NANYANG TECHNOLOGICAL UNIVERSITY

PH1011 Physics
Week 6
Lecture 12: Moment of Inertia

Moment of Inertia
- Rotational inertia (scalar quantity), resists changes in angular velocity ω
- Calculation of moment of inertia must use the rotational axis as the origin
- Parallel axis theorem: $I = I_{CM} + Md^2$
- Perpendicular axis theorem: $I_z = I_x + I_y$

Discrete Mass Case

Refers to several point masses

$$I = \sum_i m_i r_i^2$$

Continuous Mass Case
- Refers to an extended mass
- Can be viewed as “many slices” of small masses dm with distance-squared r^2 joined together

$$I = \int r^2 \, dm$$

with $r^2 = x^2 + y^2 + z^2$ in 3D.

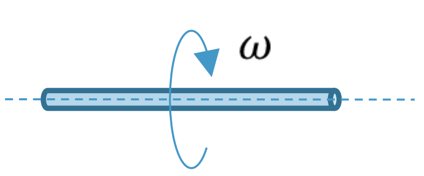
Examples
- 1D, 3 discrete masses on page 5
- 1D 2 discrete masses, Giancoli example 10-8 on page 7

Examples
- Thin Hoop on page 3
- Table of moment of inertia on page 5
- 1D, uniform rod on page 6
- 2D, triangle on page 6

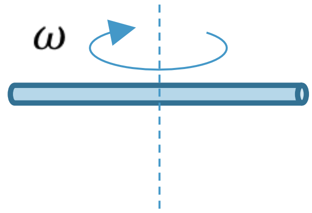
Other Examples
- Serway example 10.6 on page 8
- Serway example 10.16 on page 9
- Serway examples 10.4/10.11 on page 10

Moment of Inertia >> General Discussion:
Rotational Inertia:

Consider trying to turn the rod in two different orientations starting from rest and accelerating to angular velocity ω. Which one do you think will require more effort?

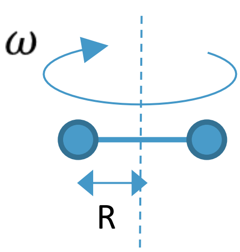


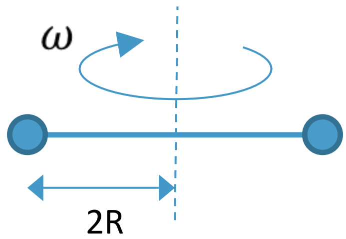
A



B

We will do another example that is easier to compute a numerical quantity for comparison. Consider two systems comprising of two masses m attached to a light rod, in one case length R and in another case $4R$. Which would have gain more kinetic energy starting from rest and accelerating to angular velocity ω about the middle of the rod?





KE = $2(\frac{1}{2}mv^2)$ then $v = R\omega$
$= mR^2\omega^2$

KE = $2(\frac{1}{2}mv_1^2)$ then $v_1 = 2R\omega$
$= 4mR^2\omega^2$

The configuration of masses with a given separation about a fixed axis of rotation can be characterized by the physical quantity mR^2, known as the moment of inertia I (in this case for point mass m). We see that the moment of inertia of an object, plays the same role for rotational motion that mass does for translational motion. Consider a mass m rotating in a circle of radius R about a fixed point and a single force F acts on m:

$$F = ma = mR\alpha \Longrightarrow FR = \tau = mR^2\alpha = I\alpha$$

Thus, we see from translational dynamics to rotational dynamics.

$$\text{mass } m \ \Longrightarrow \text{moment of inertia } I$$

Final Dictionary between Linear and Angular Motion:

Expression	Linear version	"Rotational" version
Constant acceleration	$s = ut + \frac{1}{2}at^2 + s_0$ $v = u + at$ $v^2 = u^2 + 2as$	$\theta = \omega_i t + \frac{1}{2}\alpha t^2 + \theta_0$ $\omega_f = \omega_i + \alpha t$ $\omega_f^2 = \omega_i^2 + 2\alpha\theta$
Momentum	$\vec{p} = m\vec{v}$ with conservation of linear momentum	$\vec{L} = I\vec{\omega}$ with conservation of angular momentum
Newton's 2nd Law	$\vec{F}_{net} = \frac{d\vec{p}}{dt}$ $\vec{F}_{net} = m\vec{a}$	$\vec{\tau}_{net} = \frac{d\vec{L}}{dt}$ $\tau_{net} = I\alpha$
Kinetic Energy	$\frac{1}{2}mv^2$	$\frac{1}{2}I\omega^2$

Moment of Inertia >> General Discussion:

Discrete Masses:

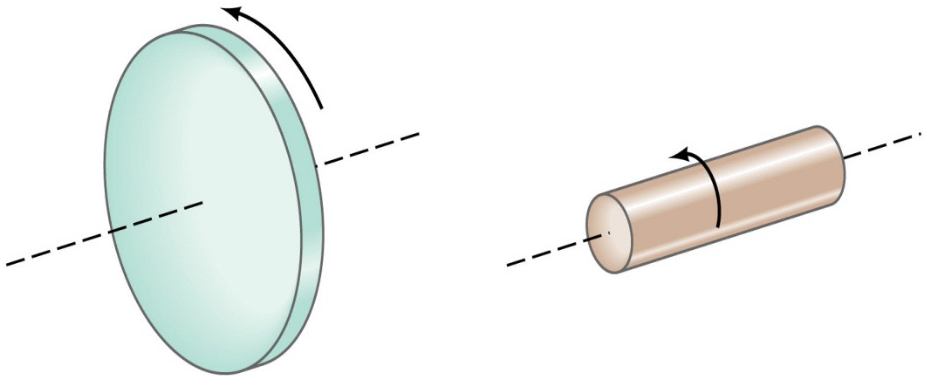
As we have seen, the quantity

$$I = \sum_i m_i r_i^2$$

is the rotational inertia for discrete point masses. We now need to generalize and get a formula for continuous mass.

Continuous Mass:

Look at these 2 continuous masses:



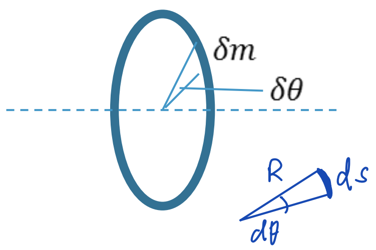
The distribution of mass matters here – these 2 objects have the same mass but the one on the left should have a greater rotational inertia because so much of its mass is far from the axis of rotation.

The generalization of discrete masses to continuous mass amounts to a “continuous summation of small slices of discrete masses”. Mathematically, “continuous summation” means “integration”.

$$I = \int r^2 \, dm$$

Quick Example:

Let’s compute the moment of inertia of a thin hoop of mass M and radius R about the cylindrical axis.



$ds = R\,d\theta$

$$I = \int R^2 \, dm$$

$$I = \int_0^{2\pi} R^2 \frac{M}{2\pi} \, d\theta$$

$$I = \frac{MR^2}{2\pi} \int_0^{2\pi} d\theta$$

$$I = MR^2 \quad \#$$

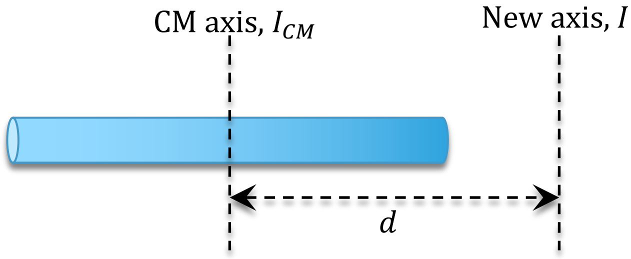
$$dm = \lambda \, ds = \lambda R \, d\theta = \frac{M}{2\pi R} R \, d\theta = \frac{M}{2\pi} \, d\theta$$

<u>Moment of Inertia >> General Discussion:</u>
<u>Theorems:</u>

<u>Parallel Axis Theorem:</u>
This theorem relates the moment of inertia I of an object of total mass M about any axis, and its moment of inertia I_{CM} about an axis passing through the centre of mass and parallel to the first axis. If the two axes are a distance d apart, then

$$I = I_{CM} + Md^2$$

Thus, if the moment of inertia about an axis through the CM is known, the moment of inertia



about any parallel axis can be easily obtained (without doing integration!).

Note: parallel axis theorem relates I_{CM} to another I. Do not use it to relate any 2 moment of inertias.

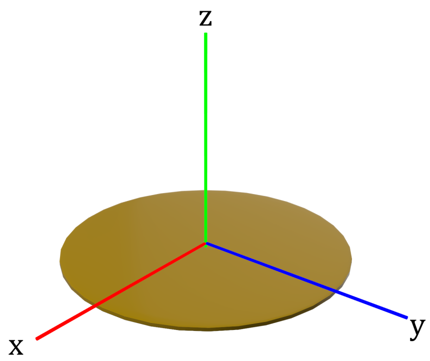
<u>Perpendicular Axis Theorem:</u>

The parallel axis theorem that we have seen earlier can be applied to any object. However, the perpendicular axis theorem can only be applied to plane (or flat) objects – objects whose thickness can be neglected compared to other dimensions. The theorem states that:

The sum of moment of inertia of a plane object about any two perpendicular axes in the plane of the object, is equal to the moment of inertia about an axis through their point of intersection perpendicular to the plane of the object. For an object in the xy plane

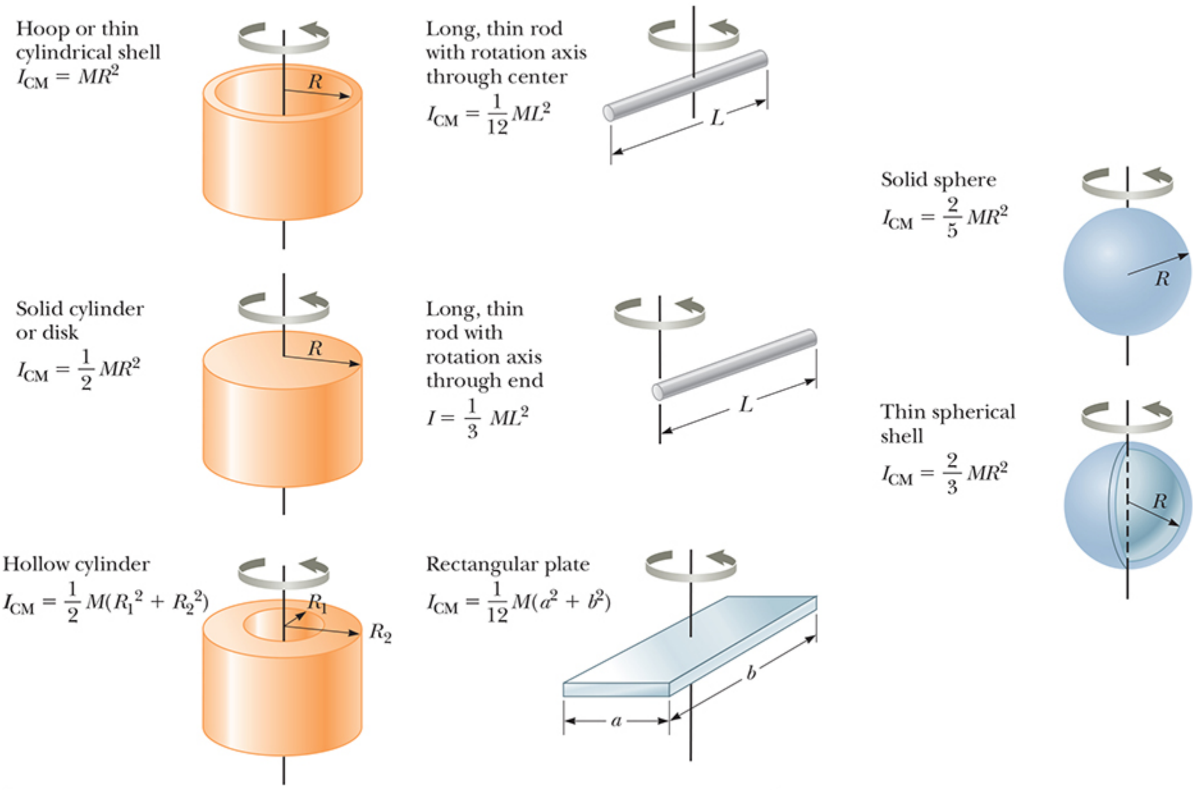
$$I_z = I_x + I_y$$

Here, I_x, I_y and I_z are the moment of inertias about the x, y and z axes. The proof is simple: since $I_x = \sum m_i y_i^2$ and $I_y = \sum m_i x_i^2$ then $I_z = \sum m_i (x_i^2 + y_i^2)$.



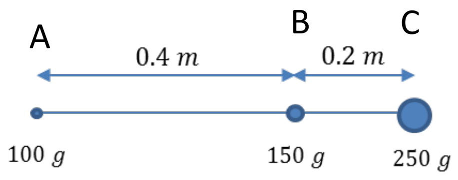
Moment of Inertia >> Example:
Table of Moment of Inertia for various objects with axis of rotation about CM

TABLE 10.2 Moments of Inertia of Homogeneous Rigid Objects with Different Geometries



Moment of Inertia >> Example:
1D, 3 Discrete Masses:

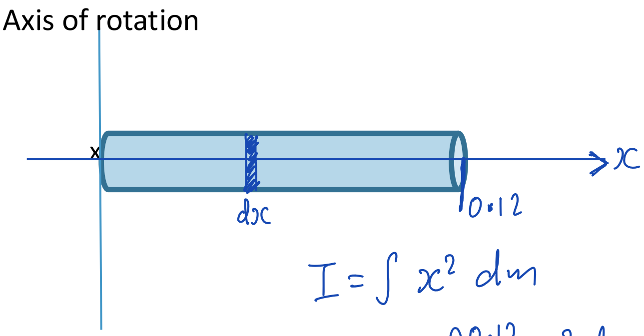
Three masses are connected by a rod of negligible mass as shown. Compute the moment of inertia of rod about the axis of rotation (A, B and C) as shown.



$$I_A = \sum_i m_i x_i^2$$
$$= 0.1 \times 0^2 + 0.15 \times 0.4^2 + 0.25 \times 0.6^2$$
$$= 0.114 \text{ kgm}^2 \ \#$$
$$I_B = 0.1 \times (-0.4)^2 + 0.15 \times 0^2 + 0.25 \times 0.2^2$$
$$= 0.026 \text{ kgm}^2 \ \#$$
$$I_C = 0.1 \times (-0.6)^2 + 0.15 \times (-0.2)^2 + 0.25 \times 0^2$$
$$= 0.042 \text{ kgm}^2 \ \#$$

Moment of Inertia >> Example:
1D, Continuous Mass:

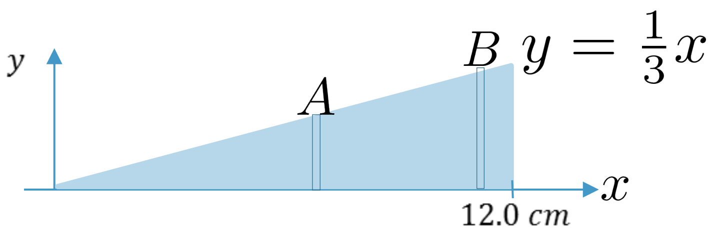
The mass per unit length λ of a rod is 15g/cm and it is 12cm long. Compute the moment of inertia of the rod about the axis of rotation as shown.



$$I = \int x^2 \, dm \qquad dm = \lambda dx$$
$$= \lambda \int_0^{0.12} x^2 \, dx$$
$$= 8.64 \times 10^{-4} \text{ kgm}^2 \ \#$$

Moment of Inertia >> Example:
2D, Continuous Mass:

Now consider a triangular sheet of metal as shown in the figure below.



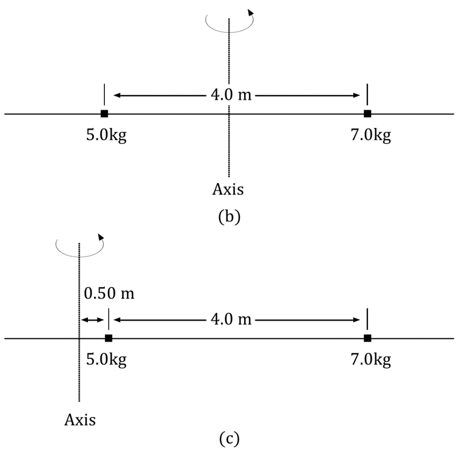
(It should be obvious that the narrow strip *A* will have a smaller mass as compared to strip *B*.) If the mass per unit area of the sheet is μ = 1.6g/cm², compute the moment of inertia of the sheet about the *y*-axis.

$$I = \int x^2 \, dm \qquad 2D \quad dm = \mu dA = \mu y dx$$

$$= \int_0^{0.12} x^2 \mu y \, dx$$
$$= \mu \int_0^{0.12} x^2 \left(\tfrac{1}{3}x\right) dx$$
$$= \frac{\mu}{3} \left[\frac{1}{4}x^4\right]_0^{0.12}$$
$$= 2.76 \times 10^{-4} \text{ kgm}^2 \ \#$$

Moment of Inertia >> Example:
Giancoli example 10-8
1D, 2 Discrete Masses:

Two small “weights” of 5 kg and 7 kg are mounted 4.0 m apart on a light rod (whose mass can be ignored). Calculate the moment of inertia of the system when rotated about an axis
a) passing its center of mass;
b) half way between the weights; and
c) 0.50 m to the left of the 5.0 kg mass.



choose 5kg as origin for CM calculation

a) $x_{CM} = \dfrac{5\times 0 + 7\times 4}{5+7} = \frac{7}{3}$ m from 5kg

Now x_{CM} has to be origin for I_{CM} calculation:

$$I_{CM} = 5\times\left(-\tfrac{7}{3}\right)^2 + 7\left(4-\tfrac{7}{3}\right)^2 = 46.7 \text{ kgm}^2 \ \#$$

b) $I_{(b)} = 5\times(-2)^2 + 7(2)^2 = 48 \text{ kgm}^2$ #

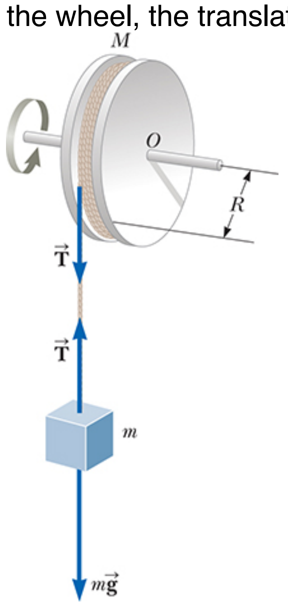
c) $I_{(c)} = 5\times(0.5)^2 + 7(4.5)^2 = 143 \text{ kgm}^2$ #

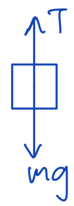
Note that these 3 moment of inertias are related by parallel axis theorem.

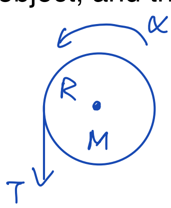
$$I_{(b)} = I_{CM} + (5+7)\left(\tfrac{7}{3}-2\right)^2 \quad \text{and} \quad I_{(c)} = I_{CM} + (5+7)\left(\tfrac{7}{3}+0.5\right)^2$$

Moment of Inertia >> Example:
Serway example 10.6

A wheel of radius R, mass M, and moment of inertia I is mounted on a frictionless, horizontal axle as in Figure. A light cord wrapped around the wheel supports an object of mass m. When the wheel is released, the object accelerates downward, the cord unwraps off the wheel, and the wheel rotates with an angular acceleration. Find expressions for the angular acceleration of the wheel, the translational acceleration of the object, and the tension in the cord.







$mg - T = ma$ $\qquad$ $\tau = I\alpha \Rightarrow TR = I\alpha$ $\qquad$ and $a = R\alpha$

$$\Rightarrow mg - T = mR\alpha$$

$$mg - T = mR\frac{TR}{I}$$

$$mg = T\left(\frac{mR^2}{I}+1\right)$$

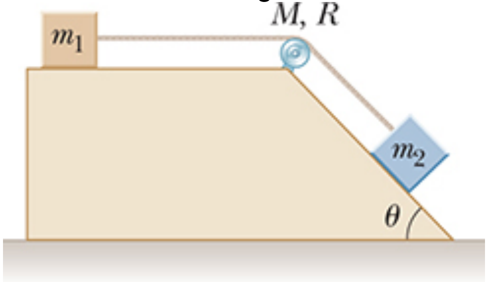
$$T = \frac{mg}{\frac{mR^2}{I}+1} \ \#$$

$$\Rightarrow \alpha = \frac{TR}{I} = \frac{mgR}{mR^2+I} = \frac{g}{R+\frac{I}{mR}} \ \#$$

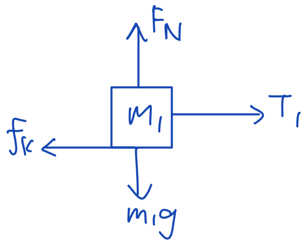
$$\Rightarrow a = R\alpha = \frac{g}{1+\frac{I}{mR^2}} \ \#$$

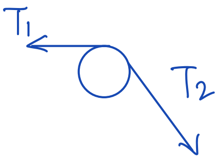
Moment of Inertia >> Example:
Serway problem 10.16

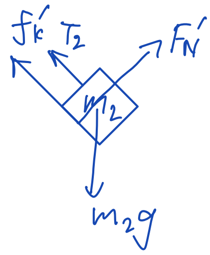
A block of mass $m_1 = 2.00$ kg and a block of mass $m_2 = 6.00$ kg are connected by a massless string over a pulley in the shape of a solid disk having radius $R = 0.250$ m and mass $M = 10.0$ kg. The fixed, wedge-shaped ramp makes an angle of $\theta = 30.0^o$ as shown in Figure. The coefficient of kinetic friction is 0.360 for both blocks. (a) Draw force diagrams of both blocks and of the pulley. Determine (b) the acceleration of the two blocks and (c) the tensions in the string on both sides of the pulley.



a)







$F_N = m_1 g$

$T_1 - f_k = m_1 a$

$f_k = \mu_k F_N = \mu_k m_1 g$

$T_2 R - T_1 R = I\alpha$

$a = R\alpha$

since string does not slip

$F_N' = m_2 g\cos\theta$

$m_2 g\sin\theta - f_k' - T_2 = m_2 a$

$f_k' = \mu_k F_N' = \mu_k m_2 g\cos\theta$

To solve for a,

$$\underbrace{T_2}_{m_2 g\sin\theta - f_k' - m_2 a} R - \underbrace{T_1}_{f_k + m_1 a} R = \underbrace{I}_{\frac{1}{2}MR^2}\underbrace{\alpha}_{\frac{a}{R}}$$

$$m_2 g\sin\theta - \mu_k m_2 g\cos\theta - m_2 a - \mu_k m_1 g - m_1 a = \frac{1}{2}Ma$$

$$a = \frac{m_2 g\sin\theta - \mu_k m_2 g\cos\theta - \mu_k m_1 g}{m_1 + m_2 + \frac{1}{2}M}$$

$$a = 0.309 \text{ m/s}^2 \ \#$$

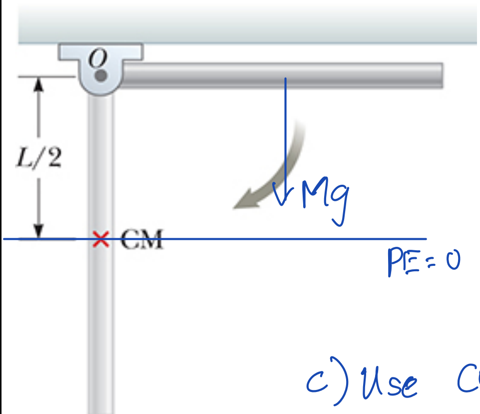
$\Rightarrow T_1 = f_k + m_1 a = \mu_k m_1 g + m_1 a = 7.67\text{N} \ \#$

$\Rightarrow T_2 = m_2 g\sin\theta - f_k' - m_2 a = m_2 g\sin\theta - \mu_k m_2 g\cos\theta - m_2 a$

$T_2 = 9.22\text{N} \ \#$

Moment of Inertia >> Example:
Serway examples 10.4/10.11

A uniform rod of length L and mass M is attached at one end to a frictionless pivot and is free to rotate about the pivot in the vertical plane as in Figure. The rod is released from rest in the horizontal position.
a) What are the initial angular acceleration of the rod and
b) the initial translational acceleration of its right end?
c) What is its angular speed when the rod reaches its lowest position?
d) Determine the tangential speed of the center of mass and the tangential speed of the lowest point on the rod when it is in the vertical position.



a) $\tau = I\alpha \Rightarrow Mg\frac{L}{2} = \frac{1}{3}ML^2\alpha$

$\alpha = \frac{3g}{2L}$ #

b) $a = L\alpha = \frac{3g}{2}$ #

c) Use COE since force method is complicated

$$Mg\frac{L}{2} = \underbrace{\frac{1}{2}I\omega^2}$$

no translational KE since this is pure rotation

$$Mg\frac{L}{2} = \frac{1}{2}\left(\frac{1}{3}ML^2\right)\omega^2$$

$\omega = \sqrt{\frac{3g}{L}}$ #

d) $V_{CM} = \frac{L}{2}\omega = \frac{1}{2}\sqrt{3gL}$ #

$V_{tan} = L\omega = \sqrt{3gL}$ #

<u>End of Lecture Reflection Questions:</u>

1) Are you able to distinguish between discrete and continuous mass problems when calculating moment of inertia?

2) Are you able to recognize situations when parallel axis theorem or perpendicular axis theorem applies?

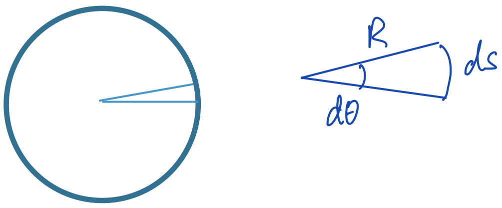
3) Are you able to deploy dm in various dimensions just like how it is used in CM calculations?

4) Are you able to form the extra equations coming from angular quantities and rotational Newton's 2^{nd} law $\tau = I\alpha$? Then are you able to juggle the algebra to solve for the quantity required by the question?

Extra Materials (Non-examinable):

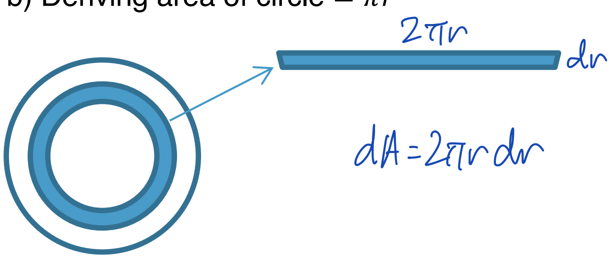
Digression (Non-examinable) >> Mathematical Digression on Integration

a) Deriving circumference $= 2\pi r$



$$\text{circumference} = \int ds = \int R\,d\theta = R\int_0^{2\pi} d\theta = 2\pi R \quad \#$$

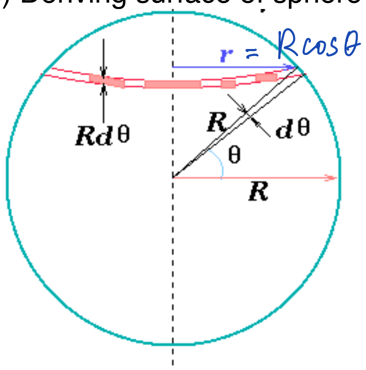
b) Deriving area of circle $= \pi r^2$



$dA = 2\pi r\,dr$

$$\text{Area} = \int dA = \int_0^R 2\pi r\,dr = \pi R^2 \quad \#$$

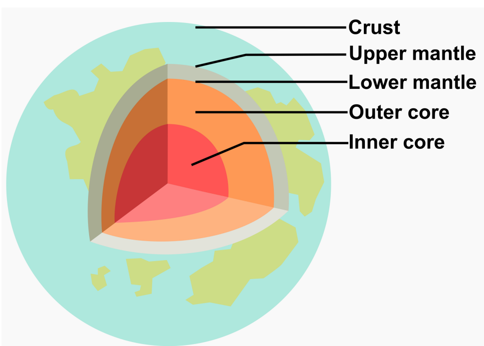
c) Deriving surface of sphere $= 4\pi r^2$



$dA = 2\pi r\,R\,d\theta$

$$\text{Area} = \int dA = \int 2\pi r\,R\,d\theta = \int 2\pi R^2 \cos\theta\,d\theta = 2\pi R^2 \int_{-\pi/2}^{\pi/2} \cos\theta\,d\theta = 4\pi R^2 \quad \#$$

d) Deriving volume of sphere $= \frac{4}{3}\pi r^3$



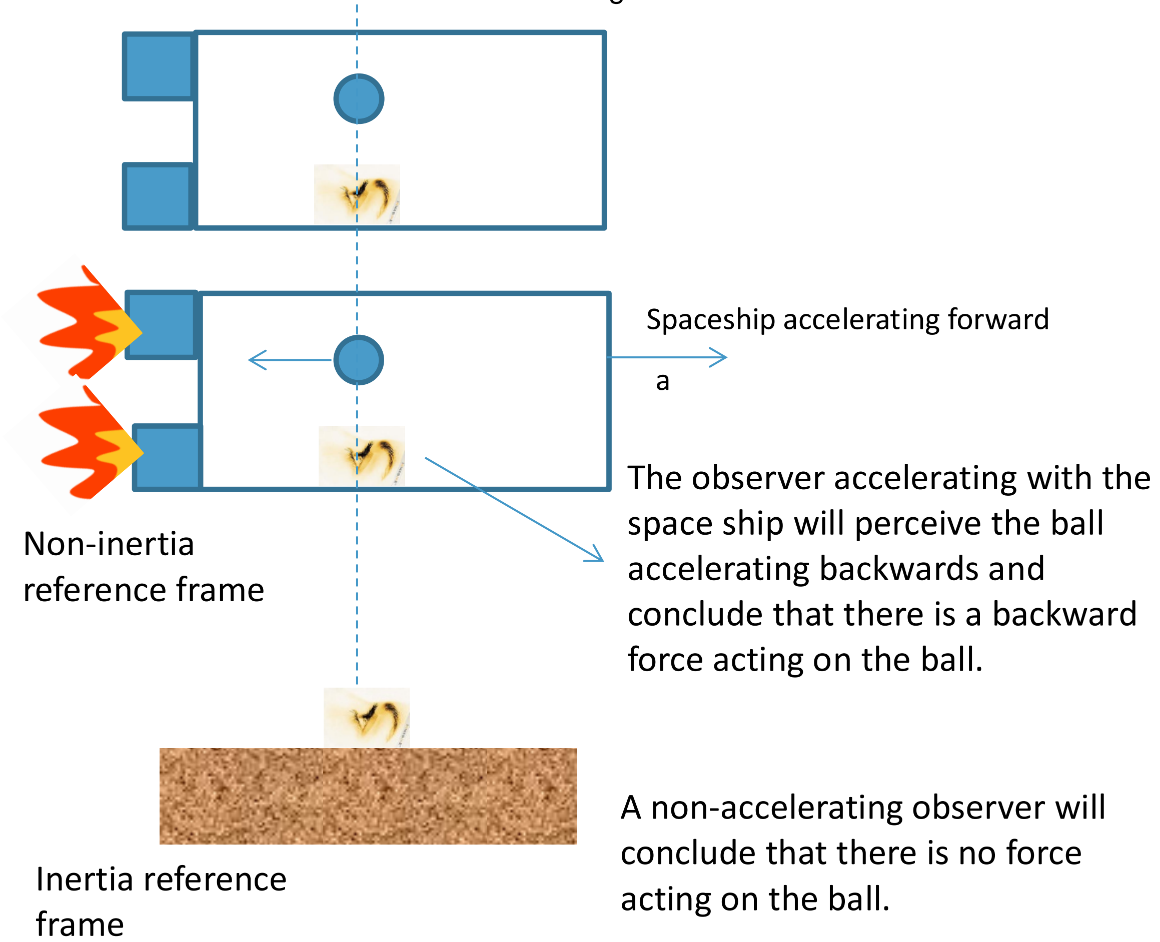
volume of a slice of "onion"

$$dV = 4\pi r^2\,dr$$

$$\text{Volume} = \int dV = \int_0^R 4\pi r^2\,dr = \frac{4}{3}\pi R^3 \quad \#$$

<u>Digression (Non-examinable) >> Inertial Reference Frames</u>

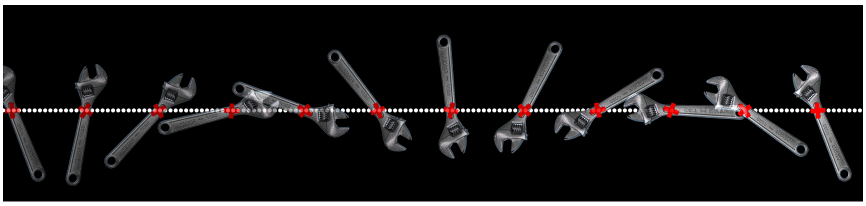
Imagine a spaceship in deep space far away from anything and there is a ball floating in it and the observer will conclude that no force is acting on the ball.



So who is correct? Is there a force acting on the ball?

Person in inertia frame is correct, there is no force.

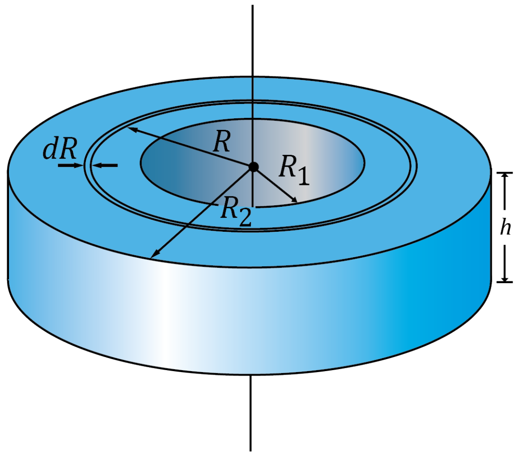
<u>Importance of Centre of Mass Reference Frame</u>



The above diagram shows the motion of a rotating spanner through space with no forces acting on it. We see that if we see that the center of mass is the only point on the spanner that is non-accelerating – making it a special and important reference.

Digression (Non-examinable) >> Moment of Inertia of a Cylinder
Giancoli pg 264 example 10-12

a) Show that the moment of inertia of a uniform hollow cylinder of inner radius R_1, outer radius R_2, and mass M, is $I = ½\ M(R_1^2 + R_2^2)$, if the rotation axis is through the center along the axis of symmetry.
b) Obtain the moment of inertia for a solid cylinder.



APPROACH We know that the moment of inertia of a thin ring of radius R is mR^2. So we divide the cylinder into thin concentric cylindrical rings or hoops of thickness dR, one of which is indicated in Fig. 10–24. If the density (mass per unit volume) is ρ, then

$$dm = \rho\, dV,$$

where dV is the volume of the thin ring of radius R, thickness dR, and height h. Since $dV = (2\pi R)(dR)(h)$, we have

$$dm = 2\pi\rho h R\, dR.$$

SOLUTION (*a*) The moment of inertia is obtained by integrating (summing) over all these rings:

$$I = \int R^2\, dm = \int_{R_1}^{R_2} 2\pi\rho h R^3\, dR = 2\pi\rho h\left[\frac{R_2^4 - R_1^4}{4}\right] = \frac{\pi\rho h}{2}(R_2^4 - R_1^4),$$

where we are given that the cylinder has uniform density, ρ = constant. (If this were not so, we would have to know ρ as a function of R before the integration could be carried out.) The volume V of this hollow cylinder is $V = (\pi R_2^2 - \pi R_1^2)h$, so its mass M is

$$M = \rho V = \rho\pi(R_2^2 - R_1^2)h.$$

Since $(R_2^4 - R_1^4) = (R_2^2 - R_1^2)(R_2^2 + R_1^2)$, we have

$$I = \frac{\pi\rho h}{2}(R_2^2 - R_1^2)(R_2^2 + R_1^2) = \tfrac{1}{2}M(R_1^2 + R_2^2),$$

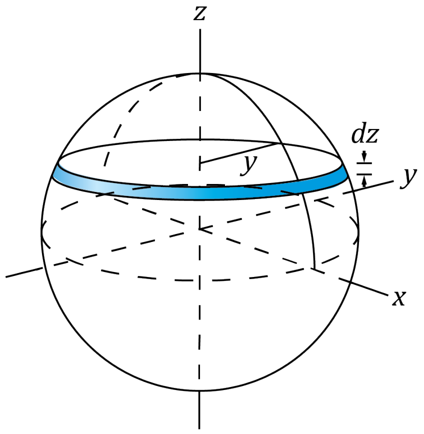
as stated in Fig. 10–20d.
(*b*) For a solid cylinder, $R_1 = 0$ and if we set $R_2 = R_0$, then

$$I = \tfrac{1}{2}MR_0^2,$$

which is that given in Fig. 10–20c for a solid cylinder of mass M and radius R_0.

Digression (Non-examinable) >> Moment of Inertia of a Sphere (Using Moment of Inertia of Discs)

http://hyperphysics.phy-astr.gsu.edu/hbase/isph.html



$$\text{Radius} = R$$
$$\text{Mass} = M$$
$$\text{Density} = \rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3}$$

Small moment of inertia:

$$dI = \frac{1}{2}y^2 dm = \frac{1}{2}y^2 \rho dV = \frac{1}{2}y^2 \rho \pi y^2 dz$$

and "summing" the small momenta of inertias gives

$$I = \frac{1}{2}\rho\pi \int_{-R}^{R} y^4 \, dz = \frac{1}{2}\rho\pi \int_{-R}^{R} (R^2 - z^2)^2 \, dz = \frac{8}{15}\rho\pi R^5$$

Substituting the density expression gives

$$I = \frac{8}{15}\left[\frac{M}{\frac{4}{3}\pi R^3}\right]\pi R^5 = \frac{2}{5}MR^2$$