

MATH 417 Homework 9

Note that this is problems: Section 15.2 #2,4,7

2. Define

$$F(x, y, z) = (xyz, x^2 + yz, 1 + 3x)$$

for all $(x, y, z) \in \mathbb{R}^3$. Find the derivative matrix of the mapping $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ at the points $(1, 2, 3)$, $(0, 1, 0)$, and $(-1, 4, 0)$.

Solution: The gradients of the component functions are

$$\nabla F_1(x, y, z) = (yz, xz, xy),$$

$$\nabla F_2(x, y, z) = (2x, z, y),$$

$$\nabla F_3(x, y, z) = (3, 0, 0).$$

Thus, the derivative matrix at $(1, 2, 3)$ is

$$DF(1, 2, 3) = \begin{bmatrix} 6 & 3 & 2 \\ 2 & 3 & 2 \\ 3 & 0 & 0 \end{bmatrix},$$

the derivative matrix at $(0, 1, 0)$ is

$$DF(0, 1, 0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 3 & 0 & 0 \end{bmatrix},$$

and the derivative matrix at $(-1, 4, 0)$ is

$$DF(-1, 4, 0) = \begin{bmatrix} 0 & 0 & -4 \\ -2 & 0 & -1 \\ 3 & 0 & 0 \end{bmatrix}.$$

4. Suppose that A is an $m \times n$ matrix. Define the mapping $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ by

$$F(\mathbf{x}) = A\mathbf{x},$$

for $\mathbf{x} \in \mathbb{R}^n$. Prove that $DF(\mathbf{x}) = A$ for all \mathbf{x} in \mathbb{R}^n .

Solution: Write

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}.$$

Then

$$F(\mathbf{x}) = F(x_1, x_2, \dots, x_n) = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n \end{bmatrix}.$$

Thus the i th component function is

$$F_i(x_1, \dots, x_n) = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \cdots + a_{in}x_n.$$

The gradient of this function is

$$\nabla F_i(x_1, \dots, x_n) = (a_{i1}, a_{i2}, a_{i3}, \dots, a_{in}).$$

Thus

$$DF(\mathbf{x}) = \begin{bmatrix} \nabla F_1(x_1, \dots, x_n) \\ \nabla F_2(x_1, \dots, x_n) \\ \vdots \\ \nabla F_m(x_1, \dots, x_n) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}.$$

7. Give a proof of the First-Order Approximation Theorem based on the Mean Value Theorem.

Solution: We assume that $\mathcal{O} \subseteq \mathbb{R}^n$ is open and that $F : \mathcal{O} \rightarrow \mathbb{R}^m$ is continuously differentiable. Let $\mathbf{x} \in \mathcal{O}$. Since \mathcal{O} is open, we can find $r > 0$ such that $B_r(\mathbf{x}) \subseteq \mathcal{O}$. We consider the limit

$$\lim_{\mathbf{h} \rightarrow 0} \frac{\|F(\mathbf{x} + \mathbf{h}) - [F(\mathbf{x}) + DF(\mathbf{x})\mathbf{h}]\|}{\|\mathbf{h}\|}. \quad (1)$$

For \mathbf{h} sufficiently close to 0, $\mathbf{x} + \mathbf{h} \in B_r(\mathbf{x})$ and hence the line segment between \mathbf{x} and $\mathbf{x} + \mathbf{h}$ is also in $B_r(\mathbf{x})$ since this set is convex. Thus for each such sufficiently small \mathbf{h} we can write the limit (1) as

$$\lim_{\mathbf{h} \rightarrow 0} \frac{\|A\mathbf{h} - DF(\mathbf{x})\mathbf{h}\|}{\|\mathbf{h}\|},$$

where the i th column A is equal to $\nabla F_i(\mathbf{x} + \theta_i \mathbf{h})$, and each $\theta_i \in (0, 1)$. By the generalized Cauchy-Schwarz inequality we have

$$\lim_{\mathbf{h} \rightarrow 0} \frac{\|A\mathbf{h} - DF(\mathbf{x})\mathbf{h}\|}{\|\mathbf{h}\|} = \lim_{\mathbf{h} \rightarrow 0} \frac{\|A - DF(\mathbf{x})\| \|\mathbf{h}\|}{\|\mathbf{h}\|} = \lim_{\mathbf{h} \rightarrow 0} \|A - DF(\mathbf{x})\|.$$

Because each ∇F_i is assumed to be continuous then as $\mathbf{h} \rightarrow 0$, $\nabla F_i(\mathbf{x} + \theta_i \mathbf{h}) \rightarrow \nabla F_i(\mathbf{x})$, and consequently $A \rightarrow DF(\mathbf{x})$, so

$$\lim_{\mathbf{h} \rightarrow 0} \|A - DF(\mathbf{x})\| = 0.$$