ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE
XLV (1) (2005), 57-70

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# On some class of exponential type operators

**Abstract.** Starting from a differential equation $\frac{\partial}{\partial t}W(\lambda,t,u) = \frac{\lambda(u-t)}{p(t)}W(\lambda,t,u) - \beta W(\lambda,t,u)$ for the kernel of an operator $S_\lambda(f,t) = \int_A^B W(\lambda,t,u)f(u)du$ with the normalization condition $\int_A^B W(\lambda,t,u)du = 1$ we prove some properties which are similar to properties proved by Ismail and May for the exponential operators. In particular, we show that all these operators are approximation operators. Moreover, a method of determining $S_\lambda$ for a given function $p$ is introduced.

*2000 Mathematics Subject Classification:* 41A36, 41A25, 35K15.

*Key words and phrases:* exponential operators, Voronovskaya type theorem, rate of convergence, limit value problems.



**1. Introduction.** C. R. May in [3], M. E. Ismail and C. R. May in [2] studied family of exponential operators. They considered the integral $S_\lambda$ defined by

$$S_\lambda(f,t) = \int_A^B W(\lambda,t,u)f(u)\,du, \quad -\infty \le A < B \le +\infty \tag{1}$$

under the following assumptions: a kernel $W$ is a positive function, satisfying the following homogenous partial differential equation

$$\frac{\partial}{\partial t}W(\lambda,t,u) = \frac{\lambda}{p(t)}W(\lambda,t,u)(u-t), \tag{2}$$

$\lambda \in \mathbb{R}$, $u,t \in (A,B)$, $p$ is analytic and positive for $t \in (A,B)$ and

$$S_\lambda(e_0,t) = 1, \quad t \in (A,B), \tag{3}$$

where $e_0(u) = 1$ for $u \in (A,B)$. They presented some well-known operators satisfying (2). For example, the Bernstein polynominals and operators of Szasz, Post-Widder, Gauss-Weierstrass and Baskakov. The above operators satisfy the condition $S_\lambda(e_1,t) = e_1(t)$, where $e_1(u) = u$ for $u \in (A,B)$.

Our purpose is to extend the results of May and Ismail to a family of operators, in which there are operators $S_\lambda$ such that $S_\lambda(e_1,t)\neq e_1(t)$. We investigate a similar family of operators. Instead of the equation (2), however, we consider the following

$$\frac{\partial}{\partial t}W(\lambda,t,u)=\frac{\lambda}{p(t)}W(\lambda,t,u)(u-t)-\beta W(\lambda,t,u), \tag{4}$$

where $\beta$ is a non-negative real number, $\lambda\in\mathbb{R}$, $u,t\in(A,B)$, $p$ is analitic and positive for $t\in(A,B)$. For these operators we obtain similar results as May and Ismail for exponential operators. We state some estimates of the rate of convergence of $S_\lambda$. We also prove the Voronovskaya type theorem for these operators. In section 4 we give examples of operators satisfying (4) and on the strength of one of these we define the function $U$ which is a solution of some limit problem.

Let $C(A,B)$ be a set of all continuous, real-valued functions on $(A,B)$. We define the space $(C_N,||\ ||_{C_N})$ as follows

$$C_N=\left\{f\in C(A,B):\exists\ M>0\ \ \forall\ t\in(A,B)\ \ |f(t)|\le Me^{N|t|}\right\}$$

$$\text{and } ||f||_{C_N}=\sup\left\{e^{-N|t|}|f(t)|:t\in(A,B)\right\}.$$

**2. Preliminaries.** In this section we give some properties of the above operators, which we use in the proofs of the main theorems. The proofs of these lemmas follow by the same method as the proofs of the appropriate properties of exponential operators in [2].

Lemma 2.1 *For each $\lambda\in\mathbb{R}$ and $t\in(A,B)$ we have*

$$S_\lambda(e_1,t)=e_1(t)+\frac{\beta p(t)}{\lambda}$$

*and*

$$S_\lambda(e_2,t)=e_2(t)+\frac{p(t)+2\beta tp(t)}{\lambda}+\frac{\beta p(t)p'(t)+\beta^2p^2(t)}{\lambda^2},$$

*where $e_2(u)=u^2$ for $u\in(A,B)$.*

Lemma 2.2 *For every $n\in\mathbb{N}^*$ we get*

$$S_\lambda(e_n,t)=e_n(t)+\sum_{k=1}^{n}\frac{\varphi_{k,n}(t)}{\lambda^k},\quad t\in(A,B),$$

*where $e_n(u)=u^n$ for $u\in(A,B)$, the function $\varphi_{k,n}$ is defined by*

$$\varphi_{k,n}(t)=t\varphi_{k,n-1}(t)+p(t)\left(\beta\varphi_{k-1,n-1}(t)+\frac{\partial}{\partial t}\varphi_{k-1,n-1}(t)\right)$$

*for $1\le k\le n$ and we put*

$$\varphi_{0,n}=e_n,\quad \varphi_{n,n-1}\equiv 0.$$

Let us denote

$$A_m(\lambda,t)=\lambda^m S_\lambda((u-t)^m,t). \tag{5}$$

LEMMA 2.3

$$A_0(\lambda,t)=1,$$

$$A_1(\lambda,t)=\beta p(t)$$

*and*

$$A_{m+1}(\lambda,t)=p(t)\frac{d}{dt}A_m(\lambda,t)+\beta p(t)A_m(\lambda,t)+m\lambda p(t)A_{m-1}(\lambda,t) \tag{6}$$

*for* $m\geq 1$.

LEMMA 2.4 *For all* $m\in\mathbb{N}$ *the operators* $A_{2m}$ *and* $A_{2m+1}$ *are polynomials in* $\lambda$, *whose the degree is* $m$. *Moreover, the coefficient of the term* $\lambda^m$ *is*

$$c_{1m}p^m(t)\quad \textit{for}\quad A_{2m}$$

*and*

$$c_{2m}p^{m+1}(t)+c_{3m}p^m(t)p'(t)\quad \textit{for}\quad A_{2m+1},$$

*where* $c_{1m}$, $c_{2m}$ *and* $c_{3m}$ *are constants.*

LEMMA 2.5 *If*

$$q(t)=\int_c^t\frac{dv}{p(v)},\quad c\in(A,B)$$

*and*

$$g(q(t))=q(g(t))=t,$$

*then the following equality holds*

$$\sum_{m=0}^{\infty}S_\lambda((u-t)^m,t)\frac{x^m}{m!}=\exp\left\{-xt+\lambda\int_t^{g\left(q(t)+\frac{x}{\lambda}\right)}\frac{vdv}{p(v)}+\beta\left(g\left(q(t)+\frac{x}{\lambda}\right)-t\right)\right\}.$$

## 3. Main results.

THEOREM 3.1 *Let* $N>0$ *and* $a$, $b$ *be such that* $A<a<b<B$, *then for a sufficiently large* $\lambda$ *we have*

$$||S_\lambda(e^{N|u|},t)||_{C[a,b]}<\infty,$$

*where* $C[a,b]$ *is the set of all continuous, real-valued functions on* $[a,b]$ *with the suppremum norm.*

Proof By the definition of $S_\lambda$ we have

$$\begin{aligned} S_\lambda(e^{Nu},t) &= e^{Nt}\int_A^B W(\lambda,t,u)e^{N(u-t)}\,du \\ &= e^{Nt}\sum_{m=0}^{\infty}\frac{N^m}{m!}S_\lambda((u-t)^m,t). \end{aligned}$$

From Lemma 2.5 we obtain

$$S_\lambda(e^{Nu},t)=\exp\left\{\lambda\int_t^{g\left(q(t)+\frac{N}{\lambda}\right)}\frac{\theta}{p(\theta)}d\theta\right\}\exp\left\{\beta\left(g\left(q(t)+\frac{N}{\lambda}\right)-t\right)\right\}.$$

Letting $\lambda\longrightarrow\infty$ and the application of de L'Hospital's formula implies that

$$\lim_{\lambda\to+\infty}S_\lambda(e^{Nu},t)=e^{Nt}\ \text{ uniformly in }\ C[a,b]. \tag{7}$$

Hence for a sufficiently large $\lambda$

$$||S_\lambda(e^{Nu},t)||_{C[a,b]}<\infty.$$

Similarly,

$$||S_\lambda(e^{-Nu},t)||_{C[a,b]}<\infty.$$

$S_\lambda$ is a positive and linear operator, so we have

$$||S_\lambda(e^{N|u|},t)||_{C[a,b]}\le||S_\lambda(e^{Nu},t)||_{C[a,b]}+||S_\lambda(e^{-Nu},t)||_{C[a,b]}<\infty,$$

because $e^{N|u|}<e^{Nu}+e^{-Nu}$. This completes the proof. ∎

Theorem 3.2 *Suppose that $m$, $\eta$ are positive numbers, $N$ is a non-negative number and $[a,b]\subset(A,B)$. Then*

$$\int_{|u-t|\ge\eta}W(\lambda,t,u)e^{Nu}du=O(\lambda^{-m})\ \textit{uniformly on } [a,b]\ \textit{as } \lambda\to+\infty. \tag{8}$$

Proof By Cauchy-Schwarz's inequality, (7) and Lemma 2.4 we have

$$\begin{aligned} &\int_{|u-t|\ge\eta}W(\lambda,t,u)e^{Nu}\,du \\ &\le \left(\int_{|u-t|\ge\eta}W(\lambda,t,u)\,du\int_{|u-t|\ge\eta}W(\lambda,t,u)e^{2Nu}\,du\right)^{\frac12} \\ &\le \left(\eta^{-4m}\lambda^{-4m}A_{4m}(\lambda,t)\right)^{\frac12}\left(S_\lambda(e^{2Nu},t)\right)^{\frac12} \\ &= O(\lambda^{-m})\quad\text{for }\ \lambda\to+\infty, \end{aligned}$$

and (8) is proved. ∎

THEOREM 3.3 *Let $f \in C_N$. Then*

$$\lim_{\lambda\to\infty} S_\lambda(f,t) = f(t), \tag{9}$$

*uniformly on every $[a,b] \subset (A,B)$.*

PROOF We have

$$\begin{aligned} |S_\lambda(f,t) - f(t)| &\le \int_{|u-t|<\delta} |f(u) - f(t)| W(\lambda,t,u)\,du \\ &\quad + \int_{|u-t|\ge\delta} |f(u) - f(t)| W(\lambda,t,u)\,du \\ &= I_1 + I_2 \end{aligned}$$

for every $\delta > 0$. Let $\epsilon > 0$. By the continuity of $f$ in $[a,b]$ there exists $\delta > 0$ such that $|f(u) - f(t)| < \epsilon$ for $|u-t| < \delta$. Hence

$$I_1 \le \frac{\epsilon}{2}\int_{|u-t|<\delta} W(\lambda,t,u)\,du \le \frac{\epsilon}{2}\int_A^B W(\lambda,t,u)\,du = \frac{\epsilon}{2}.$$

On the other hand, from (8) for a sufficiently large $\lambda$

$$I_2 \le 2||f||_{C_N}\int_{|u-t|\ge\delta} W(\lambda,t,u)e^{Nu}\,du \le M||f||_{C_N}\frac{1}{\lambda} < \frac{\epsilon}{2},$$

which proves the theorem. ∎

THEOREM 3.4 *Let $\xi \in (A,B)$. If $f \in C_N$ and $f''(\xi)$ exists, then*

$$\lim_{\lambda\to\infty} \lambda(S_\lambda(f,\xi) - f(\xi)) = \beta p(\xi) f'(\xi) + \frac{1}{2}p(\xi)f''(\xi). \tag{10}$$

PROOF By Taylor's formula, (3) and Lemma 2.1

$$\begin{aligned} S_\lambda(f,\xi) &= f(\xi) + f'(\xi)S_\lambda(e_1,\xi) - \xi f'(\xi) + \frac{1}{2}f''(\xi)S_\lambda(e_2,\xi) \\ &\quad -\xi f''(\xi)S_\lambda(e_1,\xi) + \frac{1}{2}\xi^2 f''(\xi) \\ &\quad + \int_A^B W(\lambda,\xi,u)\epsilon(u,\xi)(u-\xi)^2\,du. \end{aligned}$$

Lemma 2.1 now implies

$$\begin{aligned} S_\lambda(f,\xi) &= f(\xi) + \frac{\beta p(\xi) f'(\xi)}{\lambda} + \frac{1}{2}\frac{p(\xi)f''(\xi)}{\lambda} + \frac{1}{2}\frac{\beta p(\xi)p'(\xi)f''(\xi)}{\lambda^2} \\ &\quad + \frac{1}{2}\frac{\beta^2 p^2(\xi) f''(\xi)}{\lambda^2} + I. \end{aligned}$$

Hence

$$\begin{aligned} \lambda\left(S_\lambda(f,\xi) - f(\xi)\right) &= \beta p(\xi) f'(\xi) + \frac{1}{2}p(\xi)f''(\xi) + \frac{1}{2}\frac{\beta p(\xi)p'(\xi)f''(\xi)}{\lambda} \\ &\quad + \frac{1}{2}\frac{\beta^2 p^2(\xi) f''(\xi)}{\lambda} + I\lambda. \end{aligned}$$

On the other hand,

$$I \leq \left( \int_A^B W(\lambda,t,u)\epsilon^2(u,\xi)\,du \right)^{\frac{1}{2}} \left( \int_A^B W(\lambda,t,u)(u-\xi)^4\,du \right)^{\frac{1}{2}}.$$

Applying Theorem 3.3 we see that

$$\lim_{\lambda\to\infty} \int_A^B W(\lambda,t,u)\epsilon^2(u,\xi)\,du = 0.$$

Therefore

$$\left( \int_A^B W(\lambda,t,u)(u-\xi)^4\,du \right)^{\frac{1}{2}} = \left( \frac{A_4(\lambda,\xi)}{\lambda^4} \right)^{\frac{1}{2}} = O(\lambda^{-1})$$

by Lemma 2.4. Thus $I = o(\lambda^{-1})$, and (10) is proved. ∎

Before the next theorem we recall some notations. Let $< a, b >$ denote the interval $[a,b]$ or $(a,b)$ or $(a,b]$ etc. If $f :< a,b > \longrightarrow \mathbb{R}$, then

$$\omega(f;h,a,b) = sup\,\{|f(t) - f(t+\delta)|;\ \ t,\ t+\delta \in < a,b >,\ \ |\delta| \leq h\}$$

and

$$\begin{aligned} &\omega_2(f;h,a,b) \\ &= \ sup\,\{|f(t) - 2f(t+\delta) + f(t+2\delta)|;\ \ t,\ t+2\delta \in < a,b >,\ \ |\delta| \leq h\} \end{aligned}$$

for $h > 0$.

THEOREM 3.5 *Suppose that $f \in C_N$ and $A < a < a_1 < b_1 < b < B$. Then for every $m > 0$ there exists a constant $K_m$ such that*

$$\begin{aligned} &||S_\lambda(f,\cdot) - f||_{C[a_1,b_1]} \\ &\leq\ K_m \left[ \lambda^{-\frac{1}{2}}\omega(f;\lambda^{-\frac{1}{2}},a,b) + \omega_2(f;\lambda^{-\frac{1}{2}},a,b) + \lambda^{-m}||f||_{C_N} \right]. \end{aligned}$$

PROOF Let $\delta > 0$ and $\delta \leq \frac{1}{2}\min\{a_1 - a, b - b_1\}$. Define

$$g_\delta(x) = \frac{1}{2\delta^2} \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} [f(x+u+v) + f(x-u-v)]\,du\,dv.$$

Since $S_\lambda$ is linear we get

$$\begin{aligned} \|S_\lambda(f,\cdot) - f\|_{C[a_1,b_1]} &\leq \|S_\lambda(f - g_\delta,\cdot)\|_{C[a_1,b_1]} + \|S_\lambda(g_\delta,\cdot) - g_\delta\|_{C[a_1,b_1]} \\ &\quad + \|g_\delta - f\|_{C[a_1,b_1]} \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Let $\eta = \frac{1}{2}\min\{a_1 - a, b - b_1\}$, then for $x \in [a_1 - \eta, b_1 + \eta]$ we get

$$\begin{aligned} |f(x) - g_\delta(x)| &= \left| f(x) - \frac{1}{2\delta^2} \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} [f(x+u+v) + f(x-u-v)]\,du\,dv \right| \\ &= \left| \frac{1}{2\delta^2} \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} [f(x+u+v) - 2f(x) + f(x-u-v)]\,du\,dv \right|. \end{aligned}$$

As $|u+v| \le \delta$, the above equality shows that

$$|f(x) - g_\delta(x)| \le \frac{1}{2}\omega_2(f;\delta,a,b). \tag{11}$$

Thus

$$I_3 \le \frac{1}{2}\omega_2(f;\delta,a,b). \tag{12}$$

The estimation of $I_1$ follows from (11) and (8). Indeed, for $t \in [a_1, b_1]$ we have

$$\begin{aligned}|S_\lambda(f-g_\delta,t)| &\le \int_{|u-t|<\eta} W(\lambda,t,u)|f(u)-g_\delta(u)|\,du \\ &\quad + \int_{|u-t|\ge\eta} W(\lambda,t,u)|f(u)-g_\delta(u)|\,du \\ &= L_1 + L_2.\end{aligned}$$

Since $|u-t| < \eta$ and $\eta = \frac{1}{2}\min\{a_1-a, b-b_1\}$, (11) yields

$$|f(u) - g_\delta(u)| \le \frac{1}{2}\omega_2(f;\delta,a,b).$$

Hence

$$L_1 \le \left|\frac{1}{2}\omega_2(f;\delta,a,b)\int_{|u-t|<\eta} W(\lambda,t,u)\,du\right| \le \frac{1}{2}\omega_2(f;\delta,a,b).$$

Moreover,

$$L_2 = \left|\int_{|u-t|\ge\eta} W(\lambda,t,u)e^{Nu}|(f-g_\delta)(u)|e^{-Nu}\,du\right|.$$

As $f - g_\delta \in C_N$ we have $|(f-g_\delta)(u)|e^{-Nu} \le 4\|f\|_{C_N}$. We conclude from (8) that

$$L_2 \le 4M\lambda^{-m}\|f\|_{C_N},$$

and finally that

$$I_1 \le M_1(\omega_2(f;\delta,a,b) + \lambda^{-m}\|f\|_{C_N}). \tag{13}$$

To estimate $I_2$ we first compute

$$g_\delta^{'}(x) = \frac{1}{2\delta^2}\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} [f(x-u-\frac{\delta}{2}) - f(x-u+\frac{\delta}{2})] - [f(x+u-\frac{\delta}{2}) - f(x+u+\frac{\delta}{2})]\,du.$$

Hence for $x \in [a_1-\eta, b_1+\eta]$ we have

$$\|g_\delta^{'}\|_{C[a_1-\eta,b_1+\eta]} \le \delta^{-1}\omega(f;\delta,a,b). \tag{14}$$

Moreover,

$$g_\delta^{''}(x) = \delta^{-2}[f(x-\delta) - 2f(x) + f(x+\delta)]$$

and

$$\|g_\delta^{''}\|_{C[a_1-\eta,b_1+\eta]} \le \delta^{-2}\omega_2(f;\delta,a,b). \tag{15}$$

Condition (3) now shows that

$$|S_\lambda(g_\delta,t)-g_\delta(t)| = \left|\int_A^B W(\lambda,t,u)(g_\delta(u)-g_\delta(t))\,du\right|, \quad t\in[a,b].$$

By the above and by Taylor's formula we have

$$\begin{aligned}|S_\lambda(g_\delta,t)-g_\delta(t)| &\le \left|g_\delta^{'}(t)\int_A^B W(\lambda,t,u)(u-t)\,du\right| \\ &\quad + \left|\int_{|u-t|<\eta} W(\lambda,t,u)g_\delta^{''}(\xi)(u-t)^2\,du\right| \\ &\quad + \left|\int_{|u-t|\ge\eta} W(\lambda,t,u)g_\delta^{''}(\xi)(u-t)^2\,du\right| \\ &= J_1+J_2+J_3.\end{aligned}$$

From Lemma 2.4, (5) and (14) for $t\in[a_1-\eta,b_1+\eta]$ we obtain

$$J_1 = |g_\delta^{'}(t)\lambda^{-1}A_1(\lambda,t)| \le M\lambda^{-1}\delta^{-1}\omega(f;\delta,a,b).$$

On the other hand, (5), Lemma 2.4 and (15) show that

$$J_2 \le \delta^{-2}\omega_2(f;\delta,a,b)|\lambda^{-2}A_2(\lambda,t)| \le K\lambda^{-1}\delta^{-2}\omega_2(f;\delta,a,b), \quad t\in[a_1,b_1].$$

It is clear that $g_\delta^{''}\in C_N$ and $\|g_\delta^{''}\|_{C_N}\le 4\|f\|_{C_N}$. From this, Cauchy-Schwarz's inequality and Lemma 2.4 we deduce that

$$\begin{aligned}J_3 &\le 4\|f\|_{C_N}\left|\left(\int_{|u-t|\ge\eta} W(\lambda,t,u)e^{2N}\,du\right)^{\frac12}\left(\lambda^{-4}A_4(\lambda,t)\right)^{\frac12}\right| \\ &\le L\lambda^{-m}\|f\|_{C_N},\end{aligned}$$

and finally that

$$I_2 \le M_2[\delta^{-1}\lambda^{-1}\omega(f;\delta,a,b)+\delta^{-2}\lambda^{-1}\omega_2(f;\delta,a,b)+\lambda^{-m}\|f\|_{C_N}]. \tag{16}$$

Put $\delta=\lambda^{-\frac12}$ into (12), (13) and (16). This ends the proof. ■

Let $0<\alpha\le 1$ and $A\le a<b\le B$. The Lipschitz class $\mathrm{Lip}(\alpha,a,b)$ is defined by

$$\mathrm{Lip}(\alpha;a,b)=\{f:\omega(f;h,a,b)\le Mh^\alpha\}.$$

From Theorem 3.5 we have

Corollary 3.6 *If $A<a<a_1<b_1<b<B$ and $f\in\mathrm{Lip}(\alpha;a,b)$, then $\|S_\lambda(f,\cdot)-f\|_{C[a_1,b_1]}\le M\lambda^{-\frac{\alpha}{2}}$.*

Theorem 3.7 *Let $f\in C_0$, $0<\alpha\le 1$. If $\|S_\lambda(f,\cdot)-f\|_{C(A,B)}\le M\lambda^{-\alpha}$, then $f\in\mathrm{Lip}(\alpha,A,B)$.*

We divide the proof into a sequence of the following lemmas.

Lemma 3.8 *Let* $f \in C_0$, $0 < \alpha \leq 1$. *If* $\|S_\lambda(f,\cdot) - f\|_{C(A,B)} \leq M\lambda^{-\alpha}$, *then there exists* $M' > 0$ *such that*

$$\omega(f;h,A,B) \leq M'[\lambda^{-\alpha} + h\lambda\omega(f;\lambda^{-1},A,B)]$$

*for* $0 < h \leq 1$ *and* $\lambda > 1$.

Proof Let $x, y \in (A,B)$ be such that $|y-x| \leq h$. Then

$$\begin{aligned} |f(y)-f(x)| &\leq |f(y) - S_\lambda(f,y)| + |S_\lambda(f,y) - S_\lambda(f,x)| + |S_\lambda(f,x) - f(x)| \\ &\leq 2M\lambda^{-\alpha} + P. \end{aligned}$$

It remains to estimate the term $P$. Let us first examine $|S_\lambda^{'}(f,t)|$

$$\begin{aligned} |S_\lambda^{'}(f,t)| &= \left| \int_A^B \frac{\partial}{\partial t} W(\lambda,t,u) f(u)\,du \right| \\ &= \left| \frac{\lambda}{p(t)} \int_A^B W(\lambda,t,u)(u-t)f(u)\,du - \beta \int_A^B W(\lambda,t,u)f(u)\,du \right|. \end{aligned}$$

Since

$$\frac{\lambda}{p(t)} \int_A^B W(\lambda,t,u)(u-t)f(t)\,du = \frac{f(t)}{p(t)} A_1(\lambda,t) = \beta f(t),$$

we have

$$\begin{aligned} |S_\lambda^{'}(f,t)| &\leq \left| \frac{\lambda}{p(t)} \int_A^B W(\lambda,t,u)(u-t)[f(u)-f(t)]\,du \right| + |\beta[f(t) - S_\lambda(f,t)]| \\ &\leq \frac{\lambda}{p(t)} \int_A^B W(\lambda,t,u)|u-t|\omega(f;\lambda|u-t|\lambda^{-1},A,B)\,du + \beta M\lambda^{-\alpha} \\ &\leq \lambda M_1 \int_A^B W(\lambda,t,u)|u-t|[\lambda|u-t|+1]\omega(f;\lambda^{-1},A,B)\,du + \beta M\lambda^{-\alpha} \\ &\leq \lambda M_1 \omega(f;\lambda^{-1},A,B) \left[ \frac{A_2(\lambda,t)}{\lambda} + \int_A^B W(\lambda,t,u)|u-t|\,du \right] \\ &\quad + \beta M\lambda^{-\alpha}. \end{aligned}$$

From Cauchy-Schwarz's inequality and Lemma 2.4 we conclude that

$$|S_\lambda^{'}(f,t)| \leq M_2 \left[ \lambda^{-\alpha} + \lambda\omega(f;\lambda^{-1},A,B) \right].$$

We are now in a position to estimate $P$. We have

$$\begin{aligned} P &\leq \int_x^y |S_\lambda^{'}(f,t)|dt \\ &\leq M' \left[ \lambda^{-\alpha} + h\lambda\omega(f;\lambda^{-1},A,B) \right], \end{aligned}$$

and the lemma follows. ∎

Lemma 3.9 *Let* $f \in C_0$ *and* $0 < \alpha \leq 1$. *If*

$$\omega(f; h, A, B) \leq M'[\lambda^{-\alpha} + h\lambda\omega(f; \lambda^{-1}, A, B)]$$

*for* $0 < h \leq 1$ *and* $\lambda > 1$, *then* $f \in Lip(\alpha, A, B)$.

Proof It is sufficient to show that $\omega(f; h, A, B) \leq M''h^{\alpha}$ for $0 < h < 1$, where $M''$ is a positive constant. Let $K > 1$ be such that $2M' < K^{1-\alpha}$. Choose $M_1 = \max\{\omega(f, 1, A, B); 2M'K^{\alpha}\}$. Define $h_n = K^{-n}$, $n = 1, 2, \ldots$.

By induction it is easy to check that for every positive integer $n$

$$\omega(f; h_n, A, B) \leq M_1 h_n^{\alpha}. \tag{17}$$

On the other hand, for every $0 < h \leq 1$ exists an integer $n > 0$ such that $h_n < h \leq h_{n-1}$. From this

$$\omega(f; h, A, B) \leq \omega(f; h_{n-1}, A, B) \leq M_1 h_{n-1}^{\alpha} = M_1 K^{\alpha} h_n^{\alpha} \leq M'' h^{\alpha},$$

and the lemma is proved. ∎

Theorem 3.10 *A kernel* $W$ *can be obtained by the partial differential equation (4) and the condition (3).*

Proof Let $W$ satisfy (4) and (3). We define $\xi$ as follows

$$\xi(\lambda, t, u) = \exp\left\{-\lambda \int_c^t \frac{u-\theta}{p(\theta)} d\theta + \beta t\right\} W(\lambda, t, u).$$

Then

$$\frac{\partial \xi(\lambda, t, u)}{\partial t} = 0,$$

hence $\xi(\lambda, t, u)$ is depending only on $\lambda$ and $u$. On the other hand,

$$W(\lambda, t, u) = \exp\left\{\lambda \int_c^t \frac{u-\theta}{p(\theta)} d\theta - \beta t\right\} C(\lambda, u). \tag{18}$$

From (3) we have

$$\exp\left\{\lambda \int_c^t \frac{\theta}{p(\theta)} d\theta + \beta t\right\} = \int_A^B \exp\{\lambda u q(t)\} C(\lambda, u) du$$

and

$$\exp\left\{\lambda \int_c^{g(t)} \frac{\theta}{p(\theta)} d\theta + \beta g(t)\right\} = \int_A^B \exp\{\lambda u t\} C(\lambda, u) du. \tag{19}$$

This and (18) give $W$. ∎

**4. Examples.**

EXAMPLE 4.1 Let $p(t) \equiv 1$, $c = 0$, $A = -\infty$, $B = +\infty$. Then $q(t) = g(t) = t$. From (19) we have

$$e^{\frac{\lambda t^2}{2} + \beta t} = \int_{-\infty}^{\infty} e^{\lambda ut} C(\lambda, u) du.$$

Hence

$$e^{\frac{\lambda}{2}\left(t + \frac{\beta}{2}\right)^2} = e^{\frac{\beta^2}{2\lambda}} \int_{-\infty}^{\infty} e^{\lambda ut} C(\lambda, u) du.$$

Put $z = t + \frac{\beta}{2}$, then we obtain

$$e^{\frac{\lambda z^2}{2}} = e^{\frac{\beta^2}{2\lambda}} \int_{-\infty}^{\infty} e^{\lambda uz} e^{-u\beta} C(\lambda, u) du.$$

From this and [1] we conclude that

$$e^{\frac{\beta^2}{2\lambda} - u\beta} C(\lambda, u) = \sqrt{\frac{\lambda}{2\pi}} e^{-\frac{\lambda u^2}{2}}.$$

Therefore

$$C(\lambda, u) = \sqrt{\frac{\lambda}{2\pi}} e^{-\frac{\lambda u^2}{2} + u\beta - \frac{\beta^2}{2\lambda}}.$$

Applying (18) we get

$$W(\lambda, t, u) = \sqrt{\frac{\lambda}{2\pi}} e^{-\frac{\lambda(u-t)^2}{2} + \beta(u-t) - \frac{\beta^2}{2\lambda}}.$$

Combining this with (1) we obtain

$$\begin{aligned} S_\lambda(f, t) &= \sqrt{\frac{\lambda}{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\lambda(u-t)^2}{2} + \beta(u-t) - \frac{\beta^2}{2\lambda}} f(u) du \\ &= \sqrt{\frac{\lambda}{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\lambda\left(u - t - \frac{\beta}{\lambda}\right)^2}{2}} f(u) du. \end{aligned}$$

Now consider the function

$$U(x, t) := \int_{-\infty}^{\infty} K(u, t, x) f(u) du, \tag{20}$$

where

$$K(u, t, x) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{(u - x - 2\beta t)^2}{4t}}.$$

THEOREM 4.2 *If $f \in C_N$, then the function $U$ given by (20) belongs to $C^\infty[\Omega]$, where $\Omega = \{(x, t) : x \in \mathbb{R}, t > 0\}$. Moreover, $U$ is a solution of the equation*

$$\frac{\partial^2 U}{\partial x^2} + 2\beta \frac{\partial U}{\partial x} = \frac{\partial U}{\partial t}$$

*in $\Omega$ and $lim_{t \to 0} U(x, t) = f(x)$.*

Proof From Theorem 3.3 we conclude that $lim_{t\to 0}U(x,t)=f(x)$.

Let $t_0, T_0, x_0, X_0 \in \mathbb{R}$ be such that $0<t_0<T_0$ and $x_0<X_0$. Consider the set

$$\Omega(t_0,T_0,x_0,X_0)=\{(x,t): x_0\le x\le X_0,\ t_0\le t\le T_0\}.$$

By induction we deduce that the integral

$$\int_{-\infty}^{\infty}\frac{\partial^{n+m}}{\partial t^n\partial x^m}K(u,t,x)f(u)\,du \tag{21}$$

is a linear combination of integrals of the form:

$$\int_{-\infty}^{\infty}\frac{\beta^q(u-x)^r}{t^{s+\frac{1}{2}}}e^{-\frac{(u-x-2\beta t)^2}{4t}}f(u)\,du, \tag{22}$$

where $q,r,s\in\mathbb{N}$.

Now we prove that the integral (22) is uniformly convergent in $\Omega(t_0,T_0,x_0,X_0)$. Note that for any $p,k,l\in\mathbb{N}$ from the definition of the norm in $C_N$ we have

$$\begin{aligned}&\int_{-\infty}^{\infty}\left|\frac{\beta^p(u-x)^k}{t^{l+\frac{1}{2}}}e^{-\frac{(u-x-2\beta t)^2}{4t}}f(u)\right|du\\ &\le\ \beta^p\|f\|_{C_N}\int_{-\infty}^{\infty}\left|\frac{(u-x)^k}{t^{l+\frac{1}{2}}}e^{-\frac{(u-x-2\beta t)^2}{4t}+Nu}\right|du.\end{aligned}$$

On the other hand, for $(x,t)\in\Omega(t_0,T_0,x_0,X_0)$ we have

$$\begin{aligned}&\beta^p\|f\|_{C_N}\int_{-\infty}^{\infty}\left|\frac{(u-x)^k}{t^{l+\frac{1}{2}}}e^{-\frac{(u-x-2\beta t)^2}{4t}+Nu}\right|du\\ &\le\ M\beta^p\|f\|_{C_N}\int_{-\infty}^{\infty}\left|(u-x)^k e^{-\frac{(u-x-2\beta t)^2}{4t}+Nu}\right|du.\end{aligned}$$

Put $z=u-x$, then

$$\begin{aligned}&\int_{-\infty}^{\infty}|u-x|^k\,e^{-\frac{(u-x-2\beta t)^2}{4t}+Nu}\,du\\ &=\ e^{Nx}\int_{-\infty}^{\infty}|z|^k e^{-\frac{(z-2\beta t)^2}{4t}+Nz}dz\\ &=\ e^{Nx}\int_{-\infty}^{\infty}|z|^k e^{-\frac{1}{4t}(z-2t(\beta+N))^2+t(\beta N+N^2-\beta^2)}dz\\ &=\ e^{Nx+t(\beta N+N^2-\beta^2)}\int_{-\infty}^{\infty}|z|^k e^{-\frac{1}{4t}(z-2t(\beta+N))^2}dz.\end{aligned}$$

Hence for $v=z-2t(\beta+N)$ we get

$$\begin{aligned}&\int_{-\infty}^{\infty}|u-x|^k\,e^{-\frac{(u-x-2\beta t)^2}{4t}+Nu}\,du\\ &=\ e^{Nx+t(\beta N+N^2-\beta^2)}\int_{-\infty}^{\infty}|v+2t(\beta+N)|^k\,e^{-\frac{1}{4t}v^2}dv\\ &\le\ M_1\int_{-\infty}^{\infty}|v+\alpha|^k\,e^{-\gamma v^2}dv,\end{aligned}$$

where $M_1, \alpha, \gamma$ are positive constants depending only on the set $\Omega(x_0, X_0, t_0, T_0)$, $\beta$ and $N$. This implies that

$$\left|\int_{-\infty}^{\infty} \frac{\beta^p (u-x)^k}{t^{l+\frac{1}{2}}} e^{-\frac{(u-x-2\beta t)^2}{4t}} f(u)\, du\right| \le M_2 \int_{-\infty}^{\infty} |v+\alpha|^k e^{-\gamma v^2} dv,$$

where $M_2$ is a positive constant. Observe that the integral

$$\int_{-\infty}^{\infty} |v+\alpha|^k e^{-\gamma v^2} dv$$

is convergent, hence the integral (22) is uniformly convergent on $\Omega(t_0, T_0, x_0, X_0)$. Thus

$$\frac{\partial^{n+m}}{\partial t^n \partial x^m} \int_{-\infty}^{\infty} K(u,t,x) f(u)\, du = \int_{-\infty}^{\infty} \frac{\partial^{n+m}}{\partial t^n \partial x^m} K(u,t,x) f(u)\, du. \tag{23}$$

Consequently, the function $U$ is of the class $C^\infty$ in $\Omega$. It is easy to check that

$$\frac{\partial^2 K(u,t,x)}{\partial x^2} + 2\beta \frac{\partial K(u,t,x)}{\partial x} = \frac{\partial K(u,t,x)}{\partial t}.$$

This completes the proof. ∎

Example 4.3 For $p(t) = t$, $c = 1$, $A = -\infty$, $B = +\infty$ we have $q(t) = \ln t$, $g(t) = e^t$. From (19) we get

$$e^{\lambda e^x - \lambda + \beta e^x} = \int_{-\infty}^{\infty} e^{\lambda u t} C(\lambda, u)\, du.$$

Hence

$$e^{(\lambda+\beta)e^x} = e^{\lambda} \int_{-\infty}^{\infty} e^{\lambda u t} C(\lambda, u)\, du.$$

From this and [1] we obtain

$$e^{\lambda} C(\lambda, u) = \sum_{k=0}^{\infty} \frac{(\lambda+\beta)^k}{k!} \delta(k - \lambda u),$$

where $\delta$ is Dirac distribution. Therefore

$$C(\lambda, u) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda+\beta)^k}{k!} \delta(k - \lambda u).$$

Applying (18) we have

$$W(\lambda, t, u) = t^{\lambda u} e^{-(\lambda+\beta)t} \sum_{k=0}^{\infty} \frac{(\lambda+\beta)^k}{k!} \delta(k - \lambda u).$$

Combining this with (1) we get

$$\begin{aligned} S_\lambda(f,t) &= \int_{-\infty}^{\infty} t^{\lambda u} e^{-(\lambda+\beta)t} \sum_{k=0}^{\infty} \frac{(\lambda+\beta)^k}{k!} \delta(k-\lambda u) f(u)\, du \\ &= e^{-(\lambda+\beta)t} \sum_{k=0}^{\infty} \frac{(\lambda+\beta)^k t^k}{k!} f\left(\frac{k}{\lambda}\right). \end{aligned}$$

**Acknowledgement.** The authors is thankful to the referee for giving useful comments.

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(Received: 22.12.2003; revised: 17.10.2004)