We use Extended Kalman Filter(EKF) to estimate the position of robot. Now define the following variables to be used in EKF. $(x, y, \theta)^T$ represents the robot position with respect to world coordinate. $(b_g, b_{ax}, b_{ay})^T$ stands for the bias introduced by gyroscope and accelerometer. $(x_m, y_m, \theta_m)^T$ represents the raw data(measurements) collected by UWB(Ultra Wide Band). $(\omega_m, a_{mx}, a_{my})^T$ is the raw data comes from MPU6500 with respect to the robot's self frame. Notice that we use data from 6500 as an input to the system.

$$\mathbf{x} \triangleq \begin{bmatrix} x \\ y \\ \theta \\ \dot{x} \\ \dot{y} \\ b_g \\ b_{ax} \\ b_{ay} \end{bmatrix} \quad \mathbf{n} \triangleq \begin{bmatrix} n_g \\ n_{ax} \\ n_{ay} \\ n_{bg} \\ n_{bax} \\ n_{bay} \end{bmatrix} \quad \mathbf{z} \triangleq \begin{bmatrix} x_m \\ y_m \\ \theta_m \end{bmatrix} \quad \mathbf{u} \triangleq \begin{bmatrix} \omega_m \\ a_{mx} \\ a_{my} \end{bmatrix} \quad \mathbf{v} \triangleq \begin{bmatrix} n_x \\ n_y \\ n_\theta \end{bmatrix}$$

Notice that all distributions in Kalman filter are Gaussian Distributions.

$$\begin{aligned} \text{State estimation} : \mathbf{x_t} &\sim \mathcal{N}(\boldsymbol{\mu_t},\, \Sigma_t) \\ \text{Process noise} : \mathbf{n_t} &\sim \mathcal{N}(\mathbf{0},\, Q_t) \\ \text{Observation noise} : \mathbf{v_t} &\sim \mathcal{N}(\mathbf{0},\, R_t) \end{aligned}$$

System equation in continuous time:

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, \mathbf{n}) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \omega_m - b_g - n_g \\ cos(\theta)(a_{mx} - b_{ax} - n_{ax}) + sin(\theta)(a_{my} - b_{ay} - n_{ay}) \\ -sin(\theta)(a_{mx} - b_{ax} - n_{ax}) + cos(\theta)(a_{my} - b_{ay} - n_{ay}) \\ n_{bg} \\ n_{bax} \\ n_{bay} \end{bmatrix} \tag{1}$$

Observation function in continuous time:

$$\mathbf{z} = g(\mathbf{x}, \mathbf{v}) = \begin{bmatrix} I_{3\times3} & 0_{3\times5} \end{bmatrix} \mathbf{x} + I_{3\times3}\mathbf{v} \tag{2}$$

which is linear. Taylor expansion on (2) at $\mathbf{x} = \bar{\boldsymbol{\mu}}_t, \mathbf{v} = \mathbf{0}$:

$$g(\mathbf{x}, \mathbf{v}) = g(\bar{\boldsymbol{\mu}}_t, \mathbf{0}) + C_t(\mathbf{x} - \bar{\boldsymbol{\mu}}_t) + W_t(\mathbf{v} - \mathbf{0}) \tag{3}$$

with $C_t$ and $W_t$ defined as followed:

$$C_t = \begin{bmatrix} I_{3\times3} & 0_{3\times5} \end{bmatrix}; W_t = I_{3\times3} \tag{4}$$

Use Taylor expansion on system equation(1) at $\mathbf{x} = \boldsymbol{\mu_{t-1}}, \mathbf{u} = \mathbf{u_t}, \mathbf{n} = \mathbf{0}$

$$\dot{\mathbf{x}} \approx f(\boldsymbol{\mu_{t-1}}, \mathbf{u_t}, \mathbf{0}) + A_t(\mathbf{x} - \boldsymbol{\mu_{t-1}}) + B_t(\mathbf{u} - \mathbf{u_t}) + U_t(\mathbf{n} - \mathbf{0}) \tag{5}$$

with $A_t, B_t, U_t$ defined as followed:

$$A_t = \frac{\partial f}{\partial \mathbf{x}}\bigg|_{\boldsymbol{\mu_{t-1}},\mathbf{u_t},\mathbf{0}} ;B_t = \frac{\partial f}{\partial \mathbf{u}}\bigg|_{\boldsymbol{\mu_{t-1}},\mathbf{u_t},\mathbf{0}} ;U_t = \frac{\partial f}{\partial \mathbf{n}}\bigg|_{\boldsymbol{\mu_{t-1}},\mathbf{u_t},\mathbf{0}} \tag{6}$$

Discretize the continuous system equation by multiplying $\delta t$ on both sides of (5):

$$\mathbf{x_{\bar{t}}} \approx F_t \mathbf{x_{t-1}} + V_t \mathbf{n_t} + \delta t (f(\boldsymbol{\mu_{t-1}}, \mathbf{u_t}, \mathbf{0}) - A_t \boldsymbol{\mu_{t-1}}) \tag{7}$$

with $F_t, V_t$ defined as followed:

$$F_t = I + \delta t A_t ; V_t = \delta t U_t \tag{8}$$

$f(\mathbf{x}, \mathbf{u}, \mathbf{n})$ is not linear. From equation (1), (5) and (6), we get the following:

$$A_t = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -s(\theta)(a_{mx} - b_{ax} - n_{ax}) + c(\theta)(a_{my} - b_{ay} - n_{ay}) & 0 & 0 & 0 & -c(\theta) & -s(\theta) \\ 0 & 0 & -c(\theta)(a_{mx} - b_{ax} - n_{ax}) - s(\theta)(a_{my} - b_{ay} - n_{ay}) & 0 & 0 & 0 & s(\theta) & -c(\theta) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\Bigg|_{\mathbf{x}=\boldsymbol{\mu_{t-1}},\mathbf{u}=\mathbf{u_t},\mathbf{n}}$$

$$U_t = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -cos(\theta) & -sin(\theta) & 0 & 0 & 0 \\ 0 & sin(\theta) & -cos(\theta) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\Bigg|_{\mathbf{x}=\boldsymbol{\mu_{t-1}},\mathbf{u}=\mathbf{u_t},\mathbf{n}=\mathbf{0}}$$

Prediction step:

$$\begin{aligned} \bar{\boldsymbol{\mu}}_t &= \boldsymbol{\mu_{t-1}} + \delta t f(\boldsymbol{\mu_{t-1}}, \mathbf{u_t}, \mathbf{0}) \\ \bar{\Sigma}_t &= F_t \Sigma_{t-1} F_t^T + V_t Q_t V_t^T \end{aligned}$$

Update step:

$$\begin{aligned} \boldsymbol{\mu_t} &= \bar{\boldsymbol{\mu}}_t + K_t(\mathbf{z_t} - g(\bar{\boldsymbol{\mu}}_t, 0)) \\ \Sigma_t &= \bar{\Sigma}_t - K_t C_t \bar{\Sigma}_t \\ K_t &= \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + W_t R_t W_t^T)^{-1} \end{aligned}$$