Hello hello

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Abstract

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1 Introduction

2 Theory

2.1 Mathematical framework and notation

In the following we will use the occupation representation, making use of creation $a_p^\dagger$ and annihilation a_p operators. As a shorthand, we will write $\hat{p}^\dagger \equiv a_p^\dagger$ and $\hat{p} \equiv a_p$ when no confusion can be made. N represents the number of occupied states while L the total number of states in our calculations. The indicies $p, q, \ldots$ are reserved for the L general states, the N occupied states are indexed by $i, j, \ldots$, while the $L - N$ unoccupied (virtual) states by $a, b, \ldots$ indicies.

Since we are treating fermionic systems, the canonical anticommutation relations are used

$$\{\hat{p}^\dagger, \hat{q}^\dagger\} = \{\hat{p}, \hat{q}\} = 0 \qquad \{\hat{p}^\dagger, \hat{q}\} = \delta_{pq}.$$

One and two body matrix elements are calculated using a computational basis, with explicit expressions for one body Hamiltonians $h(\boldsymbol{x})$ and two body interaction $v(\boldsymbol{x}, \boldsymbol{x}')$ here presented in position space

$$\langle p| \hat{h} |q\rangle = \int \mathrm{d}\boldsymbol{x}\, \psi_p^*(x) \hat{h}(\boldsymbol{x}) \psi_q(x)$$
$$\langle pq| \hat{v} |rs\rangle = \int \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{x}'\, \psi_p^*(\boldsymbol{x}) \psi_q^*(\boldsymbol{x}') \hat{v}(\boldsymbol{x}, \boldsymbol{x}') \psi_r(\boldsymbol{x}) \psi_s(\boldsymbol{x}')$$

with ψ_p being a single particle wave function, often chosen to be the eigenfunction of $\hat{h}$. Note that p also contain the spin quantum number, meaning that $\mathrm{d}\boldsymbol{x}$ implicitly contains a spin component. If $\hat{h}$ or $\hat{v}$ is spin *independent*, this simply reduces to Kronecker deltas for the spin component. It is often convenient to use antisymmetrized matrix elements, defined as

$$\langle pq||rs\rangle = \langle pq| \hat{v} |rs\rangle - \langle pq| \hat{v} |sr\rangle$$

The shorthands $h_{pq} \equiv \langle p| \hat{h} |q\rangle$, $v_{rs}^{pq} \equiv \langle pq| \hat{v} |rs\rangle$ and $u_{rs}^{pq} \equiv \langle pq||rs\rangle$ will often be used. Using this formulation, a general two-body operator can be constructed

$$\hat{H} = \sum_{pq} h_{pq} \hat{p}^\dagger \hat{q} + \frac{1}{4} \sum_{pqrs} u_{rs}^{pq} \hat{p}^\dagger \hat{q}^\dagger \hat{s} \hat{r} \tag{1}$$

The simplest ground state ansatz

$$|\Phi_0\rangle = \hat{i}^\dagger \hat{j}^\dagger \ldots |0\rangle,$$

can be evaluated to calculate the simplest energy estimate using Wicks Theorem [Molinari, 2017]

$$\langle \Phi_0| \hat{H} |\Phi_0\rangle = \sum_i h_{ii} + \frac{1}{2} \sum_{ij} u_{ij}^{ij} \equiv E_{\mathrm{ref}}, \tag{2}$$

named the *reference energy*. Commonly wicks theorem is applied to Eq. 1 to pick out the Eq. 2 contribution, defining the *normal ordered* Hamiltonian

$$\hat{H} = \hat{H}_{\mathrm{N}} + E_{\mathrm{ref}} = \hat{F}_{\mathrm{N}} + \hat{V}_{\mathrm{N}} + E_{\mathrm{ref}}$$

where $\hat{F}_{\mathrm{N}}$ and $\hat{V}_{\mathrm{N}}$ is the normal ordered *Fock operator* and two body interaction respectively.

$$\hat{F}_{\mathrm{N}} = \sum_{pq} f_{pq} \{\hat{p}^\dagger \hat{q}\}, \tag{3}$$
$$\hat{V}_{\mathrm{N}} = \frac{1}{4} \sum_{pqrs} u_{rs}^{pq} \{\hat{p}^\dagger \hat{q}^\dagger \hat{s} \hat{r}\} \tag{4}$$

The operators inside the curly brackets denotes *normal ordering*. In constructing the Fock operator, the matrix elements f_{pq} are given as

$$f_{pq} = h_{pq} + \sum_i u_{qi}^{pi}$$

One major reason for doing this is the applicability of the *Generalized Wicks Theorem*, such that we only need to consider contractions between different normal ordered strings [Ferialdi and Diósi, 2021].

$$\hat{H} |\Psi\rangle = E |\Psi\rangle \tag{5}$$

2.2 Hartree-Fock

The Hartree-Fock method is one of the cheapest and most commonly applicated many-body methods. Using the reference energy equation Eq. 2, we perform a basis change to the Hartree-Fock basis, based on minimizing the ansatz expectation value. Using Greek letters to index the computational basis $\alpha, \beta \ldots$, going over all states L we can change to the Hartree-Fock basis using

$$|p\rangle = \sum_{\alpha} C_{\alpha p} |\alpha\rangle$$

where $C_{\alpha p}$ are the basis coefficients. Assuming the

We are however not free to choose an arbitrary transformation, since we re

$$\mathcal{L} = \langle \Phi_0 | \hat{H} | \Phi_0 \rangle + \sum_i \epsilon_{i\alpha} (\delta_{ij} - \langle i | j \rangle) \tag{6}$$

2.3 Coupled Cluster

The exact solution $|\Psi\rangle$ is approximated by an exponential ansatz $|\Psi_{\mathrm{CC}}\rangle$

$$|\Psi\rangle \approx |\Psi_{\mathrm{CC}}\rangle \equiv e^{\hat{T}} |\Phi_0\rangle . \tag{7}$$

The operators $\hat{T} = \hat{T}_1 + \hat{T}_2 + \ldots$ acting on the ground state ansatz $|\Phi_0\rangle$ are the so-called *cluster operators* defined as

$$\hat{T}_m = \frac{1}{(m!)^2} \sum_{\substack{ab\ldots \\ ij\ldots}} t_{ij\ldots}^{ab\ldots} \{\hat{a}^\dagger \hat{i} \hat{b}^\dagger \hat{j} \ldots\} \tag{8}$$

where $m \leq N$. The scalars $t_{ij\ldots}^{ab\ldots}$ are unknown expansion coefficients called *amplitudes*, which we need to solve for. All the creation and annihilation operators of Eq. 8 anticommute, giving the restriction that

$$t_{\hat{P}'(ij\ldots)}^{\hat{P}(ab\ldots)} = (-1)^{\sigma(\hat{P})+\sigma(\hat{P}')} t_{ij\ldots}^{ab\ldots}. \tag{9}$$

Here P and P' permutes $\sigma(P)$ and $\sigma(P')$ indices respectively. This is the reason for the prefactor of Eq. 8, since we have $m!$ ways to independently permute particle and hole indices. Instead of having $(L-N)^m N^m$ independent unknowns, we reduce this number by a factor of $(m!)^2$.

2.4 Doubles truncation

Considering N cluster operators in the exponential ansatz of Eq. 7 is not computationally feasible for realistic systems. The common practice is to include one or more $\hat{T}_m$ operators, making a truncation on $|\Psi_{\mathrm{CC}}\rangle$ as well. In the following we will include only the double excitation operator $\hat{T}_2$, know as the CCD approximation. This gives us

$$|\Psi\rangle \approx |\Psi_{\mathrm{CC}}\rangle \approx |\Psi_{\mathrm{CCD}}\rangle \equiv e^{\hat{T}_2} |\Phi_0\rangle , \tag{10}$$

$$\hat{T}_2 = \frac{1}{4} \sum_{abij} t_{ij}^{ab} \{\hat{a}^\dagger \hat{i} \hat{b}^\dagger \hat{j}\}, \tag{11}$$

with the four-fold amplitude permutation symmetry [1],

$$t_{ij}^{ab} = -t_{ij}^{ba} = -t_{ji}^{ab} = t_{ji}^{ba}. \tag{12}$$

Incorporating the CCD approximation in the Schrödinger equation (Eq. 5), we see that

$$\begin{aligned} \hat{H} e^{\hat{T}_2} |\Phi_0\rangle &= E e^{\hat{T}_2} |\Phi_0\rangle , \\ \hat{H}_{\mathrm{N}} e^{\hat{T}_2} |\Phi_0\rangle &= \Delta E_{\mathrm{CCD}} e^{\hat{T}_2} |\Phi_0\rangle , \end{aligned} \tag{13}$$

where $\Delta E_{\mathrm{CCD}} = E - E_{\mathrm{ref}}$. Expanding both sides and taking the inner product with $\langle \Phi_0 |$, we in principle get an equation for the energy. However, this approach is not amenable to practical computer implementation [Bartlett et al., 1984] since the amplitude equation will be coupled with the energy equation. Therefor, we rather apply a similarity transform to Eq. 13 by multiplying by the inverse of $e^{\hat{T}_2}$,

$$\begin{aligned} e^{-\hat{T}_2} \hat{H}_{\mathrm{N}} e^{-\hat{T}_2} |\Phi_0\rangle &= \Delta E_{\mathrm{CCD}} |\Phi_0\rangle \\ \overline{H} |\Phi_0\rangle &= \Delta E_{\mathrm{CCD}} |\Phi_0\rangle \end{aligned} \tag{14}$$

where $\overline{H} = e^{-\hat{T}_2} \hat{H}_{\mathrm{N}} e^{-\hat{T}_2}$ is the similarity transformed Hamiltonian. Using this reformulated eigenvalue problem, we can perform the inner product with different states to calculate both ΔE_{CCD} and t_{ij}^{ab}. Considering $\langle \Phi_0 |$ we get

$$\langle \Phi_0 | \overline{H} | \Phi_0 \rangle = \Delta E_{\mathrm{CCD}}, \tag{15}$$

named the *energy equation*. Considering excited states, we arrive at the *amplitude equations*

$$\left\langle \Phi_{ij\ldots}^{ab\ldots} \right| \overline{H} |\Phi_0\rangle , \tag{16}$$

used for finding the unknown amplitudes t_{ij}^{ab}. To find explicit expressions for Eq. 15 and Eq. 16, we expand $\overline{H}$ using the Hausdorff expansion

$$\overline{H} = \hat{H}_{\mathrm{N}} + \left[\hat{H}_{\mathrm{N}}, \hat{T}_2\right] + \frac{1}{2!}\left[\left[\hat{H}_{\mathrm{N}}, \hat{T}_2\right], \hat{T}_2\right].$$

The truncation at the two-fold commutator comes from the fact that we have a two-body interaction. When evaluated with a doubly excited state, at least one creation or annihilation operator from each of the cluster operators has to be contracted with $\hat{H}_{\mathrm{N}}$. Therefor having

[1] For double amplitudes, the index permutation symmetry is equal to that of antisymmetrized two-body matrix elements $\langle pq || rs \rangle$.

eight creation and annihilation operators in two $\hat{T}_2$, four of them can be contracted with the four from $\hat{H}_\mathrm{N}$, while the other four with the operators from $\left\langle \Phi_{ij}^{ab} \right|$. This gives the CCD amplitude equation calculated from

$$\left\langle \Phi_{ij}^{ab} \right| \overline{H} \left| \Phi_0 \right\rangle = 0 \tag{17}$$

To make practical use of Eq. 15 and Eq. 17

3 Method

Define

$$R \tag{18}$$

4 Results

5 Discussion

6 Concluding remarks

References

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