# MATH 417 Homework 10

Note that this is problems: Section 15.3 #1,2,3. Section 16.1 #7,8,9

1. Suppose that the function $\psi : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable. Define the function $g : \mathbb{R}^2 \to \mathbb{R}$ by

$$g(s,t) = \psi(s^2 t, s) \qquad \text{for } (s,t) \in \mathbb{R}^2.$$

Find $\frac{\partial g}{\partial s}(s,t)$ and $\frac{\partial g}{\partial t}(s,t)$.

***Solution***:

Define $F : \mathbb{R}^2 \to \mathbb{R}^2$ by $F(s,t) = (s^2 t, s)$ and to keep notation clear, write $\psi(x,y)$. Then $g = (\psi \circ F)$ with $F_1(s,t) = x$ and $F_2(s,t) = y$. In particular, observe that $F_1(s,t) = s^2 t$ and $F_2(s,t) = s$. Then we have

$$\frac{\partial F_1}{\partial s} = 2st,$$
$$\frac{\partial F_1}{\partial t} = s^2,$$
$$\frac{\partial F_2}{\partial s} = 1,$$
$$\frac{\partial F_2}{\partial t} = 0.$$

Thus, the chain rule gives us that

$$\frac{\partial g}{\partial s}(s,t) = \frac{\partial \psi}{\partial x}(s^2 t, s)(2st) + \frac{\partial \psi}{\partial y}(s^2 t, s),$$

$$\frac{\partial g}{\partial t}(s,t) = \frac{\partial \psi}{\partial x}(s^2 t, s)s^2.$$

2. Suppose that the function $h : \mathbb{R}^3 \to \mathbb{R}$ is continuously differentiable. Define the function $\eta : \mathbb{R}^3 \to \mathbb{R}$ by

$$\eta(u,v,w) = (3u+2v)h(u^2, v^2, uvw) \qquad \text{for all } (u,v,w) \in \mathbb{R}^3.$$

Find $D_1\eta(u,v,w)$, $D_2\eta(u,v,w)$, $D_3\eta(u,v,w)$.

***Solution***: Define $F : \mathbb{R}^3 \to \mathbb{R}^3$ as

$$F(u,v,w) = (u^2, v^2, uvw)$$

and to keep notation clear, write $h(x,y,z)$. Then $F_1(u,v,w) = u^2 = x$, $F_2(u,v,w) = v^2 = y$, and $F_3(u,v,w) = uvw = z$. Thus we have

$$\frac{\partial F_1}{\partial u} = 2u, \qquad \frac{\partial F_1}{\partial v} = 0, \qquad \frac{\partial F_1}{\partial w} = 0,$$
$$\frac{\partial F_2}{\partial u} = 0, \qquad \frac{\partial F_2}{\partial v} = 2v, \qquad \frac{\partial F_2}{\partial w} = 0,$$
$$\frac{\partial F_3}{\partial u} = vw, \qquad \frac{\partial F_3}{\partial v} = uw, \qquad \frac{\partial F_3}{\partial w} = uv.$$

Then the chain rule gives

$$D_1\eta(u,v,w) = 3h(u^2, v^2, uvw) + (3u+2v)\Big(D_1 h(F(u,v,w))2u + D_3 h(F(u,v,w))vw\Big),$$

$$D_2\eta(u,v,w) = 2h(u^2, v^2, uvw) + (3u+2v)\Big(D_2 h(F(u,v,w))2v + D_3 h(F(u,v,w))uw\Big),$$

$$D_3\eta(u,v,w) = (3u+2v)D_3 h(F(u,v,w))uv.$$

3. Suppose that the functions $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ have continuous second-order partial derivatives. Define the function $u : \mathbb{R}^2 \to \mathbb{R}$ by

$$u(s,t) = g(s-t) + h(s+t) \qquad \text{for all } (s,t) \in \mathbb{R}^2.$$

Prove that

$$\frac{\partial^2 u}{\partial t^2}(s,t) - \frac{\partial^2 u}{\partial s^2}(s,t) = 0 \qquad \text{for all } (s,t) \in \mathbb{R}^2.$$

***Solution***: We can actually use the chain rule for functions of single variable to prove this. We calculate,

$$\frac{\partial^2 g}{\partial t^2} = g''(s-t),$$

$$\frac{\partial^2 g}{\partial s^2} = g''(s-t),$$

$$\frac{\partial^2 h}{\partial t^2} = h''(s+t),$$

$$\frac{\partial^2 h}{\partial t^2} = h''(s+t).$$

Since the operators $\frac{\partial^2}{\partial t}$ and $\frac{\partial^2}{\partial t}$ are linear, then

$$\frac{\partial^2 u}{\partial t^2}(s,t) - \frac{\partial^2 u}{\partial s^2}(s,t) = \Big(g''(s-t) + h''(s+t)\Big) - \Big(g''(s-t) + h''(s+t)\Big) = 0$$

as desired.

7. Let $\mathcal{O}$ and $V$ be open subsets of $\mathbb{R}$ and suppose that the differentiable function $f : \mathcal{O} \to V$ is one-to-one and onto. Suppose that $x_0$ is a point in $\mathcal{O}$ at which $f'(x_0) = 0$. Show that the inverse function $f^{-1} : V \to \mathbb{R}$ cannot be differentiable at the point $f(x_0)$.

***Solution***: We use proof by contradiction. Suppose that $f^{-1}$ is differentiable. Then it follows that we can use implicit differentiation to obtain

$$\frac{df^{-1}}{dx}(f(x))\frac{df}{dx}(x) = 1$$

from

$$f^{-1}(f(x)) = x.$$

But since $\frac{df}{dx}(x) = 0$, this then tells us that $1 = 0$, a contradiction.

8. For each of the following mappings $F : \mathbb{R}^2 \to \mathbb{R}^2$, apply the Inverse Function Theorem at the point $(x_0, y_0) = (0,0)$ and calculate the partial derivatives of the components of the inverse mapping at the point $(u_0, v_0) = F(0,0)$:

   (a) $F(x,y) = (x + x^2 + e^{x^2y^2}, -x - y + \sin(xy))$
   (b) $F(x,y) = (e^{x+y}, e^{x-y})$.

   ***Solution***:

   (a) It is clear that $F$ is continuously differentiable. The derivative matrix for $F$ is

   $$DF(x,y) = \begin{bmatrix} 1 + 2x + 2xy^2e^{x^2y^2} & 2yx^2e^{x^2y^2} \\ -1 + y\cos(xy) & -1 + x\cos(xy) \end{bmatrix}.$$

   Then at $(x,y) = (0,0)$ we have

   $$DF(0,0) = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

   and $F(0,0) = (1,0)$. This matrix is invertible as its determinant is $-1$. The inverse of this matrix is,

   $$[DF(0,0)]^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}.$$

Therefore, if we write the inverse as $F^{-1} = (G_1(u,v), G_2(u,v))$ then we can calculate the partial derivatives of $G_1$ and $G_2$ at $(1,0)$ to be

$$\frac{\partial G_1}{\partial u}(1,0) = 1 \qquad \text{and} \qquad \frac{\partial G_1}{\partial v}(1,0) = 0$$

$$\frac{\partial G_2}{\partial u}(1,0) = -1 \qquad \text{and} \qquad \frac{\partial G_2}{\partial v}(1,0) = -1.$$

(b) It is clear that $F$ is continuously differentiable. The derivative matrix for $F$ is

$$DF(x,y) = \begin{bmatrix} e^{x+y} & e^{x+y} \\ e^{x-y} & -e^{x-y} \end{bmatrix}.$$

Then at $(x,y) = (0,0)$ we have

$$DF(0,0) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

and $F(0,0) = (1,1)$. This matrix is invertible as its determinant is $-2$. The inverse of this matrix is,

$$[DF(0,0)]^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

Therefore, if we write the inverse as $F^{-1} = (G_1(u,v), G_2(u,v))$ then we can calculate the partial derivatives of $G_1$ and $G_2$ at $(1,1)$ to be

$$\frac{\partial G_1}{\partial u} = \frac{1}{2} \qquad \text{and} \qquad \frac{\partial G_1}{\partial v} = \frac{1}{2}$$

$$\frac{\partial G_2}{\partial u} = \frac{1}{2} \qquad \text{and} \qquad \frac{\partial G_2}{\partial v} = -\frac{1}{2}.$$

9. Define the function $F : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(x,y) = (e^x \cos(y), e^x \sin(y)).$$

(a) Show that the Inverse Function Theorem is applicable at every point $(x_0, y_0)$ in the plane $\mathbb{R}^2$.

(b) Show that the function $F$ is not one-to-one.

(c) Does $(b)$ contradict $(a)$.

***Solution***:

(a) The derivative matrix for this function is

$$DF(x,y) = \begin{bmatrix} e^x \cos(y) & -e^x \sin(y) \\ e^x \sin(y) & e^x \cos(y) \end{bmatrix}.$$

The determinant of this matrix is $e^x(\cos^2(y) + \sin^2(y)) = e^x > 0$, and hence this matrix is invertible for any point $(x_0, y_0) \in \mathbb{R}^2$.

(b) On the other hand, $F$ is not one-to-one. For example, $F(0,0) = F(0, 2\pi)$.

(c) This is not a contradiction because the Inverse Function Theorem only tells us that a function is one-to-one locally in some neighborhood (potentially very small) of the function that we are interested in.