Chapter 6

RIEMANN INTEGRATION

In the eighteenth century the work on calculus, particularly integration, and its application, was extremely impressive and excellence, but there was no actual "theory" for it.

In a calculus, integration is introduced as "finding the area under a curve". While this interpretation is certainly useful, we instead want to think of 'integration' as more sophisticated form of summation. Geometric considerations will not be so fruitful, whereas the summation interpretation of integration will make many of its properties easy to remember.

In 1854, the Riemann integral was developed by `Bernhard Riemann`, and invented, the first rigorous definition of integration applicable to not necessarily continuous functions and then describe the relationship between integration and differentiation. In other word we can say "analytical proof of integration". More technical and also more flexible form of the Riemann integral is Lebesgue integral.

Before defining Riemann integral we must know meaning of "almost every" in terms of the concept of set of measure zero.

If I is an interval of real numbers, we denote the length of I by $|I|$.

Definition 6.1 *A subset E of $\mathbb{R}^1$ has* ***measure zero*** *if for each $\varepsilon > 0$, E can be covered by a finite or countable collection of open intervals $I_1, I_2, ...$ whose total lengths less than ε. That is, E has measurable zero if $\forall\ \varepsilon > 0\ \exists \cup\{I_n : n \in \mathbb{N}\}$ of open intervals $I_n = (a_n, b_n)$ such that $\sum\limits_{n=1}^{\infty} length\,(I_n) < \varepsilon$ where $length\,(I_n) = (b_n - a_n)$.*

Theorem 6.1 *If each of the subsets $E_1, E_2, ...$ of $\mathbb{R}^1$ is of measurable zero, then $\bigcup\limits_{n=1}^{\infty} E_n$ is also of measurable zero.*

Proof. Let $\varepsilon > 0$ be fixed. For each $n \in \mathbb{N}$, E_n has measurable zero there exists a finite or countable collection of open intervals $I_1, I_2, ...$ which cover

E_n whose total lengths less than $\varepsilon/2^n$. The union of all such open intervals $\bigcup\limits_{n=1}^{\infty} \{I_n : n \in \mathbb{N}\}$ covers $\bigcup\limits_{n=1}^{\infty} E_n$, and

$$\sum_{n=1}^{\infty} length\,(I_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + ... = \frac{\varepsilon/2}{1-1/2} = \varepsilon.$$

Hence $\bigcup\limits_{n=1}^{\infty} E_n$ has measure zero. ■

Corollary 6.1 *Every countable subset of* $\mathbb{R}^1$ *has measure zero.*

Proof. Let $A = \{x_n : n \in \mathbb{N}\}$ be a countable set. Let $\varepsilon > 0$ and

$$I_n = \left(x_n - \frac{\varepsilon}{2^{n+2}}, x_n + \frac{\varepsilon}{2^{n+2}}\right), \ \forall n \in \mathbb{N}.$$

Then $\{I_n : n \in \mathbb{N}\}$ is a countable collection of open intervals that cover A. Moreover

$$length\,(I_n) = \left(x_n + \frac{\varepsilon}{2^{n+2}}\right) - \left(x_n - \frac{\varepsilon}{2^{n+2}}\right) = \frac{2\varepsilon}{2^{n+2}} = \frac{\varepsilon}{2^{n+1}}.$$

Hence

$$\begin{aligned}\sum_{n=1}^{\infty} length\,(I_n) &= \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{4} + \frac{\varepsilon}{8} + \frac{\varepsilon}{16} + ... \\ &= \frac{\varepsilon}{4}\left(1 + \frac{1}{2} + \frac{1}{4} + ...\right) = \frac{\varepsilon}{4}.2 = \frac{\varepsilon}{2} < \varepsilon.\end{aligned}$$

Thus A has measure zero. ■

Definition 6.2 *A property which is true except for a set of measure zero is said to hold almost everywhere. That is a property is said to hold almost everywhere on* $[a, b]$ *if it holds at each point of* $[a, b] \setminus E$*, where* E *is a set of measure zero.*

First, as usual, we need to define integration before we can discuss its properties.

6.0.1 Partitions

Let $a, b \in \mathbb{R}$. A *Partition* $\mathcal{P}$ is defined as the ordered n-tuple of real numbers $\mathcal{P} = \{x_0, x_1, x_2, ..., x_n\}$ such that

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b$$

and $[x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n]$ are the sub-intervals of $[a, b]$. We write

$$\Delta x_i = x_i - x_{i-1}, \text{ for } i = 1, 2, ..., n$$

so that Δx_i is the length of $i-th$ sub-interval $[x_{i-1}, x_i]$.

Partition $\mathcal{P} = \{x_0, x_1, x_2, ..., x_n\}$ is called `Uniform Partition` of $[a, b]$, if

$$x_n - x_{n-1} = ... = x_i - x_{i-1} = ... = x_1 - x_0 = \frac{b-a}{n}.$$

6.0.2 Norm or Mesh of the partition

The maximum difference between any two consecutive points of the partition, i.e., the greatest segments(largest sub-interval) is called the norm or mesh of the partition and denoted as:

$$\mathcal{P} = \max \Delta x_i (1 \leqslant i \leqslant n) = \underset{1 \leqslant i \leqslant n}{max} |x_i - x_{i-1}|$$

or

$$\|\mathcal{P}\| = \sup\{x_i - x_{i-1} : \ 1 \leqslant i \leqslant n\}.$$

6.0.3 Tagged Partition

A Tagged Partition $\dot{\mathcal{P}}(x, t)$ of an interval $[a, b]$ is a partition together with a finite sequence of numbers $t_1, \ldots, t_n$ subject to the conditions that for each $i, t_i \in [x_{i-1}, x_i]$. In other words, it is a partition together with a distinguished point of every subinterval.

A Tagged Partition $\dot{\mathcal{P}}(x, t)$ is defined as the set of ordered pairs

$$\dot{\mathcal{P}} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n} \text{ such that } x_{i-1} < t_i < x_i,$$

and the points t_i are called Tags.

6.0.4 Refinement(Finer)

If $\mathcal{P}$ and $\mathcal{P}^*$ are partitions of $[a, b]$ with $\mathcal{P} \subset \mathcal{P}^*$, then $\mathcal{P}^*$ is said to be a *refinement* of $\mathcal{P}$, i.e., every point of $\mathcal{P}$ is a point of $\mathcal{P}^*$. In other word $\mathcal{P}^*$ *refines* $\mathcal{P}$ or $\mathcal{P}^*$ is *finer* than $\mathcal{P}$.

If $\mathcal{P}_1$ and $\mathcal{P}_2$ are two partitions, then $\mathcal{P}^*$ is their common refinement if $\mathcal{P}^* = \mathcal{P}_1 \cup \mathcal{P}_2$.

6.0.5 Riemann Sums

Suppose $f(f : [a, b] \to \mathbb{R})$ is a bounded real function defined on $[a, b]$. Evidently f is bounded real function defined on each subinterval of partition $\mathcal{P}$. For each $i = 1, 2, ..., n$ define

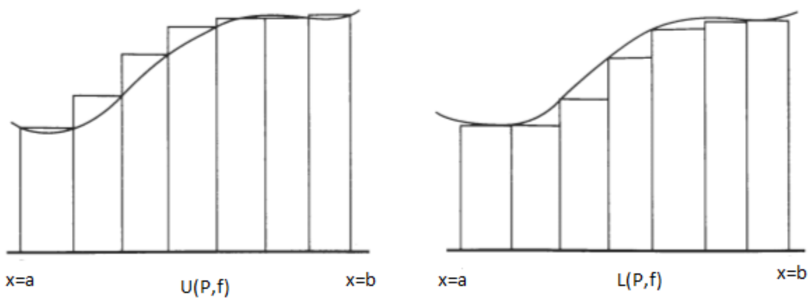
$$\begin{aligned} M_i(f) &= sup\{f(x) : x \in [x_{i-1}, x_i]\} \\ \text{and} \quad m_i(f) &= \inf\{f(x) : x \in [x_{i-1}, x_i]\}. \end{aligned}$$

Also define the two sums

$$U(\mathcal{P}, f) = \sum_{i=1}^{n} M_i \Delta x_i = M_1 \Delta x_1 + M_2 \Delta x_2 + ... + M_n \Delta x_n$$

$$\text{and} \quad L(\mathcal{P}, f) = \sum_{i=1}^{n} m_i \Delta x_i = m_1 \Delta x_1 + m_2 \Delta x_2 + ... + m_n \Delta x_n.$$

respectively called the upper and lower *Riemann(Darboux)* sums.



6.0.6 Upper and Lower Riemann Integral

Let $f(f : [a, b] \to \mathbb{R})$ be a bounded real function defined on $[a, b]$. Then the infimum of the set of upper sums is called the upper integral and supremum of the set of lower sums is called the lower integral of f. Thus

$$\int_a^{-b} f dx = \inf U(\mathcal{P}, f) \quad \text{and} \quad \int_{-a}^{b} f dx = \sup L(\mathcal{P}, f).$$

These two integrals may or may not be equal.

6.0.7 Riemann Integral

If the upper and lower integrals are equal then we say that f is *Riemann integrable* on $[a, b]$ and the common value of these integrals is called the *Riemann Integral* of f over $[a, b]$ denoted by

$$\int_a^{-b} f dx = \int_{-a}^{b} f dx = \int_a^b f dx.$$

Remark 6.1 *(i) A bounded function is said to be integrable when the upper and lower integrals are equal.*

(ii) Since $\mathcal{P} = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ *and* $\Delta x_i = x_i - x_{i-1}$ *then*

$$\begin{aligned}\sum_{i=1}^{n} \Delta x_i &= \Delta x_1 + \Delta x_2 + ... + \Delta x_n \\ &= (x_1 - x_0) + (x_2 - x_1) + ... + (x_n - x_{n-1}) \\ &= x_n - x_0 = b - a.\end{aligned}$$

$$\textit{Thus} \qquad \sum_{i=1}^{n} \Delta x_i = b - a.$$

Lemma 6.1 *If* $f : [a, b] \to \mathbb{R}$ *is bounded function and* $\mathcal{P}$ *is partition of* $[a, b]$ *then* $L(\mathcal{P}, f) \leqslant U(\mathcal{P}, f)$.

Proof. Let $\mathcal{P}$ be a partition and f is bounded function in $[a, b]$. Evidently f is bounded on each subinterval. Let m_i and M_i be the *infimum* and *supremum* of f in $[x_{i-1}, x_i]$ then

$$\begin{aligned} & m_i \leqslant M_i, \quad (i = 1, 2, 3, ..., n) \\ \Rightarrow \qquad & \sum_{i=1}^{n} m_i \Delta x_i \leqslant \sum_{i=1}^{n} M_i \Delta x_i \\ \textit{Hence} \qquad & L(P, f) \leqslant U(P, f).\end{aligned}$$

■

Lemma 6.2 *If* $\mathcal{P}^*$ *is a refinement of a partition* $\mathcal{P}$*, then for a bounded function* f*,*

$$L(\mathcal{P}^*, f) \geqslant L(\mathcal{P}, f) \quad \textit{and} \quad U(\mathcal{P}^*, f) \leqslant U(\mathcal{P}, f).$$

Proof. If $\mathcal{P}^*$ is a refinement of a partition $\mathcal{P}$, then suppose that $\mathcal{P}^*$ contains just one point more than $\mathcal{P}$. Let ξ be extra point and $\xi \in (x_{i-1}, x_i)$, *i.e.*, $x_{i-1} < \xi < x_i$.

The function f is bounded in $[a, b]$, therefore it is bounded in every subinterval $[x_{i-1}, x_i]\,(i = 1, 2, 3, ...)$. Let ω_1, ω_2, m_i be the infimum of f in the intervals $[x_{i-1}, \xi], [\xi, x_i], [x_{i-1}, x_i]$ respectively, i.e.,

$$\begin{aligned}\omega_1 &= \inf\{f(x) : x_{i-1} \leqslant x \leqslant \xi\} \\ \omega_2 &= \inf\{f(x) : \xi \leqslant x \leqslant x_i\} \\ m_i &= \inf\{f(x) : x_{i-1} \leqslant x \leqslant x_i\},\end{aligned}$$

clearly $m_i \leqslant \omega_1, m_i \leqslant \omega_2$.

$$\begin{aligned}& L(\mathcal{P}^*, f) - L(\mathcal{P}, f) \\ & = \omega_1(\xi - x_{i-1}) + \omega_2(x_i - \xi) - m_i(x_i - x_{i-1}) \\ & = \omega_1(\xi - x_{i-1}) + \omega_2(x_i - \xi) - m_i(x_i - \xi) - m_i(\xi - x_{i-1}) \\ & = (\omega_1 - m_i)(\xi - x_{i-1}) + (\omega_2 - m_i)(x_i - \xi) \geqslant 0 \ [\because \text{each bracket is positive}]\end{aligned}$$

If $\mathcal{P}^*$ contains k more points than $\mathcal{P}$, repeated this arguments k times to and arrive at

$$L(\mathcal{P}^*, f) \geqslant L(\mathcal{P}, f)$$

Let W_1, W_2, M_i be the supremum of f in the intervals $[x_{i-1}, \xi], [\xi, x_i], [x_{i-1}, x_i]$ respectively, i.e.,

$$\begin{aligned} W_1 &= \sup\{f(x) : x_{i-1} \leqslant x \leqslant \xi\} \\ W_2 &= \sup\{f(x) : \xi \leqslant x \leqslant x_i\} \\ M_i &= \sup\{f(x) : x_{i-1} \leqslant x \leqslant x_i\}, \end{aligned}$$

clearly $W_1 \leqslant M_i, W_2 \leqslant M_i$.

$$\begin{aligned} &U(\mathcal{P}, f) - U(\mathcal{P}^*, f) \\ &= M_i(x_i - x_{i-1}) - \{W_1(\xi - x_{i-1}) + W_2(x_i - \xi)\} \\ &= M_i(x_i - \xi) + M_i(\xi - x_{i-1}) - W_1(\xi - x_{i-1}) - W_2(x_i - \xi) \\ &= (M_i - W_2)(x_i - \xi) + (M_i - W_1)(\xi - x_{i-1}) \geqslant 0 \ [\because \text{each bracket is positive}] \end{aligned}$$

If $\mathcal{P}^*$ contains k more points than $\mathcal{P}$, repeated this arguments k times to and arrive at

$$U(\mathcal{P}^*, f) \leqslant U(\mathcal{P}, f).$$

■

Theorem 6.2 *If $f : [a, b] \to \mathbb{R}$ is bounded function and let $\mathcal{P}_1$ and $\mathcal{P}_2$ be partitions of $[a, b]$ and $L(\mathcal{P}, f) \leqslant U(\mathcal{P}, f)$ then*

$$L(\mathcal{P}_1, f) \leqslant U(\mathcal{P}_2, f) \quad \text{and} \quad L(\mathcal{P}_2, f) \leqslant U(\mathcal{P}_1, f),$$

i.e., no upper sum can ever be less than any lower sum.

Proof. Suppose $\mathcal{P}^*$ is a common refinement of $\mathcal{P}_1, \mathcal{P}_2$, so that

$$P^* = \mathcal{P}_1 \cup \mathcal{P}_2$$

Since $P_1 \subset P^*$ and $P_2 \subset P^*$ then from Lemma 6.2, we have

$$L(\mathcal{P}_1, f) \leqslant L(\mathcal{P}^*, f) \quad \text{and} \quad U(\mathcal{P}^*, f) \leqslant U(\mathcal{P}_2, f).$$

By Lemma 6.1, we have

$$L(\mathcal{P}^*, f) \leqslant U(\mathcal{P}^*, f)$$

Using above three inequalities then we have

$$L(\mathcal{P}_1, f) \leqslant L(\mathcal{P}^*, f) \leqslant U(\mathcal{P}^*, f) \leqslant U(\mathcal{P}_2, f)$$

Thus

$$L(\mathcal{P}_1, f) \leqslant U(\mathcal{P}_2, f)$$

Similarly, by Lemma 6.2, we have

$$L(\mathcal{P}_2, f) \leqslant L(\mathcal{P}^*, f) \quad \text{and} \quad U(\mathcal{P}^*, f) \leqslant U(\mathcal{P}_1, f).$$

By Lemma 6.1

$$L(\mathcal{P}^*, f) \leqslant U(\mathcal{P}^*, f)$$

From above three inequalities then we have

$$L(\mathcal{P}_2, f) \leqslant L(\mathcal{P}^*, f) \leqslant U(\mathcal{P}^*, f) \leqslant U(\mathcal{P}_1, f)$$

Thus

$$L(\mathcal{P}_2, f) \leqslant U(\mathcal{P}_1, f).$$

■

Theorem 6.3 *If $f : [a, b] \to \mathbb{R}$ is bounded function and let m and M are lower and upper bounds then*

$$m(b-a) \leqslant L(\mathcal{P}, f) \leqslant U(\mathcal{P}, f) \leqslant M(b-a).$$

or

If f is bounded function on $[a, b]$ then for any partition $\mathcal{P}$ of $[a, b]$, the upper sum $U(\mathcal{P}, f)$ and lower sum $L(\mathcal{P}, f)$ are bounded.

Proof. Let $\mathcal{P} = \{x_0, x_1, x_2, ..., x_n\}$ be any partition of $[a, b]$ and let m, M be the infimum, supremum in $[a, b]$ respectively. Again let m_i, M_i be the infimum, supremum in $[x_{i-1}, x_i]$ respectively so that

$$\begin{aligned} m \leqslant m_i \leqslant M_i \leqslant M \quad & (i = 1, 2, ..., n) \\ \Rightarrow m\Delta x_i \leqslant m_i \Delta x_i \leqslant M_i \Delta x_i \leqslant M \Delta x_i & \end{aligned}$$

Putting $i = 1, 2, ..., n$ and adding all the inequalities, we get

$$\begin{aligned} \sum_{i=1}^{n} m\Delta x_i \leqslant \sum_{i=1}^{n} m_i \Delta x_i \leqslant \sum_{i=1}^{n} M_i \Delta x_i \leqslant \sum_{i=1}^{n} M \Delta x_i \\ \Rightarrow m(b-a) \leqslant L(\mathcal{P}, f) \leqslant U(\mathcal{P}, f) \leqslant M(b-a), \end{aligned}$$

so that $L(\mathcal{P}, f)$ and $U(\mathcal{P}, f)$ form a bounded set. This shows that the *upper and lower Riemann sums are defined for every bounded function f.* ■
Geometrically interpretation, (see figure....) the lower sum $L(\mathcal{P}, f)$ is the area of all rectangle (like $ABCD$) under above the curve and the upper sum $U(\mathcal{P}, f)$ is the area of all rectangles (like $ABEF$) above the curve. So the lower integral approximates the area under the curve $y = f(x)$, $a \leqslant x \leqslant b$ with rectangles, constructed below the curve. The upper integral approximates to the area from above the curve. These considerations are appears in the following theorem.

Theorem 6.4 *The lower $\mathcal{R}$-integral cannot exceed the upper $\mathcal{R}$-integral, i.e.,*

$$\int_{-a}^{b} f dx \leqslant \int_{a}^{-b} f dx \quad or \quad L(f) \leqslant U(f).$$

Proof. If $f : [a, b] \to \mathbb{R}$ is bounded function. If $\mathcal{P}_1$ and $\mathcal{P}_2$ are two partitions of $[a, b]$ then by theorem

$$L(\mathcal{P}_1, f) \leqslant U(\mathcal{P}_2, f)$$

Keeping $\mathcal{P}_2$ fixed and taking the supremum over all partitions $\mathcal{P}_1$ then by above inequality

$$\int_{-a}^{b} f dx = \sup L(\mathcal{P}_1, f) \leqslant U(\mathcal{P}_2, f)$$

Now taking the infimum over all partitions $\mathcal{P}_2$ then by above inequality

$$\int_{-a}^{b} f dx \leqslant \int_{a}^{-b} f dx = \inf U(\mathcal{P}_2, f)$$

Thus

$$\int_{-a}^{b} f dx \leqslant \int_{a}^{-b} f dx.$$

■

Theorem 6.5 [Riemann's criterion for integrability] *A bounded function $f : [a, b] \to \mathbb{R}$ on $[a, b]$ is integrable if and only if for each $\varepsilon > 0$ there exists a partition $\mathcal{P}$ of $[a, b]$ such that*

$$U(\mathcal{P}, f) - L(\mathcal{P}, f) < \varepsilon.$$

Proof. Suppose that f is integrable on closed interval $[a, b]$ then

$$\int_{a}^{-b} f dx = \int_{-a}^{b} f dx = \int_{a}^{b} f dx.$$

Since $\mathcal{P}$ is partition of $[a, b]$ and now consider $\varepsilon > 0$ then

$$\int_{a}^{b} f dx = \sup L(\mathcal{P}, f),$$

therefore $\int_a^b f dx - \frac{\varepsilon}{2}$ is not an upper bound of set of all lower sums, hence there is a partition $\mathcal{P}_1$ such that

$$\int_{a}^{b} f dx - \frac{\varepsilon}{2} < L(\mathcal{P}_1, f) \ldots (i)$$

Similarly $\int_a^b f dx + \frac{\varepsilon}{2}$ is not a lower bound for set of all upper sums, therefore there is a partition $\mathcal{P}_2$ such that

$$U(\mathcal{P}_2, f) < \int_a^b f dx + \frac{\varepsilon}{2} ...(ii)$$

Let $\mathcal{P}$ be the common refinement of $\mathcal{P}_1$ and $\mathcal{P}_2$, i.e., $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ then by Lemma 6.2

$$U(\mathcal{P}, f) \leqslant U(\mathcal{P}_2, f) < \int_a^b f dx + \frac{\varepsilon}{2} \quad [by (ii)]$$

$$\text{and} \quad L(\mathcal{P}, f) \geqslant L(\mathcal{P}_1, f) > \int_a^b f dx - \frac{\varepsilon}{2} \quad [by (i)]$$

Thus

$$U(\mathcal{P}, f) - L(\mathcal{P}, f) < \varepsilon.$$

Suppose $f : [a, b] \to R$ bounded on $[a, b]$ and for each $\varepsilon > 0$ there is a partition $\mathcal{P}$ such that

$$U(\mathcal{P}, f) - L(\mathcal{P}, f) < \varepsilon.$$

We have to show that

$$\int_a^{-b} f dx = \int_{-a}^{b} f dx,$$

i.e., f is integrable. For any partition $\mathcal{P}$, we know that

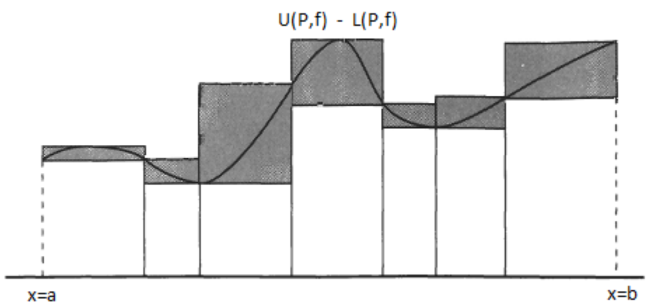
$$\begin{aligned} L(\mathcal{P}, f) \leqslant \int_{-a}^{b} f dx \leqslant \int_a^{-b} f dx &\leqslant U(\mathcal{P}, f) \\ &\leqslant U(\mathcal{P}, f) - L(\mathcal{P}, f) + L(\mathcal{P}, f) \\ &< \varepsilon + L(\mathcal{P}, f). \end{aligned}$$

Therefore,

$$0 \leqslant \int_a^{-b} f dx - \int_{-a}^{b} f dx < \varepsilon.$$

Thus

$$\int_{-a}^{b} f dx = \int_a^{-b} f dx, \text{ i.e., } f \text{ is integrable.}$$



■

Example 6.1 *Let $f(x) = x^2$ on $[0, k], k > 0$. Show that $f \in \mathcal{R}[0, k]$ and*

$$\int_0^k f dx = \frac{k^3}{3}.$$

Solution 6.1 *Given $f(x) = x^2,\ \forall x \in [0, k]$, then show that $f \in \mathcal{R}[0, k]$.*

The interval $[0, k]$ is divided in to n equal parts $\Delta x_i = \frac{k}{n};\ i = 1, 2, ..., n$, then we have partition $\mathcal{P} = \{0 = x_0, x_1, x_2, ..., x_n = k\}$ of $[0, k]$.

Let M_i, m_i be $\sup.$, $\inf.$ in Δx_i respectively.

$$M_i = \sup\{f(x) : x \in \Delta x_i\} = x_i^2 = \left(\frac{ik}{n}\right)^2,$$

$$\text{and} \qquad m_i = \inf\{f(x) : x \in \Delta x_i\} = x_{i-1}^2 = \left\{\frac{(i-1)k}{n}\right\}^2.$$

Now

$$\begin{aligned} U(\mathcal{P}, f) &= \sum_{i=0}^{n} M_i \Delta x_i \\ &= M_1 \Delta x_1 + M_2 \Delta x_2 + ... + M_n \Delta x_n \\ &= \left(\frac{k}{n}\right)^3 (1^2 + 2^2 + ... + n^2) \\ &= \left(\frac{k}{n}\right)^3 \frac{n(n+1)(2n+1)}{6} \\ &= \frac{k^3}{3}\left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{2n}\right). \end{aligned}$$

Hence

$$\begin{aligned} \int_a^{-b} f dx = \int_0^{-k} f dx &= \inf U(\mathcal{P}, f), \quad \forall \mathcal{P} \\ &= \lim_{n \to \infty} \frac{k^3}{3}\left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{2n}\right) = \frac{k^3}{3}. \end{aligned}$$

Similarly,

$$\begin{aligned}
L(\mathcal{P}, f) &= \sum_{i=0}^{n} m_i \Delta x_i \\
&= m_1 \Delta x_1 + m_2 \Delta x_2 + ... + m_n \Delta x_n \\
&= \left(\frac{k}{n}\right)^3 \left\{1^2 + 2^2 + ... + (n-1)^2\right\} \\
&= \left(\frac{k}{n}\right)^3 \frac{(n-1)n(2n-1)}{6} \\
&= \frac{k^3}{3}\left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{2n}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{-a}^{b} f dx = \int_{-0}^{k} f dx &= \sup L(\mathcal{P}, f), \quad \forall \mathcal{P} \\
&= \lim_{n \to \infty} \frac{k^3}{3}\left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{2n}\right) = \frac{k^3}{3}.
\end{aligned}$$

So that

$$\int_{0}^{-k} f dx = \int_{-0}^{k} f dx = \frac{k^3}{3}.$$

Thus

$$f \in \mathcal{R}[a, b] \quad \textit{and} \quad \int_0^k f dx = \frac{k^3}{3}.$$

Example 6.2 *Show that the function defined as:*

$$f(x) = \begin{cases} 0, & \textit{when } x \textit{ is rational} \\ 1, & \textit{when } x \textit{ is irrational} \end{cases}$$

is not integrable on any interval, i.e., $f \notin \mathcal{R}[a, b]$.

Solution 6.2 *Given* $f(x) = 0, \quad \forall x \in \mathbb{Q}$ *and* $f(x) = 1, \quad \forall x \in \mathbb{R} - \mathbb{Q}$ *then prove that* $f \notin \mathcal{R}[a, b]$. *Let* $\mathcal{P} = \{a = x_0, x_1, x_2, ..., x_n = b\}$ *be any partition of* $[a, b]$. *Hence*

$$\begin{aligned}
M_i &= \sup\{f(x) : x \in \Delta x_i\} = \sup\{0, 1\} = 1 \\
\textit{and} \quad m_i &= \inf\{f(x) : x \in \Delta x_i\} = \inf\{0, 1\} = 0 \\
\therefore \quad M_i &= 1, m_i = 0, \quad \forall i = 0, 1, 2, ..., n.
\end{aligned}$$

Now

$$U(\mathcal{P}, f) = \sum_{i=0}^{n} M_i \Delta x_i = (b - a)$$

$$\textit{and} \quad L(\mathcal{P}, f) = \sum_{i=0}^{n} m_i \Delta x_i = 0$$

Hence

$$\int_{a}^{-b} f dx = \inf U(\mathcal{P}, f) = \inf(b-a) = (b-a)$$
$$\int_{-a}^{b} f dx = \sup L(\mathcal{P}, f) = \sup\{0\} = 0$$
$$\Rightarrow \int_{a}^{-b} f dx \neq \int_{-a}^{b} f dx$$

Thus $f \notin \mathcal{R}[a,b]$.

Example 6.3 *If the function* f *defined on* $[a,b]$ *by*

$$f(x) = c, \quad \forall x \in [a,b],$$

where c *is a constant, then show that* $f \in \mathcal{R}[a,b]$ *and*

$$\int_{a}^{b} c dx = c(b-a).$$

Solution 6.3 *Given* $f(x) = c$, $\forall x \in [a,b]$, *we have to show that* $f \in \mathcal{R}[a,b]$.

Let $\mathcal{P} = \{a = x_0, x_1, x_2, ..., x_n = b\}$ *be any partition of* $[a,b]$. *Let* M_i, m_i *be* $\sup, \inf$ *in* Δx_i *respectively* $(i = 1, 2, ..., n)$. *Hence*

$$M_i = \sup\{f(x) : x \in \Delta x_i\} = \sup\{c\} = c$$
$$\text{and} \quad m_i = \inf\{f(x) : x \in \Delta x_i\} = \inf\{c\} = c$$
$$\therefore M_i = m_i = c, \quad \forall i = 1, 2, ..., n.$$

Now

$$U(\mathcal{P}, f) = \sum_{i=1}^{n} M_i \Delta x_i = c \sum_{i=1}^{n} \Delta x_i = c(b-a)$$
$$\text{and} \quad L(\mathcal{P}, f) = \sum_{i=1}^{n} m_i \Delta x_i = c \sum_{i=1}^{n} \Delta x_i = c(b-a)$$

Hence

$$\int_{a}^{-b} f dx = \inf U(\mathcal{P}, f) = \inf c(b-a) = c(b-a)$$
$$\text{and} \quad \int_{-a}^{b} f dx = \sup L(\mathcal{P}, f) = \sup c(b-a) = c(b-a).$$

Hence

$$\int_{a}^{-b} f dx = \int_{-a}^{b} f dx.$$

Thus $f \in \mathcal{R}[a,b]$.

Theorem 6.6 *"Every monotonic function* f *on* $[a,b]$ *is integrable"*

Proof. Suppose f is monotonic increasing (the proof is similar for the monotonic decreasing case) function on $[a, b]$, therefore it is bounded on $[a, b]$. Since

$$f(a) \leqslant f(x) \leqslant f(b), \quad \forall x \in [a, b].$$

Let $\varepsilon > 0$ be given and taking uniform partition $\mathcal{P} = \{a = x_0, x_1, x_2, ..., x_n = b\}$ of $[a, b]$ with $\Delta x_i = \frac{b-a}{n}$, $M_i = f(x_i)$ and $m_i = f(x_{i-1})$, then

$$\begin{aligned} U(f, \mathcal{P}) - L(f, \mathcal{P}) &= \sum_{k=1}^{n} (M_i - m_i)\Delta x_i \\ &= \sum_{k=1}^{n} [f(x_i) - f(x_{i-1})]\Delta x_i \\ &= [f(b) - f(a)] \frac{(b-a)}{n} \\ &< \varepsilon, \quad \text{if } n \text{ is sufficiently large.} \end{aligned}$$

Hence $f \in \mathcal{R}[a, b]$. ■

Theorem 6.7 *If f is continuous on $[a, b]$, then it is integrable.*

or

"Every continuous function is integrable"

Proof. By above theorem, f is bounded and $[a, b]$ is compact (since $[a, b]$ is closed and bounded interval), therefore f is uniformly continuous, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall x, y \in [a, b] \text{ and } |x - y| < \delta \text{ imply } |f(x) - f(y)| < \frac{\varepsilon}{b-a}.$$

Let $\mathcal{P} = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be any partition of $[a, b]$ such that

$$\max\{x_i - x_{i-1} : i = 1, 2, ..., n\} < \delta.$$

Since f acquire its maximum and minimum on each subintervals $[x_{i-1}, x_i]$ therefore for $x, y \in [x_{i-1}, x_i]$ we have

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) \text{ and } m_i = \inf_{y \in [x_{i-1}, x_i]} f(y).$$

Therefore

$$M_i - m_i < \frac{\varepsilon}{b-a}, \quad \forall i = 1, 2, ..., n.$$

Hence

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \leqslant \sum_{i=1}^{n} \frac{\varepsilon}{b-a}(x_i - x_{i-1}) < \varepsilon.$$

Thus f is integrable, i.e., $f \in \mathcal{R}[a, b]$. ■

6.1 Properties of the Riemann Integral

6.1.1 Linearity Property

Theorem 6.8 [Scaling] *If $f : [a, b] \to \mathbb{R}$ is integrable function on $[a, b]$ and λ is any real number, then $\lambda f \in \mathcal{R}[a, b]$ and then*

$$\int_a^b \lambda f = \lambda \int_a^b f. \tag{6.1}$$

Proof.

(*i*) If $\lambda = 0$, then the function $\lambda f = 0$ and (6.1) is obvious.

(*ii*) If $\lambda > 0$, since $f \in \mathcal{R}[a, b]$, therefore given $\varepsilon > 0$ $\exists$ a partition $\mathcal{P} = \{a = x_0, x_1, x_2, ..., x_n = b\}$ of $[a, b]$ such that

$$U(\mathcal{P}, f) - L(\mathcal{P}, f) < \frac{\varepsilon}{\lambda}.$$

Therefore f is bounded in $[a, b]$(by Theorem 6.5). Hence $|\lambda f| = \lambda |f|$ is also bounded in $[a, b]$.

Let M_i, m_i are supremum, infimum on $[x_{i-1}, x_i]$ respectively, then λM_i, λm_r are supremum, infimum on $\Delta x_i = [x_{i-1}, x_i]$.

$$U(\mathcal{P}, \lambda f) = \sum_{i=1}^{n} (\lambda M_i)\Delta x_i = \lambda \sum_{i=1}^{n} M_i \Delta x_i = \lambda U(\mathcal{P}, f).$$

Similarly

$$L(\mathcal{P}, \lambda f) = \lambda L(\mathcal{P}, f).$$

Therefore

$$U(\mathcal{P}, \lambda f) - L(\mathcal{P}, \lambda f) = \lambda [U(\mathcal{P}, f) - L(\mathcal{P}, f)] < \lambda \frac{\varepsilon}{\lambda} = \varepsilon.$$

Hence

$$\lambda f \in \mathcal{R}[a, b], \quad i.e., \quad \int_a^{-b} \lambda f = \int_{-a}^{b} \lambda f = \int_a^b \lambda f$$

Thus

$$\int_a^b \lambda f = \int_a^{-b} \lambda f = \inf U(\mathcal{P}, \lambda f) = \lambda \inf U(\mathcal{P}, f)$$
$$= \lambda \int_a^{-b} f = \lambda \int_a^b f.$$

or

$$\int_a^b \lambda f = \int_{-a}^{b} \lambda f = \sup L(\mathcal{P}, \lambda f) = \lambda \sup L(\mathcal{P}, f)$$
$$= \lambda \int_{-a}^{b} f = \lambda \int_a^b f. \tag{6.2}$$

(iii) If $\lambda < 0$, Assume $\lambda = -1$ then

$$U(\mathcal{P}, -f) = -L(\mathcal{P}, f) \quad \text{for all partitions } \mathcal{P} \text{ of } [a,b].$$

Hence we have

$$\begin{aligned}\int_{a}^{-b} (-f) &= \inf\{U(\mathcal{P}, -f) : \mathcal{P}[a,b]\} \\ &= \inf\{-L(\mathcal{P}, f) : \mathcal{P}[a,b]\} \quad [\because U(\mathcal{P}, -f) = -L(\mathcal{P}, f)] \\ &= -\sup\{L(\mathcal{P}, f) : \mathcal{P}[a,b]\} = -\int_{-a}^{b} f.\end{aligned}$$

Similarly,

$$\int_{-a}^{b} (-f) = -\int_{a}^{-b} f.$$

Since f is integrable, therefore we have

$$\int_{a}^{-b} (-f) = -\int_{-a}^{b} f = -\int_{a}^{-b} f = \int_{-a}^{b} (-f).$$

Thus $-f$ is integrable and

$$\int_{a}^{b} (-f) = -\int_{a}^{b} f. \tag{6.3}$$

For case $\lambda < 0$, using (6.3) and then (6.2) to $-\lambda$, we obtain

$$\int_{a}^{b} \lambda f = -\int_{a}^{b} (-\lambda) f = -(-\lambda)\int_{a}^{b} f = \lambda \int_{a}^{b} f.$$

■

Theorem 6.9 [Splitting integrals] *Let a, c, b be real numbers with $a < c < b$, and consider any function $f : [a,b] \to \mathbb{R}$ which is integrable over $[a,c]$ and over $[c,b]$. Then $f \in \mathcal{R}[a,b]$, and*

$$\int_{a}^{b} f = \int_{a}^{c} f + \int_{c}^{b} f$$

Proof. Let $f \in \mathcal{R}[a,c]$ and $f \in \mathcal{R}[c,b]$. Choose $\varepsilon > 0$, then $\mathcal{P}_1$ and $\mathcal{P}_2$ be a partition of $[a,c]$ and $[c,b]$, respectively such that

$$\begin{aligned} & U(\mathcal{P}_1, f) - L(\mathcal{P}_1, f) < \frac{\varepsilon}{2} \\ \text{and} \quad & U(\mathcal{P}_2, f) - L(\mathcal{P}_2, f) < \frac{\varepsilon}{2}\end{aligned}$$

Now put together the points of $\mathcal{P}_1$ and $\mathcal{P}_2$ then $\mathcal{P}$ is the combined partition of interval $[a, b]$, i.e., $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$, then

$$U(\mathcal{P}, f) = U(\mathcal{P}_1, f) + U(\mathcal{P}_2, f) \geqslant \int_a^c f + \int_c^b f \tag{6.4}$$

$$\text{and} \quad L(\mathcal{P}, f) = L(\mathcal{P}_1, f) + L(\mathcal{P}_2, f) \leqslant \int_a^c f + \int_c^b f \tag{6.5}$$

Consequently,

$$\begin{aligned} U(\mathcal{P}, f) - L(\mathcal{P}, f) &= [U(\mathcal{P}_1, f) - L(\mathcal{P}_1, f)] + [U(\mathcal{P}_2, f) - L(\mathcal{P}_2, f)] \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore, $f \in \mathcal{R}[a, b]$.

$$\begin{aligned} \int_a^b f \leqslant U(\mathcal{P}, f) = U(\mathcal{P}_1, f) + U(\mathcal{P}_2, f) &< L(\mathcal{P}_1, f) + L(\mathcal{P}_2, f) + \varepsilon \\ &\leqslant \int_a^c f + \int_c^b f + \varepsilon, \end{aligned}$$

and similarly

$$\int_a^b f \geqslant \int_a^c f + \int_c^b f - \varepsilon.$$

Thus

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

■

Theorem 6.10 [Triangle inequality] *If function* $f \in \mathcal{R}[a, b]$ *then* $|f| \in \mathcal{R}[a, b]$ *and*

$$\left| \int_a^b f \right| \leqslant \int_a^b |f|.$$

Proof. Since $f \in \mathcal{R}[a, b]$ then by definition, for $\varepsilon > 0$ $\exists$ a partition $\mathcal{P} = \{a = x_0, x_1, x_2, ..., x_n = b\}$ of $[a, b]$ such that

$$U(\mathcal{P}, f) - L(\mathcal{P}, f) < \varepsilon. \tag{6.6}$$

Let M_i and m_i respectively be the supremum and infimum of $f(x)$ in the interval $[x_{i-1}, x_i]$ and also let M_i' and m_i' respectively be the supremum and infimum of $|f(x)|$ in the same interval. For $\forall x, y \in [x_{i-1}, x_i]$ we have

$$|f(x)| - |f(y)| \leqslant |f(x) - f(y)|, \tag{6.7}$$

and

$$|f(x) - f(y)| \leqslant M_i - m_i. \tag{6.8}$$

From (6.7) and (6.8) we get

$$|f(x)| - |f(y)| \leqslant M_i - m_i. \tag{6.9}$$

Now taking supremum over x in (6.9), we obtain

$$M_i' - |f(y)| \leqslant M_i - m_i. \tag{6.10}$$

Again taking supremum over y in (6.10), we get

$$\begin{aligned} M'_i + \sup\left(-|f(y)|\right) &\leqslant M_i - m_i \\ \Rightarrow M'_i - inf|f(y)| &\leqslant M_i - m_i \\ \Rightarrow \qquad M'_i - m_i' &\leqslant M_i - m_i. \end{aligned} \tag{6.11}$$

Using (6.11) and (6.6), we have

$$\begin{aligned} U\left(\mathcal{P}, |f|\right) - L\left(\mathcal{P}, |f|\right) &= \sum_{i=0}^{n} \left(M' - m'_i\right)\Delta x_i \\ &\leqslant \sum_{i=0}^{n} (M_i - m_i)\Delta x_i = U\left(\mathcal{P}, f\right) - L\left(\mathcal{P}, f\right) < \varepsilon. \end{aligned}$$

Hence prove that $|f| \in R[a, b]$.
Now we have to show that

$$\left|\int_a^b f\right| < \int_a^b |f|.$$

Since

$$-f(x) \leqslant |f(x)| \Rightarrow -|f(x)| \leqslant f(x) \text{ and } f(x) \leqslant |f(x)|\,.$$

Therefore

$$-|f(x)| \leqslant f(x) \leqslant |f(x)|\,.$$

Hence

$$-\int_a^b |f(x)|dx \leqslant \int_a^b f(x)dx \leqslant \int_a^b |f(x)|dx.$$

Finally, we have

$$\left|\int_a^b f(x)dx\right| \leqslant \int_a^b |f(x)|dx.$$

■

6.2 Fundamental Theorem of Calculus

Fundamental theorem of calculus have two versions and both the version express the same thing, that differentiation and integration are inverse operation.

Theorem 6.11 [Part-I Integral of a Derivative] *If $f:[a,b]\to\mathbb{R}$ is differentiable on $[a,b]$, and $f'\in\mathcal{R}[a,b]$, then*

$$\int_a^b f'(x)dx = f(b)-f(a). \tag{6.12}$$

Proof. Let $\mathcal{P}=\{a=x_0,x_1,x_2,...,x_n=b\}$ be a partition of $[a,b]$. Let $\varepsilon>0$ then

$$U\left(\mathcal{P},f'\right)-L\left(\mathcal{P},f'\right)<\varepsilon \tag{6.13}$$

Applying Mean-Value theorem, f defined on $[x_{i-1},x_i]$ and the points $t_i\in[x_{i-1},x_i]$ such that

$$f\left(x_i\right)-f\left(x_{i-1}\right)=f'(t_i)\left(x_i-x_{i-1}\right).$$

Hence

$$f(b)-f(a)=\sum_{i=1}^{n}\left[f\left(x_i\right)-f\left(x_{i-1}\right)\right]=\sum_{i=1}^{n}f'(t_i)\left(x_i-x_{i-1}\right).$$

By definition of Riemann integration

$$L\left(\mathcal{P},f'\right)\leqslant f(b)-f(a)\leqslant U\left(\mathcal{P},f'\right). \tag{6.14}$$

Therefore

$$L\left(\mathcal{P},f'\right)\leqslant\int_a^b f'\leqslant U\left(\mathcal{P},f'\right),$$

From (6.13) and (6.14), we get

$$\left|\int_a^b f'-\left[f(b)-f(a)\right]\right|<\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, therefore (6.12) holds. ■

Theorem 6.12 [Part-II Derivative of an Integral] *If function $f:[a,b]\to\mathbb{R}$ is a bounded and $f\in\mathcal{R}[a,b]$, let the function F defined as*

$$F(x)=\int_a^x f(t)dt,\qquad a\leqslant x\leqslant b,$$

then F is continuous on $[a,b]$. If f is continuous at x_0 in (a,b), then F is differentiable at x_0 and $F'(x_0)=f(x_0)$.

Proof. Since f is bounded and Riemann integrable on $[a,b]$ therefore $\exists$ a number $M>0$, such that

$$|f(x)| \leqslant M, \quad \text{for all } x \in [a,b].$$

If x,y are two points of $[a,b]$ such that $a \leqslant x < y \leqslant b$ then

$$\begin{aligned} |F(y)-F(x)| &= \left| \int_a^y f(t)dt - \int_a^x f(t)dt \right| \\ &= \left| \int_x^y f(t)dt \right| \leqslant |y-x|\, M \end{aligned}$$

Choose $\varepsilon > 0$ and let $\delta = \varepsilon / M$. If $|y-x|<\delta$, then $|F(y)-F(x)|<\varepsilon$. Thus F is continuous on $[a,b]$.

Let $\varepsilon > 0$. Since f is continuous at x_0, there exists $\delta > 0$ such that

$$t \in (a,b) \quad \text{and} \quad |t-x_0|<\delta \quad \text{imply} \quad |f(t)-f(x_0)|<\varepsilon$$

$$\begin{aligned} \left| \frac{F(x)-F(x_0)}{x-x_0} - f(x_0) \right| &= \left| \frac{1}{x-x_0} \int_{x_0}^x f(t)dt - f(x_0) \right| \\ &= \frac{1}{|x-x_0|} \left| \int_{x_0}^x [f(t)-f(x_0)]dt \right| \\ &< \frac{1}{|x-x_0|} |x-x_0|\, \varepsilon < \varepsilon \end{aligned}$$

Hence

$$\lim_{x \to x_0} \frac{F(x)-F(x_0)}{x-x_0} = f(x_0).$$

In other words, $F'(x_0) = f(x_0)$. ■

Example 6.4 *Standard example from calculus $u = logx$*

$$\int_2^4 \frac{(logx)^4}{x} dx = \int_{\log 2}^{\log 4} u^4 du = \frac{(log4)^4 - (\log 2)^4}{4}.$$

By substitution formula and by integrating the chain rule, we get the integration by part formula by integrating the product rule.

6.2.1 Scope and limitations of the Fundamental theorem

The fundamental theorem leads us to certain important conclusions as to the relations of differentiation and integration. For simple functions considered in elementary calculus, they are inverse operations. If we include all functions to which these two processes apply, we shall see that they are not always the inverse of each other. It is desirable under what conditions they may be so regarded. Suppose we have a function F defined by

$$\phi(x) = \int_a^x f(x)dx \quad \forall x \in [a,b]. \tag{6.15}$$

The proposed problem involves the conditions that the given integral shall exist and that we shall have

$$f(x) = \frac{d\phi}{dx}. \tag{6.16}$$

Conversely it involves the condition that a given function ϕ shall have at each point of $[a,b]$ a unique derivative such that this derived function, say f shall have the given function ϕ as an indefinite integral.

Now assume that f is bounded and integrable on $[a,b]$. From theorem , it follows that at each point of continuity of f, the relation (6.16) holds. If f is continuous at all points of $[a,b]$ then f is the derivative of the indefinite integral F for all values of x on $[a,b]$. In other words, differentiation is in this case the inverse of integration. The continuity of f on $[a,b]$ is only a **sufficient** condition for f to have a derivative on $[a,b]$. It is however not a necessary condition. For example, the derivative of ϕ is not continuous where $\phi(x) = x^2 \sin\frac{1}{x}$ $(x \neq 0)$, $\phi(0) = 0$. On the basis of f being continuous on $[a,b]$, we have proved that

$$\int_a^x f(x)dx = \phi(x) - \phi(a)$$

where ϕ is the primitive of f. Thus we have obtained the important result that whenever f is continuous it possesses a primitive and the knowledge of the primitive is equivalent to our ability to evaluate

$$\int_a^b f(x)dx$$

for if ϕ is primitive in question, then

$$\int_a^b f(x)dx = \phi(b) - \phi(a).$$

The question as to whether a primitive exists and the question of the existence of an integral of a function f on $[a,b]$ are entirely independent questions. The fundamental theorem however shows that when f is continuous on $[a,b]$, the function F defined by

$$\phi(x) = \int_a^b f(x)dx$$

for all $x \in [a, b]$, and the function ϕ defined by the $\phi'(x) = f(x)$ are identical except for an arbitrary additive constant.

We have also proved in the proof of theorem the equality $\int_a^b f(x)dx = \phi(b) - \phi(a)$ where $\phi'(x) = f(x)$ for all $x \in [a, b]$ simply by assuming f to be bounded and integrable. Hence if we somehow are able to find ϕ such that $\phi'(x) = f(x)$ for all $x \in [a, b]$, then fundamental theorem ( ) enables us to evaluate $\int_a^b f(x)dx$.

The R-integral of f may h however exist even if f has no primitive at all on $[a, b]$. For example, the function $f$defined by $f(x) = 0$ for $x \neq 0$ and $f(0) = 1$ is R-integrable, being bounded and having one point of discontinuity at $x = 0$ but f has no primitive, that is there is no function ϕ whose derivative satisfies the above condition, *i.e.*, for which $\frac{d}{dx}\phi(x) = f(x)$.

6.2.2 Distinction between a primitive and an integral.

(1) Recall that a function ϕ which has f as its derivative on $[a, b]$ is called a primitive of $f, \phi + c$ where c is an arbitrary constant is also a primitive of f. It is the solution of differential equation $\frac{dy}{dx} = f(x)$. The integral on the other hand is the analytical solution for an area if $y = f(x)$ has a continuous graph.

(2) Finding the primitive thus means the process inverse to differentiation whereas as integral is regarded as the limit of sum of an indefinitely large number of small elements. It is possible to develop the theory of integrals completely without any reference to primitives.

(3) In practise the distinction between the two is that while integrals can be calculated, primitives cannot.

(4) Comparatively few functions have primitives, a great many more have integrals.

(5) There is some connection between primitive and integral but not a great deal. That relation is represented by fundamental theorem of integral calculus, viz.

$$\int_a^b f'(x)dx = f(b) - f(a).$$

(6) A function may have a primitive but not an integral and vice versa. For example the function f defined by

$$f(x) = 2x\sin(1/x^2) - (2/x)\cos(1/x^2), x \neq 0$$

and $f(0) = 0$ has a primitive $x^2 \sin(1/x^2)$ but in the interval $[-1,1]$ there is no R-integral since f is unbounded in the neighbourhood of zero. Again the function ϕ defined by

$$\phi(x) = 1 \quad \text{for} \quad x \neq 0 \quad \text{and} \quad \phi(0) = 0$$

is R-integrable being bounded and having only one point of discontinuity at $x = 0$ but has no primitive.

Theorem 6.13 [Integration by Parts] *Suppose f and g are continuous functions on $[a,b]$ and differentiable on (a,b), then*

$$\int_a^b f'(x)g(x)dx + \int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a).$$

Proof. We know that

$$(fg)' = f'g + g'f \tag{6.17}$$

Since all the functions are continuous, so we can integrate (6.17)

$$\int_a^b (fg)'(x)dx = \int_a^b f'(x)g(x)dx + \int_a^b f(x)g'(x)dx \tag{6.18}$$

Applying first fundamental theorem, we obtain

$$\int_a^b (fg)'(x)dx = f(b)g(b) - f(a)g(a) \tag{6.19}$$

From (6.18) and (6.19), we get

$$\int_a^b f'(x)g(x)dx + \int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a).$$

■

6.2.3 Mean-Value and Change-of-Variable Theorems

An antiderivative is often required for applying the fundamental theorem of calculus for this reason integration by substitution is an important tool of calculus. It is the counterpart to the chain rule of differentiation and the last two theorems are in a sense analogues of the Mean-Value Theorem for derivatives.

Theorem 6.14 [Change of variables] *Suppose $[a,b] \subseteq \mathbb{R}$ be an interval and $g : [a,b] \to \mathbb{R}$ be a continuously differentiable function. If $g([a,b]) \subset [c,d]$ and $f : [c,d] \to \mathbb{R}$ is a continuous function on $g([a,b])$, then*

$$\int_{g(a)}^{g(b)} f(x)\,dx = \int_a^b f(g(t))g'(t)\,dt \tag{6.20}$$

the substitution $x = g(t)$ and thus $dx = g'(t)\,dt$.

Proof. Since f is continuous on closed and bounded interval $[a,b]$, it possesses an antiderivative F, i.e., $F' = f$. Since F and g are differentiable, then chain rule gives

$$(F \circ g)'(t) = F'(g(t))g'(t) = f(g(t))g'(t).$$

Applying the second fundamental theorem of calculus twice gives

$$\begin{aligned}\int_a^b f(g(t))g'(t)dt &= (F \circ g)(b) - (F \circ g)(a) \\ &= F(g(b)) - F(g(a)) \\ &= \int_{g(a)}^{g(b)} f(x)dx.\end{aligned}$$

■

Theorem 6.15 [First Mean-Value Theorem] *If $f : [a,b] \to \mathbb{R}$ is continuous function, then there is $\xi \in [a,b]$ such that*

$$\int_a^b f(x)dx = f(\xi)(b-a).$$

Proof. Let $m = inf\,\{f(x) : x \in [a,b]\}$ and $M = \sup\{f(x) : x \in [a,b]\}$ then

$$m(b-a) \leqslant \int_a^b f(x)dx \leqslant M(b-a), \quad \text{if } b \geqslant a$$

Hence there is a number $\mu \in [m,M]$, such that

$$\int_a^b f(x)dx = \mu(b-a).$$

Since f is continuous on $[a,b]$, it attains every value between its bounds m, M. Therefore there exists a number $\xi \in [a,b]$ such that $f(\xi) = \mu$. Thus

$$\int_a^b f(x)dx = f(\xi)(b-a).$$

■

Theorem 6.16 [Second Mean-Value Theorem] *If $f : [a,b] \to \mathbb{R}$ is monotonic function, then there is $\xi \in [a,b]$ such that*

$$\int_a^b f(x)dx = f(a)(\xi - a) + f(b)(b - \xi).$$

Proof. Since f is monotonic function on $[a,b]$, therefore $f \in \mathcal{R}[a,b]$. Let $g : [a,b] \to \mathbb{R}$ be a continuous function define by $g(x) = f(a)(x-a)+f(b)(b-x)$. Since f is monotonic function, therefore

$$g(a) = f(b)(b-a) \leqslant \int_a^b f(x)dx \leqslant f(a)(b-a) = g(b)$$

By the continuity of g, there is $\xi \in [a,b]$ such that

$$g(\xi) = \int_a^b f(x)dx.$$

Thus

$$\int_a^b f(x)dx = g(\xi) = f(a)(\xi - a) + f(b)(b-\xi).$$

■

6.3 Exercise

1. Prove that the function f defined on $[a, b]$ by

$$f(x) = \begin{cases} x & \text{when } x \in \mathbb{Q} \\ -x & \text{when } x \notin \mathbb{Q}, \end{cases}$$

is not integrable over $[a, b]$ whereas $|f| \in \mathcal{R}[a, b]$

2. If f is defined on $[0, 1]$ by $f(x) = 0$ when x is irrational or zero and $f(x) = 1/q^{1/3}$ when x is any non-zero rational number p/q with least $p, q \in \mathbb{N}$, prove that $f \in \mathcal{R}[0, 1]$ and

$$\int_0^1 f = 0.$$

3. If $f \in \mathcal{R}[a, b]$ and $\int_a^b f^2(x)dx = 0$, then prove that $f(x) = 0$ at its points of continuity.

4. Distinguish between an integral and primitive of a function.

5. Give a different example of discontinuous function which admits primitive but not the integral on a closed interval.

6. If $f(x) = x[1/x]$ when $0 < x \le 1$ and $f(0) = 1$, prove that $f \in \mathcal{R}[0, 1]$ and $\int_0^1 f = \frac{\pi^2}{12}$, where $[1/x]$ denotes integral part of $1/x$.

7. If f is bounded, defined on $[0, 1]$ and $f(x) = (-1)^{n-1}$ when $1/(n+1) < x < 1/n$; $n \in \mathbb{N}$, then prove that $f \in \mathcal{R}[0, 1]$ and

$$\int_0^1 f = 2\log 2 - 1.$$

8. If f is continuous on $[a, b]$, if

$$f(x) \ge 0 \quad (a \le x \le b),$$

and if

$$\int_a^b f(x)dx = 0,$$

prove that f is identically zero on $[a, b]$.

9. Let

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ \sin x & \text{if } x \notin \mathbb{Q}, \end{cases}$$

prove that $f'(0) = 1$.

10. True or false? If f is a function on $[a, b]$, if $c \in [a, b]$, and if $f'(c) > 0$, then f is strictly increasing on some open subinterval of $[a, b]$ containing c.

11. If f is a real- valued function on $[a, b]$ and if f has a right-hand derivative at $c \in [a, b]$, prove that f is continuous on the right at c.