Practice Exam 1

1. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that

$$T\left(\begin{pmatrix}2\\0\end{pmatrix}\right) = \begin{pmatrix}8\\-2\end{pmatrix} \qquad \text{and} \qquad T\left(\begin{pmatrix}0\\3\end{pmatrix}\right) = \begin{pmatrix}3\\6\end{pmatrix}.$$

. What is the value of

$$T\left(\begin{pmatrix}1\\1\end{pmatrix}\right)?$$

Solution: Using the linearity of T we have

$$T\left(\begin{pmatrix}2\\0\end{pmatrix}\right) = \begin{pmatrix}8\\-2\end{pmatrix} \implies 2T\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}8\\-2\end{pmatrix} \implies T\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}4\\-1\end{pmatrix},$$

and

$$T\left(\begin{pmatrix}0\\3\end{pmatrix}\right) = \begin{pmatrix}3\\6\end{pmatrix} \implies 3T\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}3\\6\end{pmatrix} \implies T\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}1\\2\end{pmatrix}.$$

Thus by the linearity of T

$$T\left(\begin{pmatrix}1\\1\end{pmatrix}\right) = T\left(\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right) = T\left(\begin{pmatrix}1\\0\end{pmatrix}\right) + T\left(\begin{pmatrix}0\\1\end{pmatrix}\right)$$

$$\begin{pmatrix}4\\-1\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}5\\1\end{pmatrix}.$$

2. Use the definition of a linear transformation to explain why the function $F : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$F\left(\begin{pmatrix}x\\y\\z\end{pmatrix}\right) = \begin{pmatrix}4x^2 + y - z\\2y + 8z\end{pmatrix}$$

is NOT linear. (You would need to use the definition to get full credit on this).

Solution: Part of the definition of a linear transformation says that for any choice of x, y, z we should have

$$2F\left(\begin{pmatrix}x\\y\\z\end{pmatrix}\right) = F\left(\begin{pmatrix}2x\\2y\\2z\end{pmatrix}\right).$$

Choose $x = 1, y = 0, z = 0$. Then we have

$$2F\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right) = 2\begin{pmatrix}4\\0\end{pmatrix} = \begin{pmatrix}8\\0\end{pmatrix}.$$

On the other hand

$$F\left(\begin{pmatrix}2\\0\\0\end{pmatrix}\right) = \begin{pmatrix}16\\0\end{pmatrix}.$$

It follows that F cannot be a linear transformation.

3. Calculate the value of the determinant of A where

$$A = \begin{pmatrix}1 & -1 & 4\\2 & -5 & 3\\2 & 1 & 0\end{pmatrix}.$$

Solution: The determinant of A is 39.

4. Find the value of x which minimizes the determinant of the matrix:

$$\begin{pmatrix} x & 1 & 0 \\ 0 & x & 6 \\ 0 & 4 & 2 \end{pmatrix}.$$

Solution: The determinant of this matrix is $2x^2 - 24x$. The minimum of this function is attained at $x = 6$.

5. Suppose that for two vectors $\mathbf{u}$, $\mathbf{v}$ in $\mathbb{R}^n$, we know that $||\mathbf{u}|| = \sqrt{6}$, $||\mathbf{v}|| = \sqrt{24}$, and $\mathbf{u} \cdot \mathbf{v} = 12$. What is the angle between $\mathbf{u}$ and $\mathbf{v}$?

Solution: Using the formula

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||||\mathbf{v}||}\right).$$

we get

$$\cos^{-1}(1) = 0 = \theta.$$

So $\mathbf{u}$ and $\mathbf{v}$ actually point in the same direction.

6. Find real values x_1 and x_2 so that the vector

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

is a unit vector AND orthogonal to

$$\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Solution: This amounts to solving two problems:

$$0 = \mathbf{x} \cdot \mathbf{u} = x_1 + 2x_2$$

and

$$1 = ||\mathbf{x}|| = \sqrt{x_1^2 + x_2^2}.$$

Using basic algebra one can find two different solutions:

$$\mathbf{x} = \begin{pmatrix} -\frac{2\sqrt{5}}{2} \\ \\ \frac{\sqrt{5}}{2} \end{pmatrix} \qquad \text{and} \qquad \mathbf{x} = \begin{pmatrix} \frac{2\sqrt{5}}{2} \\ \\ -\frac{\sqrt{5}}{2} \end{pmatrix}$$

7. Recall that the set V of all real valued functions $f : \mathbb{R} \to \mathbb{R}$ with the usual addition of functions and multiplication of functions by scalars is a vector space. Consider the subset W

$$V = \{ \text{ functions } f : \mathbb{R} \to \mathbb{R} \text{ with } f(0) = 1\}$$

Is W a subspace V?

Solution: No. There are quite a few ways to see this, but one is to note that if $f : \mathbb{R} \to \mathbb{R}$ is in W (so $f(0) = 1$), then $2f$ is not in W since $2f(0) = 2(1) = 2 \neq 1$.

8. Show that the collection of all vectors

$$V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ with } 2x_1 + 3x_2 + x_3 = 1 \right\}$$

with the usual scalar multiplication and vector addition is NOT a vector space.

Solution: Note that the zero vector, which we know to be

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

does not satisfy the equation $2x_1 + 3x_2 + x_3 = 1$ and is therefore not in V.