Practice Exam 3 Solutions

1. (a) Give an example of a 2×2 matrix which is NOT positive definite. Justify your answer.

   (b) Find the 3×3 symmetric matrix A associated with quadratic form defined by

$$\langle A\mathbf{x}, \mathbf{x} \rangle = 3x_1^2 + 2x_1x_2 - x_3^2.$$

   Solution: The matrix A is

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function such that $\frac{\partial f}{\partial x_n}(\mathbf{x}) = c$ for all $\mathbf{x} \in \mathbb{R}^n$ and for some constant $c \in \mathbb{R}$. Show that we will not be able to find any extreme points using the second derivative test.

   Solution: If $\frac{\partial f}{\partial x_n}(\mathbf{x}) = c$ then the last row of the Hessian matrix is zero since this will be $\nabla(\frac{\partial f}{\partial x_n})$. It is then impossible for $\nabla^2 f$ to be positive definite since, for example, we will always have

$$\langle \nabla^2 f(\mathbf{x})\mathbf{e}_n, \mathbf{e}_n \rangle = 0.$$

   Note also that we would need $c = 0$ for the second derivative test to work to find extreme points.

3. Determine whether the following functions $F : \mathbb{R}^3 \to \mathbb{R}^3$ are linear. When they are linear, find their corresponding matrix.

   (a) $F(x_1, x_2, x_3) = (x_1, x_1 - x_2, x_2 + x_3)$,

   (b) $F(x_1, x_2, x_3) = (x_1, x_2, x_2x_3)$,

   (c) $F(x_1, x_2, x_3) = (x_1, 0, 0)$,

   (d) $F(x_1, x_2, x_3) = (1, 0, 0)$,

   (e) $F(x_1, x_2, x_3) = (3x_1 + 2x_2, x_3, |x_2|)$,

   (f) $F(x_1, x_2, x_3) = (x_1 - x_2, x_1 + x_2, x_3)$.

   Solutions:

   (a) F is linear, the corresponding matrix is:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

   (b) F is not linear because of the third component x_2x_3.

   (c) F is linear, the corresponding matrix is:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

   (d) F is not linear because of the 1st component 1.

   (e) F is not linear because of the 3rd component $|x_2|$.

   (f) F is linear, the corresponding matrix is:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

4. Define $F(x, y) = (e^{xy} + 2x, y^2 + \sin(x - y))$ for $(x, y) \in \mathbb{R}^2$. Find the derivative matrix of the mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ at the points $(0, 0)$ and $(\pi, 0)$.

   Solutions: We calculate the gradients for the component functions,

$$\nabla F_1(x, y) = (ye^{xy} + 2, xe^{xy}),$$

$$\nabla F_2(x, y) = (\cos(x - y), 2y - \cos(x - y)).$$

   It follows that

$$DF(0, 0) = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$$

   and

$$DF(\pi, 0) = \begin{bmatrix} 2 & \pi \\ -1 & 1 \end{bmatrix}.$$

5. Suppose that $F : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable and that the derivative matrix $DF(\mathbf{x})$ has all entries equal to 0 for all $\mathbf{x} \in \mathbb{R}^2$. Prove that $F : \mathbb{R}^2 \to \mathbb{R}^2$ is constant, i.e. there is some $\mathbf{c} \in \mathbb{R}^2$ such that $F(\mathbf{x}) = \mathbf{c}$.

   Solutions: We start by proving that if $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuously differentiable function and $\nabla f(x, y) = (0, 0)$ for all $(x, y) \in \mathbb{R}^2$, then $f(x, y) = c$ for some $c \in \mathbb{R}$.

   Since $\frac{\partial f}{\partial x}(x, y) = 0$, then we can integrate with respect to x to get that $f(x, y) = g(y)$. That is, f is constant with respect to x. We can also integrate $\frac{\partial f}{\partial y}(x, y) = 0$ to get that $f(x, y) = h(x)$, i.e. f is constant with respect to y. It follows then that $f(x, y)$ must be constant, or $f(x, y) = c$.

   Since $DF(\mathbf{x})$ is the zero matrix, that means each of the component functions F_i of F has the property that $\nabla F_i(x, y) = (0, 0, \ldots, 0)$. From above, this means that $F_i(x, y) = c_i$ for some $c \in \mathbb{R}$. It follows that $F(x, y) = (c_1, c_2)$.