \dfrac{144(x+1)}{x}$, we have*

$$
\left|\sum_{k=k'+1}^{\infty} b_{n,k}^{\alpha}(x) - \frac{1}{2^{\alpha}}\right| \leq \frac{12\alpha}{4^{\alpha}} \frac{\sqrt{x+1}}{\sqrt{nx}}.
$$

Proof. By the mean value theorem, we have

$$
\left|\sum_{k=k'+1}^{\infty} b_{n,k}^{\alpha}(x) - \frac{1}{2^{\alpha}}\right| = \alpha(\xi_{n,k'}(x))^{\alpha-1}\left|\sum_{k=k'+1}^{\infty} b_{n,k}(x) - \frac{1}{2}\right|, \tag{8.36}
$$

where $\xi_{n,k'}(x)$ lies between $\frac{1}{2}$ and $\sum_{k=k'+1}^{\infty} b_{n,k}(x)$. Using Lemma 5 of [253], it holds that

$$\left|\sum_{k=k'+1}^{\infty} b_{n,k}(x) - \frac{1}{2}\right| \leq \frac{3\sqrt{x+1}}{\sqrt{nx}}. \tag{8.37}$$

From (8.37), we get $\sum_{k=k'+1}^{\infty} b_{n,k} > \frac{1}{4}$, for $n > \frac{144(x+1)}{x}$. Thus, $\xi_{n,k'}(x) > \dfrac{1}{4}$, for $n > \frac{144(x+1)}{x}$. The conclusion of the lemma now holds from (8.36) and (8.37).

Lemma 8.14 ([257]). *Let* $0 < \alpha \leq 1$. *If* x *belongs to the interval* $I_{k'}$ *for some nonnegative integer* k', *then for* $n > \dfrac{144(x+1)}{x}$, *it holds that*

$$Q_{n,k'}^{(\alpha)}(x) \leq \frac{2\alpha}{4^{\alpha}} \frac{\sqrt{x+1}}{\sqrt{nx}}.$$

Proof. By using the bound given in [256], for any k, we have

$$b_{n,k}(x) \leq \frac{\sqrt{x+1}}{\sqrt{2enx}}. \tag{8.38}$$

On the other hand, by the mean value theorem, we have

$$Q_{n,k'}^{(\alpha)}(x) = \alpha(\zeta_{n,k'}(x))^{\alpha-1}[J_{n.k'}(x) - J_{n,k'+1}(x)] = \alpha(\zeta_{n,k'}(x))^{\alpha-1} b_{n,k'}(x), \tag{8.39}$$

where $J_{n,k'+1}(x) < \zeta_{n,k'}(x) < J_{n,k'}(x)$. From (8.37), we get

$$\zeta_{n,k'}(x) > J_{n,k'+1}(x) = \sum_{j=k'+1}^{\infty} b_{n,j}(x) > \frac{1}{4}, \tag{8.40}$$

for $n > \frac{144(x+1)}{x}$. Combining (8.38), (8.39), and (8.40), we get the desired result. ■

Theorem 8.13 ([257]). *Let* $0 < \alpha < 1$, *and let* $f \in \Phi_B$ *and* $f(x+)$, $f(x-)$ *exist at a fixed point* $x \in (0, \infty)$. *Then, for* $n > \dfrac{144(x+1)}{x}$, *we have*

$$\left|V_{n,\alpha}^{*}(f, x) - \frac{f(x+) + (2^{\alpha}-1)f(x-)}{2^{\alpha}}\right| \leq \frac{4C_{\alpha} + 4 + x}{nx} \sum_{k=1}^{n} \Omega_x(g_x, x/\sqrt{k})$$
$$+ \frac{14\alpha\sqrt{1+x}}{4^{\alpha}\sqrt{nx}} |f(x+) - f(x-)| + O(n^{-l}),$$

where C_{α} *is a constant depending only on* α, $l > r$, *and the auxiliary function* $g_x(t)$ *is defined in Theorem 7.7.*

Proof. As in Theorem 8.9, we have

$$\left|V_{n,\alpha}^{*}(f,x)-\frac{1}{2^{\alpha}}f(x+)-\left(1-\frac{1}{2^{\alpha}}\right)f(x-)\right|\le|V_{n,\alpha}^{*}(g_x,x)| \\ +\left|\frac{f(x+)-f(x-)}{2^{\alpha}}V_{n,\alpha}^{*}(\mathrm{sgn}_{x,\alpha},x)\right|. \tag{8.41}$$

We need to estimate $|V_{n,\alpha}^{*}(\mathrm{sgn}_{x,\alpha},x)|$ and $|V_{n,\alpha}^{*}(g_x,x)|$.

Let $x\in I_{k'}$ for some k'. Direct computation gives

$$\begin{aligned}V_{n,\alpha}^{*}(\mathrm{sgn}_{x,\alpha},x)&=(2^{\alpha}-1)\sum_{k=k'+1}^{\infty}Q_{n,k}^{(\alpha)}(x)-\sum_{k=0}^{k'-1}Q_{n,k}^{(\alpha)}(x)\\&\quad+nQ_{n,k'}^{(\alpha)}(x)\left(2^{\alpha}\left(\frac{k'+1}{n}-x\right)-\frac{1}{n}\right)\\&=2^{\alpha}\sum_{k=k'+1}^{\infty}Q_{n,k}^{(\alpha)}(x)-1+2^{\alpha}(k'+1-nx)Q_{n,k'}^{(\alpha)}(x).\end{aligned}$$

Note that $0<k'+1-nx<1$. By Lemmas 8.13 and 8.14, we have

$$\begin{aligned}|V_{n,\alpha}^{*}(\mathrm{sgn}_{x,\alpha},x)|&\le 2^{\alpha}\left|\sum_{k=k'+1}^{\infty}Q_{n,k}^{(\alpha)}(x)-\frac{1}{2^{\alpha}}\right|+2^{\alpha}Q_{n,k'}^{(\alpha)}(x)\\&=2^{\alpha}\left|\left(\sum_{k=k'+1}^{\infty}b_{n,k}(x)\right)^{\alpha}-\frac{1}{2^{\alpha}}\right|+2^{\alpha}Q_{n,k'}^{(\alpha)}(x).\\&\le\frac{12\alpha}{2^{\alpha}}\frac{\sqrt{1+x}}{\sqrt{nx}}+\frac{2\alpha}{2^{\alpha}}\frac{\sqrt{1+x}}{\sqrt{nx}}=\frac{14\alpha}{2^{\alpha}}\frac{\sqrt{1+x}}{\sqrt{nx}}.\end{aligned}\tag{8.42}$$

Next, we estimate $|V_{n,\alpha}^{*}(g_x,x)|$. Using Bojanic–Cheng decomposition (see, e.g., [45, 54, 55]), one can write

$$V_{n,\alpha}^{*}(g_x,x)=\int_{[0,\infty)}g_x(t)K_{n,\alpha}(x,t)dt=\sum_{j=1}^{4}\int_{A_j}g_x(t)K_{n,\alpha}(x,t)dt, \tag{8.43}$$

where

$$\begin{aligned}&A_1:=[0,\ x-x/\sqrt{n}],\quad &&A_2:=(x-x/\sqrt{n},\ x+x/\sqrt{n}],\\&A_3:=(x+x/\sqrt{n},\ 2x],&&A_4:=(2x,\ \infty).\end{aligned}$$

First, note that $g_x(x)=0$. Thus,

$$\left|\int_{A_2} g_x(t)K_{n,\alpha}(x,t)dt\right| \le \Omega_x(g_x, x/\sqrt{n}) \le \frac{1}{n}\sum_{k=1}^{n}\Omega_x(g_x, x/\sqrt{k}). \tag{8.44}$$

To estimate $|\int_{A_1} g_x(t)K_{n,\alpha}(x,t)dt|$, note that $\Omega_x(g_x,\eta)$ is monotone increasing with respect to η. Thus, it follows that

$$\left|\int_0^{x-x/\sqrt{n}} g_x(t)K_{n,\alpha}(x,t)dt\right| \le \int\limits_0^{x-x/\sqrt{n}} \Omega_x(g_x, x-t)K_{n,\alpha}(x,t)dt.$$

Integrating by parts with $y = x - x/\sqrt{n}$, we have

$$\int\limits_0^{x-x/\sqrt{n}} \Omega_x(g_x, x-t)H_{n,\alpha}(x,t)dt \le \Omega_x(g_x, x-y)\int_0^y K_{n,\alpha}(x,t)dt$$
$$+\int\limits_0^y \left(\int_0^t K_{n,\alpha}(x,u)du\right) d\left(-\Omega_x(g_x, x-t)\right).$$

From the preceding inequality and Lemma 8.12, it follows that

$$\left|\int_{A_1} g_x(t)d_t\lambda_{n,\alpha}(x,t)\right| \le \Omega_x\left(g_x, x/\sqrt{n}\right)\frac{3nx+1}{2n^2(x-y)^2} + \frac{3nx+1}{2n^2}$$
$$\int\limits_0^y \frac{1}{(x-t)^2}d(-\Omega_x(g_x, x-t)). \tag{8.45}$$

Because

$$\int\limits_0^y \frac{1}{(x-t)^2}d(-\Omega_x(g_x, x-t)) = -\frac{\Omega_x(g_x, x-y)}{(x-y)^2} + \frac{\Omega_x(g_x, x)}{x^2}$$
$$+\int\limits_0^y \Omega_x(g_x, x-t)\frac{2}{(x-t)^3}dt,$$

from (8.45), it is obvious that

$$\left|\int_0^{x-x/\sqrt{n}} g_x(t)d_tK_{n,\alpha}(x,t)\right|$$

$$\leq \frac{3nx+1}{2n^2x^2}\Omega_x(g_x,x) + \frac{3nx+1}{2n^2}\int\limits_0^{x-x/\sqrt{n}} \Omega_x(g_x,x-t)\frac{2}{(x-t)^3}dt.$$

Putting $t = x - x/\sqrt{u}$ for the last integral, we get

$$\int\limits_0^{x-x/\sqrt{n}} \Omega_x(g_x,x-t)\frac{2}{(x-t)^3}dt$$

$$= \frac{1}{x^2}\int\limits_1^n \Omega_x(g_x,x/x\sqrt{u})du \leq \frac{1}{x^2}\sum_{k=1}^n \Omega_x(g_x,x/\sqrt{k}).$$

Consequently,

$$\left|\int_{A_1} g_x(t)K_{n,\alpha}(x,t)dt\right| \leq \frac{3nx+1}{2n^2x^2}\left(\Omega_x(g_x,x) + \sum_{k=1}^n \Omega_x(g_x,x/\sqrt{k})\right)$$

$$\leq \frac{3nx+1}{n^2x^2}\sum_{k=1}^n \Omega_x(g_x,x/\sqrt{k}). \tag{8.46}$$

Using a similar method to estimate $\left|\int_{A_3} g_x(t)K_{n,\alpha}(x,t)dt\right|$, we get

$$\left|\int_{A_3} g_x(t)K_{n,\alpha}(x,t)dt\right| \leq C_\alpha \frac{3\alpha+1}{nx}\sum_{k=1}^n \Omega_x(g_x,x/\sqrt{k}), \tag{8.47}$$

where C_α is the constant in Lemma 8.12.

Finally, we estimate

$$\left|\int_{A_4} g_x(t)K_{n,\alpha}(x,t)dt\right|.$$

Because $f(t) = O(t^r)$, there is a constant $M > 0$ such that $|f(t)| \leq Mt^r$. Thus, we have

$$\left|\int_{A_4} g_x(t)K_{n,\alpha}(x,t)dt\right| \leq Mn\sum_{k=[2nx]}^\infty Q_{n,k}^{(\alpha)}(x)\int_{k/n}^{(k+1)/n} t^r dt$$

$$= M\sum_{[k=2nx]}^\infty Q_{n,k}^{(\alpha)}(x)\frac{(k+1)^{r+1}-k^{r+1}}{(r+1)n^r}.$$

By binomial expansion,

$$(k+1)^{r+1} - k^{r+1} = \sum_{i=0}^{r} \frac{(r+1)!}{i!(r+1-i)!} k^i.$$

If we take $M_r = \frac{M}{r+1} \sum_{i=0}^{r} \frac{(r+1)!}{i!(r+1-i)!}$, then it follows that

$$M \sum_{[k=2nx]}^{\infty} Q_{n,k}^{(\alpha)}(x) \frac{(k+1)^{r+1} - k^{r+1}}{(r+1)n^r} \leq M_r \sum_{[k=2nx]}^{\infty} Q_{n,k}^{(\alpha)}(x)(k/n)^r.$$

Now by the results of Lemma 8.13 and Equation (11) of [101], we obtain

$$\left| \int_{A_4} g_x(t) K_{n,\alpha}(x,t) dt \right| \leq \frac{M(f,\alpha,r,x)}{n^l}, \tag{8.48}$$

where $M(f,\alpha,r,x)$ is a constant depending only on f,α,r,x. The theorem follows by combining the estimates of (8.41)–(8.44), (8.46), (8.47), and (8.48). ■

8.7 Baskakov–Durrmeyer–Bézier Operators

For $\alpha \geq 1$ and $x \in [0,\infty)$, the Bézier variant of the well-known Baskakov–Durrmeyer operators was introduced by Gupta and Abel [106]. It is defined as

$$\tilde{V}_{n,\alpha}(f,x) = (n-1) \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \int_0^{\infty} b_{n,k}(t) f(t) dt, \tag{8.49}$$

where $Q_{n,k}^{(\alpha)}(x) = J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)$, $J_{n,k}(x) = \sum_{j=k}^{\infty} p_{n,j}(x)$ and

$$p_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}.$$

Obviously, $\tilde{V}_{n,\alpha}(1,x) = 1$, and as a special case if $\alpha = 1$, the operators $\tilde{V}_{n,\alpha}(f,x)$ reduce to the well-known Baskakov–Durrmeyer operators.

Remark 8.1. If $\mu_{n,m}(x) = (n-1)\sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} p_{n,k}(t)(t-x)^m dt$, then given any $x \in (0,\infty)$, $\lambda > 2$ and for all n sufficiently large, we have

$$\mu_{n,2}(x) < \lambda \frac{x(1+x)}{n}.$$

Lemma 8.15. *Given* $x \in (0,\infty)$ *and* $K_{n,\alpha}(x,t) = (n-1)\sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x)p_{n,k}(t)$, *then for* $\lambda > 2$ *and* $n \geq N(\lambda, x)$, *we have*

$$\lambda_{n,\alpha}(x,y) = \int_0^y K_{n,\alpha}(x,t)dt \leq \frac{\lambda\alpha x(1+x)}{n(x-y)^2},\ 0 \leq y < x,$$

$$1-\lambda_{n,\alpha}(x,z) = \int_z^{\infty} K_{n,\alpha}(x,t)dt \leq \frac{\lambda\alpha x(1+x)}{n(z-y)^2},\ x \leq z < \infty,$$

where

$$\int_0^1 K_{n,\alpha}(x,u)du = 1,\ \text{and}\ \lambda_{n,\alpha}(x,t) = \int_0^t K_{n,\alpha}(x,u)du.$$

Lemma 8.16 ([106]). *For all* $x \in (0,\infty)$, *and* $n,k \in N$,

$$J_{n,k}^{\alpha}(x)p_{n,k}(x) \leq Q_{n,k}^{(\alpha)}(x) \leq \alpha p_{n,k}(x) \leq \alpha\frac{\sqrt{1+x}}{\sqrt{2enx}}$$

holds, where the constant $1/\sqrt{2e}$ *is the best possible.*

Lemma 8.17. *For* $x \in (0,\infty)$ *and* $k = 0,1,2,\ldots,$

$$|J_{n,k}^{\alpha}(x) - J_{n-1,k+1}^{\alpha}(x)| \leq \frac{\alpha(10+11x)}{2\sqrt{nx(1+x)}},$$

$$|J_{n,k}^{\alpha}(x) - J_{n-1,k}^{\alpha}(x)| \leq \frac{\alpha(10+11x)}{2\sqrt{nx(1+x)}}.$$

Theorem 8.14 ([106]). *Assume that* $f \in W(0,\infty)$ *is a function of bounded variation on every finite subinterval of* $(0,\infty)$. *Furthermore, let* $\alpha \geq 1, \lambda > 2$ *and* $x \in (0,\infty)$ *be given. Then, for each* $r \in N$, *there exists a constant* $M(f,\alpha,r,x)$ *such that for sufficiently large* n, *we have*

$$\left|\tilde{V}_{n,\alpha}(f,x) - \frac{f(x+)+\alpha f(x-)}{\alpha+1}\right| \leq \frac{2\alpha\lambda(1+x)+x}{nx}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x)$$
$$+\frac{\alpha(10+11x)}{2\sqrt{nx(1+x)}}|f(x+)-f(x-)| + \frac{M(f,\alpha,r,x)}{n^r},$$

where the auxiliary function $g_x(t)$ *is defined in Theorem 7.7.*

Proof. We can easily see that

$$\left|\tilde{V}_{n,\alpha}(f,x)-\frac{f(x+)+\alpha f(x-)}{\alpha+1}\right|$$
$$\leq\left|\tilde{V}_{n,\alpha}(g_x,x)\right|+\frac{1}{2}\left|f(x+)-f(x-)\right|.\left|\tilde{V}_{n,\alpha}(\mathrm{sign}(t-x),x)+\frac{\alpha-1}{\alpha+1}\right|. \tag{8.50}$$

First, we estimate

$$\tilde{V}_{n,\alpha}(\mathrm{sign}(t-x),x)=(n-1)\sum_{j=0}^{\infty}Q_{n,j}^{(\alpha)}(x)\left(\int_x^{\infty}p_{n,j}(t)dt-\int_0^x p_{n,j}(t)dt\right)$$
$$=1-2(n-1)Q_{n,j}^{(\alpha)}(x)\int_0^x p_{n,j}(t)dt.$$

Using the identity $\sum_{j=0}^{k}p_{n-1,j}(x)=(n-1)\int_x^{\infty}p_{n,k}(t)dt$, we conclude that

$$\tilde{V}_{n,\alpha}(\mathrm{sign}(t-x),x)=1-2\sum_{j=0}^{\infty}Q_{n,j}^{(\alpha)}(x)\left(1-\sum_{k=0}^{j}p_{n-1,k}(x)\right)$$
$$=-1+2\sum_{k=0}^{\infty}p_{n-1,k}(x)\sum_{j=k}^{\infty}Q_{n,j}^{(\alpha)}(x)$$
$$=-1+2\sum_{k=0}^{\infty}p_{n-1,k}(x)J_{n,k}^{\alpha}(x),$$

because $\sum_{j=0}^{\infty}Q_{n,j}^{(\alpha)}(x)=1$. Therefore, we obtain

$$\tilde{V}_{n,\alpha}(\mathrm{sign}(t-x),x)+\frac{\alpha-1}{\alpha+1}=2\sum_{k=0}^{\infty}p_{n-1,k}(x)J_{n,k}^{\alpha}(x)-\frac{2}{\alpha+1}\sum_{k=0}^{\infty}Q_{n-1,k}^{(\alpha+1)}(x)$$

because $\sum_{k=0}^{\infty}Q_{n-1,k}^{(\alpha+1)}(x)=1$. By the mean value theorem, it follows that

$$Q_{n-1,k}^{(\alpha+1)}(x)=J_{n-1,k}^{\alpha+1}(x)-J_{n-1,k+1}^{\alpha+1}(x)=(\alpha+1)p_{n-1,k}(x)\gamma_{n,k}^{\alpha}(x),$$

where $J_{n-1,k+1}(x)<\gamma_{n,k}(x)<J_{n-1,k}(x)$. Hence,

$$\tilde{V}_{n,\alpha}(\mathrm{sign}(t-x),x)+\frac{\alpha-1}{\alpha+1}=2\sum_{k=0}^{\infty}p_{n-1,k}(x)(J_{n,k}^{\alpha}(x)-\gamma_{n,k}^{\alpha}(x)),$$

where

$$J_{n,k}^{\alpha}(x) - J_{n-1,k}^{\alpha}(x) < J_{n,k}^{\alpha}(x) - \gamma_{n,k}^{\alpha}(x) < J_{n,k}^{\alpha}(x) - J_{n-1,k+1}^{\alpha}(x).$$

Lemma 8.17 implies that

$$\left|\tilde{V}_{n,\alpha}(\mathrm{sign}(t-x), x) + \frac{\alpha-1}{\alpha+1}\right| \le \frac{\alpha(10+11x)}{\sqrt{nx(1+x)}}, x \in (0,\infty).$$

Finally, we can write

$$\tilde{V}_{n,\alpha}(g_x, x) = \left(\int_0^{x-x/\sqrt{n}} \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} + \int_{x+x/\sqrt{n}}^{\infty}\right) K_{n,\alpha}(x,t) g_x(t) dt.$$

Using Remark 8.1, Lemma 8.15, and Lemma 8.16, and applying standard techniques, we have

$$\left|\tilde{V}_{n,\alpha}(g_x, x)\right| \le \frac{2\alpha\lambda(1+x)+x}{nx} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x) + O(n^{-r}), n \to \infty.$$

The theorem follows immediately by combining the estimates of $\tilde{V}_{n,\alpha}(\mathrm{sign}(t-x), x)$ and $\tilde{V}_{n,\alpha}(g_x, x)$. ■

We have seen that by taking the weight function of Beta basis functions, some approximation properties become simpler. The same holds for the Bézier variant of Baskakov–Beta operators, which were introduced in [98]. For $\alpha \ge 1$ and $x \in [0,\infty)$, they are defined as

$$B_{n,\alpha}(f, x) = \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \int_0^{\infty} b_{n,k}(t) f(t) dt, \tag{8.51}$$

where $Q_{n,k}^{(\alpha}(x) = J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)$, $J_{n,k}(x) = \sum_{j=k}^{\infty} p_{n,j}(x)$, and

$$p_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}, \; b_{n,k}(t) = \frac{1}{B(k+1,n)} \frac{t^k}{(1+t)^{n+k+1}},$$

with $B(m,n) = \frac{(m-1)!(n-1)!}{(n+m-1)!}$. Obviously, $B_{n,\alpha}(1,x) = 1$; as a special case if $\alpha = 1$, the operators $B_{n,\alpha}(f,x)$ reduce to Baskakov–Beta operators (see, e.g., [92, 94]). Actually, the approximation properties for Baskakov–Beta operators become simpler than those of the usual Baskakov–Durrmeyer operators.

Let

$$H(0,\infty) = \{f : f \text{ is locally bounded on} (0,\infty) \text{ and}$$
$$|f(t)| \le M(1+t)^{\beta}, M > 0, \beta \in N^0\}.$$

Theorem 8.15 ([98]). *Let $f \in H(0,\infty)$, and at a fixed point $x \in (0,\infty)$, let the one-sided limits $f(x\pm)$ exist. Then, for $\alpha \geq 1, \lambda > 2, x \in (0,\infty)$ and for $n > \max\{1+\beta, N(\lambda,x)\}$, we have*

$$\left|B_{n,\alpha}(f,x) - \left[\frac{f(x+)+\alpha f(x-)}{\alpha+1}\right]\right| \leq \frac{\alpha[3\lambda+(1+3\lambda)x]}{nx}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x)$$
$$+\frac{\alpha\sqrt{1+x}}{\sqrt{2enx}}|f(x+)-f(x-)|$$
$$+M\alpha(2^\beta-1)\frac{(1+x)^\beta}{x^{2\beta}}O(n^{-\beta})$$
$$+\frac{2M\alpha\lambda(1+x)^{\beta+1}}{nx},$$

where $g_x(t)$ is defined in Theorem 7.7.

In 2005, Ispir and Yuksel [158] extended the studies on a general sequence of summation–integral-type operators that was proposed by Srivastava and Gupta [218]. They introduced the Bézier variant of (7.38) and called such operators Srivastava–Gupta operators. Let $B_r[0,\infty)$ be the class of bounded variation functions satisfying the growth condition $|f(t)| \leq M(1+t)^r, M>0, r\geq 0, t\to\infty$. For $f \in B_r[0,\infty)$ and $\alpha \geq 1$, Ispir and Yuksel [158] proposed the Bézier variant of Srivastava–Gupta operators as

$$G_{n,c,\alpha}(f;x) = n\sum_{k=1}^{\infty} Q_{n,k}^{(\alpha)}(x;c)\int_0^\infty p_{n+c,k-1}(t;c)\,f(t)\,dt + p_{n,0}(x;c)\,f(0), \tag{8.52}$$

where $Q_{n,k}^{(\alpha)}(x;c) = J_{n,k}^{\alpha}(x;c) - J_{n,k+1}^{\alpha}(x;c)$ and $J_{n,k}^{\alpha}(x;c) = \sum_{j=k}^{\infty} p_{n,j}(x;c)$.

Theorem 8.16 ([158]). *Let f be an element of $B_r[0,\infty)$. If $\alpha \geq 1, r \in N$, and $\lambda > 2$ are given, then there exists a constant $C(f,\alpha,r,x)$ such that for n sufficiently large, we have*

$$\left|G_{n,c,\alpha}(f;x) - \frac{f(x+)+\alpha f(x-)}{\alpha+1}\right| \leq \frac{6\alpha\lambda x(1+cx)}{nx}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x)$$
$$+2\alpha\sqrt{\frac{1+cx}{2enx}}|f(x+)-f(x-)| + C\alpha 2^r\frac{(1+x)^r}{x^r}O(n^{-r}),$$

where the auxiliary function $g_x(t)$ is defined in Theorem 7.7.

8.8 MKZ Bézier-Type Operators

For $\alpha \geq 1$ and a function f defined on the interval $[0,1]$, the Meyer–König–Zeller–Bézier and integrated Meyer–König–Zeller–Bézier operators $M_{n,\alpha}(f,x)$ and $\hat{M}_{n,\alpha}(f,x)$ are respectively defined (cf. [248]) by

$$M_{n,\alpha}(f,x) = \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n+k}\right) \tag{8.53}$$

and

$$\hat{M}_{n,\alpha}(f,x) = \sum_{k=0}^{\infty} \frac{(n+k)(n+k+1)}{n} Q_{n,k}^{(\alpha)}(x) \int_{I_k} f(t)dt, \tag{8.54}$$

where $Q_{n,k}^{(\alpha)}(x) = J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)$, $I_k = [k/(n+k), (k+1)/(n+k+1))$, and $J_{n,k}(x) = \sum_{j=k}^{\infty} m_{n,j}(x)$, with the MKZ basis functions

$$m_{n,j}(x) = \binom{n+j-1}{j} x^j (1-x)^n, \quad j = 0, 1, 2, \ldots.$$

For $\alpha = 1$, the above operators reduce to the operators discussed in [90]. The pointwise approximation properties of the operators $M_{n,\alpha}(f,x)$ and $\hat{M}_{n,\alpha}(f,x)$ for $\alpha \geq 1$ have been dealt with in [248].

Theorem 8.17 ([248]). *Let f be a function of bounded variation on $[0,1]$ ($f \in BV[0,1]$). Then, for $x \in (0,1)$, $\alpha \geq 1$, and for n sufficiently large, we have*

$$\begin{aligned}
&\left| M_{n,\alpha}(f,x) - \frac{1}{2^\alpha} f(x+) - \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right| \\
&\quad \leq \frac{5\alpha}{nx+1} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x) \\
&\qquad + \frac{3\alpha}{\sqrt{nx}+1} (|f(x+) - f(x-)| + \varepsilon_n(x)|f(x) - f(x-)|),
\end{aligned}$$

where $V_a^b(g_x)$ is the total variation of g_x on $[a,b]$,

$$g_x(t) = \begin{cases} f(t) - f(x+), & x < t \leq 1, \\ 0, & t = x, \\ f(t) - f(x-), & 0 \leq t < x, \end{cases}$$

and

$$\varepsilon_n(x) = \begin{cases} 1, \ x = \frac{k'}{(n+k')} \ \textit{for all} \ k' \in N, \\ 0, \ x \neq \frac{k}{(n+k)} \ \textit{for all} \ k \in N. \end{cases}$$

Theorem 8.18 ([248]). *Let f be a function of bounded variation on $[0,1]$ ($f \in BV[0,1]$). Then, for $x \in (0,1)$, $\alpha \geq 1$, and sufficiently large n, we have*

$$\left| \hat{M}_{n,\alpha}(f,x) - \frac{1}{2^\alpha} f(x+) - \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right|$$
$$\leq \frac{5\alpha}{nx+1} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x) + \frac{5\alpha}{\sqrt{nx}+1} |f(x+) - f(x-)|,$$

where the auxiliary function g_x is defined in Theorem 8.17.

Theorem 8.19 ([248]). *Let f be a bounded function on $[0,1]$ and if $x \in (0,1)$ is a continuous point of the first kind of f, then for $\alpha \geq 1$, we have*

$$\lim_{n\to\infty} M_{n,\alpha}(f,x) = \frac{1}{2^\alpha} f(x+) - \left(1 - \frac{1}{2^\alpha}\right) f(x-)$$

and

$$\lim_{n\to\infty} \hat{M}_{n,\alpha}(f,x) = \frac{1}{2^\alpha} f(x+) - \left(1 - \frac{1}{2^\alpha}\right) f(x-).$$

The above results were proved for the case $\alpha \geq 1$. The other case [i.e., $0 < \alpha < 1$ for the operators $\hat{M}_{n,\alpha}(f,x)$] was considered by Zeng and Gupta [254]. Next, we present this case.

Let

$$K_{n,\alpha}(x,t) = \sum_{k=0}^{\infty} \frac{(n+k)(n+k+1)}{n} Q_{n,k}^{(\alpha)}(x)\phi_k(t).$$

We recall the Lebesgue–Stieltjes integral representation

$$\hat{M}_{n,\alpha}(f,x) = \int_0^1 f(t)K_{n,\alpha}(x,t)dt, \tag{8.55}$$

where ϕ_k is the characteristic function of the interval $I_k = \Big[k/(n+k), (k+1)/(n+k+1)\Big]$ with respect to $I = [0,1]$.

The following lemmas are necessary for the proof of the rate of convergence.

Lemma 8.18.

(i) ([246, Theorem 2]). For every $k \in N, x \in (0, 1]$, *we have*

$$m_{n,k}(x) < \frac{1}{\sqrt{2enx}}.$$

(ii) ([248, Lemma 4]) For $x \in (0, 1)$, *we have*

$$\left| \sum_{k>nx/(1-x)} m_{n,k}(x) - \frac{1}{2} \right| \leq \frac{3}{\sqrt{nx}+1}.$$

Lemma 8.19. *Let* $0 < \alpha \leq 1$ *and* $x \in (0, 1)$. *Then for* $n > 121/x$ *and* $k' = [nx/(1-x)]$, *we have*

(i) $Q^{(\alpha)}_{n,k'}(x) < \frac{2}{\sqrt{nx}+1}$.

(ii) $\left| \left(\sum_{k>nx/(1-x)} m_{n,k}(x) \right)^{\alpha} - \frac{1}{2^{\alpha}} \right| \leq \frac{4}{\sqrt{nx}+1}$.

Lemma 8.20 ([248]). *For all* $m \in N$, *there is a constant* A_m *such that for* $n \geq 2$,

$$\hat{M}_n\big((t-x)^{2m}, x\big) \leq \frac{A_m}{n^m}.$$

Lemma 8.21. *Let* $K_{n,\alpha}(x,t)$ *be defined by (8.55),* $0 < \alpha \leq 1$. *Then, for sufficiently large n,*

(i) For $0 \leq y < x < 1$, *it holds that*

$$\int_0^y K_{n,\alpha}(x,t)dt \leq \frac{2x(1-x)^2}{n(x-y)^2}.$$

(ii) For $0 \leq x < z \leq 1$, *it holds that*

$$\int_z^1 K_{n,\alpha}(x,t)dt \leq \frac{C_\alpha}{n(z-x)^2},$$

where C_α *is a constant dependent only on* α.

Theorem 8.20 ([254]). *Let* f *be a bounded function on* $[0, 1]$, *and let* $f(x+)$, $f(x-)$ *exist at a fixed point* $x \in (0, 1)$, $0 < \alpha < 1$. *Then, for sufficiently large n, we have*

$$\left| \hat{M}_{n,\alpha}(f,x) - \frac{1}{2^{\alpha}} f(x+) - \left(1 - \frac{1}{2^{\alpha}}\right) f(x-) \right|$$

$$\leq \frac{6}{\sqrt{nx}+1}|f(x+)-f(x-)| + \frac{4+2C_\alpha}{nx(1-x)}\sum_{k=1}^{n}\omega_x\left(g_x, 1/\sqrt{k}\right),$$

where C_α is a positive constant dependent only on α, the auxiliary function g_x is as defined in Theorem 8.17, and $\omega_x(f,h) = \sup_{t\in[x-h,x+h]}|f(t)-f(x)|$.

Proof. We can write

$$\begin{aligned}
&\left|M_{n,\alpha}(f,x) - \frac{1}{2^\alpha}f(x+) - \left(1-\frac{1}{2^\alpha}\right)f(x-)\right| \\
&\quad \leq |M_{n,\alpha}(g_x,x)| + \left|\frac{f(x+)-f(x-)}{2^\alpha}M_{n,\alpha}(\mathrm{sgn}_{x,\alpha},x)\right|. \qquad (8.56)
\end{aligned}$$

Let $x \in \left[\frac{k'}{n+k'}, \frac{k'+1}{n+k'+1}\right)$. Direct calculations give

$$\begin{aligned}
\hat{M}_{n,\alpha}(\mathrm{sgn}_{x,\alpha},x) &= -\sum_{k=0}^{k'-1}Q_{nk}^{(\alpha)}(x) + (2^\alpha-1)\sum_{k'+1}^{\infty}Q_{nk}^{(\alpha)}(x) \\
&+ \frac{(n+k')(n+k'+1)}{n}\left[-\left(x-\frac{k'}{n+k'}\right)+(2^\alpha-1)\left(\frac{k'+1}{n+k'+1}-x\right)\right]Q_{nk'}^{(\alpha)}(x) \\
&= 2^\alpha\sum_{k=k'+1}^{\infty}Q_{n,k}^{(\alpha)}(x) - 1 + \frac{(n+k')(n+k'+1)}{n}2^\alpha\left(\frac{k'+1}{n+k'+1}-x\right)Q_{nk'}^{(\alpha)}(x).
\end{aligned}$$

Because

$$\frac{k'+1}{n+k'+1} - x \leq \frac{n}{(n+k')(n+k'+1)},$$

it follows that

$$\begin{aligned}
\left|\hat{M}_{n,\alpha}(\mathrm{sgn}_{x,\alpha},x)\right| &\leq \left|2^\alpha\sum_{k=k'+1}^{\infty}Q_{n,k}^{\alpha}(x) - 1\right| + 2^\alpha Q_{nk'}^{(\alpha)}(x) \\
&= 2^\alpha\left|\left(\sum_{k>nx/(1-x)}m_{n,k}(x)\right)^\alpha - \frac{1}{2^\alpha}\right| + 2^\alpha Q_{nk'}^{(\alpha)}(x).
\end{aligned}$$

Thus, using Lemma 8.19, we obtain

$$\left|\frac{f(x+)-f(x-)}{2^\alpha}M_{n,\alpha}(\mathrm{sgn}_{x,\alpha},x)\right|$$

$$= |f(x+) - f(x-)| \left(\left| \left(\sum_{k > nx/(1-x)} m_{n,k}(x) \right)^{\alpha} - \frac{1}{2^{\alpha}} \right| + Q_{nk'}^{(\alpha)}(x) \right).$$

$$\leq \frac{6}{\sqrt{nx}+1} \big| f(x+) - f(x-) \big|. \tag{8.57}$$

Next, we estimate $\hat{M}_{n,\alpha}(g_x, x)$. Let

$$H_{n,\alpha}(x,t) = \int_0^t K_{n,\alpha}(x,u)du.$$

Then

$$\begin{aligned} \hat{M}_{n,\alpha}(g_x, x) &= \int_0^1 g_x(t) K_{n,\alpha}(x,t)dt \\ &= \int_0^1 g_x(t) d_t H_{n,\alpha}(x,t) \\ &= \left(\int_{D_1} + \int_{D_2} + \int_{D_3} \right) g_x(t) d_t H_{n,\alpha}(x,t) \\ &=: \Delta_1 + \Delta_2 + \Delta_3, \end{aligned} \tag{8.58}$$

where $D_1 = \left[0,\ x - \frac{x}{\sqrt{n}}\right]$, $D_2 = \left[x - \frac{x}{\sqrt{n}},\ x + \frac{1-x}{\sqrt{n}}\right]$, and $D_3 = \left[x + \frac{1-x}{\sqrt{n}},\ 1\right]$. Note that $g_x(x) = 0$. We have

$$\begin{aligned} |\Delta_2| &\leq \int_{D_2} |g_x(t) - g_x(x)| d_t H_{n,\alpha}(x,t) \\ &\leq \omega_x\left(g_x, x/\sqrt{n}\right) \leq \frac{1}{n} \sum_{k=1}^{n} \omega_x\left(g_x, x/\sqrt{k}\right). \end{aligned} \tag{8.59}$$

To estimate Δ_1, let $y = x - x/\sqrt{n}$. Using Lebesgue–Stieltjes integration by parts, we have

$$\begin{aligned} \big|\Delta_1\big| &\leq \int_0^y \omega_x(g_x, x-t) d_t H_{n,\alpha}(x,t) \\ &= \omega_x(g_x, x-y) H_{n,\alpha}(x,y) + \int_0^y \hat{H}_{n,\alpha}(x,t) d_t\big(-\omega_x(g_x, x-t)\big), \end{aligned}$$

where $\hat{H}_{n,\alpha}(x,t)$ is the normalized form of $H_{n,\alpha}(x,t)$. Note that $\hat{H}_{n,\alpha}(x,t) \leq H_{n,\alpha}(x,t)$ on [0, 1]. Using Lemma 8.21(i), we get

$$\left|\Delta_1\right| \le \omega_x(g_x, x-y)\frac{2x(1-x)^2}{n(x-y)^2} + \frac{2x(1-x)^2}{n}\int_0^y \frac{1}{(x-t)^2} d_t\big(-\omega_x(g_x, x-t)\big). \tag{8.60}$$

Because

$$\int_0^y \frac{1}{(x-t)^2} d_t\big(-\omega_x(g_x, x-t)\big) = -\frac{\omega_x(g_x, x-t)}{(x-t)^2}\Bigg|_0^{y+} + \int_0^y \frac{2\omega_x(g_x, x-t)}{(x-t)^3} dt,$$

from (8.60) and by the change of variable $t = x - x/\sqrt{u}$, we get

$$\begin{aligned}
\left|\Delta_1\right| &\le \frac{2(1-x)^2}{n}\omega_x(g_x, x) + \frac{4x(1-x)^2}{n}\int_0^{x-x/\sqrt{n}} \frac{\omega_x(g_x, x-t)}{(x-t)^3} dt. \\
&= \frac{2(1-x)^2}{n}\omega_x(g_x, x) + \frac{2(1-x)^2}{nx}\int_1^n \omega_x\left(g_x, x/\sqrt{u}\right) du \\
&\le \frac{4(1-x)^2}{nx}\sum_{k=1}^n \omega_x\left(g_x, 1/\sqrt{k}\right).
\end{aligned} \tag{8.61}$$

Via a similar method and using Lemma 8.21 (ii) to estimate $|\Delta_3|$, we get

$$\left|\Delta_3\right| \le \frac{2C_\alpha}{n(1-x)}\sum_{k=1}^n \omega_x\left(g_x, 1/\sqrt{k}\right), \tag{8.62}$$

where C_α is the constant defined in Lemma 8.21 (ii). Combining the estimates of (8.59), (8.61), and (8.62) and with easy calculations, we obtain

$$\left|\hat{M}_{n,\alpha}(g_x, x)\right| \le \frac{4+2C_\alpha}{nx(1-x)}\sum_{k=1}^n \omega_x\left(g_x, 1/\sqrt{k}\right). \tag{8.63}$$

The theorem now follows by combining (8.56), (8.57), and (8.63). ■

Also, Gupta [100] considered the Bézier variant of certain integrated MKZ operators of the Durrmeyer type. For a function $f \in L_1[0,1]$ and $\alpha \ge 1$, the operators introduced in [100] are defined as

$$M'_{n,\alpha}(f, x) = \sum_{k=1}^\infty Q_{n,k+1}^{(\alpha)}(x)\frac{(n+k-2)(n+k-3)}{(n-2)}\int_0^1 m_{n-2,k-1}(t) f(t) dt, \tag{8.64}$$

where for $Q_{n,k+1}^{(\alpha)}(x)$ and $m_{n-2,k-1}(t)$, we refer to (8.53).

Using some results of probability theory, Gupta [100] established the estimate on the rate of convergence of $M'_{n,\alpha}(f, x)$.

Theorem 8.21 ([100]). *Let f be a function of bounded variation on $[0,1], \alpha \geq 1$ and $\mu > 2$. Then, for every $x \in (0,1)$ and for sufficiently large n, we have*

$$\left| M'_{n,\alpha}(f,x) - \left[\frac{f(x+) + \alpha f(x-)}{\alpha+1} \right] \right| \leq \frac{3\alpha}{\sqrt{n} x^{3/2}} |f(x+) - f(x-)| + \frac{(2\mu\alpha + x)}{nx} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x),$$

where $g_x(t)$ is as defined in Theorem 7.5.

For $f \in L_1[0,1]$ and $\alpha \geq 1$, Gupta [102] considered another Durrmeyer-type modification of MKZ operators as

$$\breve{M}_{n,\alpha}(f,x) = \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \int_0^1 f(t) b_{n,k}(t) dt, \tag{8.65}$$

where $b_{n,k}(t) = n \binom{n+k}{k} t^k (1-t)^{n-1}$, for $Q_{n,k}^{(\alpha)}(x)$ [see (8.53)].

Theorem 8.22 ([102]). *Let $\alpha \geq 1$, and let f be a function of bounded variation on $[0,1]$. For each $x \in (0,1)$ and each constant $\lambda > 2$ and each $\varepsilon > 0$, there exists an integer $N(\lambda, x) \geq 3$ such that*

$$\left| \breve{M}_{n,\alpha}(f,x) - \left[\frac{f(x+) + \alpha f(x-)}{\alpha+1} \right] \right| \leq \frac{\alpha}{\sqrt{2enx}} |f(x+) - f(x-)| + \frac{(2\lambda\alpha(x+\varepsilon)+x^2}{nx^2} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x),$$

where $g_x(t)$ is as defined in Theorem 7.5.

Corollary 8.1 ([102]). *Let f be a function of bounded variation on $[0,1]$, and let $x \in (0,1)$. If f has a discontinuity of the first kind in x, then we have*

$$\lim_{n\to\infty} \breve{M}_{n,\alpha}(f,x) = \frac{f(x+) + \alpha f(x-)}{\alpha+1}.$$

This limit is essentially different from the operators (8.53) and (8.53), which implies under the corollary's assumption that

$$\lim_{n\to\infty} M_{n,\alpha}(f,x) = \lim_{n\to\infty} \hat{M}_{n,\alpha}(f,x) = \frac{1}{2^\alpha} f(x+) - \left(1 - \frac{1}{2^\alpha}\right) f(x-).$$

Chapter 9
Some More Results on the Rate of Convergence

9.1 Nonlinear Operators

Shaw et al. [210] investigated the problem for the general family of positive linear operators, which includes Bernstein, Kantorovich, and Durrmeyer operators as special cases. They investigated their results for the classes of functions $BV[a,b]$ and $DBV[a,b]$. Also, Hua and Shaw [156] extended this problem for linear integral operators with a not necessarily positive kernel. We observed in [166] that the previous work on this subject has dealt with operators whose kernel functions are probability measures and do not have singularities at any point.

Let G be a locally compact abelian group with the Haar measure. By $\int\limits_G f(t)dt$, we mean the Haar integral of an extended real-valued function over G. Let Λ be a nonempty set of indices with any topology and λ_0 an accumulation point of Λ in this topology. We take a family $\mathcal{K}$ of functions $K_\lambda : G \text{ x } R \to R$, where $K_\lambda(t,0) = 0$ for all $t \in G$ and $\lambda \in \Lambda$, such that $K_\lambda(t,u)$ is integrable over G in the sense of the Haar measure, for all values of the second variable and for every $\lambda \in \Lambda$. The family $\mathcal{K}$ will be called a kernel.

In approximation theory, most of the applications are limited to linear integral operators, because the notion of the singularity of an integral operator is closely connected with its linearity. In 1981, Musielak [197] used the nonlinear integral operators, which are defined as

$$T_w f(s) = \int\limits_G K_w(t-s, f(t))\, dt, \qquad s \in G.$$

The assumption of the linearity of the operators was replaced here by a Lipschitz condition on K_w with respect to the second variable. This study allows us to use the classical method for linear integral operators [49] to obtain the convergence of the nonlinear integral operators, although the notion of the singularity of an integral

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operator is closely connected with its linearity. In this section, we mention the results studied in [166]. We extend in three directions the problem of the rate of convergence of operators acting on functions of bounded variation:

(i) The set $N = \{1, 2, 3, \ldots\}$ of indices is replaced by an abstract set Λ of indices.
(ii) The stochastic condition is replaced by a limit form (see condition c) and Example 9.2).
(iii) The family of linear integral operators is replaced by the family of nonlinear integral operators T_λ, which are defined as

$$T_\lambda(f;x) = \int_a^b K_\lambda(t-x, f(t))\,dt, \qquad x \in < a,b >, \tag{9.1}$$

where $f \in BV < a,b >$ and $< a,b >$ is an arbitrary finite interval in R.

Operators of the type (9.1) and its linear cases have been studied by Karsli [165] and Karsli and Ibikli [167]. They determined the rate of convergence at the μ-generalized Lebesgue points of functions belonging to $L_1 < a,b >$ and $L_1(R)$. We also mention the study due to Swiderski and Wachnicki [226], who estimated the pointwise approximation of nonlinear singular convolution integrals, which is the special case of (9.1), in the class of functions $f \in L_p(-\pi,\pi)$ at the p-Lebesgue points.

The aim of this section is to present the results studied in [166], that is, the behavior of the operators T_λ for functions of bounded variation on $< a,b >$, and give an estimate for the rate of pointwise convergence of the nonlinear family of singular integral operators (9.1) to the point x, having a discontinuity of the first kind of f, as $\lambda \to \lambda_0$.

Throughout this chapter, we assume that $G = R$, $\lambda_0 = \infty$, $\beta > 0$, and $\gamma \geq 1$.

First, we assume that the following conditions hold:

(1) There exists an integrable function $L_\lambda(t)$, such that

$$|K_\lambda(t,u) - K_\lambda(t,v)| \leq L_\lambda(t)\,|u-v|\,,$$

for every $t, u, v \in R$ and for any $\lambda \in \Lambda$.

(2) $\int\limits_{x-(x-a)/\lambda^{\gamma/\beta}}^{x+(b-x)/\lambda^{\gamma/\beta}} L_\lambda(t)dt \leq B(x)$, for $x \in < a,b >$ and for all $\lambda \in \Lambda$.

(3) $\left[\int\limits_R L_\lambda(t)\,dt - 1\right] \leq \lambda^{-\gamma}$ for sufficiently large λ.

According to condition (1), it is easy to see that $K_\lambda : G \text{ x } R \to R$ is a kernel.

In [166], Karsli and Gupta introduced a function $\tilde{f}$, which is defined on R, as

$$\tilde{f}\,(t) := \begin{cases} f(t), & t \in < a,b >, \\ 0, & t \notin < a,b > . \end{cases} \tag{9.2}$$

Lemma 9.1. *For all $x \in (a,b)$ and for each $\lambda \in \Lambda$, let*

$$\int_a^b L_\lambda(u-x)\,|u-x|^\beta\,du \le B(x)\lambda^{-\gamma}, \tag{9.3}$$

where $L_\lambda(t)$ is as defined above. Then we have

$$m_\lambda(x,t) := \int_a^t L_\lambda(u-x)du \le \frac{1}{\lambda^\gamma (x-t)^\beta} B(x)\ , \quad a \le t < x, \tag{9.4}$$

and

$$1 - m_\lambda(x,z) := \int_z^b L_\lambda(u-x)du \le \frac{1}{\lambda^\gamma (z-x)^\beta} B(x)\ , \quad x < z < b. \tag{9.5}$$

Theorem 9.1 ([166]). *Let f be a function of bounded variation on $< a,b >$. Then, for every $x \in (a,b)$ and λ sufficiently large, we have*

$$\begin{aligned} T_\lambda(f;x) &- \frac{|f(x+)+f(x-)|+|f(x+)-f(x-)|}{2} \\ &\le B^*(x)\lambda^{-\gamma}\left[\bigvee_a^b (f_x) + \sum_{k=1}^{[\lambda^\gamma]} \bigvee_{x-(x-a)/k^{1/\beta}}^{x+(b-x)/k^{1/\beta}} (f_x)\right] + B(x) \bigvee_{x-(x-a)/\lambda^{\gamma/\beta}}^{x+(b-x)/\lambda^{\gamma/\beta}} (f_x), \\ &\quad + \frac{|f(x+)+f(x-)|+|f(x+)-f(x-)|}{2}\lambda^{-\gamma}, \end{aligned} \tag{9.6}$$

where $[\lambda^\gamma]$ is the greatest integer less than or equal to λ^γ, $B^*(x) = B(x) \max\{(x-a)^{-\beta}, (b-x)^{-\beta}\}$,

$$f_x(t) = \begin{cases} f(t) - f(x+), & x < t \le b, \\ 0, & t = x, \\ f(t) - f(x-), & a \le t < x\ , \end{cases} \tag{9.7}$$

and $\bigvee_a^b (f_x)$ is the total variation of f_x on $< a,b >$.

Proof. For any $f(t) \in BV < a,b >$, we can easily verify from (9.7) that

$$\begin{aligned} f(t) = &\frac{f(x+)+f(x-)}{2} + f_x(t) + \frac{f(x+)-f(x-)}{2} sgn(t-x) \\ &+ \delta_x(t)\left[f(x) - \frac{f(x+)+f(x-)}{2}\right], \end{aligned} \tag{9.8}$$

where

$$\delta_x(t) = \begin{cases} 1, & x = t, \\ 0, & x \neq t. \end{cases}$$

Applying the operator (9.1) to (9.8), using (9.2), and from condition (1), we have

$$\begin{aligned} T_\lambda(f;x) &= \int_R K_\lambda(t-x, \widetilde{f}(t))dt \le \int_R L_\lambda(t-x)\left|\widetilde{f}(t)\right| dt \\ &\le \left|\frac{f(x+)+f(x-)}{2}\right| \int_R L_\lambda(t-x)\,dt + \int_R L_\lambda(t-x)\left|\widetilde{f}_x(t)\right| dt \\ &\quad + \left|\frac{f(x+)-f(x-)}{2}\right| \int_R L_\lambda(t-x)\,dt \\ &\quad + \left|f(x) - \frac{f(x+)+f(x-)}{2}\right| \int_R L_\lambda(t-x)\,|\delta_x(t)|\,dt. \end{aligned}$$

It is obvious that the last term o the right-hand side of the preceding inequality is zero, because of the definition of $\delta_x(t)$. Hence, from (9.2), we have

$$\begin{aligned} T_\lambda(f;x) &\le \frac{|f(x+)+f(x-)| + |f(x+)-f(x-)|}{2} \int_R L_\lambda(t-x)\,dt \\ &\quad + \int_a^b L_\lambda(t-x)\,|f_x(t)|\,dt. \end{aligned}$$

In order to prove our theorem, we write the following inequality:

$$\begin{aligned} T_\lambda(f;x) \le{}& \frac{|f(x+)+f(x-)| + |f(x+)-f(x-)|}{2} \int_R L_\lambda(t-x)\,dt \\ &+ \int_a^b L_\lambda(t-x)\,|f_x(t)|\,dt + \left|\frac{f(x+)+f(x-)}{2}\right| - \left|\frac{f(x+)+f(x-)}{2}\right| \\ &+ \left|\frac{f(x+)-f(x-)}{2}\right| - \left|\frac{f(x+)-f(x-)}{2}\right|. \end{aligned}$$

Thus, we get

$$
\begin{aligned}
&T_\lambda(f;x) - \frac{|f(x+)+f(x-)|+|f(x+)-f(x-)|}{2}\\
&\quad\le \int_a^b L_\lambda(t-x)\,|f_x(t)|\,dt + \frac{|f(x+)+f(x-)|+|f(x+)-f(x-)|}{2}\\
&\quad\left[\int_R L_\lambda(t-x)\,dt - 1\right].
\end{aligned}
\tag{9.9}
$$

According to condition (3), for sufficiently large λ, we have

$$
\left[\int_R L_\lambda(t-x)\,dt - 1\right] \le \lambda^{-\gamma}. \tag{9.10}
$$

In order to complete the proof of theorem, we need the estimation of

$$
\int_a^b L_\lambda(t-x)\;|f_x(t)|\,dt
$$

in (9.9). Now, we split the last integral in three parts as follows:

$$
\begin{aligned}
\int_a^b L_\lambda(t-x)\,|f_x(t)|\,dt &\le \left(\int_a^{x-(x-a)/\lambda^{\gamma/\beta}} + \int_{x-(x-a)/\lambda^{\gamma/\beta}}^{x+(b-x)/\lambda^{\gamma/\beta}} + \int_{x+(b-x)/\lambda^{\gamma/\beta}}^{b}\right) L_\lambda(t-x)\,|f_x(t)|\,dt\\
&=: |I_1(\lambda,x)| + |I_2(\lambda,x)| + |I_3(\lambda,x)|.
\end{aligned}
\tag{9.11}
$$

We observe that $I_1(\lambda,x)$ and $I_3(\lambda,x)$ can be written as a Lebesgue–Stieltjes integral as follows:

$$
|I_1(\lambda,x)| = \int_a^{x-(x-a)/\lambda^{\gamma/\beta}} |f_x(t)|\,d_t\,(m_\lambda(x,t))
$$

and

$$
|I_3(\lambda,x)| = \int_{x+(b-x)/\lambda^{\gamma/\beta}}^{b} |f_x(t)|\,d_t\,(m_\lambda(x,t)),
$$

where $m_\lambda(x,t) = \int\limits_a^t L_\lambda(u-x)du$. First, we estimate $I_2(\lambda,x)$. Since $f_x(x) = 0$, for $t \in \left[x-(x-a)/\lambda^{\gamma/\beta}, x+(b-x)/\lambda^{\gamma/\beta}\right]$, we have

$$|I_2(\lambda,x)| = \int\limits_{x-(x-a)/\lambda^{\gamma/\beta}}^{x+(b-x)/\lambda^{\gamma/\beta}} |f_x(t)-f_x(x)|\, L_\lambda(t-x)dt.$$

Also, by condition (2), we have

$$|I_2(\lambda,x)| \le \bigvee_{x-(x-a)/\lambda^{\gamma/\beta}}^{x+(b-x)/\lambda^{\gamma/\beta}} (f_x) \int\limits_{x-(x-a)/\lambda^{\gamma/\beta}}^{x+(b-x)/\lambda^{\gamma/\beta}} L_\lambda(t-x)dt \le B(x) \bigvee_{x-(x-a)/\lambda^{\gamma/\beta}}^{x+(b-x)/\lambda^{\gamma/\beta}} (f_x). \tag{9.12}$$

Next, we estimate $I_1(\lambda,x)$. Using partial Lebesgue–Stieltjes integration, we obtain

$$\begin{aligned} |I_1(\lambda,x)| &= \int\limits_a^{x-(x-a)/\lambda^{\gamma/\beta}} |f_x(t)|\, d_t\left(m_\lambda(x,t)\right) \\ &= \left|f_x(x-(x-a)/\lambda^{\gamma/\beta})\right| m_\lambda(x, x-(x-a)/\lambda^{\gamma/\beta}) \\ &\quad - \int\limits_a^{x-(x-a)/\lambda^{\gamma/\beta}} m_\lambda(x,t)d_t\left(|f_x(t)|\right). \end{aligned}$$

Let $y = x-(x-a)/\lambda^{\gamma/\beta}$. By (10.5), it is clear that

$$m_\lambda(x,y) \le B(x)\,(x-a)^{-\beta}\,. \tag{9.13}$$

Here we note that

$$\left|f_x(x-(x-a)/\lambda^{\gamma/\beta})\right| = \left|f_x(x-(x-a)/\lambda^{\gamma/\beta}) - f_x(x)\right| \le \bigvee_{x-(x-a)/\lambda^{\gamma/\beta}}^{x} (f_x).$$

Using partial integration and applying (9.13), we obtain

$$|I_1(\lambda,x)| \le \bigvee_{x-(x-a)/\lambda^{\gamma/\beta}}^{x} (f_x) \left|m_\lambda(x, x-(x-a)/\lambda^{\gamma/\beta})\right|$$

$$
\begin{aligned}
&+\int_a^{x-(x-a)/\lambda^{\gamma/\beta}} m_\lambda(x,t)d_t\left(-\bigvee_t^x(f_x)\right)\\
&\le \bigvee_{x-(x-a)/\lambda^{\gamma/\beta}}^{x}(f_x)B(x)\,(x-a)^{-\beta}\\
&+B(x)\lambda^{-\gamma}\int_a^{x-(x-a)/\lambda^{\gamma/\beta}}(x-t)^{-\beta}d_t\left(-\bigvee_t^x(f_x)\right)\\
&=\bigvee_{x-(x-a)/\lambda^{\gamma/\beta}}^{x}(f_x)B(x)\,(x-a)^{-\beta}\\
&+B(x)\lambda^{-\gamma}\left[-\,(x-a)^{-\beta}\,/\lambda^{-\gamma}\bigvee_{x-(x-a)/\lambda^{\gamma/\beta}}^{x}(f_x)\right.\\
&\left.+(x-a)^{-\beta}\bigvee_a^x(f_x)+\int_a^{x-(x-a)/\lambda^{\gamma/\beta}}\bigvee_t^x(f_x)\frac{\beta}{(x-t)^{\beta+1}}dt\right]\\
&=B(x)\lambda^{-\gamma}\left[(x-a)^{-\beta}\bigvee_a^x(f_x)+\int_a^{x-(x-a)/\lambda^{\gamma/\beta}}\bigvee_t^x(f_x)\frac{\beta}{(x-t)^{\beta+1}}dt\right].
\end{aligned}
$$

Changing the variable t by $x-(x-a)\,/u^{1/\beta}$ in the last integral, we have

$$
\begin{aligned}
\int_a^{x-(x-a)/\lambda^{\gamma/\beta}}\bigvee_t^x(f_x)\frac{\beta}{(x-t)^{\beta+1}}dt &= \frac{1}{(x-a)^{\beta}}\int_1^{\lambda^{\gamma}}\bigvee_{x-(x-a)/u^{1/\beta}}^{x}(f_x)\;du\\
&\le \frac{1}{(x-a)^{\beta}}\sum_{k=1}^{[\lambda^{\gamma}]}\bigvee_{x-(x-a)/k^{1/\beta}}^{x}(f_x).
\end{aligned}
$$

Consequently, we obtain

$$
|I_1(\lambda,x)|\le B(x)\lambda^{-\gamma}(x-a)^{-\beta}\left[\bigvee_a^x(f_x)+\sum_{k=1}^{[\lambda^{\gamma}]}\bigvee_{x-(x-a)/k^{1/\beta}}^{x}(f_x)\right]. \tag{9.14}
$$

Using a similar method, we can find

$$|I_3(\lambda, x)| \le B(x)\lambda^{-\gamma}(b-x)^{-\beta}\left[\bigvee_x^b (f_x) + \sum_{k=1}^{[\lambda^\gamma]} \bigvee_x^{x+(b-x)/k^{1/\beta}} (f_x)\right]. \tag{9.15}$$

Combining (9.12), (9.14), and (9.15) in (9.11), we obtain

$$\begin{aligned}
\int_a^b L_\lambda(t-x)\,|f_x(t)|\,dt &\le B(x)\lambda^{-\gamma}(x-a)^{-\beta}\left[\bigvee_a^x (f_x) + \sum_{k=1}^{[\lambda^\gamma]} \bigvee_{x-(x-a)/k^{1/\beta}}^{x} (f_x)\right] \\
&\quad + B(x)\lambda^{-\gamma}(b-x)^{-\beta}\left[\bigvee_x^b (f_x) + \sum_{k=1}^{[\lambda^\gamma]} \bigvee_x^{x+(b-x)/k^{1/\beta}} (f_x)\right] \\
&\quad + B(x) \bigvee_{x-(x-a)/\lambda^{\gamma/\beta}}^{x+(b-x)/\lambda^{\gamma/\beta}} (f_x) \\
&\le B^*(x)\lambda^{-\gamma}\left[\bigvee_a^b (f_x) + \sum_{k=1}^{[\lambda^\gamma]} \bigvee_{x-(x-a)/k^{1/\beta}}^{x+(b-x)/k^{1/\beta}} (f_x)\right] \\
&\quad + B(x) \bigvee_{x-(x-a)/\lambda^{\gamma/\beta}}^{x+(b-x)/\lambda^{\gamma/\beta}} (f_x).
\end{aligned} \tag{9.16}$$

Collecting (9.10) and (9.16), we get the desired result. This completes the proof of the theorem. ■

Example 9.1. A special case of the function $K_\lambda(t,u)$ is linearity with respect to the second variable, namely,

$$K_\lambda(t,u) = Z_\lambda(t)\,u.$$

This case is widely used in approximation theory [49].

Example 9.2. If we choose $\Lambda = N$ and

$$K_n(t,u) = Z_n(t)\,u$$

and replace condition (3) with

$$\int_R Z_n(t-x)\,dt = 1$$

for every $n \in N$, we get the well-known operators such as Gauss–Weierstrass and Picard operators, among others(see [210]). The rates of convergence of these operators were investigated for functions of bounded variation (see also [49, 210]).

Example 9.3. $G = R$. Let the kernel function be defined by

$$K_\lambda(t,u) = \begin{cases} \frac{\lambda u}{2} + \sin\frac{\lambda u}{2}, & t \in [0, \frac{1}{\lambda}], \\ 0, & t \notin [0, \frac{1}{\lambda}], \end{cases} \tag{9.17}$$

where $\Lambda = [1, \infty)$ is a set of indices with natural topology, and $\lambda_0 = \infty$ is an accumulation point of Λ in this topology. According to (9.17), for every $u, v \in R$ and $t \in [0, \frac{1}{\lambda}]$,

$$\begin{aligned} |K_\lambda(t,u) - K_\lambda(t,v)| &\le \frac{\lambda}{2}|u-v| + 2\left|\sin\frac{\lambda(u-v)}{4}\cos\frac{\lambda(u+v)}{4}\right| \\ &\le \frac{\lambda}{2}|u-v| + 2\left|\sin\frac{\lambda(u-v)}{4}\right| \\ &\le \lambda|u-v|, \end{aligned} \tag{9.18}$$

and for every $u, v \in R$ and $t \notin [0, \frac{1}{\lambda}]$,

$$|K_\lambda(t,u) - K_\lambda(t,v)| = 0. \tag{9.19}$$

Moreover, $K_\lambda(t,0) = 0$.Combining (9.18) and (9.19), we find

$$L_\lambda(t) = \begin{cases} \lambda, & t \in [0, \frac{1}{\lambda}], \\ 0, & t \notin [0, \frac{1}{\lambda}], \end{cases}$$

and

$$\int_R L_\lambda(t)dt = \int_{[0,\frac{1}{\lambda}]} \lambda dt = 1 < \infty.$$

Furthermore,

$$\left[\int_R L_\lambda(t)\,dt - 1\right] = \left[\int_{[0,\frac{1}{\lambda}]} \lambda dt - 1\right] = 0 \le \frac{1}{\lambda},$$

for every $\lambda \in \Lambda = [1, \infty)$.

9.2 Chanturiya's Modulus of Variation

In 1991, using this modulus of variation, Aniol and Pych-Taberska [29] established approximation properties of the Gamma and Beta operators. Also, Pych-Taberska [202] estimated the rate of pointwise convergence of the Feller operators. In 1996, Pych-Taberska [200] estimated the rate of pointwise convergence of partial sums of Fourier series for some 2π-periodic Lebesgue-integrable functions. She estimated the main results by using Chanturiya's modulus of variation. In another paper, Pych-Taberska [201] estimated the rate of pointwise convergence of Bernstein and Kantorovich polynomials.

Let j be a positive integer, and suppose g is a function bounded on a finite or infinite interval Y contained in $[0, \infty)$. The variation of order j of the function g on Y is denoted by $v_j(g; Y)$ and defined as the upper bound of the set of all numbers

$$\sum_{k=1}^{j} |g(b_k) - g(a_k)|$$

over all systems of the j nonoverlapping intervals $(a_k, b_k), k = 1, 2, \ldots, j$, contained in Y. If $j = 0$, we take $v_0(g; Y) = 0$. The sequence $\{v_j(g; Y)\}_{j=0}^{\infty}$ is called the modulus of variation of the function g on Y. For basic properties, we refer to [52]. In case $Y = [a, b]$, we usually denote it as $v_j(g; a, b)$ instead of $v_j(g; Y)$.

Aniol and Taberska [30] obtained the rate of pointwise convergence for a general sequence of Durrmeyer-type operators. Let I be a finite or infinite interval, and suppose $M_{loc}(I)$ is the class of all measurable complex-valued locally bounded functions on I. Then, for $f \in M_{loc}(I)$ and $x \in I$, the Durrmeyer-type operators are defined as

$$L_n(f, x) = \sum_{j \in J_n} q_{j,n} p_{j,n}(x) \int_I p_{j,n}(t) f(t) dt,$$

where $J_n \subseteq Z := \{0, \pm 1, \pm 2, \ldots\}$ and $\int_I p_{j,n}(t) dt = 1/q_{j,n}$. Assume for every $x \in I$ $\rho_n(x) := \sum_{j \in J_n} p_{j,n}(x) - 1 \to 0$ as $n \to \infty$ and that $\mu_{2,n}(x) := L_n((t - x)^2, x) < \infty$. Then in view of the Shisha–Mond theorem (see [59]), at every point x of continuity of $f \in M(I)$ (the class of all measurable complex-valued functions bounded on I), with $\mu_{2,n}(x) \to 0$ as $n \to \infty$, we have

$$\lim_{n \to \infty} L_n(f, x) = f(x).$$

Denote

$$s(x) = \frac{1}{2}[f(x + 0) + f(x - 0)], r(x) = \frac{1}{2}[f(x + 0) - f(x - 0)],$$

and for $t \in I$, let

$$g_x(t) = \begin{cases} f(t) - f(x+0), & t \geq x, \\ 0, & t = x, \\ f(t) - f(x-0), & t < x. \end{cases}$$

Let $f \in M(I)$ [or $f \in M_{loc}(I)$] and consider $I_x(h) := [x+h, x] \cap I$. If $h < 0$, $I_x(h) := [x, x+h] \cap I$; if $h > 0$, then

$$L_n(f, x) - s(x) = L_n(g_x, x) + r(x)\Delta_n(x) + s(x)\rho_n(x), \tag{9.20}$$

where

$$\Delta_n(x) := \sum_{j \in J_n} q_{j,n} p_{j,n}(x) \left(\int_{t>x} p_{j,n}(t)dt - \int_{t \leq x} p_{j,n} t)dt \right).$$

In order to estimate $L_n(g_x, x)$, one may write

$$\begin{aligned} L_n(g_x, x) &= \sum_{j \in J_n} q_{j,n} p_{j,n}(x) \left(\int_{I_x(-a)} + \int_{I_x(b)} \right) g_x(t) p_{j,n}(t)dt \\ &\quad + V_x(a,b) \sum_{j \in J_n} q_{j,n} p_{j,n}(x) \int_{R_x(a,b)} g_x(t) p_{j,n}(t)dt, \end{aligned} \tag{9.21}$$

where $a > 0, b > 0$, $R_x(a,b) = I \setminus [x-a, x+b]$, $V_x(a,b) = 0$ if neither of the points $x-a, x+b$ belongs to I, and $V_x(a,b) = 1$ otherwise.

Lemma 9.2 ([30]). *Suppose $x \in I$ and that f is bounded on an interval $I_x(h), h \neq 0$. Choose a positive null sequence $\{d_n\}_1^\infty$ such that $d_n \leq \frac{1}{2}$ and write $m := [1/d_n]$. Then for every $n \in N$,*

$$\begin{aligned} &\left| \sum_{j \in J_n} q_{j,n} p_{j,n}(x) \int_{I_x(h)} g_x(t) p_{j,n}(t)dt \right| \\ &\qquad \leq P_n(x,h) \left\{ \sum_{i=1}^{m-1} \frac{1}{i^3} v_i(g_x; I_x(ihd_n)) + \frac{1}{m^2} v_m(g_x; I_x(h)) \right\}, \end{aligned}$$

where $P_n(x,h) := 1 + \rho_n(x) + 8\mu_{2,n}(x)h^{-2}d_n^{-2}$.

Proof. Let $h < 0$. Set the points $t_i = x + ihd_n, i = 1, 2, \ldots, l$, where l is the largest integer such that $t_i \in I_x(h)$, and denote by t_{l+1} the left endpoint of the interval $I_x(h)$. Put $T_i = [t_i, x]$ for $i = 1, 2, \ldots, l+1$, and write $g_x = g$. Then

$$\begin{aligned}
&\sum_{j\in J_n} q_{j,n} p_{j,n}(x) \int_{I_x(h)} g(t) p_{j,n}(t)dt \\
&= \sum_{j\in J_n} q_{j,n} p_{j,n}(x) \int_{t_1}^{x} g(t) p_{j,n}(t)dt \\
&\quad + \sum_{j\in J_n} q_{j,n} p_{j,n}(x) \sum_{i=1}^{l} g(t_i) \int_{t_{i+1}}^{t_i} p_{j,n}(t)dt \\
&\quad + \sum_{j\in J_n} q_{j,n} p_{j,n}(x) \sum_{i=1}^{l} \int_{t_{i+1}}^{t_i} (g(t) - g(t_i)) p_{j,n}(t)dt \\
&:= E_1 + E_2 + E_3.
\end{aligned}$$

Obviously,

$$|E_1| \le \sum_{j\in J_n} q_{j,n} p_{j,n}(x) \int_{t_1}^{x} |g(t) - g(x)| p_{j,n}(t)dt \le (1 + \rho_n(x)) v_1(g; T_1).$$

Next, by applying Abel's transformation and using $\sum_{j\in J_n} q_{j,n} p_{j,n}(x) \int_{|t-x|\ge s} p_{j,n}(t)dt \le \frac{1}{s^2}\mu_{2,n}(x), x \in I, s > 0$, we conclude that

$$|E_2| \le \frac{\mu_{2,n}(x)}{h^2 d_n^2} \left\{ 2\sum_{i=1}^{l} v_i(g; T_i) + \frac{1}{i^2} v_i(g; T_{i+1}) \right\}$$

and

$$|E_3| \le \frac{\mu_{2,n}(x)}{h^2 d_n^2} \left\{ 6\sum_{i=1}^{l} v_i(g; T_i) + \frac{1}{i^2} v_i(g; T_{i+1}) \right\} .$$

Combining the estimates of $E_1 - E_3$ and observing that $l \le m$ and $T_i \equiv I_x(ihd_n)$, we get the desired result for $h < 0$. The result holds well for $h > 0$; the proof runs analogously. ■

Theorem 9.2 ([30]). *Suppose that for all $x \in I$ and all $n \in N$, $\sum_{j\in J_n} p_{j,n}(x) \equiv 1 + \rho_n(x) \le \varphi_1(x)$, $\mu_{2,n}(x) \le \varphi_2(x) d_n^2$, where φ_1 and φ_2 are positive functions continuous on I and $\{d_n\}_1^\infty$ is a null sequence such that $0 < d_n \le \frac{1}{2}$. If $f \in M(I)$ and if at a point $x \in I$ the one-sided limits $f(x \pm 0)$ exist, then for all positive integers n, we have*

$$|L_n(f, x) - s(x)| \le P(x, a) \left\{ \sum_{i=1}^{m-1} \frac{1}{i^3} v_i(g_x; I_x(iad_n)) + \frac{1}{m^2} v_m(g_x; I_x(-a)) \right\}$$

$$+P(x,b)\left\{\sum_{i=1}^{m-1}\frac{1}{i^3}v_i(g_x;I_x(ibd_n))+\frac{1}{m^2}v_m(g_x;I_x(b)\right\}$$
$$+V_x(a,b)\varphi_2(x)c^{-2}d_n^2v_1(g_x;I)+|r(x)\Delta_n(x)|+|s(x)\rho_n(x)|,$$

where $a>0,b>0,c=\min\{a,b\},m=[1/d_n],andP(x,h)=\varphi_1(x)+8\varphi_2(x)/h^2$ *for* $h\neq 0$*.*

Theorem 9.3 ([30]). *Suppose* $I=[0,\infty)$ *or* $I=(-\infty,\infty)$*, and let* $\sum_{j\in J_n}p_{j,n}(x)\equiv 1+\rho_n(x)\le\varphi_1(x)$*,* $\mu_{2,n}(x)\le\varphi_2(x)d_n^2$*, where* φ_1 *and* φ_2 *are positive functions continuous on* I *and* $\{d_n\}_1^\infty$ *is a null sequence such that* $0<d_n\le\frac{1}{2}$*. If* $f\in M_{loc}(I)$ *satisfy the growth condition* $|f(x)|\le\phi(x),x\in I$ *with a positive continuous function* ϕ *such that for all* $n\ge n_0\in N$*,*

$$\sum_{j\in J_n}q_{j,n}p_{j,n}(x)\int_I\phi^2(t)p_{j,n}(t)dt\le\varphi_3(x),x\in I,0<\varphi_3(x)<\infty. \quad (9.22)$$

If at a point $x\in I$ *the one-sided limits* $f(x\pm 0)$ *exist, and if* A *is an arbitrary positive number for which* $|x|\le A$*, then for every integer* $n\ge n_)$*, we have*

$$|L_n(f,x)-s(x)|\le 2P(x,a)\left\{\sum_{i=1}^{m-1}\frac{1}{i^3}v_i(g_x;J_x(iAd_n))+\frac{1}{m^2}v_m(g_x;J_x(A))\right\}$$
$$+\Lambda(x,A)d_n+|r(x)\Delta_n(x)|+|s(x)\rho_n(x)|,$$

where $J_x(h)=I_x(-h)\cup I_x(h)\equiv[x-h,x+h]\cap I$ *for* $h>0$*,*

$$\Lambda(x,A):=A^{-1}(\varphi_2(x)\varphi_3(x))^{1/2}+\frac{1}{2}A^{-2}\phi(x)\varphi_2(x).$$

Proof. Take $A>0$ and write $L_n(g_x,x)$ in the form (9.21) with $a=b=A$; in this case, $R_x(a,b)=I\backslash J_x\backslash(A)$. In view of $|f(x)|\le\phi(x),x\in I$,

$$\left|\sum_{j\in J_n}q_{j,n}p_{j,n}(x)\int_{I\backslash J_x(A)}g_x(t)p_{j,n}(t)dt\right|$$
$$\le\sum_{j\in J_n}q_{j,n}p_{j,n}(x)\int_{I\backslash J_x(A)}\{\phi(t)+\phi(t)\}p_{j,n}(t)dt$$
$$\le\frac{1}{A}\sum_{j\in J_n}q_{j,n}p_{j,n}(x)\int_I\phi(t)|t-x|p_{j,n}(t)dt+\frac{1}{A^2}\phi(x)\mu_{2,n}(x)$$
$$\le\frac{1}{A}(\varphi_3(x)\varphi_2(x))^{1/2}d_n+\frac{1}{A^2}\phi(x)\varphi_2(x)d_n^2\le\Lambda(x,A)d_n.$$

By the Cauchy–Schwarz inequality, $\mu_{2,n}(x) \leq \varphi_2(x)d_n^2$ and (9.22). The above inequality, identities (9.20), (9.21), and Lemma 9.2 (with $h = -A$ and $h = A$) give the assertion immediately. ■

Remark 9.1. By the properties of the modulus of variation and in view of the continuity of g_x at x, we have

$$\lim_{m\to\infty}\sum_{i=1}^{m-1}\frac{1}{i^3}v_i(g_x; I_x(ihd_n)) = 0, \lim_{m\to\infty}\frac{1}{m^2}v_m(g_x; I_x(h)) = 0, h \neq 0.$$

The above result is valid if $I_x(ihd_n)$ and I_x are respectively replaced by $J_x(ihd_n)$ and J_x. Thus, for Durrmeyer-type operators L_n, $\lim_{n\to\infty}\Delta_n(x) = 0$ at every $x \in I$. Consequently, Theorems 9.2 and 9.3 are convergence results.

Aniol and Taberska [30] discussed the special cases of the above theorems and obtained explicit estimates for different operators, including Bernstein–Durrmeyer, Meyer–König–Zeller, Szász–Durrmeyer, and Baskakov–Durrmeyer operators.

In this direction, Gupta and Taberska [121] established the rate of convergence for Baskakov–Beta operators in terms of Chanturiya's modulus of variation. Let $I = [0,\infty)$. Then, for $f \in M_{loc}(I)$, the Baskakov–Beta operators are defined as

$$B_n(f,x) = \sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} b_{n,k}(t)f(t)dt,$$

where

$$p_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}},\ b_{n,k}(t) = \frac{1}{B(k+1,n)}\frac{t^k}{(1+t)^{n+k+1}}.$$

Theorem 9.4 ([121]). *Let $f \in M_{loc}(I)$, and at a fixed point $x \in (0,\infty)$, let the one-sided limits $f(x\pm)$ exist. Suppose, moreover, that $|f(t)| \leq M(1+t)^{\alpha}$, for $t \in I$, where $M > 0, \alpha \in N^0$, and choose a number $\lambda > 2$. Then, for $n \geq \max\{1+\alpha, N(\lambda,x)\}$, we have*

$$\begin{aligned}&\left|B_n(f,x) - \frac{1}{2}(f(x+)+f(x-))\right|\\ &\leq Q(x)\left\{\sum_{j=1}^{m-1}\frac{v_j(g_x; x - jx/\sqrt{n}) + v_j(g_x; x + jx/\sqrt{n})}{j^3}\right.\\ &\quad\left.+\frac{v_m(g_x;0,x)+v_m(g_x;x,2x)}{m^2}\right\} + \lambda M K_{\alpha}\frac{(1+x)^{\alpha+1}}{nx}\\ &\quad+\frac{4\sqrt{1+9x(1+x)}+1}{5\sqrt{nx(1+x)}}|f(x+)-f(x-)|,\end{aligned}$$

where $m = [\sqrt{n}]$, $Q(x) = 1 + 8\lambda(1 + x)/x$,, $N(\lambda, x) > 2$, *and* K_α *is a positive constant depending only on* α, *in particular,* $K - 0 = 2, K_1 = 3, K_3 = 5$.

The preceding theorem enables us to obtain estimates for the rate of convergence of $B_n(f, x)$ for functions f of bounded variation in the Jordan sense or in the generalized sense, specifically if we choose $p \geq 1$ and let $V_p(g; a, b)$ denote the total pth-power variation of a function g on an interval $[a, b] \subset I$, defined as the upper bound of the set of all numbers

$$\left(\sum_k |g(b_k) - g(a_k)|^p \right)^{1/p}$$

over all finite systems of non-overlapping intervals $(a_k, b_k) \subset [a, b]$. Clearly, if $V_p(g; a, b) < \infty$, then for every positive integer j,

$$v_j(g; a, b) \leq j^{1-1/p} V_p(g; a, b).$$

Using the above inequality, we get the following corollary.

Corollary 9.1 ([121]). *Let* f *be a function of bounded pth-power variation* $(p \geq 1)$ *on every finite subinterval of* $[0, \infty)$, *and let it satisfy the growth* $|f(t)| \leq M(1 + t)^\alpha$, *for* $t \in I$, *where* $M > 0, \alpha \in N^0$. *Then, for every* $x \in (0, \infty)$ *and all* $n \geq \{1 + \alpha, N(\lambda, x)\}$, *we have*

$$\begin{aligned}\left| B_n(f, x) - \frac{1}{2}(f(x+) + f(x-)) \right| \leq{}& \frac{8Q(x)}{(\sqrt{n})^{1+1/p}} \sum_{k=1}^{n} \left(\frac{1}{\sqrt{k}} \right)^{1-1/p} \\ & p(g_x; x - x/\sqrt{k}, x + x/\sqrt{k}) \\ & + \lambda M K_\alpha \frac{(1 + x)^{\alpha+1}}{nx} \\ & + \frac{4\sqrt{1 + 9x(1 + x)} + 1}{5\sqrt{nx(1 + x)}} |f(x+) - f(x-)|,\end{aligned}$$

where λ, $N(\lambda, x)$, $Q(x)$, *and* K_α *have the same meanings as in Theorem 9.4.*

9.3 Functions with Derivatives of Bounded Variation

The rate of convergence for functions having derivatives of bounded variation is also an interesting area of research. Bojanic and Cheng [44] investigated the asymptotic behavior of Hermite–Fejér polynomials $H_n(f, x)$ [see (7.5)] for functions defined by

$$f(x) = f(-1) + \int_{-1}^{x} \varphi(t)dt, x \in [-1, 1], \tag{9.23}$$

where φ is a function of bounded variation on $[-1, 1]$. This class of functions can be described as the class of differentiable functions whose derivatives are of bounded variation. Usually, the class is denoted as $DBV[-1, 1]$ in this case. It is clear that this class of functions is much more general than functions with continuous derivatives.

Theorem 9.5 ([44]). *Let f be a function in $DBV[-1, 1]$ and $\varphi \in BV[-1, 1]$, so that (9.23) is satisfied. Then, for $x \in (-1, 1)$ such that $x \neq x_{kn}$ for $k = 1, 2, 3, \ldots, n$, we have*

$$\left| H_n(f, x) - f(x) - \frac{\sigma(1-x^2)^{1/2} T_n^2(x)}{\pi} \frac{\log n}{n} \right|$$
$$\leq \frac{C(|\sigma| + |\lambda|)}{2} \frac{|T_n(x)|}{n} + \frac{\pi |T_n(x)|}{n} V_{x-\pi[T_n(x)]/n}^{x+\pi[T_n(x)]/n}(\varphi_x)$$
$$+ \frac{12 T_n^2(x)}{n} \sum_{k=1}^{n} \frac{V_{x-\pi/k}^{x+\pi/k}(\varphi_x)}{k},$$

where $V_a^b(\varphi_x)$ is the total variation of φ_x on $[a, b]$, $\sigma = \varphi(x+) - \varphi(x-)$, $\lambda = \varphi(x+) + \varphi(x-)$, and $C = 7 + \pi$.

Bojanic and Cheng [45] investigated the asymptotic behavior of Bernstein polynomials for functions defined as

$$f(x) = f(0) + \int_{0}^{x} \psi(t)dt, x \in [0, 1], \tag{9.24}$$

where ψ is a function of bounded variation on $[0, 1]$. This class of functions can be defined as

$$DBV[0, 1] = \{f : f' \in BV[0, 1]\}.$$

For the Bernstein polynomials $B_n(f, x)$ defined by (7.7), Bojanic and Cheng [45] estimated the rate of convergence of $B_n(f, x)$ to $f(x)$.

Theorem 9.6 ([45]). *Let f be a function in $DBV[0, 1]$ and $\psi \in BV[0, 1]$, so that (9.24) is satisfied. Then, for any $x \in (0, 1)$ and $n \geq 1/x(1-x)$, we have*

$$\left| B_n(f, x) - f(x) - \sigma \left(\frac{2x(1-x)}{2\pi} \right)^{1/2} \right|$$
$$\leq \frac{\sigma}{2} \frac{M}{n(x(1-x))^{1/2}} + \frac{2}{n} \sum_{k=1}^{[\sqrt{n}]} V_{x-x/k}^{x+(1-x)/k}(\Psi_x), \tag{9.25}$$

where $[\sqrt{n}]$ *is the greatest integer less than or equal to* $\sqrt{n}$, $V_a^b(\Psi_x)$ *is the total variation of* Ψ_x *on* $[a,b]$, $\sigma = f'(\xi+) - f'(\xi-)$, *and* $M = \left(\left(\frac{\pi}{2}\right)^{5/2} + \frac{4}{\pi} + \frac{\pi^4\sqrt{\pi}}{54\sqrt{2}}\right)$. *If* f' *is continuous at* x, *that is,* $\sigma = 0$, *then* (9.25) *can be simplified as*

$$|B_n(f,x) - f(x)| \leq \frac{2}{n}\sum_{k=1}^{[\sqrt{n}]} V_{x-x/k}^{x+(1-x)/k}(\Psi_x),$$

which is the best possible and cannot be improved any further asymptotically.

Bai et al. [32] also worked in this direction and estimated the rate of convergence for general operators. Gupta et al. [141] estimated the rate of convergence for functions having DBV for Beta operators of the second kind. Ispir et al. [159] estimated the convergence rate for the Kantorovich-type operators. This section includes additional results on the rate of convergence for functions having derivatives of bounded variations.

1. **Stancu–Beta Operators** For Lebesgue integrable functions f on the interval $I = (0,\infty)$, Beta operators L_n of the second kind, introduced in [223], are given by

$$(L_n f)(x) = \frac{1}{(nx, n+1)}\int_0^\infty \frac{t^{nx-1}}{(1+t)^{nx+n+1}} f(t)dt. \tag{9.26}$$

Obviously, the operators L_n are positive linear operators on the space of locally integrable functions on I of polynomial growth as $t \to \infty$, provided that n is sufficiently large.

This section discusses the study of operators (9.26). We estimate their rate of convergence by the decomposition technique for absolutely continuous functions f of polynomial growth as $t \to \infty$, having a derivative f' coinciding a.e. with a function that is of bounded variation on each finite subinterval of I. For the sake of convenient notation in the proofs, we rewrite the operators (9.26) as

$$(L_n f)(x) = \int_0^\infty K_n(x,t) f(t)dt, \tag{9.27}$$

where the kernel function K_n is given by

$$K_n(x,t) = \frac{1}{(nx, n+1)}\frac{t^{nx-1}}{(1+t)^{nx+n+1}}.$$

Moreover, we put

$$\lambda_n(x,y) = \int_0^y K_n(x,t)dt, y \geq 0.$$

Note that $0 \leq \lambda_n(x, y) \leq 1, y \geq 0$. For fixed $x \in I$, define the function ψ_x by $\psi_x(t) = t - x$. The central moments for the operators L_n are given by

$$(L_n\psi_x^0)(x) = 1, (L_n\psi_x^1)(x) = 0, (L_n\psi_x^2)(x) = \frac{x(1+x)}{n-1}. \tag{9.28}$$

Moreover, let $x \in I$ be fixed. For $r = 0, 1, 2, \ldots$ and $n \in N$, the central moments for the operators L_n satisfy

$$(L_n\psi_x^r)(x) = O(n^{-[(r+1)/2]}), n \to \infty.$$

In view of this, an application of the Schwarz inequality, for $r = 0, 1, 2, \ldots$, yields

$$(L_n|\psi_x^r|)(x) \leq \sqrt{(L_n\psi_x^{2r})(x)} = O(n^{-r/2}), n \to \infty. \tag{9.29}$$

In particular, by (9.28), we have

$$(L_n|\psi_x|)(x) \leq \sqrt{\frac{x(1+x)}{n-1}}. \tag{9.30}$$

Lemma 9.3. *Let $x \in I$ be fixed and $K_n(x,t)$ be defined in (9.27). Then, for $n \geq 2$, we have*

$$\lambda_n(x, y) = \int_0^y K_n(x,t)dt \leq \frac{x(1+x)}{(n-1)(x-y)^2}(0 \leq y < x),$$

$$1 - \lambda_n(x, z) = \int_z^\infty K_n(x,t)dt \leq \frac{x(1+x)}{(n-1)(z-x)^2}(x < z < \infty).$$

For $r \geq 0$, let $DB_r(I)$ be the class of all absolutely continuous functions f defined on I

(i) having on I a derivative f' coinciding a.e. with a function that is of bounded variation on each finite subinterval of I,
(ii) satisfying $f(t) = O(t^r)$ as $t \to \infty$. Note that all functions $f \in DB_r(I)$ possess, for each $a > 0$, a representation

$$f(x) = f(a) + \int_a^x \psi(t)dt, x > a,$$

with a function ψ of bounded variation on each finite subinterval of I.

Theorem 9.7 ([141]). *Let $r \in N, x \in I$, and $f \in DB_r(I)$. Then it holds that*

$$
\begin{aligned}
&|(L_n f)(x) - f(x)| \\
&\le \frac{1}{2}\sqrt{\frac{x(1+x)}{n-1}}|f'(x+) - f'(x-)| + \frac{x}{\sqrt{n}} V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}((f')_x) \\
&+ \frac{1+x}{n-1}\left(\sum_{k=1}^{[\sqrt{n}]} V_{x-x/k}^{x+x/k}((f')_x) + x^{-1}|f(2x) - f(x)| + 2|f'(x+)|\right) \\
&+ \frac{c_{r,x} M_{r,x}(x)}{n^{r/2}},
\end{aligned}
$$

where $\bigvee_a^b f(x)$ denotes the total variation of f_x on $[a,b]$, and f_x is defined by

$$
f_x(t) = \begin{cases} f(t) - f(x^-), & 0 \le t < x, \\ 0, & t = x, \\ f(t) - f(x^+), & x < t < \infty, \end{cases}
$$

and the constants $c_{r,x}$ and $M_{r,x}(f)$ are given by

$$
c_{r,x} = \sup_{n \in N} \sqrt{n^r (L_n \psi_x^{2r})(x)}, M_{r,x}(f) = 2^r \sup_{t \ge 2x} t^{-r}|f(t) - f(x)|.
$$

Proof. For $x \in I$, we have

$$
(L_n f)(x) - f(x) = \int_0^\infty K_n(x,t)(f(t) - f(x))dt = \int_0^\infty K_n(x,t)\int_x^t f'(u)du dt.
$$

Now we take advantage of the identity

$$
\begin{aligned}
f'(u) = (f')_x(u) &+ \frac{1}{2}(f'(x+) + f'(x-)) + \frac{1}{2}(f'(x+) - f'(x-))sign(u-x) \\
&+ \left(f'(x) - \frac{1}{2}(f'(x+) - f'(x-))\right)\chi_x(u),
\end{aligned}
$$

where $\chi_x(u) = 1(u = x)$ and $\chi_x(u) = 0(u \ne x)$. Obviously, we have

$$
\int_0^\infty K_n(x,t)\int_x^t \left(f'(x) - \frac{1}{2}(f'(x+) - f'(x-))\right)\chi_x(u)du dt = 0.
$$

Furthermore, by (9.28) and (9.30), respectively, we have

$$\int_0^\infty K_n(x,t)\int_x^t \frac{1}{2}(f'(x+)+f'(x-))dudt = \frac{1}{2}(f'(x+)+f'(x-))$$
$$\int_0^\infty K_n(x,t)(t-x)dt = 0,$$
$$\left|\int_0^\infty K_n(x,t)\int_x^t \frac{1}{2}(f'(x+)+f'(x-))sign(u-x)dudt\right|$$
$$\leq \frac{1}{2}\sqrt{\frac{x(1+x)}{n-1}}|f'(x+)+f'(x-)|.$$

Collecting the above relations, we obtain the estimate

$$|(L_n f)(x)-f(x)| \leq |A_n(f,x)+B_n(f,x)+C_n(f,x)| + \frac{1}{2}\sqrt{\frac{x(1+x)}{n-1}}|f'(x+)+f'(x-)|, \tag{9.31}$$

where

$$A_n(f,x) = \int_0^x K_n(x,t)\int_x^t (f')_x(u)dudt,$$
$$B_n(f,x) = \int_x^{2x} K_n(x,t)\int_x^t (f')_x(u)dudt,$$
$$C_n(f,x) = \int_{2x}^\infty K_n(x,t)\int_x^t (f')_x(u)dudt.$$

Using integration by parts and applying Lemma 9.3 yields

$$\begin{aligned}|A_n(f,x)| &= \left|\int_0^x \int_x^t (f')_x(u)dud_t\lambda_n(x,t)\right| \\ &\leq \left(\int_0^{x-x/\sqrt{n}} + \int_{x-x/\sqrt{n}}^x\right)|\lambda_n(x,t)|V_t^x((f')_x)dt \\ &\leq \frac{x(1+x)}{n-1}\int_0^{x-x/\sqrt{n}}(x-t)^{-2}V_t^x((f')_x)dt + \frac{x}{\sqrt{n}}V_{x-x/\sqrt{n}}^x((f')_x).\end{aligned}$$

By the substitution of $u = x/(x-t)$, we obtain

$$|A_n(f,x)| \leq \frac{(1+x)}{n-1}\sum_{k=1}^{[\sqrt{n}]} V_{x-x/k}^x((f')_x)dt + \frac{x}{\sqrt{n}}V_{x-x/\sqrt{n}}^x((f')_x). \tag{9.32}$$

Furthermore, we have

$$
\begin{aligned}
|B_n(f,x)| &\le \left|-\int_x^{2x}\int_x^t (f')_x(u)du\,d_t(1-\lambda_n(x,t))\right| \\
&\le \left|\int_x^{2x}(f')_x(u)du\right||1-\lambda_n(x,2x)| + \int_x^{2x}|(f')_x(t)|.|1-\lambda_n(x,t)|dt \\
&\le \frac{(1+x)}{(n-1)x}|f(2x)-f(x)-xf'(x+)| + \int_x^{x+x/\sqrt{n}} V_x^t((f')_x)dt \\
&\quad + \frac{x(1+x)}{(n-1)}\int_{x+x/\sqrt{n}}^{2x}(t-x)^{-2}V_x^t((f')_x)dt.
\end{aligned}
$$

By the substitution of $u = x/(t-x)$, we obtain

$$
\begin{aligned}
|B_n(f,x)| \le \frac{(1+x)}{(n-1)x}|f(2x)-f(x)-xf'(x+)| + \frac{(1+x)}{(n-1)}\sum_{k=1}^{[\sqrt{n}]} V_x^{x+x/k}((f')_x) \\
+\frac{x}{\sqrt{n}}V_x^{x+x/\sqrt{n}}((f')_x). \qquad (9.33)
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
|C_n(f,x)| &= \left|\int_{2x}^{\infty} K_n(x,t)(f(t)-f(x)-(t-x)f'(x+))dt\right| \\
&\le 2^{-r}M_{r,x}(f)\int_{2x}^{\infty}K_n(x,t)t^r\,dt + |f'(x+)|\int_{2x}^{\infty}K_n(x,t)|t-x|dt.
\end{aligned}
$$

Using the obvious inequalities $t \le 2(t-x)$ and $x \le t-x$ for $t \ge 2x$, and applying (9.29), we obtain

$$
|C_n(f,x)| \le M_{r,x}(f)c_{r,x}n^{-r/2} + \frac{1+x}{n-1}|f'(x+)|. \qquad (9.34)
$$

Combining the estimates (9.31)–(9.34), we get the desired result. This completes the proof of the theorem. ■

2. **Kantorovich Operators** The Kantorovich-type operators discussed in [159] are defined as

$$
(G_n^* f)(x) = n^2\psi_n(0)\sum_{v=0}^{\infty}P_v(\alpha_n,\psi_n;K_n)\int_{I_{n,v}}f(t)dt, \qquad (9.35)
$$

where $I_{n,v} := [v/n^2\psi_n(0), (v+1)/n^2\psi_n(0)], n \in N, v \in N_0$, $P_v(\alpha_n, \psi_n; K_n) = \left(\frac{\partial^v}{\partial u^v} K_n(x,t,u)|_{u=\alpha_n\psi_n(t),t=0}\right) \frac{(-\alpha_n\psi_n(0))^v}{v!}$, and $f \in M_{loc}[0,\infty)$, the class of all measurable functions on $[0,\infty)$ and bounded on every compact subinterval of $[0,\infty)$. The operators (9.35) can be written alternatively as

$$(G_n^* f)(x) = \int_0^\infty W_n(x,t) f(t) dt,$$

where $W_n(x,t) = n^2\psi_n(0) \sum_{v=0}^{\infty} P_v(\alpha_n, \psi_n; K_n) \chi_{n,v}(t)$, and $\chi_{n,v}$ is the characteristic function of the interval $I_{n,v}$ with respect to $[0,\infty)$.

Theorem 9.8 ([159]). *Let $r \in N, x \in I$, and $f \in DB_r(0,\infty)$. Then, for n sufficiently large, it holds that*

$$\begin{aligned}
&|(G_n^* f)(x) - f(x)| \\
\le\ & \frac{\eta}{n^2\psi_n(0)}(x+1)|f'(x+) + f'(x-)| \\
&+ \sqrt{\frac{\lambda}{n^2\psi_n(0)}}(x+1)\left(\left|\frac{f'(x+) - f'(x-)}{2}\right| + x^{-1}|f'(x+)|\right) \\
&+ \lambda\frac{(x+1)^2}{n^2\psi_n(0)}\left(\sum_{v=1}^{[\sqrt{n}]} V_{x-x/x}^{x+x/v}((f')_x) + x^{-1}|f(2x) - f(x)| + |f'(x+)|\right) \\
&+ \frac{x}{\sqrt{n}} V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}((f')_x) + T_{\gamma,x}(f) k_{\gamma,x} n^{-\gamma/2},
\end{aligned}$$

where $T_{\gamma,x}(f) = 2^\gamma \sup_{t\ge 2x} t^{-\gamma}|f(t) - f(x)|$, $k_{\gamma,x} = \sup_{n\in N}\sqrt{n^\gamma G_n * ((e_1 - xe_0)^{2\gamma}, x)}$, $V_a^b(f)$ denotes the total variation of f, and the auxiliary function is as defined in Theorem 9.7.

9.4 Convergence for Bounded and Absolutely Continuous Functions

In this section, we present approximation properties on locally bounded functions and the absolutely continuous functions, which were studied in [249].

Let f be a function defined on $[0,\infty)$ and satisfying the following growth condition:

$$|f(t)| \le Me^{\beta t} \qquad (M > 0, \beta \ge 0, t \to \infty). \tag{9.36}$$

Then, for such functions, the Gamma operators $G_n(f, x)$ are defined as

$$G_n(f, x) = \frac{1}{x^n \Gamma(n)} \int_0^{+\infty} f(t/n) t^{n-1} e^{-t/x} dt. \tag{9.37}$$

The class of locally bounded functions Φ_B and the class of absolutely continuous functions Φ_{DB} are defined as follows:

$$\Phi_B = \{f : f \text{ is bounded on every finite subinterval of } [0, \infty)\},$$

$$\Phi_{DB}=\{f : f(x)-f(0)=\int_0^x h(t)dt; h \text{ is bounded on every finite subinterval of } [0, \infty)\}.$$

Furthermore, for a function $f \in \Phi_B$, we introduce the following metric form:

$$\Omega_x(f, \lambda) = \sup_{t\in[x-\lambda, x+\lambda]} |f(t) - f(x)|,$$

where $x \in [0, \infty)$ is fixed, $\lambda \geq 0$. The rate of convergence for functions belonging to the class of locally bounded functions for Gamma operators is given as follows:

Theorem 9.9 ([249]). *Let $f \in \Phi_B$ and let $f(t) = O(e^{\beta t})$ for some $\beta \geq 0$ as $t \to \infty$. If $f(x+)$ and $f(x-)$ exist at a fixed point $x \in (0, \infty)$, then for $n > 4\beta x$, we have*

$$\left|G_n(f, x) - \frac{f(x+) + f(x-)}{2} + \frac{f(x+) - f(x-)}{3\sqrt{2\pi n}}\right|$$
$$\leq \frac{5}{n} \sum_{k=1}^{n} \Omega_x(g_x, x/\sqrt{k}) + O(n^{-1}),$$

$$g_x(t) = \begin{cases} f(t) - f(x+), & x < t < \infty, \\ 0, & t = x, \\ f(t) - f(x-), & 0 \leq t < x. \end{cases}$$

Corollary 9.2 ([249]). *Let f be a function of bounded variation on every subinterval of $[0, \infty)$, and let $f(t) = O(e^{\beta t})$ for some $\beta \geq 0$ as $t \to \infty$. Then, for $x \in (0, \infty)$ and $n > 4\beta x$, we have*

$$\left|G_n(f, x) - \frac{f(x+) + f(x-)}{2} + \frac{f(x+) - f(x-)}{3\sqrt{2\pi n}}\right|$$
$$\leq \frac{5}{n} \sum_{k=1}^{n} \Omega_x(g_x, x/\sqrt{k}) + O(n^{-1}) \leq \frac{5}{n} \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}} (g_x) + O(n^{-1}).$$

Corollary 9.3 ([249]). *Under the conditions of Theorem (9.9), if* $\Omega_x(g_x, \lambda) = o(\lambda)$, *then*

$$G_n(f, x) = \frac{f(x+) + f(x-)}{2} + \frac{f(x+) - f(x-)}{3\sqrt{2\pi n}} + o(n^{-1/2}).$$

The next result on the rate of convergence of Gamma operators is for absolutely continuous functions.

Theorem 9.10 ([249]). *Let* f *be a function in* Φ_{DB}, *and let* $f(t) \leq Me^{\beta t}$ *for some* $M > 0$ *and* $\beta \geq 0$ *as* $t \to \infty$. *If* $h(x+)$ *and* $h(x-)$ *exist at a fixed point* $x \in (0, \infty)$, *then for* $n > 4\beta x$, *we have*

$$\left| G_n(f, x) - f(x) - \frac{\tau x}{\sqrt{2\pi n}} \right|$$
$$\leq \frac{x}{n} \sum_{k=1}^{[\sqrt{n}]} \Omega_x(g_x, x/\sqrt{k}) + \frac{|\tau| x + 17M(2e)^{\beta x}}{n^{3/2}},$$

where $\tau = h(x+) - h(x-)$ *and*

$$\phi_x = \begin{cases} h(t) - h(x+), & x < t < \infty, \\ 0, & t = x, \\ h(t) - h(x-), & 0 \leq t < x. \end{cases}$$

Remark 9.2. If f is a function with a derivative of bounded variation, then $f \in \Phi_{DB}$. Thus, the approximation of functions with derivatives of bounded variation is a special case of Theorem 9.10.

In 2012, Zeng [250] obtained an exact estimate for the first-order absolute moment of Stancu–Beta operators using the Stirling formula and established an estimate on convergence for Stancu–Beta operators for absolutely continuous functions.

For Lebesgue integrable functions f on the interval $(0, \infty)$, the Stancu–Beta operators introduced in [223] are defined by

$$L_n(f, x) = \frac{1}{B(nx, n+1)} \int_0^\infty \frac{t^{nx-1}}{(1+t)^{nx+n+1}} f(t) dt.$$

The class of functions Φ_{DB} is defined as

$$\Phi_{DB} = \{f : f(x) - f(0) = \int_0^x \phi(t) dt; \phi \text{ is bounded on}$$
$$\text{every finite subinterval of } [0, \infty); f(t) = O(t^r), t \to \infty\}.$$

Furthermore, for a function $f \in \Phi_B$, the metric is defined as

$$\omega_x(f, \lambda) = \sup_{t \in [x-\lambda, x+\lambda] \cap [0,\infty)} |f(t) - f(x)|,$$

where $x \in [0, \infty)$ is fixed, $\lambda \geq 0$.

Theorem 9.11 ([250]). *Let f be a function in Φ_{DB}. If $\phi(x+)$ and $\phi(x-)$ exist at a fixed point $x \in (0, \infty)$, we write $\tau = (\phi(x+) - \phi(x-))/2$. Then for $n \geq 2$, we have*

$$\left| L_n(f, x) - f(x) - \tau \left(\frac{2x(1+x)}{n\pi} \right)^{1/2} \right|$$

$$\leq \frac{3+7x}{n-1} \sum_{k=1}^{[\sqrt{n}]} \omega_x(g_x, x/k) + \frac{(1+x)^{3/2}}{\sqrt{72x\pi}} n^{-3/2} + \frac{(2x)^r}{x^{2m}} O(n^{-[(m+1)/2]}),$$

where

$$\phi_x(t) = \begin{cases} \phi(t) - \phi(x+), & x < t < \infty, \\ 0, & t = x, \\ \phi(t) - \phi(x-), & 0 \leq t < x. \end{cases}$$

Chapter 10
Rate of Convergence in Simultaneous Approximation

In the theory of approximation, the study of the rate of convergence in simultaneous approximation is also an interesting area of research. Several researchers have worked in this direction; some of them have obtained the rate of convergence for bounded/bounded variation functions in simultaneous approximation.

Gupta et al. [142] estimated the rate of convergence for the well-known Szász–Mirakyan–Durrmeyer operators $M_n(f, x)$ defined in (2.17). They considered the class $B_{r,\alpha}$ by

$$B_{r,\alpha} = \{f : f^{(r-1)} \in C[0,\infty),\ f_{\pm}^{(r)}(x) \text{ exist everywhere and are bounded on every finite subinterval of } [0,\infty) \text{ and } f_{\pm}^{(r)}(t) = O(e^{\alpha t})(t \to \infty) \text{ for some } \alpha > 0\}.$$

Theorem 10.1 ([142]). *Let $f \in B_{r,\alpha}, r \in N^0$. Then, for every $x \in (0,\infty)$ and $n \geq \max\{r^2 + 3r + 2, 4\alpha\}$, we have*

$$\begin{aligned}\left|M_n^{(r)}(f,x) - \frac{1}{2}[f_+^{(r)}(x) + f_-^{(r)}(x)]\right| \leq{}& \frac{x^2 + 3(1+2x)}{nx^2} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x)/\sqrt{k}}(g_{r,x}) \\ &+ \frac{|2r+1|}{\sqrt{8enx}} |f_+^{(r)}(x) - f_-^{(r)}(x)| \\ &+ 2^{(r-1)/2} e^{2\alpha x} \sqrt{\frac{2x+1}{nx^2}},\end{aligned}$$

where the auxiliary function is defined as

$$g_{r,x}(t) = \begin{cases} f^{(r)}(t) - f_-^{(r)}(x), & 0 \leq t < x, \\ 0, & t = x, \\ f^{(r)}(t) - f_+^{(r)}(x), & x < t < \infty. \end{cases}$$

$V_a^b(g_{r,x}(t))$ is the total variation of $g_{r,x}(t)$ on $[a,b]$. In particular, $g_{0,x}(t) = g_x(t)$.

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10.1 Bernstein–Durrmeyer–Bézier-Type Operators

The family of Durrmeyer-type operators discussed in [115] is defined as

$$P_{n,r}(f,x)=\begin{cases} n\sum_{k=1}^{n} p_{n,k}(x)\int_0^1 p_{n-1,k-1}(t)f(t)dt+p_{n,0}(x)f(0), & r=0,\\ \dfrac{n^{\underline{r}}}{(n+r-1)^{\underline{r-1}}}\sum_{k=0}^{n-r} p_{n-r,k}(x)\int_0^1 p_{n+r-1,k+r-1}(t)f(t)dt, & r>0,\end{cases}$$

where $r,n\in N_0$ with $r\le n$, and for any $a,b\in N_0$, $a^{\underline{b}}=a(a-1)\cdots(a-b+1)$, $a^{\underline{0}}=1$ is the falling factorial. The operators of this family are linear and positive. Notice that the family $P_{n,\alpha}$, as a particular case ($r=0$), contains the sequence $P_{n,0}$ introduced by Srivastava and Gupta in [218]. For any $\alpha\ge 1$, the Bézier modification of the family $P_{n,r}$ is given by

$$P_{n,r,\alpha}(f,x)=\begin{cases} n\sum_{k=1}^{n} Q_{n,k}^{(\alpha)}(x)\int_0^1 p_{n-1,k-1}(t)f(t)dt+Q_{n,0}^{(\alpha)}(x)f(0), & r=0,\\ \dfrac{n^{\underline{r}}}{(n+r-1)^{\underline{r-1}}}\sum_{k=0}^{n-r} Q_{n-r,k}^{(\alpha)}(x)\int_0^1 p_{n+r-1,k+r-1}(t)f(t)dt, & r>0,\end{cases}$$

where $Q_{n,k}^{(\alpha)}(x)=J_{n,k}^{\alpha}(x)-J_{n,k+1}^{\alpha}(x)$ and $J_{n,k}(x)=\sum_{j=k}^{n} p_{n,j}(x)$.

This new family of Bézier-type operators, for $\alpha=1$, again yields the family $P_{n,r}$. In particular, $P_{n,0,\alpha}$ is the sequence introduced by Gupta and Maheshwari in [118].

The aim of this section is to study the approximation properties of the operators $P_{n,r,\alpha}$ in the space of functions

$$H=\{f:[0,1]\to R: f_{\pm}(x) \text{ exists everywhere and be bounded on } [0,1]\}.$$

Usually, the rate of approximation for this kind of linear positive operator is analyzed for the class of continuous functions or, in a more general setting, for the class of functions of bounded variation (see, for example, [118, 252]). We would like to stress that the space H is larger than the space of bounded variation functions on $[0,1]$. For instance, the function

$$f(x)=\begin{cases} 0, & \text{if } x=0,\\ x\sin\left(\dfrac{1}{x^2}\right), & \text{if } 0<x\le 1,\end{cases}$$

is not of bounded variation, but $f\in H$. Moreover, as a consequence of our results for $P_{n,r,\alpha}$, we will also establish estimates of the simultaneous approximation error for the operators $P_{n,r}$. The operators $P_{n,r,\alpha}$ also admit the integral representation

$$P_{n,r,\alpha}(f,x) = \int_0^1 K_{r,n,\alpha}(x,t) f(t)dt,$$

where the kernel $K_{r,n,\alpha}$ is given by

$$K_{r,n,\alpha}(x,t) = \begin{cases} n \sum_{k=1}^{n} Q_{n,k}^{(\alpha)}(x) p_{n-1,k-1}(t) + Q_{n,0}^{(\alpha)}(x)\delta(t), & r = 0, \\ \dfrac{n^{\underline{r}}}{(n+r-1)^{\underline{r-1}}} \sum_{k=0}^{n-r} Q_{n-r,k}^{(\alpha)}(x) p_{n+r-1,k+r-1}(t), & r > 0. \end{cases}$$

Here, $\delta(t)$ denotes the Dirac delta function. This section is based on [115].

For the class of bounded functions on the interval $[0,1]$, we consider the following quantities:

$$\Omega_{x-}(f,\delta_1) = \sup_{t\in[x-\delta_1,x]} |f(t) - f(x)|,$$

$$\Omega_{x+}(f,\delta_2) = \sup_{t\in[x,x+\delta_2]} |f(t) - f(x)|,$$

$$\Omega_{x}(f,\lambda) = \sup_{t\in[x-x/\lambda,x+(1-x)/\lambda]} |f(t) - f(x)|,$$

where $x \in [0,1]$ and $0 \le \delta_1 \le x$, $0 \le \delta_2 \le 1-x$, and $\lambda \ge 1$. We summarize several properties of these moduli in the following items:

i. $\Omega_{x-}(f,\delta_1)$ and $\Omega_{x+}(f,\delta_2)$ are monotone nondecreasing with respect to δ_1 and δ_2, respectively. Also, $\Omega_x(f,\lambda)$ is monotone nonincreasing with respect to λ.
ii. $\lim_{\delta_1\to 0+} \Omega_{x-}(f,\delta_1) = 0$, $\lim_{\delta_2\to 0+} \Omega_{x+}(f,\delta_2) = 0$, and $\lim_{\lambda\to\infty} \Omega_x(f,\lambda) = 0$ if f is continuous on the left, continuous on the right, or continuous at the point x, respectively.
iii. $\Omega_{x-}(f,\delta_1) \le \Omega_x(f,x/\delta_1)$ and $\Omega_{x+}(f,\delta_2) \le \Omega_x(f,(1-x)/\delta_2)$.
iv. $\Omega_{x-}(f,\delta_1) \le V_{x-\delta_1}^{x}(f)$, $\Omega_{x+}(f,\delta_2) \le V_x^{x+\delta_2}(f)$, and $\Omega_x(f,\lambda) \le V_{x-x/\lambda}^{x+(1-x)/\lambda}(f)$.

All these properties can be found in [252].

In the sequel, to obtain the rate of convergence, we shall need several lemmas.

Lemma 10.1. *For* $T_{n,r,m}(x) = \frac{(n+r)^{\underline{r}}}{n^{\underline{r}}} P_{n,r}((t-x)^m, x)$, *the following claims hold:*

i. For any $r, m \in N_0$ *and* $x \in [0,1]$, *the following recurrence relation is satisfied:*

$$(n+m+r+1)T_{n,r,m+1}(x) = x(1-x)[DT_{n,r,m}(x) + 2mT_{r,n,m-1}(x)] + [(m+r) - x(1+2m+2r)]T_{n,r,m}(x),$$

where, for $m = 0$, *we denote* $T_{n,r,-1}(x) = 0$.

ii. *For all* $r \in N_0$ *and* $x \in [0,1]$,

$$T_{n,r,0}(x) = 1, \quad T_{n,r,1}(x) = \frac{r - x(1+2r)}{n+r+1},$$

$$T_{n,r,2}(x) = \frac{2nx(1-x) + r(1+r) - 2rx(2r+3) + 2x^2(2r^2+4r+1)}{(n+r+1)(n+r+2)}.$$

iii. *For all* $r, m \in N_0$ *and* $x \in [0,1]$,

$$T_{n,r,m}(x) = O\left(n^{-[(m+1)/2]}\right).$$

Lemma 10.2. *For any* $r, s \in N_0$ *and* $x \in [0,1]$,

$$D^s P_{n,r}(f,x) = P_{n,r+s}(D^s f, x).$$

Lemma 10.3. *The following claims hold:*

i. *For every* $x \in (0,1)$ *and* $k, n_1, n_2 \in N_0$, *with* $0 < k \le n_1 < n_2$, *we have*

$$J_{n_1,k}(x) < J_{n_2,k}(x).$$

ii. *For every* $x \in [0,1]$, $r \in N$,

$$\left| \sum_{j=0}^{n} \left(J^{\alpha}_{n+r,j}(x) - J^{\alpha}_{n,j}(x) \right) \right| \le \alpha r x.$$

Proof. Let us first prove claim (i). It is clear that it is sufficient to prove the inequality for $J_{n,k}$ and $J_{n+1,k}$. For this purpose, let's take into account first that from the definition of $J_{n,k}$, we know that $(J_{n+1,k} - J_{n,k})(0) = 0$ and $(J_{n+1,k} - J_{n,k})(1) = 0$ and also that

$$DJ_{n,k}(x) = np_{n-1,k-1}(x) \qquad \text{and} \qquad DJ_{n+1,k}(x) = (n+1)p_{n,k-1}(x).$$

In this case,

$$\begin{aligned} &D(J_{n+1,k} - J_{n,k})(x) \\ &= (n+1)\binom{n}{k-1} x^{k-1}(1-x)^{n-k+1} - n\binom{n-1}{k-1} x^{k-1}(1-x)^{n-k} \\ &= \binom{n}{k-1} x^{k-1}(1-x)^{n-k}\left(k - (n+1)x\right). \end{aligned}$$

Then it is immediate that

(a) for $x \in (0,1)$, $D(J_{n+1,k} - J_{n,k})(x) = 0 \Leftrightarrow x = \dfrac{k}{n+1}$,

(b) for $x \in \left(0, \dfrac{k}{n+1}\right)$, $D(J_{n+1,k} - J_{n,k})(x) > 0$.

From (b), it is clear that $(J_{n+1,k} - J_{n,k})(\frac{k}{n+1}) > 0$, and then (a) together with the mean value theorem leads us to the conclusion.

To prove claim (ii), we know that, provided $\alpha \geq 1$, $b^\alpha - a^\alpha \leq \alpha(b-a)$ whenever $0 \leq a \leq b \leq 1$. Then, if we use claim (i),

$$\begin{aligned}
&\left|\sum_{j=0}^{n}\left(J^{\alpha}_{n+r,j}(x) - J^{\alpha}_{n,j}(x)\right)\right| \\
&= \sum_{j=0}^{n}\left(J^{\alpha}_{n+r,j}(x) - J^{\alpha}_{n,j}(x)\right) \leq \alpha \sum_{j=0}^{n}\left(J_{n+r,j}(x) - J_{n,j}(x)\right) \\
&\leq \alpha\left(\sum_{j=0}^{n+r} J_{n+r,j}(x) - \sum_{j=0}^{n} J_{n,j}(x)\right) = \alpha\left(1 + (n+r)x - (1+nx)\right) = \alpha r x.
\end{aligned}$$

■

Since for all $n \in N_0$, $J_{n,0}(x) = 1$, we can conclude from the preceding lemma that for $x \in [0,1]$ and $0 \leq k \leq n_1 \leq n_2$, we have that $J_{n_1,k}(x) \leq J_{n_2,k}(x)$.

Remark 10.1. For every $0 \leq k \leq n, x \in (0,1), \alpha \geq 1$, and for all n, if we apply the mean value theorem, we know that there exists $\xi \in [0,1]$ with $J_{n,k+1}(x) \leq \xi \leq J_{n,k}(x)$ such that

$$Q^{(\alpha)}_{n,k}(x) = J^{\alpha}_{n,k}(x) - J^{\alpha}_{n,k+1}(x) = \alpha\xi^{\alpha-1}\left(J_{n,k}(x) - J_{n,k+1}(x)\right) \leq \alpha p_{n,k}(x).$$

Then, considering the results in [246, Theorem 1], we can write the following inequalities:

$$Q^{(\alpha)}_{n,k}(x) \leq \alpha p_{n,k}(x) \leq \frac{\alpha}{\sqrt{2enx(1-x)}}.$$

As a consequence, we also have $K_{n,r,\alpha}(x,t) \leq \alpha K_{n,r,1}(x,t)$.

Remark 10.2. For a fixed $x \in (0,1)$, when n is sufficiently large, it can be seen from Lemma 10.1, that

$$\frac{x(1-x)}{n} \leq T_{n,r,2}(x) \leq \frac{Cx(1-x)}{n},$$

for any $C > 2$.

Lemma 10.4. *Let $x \in (0,1)$ and $C > 2$; then for n sufficiently large, we have*

$$\lambda_{n,r,\alpha}(x,y) = \int_0^y K_{n,r,\alpha}(x,t)dt \le \frac{C\alpha x(1-x)}{n(x-y)^2}, \quad 0 \le y < x,$$
$$1-\lambda_{n,r,\alpha}(x,z) = \int_z^1 K_{n,r,\alpha}(x,t)dt \le \frac{C\alpha x(1-x)}{n(z-x)^2}, \quad x < z < 1.$$

Theorem 10.2 ([115]). *Let $f \in H$, then for $\alpha \ge 1$, $r \in N_0$, $x \in (0,1)$, any $C > 2$, and n sufficiently large, we have*

$$\left| P_{n,r,\alpha}(f,x) - \frac{1}{\alpha+1}[f_+(x) + \alpha f_-(x)] \right| \le |f_+(x) - f_-(x)| \frac{5\alpha(1+\alpha)(r+1)}{\sqrt{2e(n+r)x(1-x)}}$$
$$+ \frac{|f_+(x) + \alpha f_-(x)|}{\alpha+1}\frac{r^2}{n} + \frac{2C\alpha}{nx(1-x)}$$
$$\sum_{k=1}^{n} \Omega_x(g_x, \sqrt{k}),$$

where the auxiliary function g_x is defined as

$$g_x(t) = \begin{cases} f(t) - f_-(x), & 0 \le t < x, \\ 0, & t = x, \\ f(t) - f_+(x), & x < t \le 1. \end{cases}$$

Proof. We can write

$$f(t) = \frac{f_+(x) + \alpha f_-(x)}{\alpha+1} + g_x(t) + \frac{f_+(x) - f_-(x)}{\alpha+1}\operatorname{sign}_\alpha(t-x)$$
$$+ \delta_x(t)\left(f(x) - \frac{f_+(x) + \alpha f_-(x)}{\alpha+1} \right), \tag{10.1}$$

where

$$\operatorname{sign}_\alpha(t) = \begin{cases} \alpha, & t > 0, \\ 0, & t = 0, \\ -1, & t < 0, \end{cases}$$

and

$$\delta_x(t) = \begin{cases} 1, & t = x, \\ 0, & t \ne x. \end{cases}$$

Thus, using Remark 10.1, identity (6.10), and the fact that $P_{n,r,\alpha}(\delta_x, x) = 0$, we have

$$\begin{aligned}&\left|P_{n,r,\alpha}(f,x) - \frac{1}{\alpha+1}[f_+(x) + \alpha f_-(x)]\right| \\ &\leq \alpha P_{n,r}(|g_x|, x) + \frac{|f_+(x) - f_-(x)|}{\alpha+1} P_{n,r,\alpha}(\mathrm{sign}_\alpha(t-x), x) \\ &\quad + |P_{n,r,\alpha}(1,x) - 1| \frac{|f_+(x) + \alpha f_-(x)|}{\alpha+1}. \end{aligned} \tag{10.2}$$

Let us analyze the three summands of the right-side term of this inequality.

For the first summand of (10.2), we have

$$\begin{aligned} P_{n,r}(|g_x|, x) &= \left(\int_0^{x-x/\sqrt{n}} + \int_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}} + \int_{x+(1-x)/\sqrt{n}}^{1}\right) |g_x(t)| K_{n,r,\alpha}(x,t)\,dt \\ &= E_1 + E_2 + E_3. \end{aligned}$$

First, we estimate E_1. Note that $\Omega_{x-}(g_x, \delta_1)$ is monotone nondecreasing with respect to δ_1, and so it follows that

$$|E_1| = \left|\int_0^{x-x/\sqrt{n}} |g_x(t)| d_t \lambda_{n,r,\alpha}(x,t)\right| \leq \int_0^{x-x/\sqrt{n}} \Omega_{x-}(g_x, x-t) d_t \lambda_{n,r,\alpha}(x,t).$$

Integrating by parts with $y = x - x/\sqrt{n}$, we have

$$\begin{aligned} &\int_0^{x-x/\sqrt{n}} \Omega_{x-}(g_x, x-t) d_t \lambda_{n,r,\alpha}(x,t) \\ &\leq \Omega_{x-}(g_x, x-y) \lambda_{n,r,\alpha}(x,y) + \int_0^y \lambda_{n,r,\alpha}(x,t)\, d(-\Omega_{x-}(g_x, x-t)). \end{aligned}$$

Applying Remark 10.3, we have

$$|E_1| \leq \Omega_{x-}(g_x, x-y) \frac{C\alpha x(1-x)}{n(x-y)^2} + \frac{C\alpha x(1-x)}{n} \int_0^y \frac{1}{(x-t)^2} d(-\Omega_{x-}(g_x, x-t)). \tag{10.3}$$

Since

$$\begin{aligned} &\int_0^y \frac{1}{(x-t)^2} d(-\Omega_{x-}(g_x, x-t)) \\ &= \frac{-\Omega_{x-}(g_x, x-y)}{(x-y)^2} + \frac{\Omega_{x-}(g_x, x)}{x^2} + \int_0^y \Omega_{x-}(g_x, x-t) \frac{2}{(x-t)^3} dt, \end{aligned}$$

from (10.3), we obtain

$$|E_1| \leq \Omega_{x-}(g_x, x)\frac{C\alpha x(1-x)}{nx^2} + \frac{C\alpha x(1-x)}{n}\int_0^{x-x/\sqrt{n}} \Omega_{x-}(g_x, x-t)\frac{2}{(x-t)^3}dt.$$

Substituting $t = x - x/\sqrt{u}$ in the last integral, we get

$$\int_0^{x-x/\sqrt{n}} \Omega_{x-}(g_x, x-t)\frac{2}{(x-t)^3}dt \leq \frac{1}{x^2}\int_1^n \Omega_{x-}(g_x, x/\sqrt{u})du.$$

Thus, we have

$$|E_1| \leq \frac{C\alpha(1-x)}{nx}\left(\Omega_{x-}(g_x, x) + \int_1^n \Omega_{x-}(g_x, x/\sqrt{u})\right). \tag{10.4}$$

As $g_x(x) = 0$,

$$|E_2| \leq \int_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}} |g_x(t) - g_x(x)|d_t\lambda_{n,r,\alpha}(x,t) \leq \Omega_x(g_x, \sqrt{n}). \tag{10.5}$$

Using similar methods as in the estimation of E_1, we obtain

$$|E_3| \leq \frac{C\alpha x}{n(1-x)}\left(\Omega_{x+}(g_x, 1-x) + \int_1^n \Omega_{x+}(g_x, (1-x)/\sqrt{u})\right). \tag{10.6}$$

Combining the estimates (10.4)–(10.6) with property (iii) of the modulus Ω_x, which we can find on p. 315, we get

$$\begin{aligned}
&E_1 + E_2 + E_3\\
&\leq \Omega_x(g_x, \sqrt{n}) + C\alpha\left(\frac{(1-x)}{nx} + \frac{x}{n(1-x)}\right)\left(\Omega_x(g_x, 1) + \int_1^n \Omega_x(g_x, \sqrt{u})\right).
\end{aligned}$$

By the monotonocity of $\Omega_x(g_x, \lambda)$ and the facts $(1-x)^2+x^2 \leq 1$ and $\frac{1}{n-1} \leq \frac{C\alpha}{nx(1-x)}$ for $n > 1$, we have

$$\begin{aligned}
P_{n,r}(|g_x|, x) &= E_1 + E_2 + E_3\\
&\leq \frac{1}{n-1}\sum_{k=2}^{n}\Omega_x(g_x, \sqrt{k})\\
&\quad + \frac{C\alpha}{nx(1-x)}\Omega_x(g_x, 1) + \frac{C\alpha}{nx(1-x)}\sum_{k=1}^{n-1}\Omega_x(g_x, \sqrt{k})
\end{aligned}$$

$$\leq \frac{2C\alpha}{nx(1-x)} \sum_{k=1}^{n} \Omega_x(g_x, \sqrt{k}). \tag{10.7}$$

To estimate the second summand of (10.2), we proceed as follows: For $r > 0$,

$$\begin{aligned}
&P_{n,r,\alpha}(\mathrm{sign}_\alpha(t-x), x)\\
&= \frac{n^{\underline{r}}}{(n+r-1)^{\underline{r-1}}} \sum_{k=0}^{n-r} Q_{n-r,k}^{(\alpha)}(x) \left(\int_x^1 \alpha p_{n+r-1,k+r-1}(t)dt - \int_0^x p_{n+r-1,k+r-1}(t)dt \right)\\
&= \frac{n^{\underline{r}}}{(n+r-1)^{\underline{r-1}}} \sum_{k=0}^{n-r} Q_{n-r,k}^{(\alpha)}(x) \Bigg(\int_0^1 \alpha p_{n+r-1,k+r-1}(t)dt\\
&\qquad -(1+\alpha) \int_0^x p_{n+r-1,k+r-1}(t)dt \Bigg).
\end{aligned}$$

Using the identity $(n+r) \int_0^x p_{n+r-1,k+r-1}(t)dt = 1 - \sum_{j=0}^{k+r-1} p_{n+r,j}(x)$, we have

$$\begin{aligned}
&P_{n,r,\alpha}(\mathrm{sign}_\alpha(t-x), x)\\
&= \frac{n^{\underline{r}}}{(n+r)^{\underline{r}}} \left(\alpha - (1+\alpha) \sum_{k=0}^{n-r} Q_{n-r,k}^{(\alpha)}(x) \left(1 - \sum_{j=0}^{k+r-1} p_{n+r,j}(x) \right) \right)\\
&= \frac{n^{\underline{r}}}{(n+r)^{\underline{r}}} \left((1+\alpha) \sum_{k=0}^{n-r} Q_{n-r,k}^{(\alpha)}(x) \sum_{j=0}^{k+r-1} p_{n+r,j}(x) - 1 \right).
\end{aligned}$$

For $r = 0$, if we follow the same steps, we obtain

$$\begin{aligned}
P_{n,0,\alpha}(\mathrm{sign}_\alpha(t-x), x) &= (1+\alpha) \sum_{k=1}^{n} Q_{n,k}^{(\alpha)}(x) \sum_{j=0}^{k-1} p_{n,j}(x) - \sum_{k=1}^{n} Q_{n,k}(x) - Q_{n,0}(x)\\
&= (1+\alpha) \sum_{k=1}^{n} Q_{n,k}^{(\alpha)}(x) \sum_{j=0}^{k-1} p_{n,j}(x) - 1.
\end{aligned}$$

Therefore, for $r \geq 0$, we can write

$$P_{n,r,\alpha}(\mathrm{sign}_\alpha(t-x), x) = \frac{n^{\underline{r}}}{(n+r)^{\underline{r}}} \left((1+\alpha) \sum_{k=0}^{n-r} Q_{n-r,k}^{(\alpha)}(x) \sum_{j=0}^{k+r-1} p_{n+r,j}(x) - 1 \right).$$

Now, if we use Remark 10.1, we have

$$\begin{aligned}
&\left|\frac{(n+r)^{\underline{r}}}{n^{\underline{r}}} P_{n,r,\alpha}(\mathrm{sign}_\alpha(t-x),x)\right| \\
&\leq \left|(1+\alpha)\sum_{k=0}^{n-r} Q_{n-r,k}^{(\alpha)}(x)\sum_{j=0}^{k} p_{n+r,j}(x)-1\right| \\
&\quad+\left|(1+\alpha)\sum_{k=0}^{n-r} Q_{n-r,k}^{(\alpha)}(x)\left(\sum_{j=k+1}^{k+r-1} p_{n+r,j}(x)\right)\right| \\
&\leq \left|(1+\alpha)\sum_{k=0}^{n-r} Q_{n-r,k}^{(\alpha)}(x)\sum_{j=0}^{k} p_{n+r,j}(x)-1\right| + \frac{|r-1|(1+\alpha)}{\sqrt{2e(n+r)x(1-x)}} \\
&= \left|(1+\alpha)\sum_{j=0}^{n-r} p_{n+r,j}(x)\sum_{k=j}^{n-r} Q_{n-r,k}^{(\alpha)}(x)-1\right| + \frac{|r-1|(1+\alpha)}{\sqrt{2e(n+r)x(1-x)}} \\
&= \left|(1+\alpha)\sum_{j=0}^{n-r} p_{n+r,j}(x)J_{n-r,j}^{\alpha}(x)-1\right| + \frac{|r-1|(1+\alpha)}{\sqrt{2e(n+r)x(1-x)}}. \qquad (10.8)
\end{aligned}$$

Using Lemma 10.3, we can write

$$\begin{aligned}
&\left|(1+\alpha)\sum_{j=0}^{n-r} p_{n+r,j}(x)J_{n-r,j}^{\alpha}(x)-1\right| \\
&\leq \left|(1+\alpha)\sum_{j=0}^{n-r} p_{n+r,j}(x)J_{n+r,j}^{\alpha}(x)-1\right| \\
&\quad+(1+\alpha)\sum_{j=0}^{n-r} p_{n+r,j}(x)\left(J_{n+r,j}^{\alpha}(x)-J_{n-r,j}^{\alpha}(x)\right) \\
&\leq \left|(1+\alpha)\sum_{j=0}^{n-r} p_{n+r,j}(x)J_{n+r,j}^{\alpha}(x)-1\right| \\
&\quad+\frac{1+\alpha}{\sqrt{2e(n+r)x(1-x)}}\sum_{j=0}^{n-r}\left(J_{n+r,j}^{\alpha}(x)-J_{n-r,j}^{\alpha}(x)\right) \\
&\leq \left|(1+\alpha)\sum_{j=0}^{n+r} p_{n+r,j}(x)J_{n+r,j}^{\alpha}(x)-1\right|
\end{aligned}$$

$$+\frac{2(1+\alpha)r}{\sqrt{2e(n+r)x(1-x)}}+\frac{\alpha(1+\alpha)rx}{\sqrt{2e(n+r)x(1-x)}}$$
$$\le \left|(1+\alpha)\sum_{j=0}^{n+r} p_{n+r,j}(x)J_{n+r,j}^{\alpha}(x)-\sum_{j=0}^{n+r} Q_{n+r,j}^{(\alpha)}(x)\right|$$
$$+\frac{3\alpha(1+\alpha)r}{\sqrt{2e(n+r)x(1-x)}}. \tag{10.9}$$

By the mean value theorem, we have

$$Q_{n+r,j}^{(\alpha+1)}(x)=J_{n+r,j}^{\alpha+1}(x)-J_{n+r,j+1}^{\alpha+1}(x)=(\alpha+1)p_{n+r,j}(x)\gamma_{n+r,j}^{\alpha}(x),$$

where $J_{n+r,j+1}(x) < \gamma_{n+r,j}(x) < J_{n+r,j}(x)$. Therefore, for n sufficiently large, we have

$$(1+\alpha)\sum_{j=0}^{n+r} p_{n+r,j}(x)J_{n+r,j}^{\alpha}(x)-\sum_{j=0}^{n+r} Q_{n+r,j}^{(\alpha)}(x)$$
$$=(1+\alpha)\sum_{j=0}^{n+r} p_{n+r,j}(x)\left(J_{n+r,j}^{\alpha}(x)-\gamma_{n+r,j}^{\alpha}(x)\right)$$
$$\le (1+\alpha)\sum_{j=0}^{n+r} p_{n+r,j}(x)Q_{n+r,j}^{(\alpha)}(x)$$
$$\le \frac{1+\alpha}{\sqrt{2e(n+r)x(1-x)}}\sum_{j=0}^{n+r} Q_{n+r,j}^{(\alpha)}(x)=\frac{1+\alpha}{\sqrt{2e(n+r)x(1-x)}}. \tag{10.10}$$

Then, if we put together (10.8)–(10.10), we finally obtain

$$|P_{n,r,\alpha}(\mathrm{sign}_{\alpha}(t-x),x)|$$
$$\le \frac{n^{\underline{r}}}{(n+r)^{\underline{r}}}\left(\frac{|r-1|(1+\alpha)}{\sqrt{2e(n+r)x(1-x)}}+\frac{3\alpha(1+\alpha)r}{\sqrt{2e(n+r)x(1-x)}}+\frac{1+\alpha}{\sqrt{2e(n+r)x(1-x)}}\right)$$
$$\le \frac{5\alpha(1+\alpha)(r+1)}{\sqrt{2e(n+r)x(1-x)}}. \tag{10.11}$$

With respect to the third summand of (10.2), we have

$$|P_{n,r,\alpha}(1,x)-1|=\left|\frac{n^{\underline{r}}}{(n+r)^{\underline{r}}}-1\right|=1-\frac{n^{\underline{r}}}{(n+r)^{\underline{r}}}\le \frac{r^2}{n}. \tag{10.12}$$

To prove the last inequality, first notice that for $n \leq r^2$, it is immediate (take into account that we always have $r \leq n$). For the rest of the cases, we can consider the equivalent inequality $(n - r^2)(n + r)^{\underline{r}} \leq n n^{\underline{r}}$, which can be written in the form

$$\prod_{s=1}^{r} (n - r^2)^{\frac{1}{r}} (n + s) \leq \prod_{s=1}^{r} n^{\frac{1}{r}} (n - (r - s)).$$

Now, for any $s = 1, \ldots, r$, we only need to prove that $(n - r^2)^{\frac{1}{r}}(n + s) \leq n^{\frac{1}{r}}(n - (r - s))$, or analogously, $(n - r^2)(n + s)^r \leq n(n - (r - s))^r$. But it is clear that $\lim_{n\to\infty}[(n - r^2)(n + s)^r]/[n(n - (r - s))^r] = 1$, and by simple differentiation, it is possible to check that, for $n \geq r^2$, $[(n - r^2)(n + s)^r]/[n(n - (r - s))^r]$ is an increasing function of n.

Finally, combining (10.7), (10.11), and (10.12), we get the desired result. This completes the proof of the theorem. ∎

In particular, for the operators $P_{n,0,\alpha}$, the preceding result yields

$$\left| P_{n,0,\alpha}(f,x) - \frac{1}{\alpha+1}[f_+(x) + \alpha f_-(x)] \right| \leq |f_+(x) - f_-(x)| \frac{2(1+\alpha)}{\sqrt{2e(n+r)x(1-x)}}$$
$$+1 \frac{2C\alpha}{nx(1-x)} \sum_{k=1}^{n} \Omega_x(g_x, \sqrt{k}).$$

More precisely, we have again considered inequality (10.11), which for $\alpha = 0$ provided us with a slight refinement of the inequality of Theorem 10.2.

Corollary 10.1 ([115]). *Let $f : [0,1] \to R$ be such that $D^{r-1} f \in C[0,1]$ and $D^r f_{\pm}^{(r)}(x)$ exists everywhere and is bounded on $[0,1]$. Then, for $x \in (0,1)$, $r, s \in N_0$, $C > 2$, and n large enough, we have*

$$\left| D^s P_{n,r}(f,x) - \frac{1}{2}[D^s f_+(x) + \alpha D^s f_-(x)] \right|$$
$$\leq |D^s f_+(x) - D^s f_-(x)| \frac{10(r+s+1)}{\sqrt{2e(n+r+s)x(1-x)}}$$
$$+ \frac{|D^s f_+(x) + \alpha D^s f_-(x)|}{\alpha+1} \frac{(r+s)^2}{n} + \frac{2C}{nx(1-x)} \sum_{k=1}^{n} \Omega_x(g_{s,x}, \sqrt{k}),$$

where the auxiliary function $g_{s,x}$ is defined as

$$g_{s,x}(t) = \begin{cases} D^s f(t) - D^s f_-(x), & 0 \leq t < x, \\ 0, & t = x, \\ D^s f(t) - D^s f_+(x), & x < t \leq 1. \end{cases}$$

10.2 General Class of Operators for DBV

In 1967, Durrmeyer [65] introduced an integral modification of the well-known Bernstein polynomials with the purpose of approximating Lebesgue integrable functions on [0, 1]. Later, in 1989, Heilmann [149] considered a general sequence of Durrmeyer operators defined on $[0,\infty)$ for $n > c$ and $x \in [0,\infty)$ as

$$V_{n,r}(f,x) = \begin{cases} (n-c)\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k}(t)f(t)dt, & r = 0, \\ (n-c)\beta(n,r,c)\sum_{k=0}^{\infty} p_{n+cr,k}(x)\int_0^{\infty} p_{n-cr,k+r}(t)f(t)dt, & r > 0, \end{cases}$$

where $r, n \in N_0$, $p_{n,k}(x) = \frac{(-x)^k}{k!}\phi_n^{(k)}(x)$, and $\beta(n,r,c) = \prod_{l=0}^{r-1}\frac{n+cl}{n-c(l+1)}$. The special cases of these operators are as follows:

For $c = 0$, we get the Szász–Durrmeyer operators when $\phi_n(x) = e^{-nx}$.

For $c > 0$, we get the Baskakov–Durrmeyer operators when $\phi_n(x) = (1 + cx)^{-n/c}$. The special case $c = 1$ and $r = 0$ was considered in [215]. The family of operators $V_{n,r}(f,x)$ is linear and positive.

The operators $V_{n,r}$ can also be written in the form of a kernel as

$$V_{n,r}(f,x) = \int_0^{\infty} K_{n,r}(x,t)f(t)dt,$$

where the kernel $K_{r,n}$ is given by

$$K_{n,r}(x,t) = \begin{cases} (n-c)\sum_{k=0}^{\infty} p_{n,k}(x)p_{n,k}(t), & r = 0, \\ (n-c)\beta(n,r,c)\sum_{k=0}^{\infty} p_{n+cr,k}(x)p_{n-cr,k+r}(t), & r > 0. \end{cases}$$

In this section, we present the rate of convergence for the operators $V_{n,r}(f,x)$ for functions having derivatives of bounded variation that were studied by Gupta et al. [144].

In the sequel, the following lemmas are necessary.

Lemma 10.5 ([149]). *Let $m, r \in N_0, n > cr + c, x \in [0,\infty)$, and suppose*

$$\hat{K}_{n,r}(x,t) = (n - cr - c)\sum_{k=0}^{\infty} p_{n+cr,k}(x)p_{n-cr,k+r}(t),$$

and then

$$T_{n,r,m}(x) = \int_0^\infty \hat{K}_{n,r}(x,t)(t-x)^m dt,$$

With $\phi(x) = \sqrt{x(1+cx)}$ *and for* $n > c(r+m+2)$, *we have the recurrence formula*

$$(n - c(r+m+2)T_{n,r,m+1}(x) = \phi^2(x)[T^{(1)}_{n,r,m}(x) + 2mT_{r,n,m-1}(x)] + (r+m+1)(1+2cx)T_{r,n,m}(x), m \in N.$$

Also,

$$T_{n,r,0}(x) = 1, \quad T_{n,r,1}(x) = \frac{(r+1)(1+2cx)}{n-c(r+2)},$$

$$T_{n,r,2}(x) = \frac{2x(1+cx)(n-c) + (r+1)(r+2)(1+2cx)^2}{[n-c(r+2)][n-c(r+3)]}.$$

For all $r, m \in N_0$ *and* $x \in [0,\infty)$,

$$T_{n,r,m}(x) = O\left(n^{-[(m+1)/2]}\right).$$

Remark 10.3. For n sufficiently large, $c \geq 0$, and $x \in (0,\infty)$, it can be seen from Lemma 10.5 that

$$\frac{2x(1+cx)}{n} \leq T_{n,r,2}(x) \leq \frac{Cx(1+cx)}{n},$$

for any $C > 2$.

Remark 10.4. If we use the Cauchy–Schwarz inequality, it follows from Lemma 10.5 that for n sufficiently large, $c \geq 0$, and $x \in (0,\infty)$,

$$\int_0^\infty \hat{K}_{n,r}(x,t)|t-x|dt \leq [T_{n,r,2}(x)]^{1/2} \leq \sqrt{\frac{Cx(1+cx)}{n}},$$

for any $C > 2$.

Lemma 10.6. *Let* $x \in (0,\infty)$ *and* $C > 2$. *Then for* n *sufficiently large, we have*

$$\lambda_{n,r}(x,y) = \int_0^y \hat{K}_{n,r}(x,t)dt \leq \frac{Cx(1+cx)}{n(x-y)^2}, \quad 0 \leq y < x,$$

$$1 - \lambda_{n,r}(x,z) = \int_z^\infty \hat{K}_{n,r}(x,t)dt \leq \frac{Cx(1+cx)}{n(z-x)^2}, \quad x < z < 1.$$

The proof of the preceding lemma follows easily by using Remark 10.3.

Lemma 10.7. *Suppose f is s-times differentiable on $[0,\infty)$ such that $f^{(s-1)}(t) = O(t^{\alpha})$, for some $\alpha > 0$ as $t \to \infty$. Then, for any $r, s \in N_0$ and $n > \alpha + cs$, we have*

$$D^s V_{n,r}(f,x) = V_{n,r+s}(D^s f,x).$$

Using the identity [see, e.g., (3.5) of [148]] $Dp_{n,k}(x) = n[p_{n+c,k-1}(x) - p_{n+c,k}(x)]$, and applying integration by parts, we see that the result follows by mathematical induction.

Let $DB_{\gamma}(0,\infty)$, $\gamma \geq 0$, be the class of absolutely continuous functions f defined on $(0,\infty)$ satisfying

(i) $f(t) = O(t^{\gamma}), t \to \infty$.
(ii) We have a derivative f' on the interval $(0,\infty)$ coinciding a.e. with a function that is of bounded variation on every finite subinterval of $(0,\infty)$. It can be observed that all functions $f \in DB_{\gamma}(0,\infty)$ possess for each $c > 0$ the representation

$$f(x) = f(c) + \int_c^x \psi(t)dt, \quad x \geq c.$$

Theorem 10.3 ([144]). *Let $f \in DB_{\gamma}(0,\infty)$, $\gamma > 0$, and $x \in (0,\infty)$. Then, for $C > 2$ and n sufficiently large, we have*

$$\begin{aligned}
&\left|\frac{[n-c(r+1)]}{(n-c)\beta(n,r,c)} V_{n,r}(f,x) - f(x)\right| \\
&\quad \leq \frac{C(1+cx)}{n}\left(\sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-x/k}^{x+x/k}((f')_x) + \frac{x}{\sqrt{n}} \bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}((f')_x)\right) \\
&\quad\quad + \frac{C(1+cx)}{nx}(|\, f(2x) - f(x) - xf'(x^+) \,| + |\, f(x) \,|) \\
&\quad\quad + \sqrt{\frac{Cx(1+cx)}{n}}\left(C2^{\gamma} O(n^{\frac{-\gamma}{2}}) + |\, f'(x^+) \,|\right) \\
&\quad\quad + \frac{1}{2}\sqrt{\frac{Cx(1+cx)}{n}}\, |\, f'(x^+) - f'(x^-) \,| \\
&\quad\quad + \frac{1}{2}\, |\, f'(x^+) + f'(x^-) \,|\, \frac{(r+1)(1+2cx)}{[n-c(r+2)]},
\end{aligned}$$

where $\bigvee_a^b f(x)$ denotes the total variation of f_x on $[a,b]$, and f_x is defined by

$$f_x(t) = \begin{cases} f(t) - f(x^-), & 0 \le t < x, \\ 0, & t = x, \\ f(t) - f(x^+), & x < t < \infty. \end{cases}$$

Proof. Using the mean value theorem, we can write

$$\begin{aligned} &\left| \frac{[n - c(r+1)]}{(n-c)\beta(n,r,c)} V_{n,r}(f,x) - f(x) \right| \\ &\le [n - c(r+1)] \sum_{k=0}^{\infty} p_{n+cr,k}(x) \int_0^{\infty} p_{n-cr,k+r}(t) |f(t) - f(x)| dt \\ &= \int_0^{\infty} | \int_x^t \hat{K}_{n,r}(x,t) f'(u) du | dt. \end{aligned}$$

Also, using the identity

$$\begin{aligned} f'(u) = \frac{f'(x^+) + f'(x^-)}{2} + (f')_x(u) + \frac{f'(x^+) - f'(x^-)}{2} sgn(u - x) \\ + \left[f'(x) - \frac{f'(x^+) + f'(x^-)}{2} \right] \chi_x(u), \end{aligned}$$

where

$$\chi_x(u) = \begin{cases} 1, & \mathrm{u} = x, \\ 0, & \mathrm{u} \ne x, \end{cases}$$

we obviously have

$$\int_0^{\infty} \left(\int_x^t \left(f'(x) - \frac{f'(x^+) + f'(x^-)}{2} \right) \chi_x(u) du \right) \hat{K}_{n,r}(x,t) dt = 0.$$

Thus, using above identities, we can write

$$\begin{aligned} &\left| \frac{[n - c(r+1)]}{(n-c)\beta(n,r,c)} V_{n,r}(f,x) - f(x) \right| \\ &\le \left| \int_x^{\infty} (\int_x^t (f')_x(u) du) \hat{K}_{n,r}(x,t) dt + \int_0^x (\int_x^t (f')_x(u) du) \hat{K}_{n,r}(x,t) dt \right| \\ &\quad + \frac{1}{2} \mid f'(x^+) - f'(x^-) \mid [T_{n,r,2}(x)]^{1/2} + \frac{1}{2} \mid f'(x^+) + f'(x^-) \mid T_{n,r,1}(x) \end{aligned}$$

$$=| A_{n,r}(f,x) + B_{n,r}(f,x) | + \frac{1}{2} | f'(x^+) - f'(x^-) | [T_{n,r,2}(x)]^{1/2}$$
$$+ \frac{1}{2} | f'(x^+) + f'(x^-) | T_{n,r,1}(x). \tag{10.13}$$

Applying Remarks 10.2 and 10.4, in (10.13), we have

$$\left| \frac{[n-c(r+1)]}{(n-c)\beta(n,r,c)} V_{n,r}(f,x) - f(x) \right|$$
$$\leq |A_{n,r}(f,x)| + |B_{n,r}(f,x)| + \frac{1}{2} | f'(x^+) - f'(x^-) | \sqrt{\frac{Cx(1+cx)}{n}}$$
$$+ \frac{1}{2} | f'(x^+) + f'(x^-) | \frac{(r+1)(1+2cx)}{[n-c(r+2)]}. \tag{10.14}$$

In order to complete the proof of the theorem, we need to estimate the terms $A_{n,r}(f,x)$ and $B_{n,r}(f,x)$. Applying integration by parts and Lemma 10.6, with $y = x - x/\sqrt{n}$, we have

$$| B_{n,r}(f,x) | = \left| \int_0^x \int_x^t (f')_x(u) du d_t(\lambda_{n,r}(x,t)) \right|$$

$$\int_0^x \lambda_{n,r}(x,t)(f')_x(t) dt \leq \left(\int_0^y + \int_y^x \right) | (f')_x(t) || \lambda_{n,r}(x,t) | dt$$
$$\leq \frac{Cx(1+cx)}{n} \int_0^y \bigvee_t^x ((f')_x) \frac{1}{(x-t)^2} dt$$
$$+ \int_y^x \bigvee_t^x ((f')_x) dt$$
$$\leq \frac{Cx(1+cx)}{n} \int_0^y \bigvee_t^x ((f')_x) \frac{1}{(x-t)^2} dt$$
$$+ \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^x ((f')_x).$$

Let $u = \frac{x}{x-t}$. Then we have

$$\frac{Cx(1+cx)}{n}\int_0^y \bigvee_t^x ((f')_x)\frac{1}{(x-t^2)}dt = \frac{Cx(1+cx)}{n}\int_1^{\sqrt{n}} \bigvee_{x-\frac{x}{u}}^{x} ((f')_x)du$$

$$\leq \frac{C(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x} ((f')_x).$$

Thus,

$$\mid B_{n,r}(f,x)\mid \leq \frac{C(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x} ((f')_x) + \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^{x} ((f')_x). \tag{10.15}$$

On the other hand, we have

$$\begin{aligned}
\Big|A_{n,r}(f,x)\Big| &= \Big|\int_x^\infty (\int_x^t (f')_x(u)du)\hat{K}_{n,r}(x,t)dt\Big| \\
&= \Big|\int_{2x}^\infty (\int_x^t (f')_x(u)du)\hat{K}_{n,r}(x,t)dt \\
&\quad + \int_x^{2x} (\int_x^t (f')_x(u)du)dt(1-\lambda_{n,r}(x,t))\Big|dt \\
&\leq \Big|\int_{2x}^\infty (f(t)-f(x))\hat{K}_{n,r}(x,t)dt\Big| + \Big|f'(x^+)\Big|\Big|\int_{2x}^\infty (t-x)\hat{K}_{n,r}(x,t)dt\Big| \\
&\quad + \Big|\int_x^{2x} (f')_x(u)du)\Big|\Big|(1-\lambda_{n,r}(x,2x)\Big| \\
&\quad + \int_x^{2x} \mid (f')_x(t)\Big|\Big|(1-\lambda_{n,r}(x,t)\Big|dt \\
&\leq \frac{C}{x}\int_{2x}^\infty \hat{K}_{n,r}(x,t)t^\gamma\Big|t-x\Big|dt + \frac{\Big|f(x)\Big|}{x^2}\int_{2x}^\infty \hat{K}_{n,r}(x,t)(t-x)^2dt \\
&\quad + \Big|f'(x^+)\Big|\int_{2x}^\infty \hat{K}_{n,r}(x,t)\Big|(t-x)\mid dt \\
&\quad + \frac{C(1+cx)}{nx}(\Big|f(2x)-f(x)-xf'(x^+)\Big| \\
&\quad + \frac{C(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}]} \bigvee_{x}^{x+\frac{x}{k}} ((f')_x) + \frac{x}{\sqrt{n}} \bigvee_{x}^{x+\frac{x}{\sqrt{n}}} ((f')_x).
\end{aligned} \tag{10.16}$$

Next, applying Hölder's inequality, we proceed as follows for the estimation of the first two terms on the right-hand side of (10.16):

$$\frac{C}{x}\int_{2x}^{\infty}\hat{K}_{n,r}(x,t)t^{\gamma}\mid t-x\mid dt+\frac{\mid f(x)\mid}{x^{2}}\int_{2x}^{\infty}\hat{K}_{n,r}(x,t)(t-x)^{2}dt$$
$$\leq\frac{C}{x}(\int_{2x}^{\infty}\hat{K}_{n,r}(x,t)t^{2\gamma}dt)^{\frac{1}{2}}(\int_{0}^{\infty}\hat{K}_{n,r}(x,t)(t-x)^{2}dt)^{\frac{1}{2}}$$
$$+\frac{\mid f(x)\mid}{x^{2}}(\int_{2x}^{\infty}\hat{K}_{n,r}(x,t)(t-x)^{2}dt).$$

In order to estimate the integral $\int_{2x}^{\infty}\hat{K}_{n,r}(x,t)t^{2\gamma}\,dt$, we proceed as follows:

$$\int_{2x}^{\infty}\hat{K}_{n,r}(x,t)t^{2\gamma}\,dt\leq 2^{2\gamma}\int_{2x}^{\infty}\hat{K}_{n,r}(x,t)(t-x)^{2\gamma}\,dt$$
$$\leq 2^{2\gamma}\int_{|t-x|\geq\delta}\hat{K}_{n,r}(x,t)(t-x)^{2\gamma}\,dt,\ \ \text{on choosing }\ 0<\delta\leq x$$
$$\leq\frac{2^{2\gamma}}{\delta^{2m-2\gamma}}\int_{0}^{\infty}\hat{K}_{n,r}(x,t)(t-x)^{2m}\,dt\ \ \ m\text{ being a positive integer }>\gamma$$
$$=\frac{2^{2\gamma}}{\delta^{2m-2\gamma}}O(n^{-m})=\frac{2^{2\gamma}}{\delta^{2m-2\gamma}}O(n^{-\gamma}).$$

Hence, in view of Lemma 10.5,

$$\frac{C}{x}\int_{2x}^{\infty}\hat{K}_{n,r}(x,t)t^{\gamma}\Big|t-x\Big|dt+\frac{\mid f(x)\mid}{x^{2}}\int_{2x}^{\infty}\hat{K}_{n,r}(x,t)(t-x)^{2}dt$$
$$\leq C2^{\gamma}O(n^{-\gamma/2})\frac{\sqrt{Cx(1+cx)}}{\sqrt{n}}+\Big|f(x)\Big|\frac{C(1+cx)}{nx}. \tag{10.17}$$

By using the Schwarz inequality and Remark 10.3, we estimate the third term on the right side of (10.16) as follows:

$$\Big|f'(x^{+})\Big|\Big|\int_{2x}^{\infty}\hat{K}_{n,r}(x,t)\Big|t-x\Big|dt\leq\Big|f'(x^{+})\Big|\Big|\int_{0}^{\infty}\hat{K}_{n,r}(x,t)\mid t-x\mid dt$$
$$\leq\Big|f'(x^{+})\Big|\Big|(\int_{0}^{\infty}\hat{K}_{n,r}(x,t)(t-x)^{2}dt)^{\frac{1}{2}}(\int_{0}^{\infty}\hat{K}_{n,r}(x,t)dt)^{\frac{1}{2}}$$
$$=\Big|f'(x^{+})\Big|\frac{\sqrt{Cx(1+cx)}}{\sqrt{n}}. \tag{10.18}$$

By using (10.16)–(10.18), we have

$$| A_{n,r}(f,x) |$$
$$\leq C2^{\gamma}O(n^{-\gamma/2})\frac{\sqrt{Cx(1+cx)}}{\sqrt{n}}+| f(x) |\frac{C(1+cx)}{nx}+| f'(x^+) | \frac{\sqrt{Cx(1+cx)}}{\sqrt{n}}$$
$$+\frac{C(1+cx)}{nx}(| f(2x)-f(x)-xf'(x^+) | +\frac{C(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{x}{k}}((f')_x)$$
$$+\frac{x}{\sqrt{n}}\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}((f')_x). \tag{10.19}$$

By collecting the estimates (10.14), (10.15), and (10.19), we get the required result.
This completes the proof of Theorem 10.3. ■

Remark 10.5. For $r = 0$ and $c = 0$, we can easily obtain the result, which was established by Gupta et al. in 2008 [143]

Furthermore, if we apply Lemma 10.7, we immediately have the following result for the derivatives of the operators $V_{n,r}$.

Theorem 10.4 ([144]). *Let $f^{(s)} \in DB_{\gamma}(0,\infty), \gamma > 0$, and $x \in (0,\infty)$. Then for $C > 2$ and for n sufficiently large, we have*

$$\left|\frac{[n-c(r+s+1)]}{(n-c)\beta(n,r+s,c)}D^sV_{n,r}(f,x)-f^{(s)}(x)\right|$$
$$\leq\frac{C(1+cx)}{n}(\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-x/k}^{x+x/k}((D^{s+1}f)_x)+\frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}((D^{s+1}f)_x))$$

$$+\frac{C(1+cx)}{nx}(| D^sf(2x)-D^sf(x)-xD^{s+1}f(x^+) | + | D^sf(x) |)$$
$$+\sqrt{\frac{Cx(1+cx)}{n}}\left(C2^{\gamma}O(n^{\frac{-\gamma}{2}})+| D^{s+1}f(x^+) |\right)$$
$$+\frac{1}{2}\sqrt{\frac{Cx(1+cx)}{n}}\, | D^{s+1}f(x^+)-D^{s+1}f(x^-) |$$
$$+\frac{1}{2}\, | D^{s+1}f(x^+)+D^{s+1}f(x^-) | \,\frac{(r+s+1)(1+2cx)}{[n-c(r+s+2)]},$$

where $\bigvee_a^b f(x)$ denotes the total variation of f_x on $[a,b]$, and f_x is defined by

$$D^{s+1} f_x(t) = \begin{cases} D^{s+1} f(t) - D^{s+1} f(x^-), & 0 \le t < x, \\ 0, & t = x, \\ D^{s+1} f(t) - D^{s+1} f(x^+), & x < t < \infty. \end{cases}$$

10.3 Baskakov–Beta Operators for DBV

We [134] studied the Baskakov–Beta operators in generalized form as

$$V_{n,r}(f,x) = \frac{(n+r-1)!\,(n-r-1)!}{((n-1)!)^2} \sum_{k=0}^{\infty} p_{n+r,k}(x) \int\limits_0^{\infty} b_{n-r,k+r}(t) f(t)dt, \tag{10.20}$$

where $n \in N, r \in N^0, n > r$, and the Baskakov and Beta basis functions are respectively defined as

$$p_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}, b_{n,k}(t) = \frac{1}{B(k+1,n)} \frac{t^k}{(1+t)^{n+k+1}},$$

and $B(m,n) = \frac{(m-1)!(n-1)!}{(n+m-1)!}$.

The class of absolutely continuous functions f defined on $(0,\infty)$ is defined by $B_q(0,\infty), q > 0$, and satisfies

(i) $|f(t)| \le C_1 t^q, C_1 > 0$,

(ii) having a derivative f' on the interval $(0,\infty)$ that coincides a.e. with a function that is of bounded variation on every finite subinterval of $(0,\infty)$. It can be observed that all functions $f \in B_q(0,\infty)$ possess for each $C > 0$ the representation

$$f(x) = f(c) + \int_c^x \psi(t)dt, \quad x \ge c.$$

Theorem 10.5 ([134]). *Let $f \in B_q(0,\infty)$, $q > 0$, and $x \in (0,\infty)$. Then for $C > 2$ and n sufficiently large, we have*

$$\left| \frac{((n-1)!)^2}{(n+r-1)!(n-r-1)!} V_{n,r}(f,x) - f(x) \right|$$

$$\leq \frac{C(1+x)}{n-r-2}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-x/k}^{x+x/k}((f')_x) + \frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}((f')_x)$$

$$+ \frac{C(1+x)}{(n-r-2)x}\left(\left|f(2x) - f(x) - xf'(x^+)\right| + |f(x)|\right) + O(n^{-q})$$

$$+ |f'(x^+)|\frac{C(1+x)}{n-r-2} + \frac{1}{2}\sqrt{\frac{Cx(1+x)}{n-r-2}}\left|f'\left(x^+\right) - f'\left(x^-\right)\right|$$

$$+ \frac{1}{2}\left|f'\left(x^+\right) + f'\left(x^-\right)\right|\frac{(1+r)+x(1+2r)}{n-r-1},$$

where $\bigvee_a^b f(x)$ *denotes the total variation of* f_x *on* $[a,b]$, *and the auxiliary function* f_x *is defined by Theorem 10.3.*

Corollary 10.2 ([134]). *Let* $f^{(s)} \in DB_q(0,\infty)$, $q > 0$, *and* $x \in (0,\infty)$. *Then, for* $C > 2$ *and for n sufficiently large, we have*

$$\left|\frac{((n-1)!)^2}{(n+r-1)!(n-r-1)!}D^s V_{n,r}(f,x) - f^{(s)}(x)\right|$$

$$\leq \frac{C\,(1+x)}{n-r-2}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-x/k}^{x+x/k}((D^{s+1}f)_x) + \frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}((D^{s+1}f)_x)$$

$$+\frac{C\,(1+x)}{x\,(n-r-2)}(|\,D^s f(2x) - D^s f(x) - xD^{s+1}f(x^+)\,| + |\,D^s f(x)\,|) + O(n^{-q})$$

$$+\frac{C(1+x)}{n-r-2}\,|\,D^{s+1}f(x^+)\,| + \frac{1}{2}\sqrt{\frac{Cx\,(1+x)}{n}}\,|\,D^{s+1}f(x^+) - D^{s+1}f(x^-)\,|$$

$$+\frac{1}{2}\,|\,D^{s+1}f(x^+) + D^{s+1}f(x^-)\,|\,\frac{(1+r)+x(1+2r)}{n-r-1},$$

where $\bigvee_a^b f(x)$ *denotes the total variation of* f_x *on* $[a,b]$, *and* f_x *is defined by Theorem 10.4.*

10.4 Szász–Mirakian–Stancu–Durrmeyer Operators

Let α and β be two nonnegative parameters satisfying the condition $0 \leq \alpha \leq \beta$. For any nonnegative integer n,

$$f \in C\,[0,\infty) \to S_n^{(\alpha,\beta)} f,$$

the Stancu-type Szász–Mirakian–Durrmeyer operators recently introduced in [147] are defined by

$$S_{n,r}^{(\alpha,\beta)}(f,x) = n\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^{\infty} s_{n,k+r}(t) f\left(\frac{nt+\alpha}{n+\beta}\right)dt, \tag{10.21}$$

where

$$s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}.$$

For $\alpha=\beta=r=0$, these operators become the well-known Szász –Mirakian–Durrmeyer operators $S_n^{(0,0)}(f,x) = S_n(f,x)$ introduced by Mazhar and Totik [191]. We need the following basic results to obtain the rate of convergence:

Lemma 10.8 ([147]). *For $r, m \in \mathbb{N}\cup\{0\}$, $0\le\alpha\le\beta$, let us consider*

$$\mu_{n,m,r}^{(\alpha,\beta)}(x) = S_{n,r}^{(\alpha,\beta)}((t-x)^m, x) = n\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^{\infty} s_{n,k+r}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt.$$

We get

$$\mu_{n,0,r}^{(\alpha,\beta)}(x)=1,\ \ \mu_{n,1,r}^{(\alpha,\beta)}(x) = \frac{\alpha+r+1-\beta x}{n+\beta},$$

$$\mu_{n,2,r}^{(\alpha,\beta)}(x) = \frac{\beta^2x^2+2(n-\alpha\beta-\beta-\beta r)x+(\alpha+r+1)(\alpha+r+2)-\alpha}{(n+\beta)^2},$$

and

$$\begin{aligned}(n+\beta)\mu_{n,m+1,r}^{(\alpha,\beta)}(x) &= x[\mu_{n,m,r}^{(\alpha,\beta)}(x)]' + (m+\alpha+r+1-\beta x)\mu_{n,m,r}^{(\alpha,\beta)}(x)\\ &\quad + m\left(\frac{2(n+\beta)x-\alpha}{n+\beta}\right)\mu_{n,m-1,r}^{(\alpha,\beta)}(x).\end{aligned}$$

Remark 10.6 ([147]). From Lemma 10.8, for $n\ge\beta^2+(\alpha+r)^2+3r+2\alpha+2$ and any $x\in(0,\infty)$, we have

$$\mu_{n,2,r}^{(\alpha,\beta)}(x) \le \frac{(x+1)^2}{n+\beta}.$$

Remark 10.7 ([147]). Applying the Cauchy–Schwarz inequality and Remark 10.6, for $n\ge\beta^2+(\alpha+r)^2+3r+2\alpha+2$, we have

$$S_{n,r}^{(\alpha,\beta)}(|t-x|,x) \leq \left[\mu_{n,2,r}^{(\alpha,\beta)}(x)\right]^{\frac{1}{2}} \leq \frac{x+1}{\sqrt{n+\beta}}.$$

Lemma 10.9 ([147]). *Suppose that $x \in (0,\infty)$. Then for $n \geq r^2+3r+2$, we have*

$$\lambda_{n,r}(x,y) = n\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^y s_{n,k+r}(t)dt \leq \frac{(x+1)^2}{n(x-y)^2},\ 0 \leq y < x,$$

$$1-\lambda_{n,r}(x,z) = n\sum_{k=0}^{\infty} s_{n,k}(x)\int_z^{\infty} s_{n,k+r}(t)dt \leq \frac{(x+1)^2}{n(z-x)^2},\ \ x < z < \infty.$$

Proof. The proof follows directly from Remark 10.6 in the case $\alpha = \beta = 0$. As for the first inequality, we have

$$\begin{aligned}\lambda_{n,r}(x,y) &= n\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^y s_{n,k+r}(t)dt\\ &= \frac{S_{n,r}^{(0,0)}\left((t-x)^2,x\right)}{(y-x)^2} \leq \frac{(x+1)^2}{n(x-y)^2}.\end{aligned}$$

We can prove the second inequality similarly. ∎

Lemma 10.10 ([147]). *Let us consider that f is s-times differentiable on $[0,\infty)$ such that $f^{(s-1)}(t) = O(t^q)$, as $t \to \infty$, where q is a positive integer. Then, for any $r,s \in N^0$ and $n >$ $\max\{q, r+s+1\}$, we have*

$$D^s S_{n,r}^{(\alpha,\beta)}(f,x) = \left(\frac{n}{n+\beta}\right)^s S_{n,r+s}^{(\alpha,\beta)}(D^s f,x),\ D \equiv \frac{d}{dx}.$$

Proof. First, by simple computation, we have

$$D\left[s_{n,k}(x)\right] = n\left[s_{n,k-1}(x) - s_{n,k}(x)\right]. \tag{10.22}$$

The identity (10.22) is true even for the case $k = 0$, as we observe for $r < 0$, $s_{n,r}(x) = 0$. We shall prove the result by using a principle of mathematical induction. Using (10.22), we have

$$\begin{aligned}D\left[S_{n,r}^{(\alpha,\beta)}(f,x)\right] &= n\sum_{k=0}^{\infty} Ds_{n,k}(x)\int_0^{\infty} s_{n,k+r}(t)f\left(\frac{nt+\alpha}{n+\beta}\right)dt\\ &= n\sum_{k=0}^{\infty} n\left[s_{n,k-1}(x)-s_{n,k}(x)\right]\int_0^{\infty} s_{n,k+r}(t)f\left(\frac{nt+\alpha}{n+\beta}\right)dt\end{aligned}$$

$$= n^2 \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} [s_{n,k+r+1}(t) - s_{n,k+r}(t)]\, f\left(\frac{nt+\alpha}{n+\beta}\right) dt.$$

Using (10.22) and integrating by parts, we have

$$\begin{aligned} DS_{n,r}^{(\alpha,\beta)}(f,x) &= n^2 \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} -\frac{D\,[s_{n,k+r+1}(t)]}{n} f\left(\frac{nt+\alpha}{n+\beta}\right) dt \\ &= \frac{n^2}{n+\beta} \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} s_{n,k+r+1}(t) f^{(1)}\left(\frac{nt+\alpha}{n+\beta}\right) dt \\ &= \frac{n}{n+\beta} S_{n,r+1}^{(\alpha,\beta)}(Df,x), \end{aligned}$$

which means that the identity is satisfied for $s = 1$. Let us suppose that the result holds for $s = l$; that is,

$$\begin{aligned} D^l S_{n,r}^{(\alpha,\beta)}(f,x) &= \left(\frac{n}{n+\beta}\right)^l S_{n,r+l}^{(\alpha,\beta)}\left(D^l f,x\right) \\ &= n\left(\frac{n}{n+\beta}\right)^l \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} s_{n,k+r+l}(t) D^l f\left(\frac{nt+\alpha}{n+\beta}\right) dt. \end{aligned}$$

Now,

$$\begin{aligned} D^{l+1} S_{n,r}^{(\alpha,\beta)}(f,x) &= n\left(\frac{n}{n+\beta}\right)^l \sum_{k=0}^{\infty} D s_{n,k}(x) \int_0^{\infty} s_{n,k+r+l}(t) D^l f\left(\frac{nt+\alpha}{n+\beta}\right) dt \\ &= n\left(\frac{n}{n+\beta}\right)^l \sum_{k=0}^{\infty} n\,[s_{n,k+r+l-1}(x) - s_{n,k+r+l}(x)] \\ &\qquad \int_0^{\infty} s_{n,k+r+l}(t) D^l f\left(\frac{nt+\alpha}{n+\beta}\right) dt \\ &= n^2\left(\frac{n}{n+\beta}\right)^l \sum_{k=0}^{\infty} s_{n,k}(x) \\ &\qquad \int_0^{\infty} [s_{n,k+r+l+1}(t) - s_{n,k+r+l}(t)]\, D^l f\left(\frac{nt+\alpha}{n+\beta}\right) dt \\ &= n^2\left(\frac{n}{n+\beta}\right)^l \sum_{k=0}^{\infty} s_{n,k}(x) \end{aligned}$$

$$\int_0^\infty -\frac{D\left[s_{n,k+r+l+1}(t)\right]}{n} D^l f\left(\frac{nt+\alpha}{n+\beta}\right) dt.$$

Integrating by parts the last integral, we get

$$D^{l+1} S_{n,r}^{(\alpha,\beta)}(f,x) = n\left(\frac{n}{n+\beta}\right)^{l+1} \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^\infty s_{n,k+r+l+1}(t) D^{l+1} f\left(\frac{nt+\alpha}{n+\beta}\right) dt.$$

Therefore,

$$D^{l+1} S_{n,r}^{(\alpha,\beta)}(f,x) = \left(\frac{n}{n+\beta}\right)^{l+1} S_{n,r+l+1}^{(\alpha,\beta)}\left(D^{l+1} f(x)\right).$$

Thus, the result is true for $s = l+1$, and hence, by mathematical induction, the proof of the lemma is complete. ∎

The class of absolutely continuous functions f defined on $(0,\infty)$ is defined by $B_q(0,\infty), q > 0$, and satisfies

(i) $|f(t)| \leq C_1 t^q, C_1 > 0$,

(ii) having a derivative f' on the interval $(0,\infty)$ that coincides a.e. with a function that is of bounded variation on every finite subinterval of $(0,\infty)$. It can be observed that for all functions $f \in B_q(0,\infty)$ possess for each $C > 0$ the representation

$$f(x) = f(c) + \int_c^x \psi(t)dt, \quad x \geq c.$$

Theorem 10.6 ([147]). *Let* $f \in B_q(0,\infty)$*,* $q > 0$*, and* $x \in (0,\infty)$*. Then for* n *sufficiently large, we have*

$$\begin{aligned}\left|S_{n,r}^{(\alpha,\beta)}(f,x) - f(x)\right| \leq{} & \frac{(x+1)^2}{nx} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-x/k}^{x+x/k}((f')_x) + \frac{x}{\sqrt{n}} \bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}((f')_x) \\ & + \frac{(1+1/x)^2}{n}(|\, f(2x) - f(x) - xf'(x^+)\,| \\ & + |f(x)|) + O(n^{-q}) + |f'(x^+)| \frac{(x+1)^2}{n} \\ & + \frac{1}{2}\frac{x+1}{\sqrt{n+\beta}}\,|\, f'\left(x^+\right) - f'\left(x^-\right)| \\ & + \frac{\alpha+r+1-\beta x}{2(n+\beta)}\,|\, f'\left(x^+\right) + f'\left(x^-\right)|,\end{aligned}$$

where $\bigvee_a^b f(x)$ *denotes the total variation of* f_x *on* $[a, b]$, *and the auxiliary function* f_x *is defined by*

$$f_x(t) = \begin{cases} f(t) - f(x^-), & 0 \le t < x, \\ 0, & t = x, \\ f(t) - f(x^+), & x < t < \infty. \end{cases}$$

Proof. We'll use the identity

$$f'(u) = (f')_x(u) + \frac{f'\left(x^+\right) + f'\left(x^-\right)}{2} + \frac{f'\left(x^+\right) - f'\left(x^-\right)}{2} sgn(u - x)$$
$$+ \left[f'(x) - \frac{f'\left(x^+\right) + f'\left(x^-\right)}{2} \right] \chi_x(u), \tag{10.23}$$

where

$$\chi_x(u) = \begin{cases} 1, & u = x, \\ 0, & u \neq x. \end{cases}$$

On applying the mean value theorem, we get

$$S_{n,r}^{(\alpha,\beta)}(f, x) - f(x) = S_{n,r}^{(\alpha,\beta)}\left(\int_x^t f'(u)du, x \right). \tag{10.24}$$

Now, by using above identity (10.23) in (10.24) and the fact that $S_{n,r}^{\alpha,\beta}$ $\left(\int_x^t \chi_x(u)du, x \right) = 0$, after simple computation, we have

$$\begin{aligned} \left| S_{n,r}^{\alpha,\beta}(f, x) - f(x) \right| \le & \left| \int_x^\infty \left(\int_x^t (f')_x(u)du \right) n \sum_{k=0}^{\infty} s_{n,k}(x) s_{n,k+r}(t)dt \right. \\ & \left. + \int_0^x \left(\int_x^t (f')_x(u)du \right) n \sum_{k=0}^{\infty} s_{n,k}(x) s_{n,k+r}(t)dt \right| \\ & + \frac{\left| f'\left(x^+\right) + f'\left(x^-\right) \right|}{2} \mu_{n,1,r}^{(\alpha,\beta)}(x) \\ & + \frac{\left| f'\left(x^+\right) - f'\left(x^-\right) \right|}{2} [\mu_{n,2,r}^{(\alpha,\beta)}(x)]^{1/2} \end{aligned}$$

$$=\mid A_{n,r}(f,x)+B_{n,r}(f,x)\mid+\frac{\left|f'(x^+)+f'(x^-)\right|}{2}\mu_{n,1,r}^{(\alpha,\beta)}(x).$$

$$+\frac{\left|f'(x^+)-f'(x^-)\right|}{2}[\mu_{n,2,r}^{(\alpha,\beta)}(x)]^{1/2}. \tag{10.25}$$

Applying Remarks 10.6 and 10.7 in (10.25), we have

$$\begin{aligned}\left|S_{n,r}^{\alpha,\beta}(f,x)-f(x)\right| &\le |A_{n,r}(f,x)|+|B_{n,r}(f,x)|\\ &\quad+\frac{\left|f'(x^+)-f'(x^-)\right|}{2}\frac{x+1}{\sqrt{n+\beta}}\\ &\quad+\frac{\left|f'(x^+)+f'(x^-)\right|}{2}\frac{\alpha+r+1-\beta x}{(n+\beta)}.\end{aligned} \tag{10.26}$$

Estimating the terms $A_{n,r}(f,x)$ and $B_{n,r}(f,x)$ will lead to proof of the theorem. First,

$$\begin{aligned}\mid A_{n,r}(f,x)\mid &=\left|\int_x^\infty\left(\int_x^t(f')_x(u)du\right)n\sum_{k=0}^\infty s_{n,k}(x)s_{n,k+r}(t)dt\right|\\ &=\left|\int_{2x}^\infty(\int_x^t(f')_x(u)du)n\sum_{k=0}^\infty s_{n,k}(x)s_{n,k+r}(t)dt\right.\\ &\quad\left.+\int_x^{2x}(\int_x^t(f')_x(u)du)d_t(1-\lambda_{n,r}(x,t))\right|\\ &\le\left|n\sum_{k=0}^\infty s_{n,k}(x)\int_{2x}^\infty(f(t)-f(x))s_{n,k+r}(t)dt\right|\\ &\quad+\left|f'(x^+)\right|\left|n\sum_{k=0}^\infty s_{n,k}(x)\int_{2x}^\infty s_{n,k+r}(t)(t-x)dt\right|\\ &\quad+\left|\int_x^{2x}(f')_x(u)du\right|\mid 1-\lambda_{n,r}(x,2x)\mid\\ &\quad+\int_x^{2x}\mid(f')_x(t)\mid\mid 1-\lambda_{n,r}(x,t)\mid dt.\end{aligned}$$

Applying Remark 10.6, with $\alpha=\beta=0$, we have

$$\begin{aligned}
\mid A_{n,r}(f,x)\mid \;\le\;& n\sum_{k=0}^{\infty}s_{n,k}(x)\int_{2x}^{\infty}s_{n,k+r}(t)C_1t^{2q}dt\\
&+\frac{\mid f(x)\mid}{x^2}n\sum_{k=0}^{\infty}s_{n,k}(x)\int_{2x}^{\infty}s_{n,k+r}(t)(t-x)^2dt\\
&+\mid f'\left(x^+\right)\mid\int_{2x}^{\infty}n\sum_{k=0}^{\infty}s_{n,k}(x)s_{n,k+r}(t)\mid t-x\mid dt\\
&+\frac{(1+1/x)^2}{n}\mid f(2x)-f(x)-xf'(x^+)\mid\\
&+\frac{(x+1)^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{x}{k}}((f')_x)+\frac{x}{\sqrt{n}}\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}((f')_x). \qquad (10.27)
\end{aligned}$$

To estimate the integral $n\sum_{k=0}^{\infty}s_{n,k}(x)\int_{2x}^{\infty}s_{n,k+r}(t)C_1t^{2q}dt$ in (10.27), we proceed as follows:

Obviously, $t\ge 2x$ implies that $t\le 2(t-x)$, and it follows from Lemma 10.8 that

$$\begin{aligned}
&n\sum_{k=0}^{\infty}s_{n,k}(x)\int_{2x}^{\infty}s_{n,k+r}(t)t^{2q}dt\le 2^{2q}\sum_{k=0}^{\infty}s_{n,k}(x)\int_{0}^{\infty}s_{n,k+r}(t)(t-x)^{2q}dt\\
&=2^{2q}\mu_{n,2q,r}^{(\alpha,\beta)}(x)=O(n^{-q})(n\to\infty).
\end{aligned}$$

Applying the Schwarz inequality and Remark 10.1 ($\alpha=\beta=0$), we can estimate the third term on the right-hand side of (10.27) as follows:

$$\begin{aligned}
&\mid f'\left(x^+\right)\mid n\sum_{k=0}^{\infty}s_{n,k}(x)\int_{2x}^{\infty}s_{n,k+r}(t)\mid t-x\mid dt\\
&\le\frac{\mid f'\left(x^+\right)\mid}{x}n\sum_{k=0}^{\infty}s_{n,k}(x)\int_{0}^{\infty}s_{n,k+r}(t)(t-x)^2dt=\mid f'\left(x^+\right)\mid\frac{(x+1)^2}{nx}.
\end{aligned}$$

Thus, by Lemma 10.8 and Remark 10.6 ($\alpha=\beta=0$), we have

$$\begin{aligned}
\mid A_{n,r}(f,x)\mid\le\;&\mathcal{O}(n^{-q})+\mid f'\left(x^+\right)\mid.\frac{(x+1)^2}{nx}\\
&+\frac{(1+1/x)^2}{n}(\mid f(2x)-f(x)-xf'(x^+)\mid+\mid f(x)\mid)
\end{aligned}$$

$$+\frac{(x+1)^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{x}{k}}((f')_x)+\frac{x}{\sqrt{n}}\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}((f')_x). \qquad (10.28)$$

On applying Lemma 10.9 with $y = x - \frac{x}{\sqrt{n}}$, and integrating by parts, we have

$$\begin{aligned}
|\, B_{n,r}(f,x)\,| &= \left|\int_0^x \int_x^t (f')_x(u)du\,d_t(\lambda_{n,r}(x,t))\right| \\
&= \int_0^x \lambda_{n,r}(x,t)(f')_x(t)dt \le \left(\int_0^y + \int_y^x\right) |\,(f')_x(t)\,||\,\lambda_{n,r}(x,t)\,|\,dt \\
&\le \frac{(x+1)^2}{n}\int_0^y \bigvee_t^x((f')_x)\frac{1}{(x-t)^2}dt + \int_y^x \bigvee_t^x((f')_x)dt \\
&\le \frac{(x+1)^2}{n}\int_0^y \bigvee_t^x((f')_x)\frac{1}{(x-t)^2}dt + \frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}((f')_x).
\end{aligned}$$

Let $u = \frac{x}{x-t}$. Then we have

$$\begin{aligned}
\frac{(x+1)^2}{n}\int_0^y \bigvee_t^x((f')_x)\frac{1}{(x-t)^2}dt &= \frac{(x+1)^2}{n}\int_1^{\sqrt{n}} \bigvee_{x-\frac{x}{u}}^{x}((f')_x)du \\
&\le \frac{(x+1)^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x}((f')_x).
\end{aligned}$$

Thus,

$$|\, B_{n,r}(f,x)\,| \le \frac{(x+1)^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x}((f')_x)+\frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}((f')_x). \qquad (10.29)$$

The required result is obtained on combining (10.26), (10.28), and (10.29). ■

As a consequence of Lemma 10.10, we have the following corollary:

Corollary 10.3 ([147]). *Let* $f^{(s)} \in DB_q(0,\infty)$, $q > 0$, *and* $x \in (0,\infty)$. *Then, for n sufficiently large, we have*

$$\left|D^s S_{n,r}^{(\alpha,\beta)}(f,x) - f^{(s)}(x)\right| \le \frac{(x+1)^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-x/k}^{x+x/k}((D^{s+1}f)_x)$$

$$+\frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}((D^{s+1}f)_x)$$
$$+\frac{(1+1/x)^2}{n}(\mid D^s f(2x)-D^s f(x)-xD^{s+1}f(x^+)\mid$$
$$+\mid D^s f(x)\mid)+O(n^{-q})$$
$$+\frac{(x+1)^2}{nx}\mid D^{s+1}f(x^+)\mid$$
$$+\frac{1}{2}\frac{x+1}{\sqrt{n+\beta}}\mid D^{s+1}f(x^+)-D^{s+1}f(x^-)\mid$$
$$+\frac{1}{2}\mid D^{s+1}f(x^+)+D^{s+1}f(x^-)\mid\frac{\alpha+r+1-\beta x}{n+\beta},$$

where $\bigvee_a^b f(x)$ *denotes the total variation of* f_x *on* $[a,b]$, *and* f_x *is defined by*

$$D^{s+1}f_x(t)=\begin{cases}D^{s+1}f(t)-D^{s+1}f(x^-), & 0\leq t<x,\\ 0, & t=x,\\ D^{s+1}f(t)-D^{s+1}f(x^+), & x<t<\infty.\end{cases}$$

If we consider the class $L[0,\infty)$ of all measurable functions defined on $[0,\infty)$ such that

$$L[0,\infty):=\left\{f:\int_0^\infty e^{-nt}f(t)dt<\infty \text{ for some positive integer } n\right\},$$

we can observe that this class is bigger than the class of all integrable functions on $[0,\infty)$.

Further, we consider

$$L_\alpha[0,\infty):=\left\{f\in L[0,\infty):f(t)=O(e^{\alpha t}),t\to\infty,\alpha>0\right\}.$$

We have the following asymptotic formula by using Lemma 10.8.

Theorem 10.7 ([147]). *Let* $f\in L_\alpha[0,\infty)$, *and suppose it is bounded on every finite subinterval of* $[0,\infty)$ *having a derivative of order* $r+2$ *at a point* $x\in(0,\infty)$. *Then we have*

$$\lim_{n\to\infty}n[(S_{n,r}^{(\alpha,\beta)})^{(r)}(f,x)-f^{(r)}(x)]=(\alpha+r+1-\beta x)f^{(r+1)}(x)+xf^{(r+2)}(x).$$

Chapter 11
Future Scope and Open Problems

In 1983, based on two parameters α, β satisfying the conditions $0 \le \alpha \le \beta$, Stancu [222] proposed a generalization of the classical Bernstein polynomials. In more recent papers, some approximation properties of the Stancu-type generalization on different operators were discussed (see, e.g., [50, 133, 187, 238]). Future studies could address defining the Stancu-type generalization of other operators and the convergence behavior, asymptotic formulas, and rate of convergence for functions of BV and for functions having derivatives of BV.

Lenze [175] studied the multivariate functions of bounded variation. In future studies, one can find the results on the rate of convergence for BV functions for multivariate operators. To the best of our knowledge, this problem has not been done to date, and there may be a future scope in this direction.

It has been observed that Bernstein polynomials reproduce constant as well as linear functions. To make the convergence faster, King [172] modified the well-known Bernstein polynomials to preserve the test functions e_0 and e_2, where $e_i = t^i$. Gupta and Duman [111] modified the Bernstein–Durrmeyer polynomials, to preserve the test functions e_0 and e_1. In [111], the authors achieved a better approximation on some compact interval. Thus, one can study in this direction for analogous results presented in Gupta and Duman [111], that is, direct, inverse results in ordinary and simultaneous approximation and in the rate of approximation for functions of BV and DBV.

Also, in order to study the overconvergence phenomenon by considering the operators in a complex domain, one can extend the studies for the other mixed summation–integral-type operators, as mentioned in Chap. 2. As of this publication, no result has been available for MKZ operators and mixed summation–integral-type operators. For MKZ operators, there is a recurrence formula available for moments, which is required in analysis; one can find some other techniques to overcome this difficulty.

The approximation properties of Durrmeyer-type mixed operators have been found to hold well for direct, inverse, and saturation results in ordinary and simultaneous approximation. We can even easily define their q analog and Bézier variants. But the rate of convergence on functions of bounded variation for such

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mixed operators have difficulties in analysis. For example, if we consider the Baskakov–Szász operators (see [117]), which for $x \in [0, \infty)$ are defined by

$$BS_n(f, x) = n \sum_{k=0}^{\infty} b_{n,k}(x) \int_0^{\infty} s_{n,k}(t) f(t) dt, \tag{11.1}$$

where

$$s_{n,k}(x) = \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!}, b_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}},$$

the main problem is in obtaining the value of $\text{sign}_\alpha(t-x)$ as it is needed in analysis to relate summation and integration basis functions. If they are of the same type, then one can relate the integral with the summation of the basis function. As for Szász–Durrmeyer operators in Theorem 7.10, we use the equality $n \int_x^{\infty} s_{n,k}(t) dt = \sum_{j=0}^{k} s_{n,j}(x)$. Also, for Baskakov–Durrmeyer operators, in the proof of Theorem 7.12, we need the equality $(n-1) \int_x^{\infty} b_{n,k}(t) dt = \sum_{j=0}^{k} b_{n-1,j}(x)$. Even for Baskakov–Beta operators (8.51), whose bases are of a similar type, we have the relation $\int_x^{\infty} v_{n,k}(t) dt = \sum_{j=0}^{k} b_{n,j}(x)$. But for the mixed operators of the form (11.1), it is not possible to have an equality that relates different basis functions. We raised this problem in earlier papers, but to date, this has not been resolved. Thus, one can consider as an open problem the rate of convergence for bounded variation functions for the mixed operators, such as Szász–Baskakov, Szász–Beta, and Beta–Szász . We have observed that there may not be any problem in obtaining the results on the rate of convergence for functions having derivatives of bounded variation on mixed operators.

In 1972, Jain [161] introduced the following operators:

$$J_n^{\beta}(f, x) = \sum_{k=0}^{\infty} h_{n,k}^{(\beta)}(x) f(k/n), \tag{11.2}$$

where $0 \le \beta < 1$, and the basis function is defined as

$$h_{n,k}^{(\beta)}(x) = \frac{nx(nx + k\beta)^{k-1}}{k!} e^{-(nx+k\beta)}.$$

Jain observed [161] that $\sum_{k=0}^{\infty} h_{n,k}^{(\beta)}(x) = 1$. As a special case when $\beta = 0$, the operators (11.2) reduce to the well-known Szász–Mirakyan operators.

Also, Lupas [186] introduced the operator $L_n(f, x)$ for $f \in [0, \infty) \to \mathbb{R}$ as

$$L_n(f, x) = \sum_{k=0}^{\infty} l_{n,k}(x) f\left(\frac{k}{n}\right), \tag{11.3}$$

where

$$l_{n,k}(x) = 2^{-nx}\binom{nx+k-1}{k}2^{-k}.$$

In order to approximate the integral functions, Agratini [16] introduced the Kantorovich and Durrmeyer variant of Lupas operators defined by (11.3).

These operators have interesting approximation properties; to date we have not found much on such operators, so one can study the convergence properties of these operators in local and global settings. Also, integral modifications of Kantorovich and Durrmeyer types can be defined, and some results for functions of bounded variation for the operators (11.2) and (11.3) may be one of the open problems for the readers.

Another problem is to study the rate of convergence for the q operators on functions of bounded variation. In fact, in q-calculus, continuity is required; no result is available for point of discontinuity. Thus, it is difficult to study analogous results of the book for q operators. Many q operators are available in the literature; their inverse theorems can be discussed in future studies.

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