Discussion Questions:

1. For the given system:

$$A = \begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D = [0]$$

the controller structure: $u = \theta_1 x_1 + \theta_2 x_2$

the reference model: $\omega_n = 2$ rad/sec, $\zeta = 0.8$

Design an adaptive controller using Lyapunov and MIT Techniques.

Solution:

MIT Rule: $e = y - y_m$, $\dfrac{d\theta}{dt} = -\gamma e \dfrac{\partial e}{\partial \theta}$, $\dfrac{\partial e}{\partial \theta}$ ← to be found

Lyapunov: $e = y - y_m$, $\dfrac{de}{dt} = e + (\theta - \theta^\circ)$, $V = \frac{1}{2}\left[e^2 + (\theta - \theta^\circ)^2\right]$

$$\dot{x} = Ax + Bu \Rightarrow \dot{x}_1 = -a x_1 + u \Rightarrow \dot{x}_1 = -a x_1 + \theta_1 x_1 + \theta_2 x_2$$

$$\Rightarrow \boxed{\dot{x}_1 = (\theta_1 - a) x_1 + \theta_2 x_2}$$

$$\dot{x}_2 = -b x_2 + u \Rightarrow \dot{x}_2 = -b x_2 + \theta_1 x_1 + \theta_2 x_2$$

$$\Rightarrow \boxed{\dot{x}_2 = \theta_1 x_1 + (\theta_2 - b) x_2}$$

The closed-loop system:

$$\dot{x} = A_c x = \begin{bmatrix} \theta_1 - a & \theta_2 \\ \theta_1 & \theta_2 - b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The desired system: $\zeta = 0.8$ and $\omega_n = 2$ rad/sec

$\Rightarrow$ Characteristic Eqn: $P_m(s) = s^2 + 2\zeta\omega_n s + \omega_n$

$$\boxed{P_m(s) = s^2 + 3.2s + 4}$$

(2)

Then, $\dot{X}_m = \underbrace{\begin{bmatrix} 0 & 1 \\ -4 & -3.2 \end{bmatrix}}_{A_m} X_m$ [This is the cononical controllable form]

$$e = X - X_m \Rightarrow \dot{e} = \dot{X} - \dot{X}_m = \underbrace{\begin{bmatrix} \theta_1 - a & \theta_2 \\ \theta_1 & \theta_2 - b \end{bmatrix}}_{A_c(\theta)} X - \underbrace{\begin{bmatrix} 0 & 1 \\ -4 & -3.2 \end{bmatrix}}_{A_m} X_m$$

$$\Rightarrow \dot{e} = A_c X - A_m X_m + \underbrace{A_m X - A_m X}$$

$$= A_m \underbrace{(X - X_m)}_{e} + A_c X - \underset{\uparrow A_c(\theta^\circ)}{A_m} X = A_m e + A_c X - A_m X$$

$$= A_m e + \begin{bmatrix} \theta_1 - a & \theta_2 \\ \theta_1 & \theta_2 - b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} \theta_1^\circ - a & \theta_2^\circ \\ \theta_1^\circ & \theta_2^\circ - b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= A_m e + \begin{bmatrix} (\theta_1 - a) x_1 + \theta_2 x_2 \\ \theta_1 x_1 + (\theta_2 - b) x_2 \end{bmatrix} - \begin{bmatrix} (\theta_1^\circ - a) x_1 + \theta_2^\circ x_2 \\ \theta_1^\circ x_1 + (\theta_2^\circ - b) x_2 \end{bmatrix}$$

$$= A_m e + \begin{bmatrix} (\theta_1 - a) x_1 + \theta_2 x_2 - (\theta_1^\circ - a) x_1 - \theta_2^\circ x_2 \\ \theta_1 x_1 + (\theta_2 - b) x_2 - \theta_1^\circ x_1 - (\theta_2^\circ - b) x_2 \end{bmatrix}$$

$$= A_m e + \begin{bmatrix} \theta_1 x_1 \cancel{- a x_1} + \theta_2 x_2 - \theta_1^\circ x_1 \cancel{+ a x_1} - \theta_2^\circ x_2 \\ \theta_1 x_1 + \theta_2 x_2 \cancel{- b x_2} - \theta_1^\circ x_1 - \theta_2^\circ x_2 \cancel{+ b x_2} \end{bmatrix}$$

$$= A_m e + \begin{bmatrix} (\theta_1 - \theta_1^\circ) x_1 + (\theta_2 - \theta_2^\circ) x_2 \\ (\theta_1 - \theta_1^\circ) x_1 + (\theta_2 - \theta_2^\circ) x_2 \end{bmatrix}$$

$$= A_m e + \begin{bmatrix} x_1 & x_2 \\ x_1 & x_2 \end{bmatrix} \begin{bmatrix} \theta_1 - \theta_1^\circ \\ \theta_2 - \theta_2^\circ \end{bmatrix}$$

$$= A_m e + \psi [\theta - \theta^\circ]$$

Assume

$$V(\theta, x) = \frac{1}{2}\left[\gamma e^T P e + (\theta-\theta^\circ)^T(\theta-\theta^\circ)\right]$$

$$\dot{V}(\theta, x) = \frac{\gamma}{2}\left[\dot{e}^T P e + e^T P \dot{e}\right] + \frac{1}{2}\left[\underbrace{\dot{\theta}^T(\theta-\theta^\circ) + (\theta-\theta^\circ)^T\dot{\theta}}_{2(\theta-\theta^\circ)^T\dot{\theta}}\right]$$

$$= \frac{\gamma}{2}\left[(A_m e + \psi(\theta-\theta^\circ))^T P e + e^T P(A_m e + \psi(\theta-\theta^\circ))\right] + (\theta-\theta^\circ)^T\dot{\theta}$$

$$= \frac{\gamma}{2}\left[e^T A_m^T P e + (\theta-\theta^\circ)^T\psi^T P e + e^T P A_m e + e^T P\psi(\theta-\theta^\circ)\right] + (\theta-\theta^\circ)^T\dot{\theta}$$

$$= \frac{\gamma}{2}\left[-e^T Q e + 2(\theta-\theta^\circ)^T\psi^T P e\right] + (\theta-\theta^\circ)^T\dot{\theta}$$

$$= -\frac{\gamma}{2} e^T Q e + \gamma(\theta-\theta^\circ)^T\psi^T P e + (\theta-\theta^\circ)^T\dot{\theta}$$

$$= -\frac{\gamma}{2} e^T Q e + (\theta-\theta^\circ)^T\left[\underbrace{\gamma\psi^T P e + \dot{\theta}}_{0}\right]$$

$$\Rightarrow \boxed{\dot{\theta} = -\gamma\psi^T P e}$$

P can be obtained by solving $A_m^T P + P A_m = -Q$

$$A_m^T P + P A_m = -Q$$

$$\begin{bmatrix} 0 & -4 \\ 1 & -3.2 \end{bmatrix}\begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} + \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -4 & -3.2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -4P_2 & -4P_3 \\ P_1 - 3.2P_2 & P_2 - 3.2P_3 \end{bmatrix} + \begin{bmatrix} -4P_2 & P_1 - 3.2P_2 \\ -4P_3 & P_2 - 3.2P_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -8P_2 & P_1 - 3.2P_2 - 4P_3 \\ P_1 - 3.2P_2 - 4P_3 & 2P_2 - 6.4P_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow \quad -8P_2 = -1 \qquad\qquad \Rightarrow \quad P_1 = 1.1812$$

$$P_1 - 3.2P_2 - 4P_3 = 0 \qquad\qquad P_2 = 0.125$$

$$2P_2 - 6.4P_3 = -1 \qquad\qquad P_3 = 0.1953$$

$$\Rightarrow P = \begin{bmatrix} 1.1812 & 0.125 \\ 0.125 & 0.1953 \end{bmatrix}$$

$\boxed{2}$ Consider a position servo described by:

$$\frac{dv}{dt} = -av + bu$$

$$\frac{dy}{dt} = v$$

where parameters a and b are unknowns. Assume that the control Law

$$u = \theta_1(u_c - y) - \theta_2 v$$

is used and that it is desired to control the system in such away that the transfer function from command signal to process output is given by:

$$G_m(s) = \frac{\omega^2}{s^2 + 2\zeta\omega s + \omega^2}$$

Determine an adaptive control Law that adjust parameters θ_1 and θ_2 so that the desired objective is obtained.

Solution:

$$\frac{dv}{dt} = -av + bu \quad \Rightarrow \quad \frac{V(s)}{U(s)} = \frac{b}{s+a}$$

$$\frac{dy}{dt} = v \quad \Rightarrow \quad V(s) = sY(s) \quad \Rightarrow \quad \frac{Y(s)}{V(s)} = \frac{1}{s}$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{V(s)}{U(s)} \times \frac{Y(s)}{V(s)} = \frac{b}{s(s+a)}$$

Let us use MIT Rule:

$$u = \theta_1 u_c - \theta_1 y - \theta_2 (s y)$$

$$y = G(p)\, u \quad , \quad G(p) = \frac{b}{p(p+a)}$$

$$y = G(p)\left[\theta_1 u_c - \theta_1 y - \theta_2 (s y)\right]$$

$$y = G(p)\,\theta_1 u_c - G(p)\,\theta_1 y - G(p)\,\theta_2 (s y)$$

$$\Rightarrow \frac{y}{u_c} = \frac{\theta_1 G(p)}{1 + \theta_1 G(p) + \theta_2 s\, G(p)}$$

$$\frac{y}{u_c} = \frac{b\theta_1}{p^2 + (a + b\theta_2)p + b\theta_1}$$

and $$G_m(s) = \frac{\omega^2}{s^2 + 2\zeta\omega_n s + \omega^2}$$

$$\Rightarrow a + b\theta_2 = 2\zeta\omega \quad \Rightarrow \boxed{\theta_2^\circ = \frac{2\zeta\omega - a}{b}}$$

$$\Rightarrow b\theta_1 = \omega^2 \quad \Rightarrow \boxed{\theta_1^\circ = \frac{\omega^2}{b}}$$

$$e = y - y_m$$

$$e = \frac{b\theta_1}{p^2 + (a + b\theta_2)p + b\theta_1} u_c - y_m$$

$$\frac{\partial e}{\partial \theta_1} = \frac{b u_c}{p^2 + (a + b\theta_2)p + b\theta_1} - \frac{b y}{p^2 + (a + b\theta_2)p + b\theta_1}$$

$$= \frac{b}{p^2 + (a + b\theta_2)p + b\theta_2}\left[u_c - y\right]$$

$$\frac{\partial e}{\partial \theta_2} = \frac{-b^2 \theta_1 u_c}{[p^2 + (a+b\theta_2)p + b\theta_1]^2}$$

$$= \frac{-b\,y}{p^2 + (a+b\theta_2)p + b\theta_1}$$

Approximation: $p^2 + (a+b\theta_2)p + b\theta_1 \cong p^2 + 2\zeta\omega_n p + \omega^2$

$$\frac{d\theta_1}{dt} = -\gamma \frac{\partial e}{\partial \theta_1} e = -\gamma * \left[\frac{1}{p^2 + 2\zeta\omega\, p + \omega^2}(u_c - y)\right] * e$$

$$\frac{d\theta_2}{dt} = -\gamma \frac{\partial e}{\partial \theta_2} e = -\gamma * \left[\frac{1}{p^2 + 2\zeta\omega p + \omega^2} \cdot y\right] * e$$

where: $\gamma = \gamma' b$

$\boxed{3}$ An Integrator

$$G_p(s) = \frac{b}{s}$$

is to be controlled by a zero-order continuous-time controller

$$u(t) = -s_o\, y(t) + t_o\, u_c(t)$$

The desired response model is given by:

$$G_m(s) = \frac{b_m}{s + a_m}$$

Derive, using the Lyapunov theory, a parameter update law of an MRAS guaranteeing that the error $e = y - y_m$ goes to zero. Try the Lyapunov function:

$$V(x) = \frac{1}{2}\left[e^2 + \frac{1}{b}(bs_o - a_m)^2 + \frac{1}{b}(bt_o - b_m)^2\right]$$

where, $e = y(t) - y_m(t)$

Solution:

$$\frac{y}{u} = \frac{b}{s} \Rightarrow \frac{dy}{dt} = bu = -bs_o y + bt_o u_c$$

$$\frac{dy_m}{dt} = -a_m y_m + b_m u_c$$

$$\Rightarrow bs_o = a_m \Rightarrow \boxed{s_o = \frac{a_m}{b}}, \quad bt_o = b_m \Rightarrow \boxed{t_o = \frac{b_m}{b}}$$

$$e = y - y_m, \quad \frac{de}{dt} = \frac{dy}{dt} - \frac{dy_m}{dt}$$

$$= -bs_o y + bt_o u_c + a_m y_m - b_m u_c \quad + \underbrace{a_m y - a_m y}$$

$$= (bt_o - b_m)u_c - (bs_o - a_m)y + a_m\underbrace{(y_m - y)}_{-e}$$

$$\boxed{\frac{de}{dt} = -a_m e - (bs_o - a_m)y + (bt_o - b_m)u_c}$$

$$V(x) = \frac{1}{2}\left[e^2 + \frac{1}{b}(bs_o - a_m)^2 + \frac{1}{b}(bt_o - b_m)^2\right]$$

$$\dot{V} = \frac{1}{\cancel{2}}\left[\cancel{2}.e.\dot{e} + \frac{\cancel{2}}{\cancel{b}}(bs_o - a_m).\cancel{b}.\dot{s}_o + \frac{\cancel{2}}{\cancel{b}}(bt_o - b_m)\cancel{b}\,\dot{t}_o\right]$$

$$= e.\dot{e} + (bs_o - a_m)\dot{s}_o + (bt_o - b_m)\dot{t}_o$$

$$= e\left[-a_m e - (bs_o - a_m)y + (bt_o - b_m)u_c\right] + (bs_o - a_m)\dot{s}_o + (bt_o - b_m)\dot{t}_o$$

$$= -a_m e^2 - e(bs_o - a_m)y + e(bt_o - b_m)u_c + (bs_o - a_m)\dot{s}_o + (bt_o - b_m)\dot{t}_o$$

$$= -a_m e^2 + (bs_o - a_m)\Big[\underbrace{-ey + \dot{s}_o}_{\text{zero}}\Big] + (bt_o - b_m)\Big[\underbrace{eu_c + \dot{t}_o}_{\text{zero}}\Big]$$

$$\Rightarrow \boxed{\dot{s}_o = ey}, \quad \boxed{\dot{t}_o = -eu_c}, \quad \boxed{\dot{V} = -a_m e^2}$$