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A geometric model for active contours in image processing

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Summary. We propose a new model for active contours based on a geometric partial differential equation. Our model is intrinsec, stable (satisfies the maximum principle) and permits a rigorous mathematical analysis. It enables us to extract smooth shapes (we cannot retrieve angles) and it can be adapted to find several contours simultaneously. Moreover, as a consequence of the stability, we can design robust algorithms which can be engineed with no parameters in applications. Numerical experiments are presented.

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1. Introduction

One of the problems in image processing is the edge detection problem which is the problem of finding lines separating homogeneous regions. An edge must have two properties: to be smooth or piecewise smooth and the gradient of the image should be large along the edge [31, 32]. To solve this problem, one would like to realize models which try to provide a satisfactory answer to the following issues:

i) Find global contours which best reproduce the apparent contours in the image

ii) Ability to extract smooth shapes: we get regular contours.

iii) Respect of some singularities: angles, sharp corners, …

iv) Robustness of the algorithms: stable, convergent with a minimum set of parameters

v) Eventual adaptation to find several contours simultaneously

vi) Theoretical correctness of the method

We are going to study and compare two different types of methods which provide, each of them, a partial answer to the above issues:

I) The classical method of snakes

II) A new geometric PDE model based on the mean curvature motion equation

Let us briefly describe both methods and summarize how do they behave with respect to the above demands. We start with the classical snake method.

I) The classical method of snakes starts with an initial contour C_0 called "active contour" or "snake" near some contour γ_0 in the image and one looks for admissible deformations of C_0 which let it move towards the desired contour γ_0. These deformations are obtained by trying to minimize an energy functional designed in such a way that the set of local minima constitutes the searched image features. This represents a global view on edge detection, differing from the traditional approach of detecting edges and then linking them. The original idea was due to Kass-Witkin-Terzopoulos [26, 27, 37], Blake-Zisserman [7] and further improvement of this model was successively done by Terzopoulos [36], Cohen [12], Cohen-Cohen [13] and many other contributors [2, 3, 5, 6, 9, 10, 11, 14, 29, 38, 39].

The energy functional always consists in the sum of two terms:

a) The internal energy E_{int} serves to impose a smoothness constraint. In the basic snake model, the curve is a controlled continuity spline [35]. Representing the position of the snake parametrically by $v(x)=(x(s), y(s))$, $s\in[0,1]$, we can write the internal energy as:

$$E_{\text{int}}=\int_0^1 (\alpha|v'(t)|^2+\beta|v''(t)|^2)\,dt. \tag{1.1}$$

According to the above mentioned authors, parameters $\alpha>0$ and $\beta>0$ impose the elasticity and rigidity coefficients of the curve. We shall later discuss this interpretation.

b) The external energy E_{ext} depends on the features which are searched for in the image. Its function is to push the snake towards the salient image features (dark lines, white lines, edges, termination of line segments, ...). For instance, in edge detection, E_{ext} is defined as:

$$E_{\text{ext}}=-\lambda\int_0^1 |\nabla I(v(t))|\,dt \tag{1.2}$$

where ∇I is the gradient of the image intensity and $\lambda>0$.

Then, in the snake method, the moving curve tries to minimize the global energy

$$E=E_{\text{int}}+E_{\text{ext}}. \tag{1.3}$$

Shortly, the curves for which $|\nabla I(v(t))|$ is maximal are searched for in the class of splines under tension (controlled continuity splines). Such a method is a way of regularizing the edge detection problem, which is ill posed [34].

Concerning the set of questions i)–vi) and to make a long story short one could say that the method of snakes provides an accurate localization of edges near a given initialization of the curve and they are able to extract the smooth shapes. However, we shall prove (see Sect. 2 and the appendix below) that the snakes model can retrieve angles and this is true for the general model with $\alpha>0$, $\beta>0$ or for the simplified models with $\alpha=0$, $\beta>0$ or $\alpha>0$, $\beta=0$. This raises the question of the role of the term $\int_0^1 |v''(t)|^2 dt$, introduced to control the rigidity of the moving curve. From the algorithmic point of view this is related to the adaptation of the set of parameters α, β, λ to the problem at hand. Question iv) has not yet a definite answer. Moreover, the snake model does not permit the simultaneous treatment of several contours and question vi) has not yet been addressed from a theoretical point of view. To conclude, we think that the theoretical difficulties raised by the snake model are due to the fact that it is not an intrinsec model: the parametrization of the curves does not permit to get the geometrical regularity of the contours.

II) That is why we propose a different model based on the mean curvature motion equation

$$\frac{\partial u}{\partial t}=|\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) \quad (t, x)\in [0, \infty[\times \mathbb{R}^2 \tag{1.4}$$

which describes the motion of the level sets of the function u, $\{x\in\mathbb{R}^2 : u(t, x)=k\}$, $k\in\mathbb{R}$, which evolve following the normal direction with speed depending on the mean curvature. In fact, we add to (1.4) a term representing a constant force in the direction of the normal to the level sets and then multiply it by a function $g(x)$ to stop the followed level set there where we want to reproduce the desired contour. Thus our model is

$$\frac{\partial u}{\partial t}=g(x)|\nabla u| \left(\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)+v\right) \quad (r, x)\in [0, \infty[\times \mathbb{R}^2 \tag{1.5a}$$

$$u(0, x)=u_0(x) \quad x\in\mathbb{R}^2 \tag{1.5b}$$

where

$$g(x)=\frac{1}{1+(\nabla G_\sigma * g_0)^2}. \tag{1.6}$$

v is a positive real constant, $G_\sigma * g_0$ is the convolution of the image g_0 where we are looking for the contour of an object O with the Gaussian $G_\sigma(x) = C\sigma^{-1/2}\exp(-|x|^2/4\sigma)$ and u_o is the initial data which is taken as a smoothed version of the function $1-\chi_C$, where χ_C is the characteristic function of a set C containing O.

Let us discuss the geometrical interpretation of this model. Suppose that $u(t, x, y)$ is a C^2 function and let k be a real number such that

$$\partial C=\{(x, y): u(t, x, y)=k\}.$$

We parametrize ∂C by its arclength so that C is to the left when we follow ∂C in the positive sense

$$\partial C=\{(x(s), y(s)): s\in[0, L(\partial C)]\}$$

where $L(\partial C)$ is the length of C. We have

$$u(t, x(s), y(s)) = k.$$

Hence

$$(1.7) \qquad \begin{aligned} u'_x\, x'_s + u'_y\, y'_s &= 0 \\ y'_s &= \lambda u'_x \\ x'_s &= -\lambda u'_y \end{aligned}$$

with $\lambda > 0$. Differentiating (1.7) with respect to s we have

$$u''_{xx}(x'_s)^2 + 2u''_{xy}\, x'_s\, y'_s + u''_{yy}(y'_s)^2 + u'_x\, x''_s + u'_y\, y''_s = 0$$

we see that the curvature ρ of ∂C at the point $(x(s), y(s))$ which is defined as it is classical by

$$\rho(s) = \frac{x'_s\, y''_s - y'_s\, x''_s}{((x'_s)^2 + (y'_s)^2)^{1/2}}$$

may be written as

$$\rho(s) = \lambda^2 \frac{u''_{xx}(u'_y)^2 - 2u''_{xy}\, u'_x\, u'_y + u''_{yy}(u'_x)^2}{((x'_s)^2 + (y'_s)^2)^{1/2}}.$$

Finally, since $(x'_s)^2 + (y'_s)^2 = 1 = \lambda^2((u'_x)^2 + (u'_y)^2)$, the curvature of ∂C at the point $(x(s), y(s))$ is given by

$$\rho = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right).$$

Let us interpret geometrically the proposed model (1.5a, b),

1) The term $\operatorname{div}\left(\frac{\nabla u)}{|\nabla u|}\right)$ ensures that the grey level at a point in ∂C increases proportionally to the algebraic curvature of ∂C at this point. This term is responsible for the regularizing effect of the model and plays the role of the energy term (1.1) in the snake model. The constant v is a correction term chosen so that $\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + v$ remains always positive. Hence, the grey level at a point at ∂C will increase (going from black to grey) and one expects ∂C reproduce the boundary of the object O. The constant v may be interpreted as a force pushing ∂C towards O when the curvature of ∂C becomes null or negative.

2) The term $|\nabla u|$ controls what happens at the interior and exterior of C. The function u hardly changes its grey level except on a neighborhood of ∂C.

3) The term $g(x)$ given by (1.6) controls the speed at which ∂C moves. When ∂C is near the boundary of the object O, $|\nabla G_\sigma * g_0|$ is big and ∂C stops. We convolved the image g_0 to eliminate the effect of noise on the motion of ∂C. This coefficient is the point where the image comes in our model. It slows down the growth of the function u near the boundary of the object O and it stops it exactly on the boundary of it, if this boundary was a regular curve (see Sect. 4 below). Thus the energy criterium (1.2) used to push the snake to-

wards the desired contour in the snakes model has been replaced by a slowing down criterium represented by the coefficient $g(x)$ in our model.

We summarize the behavior of this model with respect to the demands above. Our model also gives an accurate localization of the edges and is able to extract smooth shapes, thus giving a satisfactory answer to the issues i) and ii) above. But it regularizes the angles (see iii)). This is observed experimentally and it is due to the presence of the elliptic term $|\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$ on the PDE (1.5a) which has a regularizing effect on the level set curves of the function $u(t, x)$ (see [18, 20, 21, 22, 24]). Concerning iv), one can design stable numerical algorithms with a minimum set of adjustment parameters. As a matter of fact, our method removes all adjustment parameters. Moreover, a nice aspect of the model is that it can provide several contours at no additional computational expense time (see v)). Finally concerning vi): it permits a theoretical analysis and proof of the correctness of the method.

Let us again point out that the philosophy of both methods is not too far. The "active contour" or "snake" in the classical snake model has been replaced by a level set curve of a function u whose graph is moving upwards except on the precise object O whose boundary we are looking for. If we initialize (1.5a, b) with a smoothed version of the function $1-\chi_C$, where χ_C is the characteristic function of a set C containing O, we see how the zero level set squeezes the boundary of O. The elliptic term $|\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$ on the PDE (1.5a) plays the role of the regularizing term (1.1) in the energy functional (1.3) and the energy term (1.3) has been replaced by the stopping coefficient $g(x)$ in this geometric model. The main difference being that the geometric PDE model is an intrinsec one., i.e., *the evolution of the level set curves does not depend on its particular parametrization*, while the snake model does not.

Geometric equations like (1.5) have recently been studied in [4, 8, 17, 18, 23, 33]. In particular, a related model of anisotropic diffusion with application to images was studied in [1]. In this paper the smoothing effect of the operator of anisotropic diffusion on the image was put in evidence and a stable algorithm was designed for it. It will be the basis for the numerical treatment of our model (1.5) to be discussed in Sect. 5 below. All these equations satisfy a maximum principle and the basic theory used to study them is the theory of viscosity solutions.

Let us finally explain the plan of this paper. In Sect. 2, we discuss the issue iii) for the snakes functional (1.3) for a particular and simple image intensity function whose gradient has a discontinuity at a corner and prove that functionals like (1.3) reproduce corners in a soft edge like ours. Functionals like (1.3) with $\alpha=0$, β, $\lambda>0$ and $\beta=0$, α, $\lambda>0$ also permit the accurate reconstruction of corners. As a matter of fact, the model (1.3) implies smooth parametrizations but not smooth shapes. This raises the question of whether the simplest functional of the family with $\beta=0$, α, $\lambda>0$ would be sufficient for the treatment of snakes. To simplify the presentation of the paper, in Sect. 2 only the proof for the case $\beta=0$, α, $\lambda>0$ will be given (the other cases are treated on the appendix in Sect. 7). In Sect. 3 we prove the existence and uniqueness of viscosity solutions of (1.5) for any initial condition in $W^{1,\infty}$. The asymptotic behavior of the level zero set curve ∂C (with $k=0$) will be discussed in Sect. 4 where we prove that ∂C converges to the desired contour ∂O as $t \to +\infty$ provided that we suppose it to be smooth (C^2, in fact), hence, justifying the issue vi) above for the geometric PDE model. In Sect. 5 we discuss the numerical algorithm and present the experimental results obtained using the geometric PDE model (1.5a, b).

Let us mention that the basic mathematical tools employed are the theory of viscosity solutions for second order degenerate elliptic equations (see [15, 16, 17, 30]) and a lot of standard calculations.

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2. Snake model can retrieve angles

Let us consider the image intensity function $I: [-1, 1]^2 \to \mathbb{R}$ given by

$$I(x, y) = y \quad \text{for } y \geqq 0, x+y \geqq 0; \qquad -x \quad \text{for } x \leqq 0, y+x \leqq 0;$$
$$0 \quad \text{for } x \geqq 0, y \leqq 0; \qquad (x, y) \in [-1, 1]^2.$$

I is a Lipschitz function such that

$$|\nabla I(x, y)| = 1 \quad \text{for } x<0 \text{ or } y>0; \qquad 0 \quad \text{for } x>0, \ y<0$$

and it is discontinuous on the corner $C = \{(x, y) \in [-1, 1]^2 : x=0, y \in [-1, 0] \text{ or } x \in [0, 1], y=0\}$. For technical reasons we make $|\nabla I(x, y)|$ upper semicontinuous by defining it to be 1 on the corner C. We want to study whether the minimization of the functional

$$E(v; \alpha, \beta, \lambda) = \int_0^1 (\alpha |v'(t)|^2 + \beta |v''(t)|^2 \, dt - \lambda \int_0^1 |\nabla I(v(t))| \, dt \tag{2.1}$$

on a suitable space of admissible functions $Ad(\alpha, \beta)$ will give an accurate description of the corner C. Our minimization problem can be written as

$$\operatorname{Min} E(v; \alpha, \beta, \lambda) \quad \text{on } Ad(\alpha, \beta) \tag{2.2}$$

where $Ad(\alpha, 0) = \{v \in H^1([0, 1], \mathbb{R}^2) : v(0) = (0, -1), v(1) = (1, 0), v(t) \in]-1, 1[^2\}$, $Ad(0, \beta) = \{v \in H^2([0, 1], \mathbb{R}^2) : v(0) = (0, -1), v(1) = (1, 0), v'(0) = (0, 0), v'(1) = (0, 0), v(t) \in]-1, 1[^2\}$ and $Ad(\alpha, \beta) = Ad(0, \beta)$, $\alpha > 0$, $\beta > 0$. We are going to prove the following

Proposition 2.1. *For λ big enough, there exists a unique function $\phi \in Ad(\alpha, \beta)$ such that $E(\phi; \alpha, \beta, \lambda) = \operatorname{Min} E(v; \alpha, \beta, \lambda)$ on $Ad(\alpha, \beta)$, $\alpha \geqq 0$, $\beta \geqq 0$. Moreover ϕ is a parametrization of the corner.*

The solution of the minimization problem (2.2) is an accurate parametrization of the corner belonging to $Ad(\alpha, \beta)$. The proof of Proposition 2.1 is elementary but long. Therefore, we give the complete proof only in the cases $\alpha > 0$, $\beta = 0$ and $\alpha = 0$, $\beta > 0$. The case $\alpha > 0$, $\beta > 0$ is rather tedious and we give only a sketch of the proof. To simplify the presentation, this will be done in the appendix (Sect. 6).

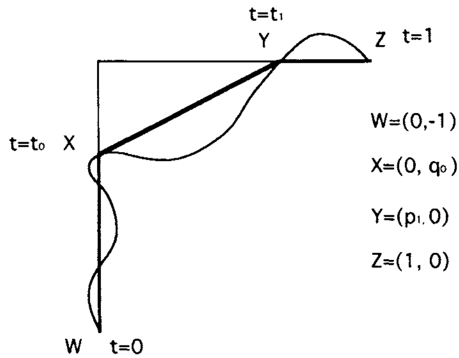

Fig. 1. A path joining W and Z

Proof for $\alpha>0$, $\beta=0$. Without loss of generality we may assume that $\alpha=1$ (changing λ by λ/α).

Before giving the complete proof, let us briefly explain the result. Let, for instance, v be a parametrization of a path joining the points W and Z as represented in Fig. 1.

Then,

$$E(v;1,0,\lambda)=\int_0^1 |v'(t)|^2\,dt-\lambda\int_0^1 |\nabla I(v(t))|\,dt$$
$$\leqq \int_0^1 |v'(t)|^2\,dt-\lambda t_0-\lambda(1-t_1)$$

and by the Cauchy-Schwarz inequality:

$$\int_0^1 |v'(t)|^2\,dt \geqq \frac{L_0^2}{t_0}+\frac{L_1^2}{t_1-t_0}+\frac{L_2^2}{1-t_1}$$

where L_0, L_1, L_2 are, respectively, the length of the path joining W to X, X to Y, Y to Z. On the other hand, consider the uniform parametrization (constant speed) w of the segments WX, XY, YZ. We have:

$$E(w;1,0,\lambda)=\frac{(1+q_0)^2}{t_0}+\frac{q_0^2+p_1^2}{t_1-t_0}+\frac{(1-p_1)^2}{1-t_1}-\lambda t_0-\lambda(1-t_1)\leqq E(v;1,0,\lambda).$$

So it is enough to search inside uniform parametrization of at most 3 segments making a path joining W to Z. See the end of the proof for an illustration of the result.

Step 1. Existence of a global minimum in $Ad(1,0)$.

Since $|\nabla I(x,y)|$ is upper semicontinuous and the injection of $H^1[0,1]$ in $C[0,1]$ is compact, $E(.,1,0,\lambda)$ is a weakly lower semicontinuous functional defined on $Ad(1,0)$. Moreover, it is the sum of a convex functional and a bounded one. From these remarks it follows easily that the minimum in (2.2) is attained by some function $\phi\in Ad(1,0)$.

Let $x_0(t)$, $y_0(t)$ be the coordinate functions of $\phi(t)$. Now we are going to study how does the function ϕ look like. To this aim, let us introduce some notation: let$R_0=\{(x,y)\in]-1,1[^2: x>0, y<0\}$, $R_i=\{(x,y)\in]-1,1[^2: x<0$ or $y>0\}$ and let $A=\{t\in[0,1]: \phi(t)\in R_0\}$, $B=\{t\in[0,1]: \phi(t)\in R_i\}$, $Z_-=\{t\in[0,1]: \phi(t)=(0,y_0(t))$ with $y_0(t)\leqq 0\}$, $Z_+=\{t\in[0,1]: \phi(t)=(x_0(t),0)$ with $x_0(t)\geqq 0\}$. Since $\phi\in Ad(1,0)$ is a continuous function, A, B are open subsets of $[0,1]$. Then it follows using standard methods [19] that

Lemma 2.2. *$\phi''=0$ in A, B and in the open intervals of Z_- and Z_+.*

Sketch of the proof. Since A is open, A is the union of its connected components which are intervals. Let $]a,b[$ be a connected component of A and let $\xi\in C^\infty(\mathbb{R})$ with support $\operatorname{supp}\xi\subseteq]a,b[$. Then $E(\phi+\varepsilon\xi,1,0,\lambda)\geqq E(\phi,1,0,\lambda)$ for ε small enough. Writing explicitly this inequality, our assertion follows immediately. □

This lemma means that once ϕ enters R_0 or R_i it moves on a straight line. Similarly, if for each $t\in[t_0,t_1]$, $\phi(t)$ is in Z_- (resp. in Z_+), then $\phi(t)$ is a parametrization of the line segment joining $\phi(t_0)$ to $x(t_1)$.

Step 2. The trajectory ϕ does not oscillate and never travels back.

For each $v(t)=(x(t),y(t))\in H^1([0,1],\mathbb{R}^2)$, $t_0<t_1$ in $[0,1]$ let us define

$$E(v;t_0,t_1)=\int_{t_0}^{t_1}|x'(t)|^2+|y'(t)|^2\,dt-\lambda\int_{t_0}^{t_1}|\nabla I(v(t))|\,dt. \tag{2.3}$$

Lemma 2.3. *If there are two points $t_0<t_1$ in $[0,1]$ such that $x_0(t_0)=0$, $x_0(t_1)=0$ then $x_0(t)=0$ for all $t\in[t_0,t_1]$. A similar statement holds for y_0.*

This information must be complemented with the

Lemma 2.4. *If $0\leqq t_0<t_1\leqq 1$ and $y_0(t_0)=y_0(t_1)=0$, then $x_0(t_0)<x_0(t_1)$ (resp. if $x_0(t_0)=x_0(t_1)=0$, then $y_0(t_0)<y_0(t_1)$).*

Proof of Lemma 2.3. If $x_0(t)$ is not constant in $[t_0,t_1]$, *then* $\int_{t_0}^{t_1}|x_0'(t)|^2>0$. Then, replacing $x_0(t)$ by $x_0^*(t)=0$ in $[t_0,t_1]$ we get a trajectory ϕ^* such that $E(\phi^*;t_0,t_1)<E(\phi;t_0,t_1)$ which is a contradiction. □

Proof of Lemma 2.4. Since $y_0(t_0)=y_0(1)=0$, $y_0(t)=0$ for all $t\in[t_0,1]$ by the previous lemma. On the other hand $x_0(t_0)$, $x_0(1)=1$ are given. By Lemma 21.2: $x_0(t)=\dfrac{t-t_0}{1-t_0}+\dfrac{1-t}{1-t_0}x_0(t_0)$. Obviously $x_0(t_0)<x_0(t_1)$. □

Combining these lemmas we see that ϕ belongs to the set of model admissible trajectories $\boldsymbol{M}$ defined by: there exist $0\leqq t_0\leqq t_1\leqq 1$, $-1\leqq q_0\leqq 0$ and $0\leqq p_1\leqq 1$ such that

in $[0,t_0]$ $\phi(t)$ is the line segment joining $(0,-1)$ to $(0,q_0)$,
in $[t_0,t_1]$ $\phi(t)$ is the line segment joining $(0,q_0)$ to $(p_1,0)$,
in $[t_1,1]$ $\phi(t)$ is the line segment joining $(p_1,0)$ to $(1,0)$.

Step 3. The final step.

We have reduced the number of parameters describing the minimum ϕ to four. Let us call by $\boldsymbol{P}$ the set of parameters

$$\boldsymbol{P}=\{(t_0,t_1,q_0,p_1): 0\leqq t_0\leqq t_1\leqq 1,\ -1\leqq q_0\leqq 0, 0\leqq p_1\leqq 1\}.$$

If $(t_0,t_1,q_0,p_1)\in \mathrm{int}(\boldsymbol{P})$($=$ the interior of $\boldsymbol{P}$) they describe an admissible trajectory in $\boldsymbol{M}$. At the boundary of $\boldsymbol{P}$ some identifications have to be made: if $t_0=t_1$ then $q_0=p_1=0$. In this case, the parameters correspond to an admissible trajectory in $\boldsymbol{M}$ if $0<t_0=t_1<1$. In case $t_0=t_1=0$ or 1, the parameters describe a non admissible trajectory which can be associated to a parametrization of the corner on which one of its segments is travelled with infinite speed. Finally, $q_0=-1$ iff $t_0=0$; $q_0=0$ iff $p_1=0$ iff $t_0=t_1$ and $p_1=1$ iff $t_1=1$. Thus, if we add to $\boldsymbol{M}$ the trajectories with infinite energy described above by $t_0=t_1=0$ or 1, we get an enlarged set of model trajectories which, if no confusion arises, we call it again by $\boldsymbol{M}$. This new set $\boldsymbol{M}$ can be put in correspondence with the quotient of $\boldsymbol{P}$ given by the identifications described above. We know that the optimal trajectory ϕ belongs to $\boldsymbol{M}$ and we want to compute the parameters of $\boldsymbol{P}$ describing it.

Let us define the function $H: \boldsymbol{P}\to \mathbb{R}\cup\{+\infty\}$ by $H(t_0,t_1,q_0,p_1)$ being equal to the energy of the trajectory in $\boldsymbol{M}$ associated with the parameters (t_0,t_1,q_0,p_1) computed using (2.1). We know that the parameters $(t_0^*,t_1^*,q_0^*,p_1^*)$ describing ϕ realize a minimum of H. Since H is a C^∞ function in $\mathrm{int}(\boldsymbol{P})$, if $(t_0^*,t_1^*,q_0^*,p_1^*)\in\mathrm{int}(\boldsymbol{P})$, then

$$\frac{\partial H}{\partial t_0}=\frac{\partial H}{\partial t_1}=\frac{\partial H}{\partial p_0}=\frac{\partial H}{\partial q_1}=0 \quad \text{at } (t_0^*,t_1^*,q_0^*,p_1^*). \tag{2.4}$$

An easy computation gives

$$H(t_0,t_1,q_0,p_1)=\frac{(1+q_0)^2}{t_0}+\frac{q_0^2+p_1^2}{t_1-t_0}+\frac{(1-p_1)^2}{1-t_1}-\lambda t_0-\lambda(1-t_1). \tag{2.5}$$

Easy computations using (2.4) and (2.5) give that

$$t_0^*=1-\frac{1}{\lambda^{1/2}},\quad t_1^*=\frac{1}{\lambda^{1/2}},\quad q_0^*=\lambda^{1/2}-2,\quad p_1^*=2-\lambda^{1/2}. \tag{2.6}$$

Since $0<t_0^*<t_1^*<1$, $-1<q_0^*<0$, $0<p_1^*<1$, these restrictions impose that $1<\lambda<4$. Using (2.5) and (2.6), the value of the energy associated with these parameters is

$$H(t_0^*,t_1^*,q_0^*,p_1^*)=4\lambda^{1/2}-2\lambda,\qquad 1<\lambda<4. \tag{2.7}$$

Let us analyze the situation if $(t_0^*,t_1^*,q_0^*,p_1^*)$ is in the boundary of $\boldsymbol{P}$:

a) If $t_0^*=0$ and $t_1^*=1$ then $\phi(t)=(t,t-1)$.

b) If $t_0^*=0$, $0<t_1^*<1$ then $q_0^*=-1$ and $p_1^*<1$. We have two independent parameters: t_1^* and p_1^*. If $p_1^*=0$, then ϕ is an admissible parametrization of the corner. If $0<p_1^*<1$, then (t_1^*, p_1^*) is a local minimum of the function

$$H(t_1,p_1)=\frac{p_1^2}{t_1}+\frac{(1-p_1)^2}{1-t_1}-\lambda(1-t_1). \tag{2.8}$$

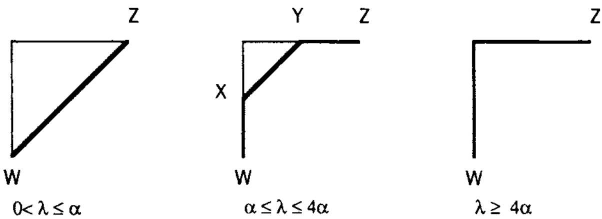

Fig. 2. The paths defined by the parametrization which realize the minimum of the energy

Setting $\frac{\partial H}{\partial t_1}=\frac{\partial H}{\partial q_1}=0$ at (t_1^*, p_1^*) we get that $\lambda=0$, a contradiction. Hence $p_1^*=0$ and ϕ is an admissible parametrization of the corner C.

c) The case $t_0^*=t_1^*=0$ does not give an admissible trajectory.

d) If $0<t_0^*=t_1^*<1$, ϕ is an admissible parametrization of the corner C.

e) If $0<t_0^*<t_1^*=1$, we define $\phi^*(t)=(x_0^*(t), y_0^*(t))$ by $x_0^*(t)=-y_0^*(1-t)$, $y_0^*(t)=-x_0^*(1-t)$. ϕ^* is also a global minimum of (2.2). We fall again into case b) already discussed.

f) If $q_0^*=-1$, then $t_0^*=0$, case already discussed. If $q_0^*=0$, we are in the presence of a corner. Similarly, if $p_1^*=0$ we have a parametrization of the corner and if $p_1^*=1$, then $t_1^*=1$, a case already discussed.

As a conclusion of this discussion, if $(t_0^*, t_1^*, q_0^*, p_1^*)$ is in the boundary of $\boldsymbol{P}$, either $\phi(t)=(t, t-1)$ with energy $E=2$ or ϕ parametrizes the corner C. Optimizing the energy functional with respect to all admissible parametrizations of C, the best one corresponds to $t_0^*=t_1^*=1/2$ with energy $E=4-\lambda$. Comparing this with (2.7), one gets:

Proposition 2.5. *If* $0<\lambda\leqq 1$, *then* $\phi(t)=(t, t-1)$. *If* $1<\lambda<4$ *then* $\phi(t)$ *is the model trajectory with parameters given by* (2.6). *Finally, if* $\lambda\geqq 4$, *then* $\phi(t)=(0, 2t-1)$ *for* $t\in[0, 1/2]$ *and* $(2t-1, 0)$ *for* $t\in[1/2, 1]$ (Fig. 2).

Hence, for $\lambda\geqq 4\alpha$, Proposition 2.1 holds when $\alpha>0$ and $\beta=0$.

2. Existence and uniqueness results for the geometric model

In this section we discuss the wellposedness on the mathematical sense of the geometric PDE model (1.5a, b), that is, we prove existence and uniqueness of solutions of (1.5a, b) in the viscosity sense for bounded Lipschitz continuous initial data. First, we rewrite the Eq. (1.5a, b) in the form

$$\text{(3.1 a)} \qquad \frac{\partial u}{\partial t}-g(x)\,a_{ij}(\nabla u)\,\partial_{ij}u-\nu g(x)|\nabla u|=0 \qquad (t,x)\in[0,\infty[\times\mathbb{R}^2$$

$$\text{(3.1 b)} \qquad u(0,x)=u_0(x) \qquad x\in\mathbb{R}^2$$

where

$$a_{ij}(p)=\delta_{ij}-\frac{p_i p_j}{|p|^2} \quad \text{if } p\neq 0, \qquad g(x)\in W^{1,\infty}(\mathbb{R}^2),$$
$$g(x)\geqq 0 \quad \text{with } g(x)^{1/2}\in W^{1,\infty}(\mathbb{R}^2).$$

We use the usual notations $\partial_i u = \frac{\partial u}{\partial x_i}$, $\partial_{ij} u = \frac{\partial^2 u}{\partial x_i \partial x_j}$ and the classical Einstein sommation convention in (3.1 a) and in all what follows.

This equation should be solved in $R=[0,1]^2$ with Neumann boundary conditions. But to simplify the presentation and as is usual in the literature [1] we extend the images by reflection to $\mathbb{R}^2$ and we look for solutions of (3.1 a, b) verifying $u(x+2h)=u(x)$ for all $x\in\mathbb{R}^2$ and all $h\in\mathbb{Z}$. The initial condition $u_0(x)$ and the data $g(x)$ are taken extended to $\mathbb{R}^2$ with the same periodicity as u.

Let us recall the definition of viscosity solutions [17]. Let $u\in C([0,T]\times\mathbb{R}^2)$ for some $T\in\,]0,\infty[$. We say that u is a viscosity subsolution of (3.1 a, b) if for any function $\phi\in C^2(\mathbb{R}\times\mathbb{R}^2)$ and any local maximum $(t_0,x_0)\in\,]0,T]\times\mathbb{R}^2$ of $u-\phi$ we have:

if $\nabla\phi(t_0,x_0)\neq 0$, then

$$(3.2)\qquad \frac{\partial\phi}{\partial t}(t_0,x_0)-g(x_0)\,a_{ij}(\nabla\phi(t_0,x_0))\,\partial_{ij}\phi(t_0,x_0)-v\,g(x)|\nabla\phi(t_0,x_0)|\leqq 0$$

if $\nabla\phi(t_0,x_0)=0$, then

$$(3.3)\qquad \frac{\partial\phi}{\partial t}(t_0,x_0)-g(x_0)\limsup_{p\to 0}\; a_{ij}(p)\,\partial_{ij}\phi(t_0,x_0)\leqq 0$$

and $u(0,x)\leqq u_0(x)$ for all $x\in\mathbb{R}^2$.

In the same way we define the notion of viscosity supersolution changing "local maximum" by local minimum, "$\leqq 0$" by "$\geqq 0$" and "lim sup" by "lim inf" in the expressions above. A viscosity solution is a function which is a viscosity subsolution and a viscosity supersolution.

Theorem 3.1. *Let $u_0, v_0\in C(R)\cap W^{1,\infty}(R)$. Then*
1) The Eq. (3.1 a, b) *admits a unique viscosity solution*

$$u\in C([0,\infty[\times\mathbb{R}^2)\cap L^\infty(0,T;W^{1,\infty}(\mathbb{R}^2)) \quad \textit{for all } T<\infty.$$

Moreover, it satisfies

$$\inf_{\mathbb{R}^2} u_0\leqq u(t,x)\leqq \sup_{\mathbb{R}^2} u_0.$$

2) Let $v\in C([0,\infty[\times\mathbb{R}^2)$ the viscosity solution of (3.1 a) *with initial data v_0. Then for all $T\in[0,\infty[$ we have*

$$\sup_{0\leqq t\leqq T}\|u(t,x)-v(t,x)\|_{L^\infty(\mathbb{R}^2)}\leqq \|u_0(x)-v_0(x)\|_{L^\infty(\mathbb{R}^2)}.$$

($W^{1,\infty}(\mathbb{R}^2)$ denotes the space of bounded Lipschitz functions on $\mathbb{R}^2$).

Proof. The proof is based on the proof of the theorem of Sect. 3 in [1]. First we prove 2). Let (t_0,x_0,y_0) be a maximum point of

$$u(t,x)-v(t,y)-(4\varepsilon)^{-1}|x-y|^4-\lambda t, \qquad t\in[0,T],\ x,y\in\mathbb{R}^2.$$

First we prove that $t_0=0$. For that, suppose that $t_0>0$. We may find for all $\mu>0$ real numbers a, b and two symmetric matrices $2\times 2\, X$, Y such that

$$a-b=\lambda, \quad \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leqq \begin{bmatrix} B+\mu B^2 & -B-\mu B^2 \\ -B-\mu B^2 & B+\mu B^2 \end{bmatrix} \tag{3.4}$$

$$a-g(x_0)\, a_{ij}(\varepsilon^{-1}|x_0-y_0|^2(x_0-y_0))\, X_{ij}-\nu g(x_0)\, \varepsilon^{-1}|x_0-y_0|^3 \leqq 0 \tag{3.5}$$

$$b-g(y_0)\, a_{ij}(\varepsilon^{-1}|x_0-y_0|^2(x_0-y_0))\, Y_{ij}-\nu g(y_0)\, \varepsilon^{-1}|x_0-y_0|^3 \geqq 0 \tag{3.6}$$

where

$$B=\varepsilon^{-1}|x_0-y_0|^2 I_2+2\varepsilon^{-1}(x_0-y_0)\otimes(x_0-y_0).$$

Hence

$$B^2=\varepsilon^{-2}|x_0-y_0|^4 I_2+8\varepsilon^{-2}|x_0-y_0|^2(x_0-y_0)\otimes(x_0-y_0).$$

In fact, (3.5)–(3.6) have to be interpreted if $x_0=y_0$. In this case, $B=0$ so that by (3.4), $X\leqq 0$ and $Y\geqq 0$. We have then

$$a-g(x_0)\limsup_{p\to 0}(a_{ij}(p)\, X_{ij})\leqq 0, \tag{3.5'}$$

$$b-g(y_0)\limsup_{p\to 0}(a_{ij}(p)\, Y_{ij})\geqq 0. \tag{3.6'}$$

Hence, in particular, $a\leqq 0$, $b\geqq 0$ a contradiction with $a-b=\lambda>0$. Therefore $x_0 \neq y_0$ and we may write and use (3.5)–(3.6). We next choose $\mu=\varepsilon|x_0-y_0|^{-2}$ and we deduce

$$\begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leqq 2\varepsilon^{-1} \begin{bmatrix} C & -C \\ -C & C \end{bmatrix} \tag{3.7}$$

where $C=|x_0-y_0|^2 I_2+5(x_0-y_0)\otimes(x_0-y_0)$.

We now set

$$G=\begin{bmatrix} g(x_0)\, A & (g(x_0)\, g(y_0))^{1/2} A \\ (g(x_0)\, g(y_0))^{1/2} A & g(y_0)\, A \end{bmatrix}$$

where $A=(a_{ij}(\varepsilon^{-1}|x_0-y_0|^2(x_0-y_0)))$.

Obviously G is a nonnegative symmetric matrix so that multiplying (3.7) to the left by G and taking the trace we find

$$g(x_0)\, a_{ij} X_{ij}-g(y_0)\, a_{ij} Y_{ij} \leqq 2\varepsilon^{-1}(g(x_0)^{1/2}-g(y_0)^{1/2})^2 \operatorname{trace}(aC).$$

Hence

$$g(x_0)\, a_{ij} X_{ij}-g(y_0)\, a_{ij} Y_{ij} \leqq C_0\, \varepsilon^{-1}(g(x_0)^{1/2}-g(y_0)^{1/2})^2 |x_0-y_0|^2 \tag{3.8}$$

where C_0 is a constant depending only on $a_{ij}(p)$. Next, if we combine (3.4), (3.5), (3.6) and (3.8) we obtain

$$a-b=\lambda \leqq C_0\, \varepsilon^{-1}(g(x_0)^{1/2}-g(y_0)^{1/2})^2|x_0-y_0|^2+B(g(x_0)-g(y_0))\, \varepsilon^{-1}|x_0-y_0|^3.$$

Now, since g and $g^{1/2}$ are Lipschitz functions we obtain

$$\lambda \leqq C_1\, \varepsilon^{-1}|x_0-y_0|^4.$$

Next, we stimate $|x_0 - y_0|$. To this aim, we observe that

$$u(t_0, x_0) - v(t_0, y_0) - (4\varepsilon)^{-1}|x_0 - y_0|^4 - \lambda t_0 \geqq u(t_0, y_0) - v(t_0, y_0) - \lambda t_0$$

and

$$(4\varepsilon)^{-1}|x_0 - y_0|^4 \leqq u(t_0, x_0) - v(t_0, y_0) \leqq L|x_0 - y_0|$$

where L is the Lipschitz constant of $u(t_0, .)$ on $[0, T] \times \mathbb{R}^2$. Therefore

$$\lambda \leqq M\varepsilon^{1/3} L^{4/3}. \tag{3.9}$$

Without loss of generality, we may suppose that $\sup_{[0,T]\times\mathbb{R}^2} |u - v| \neq 0$ (otherwise, the conclusion is immediate) and we choose

$$\varepsilon^{1/3} = \delta \sup_{[0,T]\times\mathbb{R}^2} |u - v|, \tag{3.10}$$

$$\lambda = 2M\delta \sup_{[0,T]\times\mathbb{R}^2} |u - v| \, L^{4/3}. \tag{3.11}$$

This choice contradicts (3.9). This contradiction proves that $t_0 = 0$. Let us now estimate $\sup_{[0,T]\times\mathbb{R}^2} |u - v|$. Since $t_0 = 0$

(3.12)
$$u(t, x) - v(t, y) - (4\varepsilon)^{-1}|x - y|^4 - \lambda t \leqq \sup_{x, y \in \mathbb{R}^2} (u_0(x) - v_0(x) - (4\varepsilon)^{-1}|x - y|^4).$$

In particular, we may choose $x = y$ in (3.12) while the right hand side can be estimated by $\sup_{\mathbb{R}^2} |u_0 - v_0| + \sup_{r \geqq 0}(Lr - r^4/4\varepsilon)$. We finally obtain

$$-\lambda t + \sup_{[0,T]\times\mathbb{R}^2} (u - v) \leqq \sup_{\mathbb{R}^2} |u_0 - v_0| + \sup_{r \geqq 0}(Lr - r^4/4\varepsilon),$$
$$-\lambda t + \sup_{[0,T]\times\mathbb{R}^2} (u - v) \leqq \sup_{\mathbb{R}^2} |u_0 - v_0| + \tfrac{3}{4} L^{4/3} \varepsilon^{1/3}.$$

By (3.10) and (3.11) we have

$$\sup_{[0,T]\times\mathbb{R}^2} (u - v) \leqq \sup_{\mathbb{R}^2} |u_0 - v_0| + \tfrac{3}{4} L^{4/3} \delta \sup_{[0,T]\times\mathbb{R}^2} |u - v| + 2M\delta \sup_{[0,T]\times\mathbb{R}^2} |u - v| \, L^{4/3} T.$$

Exchanging the role of u and v and letting $\delta \to 0$ we obtain

$$\sup_{[0,T]\times\mathbb{R}^2} |u - v| \leqq \sup_{\mathbb{R}^2} |u_0 - v_0|. \tag{3.13}$$

This proves part 2) of the theorem and the uniqueness claim in part 1). We next prove the existence claim in part 1). We begin by remarking that the definition of viscosity solutions immediately implies that if u is a solution, then

$$\inf_{\mathbb{R}^2} u_0 - \delta t \leqq \sup_{\mathbb{R}^2} u_0 + \delta t \quad \text{on } [0, \infty[\times \mathbb{R}^2.$$

Therefore, we have

$$\inf_{\mathbb{R}^2} u_0 \leqq u \leqq \sup_{\mathbb{R}^2} u_0 \quad \text{on } [0, \infty[\times \mathbb{R}^2. \tag{3.14}$$

Indeed, set $\phi(t,x) = \sup_{\mathbb{R}^2} u_0 + \delta t$ in (3.4), $\delta > 0$ and assume that $u - \phi$ has a local maximum at a point (t_0, x_0) with $t_0 > 0$. Then by the definition of subsolution, we get by the second relation (3.3) that $\frac{\partial \phi}{\partial t}(t_0, x_0) \leqq 0$. Thus $\delta \leqq 0$, which yields a contradiction and therefore $u - \phi$ attains its maximum, 0, for $t_0 = 0$.

Let us now prove an a priori estimate on ∇u. This estimate will be formal at that level and will be justified later on. In fact, we consider a smooth solution of

$$\frac{\partial u}{\partial t} - g(x)[a_{ij}(\nabla u)\, u_{ij} + v H_\varepsilon(\nabla u)] = 0 \qquad (t,x) \in [0, \infty[\times \mathbb{R}^2 \tag{3.15}$$

where g, a_{ij} are now supposed smooth and $g(x)$ is bounded away from zero on $\mathbb{R}^2$, $H_\varepsilon(p) = (|p|^2 + \varepsilon)^{1/2}$ and u_{ij} is a notation for $\partial_{ij} u$. We use the classical Bernstein method and derive a parabolic inequality for $|\nabla u|^2$. To this end, we differentiate (3.15) with respect to x_I and we find

$$\begin{aligned} &\frac{\partial u_I}{\partial t} - \frac{\partial g(x)}{\partial I}[a_{ij}(\nabla u)\, u_{ij} + v H_\varepsilon(\nabla u)] \\ &\quad - g(x)\left[\frac{\partial a_{ij}}{\partial k}(\nabla u)\, u_{kI}\, u_{ij} + a_{ij}(\nabla u)\, u_{ijI} + v \frac{\partial H}{\partial k}(\nabla u)\, u_{kI}\right] = 0. \end{aligned} \tag{3.16}$$

Hence, we obtain by multiplying by $2u_I$ and adding on I

$$\begin{aligned} &\frac{\partial |\nabla u|^2}{\partial t} - g(x)\left[a_{ij}(\nabla u)\, \partial_{ij} |\nabla u|^2 \right.\\ &\qquad \left. + v \frac{\partial H_\varepsilon}{\partial k}(\nabla u)\, \partial_k |\nabla u|^2 + \frac{\partial a_{ij}}{\partial k}(\nabla u)\, u_{ij}\, \partial_k |\nabla u|^2\right] \\ &\quad = -2g(x)\, a_{ij}(\nabla u)\, u_{Ii}\, u_{Ij} + 2\frac{\partial g(x)}{\partial I}[a_{ij}(\nabla u)\, u_{ij}\, u_I + v H_\varepsilon(\nabla u)\, u_I]. \end{aligned} \tag{3.17}$$

Since we have

$$\left|\frac{\partial g(x)}{\partial I}\right| \leqq C(g(x))^{1/2} \tag{3.18}$$

$$|a_{ij}(\nabla u)\, u_{ij}| \leqq C(a_{ij}(\nabla u)\, u_{Ii}\, u_{Ii})^{1/2}. \tag{3.19}$$

The last inequality is purely algebraic and only uses the fact that $a_{ij} x_i x_j$ is nonnegative. From (3.18), (3.19) and the Cauchy-Schwartz inequality, we deduce from (3.17) that

$$\begin{aligned} &\frac{\partial |\nabla u|^2}{\partial t} - g(x)\left[a_{ij}(\nabla u)\, \partial_{ij} |\nabla u|^2 \right.\\ &\qquad \left. + v \frac{\partial H_\varepsilon}{\partial k}(\nabla u)\, \partial_k |\nabla u|^2 + \frac{\partial a_{ij}}{\partial k}(\nabla u)\, u_{ij}\, \partial_k |\nabla u|^2\right] \\ &\quad \leqq -2g(x)\, a_{ij}(\nabla u)\, u_{Ii}\, u_{Ij} + g(x)\, a_{ij}(\nabla u)\, u_{Ii}\, u_{Ij} + C|\nabla u|^2 \\ &\quad \leqq C(|\nabla u|^2 + \varepsilon^2). \end{aligned} \tag{3.20}$$

Then, by applying the maximum principle, we deduce easily that

$$\|\nabla u(t,x)\|_{L^\infty(\mathbb{R}^2)} \leqq e^{Ct}(\|\nabla u_0\|_{L^\infty(\mathbb{R}^2)} + \varepsilon^2) \tag{3.21}$$

where the constant C only depends on $\sup_p |a_{ij}(p)|$, on g through $\|g\|_{L^\infty(\mathbb{R}^2)}$ and $\|\nabla(g)^{1/2}\|_{L^\infty(\mathbb{R}^2)}$ and ν.

In order to conclude, we only have to approximate (3.1) by a (slightly) simpler equation of a similar form for which we are able to produce smooth solutions. Then, we will conclude using the above a priori estimates (which will be valid on the approximated solutions). To this end, we consider u_0^ε in $C^\infty(\mathbb{R}^2)$ (periodic) such that $u_0^\varepsilon \to u_0$ uniformly, $\|\nabla u_0^\varepsilon\|_{L^\infty(\mathbb{R}^2)} \leqq \|\nabla u_0\|_{L^\infty(\mathbb{R}^2)}$, $\|u_0^\varepsilon\|_{L^\infty(\mathbb{R}^2)} \leqq \|u_0\|_{L^\infty(\mathbb{R}^2)}$. We also introduce $g_\varepsilon = g + \varepsilon$, $H_\varepsilon(p) = (|p|^2 + \varepsilon)^{1/2}$, $a_{ij}^\varepsilon = \varepsilon\delta_{ij} + \alpha_{ij}^\varepsilon$, where the $\alpha_{ij}^\varepsilon = \delta_{ij} - \frac{p_i p_j}{|p|^2 + \varepsilon}$. Using the general theory of quasilinear parabolic equations [28], one knows that there exists u^ε smooth solution of

$$\frac{\partial u^\varepsilon}{\partial t} - g_\varepsilon(x)[a_{ij}^\varepsilon(\nabla u^\varepsilon) u_{ij}^\varepsilon + \nu H_\varepsilon(\nabla u^\varepsilon)] = 0 \qquad (t,x) \in [0, \infty[\times \mathbb{R}^2 \tag{3.22}$$

such that $u^\varepsilon(0,x) = u_0^\varepsilon(x)$. By (3.21)

$$\begin{aligned} \|\nabla u^\varepsilon(t,.)\|_{L^\infty(\mathbb{R}^2)} &\leqq e^{Ct}(\|\nabla u_0^\varepsilon\|_{L^\infty(\mathbb{R}^2)} + 1) \\ &\leqq e^{Ct}(\|\nabla u_0\|_{L^\infty(\mathbb{R}^2)} + 1) \leqq C_T. \end{aligned} \tag{3.23}$$

This means

$$|u^\varepsilon(t,x) - u^\varepsilon(t,y)| \leqq C_T |x-y| \tag{3.24}$$

for all $x \in \mathbb{R}^2$ and all $t \in [0, T]$. From this follows the inequality

$$|u^\varepsilon(t,x) - u^\varepsilon(s,x)| \leqq C_T |t-s|^{1/2} \qquad \text{for all } x \in \mathbb{R}^2 \text{ and all } s, t \in [0,T] \tag{3.25}$$

from regularity results for solutions of (3.22) [28]. The inequalities (3.24), (3.25) together with (3.14) allow us to conclude by means of Arzela-Ascoli theorem that a subsequence of u^ε converges uniformly on $[0,T] \times [0,1]^2$ to a function

$$u \in C([0,T] \times \mathbb{R}^2) \cap L^\infty(0,T; W^{1,\infty}(\mathbb{R}^2))$$

for all $T < \infty$. Now, one easily proves that u is a viscosity solution of (3.1). □

The next result will be needed in Sect. 4 to study the assymptotic behavior of the model (3.1 a, b).

Theorem 3.2. *Suppose that $\Gamma \equiv \{x \in [0,1]^2 : g(x) = 0\}$ is a simple Lipschitz Jordan curve and suppose that $u_0(x) \in W^{1,\infty}(\mathbb{R}^2)$ is periodic with fundamental domain $[0,1]^2$, vanishing in an open neighborhood of Γ. Let $u(t,x)$ be the viscosity solution of* (3.1 a, b) *given by Theorem 3.1. Then*

$$u(t,x) = 0 \quad \textit{for all } x \in \Gamma \textit{ and all } t \geqq 0. \tag{3.26}$$

Before giving the proof of Theorem 3.2, let us recall the following result (see [18] or [17])

Lemma 3.3. *(i) Let u be a viscosity subsolution of*

$$(3.27)\qquad \frac{\partial u}{\partial t}-g(x)\,a_{ij}(\nabla u)\,\partial_{ij}u-g(x)\,v|\nabla u|=0 \quad \text{on } [0,+\infty[\times\mathbb{R}^2.$$

For each $\varepsilon>0$, let

$$(3.28)\qquad u^\varepsilon(t,x)=\sup\left\{u(s,y)-\frac{1}{\varepsilon}(|x-y|^2+(t-s)^2):(s,y)\in[0,+\infty[\times\mathbb{R}^2\right\}$$

for $(t,x)\in[0,+\infty[\times\mathbb{R}^2$. Then
a) $u\leqq u^\varepsilon$ on $[0,+\infty[\times\mathbb{R}^2$,
b) $u^\varepsilon\to u$ uniformly on compact subsets of $[0,+\infty[\times\mathbb{R}^2$ as $\varepsilon\to 0$,
c) u^ε is semiconvex, hence twice differentiable a.e.,
d) u^ε is also a subsolution of (3.27) on $]\sigma(\varepsilon),+\infty[\times\mathbb{R}^2$
where $\sigma(\varepsilon)=C\varepsilon^{1/2}$ for some constant C independent of ε.

(ii) Let u be a viscosity supersolution of (3.27)
For each $\varepsilon>0$, let

$$(3.29)\qquad u_\varepsilon(t,x)=\inf\left\{u(s,y)+\frac{1}{\varepsilon}(|x-y|^2+(t-s)^2):(s,y)\in[0,+\infty[\times\mathbb{R}^2\right\}$$

for $(t,x)\in[0,+\infty[\times\mathbb{R}^2$. Then
a) $u^\varepsilon\leqq u$ on $[0,+\infty[\times\mathbb{R}^2$
b) $u^\varepsilon\to u$ uniformly on compact subsets of $[0,+\infty[\times\mathbb{R}^2$ as $\varepsilon\to 0$,
c) u^ε is semiconcave, hence twice differentiable a.e.,
d) u^ε is also supersolution of (3.27) on $]\sigma(\varepsilon),+\infty[\times\mathbb{R}^2$.

Proof of Theorem 3.2. Let $g_\eta\in C_p(\mathbb{R}^2)\cap C^2(\mathbb{R}^2)$, $g_\eta\geqq 0$ be such that
i) g_η vanishes on a neighborhood V_η of Γ,
ii) $(g_\eta)^{1/2}\in W^{1,\infty}(\mathbb{R}^2)$ with Lipschitz constant L independent of η,
iii) g_η converges to g uniformly on compact sets as $\eta\to 0$.

Since g_η converges to g uniformly on compact sets as $\eta\to 0$, we know that the viscosity solution u_η of

$$(3.30\text{a})\qquad \frac{\partial v}{\partial t}-g_\eta(x)\,a_{ij}(\nabla v)\,\partial_{ij}v-v\,g_\eta(x)|\nabla v|=0 \qquad (t,x)\in[0,\infty[\times\mathbb{R}^2.$$

$$(3.30\text{b})\qquad v(0,x)=u_0(x) \qquad x\in\mathbb{R}^2$$

converges to u uniformly on compact subsets of $[0,\infty[\times\mathbb{R}^2$ as $\eta\to 0$ (see [17]). Hence, it is sufficient to prove that $u_\eta(t,x)=u_0(x)$ for all $x\in\Gamma$ all $t\geqq 0$. For each $\varepsilon>0$, let u_η^ε be given by (3.28). Since u_η^ε is a subsolution of (3.30a) on

$]\sigma(\varepsilon), +\infty[\times \mathbb{R}^2$ which is twice differentiable a.e. on $[0, \infty[\times \mathbb{R}^2$ (Lemma 3.3(i)) it follows that u_η^ε is a classical subsolution a.e. of (3.30a). Hence

$$\frac{\partial u_\eta^\varepsilon}{\partial t}(t, x) \leqq 0 \qquad \text{a.e. on }]\sigma(\varepsilon), +\infty[\times V_\eta. \tag{3.31}$$

Hence

$$u_\eta^\varepsilon(t, x) \leqq u_\eta^\varepsilon(\sigma(\varepsilon), x) \qquad \text{a.e. on }]\sigma(\varepsilon), +\infty[\times V_\eta. \tag{3.32}$$

Since $u_\eta^\varepsilon(t, x)$ is a continuous functions, (3.32) holds for all $(t, x) \in]\sigma(\varepsilon), +\infty[\times V_\eta$. Then letting $\varepsilon \to 0$, we get that $u_\eta(t, x) \leqq u_0(x)$ for all $(t, x) \in]0, +\infty[\times V_\eta$. Now, letting $\eta \to 0$, we finally get

$$u(t, x) \leqq u_0(x) \qquad \text{for all } x \in \Gamma \text{ and all } t \in [0, +\infty[.$$

The other inequality follows in the same way. Now, (3.26) follows because $u_0(x) = 0$ for all $x \in \Gamma$. □

4. A proof of correctness of the geometric model

To study the correctness of the model (1.5a, b) we shall prove that the zero level set of the function u asymptotically fits the desired contour we are looking for, provided that we suppose it to be smooth (C^2, in fact).

To study the asymptotic behavior of the equation

$$\frac{\partial u}{\partial t} = g(x)|\nabla u| \left(\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + v\right) \qquad (t, x) \in [0, \infty[\times \mathbb{R}^2, \tag{4.1a}$$

$$u(0, x) = u_0(x) \qquad x \in \mathbb{R}^2. \tag{4.1b}$$

We suppose that $\Gamma \equiv \{x \in [0, 1]^2 : g(x) = 0\}$ is a simple Jordan curve of class C^2. As it is well known Γ divides $[0, 1]^2$ into two connected components: the region inside the curve Γ and the exterior region. Denote them respectively by $I(\Gamma)$ and $E(\Gamma)$. The initial datum $u_0(x)$ is taken in $C^2(\mathbb{R}^2)$ periodic with fundamental domain $[0, 1]^2$ and vanishing in an open neighborhood of $\Gamma \cup I(\Gamma)$. Moreover, we take $u_0(x)$ such that its level curves have uniformly bounded curvatures. Let $u(t, x)$ be the unique viscosity solution of (4.1a, b) given by Theorem 3.1 above. We follow the evolution of the set $G(t) = \{x \in [0, 1]^2 : u(t, x) = 0\}$ whose boundary $\Gamma(t)$ we are interested in. In fact we are going to prove that if the constant v in (4.1a) is taken sufficiently large then $\Gamma(t)$ converges to Γ as $t \to \infty$. In fact

Theorem 4.1. *If the constant v is sufficiently large, then $\Gamma(t)$ converges to Γ in the Hausdorff distance as $t \to \infty$.*

Essentially, the proof consists in constructing a subsolution (or a family of subsolutions) of u which become strictly positive as $t \to \infty$. If we suppose Γ to be a convex C^2 curve this is not a difficult task. In this case the distance function to Γ, $d(x) = d(x, \Gamma)$, $x \in \mathbb{R}^2$ is of class C^2 in $E(\Gamma)$. The function $d(x)$ is the main tool to construct the desired subsolution. Since we are only interested in what happens inside $[0,1]^2$, we look for some C^1 functions $f(t)$, $h(t)$ defined for $t \geqq 0$ such that

$$w(t,x)=f(t)\,d(x)+h(t) \qquad (t,x)\in[0,\infty[\times[0,1]^2 \tag{4.2}$$

is a subsolution of (4.1 a, b) in $[0,\infty[\times E(\Gamma)$. Hence we look for functions $f(t)$, $h(t)$ such that

$$\frac{\partial w}{\partial t}-g(x)|\nabla w|\left(\operatorname{div}\left(\frac{\nabla w)}{|\nabla w|}+v\right)\leqq 0 \quad \text{on } [0,+\infty[\times E(\Gamma) \tag{4.3}$$

and

$$w(0,x)\leqq u_0(x) \qquad \text{on } E(\Gamma), \tag{4.4}$$

$$w(t,x)\leqq u(t,x) \qquad \text{on } [0,+\infty[\times\partial E(\Gamma). \tag{4.5}$$

Moreover, we want $w(t,x)$ to be such that

$$w(t,x) \text{ becomes asymptotically positive as } t\to\infty \quad \text{for } x\in E(\Gamma). \tag{4.6}$$

Since $|\nabla d|=1$ on $E(\Gamma)$ [30, 18], (4.3) may be written as

$$f'(t)\,d(x)+h'(t)-g(x)\,f(t)(\Delta d+v)\leqq 0 \quad \text{on } [0,+\infty[\times E(\Gamma). \tag{4.7}$$

If we take $h(t)=0$, to have (4.4), we need to take $f(0)=0$. Then, it is difficult to realize (4.6) for t near 0. Hence it is simpler to choose a function $w(t,x)$ as in (4.2) with $h(t)$ verifying: $h(0)=0$, $h'(t)\leqq 0$ (the inequality being strict in some interval $[0,t_0]$ near zero), $h(t)$ becoming asymptotically constant as $t\to\infty$ and then adjust $f(t)$ so that (4.4), (4.5), (4.6), (4.7) are satisfied. In fact, we choose $f(t)$, $h(t)$ to be of the form

$$f(t)=\lambda\left(1-\frac{1}{(1+t)^m}\right) \tag{4.8}$$

with λ, $m>0$ and

$$g_m(t)=g_m t \quad \text{for } t\in[0,t_m]; \qquad g_m t_m \quad \text{for } t>t_m. \tag{4.9}$$

Then, for any $\eta>0$, one chooses λ, m, g_m and t_m such that $w(t,x)$ verifies (4.4), (4.5), (4.6) and

$$\begin{gathered} w(t,x) \text{ becomes asymptotically positive as } t\to\infty \\ \text{for all } x\in E(\Gamma) \quad \text{with} \quad d(x,\Gamma)>\eta. \end{gathered} \tag{4.10}$$

To be able to satisfy (4.5), we suppose that v has been chosen such that

$$\operatorname{div}\left(\frac{\nabla u_0}{|\nabla u_0|}\right)+v\geqq 0 \tag{4.11}$$

which is possible by our assumptions on $u_0(x)$. This implies that $u_0(x)$ is a subsolution of (4.1 a, b). Then, using a comparison principle [23], Theorem 4.2 (or see Theorem 4.4 below), we have

$$u(t,x) \geqq u_0(x) \quad \text{for all } (t,x) \in [0,\infty[\times \mathbb{R}^2. \tag{4.12}$$

From this it follows that

$$\inf\{u(t,x): t\in[0,\infty[, x\in\partial[0,1]^2\} \geqq \inf\{u_0(x): x\in\partial[0,1]^2\} > 0. \tag{4.13}$$

Now, choosing λ small enough, (4.5) may be satisfied. Since $w(t,x)$ and $u(t,x)$ are, respectively, a subsolution and a solution of (4.3), (4.4), (4.5), from the comparison principle given in [23], Theorem 4.2 (see Theorem 4.4 below) it follows that

$$w(t,x) \leqq u(t,x) \quad \text{for all } (t,x) \in [0,\infty[\times E(\Gamma). \tag{4.14}$$

Summarizing the previous discussion, we get that: for any $\eta > 0$ there exists some $T_\eta > 0$ such that

$$\{x\in[0,1]^2 : u(t,x)=0\} \subset \{x\in[0,1]^2 : d(x, I(\Gamma)) < 2\eta\} \tag{4.15}$$

for all $t \geqq T_\eta$. If we prove that $I(\Gamma) \subset \{x\in[0,1]^2 : u(t,x)=0\}$ for all $t>0$, it will follow:

$$\{x\in[0,1]^2 : u(t,x)=0\} \to \overline{I(\Gamma)} \quad \text{in the Hausdorff distance as } t\to+\infty. \tag{4.16}$$

The proof of Theorem 4.1 in the general case follows the same lines as above but it is a bit more technical because, in general $I(\Gamma)$ need not be convex. In the general case, we need a geometrical construction. The proof of it is given at the end of this section. For any $\eta > 0$, we can construct a family of curves $\Gamma_1, \ldots, \Gamma_n$ with $n = n(\eta)$, satisfying:

(i) Each Γ_i is a C^2 simple Jordan curve with interior region $I(\Gamma_i)$ and exterior region $E(\Gamma_i)$, $i = 1, \ldots, n$. All of them are contained in $[0,1]^2$ and $\Gamma_i \subset I(\Gamma_{i-1})$, $i = 2, \ldots, n$.
(ii) $\Gamma_1 \subset \{x\in[0,1]^2 : u(0,x) > 0\}$ and $\Gamma_n \subset \{x \in E(\Gamma): 0 < d(x,\Gamma) < \eta\}$.
(iii) For each $x \in \Gamma_i$, let $k_i(x)$ be the curvature of Γ_i at the point x. Let $K_i = \max\{|k_i(x)| : x \in \Gamma_i\}$. We suppose that

$$\Gamma_i \subset \left\{x \in E(\Gamma_{i+1}): d(x, \Gamma_{i+1}) < \frac{1}{2K_{i+1}}\right\}, \qquad i = 1, \ldots, n-1,$$

and such that
(iv) $\sup\{K_i : i = 1, \ldots, n\} \leqq M$ for a constant M independent of η.

Set $\Gamma_1^* = \Gamma_1$. For each $i = 2, \ldots, n-1$, let Γ_i^* be a simple Jordan curve of class C^2 contained in $\left\{x \in E(\Gamma_i) \cap I(\Gamma_{i-1}) : d(x,\Gamma_i) < \dfrac{1}{4K_{i+1}}\right\}$. Finally let Γ_n^* be a simple Jordan curve of class C^2 contained in $\{x \in E(\Gamma_n) \cap I(\Gamma_{n-1}): d(x,\Gamma) < 2\eta\}$. Obvious-

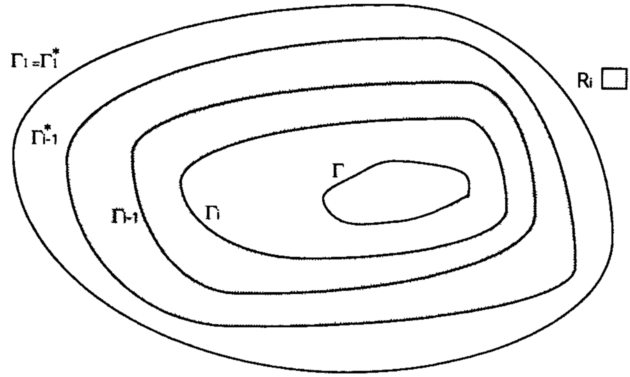

Fig. 3. Construction of the regions R_i

ly each curve Γ_i^* is in a neighborhood of Γ_{i+1} of radius $\frac{3}{4K_{i+1}}$, $i=1, \ldots, n-1$. Let R_i be the region between the curves Γ_{i-1}^* and Γ_i, $i=2, \ldots, n$ (Fig. 3).

Let $d_i(x)=d(x, \Gamma_i)$ and let $C_i>0$ be constants such that $d_i(x) \leqq C_i\, g(x)$ in R_i, $i=2, \ldots, n$. Finally, by (iv) and our assumptions on u_0, we may choose the constant v in (4.1 a) such that for some $\delta>0$

$$\Delta d_i + v \geqq \delta > 0 \quad \text{in } R_i \quad i=2, \ldots, n, \tag{4.17}$$

$$\operatorname{div}\left(\frac{\nabla u_0}{|\nabla u_0|}\right) + v \geqq 0. \tag{4.18}$$

Hence, the constant v may be taken independent of the geometrical construction above. Since $|\nabla d_i|=1$, the term $\operatorname{div}\left(\frac{\nabla d_i}{|\nabla d_i|}\right)$ coincides with Δd_i and (4.17) means that v is a lower bound for the curvatures of the level set curves of d_i.

In each region R_i, $i=2, \ldots, n$, we consider the problem:

$$\frac{\partial z}{\partial t} = g(x)|\nabla z|\left(\operatorname{div}\left(\frac{\nabla z}{|\nabla z|}\right)+v\right) \quad (t, x) \in [T_{i-1}, \infty[\times R_i \tag{4.19a}$$

$$z(t, x) = u(t, x) \quad (t, x) \in [T_{i-1}, \infty[\times \Gamma_{i-1}^* \tag{4.19b}$$

$$z(t, x) = 0 \quad (t, x) \in [T_{i-1}, \infty[\times \Gamma_i \tag{4.19c}$$

$$z(T_{i-1}, x) = 0 \quad x \in R_i \tag{4.19d}$$

where T_i will be specified later. Let us construct for each $i=1, \ldots, n$ a subsolution of (4.19a, b, c, d) which becomes positive as $t \to \infty$. We will use the following comparison principle which follows from Theorem 4.2 in [23]

Theorem 4.4 [23]. *Let w, $v \in C([0, +\infty[, C(R_i))$ be respectively a bounded sub and supersolution of* (4.19a, b, c, d). *Then $w(t, x) \leqq v(t, x)$ for all* $(t, x) \in [0, +\infty[\times R_i$. *The same comparison principle holds for* (4.1 a, b).

Observe that by (4.18) $w(t,x)=u_0(x)$ is a subsolution of (4.1 a, b). The previous result implies that $u(t,x)\geqq u_0(x)$ for all $(t,x)\in[0,\infty[\times\mathbb{R}^2$. In particular

$$\inf\{u(t,x): t\in[0,\infty[, x\in[0,1]^2\cap E(\Gamma_1^*)\}\geqq\inf\{u_0(x): x\in\Gamma_1\}>0. \tag{4.20}$$

Suppose that we have already shown that for all $j<i$ there exists some T_j such that

$$\inf\{u(t,x): t\in[T_j,\infty[, x\in[0,1]^2\cap E(\Gamma_j^*)\}\geqq\beta>0. \tag{4.21}$$

Notice that, by (4.20), this is in fact true in the case $i=2$ with $T_1=0$. Now let us construct a subsolution of (4.19 a, b, c, d) in $[T_{i-1},\infty[\times R_i, i=2,\ldots,n$. By changing variables we may take $T_{i-1}=0$. Let

$$w_m(t,x)=f_m(t)\,d_i(x)+g_m(t)$$

where $(t,x)\in[0,\infty[\times R_i, m>0$,

$$f_m(t)=\lambda\left(1-\frac{1}{(1+t)^m}\right),\qquad \lambda>0 \tag{4.23}$$

$$g_m(t)=g_m t\quad\text{for } t\in[0,t_m];\qquad g_m t_m\quad\text{for } t>t_m \tag{4.24}$$

where

$$g_m=-2m\lambda\quad\text{and}\quad t_m=\left(1+\frac{mC_i}{\delta}\right)^{1/m}-1.$$

The function w_m has been defined so that it is easy to check that

$$\frac{\partial w_m}{\partial t}-g(x)|\nabla w_m|\left(\operatorname{div}\left(\frac{\nabla w_m}{|\nabla w_m|}\right)+v\right)\leqq 0\quad\text{in } [0,+\infty[\times R_i. \tag{4.25}$$

We have checked that

Lemma 4.5. *For λ small enough and all $m>0$, w_m is a subsolution of* (4.19 a, b, c, d).

Proof. Observe that

$$\begin{aligned} w_m(t,x)&\leqq 0 & (t,x)&\in[T_{i-1},\infty[\times\Gamma_i\\ w_m(T_{i-1},x)&\leqq 0 & x&\in R_i \end{aligned}$$

hold by construction of w_m. Using (4.21),

$$w_m(t,x)\leqq u(t,x)\qquad (t,x)\in[T_{i-1},\infty[\times\Gamma_{i-1}^*$$

if we take $\lambda>0$ sufficiently small. Finally, (4.25) completes the proof of the statement. □

Remark. A minor detail in the last lemma is that $g_m(t)$ is not differentiable at the point $t=t_m$. A slight modification of $g_m(t)$ around this point permits us to regularize this function so that $w_m(t,x)$ is still a subsolution of (4.19 a, b, c, d).

Lemma 4.6. *$u(t,x) \geqq w_m(t,x)$ for all $(t,x) \in [T_{i-1}, \infty[\times R_i$. Hence there exist $m_0 > 0$ and $T_i > 0$ such that*

$$u(t,x) \geqq \inf\{w_m(t,x) : t \in [T_i, \infty[, x \in R_i \cap E(\Gamma_i^*\} > 0 \tag{4.26}$$

for all $t \in [T_i, \infty[, x \in R_i \cap E(\Gamma_i^*\}$ and all $m < m_0$.

Proof. The first inequality follows from Lemma 4.5 and Theorem 4.4. On the other hand, observe that

$$w_m(t,x) \to \lambda d_i(x) + g_m t_m \quad \text{as } t \to \infty. \tag{4.27}$$

Since t_m is bounded and $g_m \to 0$ as $m \to 0$, there exists some $m_0 > 0$ such that

$$\inf\{\lambda d_i(x) + g_m t_m : x \in R_i \cap E(\Gamma_i^*)\} > 0 \tag{4.28}$$

for all $0 < m < m_0$. The lemma follows from (4.27 and (4.28). $\square$

Collecting the results above, we have proved:

Lemma 4.7. *For any $\eta > 0$ there exists some $T_\eta > 0$ such that*

$$\{x \in [0,1]^2 : u(t,x) = 0\} \subset \{x \in [0,1]^2 : d(x, I(\Gamma)) < 2\eta\} \tag{4.29}$$

for all $t \geqq T_\eta$.

Let us finally prove that

Lemma 4.8. *$I(\Gamma) \subset \{x \in [0,1]^2 : u(t,x) = 0\}$ for all $t > 0$.*

Proof. In Theorem 3.2 of the previous section we proved that $u(t,x) = 0$ for all $x \in \Gamma$ and all $t > 0$. Consider the problem

$$\frac{\partial z}{\partial t} = g(x)|\nabla z|\left(\operatorname{div}\left(\frac{\nabla z}{|\nabla z|}\right) + v\right) \quad (t,x) \in [0, \infty[\times I(\Gamma) \tag{4.30a}$$

$$z(t,x) = 0 \quad (t,x) \in [0, \infty[\times \Gamma \tag{4.30b}$$

$$z(0,x) = 0 \quad x \in I(\Gamma). \tag{4.30c}$$

We know that $z(t,x) = 0$ and $z(t,x) = u(t,x)$ are two viscosity solutions of (4.30a, b, c). Since there is uniqueness of viscosity solutions of this problem, it follows that $u(t,x) = 0$ for all $(t,x) \in [0, \infty[\times I(\Gamma)$. The lemma is proved. $\square$

The last two lemmas together say that

$$\{x \in [0,1]^2 : u(t,x) = 0\} \to \overline{I(\Gamma)} \quad \text{in the Hausdorff distance as } t \to +\infty. \tag{4.31}$$

Theorem 4.1 follows immediately from this. To finish the proof, let us justify the geometrical construction we made above.

Construction of the family of curves with properties (i)–(iv). Let Γ_1 be a simple Jordan curve of class C^2 contained in $\{x \in [0,1]^2 : u(0,x) > 0\}$. Consider the set $\boldsymbol{E} = \{x \in [0,1]^2 \colon x \in I(\Gamma_1) \cap E(\Gamma)\}$. It is homeomorphic to the closed annulus $\boldsymbol{R}$

$=\{x\in\mathbb{R}^2 : r_1 \leqq |x| \leqq 1\}$. Moreover, since Γ_1 and Γ are of class C^2, it is even diffeomorphic to A with a diffeomorphism

$$\phi\colon A \to E \tag{4.32}$$

of class C^2 (hence, mapping the interior of **A** in the interior of **E** and the boundary onto the boundary) [25]. Consider the family γ_r of curves

$$\gamma_r(\theta) = r(\cos(\theta), \sin(\theta)) \qquad r_1 \leqq r \leqq 1,\ 0 \leqq \theta < 2\pi. \tag{4.33}$$

They are mapped by ϕ into a family of curves $\Gamma(r)$ in E of class C^2, i.e. let $\Gamma(r) = \phi \circ \gamma_r$. Without loss of generality we may suppose that $\Gamma(r_1) = \Gamma_1$ and $\Gamma(1) = \Gamma$. Since the family of curves γ_r with $r_1 \leqq r \leqq 1$ have uniformly bounded curvatures and $\phi'(z) \neq 0$ for all $z \in A$ it follows that the family of curves $\Gamma(r)$, $r_1 \leqq r \leqq 1$, have uniformly bounded curvatures. By taking $r_1 < r_2 < \ldots < r_n$ sufficiently close to each other and r_n sufficiently close to 1, one can see that the family of curves $\Gamma(r_1), \Gamma(r_2), \ldots, \Gamma(r_n)$ satisfies the properties (i), (ii), (iii), (iv) above. This completes the proof of Theorem 4.1.

5. Numerical scheme and experimental results

In order to discretize the degenerate diffusion operator $|\nabla u| \left(\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + \nu\right)$, the numerical scheme is based on the one used by Alvarez-Lions-Morel in their work [1]. Thus, we only mention the general idea that was explained in more detail in their paper. First of all, the image is sampled on a grid. Of course, the discretizations of the differential operators at a point (i,j) of the grid, for obvious fastness and simplicity reasons must involve a few points around it. Typically, one would consider four other points for the discretization of the Laplacian, namely $(i\pm 1, j)$ and $(i, j\pm 1)$. Now, in our case, the differential operator can hardly be represented by two directions. Denote by $\xi = -x \sin\eta + y\cos\eta$, where $(\cos\eta, \sin\eta) = \frac{\nabla u}{|\nabla u|}$, the coordinate in the diffusion direction (which is orthogonal to the gradient), the anisotropic term of the equation is therefore $\partial^2 u/\partial \xi^2$ and can easily be discretized only if in the direction $(-\sin\eta, \cos\eta)$ one can find points of the grid near (i,j). Anyway, we are led to a new formulation of the equation, which will take into account the discrete number of diffusion directions.

Let $0 \leqq \eta_1 < \eta_2 < \ldots < \eta_n < \pi$ be n angles and $x_1, \ldots, x_n$ the coordinates defined by $x_j = -x\sin\eta_j + y\cos\eta_j$. In other terms, x_j is the coordinate orthogonal to the direction given by the angle η_j. We shall decompose the variable diffusion operator

$$\frac{\partial^2 u}{\partial \xi^2} = (\sin^2\eta)\frac{\partial^2 u}{\partial x^2} - 2(\sin\eta\cos\eta)\frac{\partial^2 u}{\partial x\,\partial y} + (\cos^2\eta)\frac{\partial^2 u}{\partial y^2}$$

into a linear nonnegative combination of the fixed directional diffusion operators $\frac{\partial^2 u}{\partial x_j^2}$.

Consider the operator

$$A u=\sum_{j=1}^{n} f_j\left(\frac{\nabla u}{|\nabla u|}\right)\frac{\partial^2 u}{\partial x_j^2}$$

where the $f_j \geqq 0$ are designed to be "active" only if $\frac{\nabla u}{|\nabla u|}$ is close to η_j. Then in order to discretize the degenerate diffusion operator $|\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$ we use the approximated operator defined before. The functions f_j define a "partition of unity". If the directions are given by $\eta_j=(n-1)\pi/2N$, $j=1, \ldots, 2N$, then define an even smooth function f with support in $[-\pi/2N, \pi/2N]$ verifying $f(\pi/2N-\eta)+f(\eta)=1$. Finally, the functions f_j are defined by $f_j(\eta)=f(\eta-\eta_j)$, where for simplicity the boundary points of the interval $[0, \pi]$ are identified.

After this approximation, the algorithm follows and we obtain with this discretization a linear system in u^{k+1} which can be solved by any iterative method and for which $\sup|u^{k+1}| \leqq \sup|u^k|$ (see [1] for more details).

Experimental results

They have been made on WorkStation SUN IPC, with an image processing environment called MEGAWAVE, whose author is Jacques Froment.

The parameters a priori necessary for the method are:

1) The value of the parameter v, which represents the weight of the constant force in the direction of the normal to the level sets

2) The choice of the level set which we want to follow through the motion. Depending on this choice, we find the edge of an object in different iterations and as a consequence of that, in different times

3) The stopping time of the evolution.

Let us explain how all of these parameters can be automatically estimated in the method. For engineering devices, we assume that the operator of the method is adviced to define a suitable initial contour which should be not much larger (say twice) than the apparent contour. We claim that the knowledge of the initial contour is enough in practice to estimate v. The mean curvature of the initial polygon, roughly speaking, its length divided by its area gives a lower bound for v. In most cases v can be fixed to be a small multiple of this rough estimate. Finally, the stopping time is automatic because we naturally impose a control coefficient constructed in the following form: If γ_t is the curve which represents the evolution of the initial curve given by the operator, we compute at each iteration the function

$$E(\gamma_t)=\frac{1}{L(\gamma_t)} \int_{\gamma_t} |\nabla G_\sigma * g_0(x(s), y(s))|\, ds$$

where $L(\gamma_t)$ is the length of γ_t and we wait until $E(\gamma_t)$ decreases below a certain level, say $1/2E(\gamma_0)$. Then, we take as an optimal stopping time the time at which the function $E(\gamma_t)$ attains its maximum. This represents the time of best fitting between the evolving curve γ_t and the searched contour.

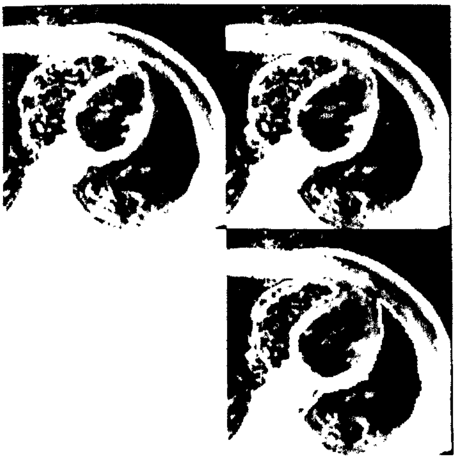

Fig. 4. "Le cœur" initial contour and iterations 2, 4

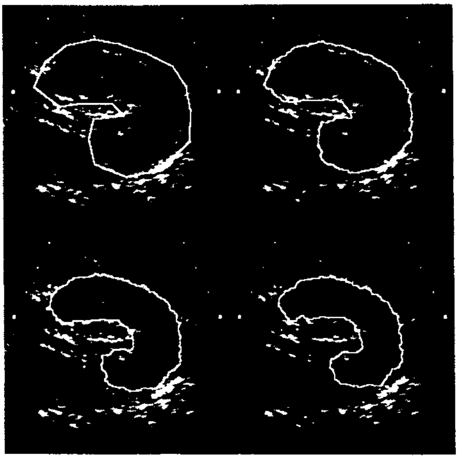

Fig. 5. "L'echographie", initial contour and iterations 2, 6, 8

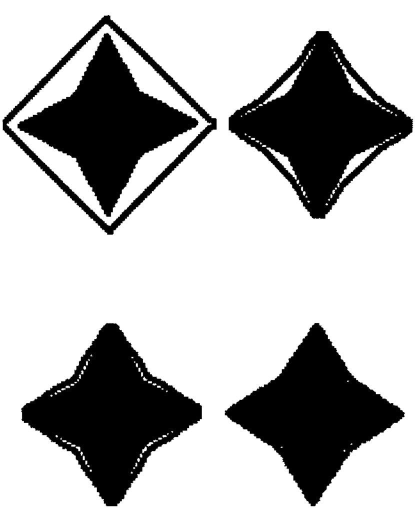

Fig. 6. "Les etoiles", initial contour and iteration 3, 6, 10

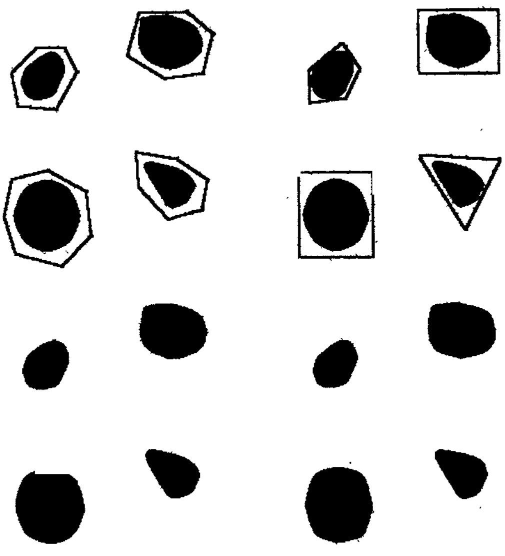

Fig. 7. "Les pommes", initial contour and iteration 13 for both of them

Some comments on the images

1. "Le cœur" (Fig. 4) and "l'echographie" (Fig. 5) are medical images. "Le cœur" shows a multisnakes experience. One can see from left to right and from top to bottom the initial contour and the results of successive iterations.

2. "Les etoiles" (Fig. 6) shows an experience of detection of the contour of a convex object. This is only possible with $\nu > 0$.

3. "Les pommes" (Fig. 7) shows an experience of simultaneous detection of several contours. The results of two different initializations are given.

The computational time on a WorkStation SUN IPC is of the order of 6 seconds by iteration for a 256×256 image.

6. Conclusion

In this paper we have examined two methods for finding automatically contours on a picture. We have proved that the first one, which is parametrized, could retrieve edges with angles but this has not yet been done with stable methods, remaining as an open problem. Moreover, the snake method depends on many adjustment parameters. Therefore, it cannot become easily an engineering tool. Then we have proposed a new method which is geometrically based which cannot retrieve angles any more but has the following advantages: first, it is stable (satisfies the maximum principle); second, it can retrieve simultaneously several contours and, as a consequence of the stability, it can be engineed as a zero parameter method in applications.

7 Appendix

In this appendix we complete the proof of Proposition 2.1 for the cases $\alpha=0$, $\beta>0$ and $\alpha>0$, $\beta>0$. We follow exactly the notations and continue the numerotation of Sect. 2.

Proof for $\alpha=0$, $\beta>0$. Without loss of generality we may suppose as above that $\beta=1$.

Step 1'. The existence of a global minimum $\phi \in Ad(0,1)$ for $E(.;0,1,\lambda)$ follows the same argument as in step 1 above.

We use the same notation $\phi(t)=(x_0(t), y_0(t))$, R_0, R_i, A, B, Z_-, Z_+ with the same meaning as in above. Again, following standard methods [19] we have:

Lemma 2.2'. $\phi^{iv}=0$ *in* A, B *and in the open intervals of* Z_- *and* Z_+.

This lemma means that once ϕ enters R_0 or R_i it moves in a curve whose coordinates are cubics. Similarly, if for each $t \in [t_0, t_1]$, $\phi(t)$ is in Z_- (resp. in Z_+), then $\phi(t)$ is a parametrization by a cubic function of t of the line segment joining $\phi(t_0)$ to $x(t_1)$.

Step 2'. The trajectory ϕ does not oscillate and never travels back.

For each $v(t)=(x(t), y(t)) \in H^2([0,1], \mathbb{R}^2)$, $t_0<t_1$ in $[0,1]$ let us define

$$E(v;t_0,t_1)=\int_{t_0}^{t_1} |x''(t)|^2+|y''(t)|^2\,dt-\lambda\int_{t_0}^{t_1} |\nabla I(v(t))|\,dt. \tag{2.9}$$

The role of Lemmas 2.3 and 2.4 is played by the following lemmas:

Lemma 2.6. *Let* $v \in Ad(0,1)$, $v(t)=(x(t), y(t))$. *Let* $0<t_0<1$. *If* $y(t_0)=a$, $y'(t_0)=0$ *with* $a>0$ *or* $x(t_0)=a$, $x'(t_0)=0$ *with* $a<0$ *then* v *cannot be a global minimum of* $E(.;0,1,\lambda)$.

Lemma 2.7. *If for some* $0<t_0<1$ $x_0(t_0)=0$, *then* $x_0'(t_0)=0$. *Similarly for* y_0.

Proof of Lemma 2.6. Let us give the proof in the first case. Let $v^*(t)=(x(t), y^*(t))$ with $y^*(t)=\mu y(t)+\gamma$ on $[0,t_0]$, $y^*(t)=0$ on $[t_0,1]$ with μ, γ chosen such that $y^*(0)=-1$, $y^*(t_0)=0$. In fact, $\mu=\frac{1}{1+a}$, $\gamma=-\frac{a}{1+a}$. Notice that $v^* \in Ad(0,1)$ and $E(v^*;0,1,\lambda)<E(v;0,1,\lambda)$. □

Proof of Lemma 2.7. By Lemma 2.6 the trajectory ϕ cannot enter into R_i. Hence $x_0(t) \geq 0$ in a neighborhood of t_0. Let $h_n \to 0+$. Then

$$0 \leq \lim_{n\to\infty} \frac{x_0(t_0+h_n)}{h_n} = x_0'(t_0) = \lim_{n\to\infty} \frac{x_0(t_0-h_n)}{-h_n} \leq 0. \quad \square$$

Combining these lemmas we see that ϕ belongs to the set of model admissible trajectories M for which there exist $0 \leq t_0 \leq t_1 \leq 1$, $-1 \leq q \leq 0$, $q' \in \mathbb{R}$ and $0 \leq p \leq 1$, $p' \in \mathbb{R}$ such that

in $[0, t_0]$ $\phi(t) = (0, y_0(t))$ where $y_0(t)$ is a cubic with $y_0(0) = -1$, $y_0'(0) = 0$, $y_0(t_0) = q$, $y_0'(t_0) = q'$,

in $[t_0, t_1]$ $\phi(t) = (x_0(t), y_0(t))$ where $x_0(t)$ is a cubic with $x_0(t_0) = 0$, $x_0'(t_0) = 0$, $x_0(t_1) = p$, $x_0'(t_1) = p'$ and $y_0(t)$ is a cubic with $y_0(t_0) = q$, $y_0'(t_0) = q'$, $y_0(t_1) = 0$, $y_0'(t_1) = 0$,

in $[t_1, 1]$ $\phi(t) = (x_0(t), 0)$ where $x_0(t)$ is a cubic with $x_0(t_1) = p$, $x_0'(t_1) = p'$, $x_0(1) = 1$, $x_0'(1) = 0$.

Let $(t_0^*, t_1^*, q^*, q^{*\prime}, p^*, p^{*\prime})$ be the parameters corresponding to ϕ.

Step 3'. The final step.

Before starting the discussion, let us prove the lemma:

Lemma 2.8. *For each $\lambda > 0$ denote by ϕ_λ a solution minimizing the energy $E(.; 0, 1, \lambda)$. Then $\|\phi_\lambda\|_{H^2}$ is uniformly bounded by a constant independent of λ. Moreover, $t_{1\lambda} - t_{0\lambda} \to 0$, $t_{1\lambda}$ are bounded away from 0 and $t_{1\lambda}$ are bounded away from 1 as $\lambda \to \infty$.*

Proof. Let φ be a parametrization of the corner in $Ad(0, 1)$. The first assertion of the lemma follows from the inequality $E(\phi; 0, 1, \lambda) < E(\varphi; 0, 1, \lambda)$. From this inequality it follows also that

$$\lambda(t_{1\lambda} - t_{0\lambda}) \leq \int_{t_0}^{t_1} |\varphi''(t)|^2 \, dt.$$

Hence, $t_{1\lambda} - t_{0\lambda} \to 0$ as $\lambda \to \infty$. Since $\|\phi_\lambda\|_{H^2}$ is uniformly bounded by a constant independent of λ,

$$|\phi(t_{1\lambda}) - \phi(t_{0\lambda})| \leq k|t_{1\lambda} - t_{0\lambda}| \tag{2.10}$$

where k is independent of λ. If $t_{0\lambda} \to 0$ and $\phi(t_{0\lambda})$ is bounded away from $(0, -1)$ as $\lambda \to \infty$ we have a trajectory that travels a positive distance in zero time, a contradiction with the first assertion of the lemma. If $\phi(t_{0\lambda}) \to (0, -1)$ as $\lambda \to \infty$, we get a contradiction using inequality (2.10). Therefore, $t_{0\lambda}$ is bounded away from 0 as $\lambda \to \infty$. In a similar way we prove that $t_{1\lambda}$ is bounded away from 1 as $\lambda \to \infty$. $\square$

We have reduced the number of parameters describing the minimum ϕ to six. Observe, that, by Lemma 2.8, q' and p' are a priori bounded by a constant M independent of λ. Let $\tau = t_1 - t_0$. Let us call by $\boldsymbol{P}$ the set of parameters

$$\boldsymbol{P} = \{(t_0, \tau, q, q', p, p') \colon 0 \leq t_0 \leq t_0 + \tau \leq 1,\ -1 \leq q \leq 0,\ -M-1 \leq q' \leq M+1,$$
$$0 \leq p \leq 1,\ -M-1 \leq p' \leq M+1\}.$$

Notice that, for λ large enough, $q^{*\prime}$ and $p^{*\prime}$ are in the interior of its range of variation (Lemma 2.8). Let $\tau^* = t_1^* - t_0^*$. Again, by Lemma 2.8, $0 < t_0^* \leq t_0^* + \tau^* < 1$ (λ large enough). If $\tau^* = 0$, then $q^* = p^* = 0$; ϕ is a parametrization of the corner. If $p^* = 0$, then $p^{*\prime} = 0$ (Lemma 2.7). Similarly, if $q^* = 0$, then $q^{*\prime} = 0$. In both cases it is easy to see that ϕ parametrizes the corner. The cases $q^* = -1$ or $p^* = 1$ are excluded (Lemma 2.8). Finally if $(t_0^*, \tau^*, q^*, q^{*\prime}, p^*, p^{*\prime}) \in \mathrm{int}(\boldsymbol{P})$, this set of parameters realize a local minimum of the function $H \colon \boldsymbol{P} \to \mathbb{R} \cup \{+\infty\}$ given by $H(t_0, \tau, q, q', p, p')$ = the energy of the trajectory in $\boldsymbol{M}$ associated with the parameters $(t_0, \tau, q, q', p, p')$ computed using (2.1). Hence

$$\frac{\partial H}{\partial t_0} = \frac{\partial H}{\partial \tau} = \frac{\partial H}{\partial q} = \frac{\partial H}{\partial q'} = \frac{\partial H}{\partial p} = \frac{\partial H}{\partial p'} = 0 \quad \text{at } (t_0^*, \tau^*, q^*, q^{*\prime}, p^*, p^{*\prime}). \tag{2.11}$$

If we write the energy function H and solve (2.8) we get that $\tau^*=0$, a contradiction. It follows that $(t_0^*, \tau^*, q^*, q^{*\prime}, p^*, p^{*\prime})$ is in the boundary of **P**. The previous discussion shows that ϕ is a parametrization of the corner with $\tau^*=q^*=q^{*\prime}=p^*=p^{*\prime}=0$. Optimizing the energy with respect to the parameter t_0 we get that $t_0^*=1/2$. This finishes the proof of Proposition 2.1 for the case $\alpha=0$, $\beta>0$.

Proof for $\alpha>0$, $\beta>0$. Since the computations needed to give the proof in some detail are elementary and rather tedious we only give a sketch of it. The existence of a global minimum for $E(.;\alpha,\beta,\lambda)$ *in* $Ad(\alpha,\beta)$ follows as in the previous cases.

Let us write

$$E^x(v;t_0,t_1)=\int_{t_0}^{t_1}\alpha|x'(t)|^2+\beta|x''(t)|^2\,dt \tag{2.12}$$

with a similar expression for $E^y(v;t_0,t_1)$. Finally, let us write

$$E(v;t_0,t_1)=E^x(v;t_0,t_1)+E^y(v;t_0,t_1)-\lambda\int_{t_0}^{t_1}|\nabla I(v(t))|\,dt. \tag{2.13}$$

We use the same notation $\phi(t)=(x_0(t), y_0(t))$, R_0, R_i, A, B, Z_-, Z_+ with the same meaning as in above. As before, we have:

Lemma 2.2″. $\phi^{\mathrm{iv}}-\mu\phi''=0$ *in* A, B *and in the open intervals of* Z_- *and* Z_+ *where* $\mu=\alpha/\beta$.

The solution of

$$\begin{gathered}v^{\mathrm{iv}}-\mu v''=0 \quad \text{in } [t_0,t_1]\\ v(t_0),\, v'(t_0),\, v(t_1),\, v'(t_1) \quad \text{given}\end{gathered} \tag{2.14}$$

is given by

$$v(t)=h'''(t-t_0)\,v(t_0)+h''(t-t_0)\,v'(t_0)+\gamma_1\,h'(t-t_0)+\gamma_2\,h(t-t_0) \tag{2.15}$$

where $h(t)=\dfrac{1}{\mu}\left\{\dfrac{sh\mu^{1/2}t}{\mu^{1/2}}-t\right\}$ and

$$\begin{aligned}\gamma_1&=\frac{1}{(h')^2-hh''}\{h'[v(t_1)-h'''v(t_0)-h''v'(t_0)]-\mathrm{h}[v'(t_1)-\mu h''v(t_0)-h'''v'(t_0)]\}\\ \gamma_2&=\frac{1}{(h')^2-hh''}\{-h''[v(t_1)-h'''v(t_0)-h''v'(t_0)]+\mathrm{h}'[v'(t_1)-\mu h''v(t_0)-h'''v'(t_0)]\}\end{aligned} \tag{2.16}$$

where the functions h, h', h'', h''' are evaluated at t_1-t_0.

Step 2″. Reduction to a set of model trajectories.

Observe that Lemmas 2.6 and 2.7 also hold in this case. Combining these lemmas we see that ϕ belongs to the set of model admissible trajectories $\boldsymbol{M}$ for which there exist $0\leqq t_0\leqq t_1\leqq 1$, $-1\leqq q\leqq 0$, $q'\in\mathbb{R}$ and $0\leqq p\leqq 1$, $p'\in\mathbb{R}$ such that

in $[0,t_0]$ $\phi(t)=(0,y_0(t))$ where $y_0(t)$ is a solution of (2.14) with $y_0(0)=-1$, $y_0'(0)=0$, $y_0(t_0)=q$, $y_0'(t_0)=q'$,

in $[t_0,t_1]$ $\phi(t)=(x_0(t), y_0(t))$ where $x_0(t)$ is a solution of (2.14) with $x_0(t_0)=0$, $x_0'(t_0)=0$, $x_0(t_1)=p$, $x_0'(t_1)=p'$ and $y_0(t)$ is also a solution of (2.14) with $y_0(t_0)=q$, $y_0'(t_0)=q'$, $y_0(t_1)=0$, $y_0'(t_1)=0$,

in $[t_1,1]$ $\phi(t)=(x_0(t),0)$ where $x_0(t)$ is a solution of (2.14) with $x_0(t_1)=p$, $x_0'(t_1)=p'$, $x_0(1)=1$, $x_0'(1)=0$.

Let $(t_0^*, t_1^*, q^*, q^{*\prime}, p^*, p^{*\prime})$ be the parameters corresponding to ϕ.

Step 3″. The final step.

Let us first mention that the analogous of Lemma 2.8 also holds in this case. As above, we introduce the set of parameters $\boldsymbol{P}$:

$$\boldsymbol{P}=\{(t_0,\tau,q,q',p,p''):0\leqq t_0\leqq t_0+\tau\leqq 1,\ -1\leqq q\leqq 0,\ -M-1\leqq q'\leqq M+1,\\ 0\leqq p\leqq 1,\ -M-1\leqq p'\leqq M+1\}$$

where $\tau=t_1-t_0$. Let $\tau^*=t_1^*-t_0^*$. Suppose that $\tau^*>0$. If $(t_0^*,\tau^*,q^*,q^{*\prime},p^*,p^{*\prime})$ is in the boundary of $\boldsymbol{P}$, by Lemma 2.8, either $q^*=0$ or $p^*=0$. In any case, repeating the discussion of step 3′ above we see that ϕ parametrizes the corner. Suppose now that $(t_0^*,\tau^*,q^*,q^{*\prime},p^*,p^{*\prime})\in\mathrm{int}(\boldsymbol{P})$. Instead of writing $E(\phi;\alpha,\beta,\lambda)$, $E^x(\phi;t_0,t_1)$, etc. let us write simply $E(\tau^*)$, $E^x(\tau^*;t_0,t_1)$, etc. to stress the dependence of the energies on τ^* and on the extremes of the interval. Then

$$\begin{aligned}(2.17)\qquad E(\tau^*)=&E^x(\tau^*;0,t_0)+E^x(\tau^*;t_0,t_1)+E^x(\tau^*;t_1,1)\\ &+E^y(\tau^*;0,t_0)+E^y(\tau^*;t_0,t_1)+E^y(\tau^*;t_1,1)-\lambda(1-\tau^*).\end{aligned}$$

Take λ big enough so that, by Lemma 2.8, the denominators in the expressions for $E^x(\tau^*;t_1,1)$, $E^y(\tau^*;0,t_0)$ do not vanish. They are analytic functions of τ^* in a neighborhood of $\tau^*=0$. $E^x(\tau^*;t_0,t_1)$ and $E^y(\tau;t_0,t_1)$ have a pole at $\tau^*=0$. Moreover, $E^x(\tau^*;0,t_0)=E^y(\tau^*;0,t_0)=0$. Hence, $E(\tau^*)$ has a pole at $\tau^*=0$. Keep t_0^*, $\tau^*>0$ (but small) fixed. Then the parameters $q^*,q^{*\prime},p^*,p^{*\prime}$ optimize the energy with t_0^*, $\tau^*>0$ fixed. Hence

$$(2.18)\qquad \frac{\partial E(\tau^*)}{\partial q}=\frac{\partial E(\tau^*)}{\partial q'}=\frac{\partial E(\tau^*)}{\partial p}=\frac{\partial E(\tau^*)}{\partial p'}=0 \quad \text{at } (q^*,q^{*\prime},p^*,p^{*\prime}).$$

The exact resolution of (2.18) being too tedious, we content ourselves with a rougher estimate. Doing Laurent expansions of $E(\tau^*)$ at $\tau^*=0$ and using (2.18) we get that

$$(2.19)\qquad p^*=O(\tau^{*2}),\qquad p^{*\prime}=O(\tau^*),\qquad q^*=O(\tau^{*2}),\qquad q^{*\prime}=O(\tau^*).$$

We also get that

$$(2.20)\qquad 2p^*-\tau^*p^{*\prime}=O(\tau^{*3})\,p^{*\prime}+O(\tau^{*4}),\qquad E^x(\tau^*;t_0,t_1)=E^x(\tau^*;t_0,t_1)=O(\tau^*)$$

where the constants appearing in the $O(\tau^*)$ expressions above do not depend on λ. Let φ be the parametrization of the corner with parameters $t_0=t_0^*$, $\tau=0$, $q=0$, $q'=0$, $p=0$, $p'=0$ and let $E^x(\varphi;t_0^*,1)$, $E^y(\varphi;0,t_0^*)$ be the corresponding expressions of the energy given by (2.12). Since

$$(2.21)\quad E^x(\tau^*;t_0^*,1)-E^x(\varphi;t_0^*,1)=E^x(\tau^*;t_0^*,1)-E^x(0;t_0^*,1)+E^x(0;t_0^*,1)-E^x(\varphi;t_0^*,1)$$

using (2.19) and the analyticity of $E^x(\tau^*;t_0^*,1)$ we get that the expression in (2.21) is an $O(\tau^*)$. In the same way one proves that $E^y(\tau^*;0,t_0^*)-E^y(\varphi;0,t_0^*)=O(\tau^*)$. Collecting all these facts, one gets that

$$(2.22)\qquad E(\tau^*)-E(\varphi)=O(\tau^*)+\lambda\tau^*.$$

We state:

Proposition 2.9. *For λ big enough $E(\tau^*)-E(\varphi)>0$.*

This proposition contradicts the fact that ϕ is a minimum for $E(.;\alpha,\beta,\lambda)$ in $Ad(\alpha,\beta)$. The parameters $(t_0^*, \tau^*, q^*, q^{*\prime}, p^*, p^{*\prime})$ must be at the boundary of $\boldsymbol{P}$. Then we know that $q^* = q^{*\prime} = p^* = p^{*\prime} = 0$. It is easy to see that $\tau^* = 0$. An optimization of the energy with respect to t_0 will give the exact value of t_0^*. This finishes the proof of Proposition 2.1 above.

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