The influence of the Kirillov-Reshetikhin crystal $B^{1,1}$ on the structure of simple cyclotomic KLR modules, arXiv:1508.04182

Henry Kvinge, UC Davis

(Joint with Monica Vazirani)

University of Oregon Algebra Seminar

$$\begin{tikzcd} \boxed{0} \arrow[r, "1"] & \boxed{1} \arrow[r, "2"] & \cdots \arrow[r, "\ell\text{-}1"] & \boxed{\ell\text{-}1} \arrow[r, "\ell"] & \boxed{\ell} \arrow[llll, bend right, "0"'] \end{tikzcd}$$

Reminder on crystals

Let $U_q(\mathfrak{g})$ be the quantum group associated to Kac-Moody algebra $\mathfrak{g}$ with Dynkin indexing set I.

A *crystal* is a combinatorial object that we can attach to certain $U_q(\mathfrak{g})$ representations V:

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vertices	$\sim$	weight spaces of V
i-directed edges	$\sim$	action of $\widetilde{f}_i$ between weight spaces

Example

Representations of $U_q(\mathfrak{sl}_2)$ already look like directed graphs,

$$V(2),\ \dim(V(2)) = 3$$

$$\bullet \xrightarrow{f} \bullet \xrightarrow{f} \bullet$$

$$V(3),\ \dim(V(3)) = 4$$

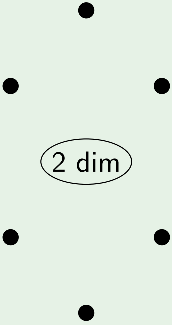
$$\bullet \xrightarrow{f} \bullet \xrightarrow{f} \bullet \xrightarrow{f} \bullet$$

$$V(4),\ \dim(V(4)) = 5$$

$$\bullet \xrightarrow{f} \bullet \xrightarrow{f} \bullet \xrightarrow{f} \bullet \xrightarrow{f} \bullet$$

Example

Adjoint representation V for $\mathfrak{sl}_3$:



- V has six 1-dimensional weight spaces, one 2-dimensional weight space.

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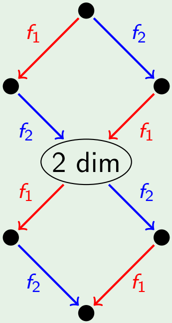

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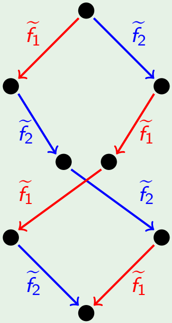


- V has six 1-dimensional weight spaces, one 2-dimensional weight space.
- f_1 and f_2 map between weight spaces.
- If we use $U_q(\mathfrak{sl}_3)$ and "rescale" operators f_i to $\widetilde{f_i}$, then "at $q = 0$" can find basis so that representation behaves like $\{1, 2\}$-colored graph.

Tensor Product of Crystals

Example

Crystals $B(2)$ and $B(3)$ associated to $\mathfrak{sl}_2$ representations $V(2)$ and $V(3)$

$B(2)$

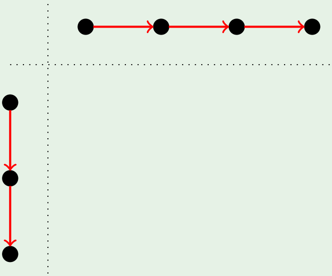
and

$B(3)$

Tensor Product of Crystals

Example

Then crystal $B(2) \otimes B(3)$ associated to $V(2) \otimes V(3)$ is



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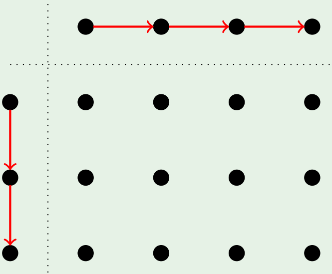

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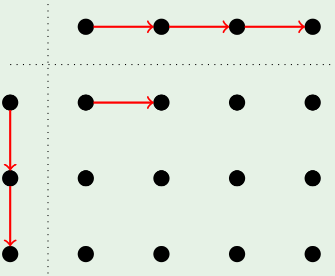

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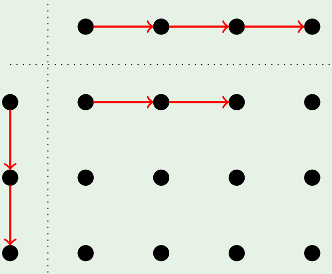

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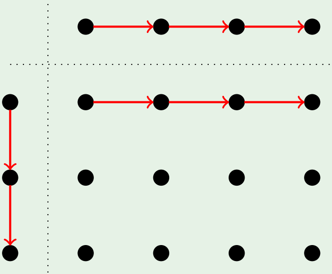

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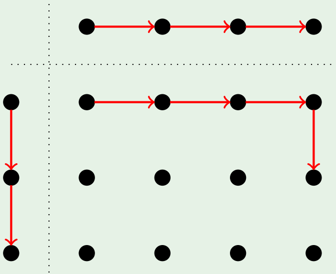

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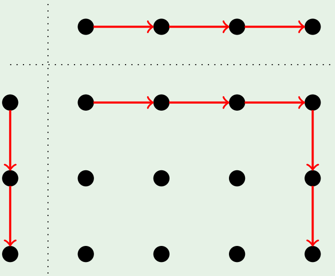

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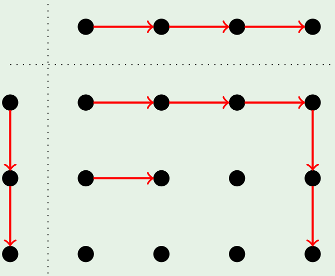

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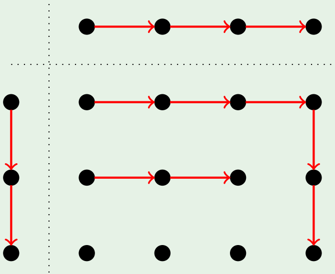

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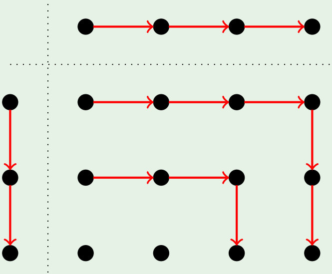

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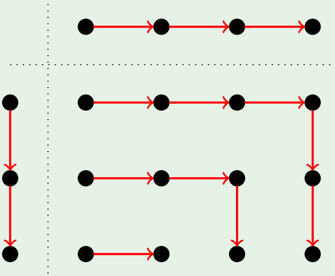

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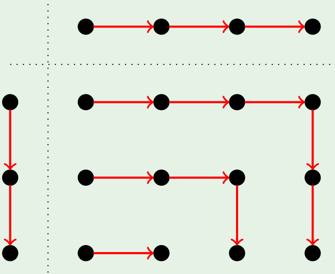

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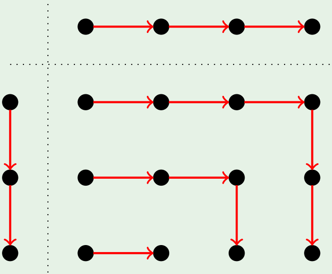


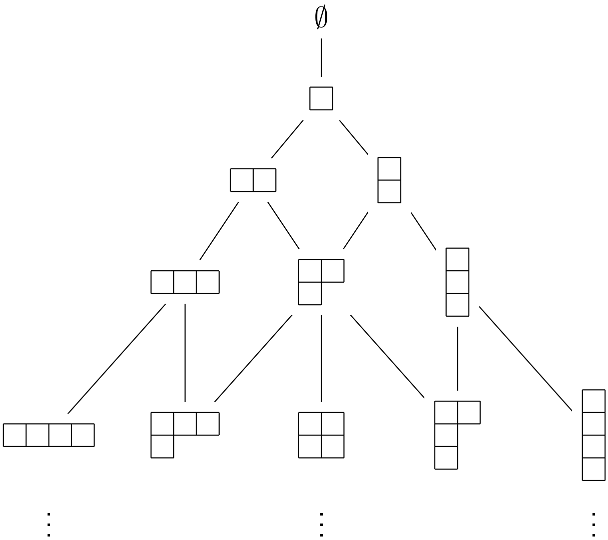
$$B(2) \otimes B(3) \cong B(1) \oplus B(3) \oplus B(5)$$

and

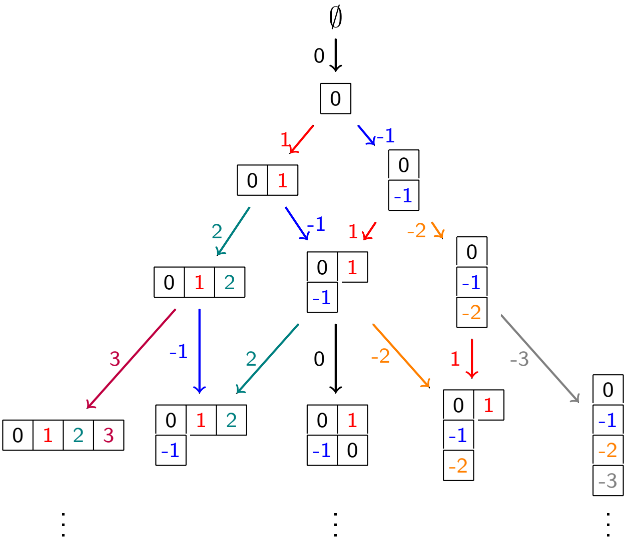
$$V(2) \otimes V(3) \cong V(1) \oplus V(3) \oplus V(5)$$

There are many useful combinatorial models for crystals...

Young's lattice of partitions



Young's lattice as a directed graph



Rich in connections to representation theory

Young's lattice as a directed graph:

Gives branching rule for symmetric groups:

- partitions in row n are simple $\mathcal{S}_n$ representations
- directed, i-colored edges correspond to induction (and restriction) functors

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Model for crystal $B(\Lambda_0)$ in type $\mathfrak{sl}_\infty$:

- partitions are nodes
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Model for crystal $B(\Lambda_0)$ in type $\mathfrak{sl}_\infty$:

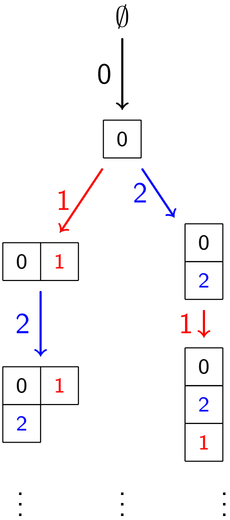
- partitions are nodes
- i-arrows correspond to $\widetilde{f}_i$ ($i \in I = \mathbb{Z}$)

Can we say something similar for $\widehat{\mathfrak{sl}_{\ell+1}}$ (type $A_\ell^{(1)}$)?

Example: $B(\Lambda_0)$ in type $A_2^{(1)}$ ($\widehat{\mathfrak{sl}_3}$)

Similar model but...

- Nodes are now 3-restricted partitions
- Gives partial branching for:
  - Symmetric group algebras $\mathbb{F}_3 \mathcal{S}_n$,
  - Cyclotomic Hecke algebras $H_n^{\Lambda_0}$,
  - Type $A_2^{(1)}$ cyclotomic KLR algebra R^{Λ_0}.



$B(\Lambda_0)$ for type $A_\ell^{(1)}$

Model of $B(\Lambda_0)$ has

- Nodes are now $(\ell+1)$-restricted partitions
- Gives partial branching for:

–Symmetric group algebras $\mathbb{F}_{\ell+1}\mathcal{S}_n$, (when $\ell+1$ prime)

–Cyclotomic Hecke algebras $H_n^{\Lambda_0}$, (with q an $\ell+1$ root of unity)

–Type $A_\ell^{(1)}$ cyclotomic KLR algebra R^{Λ_0}.

I will work today in language of cyclotomic Khovanov-Lauda-Rouquier (KLR) algebras (or cyclotomic quiver Hecke algebras).

...But results hold for both $H_n^{\Lambda_i}$ and $\mathbb{F}_{\ell+1}\mathcal{S}_n$.

Brief review of KLR algebras

Khovanov-Lauda and independently Rouquier invented associative, graded algebra R attached to any symmetrizable Cartan matrix A.

- R categorifies lower part of quantum group $U_q(\mathfrak{g})$.
- Twisted $\mathbb{Q}(q)$-bialgebra isomorphism,

$$U_q^-(\mathfrak{g}) \xrightarrow{\cong} \mathbb{Q}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} K_0(R\text{ - pmod})$$

  where R - pmod is category of finitely-generated, graded, projective R-modules.

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- $A = [a_{ij}]$, Cartan matrix for classical affine type X_ℓ,
- $I = \{0, 1, \ldots, \ell\}$ is Dynkin indexing set for A,
- $\{\alpha_i\}_{i \in I}$ are simple roots and positive root lattice is

$$Q^+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i,$$

Brief review of KLR algebras

- for $\nu \in Q^+$, $\nu = \sum_{i \in I} c_i \alpha_i$, set

$$\mathrm{ht}(\nu) = \sum_{i \in I} c_i, \tag{1}$$

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$$\mathrm{ht}(\nu) = \sum_{i \in I} c_i, \tag{1}$$

- the set $\mathrm{Seq}(\nu)$ contains all ordered sequences of elements of I such that i appears c_i times.

Example

For $1, 2 \in I$, $\mathrm{ht}(\alpha_1 + 2\alpha_2) = 3$, and

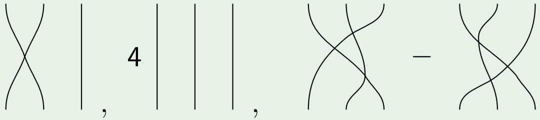
$$\mathrm{Seq}(\alpha_1 + 2\alpha_2) = \{(122), (212), (221)\}.$$

Brief review of KLR algebras

For $\nu \in Q^+$, algebra $R(\nu)$ can be presented by $\mathbb{C}$-linear combinations of braid-like planar diagrams with interacting strings:

Example

Some elements in $R(\alpha_1 + 2\alpha_2)$ are:



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- Between $\text{ht}(\nu)$ points on top and $\text{ht}(\nu)$ points on bottom,

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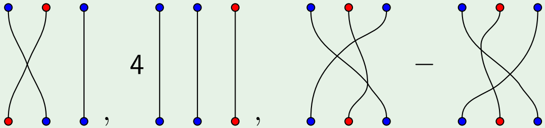

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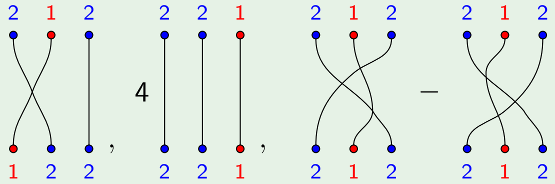

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- Between $\mathrm{ht}(\nu)$ points on top and $\mathrm{ht}(\nu)$ points on bottom, labelled by elements of $\mathrm{Seq}(\nu)$.
- Can add beads to strings.

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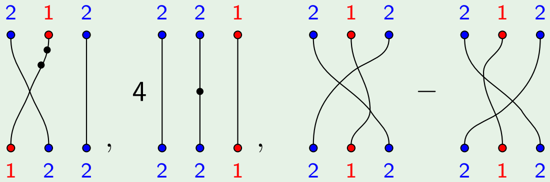

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- Between $\mathrm{ht}(\nu)$ points on top and $\mathrm{ht}(\nu)$ points on bottom, labelled by elements of $\mathrm{Seq}(\nu)$.
- Can add beads to strings.
- Modulo local relations.

Example

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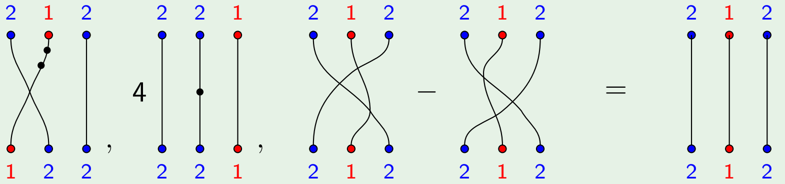
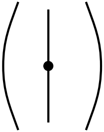
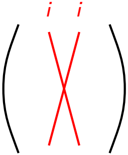
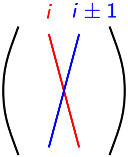


Brief review of KLR algebras

Grading is given by:

deg  $= 2$, deg 

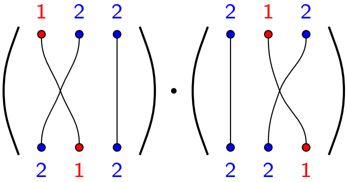
$= -2$,

deg 

$= 1$,

Brief review of KLR algebras

Multiplication is given my placing first diagram above second diagram,



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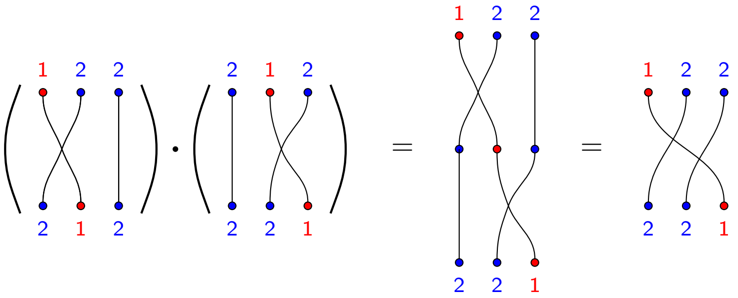
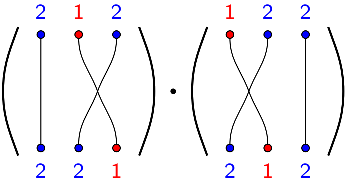


Brief review of KLR algebras

When labels of bottom of first diagram and top of second diagram do not agree, product is zero,



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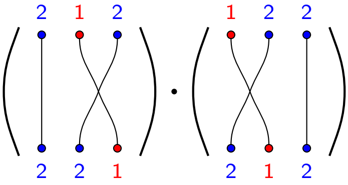
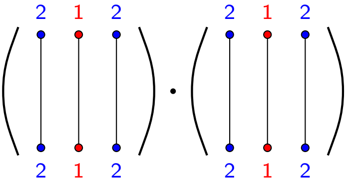


$$= \mathbf{0}$$

Brief review of KLR algebras

The elements of $R(\nu)$ in which no strings cross and which have no beads are idempotents



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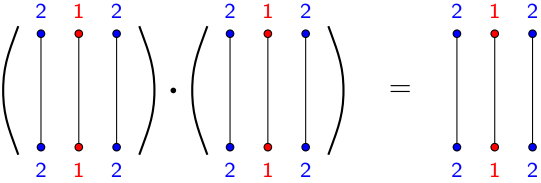
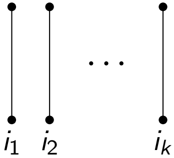


Brief review of KLR algebras

For $\underline{i} = (i_1, i_2, \ldots, i_k) \in \mathrm{Seq}(\nu)$ we write

$1_{\underline{i}} =$



Brief review of KLR algebras

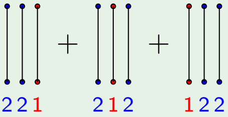
$R(\nu)$ has identity,

$$1 = \sum_{\underline{i} \in \mathrm{Seq}(\nu)} 1_{\underline{i}} \tag{2}$$

Example

For $\nu = \alpha_1 + 2\alpha_2$

$$1 = 1_{221} + 1_{212} + 1_{122} =$$



Brief review of KLR algebras

If M is an $R(\nu)$-module, we can decompose it into *weight spaces*,

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$$M = \bigoplus_{\underline{i} \in \mathrm{Seq}(\nu)} 1_{\underline{i}} M$$

Because $R(\nu)$ is graded, we can take the *graded dimension* of each component, $\mathrm{gdim}(1_{\underline{i}} M) \in \mathbb{N}[q, q^{-1}]$. The *character* is defined as

$$\mathrm{Char}(M) = \sum_{\underline{i} \in \mathrm{Seq}(\nu)} \mathrm{gdim}(1_{\underline{i}} M)[\underline{i}].$$

Brief review of KLR algebras

Example

For type $A_2^{(1)}$, $1 \in I = \{0, 1, 2\}$, $R(2\alpha_1)$ has exactly one simple representation, $L(1^2)$,

$$\mathrm{Char}(L(1^2)) = (1 + q^{-2})[1\ 1] = q[2]_q![1\ 1].$$

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$R(3\alpha_1)$ has exactly one simple representation, $L(1^3)$ with

$$\mathrm{Char}(L(1^3)) = (1 + q^{-2} + q^{-4})(1 + q^{-2})[1\ 1\ 1]$$

$$= q^{-3}[3]_q![1\ 1\ 1]$$

Brief review of KLR algebras

Example

In type $A_2^{(1)}$, $\nu = \alpha_1 + 2\alpha_2$ there are 2 simple representations M_1 and M_2 with characters:

$$\mathrm{Char}(M_1) = (1 + q^{-2})[221] + [212]$$

$$\mathrm{Char}(M_2) = (1 + q^{-2})[122] + [212]$$

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For each integral dominant weight Λ, R has a finite-dimensional quotient R^{Λ}, called the *cyclotomic KLR algebra*.

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Two key points for this presentation:

- Simple R^{Λ}-modules carry structure of $B(\Lambda)$.
- Simple R-modules carry structure of $B(\infty)$.

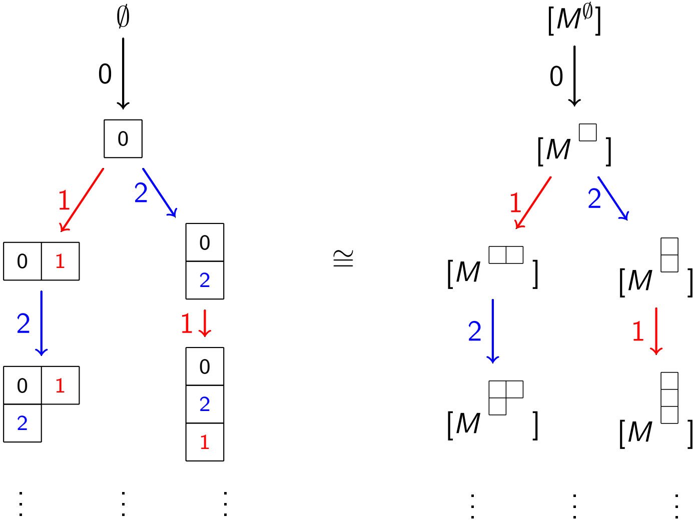
Brief review of KLR algebras

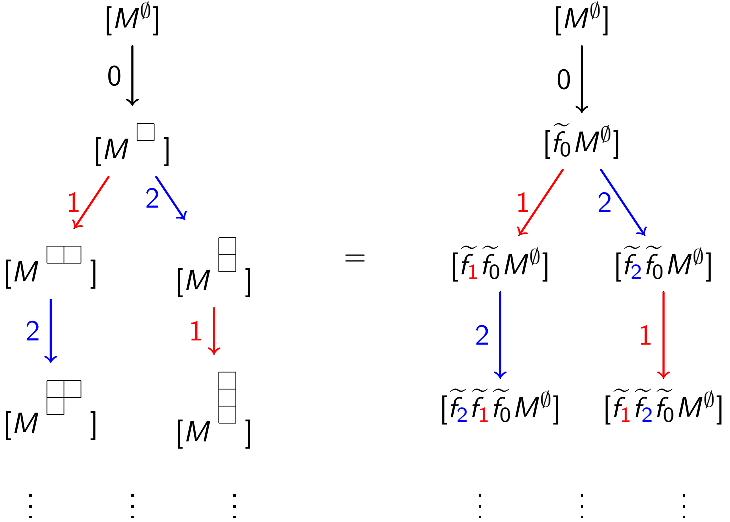
Crystal models:

	$B(\Lambda)$	$B(\infty)$
Nodes	simple R^Λ-modules	simple R-modules
Arrows, $\widetilde{f}_i$	refined induction functors	refined induction functors

For simple $M, N \in R\text{-mod}$ (or $M, N \in R^\Lambda\text{-mod}$),

$$[M] \xrightarrow{i} [N] \qquad \Leftrightarrow \qquad \widetilde{f}_i M \cong N.$$


$\emptyset$
$[M^{\emptyset}]$
$\cong$


$[M^{\emptyset}]$
0
1
2
=
$[M^{\emptyset}]$
$[\widetilde{f}_0 M^{\emptyset}]$
$[\widetilde{f}_1 \widetilde{f}_0 M^{\emptyset}]$
$[\widetilde{f}_2 \widetilde{f}_0 M^{\emptyset}]$
$[\widetilde{f}_2 \widetilde{f}_1 \widetilde{f}_0 M^{\emptyset}]$
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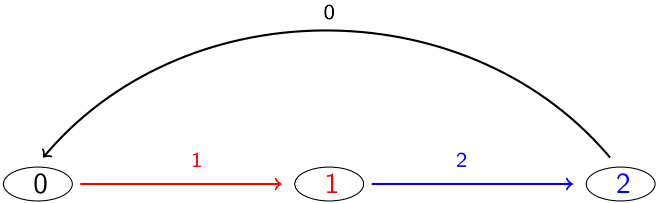
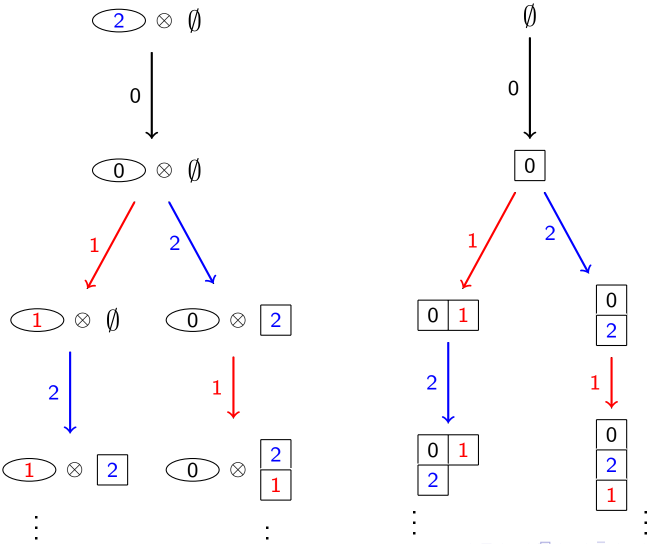
A crystal isomorphism

Kirillov-Reshetikhin crystal $B^{1,1}$ in type $A_2^{(1)}$



Compare

$B^{1,1} \otimes B(\Lambda_2)$ $\qquad\qquad$ $B(\Lambda_0)$



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$$B^{1,1} \otimes B(\Lambda_2) \cong B(\Lambda_0)$$

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- How to define this isomorphism in terms of Young diagram model for $B(\Lambda_0)$?

Notice,

$$1 \otimes \boxed{2} \quad \mapsto \quad \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 2 & \\ \cline{1-1}\end{array}$$

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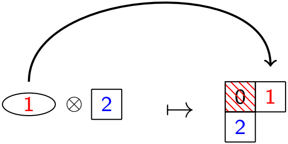

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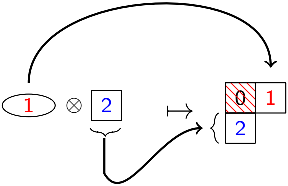

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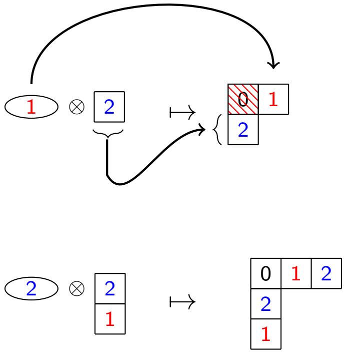

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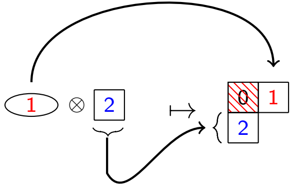

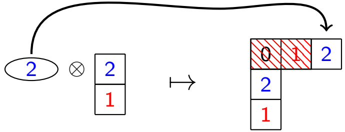

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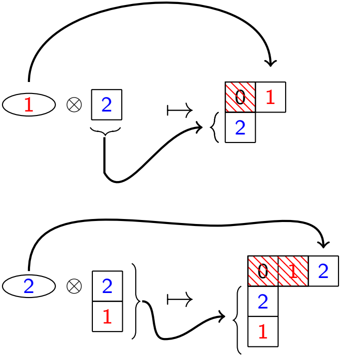

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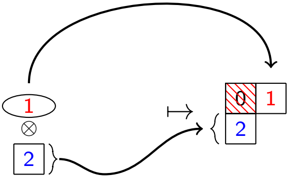
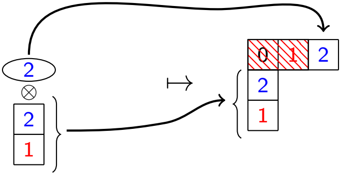
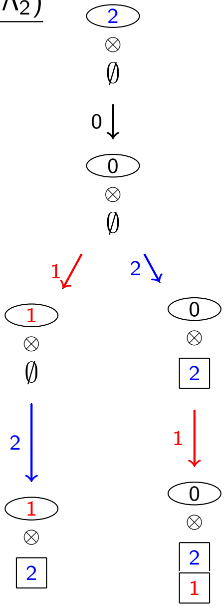
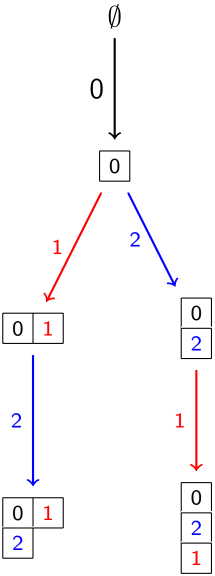




$B^{1,1} \otimes B(\Lambda_2)$



$B(\Lambda_0)$



Because diagrams are $(\ell+1)$-restricted, this map is a well-defined bijection.

More generally

- $B^{1,1}$ in type $A_\ell^{(1)}$ is an example of a perfect crystal of level 1 (also a Kirillov-Reshetikhin crystal).

$$\begin{tikzcd} \boxed{0} \arrow[r, "1"] & \boxed{1} \arrow[r, "2"] & \cdots \arrow[r, "\ell\text{-}1"] & \boxed{\ell\text{-}1} \arrow[r, "\ell"] & \boxed{\ell} \arrow[llll, bend right, "0"'] \end{tikzcd}$$

- There is a crystal isomorphism

$$B^{1,1} \otimes B(\Lambda_{i-1}) \xrightarrow{\sim} B(\Lambda_i).$$

- $B(\Lambda_i)$ is complicated, but $B^{1,1}$ is easy to understand.
- Crystals behave nicely under tensor products. If we understand crystals B_1, B_2, it is easy to understand $B_1 \otimes B_2$.

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Study $B(\Lambda_0)$ by iterating isomorphism:

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Main Question for this talk: Does the crystal isomorphism

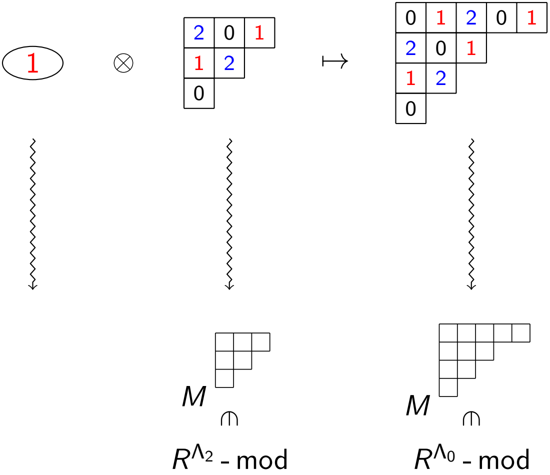
$$B^{1,1} \otimes B(\Lambda_{i-1}) \xrightarrow{\sim} B(\Lambda_i)$$

have a higher module-theoretic analogue for representation theory of KLR algebras?

For case $A_2^{(1)}$, $\Lambda_i = \Lambda_0$, what is R-module analogue of this?...

$$1 \quad \otimes \quad \begin{array}{|c|c|c|} \hline 2 & 0 & 1 \\ \hline 1 & 2 & \\ \cline{1-2} 0 & & \\ \cline{1-1} \end{array} \quad \mapsto \quad \begin{array}{|c|c|c|c|c|} \hline 0 & 1 & 2 & 0 & 1 \\ \hline 2 & 0 & 1 & & \\ \cline{1-3} 1 & 2 & & & \\ \cline{1-2} 0 & & & & \\ \cline{1-1} \end{array}$$

Should be the R^{Λ_0}-module homomorphism,



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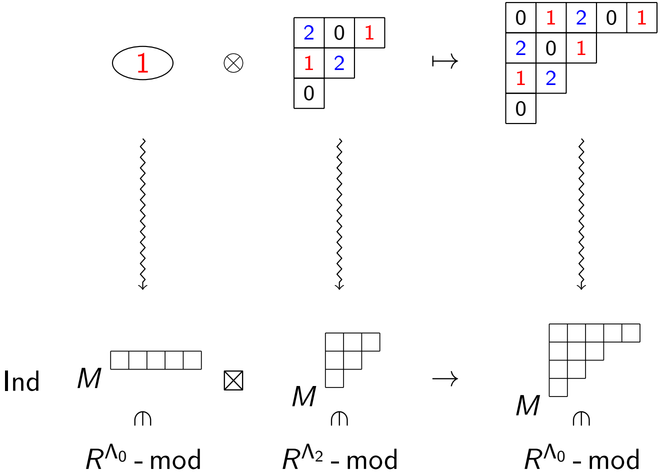

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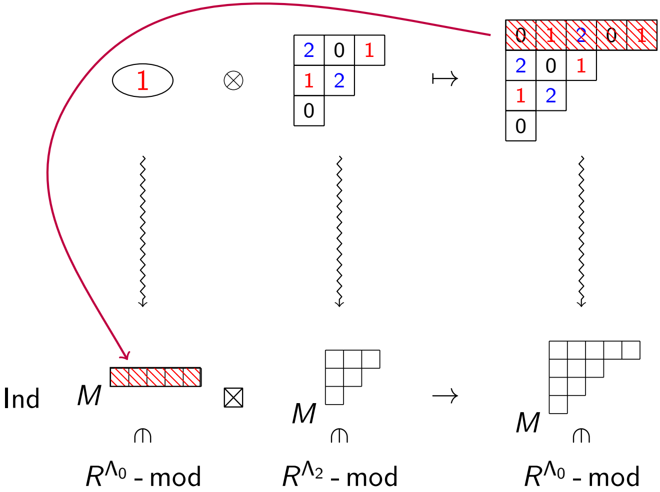

As with $\mathcal{S}_5$:

$$M^{\square\square\square\square\square} \cong \text{1-dimensional ``trivial'' } R^{\Lambda_0}\text{-module}$$

and

$$\mathrm{Char}(M^{\square\square\square\square\square}) = [0\ 1\ 2\ 0\ 1].$$

Easiest possible representation to work with!

But crystals are about much more than nodes.

Since

$$1 \otimes \begin{array}{ccc} 2 & 0 & 1 \\ 1 & 2 & \\ 0 & & \end{array} \mapsto \begin{array}{ccccc} 0 & 1 & 2 & 0 & 1 \\ 2 & 0 & 1 & & \\ 1 & 2 & & & \\ 0 & & & & \end{array}$$

tensor product rule for crystals gives

$$\widetilde{f}_1 \left(1 \otimes \begin{array}{ccc} 2 & 0 & 1 \\ 1 & 2 & \\ 0 & & \end{array} \right) = 1 \otimes \widetilde{f}_1 \left(\begin{array}{ccc} 2 & 0 & 1 \\ 1 & 2 & \\ 0 & & \end{array} \right) \mapsto \widetilde{f}_1 \left(\begin{array}{ccccc} 0 & 1 & 2 & 0 & 1 \\ 2 & 0 & 1 & & \\ 1 & 2 & & & \\ 0 & & & & \end{array} \right)$$

With

$$\textcircled{1} \quad \otimes \quad \widetilde{f}_1\left(\begin{ytableau} 2 & 0 & 1 \\ 1 & 2 \\ 0 \end{ytableau} \right) \quad \mapsto \quad \widetilde{f}_1\left(\begin{ytableau} 0 & 1 & 2 & 0 & 1 \\ 2 & 0 & 1 \\ 1 & 2 \\ 0 \end{ytableau} \right)$$

should also have

$$\operatorname{Ind} M^{\ydiagram{5}} \boxtimes \widetilde{f}_1\left(M^{\ydiagram{3,2,1}}\right) \quad \to \quad \widetilde{f}_1\left(M^{\ydiagram{5,3,2,1}}\right)$$

Given $b_1 \in B(\Lambda_i)$, $b_2 \in B(\Lambda_{i-1})$ with

$$\textcircled{j} \boxtimes b_2 \mapsto b_1$$

and corresponding $M^{b_1} \in R^{\Lambda_i}$-mod, $M^{b_2} \in R^{\Lambda_{i-1}}$-mod, K.-Vazirani show:

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- Action of $\widetilde{f}_i$ and $\widetilde{e}_i$ agree in module <u>and</u> crystal setting.

$B^{1,1} \otimes B(\Lambda_{i-1}) \cong B(\Lambda_i)$ **is the shadow of richer** R - **mod structure**.

Generalizing to other types

Question: How can we interpret nodes of $B^{1,1}$ without intuition from Young diagrams?

Is there another way to see that in $A_2^{(1)}$

$$1 \quad \text{corresponds to} \quad M^{\square\square\square\square\square}\,?$$

Is there another way to see that in $A_2^{(1)}$

$$\textcolor{red}{1} \quad \text{corresponds to} \quad M^{\square\square\square\square\square} \; ?$$

- $\square\square\square\square\square$ has residues

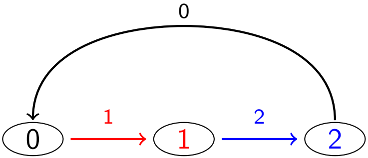
0	1	2	0	1

- the crystal $B^{1,1}$ has crystal graph

$$\begin{tikzcd} 0 \arrow[r, "1"] & 1 \arrow[r, "2"] & 2 \arrow[ll, bend right, "0"'] \end{tikzcd}$$

...Residues give directed walk in $B^{1,1}$

0	1	2	0	1



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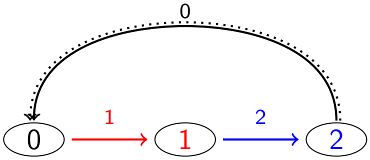

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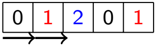

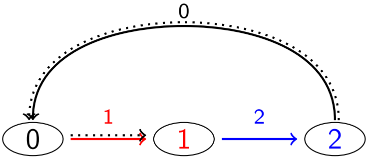

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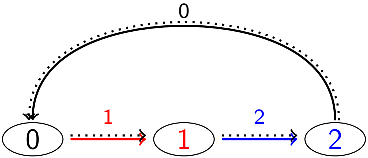

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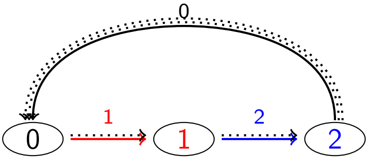

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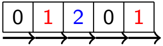

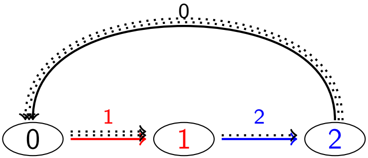

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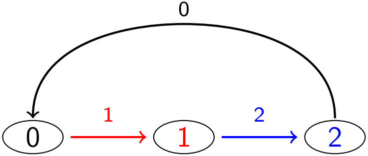

And walk describes how to build $M^{\square\square\square\square\square}$ with functors $\widetilde{f}_i$,

$$M^{\emptyset} \cong M^{\emptyset} =: \mathbb{1}$$

...Residues give directed walk in $B^{1,1}$

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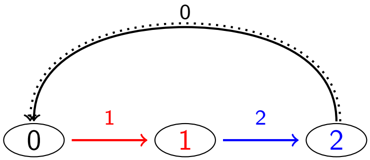

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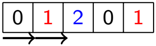

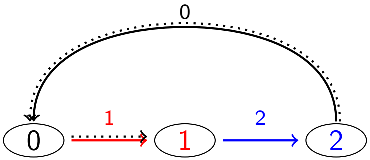

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$$M^{\square\square} \cong \widetilde{f}_1 \widetilde{f}_0 \mathbb{1}$$

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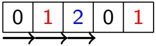

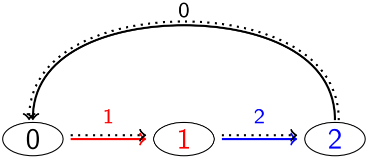

And walk describes how to build $M^{\square\square\square\square\square}$ with functors $\widetilde{f}_i$,

$$M^{\square\square\square} \cong \widetilde{f}_{\color{blue}2}\widetilde{f}_{\color{red}1}\widetilde{f}_0\, \mathbb{1}$$

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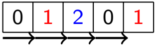

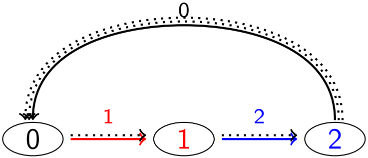

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$$M^{\square\square\square\square} \cong \widetilde{f_0}\,\widetilde{f_2}\,\widetilde{f_1}\,\widetilde{f_0}\,\mathbb{1}$$

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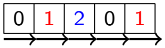

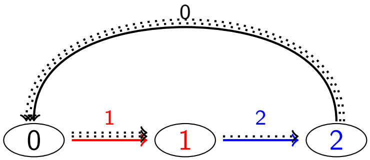

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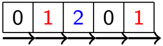
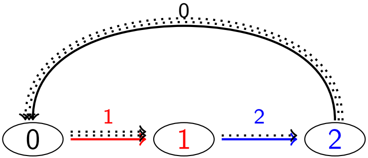




And walk describes how to build $M^{\square\square\square\square\square}$ with functors $\widetilde{f}_i$,

$$M^{\square\square\square\square\square} \cong \widetilde{f}_{1}\widetilde{f}_{0}\widetilde{f}_{2}\widetilde{f}_{1}\widetilde{f}_{0}\,\mathbb{1}$$

Recall:

$$\mathrm{Char}(M^{\square\square\square\square\square}) = [0\ 1\ 2\ 0\ 1].$$

Using functors $\widetilde{f_i}$, we can build an R-module from any walk in $B^{1,1}$.

For a directed walk p in $B^{1,1}$ of length k which traverses edges colored

$$i_1, i_2, \ldots, i_k$$

set

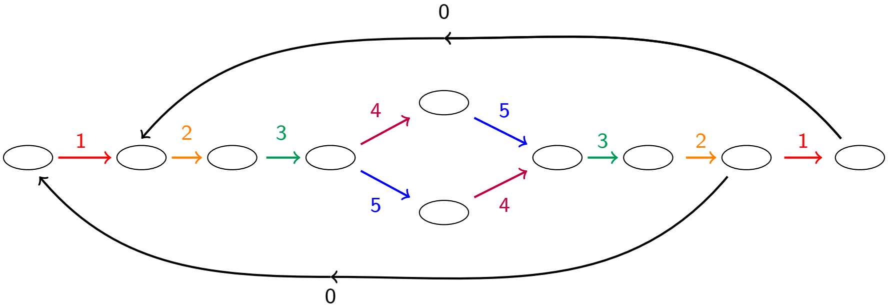
$$T_{p;k} := \widetilde{f_{i_k}} \ldots \widetilde{f_{i_2}} \widetilde{f_{i_1}} \mathbb{1}$$

$$\uparrow \tag{3}$$

Analogue for “trivial” modules in other types.

$B^{1,1}$ in type $D_5^{(1)}$

Directed walk

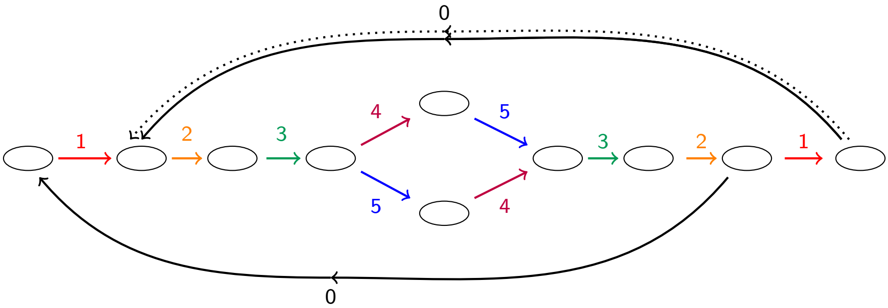


This corresponds to R-module

$$\mathbb{1}$$

$B^{1,1}$ in type $D_5^{(1)}$

Directed walk



This corresponds to R-module

$$\widetilde{f}_0 \, \mathbb{1}$$

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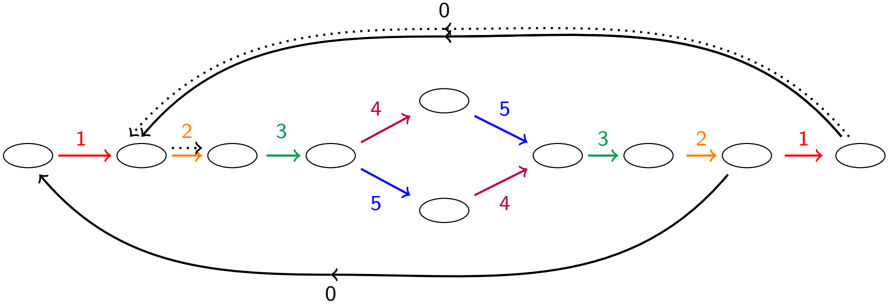

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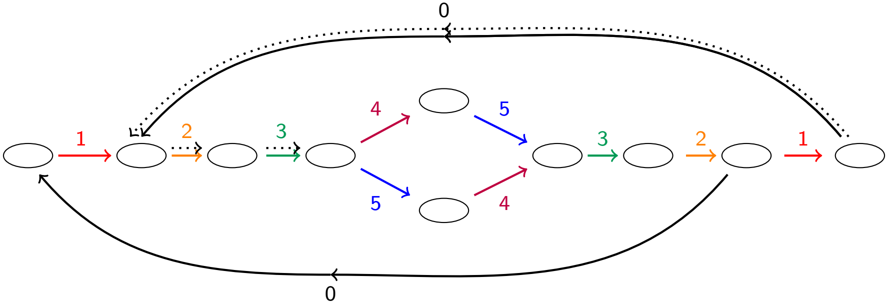

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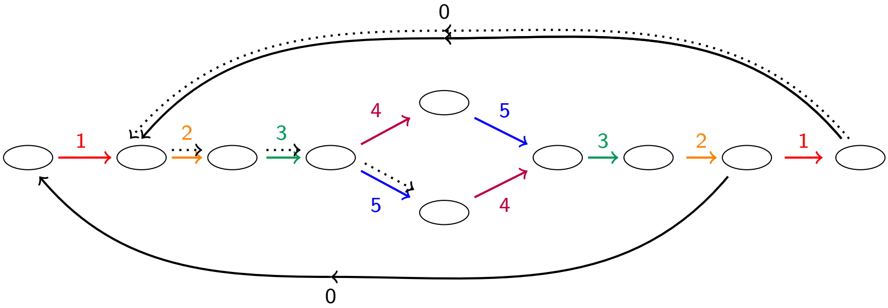

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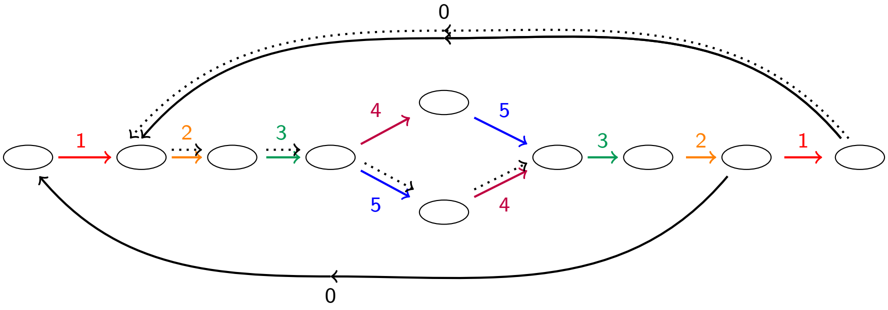

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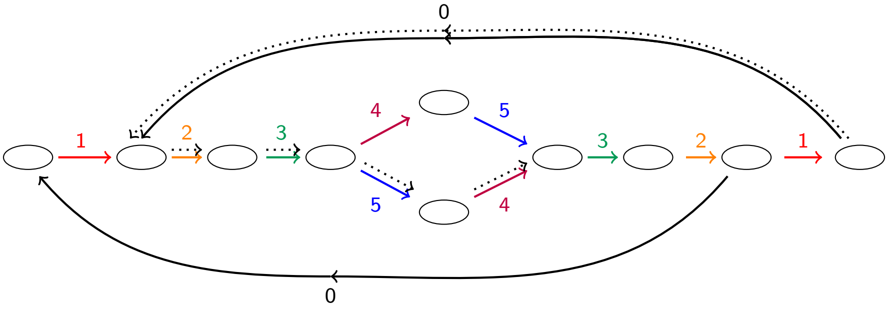

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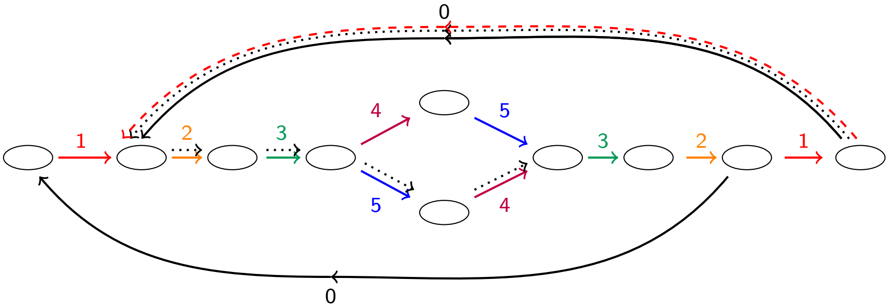
Directed walk p



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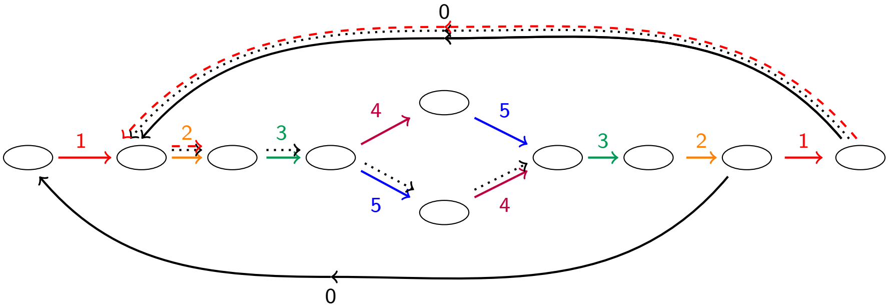

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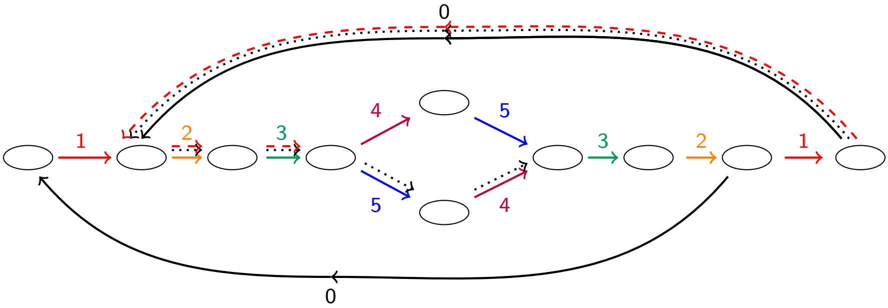

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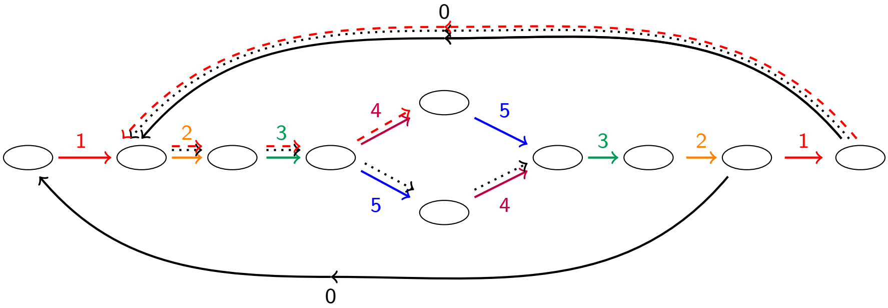

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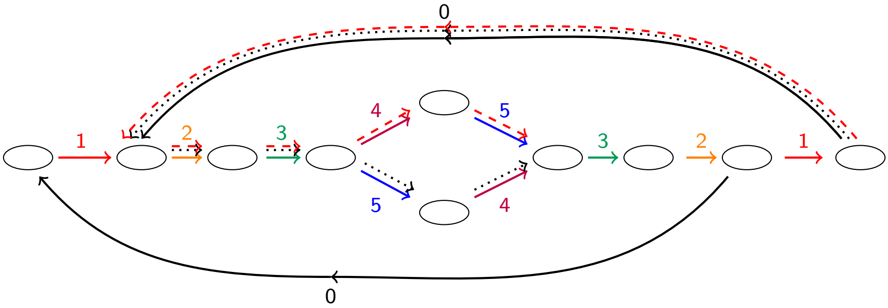

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Type X_ℓ

Question: Is there a module-theoretic interpretation of the crystal isomorphism

$$B^{1,1} \otimes B(\Lambda_{\sigma(i)}) \xrightarrow{\cong} B(\Lambda_i)$$

in other classical affine types X_ℓ? (when $B^{1,1}$ is perfect and Λ_i and $\Lambda_{\sigma(i)}$ is level 1).

Story is exactly same to type $A_\ell^{(1)}$ case, but "trivial" modules replaced by $T_{p;k}$.

If,

$$c \otimes b_1 \mapsto b_2$$

where $c \in B^{1,1}$, $b_1 \in B(\Lambda_{\sigma(i)})$, and $b_2 \in B(\Lambda_i)$, we showed:

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where $c \in B^{1,1}$, $b_1 \in B(\Lambda_{\sigma(i)})$, and $b_2 \in B(\Lambda_i)$, we showed:

- $$\mathrm{Ind}\, T_{p;k} \boxtimes M^{b_1} \twoheadrightarrow M^{b_2}.$$

  for appropriate directed walk p in $B^{1,1}$.
- Action of $\widetilde{e}_i$ and $\widetilde{f}_i$ agree in both crystal <u>and</u> module settings.

What differs in other classical types?

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- Type $\mathcal{D}$ structures,
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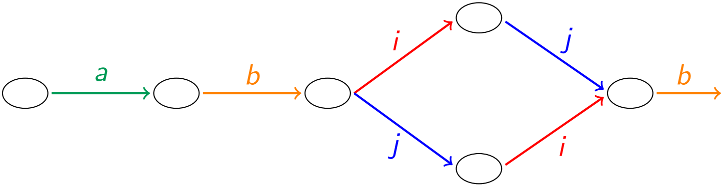
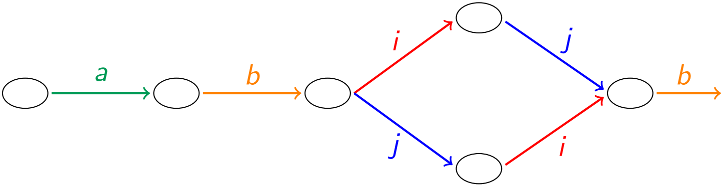
Type $\mathcal{D}$ structure

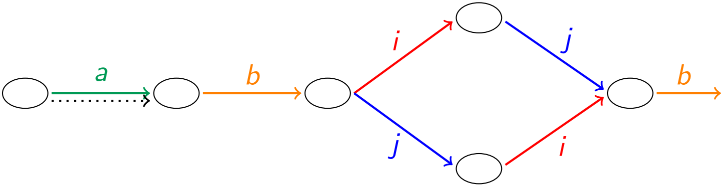


Type $\mathcal{D}$ structure



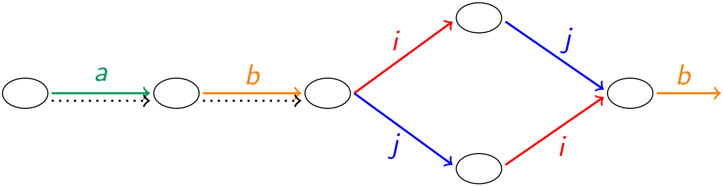
$\mathbb{1}$

Type $\mathcal{D}$ structure



$$\mathrm{Char}(\widetilde{f_a}\,\mathbb{1}) = [a]$$

Type $\mathcal{D}$ structure



$$\mathrm{Char}(\widetilde{f_b}\widetilde{f_a}\,\mathbb{1}) = [a\ b]$$

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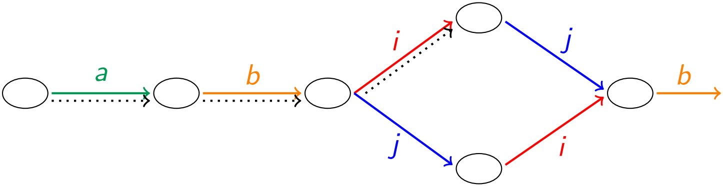
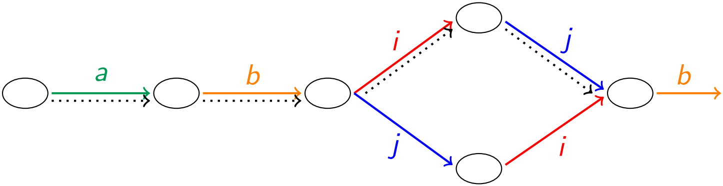


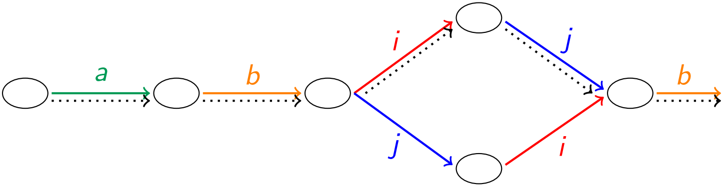
$$\mathrm{Char}(\widetilde{f_i}\widetilde{f_b}\widetilde{f_a}\,\mathbb{1}) = [a\ b\ i]$$

Type $\mathcal{D}$ structure



$$\mathrm{Char}(\widetilde{f_j}\widetilde{f_i}\widetilde{f_b}\widetilde{f_a}\,\mathbb{1}) = [a\ b\ i\ j] + [a\ b\ j\ i]$$

Type $\mathcal{D}$ structure



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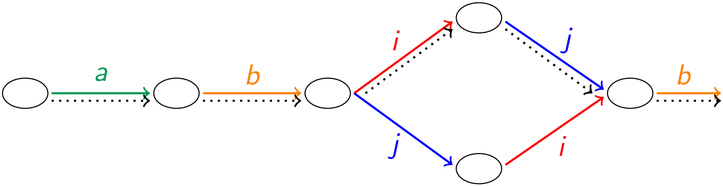

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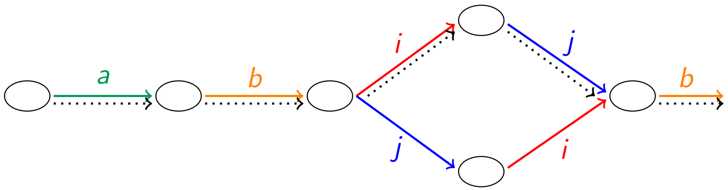
- Bifurcations double dimension.

Type $\mathcal{D}$ structure



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- Bifurcations double dimension.
- Module does not see difference between two paths around bifurcation.

Type $\mathcal{B}$ structure

$\mathbb{1}$

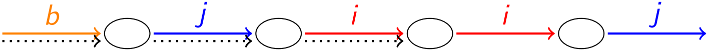
Type $\mathcal{B}$ structure

$$\mathrm{Char}(\widetilde{f}_b\, \mathbb{1}) = [b]$$

Type $\mathcal{B}$ structure

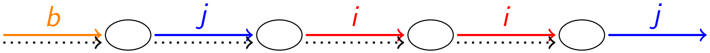
$$\mathrm{Char}(\widetilde{f_j}\widetilde{f_b}\,\mathbb{1}) = [b\ j]$$

Type $\mathcal{B}$ structure



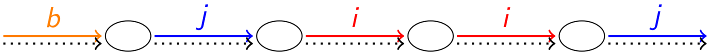
$$\mathrm{Char}(\widetilde{f_i}\widetilde{f_j}\widetilde{f_b}\,\mathbb{1}) = [b\ j\ i]$$

Type $\mathcal{B}$ structure



$$\mathrm{Char}(\widetilde{f_i}\widetilde{f_i}\widetilde{f_j}\widetilde{f_b}\,\mathbb{1}) = (1+q^{-2})[b\ j\ i\ i]$$

Type $\mathcal{B}$ structure



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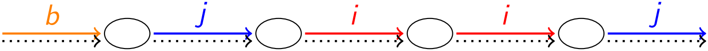

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Traveling over adjacent i-arrows, multiply character by $[2] = (q^{-1}+q)$.

Iterating the construction

Recall, we can iterate:

$$
\begin{aligned}
B^{1,1} \otimes B(\Lambda_{i-1}) &\cong B(\Lambda_i) \\
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In KLR case, process must terminate and we get decomposition,

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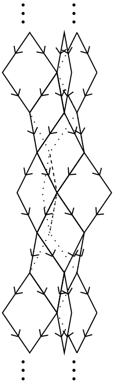
For other types, this seems to be new.

Work in progress

- Would like to generalize to other Kirillov-Reshetikhin crystals $B^{r,s}$

Work in progress

Similar arguments appear to work for $B^{r,1}$ in type $A_\ell^{(1)}$.



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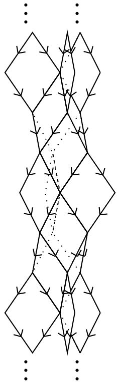

$B^{3,1}$ in type $A_5^{(1)}$

Key: "Trivial modules" $T_{p;k}$ arising from $B^{r,1}$ are homogeneous.

Work in progress

Homogeneous R-modules for simply-laced type fully classified by Kleshchev-Ram.

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Homogeneous R-modules for simply-laced type fully classified by Kleshchev-Ram.

When $s > 1$ in $B^{r,s} \implies$, type $A_\ell^{(1)}$, $T_{p;k}$ are in general not homogeneous.

...New methods will be needed.

Future directions

- Can R representation theory provide new models for KR crystals?

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Example

Kleshchev and Ram have a beautiful combinatorial model for homogeneous R-modules for simply-laced type. Can we use this to construct a new combinatorial model for KR crystals of simply-laced type?

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Example

Kleshchev and Ram have a beautiful combinatorial model for homogeneous R-modules for simply-laced type. Can we use this to construct a new combinatorial model for KR crystals of simply-laced type?

- Is there any relationship between $T_{p;k}$ and cuspidal representations?

Thank you.