* Lyapunov-based MRAC Design:

- The main advantage of Lyapunov design is that it guarantees a stable closed-loop system.

- For a linear, asymptotically stable governed by a matrix 'A', a positive symmetric matrix 'Q' yields a positive symmetric matrix 'P' by the equation:

$$\boxed{A^T P + PA = -Q}$$

This equation is known as "Lyapunov's equation".

- The main drawback of Lyapunov design is that there is no symmetric way of finding a suitable Lyapunov function 'V' Leading to a specific adaptive Law.

- For example, if one wants to add a proportional term to adaptive Law, it is not trivial to find the corresponding Lyapunov function.

- The hyperstability approach is more flexible than the Lyapunov approach.

* The Lyapunov Equation:

- Let the linear system : $\boxed{\dot{x} = Ax}$ be stable.
- Let 'Q' be an arbitrary positive definite matrix.
- Then, the Lyapunov equation: $\boxed{A^T P + PA = -Q}$ always has a unique solution where 'P' is positive definite.
- The function $\boxed{V(x) = x^T P x}$ is then a Lyapunov function.

* State Feedback:

- Process: $\dot{X} = AX + Bu$
- Desired response: $\dot{X}_m = A_m X_m + B_m u_c$
- Control Law: $u = M u_c - LX$
- Closed-loop system:

$$\begin{aligned}\dot{X} &= AX + B[Mu_c - LX] \\ &= \underbrace{[A - BL]}\, X + B\underbrace{M}\, u_c \\ &= A_c(\theta)\, X + B_c(\theta)\, u_c\end{aligned}$$

- The parameterization is: $A_c(\theta^\circ) = A_m$ and $B_c(\theta^\circ) = B_m$
- For compatibility we need: $A - A_m = BL$ and $B_m = BM$

or $L = (B^T B)^{-1} B^T [A - A_m]$

$M = (B^T B)^{-1} B^T B_M$

* Error Equation:

The error $e = X - X_m$ satisfies

$$\begin{aligned}\dot{e} = \dot{X} - \dot{X}_m &= AX + Bu - A_m X_m - B_m u_c \\ &= AX + B[Mu_c - LX] - A_m X_m - B_m u_c + \underbrace{A_m X - A_m X}\end{aligned}$$

Hence:

$$\dot{e} = [A - A_m - BL]\, X + [BM - B_m]\, u_c + [\underbrace{X - X_m}_{e}]\, A_m$$

$$= A_m e + [A - A_m - BL]X + [BM - B_m]\, u_c$$

$$= A_m e + [\underbrace{A - BL}_{A_c(\theta)} - A_m] x + [\underbrace{BM}_{B_c(\theta)} - B_m] u_c$$

$$= A_m e + [A_c(\theta) - \underset{\uparrow \atop A_c(\theta^\circ)}{A_m}] x + [B_c(\theta) - \underset{\uparrow \atop B_c(\theta^\circ)}{B_m}] u_c$$

$$= A_m e + \psi [\theta - \theta^\circ]$$

- Try:

$$V(e, \theta) = \frac{1}{2}\left(\gamma e^T P e + [\theta - \theta^\circ]^T [\theta - \theta^\circ]\right)$$

- Differentiating V:

$$\dot{V} = \frac{\gamma}{2}\left[e^T P \dot{e} + \dot{e}^T P e + [\theta - \theta^\circ]^T \dot{\theta} + \dot{\theta}^T [\theta - \theta^\circ]\right]$$

= since scalar

$$= \frac{\gamma}{2}\left[e^T P (A_m e + \psi [\theta - \theta^\circ]) + e^T A_m^T P e + \psi^T [\theta - \theta^\circ] P e\right] + [\theta - \theta^\circ]^T \dot{\theta}$$

$$= \frac{\gamma}{2}\left[e^T P A_m e + e^T A_m^T P e + e^T P \psi [\theta - \theta^\circ] + \psi^T [\theta - \theta^\circ] P e\right] + [\theta - \theta^\circ] \dot{\theta}$$

- Now, consider the Lyapunov equation: $\boxed{A_m^T P + P A_m = -Q}$

It is possible because refrence model always stable.

$$\Rightarrow \dot{V} = \frac{-\gamma}{2} e^T Q e + \gamma [\theta - \theta^\circ]^T \psi^T P e + [\theta - \theta^\circ]^T \dot{\theta}$$

$$= -\frac{\gamma}{2} e^T Q e + [\theta - \theta^\circ]^T (\underbrace{\gamma \psi^T P e + \dot{\theta}}_{0}) \Rightarrow \boxed{\dot{\theta} = -\gamma \psi^T P e}$$

Parameter adjustment

If parameter adjustment law is chosen as:

$$\boxed{\dot{\theta} = -\gamma\, \varphi^T P e}$$

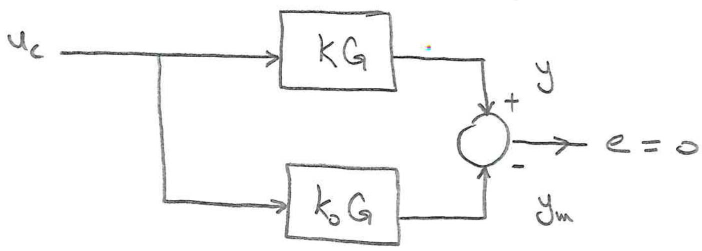
we obtain:

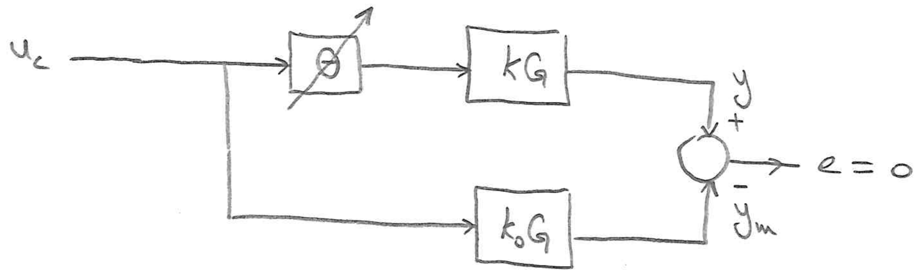
$$\dot{V} = -\frac{\gamma}{2} e^T Q e$$

which is negative sem-definite

- note that this assumes knowledge of the state.
- This does not mean that $\theta - \theta^{\circ}$ converges to zero, but $e \to 0$.

* Adaptation of Feedforward Gain:





- Process: $\hat{G} = k\,G(s)$
- Desired response: $\hat{G}_m = k_o\,G(s)$
- Control law: $u = \theta\, u_c$

$$e = y - y_m$$

$$= k\,G(p)\,u - k_o\,G(p)\,u_c$$

$$= k\,G(p)\,\theta\,u_c - k_o\,G(p)\,u_c$$

$$= k\left[G(p)\,\theta\,u_c - \frac{k_o}{k}\,G(p)\,u_c\right] \quad , \quad \frac{k_o}{k} = \theta^\circ$$

$$= k\,G(p)\,u_c\,[\theta - \theta^\circ]$$

with

$$G(s) = C\,(sI - A)^{-1}B$$

$$\dot{x} = Ax + \underbrace{\hat{B}\,\underset{\theta}{k}}_{B}\,u_c \quad , \quad y = Cx$$

$$y = \underbrace{C\,(sI-A)^{-1}B}_{G(s)}\,u \;\Rightarrow\; e = k\,C\,(sI-A)^{-1}B\,u_c\,\underbrace{(\theta-\theta^\circ)}_{\hat{B}}$$

$$\Rightarrow \dot{x} = Ax + B\,(\theta-\theta^\circ)\,u_c$$

$$e = Cx \;\Rightarrow\; \dot{e} = C\dot{x}$$

A candidate Lyapunov function is:

$$V = \frac{1}{2}\left(\gamma\, x^T P x + (\theta-\theta^\circ)^2\right)$$

$$\dot{V} = \frac{\gamma}{2}\left(\dot{x}^T P x + x^T P \dot{x}\right) + (\theta-\theta^\circ)\,\dot{\theta}$$

$$\dot{x} = Ax + B\,(\theta-\theta^\circ)\,u_c$$

$$= \frac{\gamma}{2}\Big[x^T A^T P x + u_c\,(\theta-\theta^\circ)\,B^T P x + x^T P A x + x^T P B\,(\theta-\theta^\circ)\,u_c$$

$$+ (\theta-\theta^\circ)\,\dot{\theta}\Big]$$

Let $A^T P + PA = -Q$

Then,

$$\dot{V} = -\frac{\gamma}{2} x^T Q x + \gamma(\theta - \theta^\circ) B^T P x u_c + (\theta - \theta^\circ)\dot{\theta}$$

$$= -\frac{\gamma}{2} x^T Q x + (\theta - \theta^\circ)\left[\underbrace{\gamma B^T P x u_c + \dot{\theta}}_{0}\right]$$

update Law $\Rightarrow$ $\boxed{\dot{\theta} = -\gamma u_c B^T P x}$

gives $\dot{V} = -\frac{\gamma}{2} x^T Q x$ $\quad x \to 0$

$e = Cx$ as $x \to 0$ then $e \to 0$

If we can find P such that $B^T P = C$

then, the adaptation law becomes

$$\boxed{\dot{\theta} = -\gamma u_c e}$$

i.e. this is now output feedback and the state x is not required.

* Definition:

- A rational transfer function 'G' with real coefficients is positive real (PR) if:

$$Re[G(s)] \geq 0 \quad \text{for} \quad Re[s] \geq 0$$

- A rational transfer function 'G' with real coefficients is strictly positive real (SPR) if $G(s-\epsilon)$ is PR for some $\epsilon > 0$.

$\rightarrow G(s) = \frac{1}{s+1}$ is SPR.

$\rightarrow G(s) = \frac{1}{s}$ is PR but not SPR.

Proof:

- $G(s) = \frac{1}{s+1} \Rightarrow G(j\omega) = \frac{1}{j\omega+1} \times \frac{1-j\omega}{1-j\omega} = \frac{1-j\omega}{\omega^2+1}$

$$= \underbrace{\frac{1}{\omega^2+1}}_{\text{Real}} - j\underbrace{\frac{\omega}{\omega^2+1}}_{\text{Img}}.$$

$\Rightarrow Re[G(j\omega)] = \frac{1}{\omega^2+1} > 0 \quad \forall \omega$, then SPR.

- $G(s) = \frac{1}{s} \Rightarrow G(j\omega) = \frac{1}{j\omega} \times \frac{-j\omega}{-j\omega} = -j\frac{\omega}{\omega^2} = -j\frac{1}{\omega}$, no real part

$\Rightarrow Re[G(j\omega)] \geq 0 \quad \forall \omega$, then PR.

Theorem [Kalman-Yakubovich Lemma]:

Let the Linear time-invariant system:

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

be completely controllable and completely observable. The transfer function

$$G(s) = C(sI - A)^{-1}B$$

is strictly positive real if and only if there exist positive definit matrices P and Q such that

$$\boxed{\begin{aligned} A^T P + PA &= -Q \\ B^T P &= C \end{aligned}}$$