* Introduction to Lyapunov Theorem (Stability Definition):

Consider the nonlinear differential eqn: $\dot{x} = f(x)$, $f(0) = 0$. Since $f(0) = 0$, $f(x)$ has a solution $x = 0$.

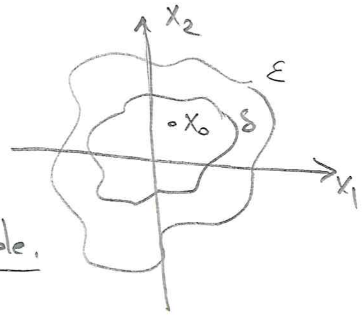
To guarantee that a solution exists and is unique, a sufficient condition is that $f(x)$ is Lipschitz:

i.e. $\|f(x) - f(y)\| \leq L\|x - y\|$, $L > 0$ in the neighborhood of the origin.

Definition:

The solution $x = 0$ to the differential eqn $\dot{x} = f(x)$ is called stable if for $\varepsilon > 0$ there exist a number $\delta(\varepsilon)$ such that all solutions with initial conditions $\|x(0)\| < \delta$ have the property $\|x(t)\| < \varepsilon$ for $0 \leq t \leq \infty$

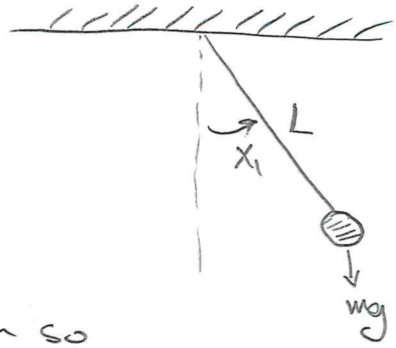
- The solution is unstable if it is not stable.
- The solution is asymptotically stable if it is stable and δ can be found such that all solutions with $\|x(0)\| < \delta$ have the property that $\|x(t)\| \to 0$ as $t \to \infty$.
- If the solution is asymptotically stable for any initial value, it is called globally asymptotically stable.



Example:

Consider the inverted pendulum equation:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{L}\sin(x_1) - \frac{k}{m}x_2$$



The reference of the potential energy is chosen so that $E(0) = 0$

$$E = \frac{1}{2}m x_2^2 + mg(L - L\cos x_1)$$

if there is no friction, $E(x) = \text{constant}$, otherwise $\frac{dE}{dt} \leq 0$.

Conclusion: $x = 0$ is a stable equilibrium point.

* Introduction to Lyapunov Theorem (Positive Functions):

Definition:

A continuously differential function $V: R^n \to R$ is called positive definite in a region $U \subset R^n$ containing the origin if:

- $V(0) = 0$
- $V(x) > 0$ for all $x \in U$ and $x \neq 0$.

$\Rightarrow$ a function is called positive semi-definite if $V(x) \geq 0$.

Note:

If we can find a function such that the velocity vector $\dot{x} = f(x)$ always points towards the interior of level curves, it seems intuitively clear that a solution that starts inside a given level curve can never pass to the outside of the same level curve.

* Lyapunov Theorem (for time invariant systems):

- If there exists a function $V: R^n \to R$ that is positive definite such that its derivative along the solution of $\dot{x} = f(x)$, $\frac{dV}{dt} = \frac{\partial V}{\partial x}^T \frac{dx}{dt} = \frac{\partial V}{\partial x}^T f(x)$ is negative semi-definite, then the solution $x(t) = 0$ to $\dot{x} = f(x)$ is stable.
- If $\frac{dV}{dt}$ is negative definite, the solution is asymptotically stable.
- The function V is called the Lyapunov function for the system $\dot{x} = f(x)$.
- Moreover, if $\frac{dV}{dt} < 0$ and $V(x) \to \infty$ when $\|x\| \to \infty$, the solution is globally asymptotically stable.

notes:

- The surface $V(x) = c$ for some $c > 0$ is called a Lyapunov surface or a Lyapunov level surface.
- The condition $\dot{V} \leq 0$ implies that when a trajectory crosses a Lyapunov surface $V(x) = c$, it moves inside the set $\Omega_c = \{x \in R^n : V(x) \leq c\}$ and cannot come out. The system is stable.
- If $\dot{V} < 0$, the trajectory moves from one Lyapunov surface to an inner Lyapunov surface with a smaller "c". As c decreses, the Lyapunov surface $V(x) = 0$ shrinks to the origin. The system is asymptotically stable.

$\Rightarrow$ Lyapunov theorem gives only sufficient stability conditions.

Example:

Consider the pendulum equations:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{-g}{L}\sin(x_1)$$

A candidate Lyapunov function is $V(x) = \frac{g}{L}(1-\cos x_1) + \frac{1}{2}x_2^2$

note that:

- $V(0) = 0$
- $V(x) > 0$ for $-2\pi < x_1 < 2\pi$
- $\dot{V}(x) = \frac{g}{L}\sin x_1 \cdot \dot{x}_1 + x_2 \cdot \dot{x}_2$

$$= \frac{g}{L}x_2\sin x_1 - \frac{g}{L}x_2\sin x_1 = 0$$

we conclude that the origin is stable.

Example:

Let $\dot{x}_1 = x_2$

$$\dot{x}_2 = -\frac{g}{L}\sin(x_1) - \frac{k}{m}x_2$$

A candidate Lyapunov function is: $V(x) = \frac{g}{L}(1-\cos x_1) + \frac{1}{2}x_2^2$

- $V(0) = 0$
- $V(x) > 0$ for $-2\pi < x_1 < 2\pi$
- $\dot{V}(x) = \frac{g}{L}\sin x_1 \cdot \dot{x}_1 + x_2 \cdot \dot{x}_2$

$$= \frac{g}{L}x_2\sin x_1 - \frac{g}{L}x_2\sin x_1 - \frac{k}{m}x_2^2 = \frac{-k}{m}x_2^2 \leq 0$$

$\dot{V}$ is negative semi-definite since $\dot{V}(x) = 0$ for $x_2 = 0$ irrespective of x_1.

$\Rightarrow$ we can only conclude that the system is stable.

However, $x_2 = 0 \Rightarrow \dot{x}_1 = 0 \Rightarrow x_1 = \text{constant}$

Also $x_2 = 0 \Rightarrow \dot{x}_2 = 0 \Rightarrow \sin x_1 = 0 \Rightarrow x_1 = 0$

$\hookrightarrow$ The system can maintain $\dot{V}(x) = 0$ only at $x = 0$, we conclude asymptotically stable.

* LaSalle's Invariance Principle:

If in a domain about the origin, it is possible to find a Lyapunov fn whose derivatives along the trajectories of the system is negative semi-definite and if it can be stablished that no system trajectory can stay forever at points where $\dot{V}(x) = 0$ except at the origin, the origin is said to be asymptotically stable.

* Design steps Using Lyapunov Method:

1. Derivation of the error equation: This is a differential equation of the error signal, e (which may be the output error or the state errors). The error should be written as a known linear function controlled by a nonlinear input term. Usually the linear term contains the reference model transfer fn.

② A candidate Lyapunov fn is formed as a fn of both the signal error and parameters error. Typically:

$$V = e^T P e + \phi^T \Gamma^{-1} \phi .$$

③ The time derivative of V is calculated. Typically, we will get

$$\dot{V} = -e^T Q e + \{\text{terms including } \phi\}$$

- Putting the terms including ϕ to zero, $\dot{V}$ is guaranteed to be negative definite with respect to e if $Q > 0$.
- P and Q satisfy $A^T P + PA = -Q$ where A represents the linear part of the error model. Barbalat's lemma leads to $e \to 0$.

Proof:

$$V = X^T P X \quad , \quad \dot{X} = AX$$

$$\dot{V} = \underbrace{\dot{X}^T}_{X^T A^T} P X + X^T P \underbrace{\dot{X}}_{AX} = X^T A^T P X + X^T P A X$$

$$= X^T [A^T P + PA] X$$

$$\leq -X^T Q X$$

$$\Rightarrow \boxed{A^T P + PA = -Q}$$

④ Putting the extra terms in $\dot{V}$ to zero provides the adaptation equations.

Example: [Adaptation of Feedforward Gain]

Process model: $\frac{dy}{dt} = -ay + ku$

Desired response: $\frac{dy_m}{dt} = -ay_m + k_o u_c$

Controller: $u = \theta u_c$

$\Rightarrow$ $e = y - y_m$

$$\dot{e} = \dot{y} - \dot{y}_m = -ay + ku + ay_m - k_o u_c$$
$$= -ay + k\theta u_c + ay_m - k_o u_c$$
$$= -a\underbrace{[y - y_m]}_{e} + [k\theta - k_o]u_c$$
$$= \boxed{-ae + [k\theta - k_o]u_c}$$

$\Rightarrow$ System model: $\frac{dy}{dt} = -ay + ku$

$$\frac{d\theta}{dt} = ??!!$$

$\Rightarrow$ Desired equilibrium: $e = 0$, $\theta = \theta_o = \frac{k_o}{k}$

$\Rightarrow$ Consider the Lyapunov function:

$$V(e,\theta) = \frac{\gamma}{2}e^2 + \frac{k}{2}(\theta - \theta_o)^2$$

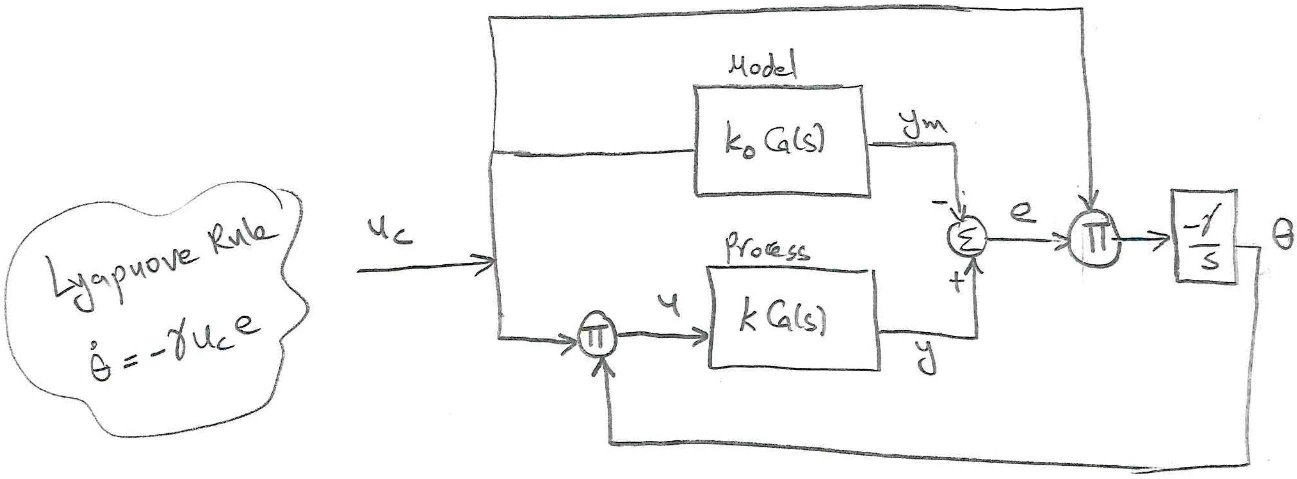
$$\frac{dV}{dt} = \frac{\partial V}{\partial e}\dot{e} + \frac{\partial V}{\partial \theta}\cdot\dot{\theta}$$
$$= \gamma e[-ae + (k\theta - k_o)u_c] + k(\theta - \theta_o)\dot{\theta}$$
$$= -a\gamma e^2 + \gamma e u_c(k\theta - k_o) + k\left(\theta - \frac{k_o}{k}\right)\dot{\theta}$$

$$\dot{V} = -a\gamma e^2 + (k\theta - k_0)\left[\underbrace{\dot{\theta} + \gamma u_c e}_{0}\right]$$

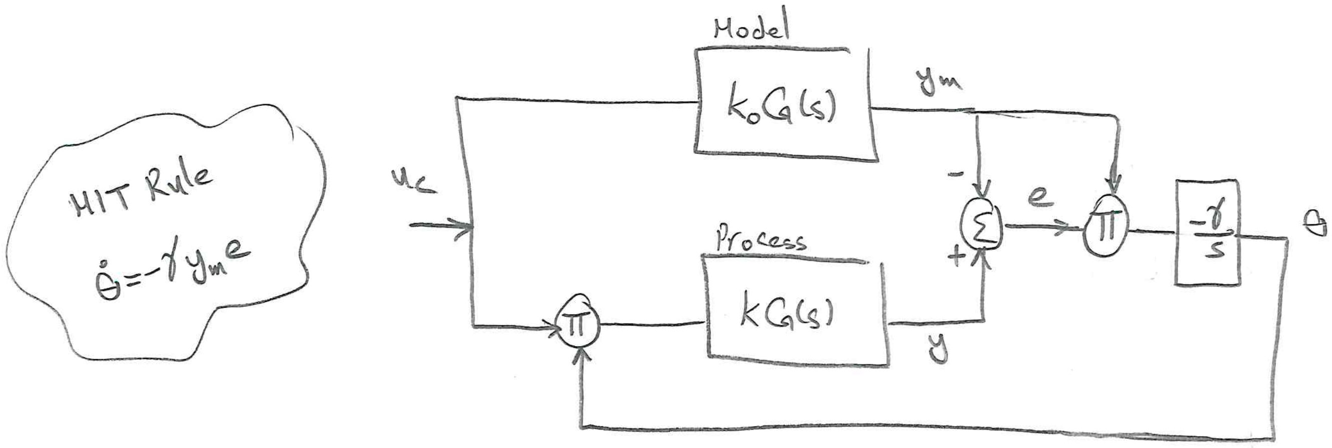
⇒ choosing the adjustment rule:

$$\dot{\theta} = -\gamma u_c e$$

gives: $\dot{V} = -a\gamma e^2$



Remember:



Example [First order system] :

- Process Model : $\frac{dy}{dt} = -ay + bu$

- Desired response : $\frac{dy_m}{dt} = -a_m y_m + b_m u_c$

- Controller : $u = \theta_1 u_c - \theta_2 y$

$$\begin{aligned}
\dot{y} &= -ay + b[\theta_1 u_c - \theta_2 y] \\
&= -[a + \theta_2]y + b\theta_1 u_c \\
&= -a_m y_m + b_m u_c
\end{aligned}$$

$$\Rightarrow a + \theta_2 = a_m \quad \Rightarrow \boxed{\theta_2 = a_m - a}$$

$$\Rightarrow b\theta_1 = b_m \quad \Rightarrow \boxed{\theta_1 = \frac{b_m}{b}}$$

$$e = y - y_m$$

$$\begin{aligned}
\dot{e} &= \dot{y} - \dot{y}_m \\
&= -ay + bu - (-a_m y_m + b_m u_c) \\
&= -ay + b[\theta_1 u_c - \theta_2 y] + a_m y_m - b_m u_c \\
&= -[a + b\theta_2]y + [b\theta_1 - b_m]u_c + a_m y_m \quad + a_m y - a_m y
\end{aligned}$$

$$\boxed{\dot{e} = -a_m e - (a + b\theta_2 - a_m)y + (b\theta_1 - b_m)u_c}$$

$$V(e,\theta_1,\theta_2)=\frac{1}{2}\left[e^2+\frac{1}{b\gamma}(b\theta_2+a-a_m)^2+\frac{1}{b\gamma}(b\theta_1-b_m)^2\right]$$

$$\dot{V}=e.\dot{e}+\frac{1}{b\gamma}(b\theta_2+a-a_m)b\cdot\dot{\theta}_2+\frac{1}{b\gamma}(b\theta_1-b_m)\cdot b\cdot\dot{\theta}_1$$

$$=e\left[-a_m e-(a+b\theta_2-a_m)y+(b\theta_1-b_m)u_c\right]+\frac{1}{\gamma}(b\theta_2+a-a_m)\dot{\theta}_2$$

$$+\frac{1}{\gamma}(b\theta_1-b_m)\dot{\theta}_1$$

$$=-a_m e^2-e(a+b\theta_2-a_m)y+e(b\theta_1-b_m)u_c$$

$$+\frac{1}{\gamma}(b\theta_2+a-a_m)\dot{\theta}_2+\frac{1}{\gamma}(b\theta_1-b_m)\dot{\theta}_1$$

$$=-a_m e^2+\left[b\theta_2+a_m-a\right]\left[\underbrace{\frac{1}{\gamma}\dot{\theta}_2-ey}_{0}\right]+\left[b\theta_1-b_m\right]\left[\underbrace{\frac{1}{\gamma}\dot{\theta}_1+eu_c}_{0}\right]$$

adjustment rules $\Rightarrow$ $\boxed{\dot{\theta}_1=-\gamma e u_c}$, $\boxed{\dot{\theta}_2=\gamma e y}$

$$\Rightarrow\quad \dot{V}=-a_m e^2$$

Let us compare Lyapunov to MIT rule:

$$\dot{\theta}=\gamma\varphi e$$

Lyapunov	MIT
$\varphi=[-u_c \quad y]$	$\varphi=\left[\frac{-a_m}{p+a_m}u_c \quad \frac{a_m}{p+a_m}y\right]$
$\Rightarrow$ no filtering for u_c and y	$=\frac{a_m}{p+a_m}[-u_c \quad y]$

- The adjustment rule obtained by Lyapunov fn is simpler because it does not require filtering of the signal.

- However, arbitrary large value of adaptation gain "γ" can be used with the Lyapunov theory.