MATH 417 Homework 4

Due: Note that this is problems: Section 13.1: #3, #8,#11, Section 13.2: #1, #3, #6.

3. Analyze the following limits:

   a. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+y}{x^2+y^2}$

   b. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x+y+z}{x^2+y^2+z^2}$

   c. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^{x^2+y^2}-1}{x^2+y^2}$

   Solution:

   a. This limit does not exist. To prove this consider the sequence $\{(\frac{1}{k}, -\frac{1}{k})\}_{k\geqslant 1}$. Observe that

   $$\lim_{k\to\infty} (\frac{1}{k}, -\frac{1}{k}) = (0,0).$$

   and

   $$\lim_{k\to\infty} \frac{\frac{1}{k} - \frac{1}{k}}{\frac{1}{k^2} + \frac{1}{k^2}} = 0.$$

   On the other hand, the sequence $\{(\frac{1}{k}, 0)\}_{k\geqslant 1}$ also converges to $(0,0)$ but

   $$\lim_{k\to\infty} \frac{\frac{1}{k} + 0}{\frac{1}{k^2} + 0} = \lim_{k\to\infty} k = \infty.$$

   Hence the limit cannot exist.

   b. This limit also does not exist and an argument similar to above works in this case as well, replacing our first sequence above by $\{(\frac{1}{k}, -\frac{1}{k}, 0)\}_{k\geqslant 1}$ and our second sequence by $\{(\frac{1}{k}, 0, 0)\}_{k\geqslant 1}$.

   c. This limit does exist and is equal to 1. To prove this, notice that via the series definition of the exponential function, we have

   $$\begin{aligned} e^{x^2+y^2} - 1 &= \Big(1 + (x^2+y^2) + \frac{(x^2+y^2)^2}{2!} + \frac{(x^2+y^2)^3}{3!} + \dots\Big) - 1 \\ &= (x^2+y^2) + \frac{(x^2+y^2)^2}{2!} + \frac{(x^2+y^2)^3}{3!} + \dots \end{aligned}$$

   Thus

   $$\frac{e^{x^2+y^2}-1}{x^2+y^2} = \frac{(x^2+y^2) + \frac{(x^2+y^2)^2}{2!} + \frac{(x^2+y^2)^3}{3!} + \dots}{x^2+y^2}$$

   $$1 + \frac{(x^2+y^2)}{2!} + \frac{(x^2+y^2)^2}{3!} + \dots$$

   It is clear that each term of the sum is continuous everywhere and hence for any sequence $\{(x_k, y_k)\}_{k\geqslant 1}$ that converges to $(0,0)$, each of these terms converges to 0, except for the first, which is identically 1. Some care is needed here since we haven't discussed how to pass a limit into an infinite series, but in this case your intuition is correct and this limit converges to 1.

6. Let A be a subset of $\mathbb{R}^n$ and let $\mathbf{x}$ be a point in $\mathbb{R}^n$. Show that $\mathbf{x}$ is a limit point of A if and only if every open ball about $\mathbf{x}$ contains a point of A that is not equal to $\mathbf{x}$.

   Solution: We first prove that if $\mathbf{x}$ is a limit point of A then every open ball about $\mathbf{x}$ contains a point of A that is not equal to $\mathbf{x}$. Suppose for a contradiction that this is not true. Then there is $r > 0$ such that $B_r(\mathbf{x})$ contains no points from A other than $\mathbf{x}$. Since $\mathbf{x}$ is a limit point, there is at least one sequence $\{\mathbf{x}_k\}_{k\geqslant 1}$ in A

that converges to $\mathbf{x}$ and for which no terms are equal to $\mathbf{x}$. By the definition of convergence of a series, there is some K such that for all $k \geqslant K$, $\mathbf{x}_k \in B_r(\mathbf{x})$. But this is a contradiction since such $\mathbf{x}_k$ belong to A and we had assumed that $B_r(\mathbf{x})$ had no other points not in A.

Next we prove that if every open ball around $\mathbf{x}$ has a point from A in it not equal to $\mathbf{x}$, then it is a limit point. Then for each $k \geqslant 1$, the ball $B_{\frac{1}{k}}(\mathbf{x})$ contains a point in A, which we call $\mathbf{x}_k$, which is not equal to $\mathbf{x}$. It is easy to show that the sequence $\{\mathbf{x}_k\}_{k\geqslant 1}$ converges to $\mathbf{x}$, and is in A by construction. Hence $\mathbf{x}$ is a limit point of A.

11. Let A be a subset of $\mathbb{R}^n$ and suppose that $\mathbf{0}$ is a limit point of A. Suppose that the function $f : A \to \mathbb{R}$ has the property that there is a positive c such that

$$f(\mathbf{x}) \geqslant c||\mathbf{x}||^2 \quad \text{for all } \mathbf{x} \in A$$

and that the function $g : A \to \mathbb{R}$ has the property that

$$\lim_{\mathbf{x}\to 0} \frac{g(\mathbf{x})}{||\mathbf{x}||^2} = 0.$$

Prove that there is a positive number r such that

$$f(\mathbf{x}) - g(\mathbf{x}) \geqslant (c/2)||\mathbf{x}||^2 \quad \text{for all } x \in A \text{ with } 0 < ||\mathbf{x}|| < r.$$

Solution: Since we know that

$$\lim_{\mathbf{x}\to 0} \frac{g(\mathbf{x})}{||\mathbf{x}||^2} = 0$$

then by definition there is some $r > 0$ such that for all $\mathbf{x} \in B_r(\mathbf{0})$ we have that

$$\left|\frac{g(\mathbf{x})}{||\mathbf{x}||^2}\right| < \frac{c}{2}.$$

Since $||\mathbf{x}||^2$ is always non-negative, it also follows that for $\mathbf{x} \in B_r(\mathbf{0})$ we have

$$g(\mathbf{x}) \leqslant |g(\mathbf{x})| < \frac{c||\mathbf{x}||^2}{2}.$$

Multiplying by -1 then gives

$$\frac{-c||\mathbf{x}||^2}{2} < -g(\mathbf{x}).$$

Finally, we can add this to the inequality for $f(\mathbf{x})$ given in the problem statement to get

$$f(\mathbf{x}) - g(\mathbf{x}) \geqslant c||\mathbf{x}||^2 + \frac{-c||\mathbf{x}||^2}{2} = \frac{-c||\mathbf{x}||^2}{2}.$$

1. Calculate the first-order partial derivatives of the following functions:

a. $f(x, y, z) = x + yz + xy + x\sin(xy)$ for $(x, y, z) \in \mathbb{R}^3$
b. $f(x, y, z) = \sin(x^2y^2)/(1 + x^2 + y^3)$ for $(x, y, z) \in \mathbb{R}^3$
c. $f(x, y, z) = \sqrt{1 + \cos^2(xy)}$ for $(x, y, z) \in \mathbb{R}^3$.

Solution:

a. The first-order partials are defined everywhere in $\mathbb{R}^3$ and are equal to:
   - $\frac{\partial f}{\partial x}(x, y, z) = 1 + y + \sin(xy) + xy\cos(xy)$,
   - $\frac{\partial f}{\partial y}(x, y, z) = z + x + x^2\sin(xy)$,
   - $\frac{\partial f}{\partial z}(x, y, z) = y$.

b. For all points in $\mathbb{R}^2$ the partial derivatives are equal to
   - $\frac{\partial f}{\partial x}(x, y) = \frac{2y^2x\cos(x^2y^2)(1+x^2+y^3)-2x\sin(x^2y^2)}{(1+y^3+x^2)^2}$,
   - $\frac{\partial f}{\partial y}(x, y) = \frac{2x^2y\cos(x^2y^2)(1+x^2+y^3)-3y\sin(x^2y^2)}{(1+y^3+x^2)^2}$.

c. Since $1+\cos^2(xy)$ is everywhere positive, then the partial derivatives are defined in all $\mathbb{R}^2$ as

- $\frac{\partial f}{\partial x}(x,y) = \frac{-2y\sin(xy)}{\sqrt{1+\cos^2(xy)}}$,
- $\frac{\partial f}{\partial y}(x,y) = \frac{-2x\sin(xy)}{\sqrt{1+\cos^2(xy)}}$.

3. For the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined in Example 13.9, show that neither the function $\partial f/\partial x : \mathbb{R}^2 \to \mathbb{R}$ nor the function $\partial f/\partial y : \mathbb{R}^2 \to \mathbb{R}$ is continuous at the point $(0,0)$.

   Solution: To show that $\frac{\partial f}{\partial x}$ is not continuous at $(0,0)$, consider the sequence $\{(0,\frac{1}{k})\}_{k\geqslant 0}$ which converges to $(0,0)$. Then
$$\lim_{k\to\infty} \frac{y_k^3 - x_k^2 y_k}{(x_k^2+y_k^2)^2} = \lim_{k\to\infty} k = \infty.$$
   But $\frac{\partial f}{\partial x}(0,0) = 0$. Hence $\frac{\partial f}{\partial x}$ is not continuous. To show that $\frac{\partial f}{\partial y}$ is not continuous at $(0,0)$, similarly consider the sequence $\{(\frac{1}{k},0)\}_{k\geqslant 0}$ and follow the same argument.

6. Define
$$g(x,y) = \begin{cases} x^2y^2/(x^2+y^2) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$
   Prove that the function $g : \mathbb{R}^2 \to \mathbb{R}$ has first-order partial derivatives. Is the function $g : \mathbb{R}^2 \to \mathbb{R}$ continuously differentiable?

   Solution: When $(x,y) \neq (0,0)$, then there is a neighborhood such that $g(x,y)$ is the quotient of two polynomials and hence for such points
$$\frac{\partial g}{\partial x}(x,y) = \frac{2xy^4}{(x^2+y^2)^2}$$
   and
$$\frac{\partial g}{\partial y}(x,y) = \frac{2yx^4}{(x^2+y^2)^2}.$$
   On the other hand, we have
$$\frac{\partial g}{\partial x}(0,0) = \lim_{t\to 0} \frac{\frac{t^2(0)^2}{t^2+0^2} - 0}{t} = 0,$$
   and
$$\frac{\partial g}{\partial y}(0,0) = \lim_{t\to 0} \frac{\frac{t^2(0)^2}{0^2+t^2} - 0}{t} = 0.$$
   So g has first-order partial derivatives. Both of these functions are continuous. It is clear that they are continuous everywhere except possibly $(0,0)$ since they are the quotient of two polynomials and the polynomials in the denominator only vanish at $(0,0)$. To show that $\frac{\partial g}{\partial x}$ is continuous at $(0,0)$ note that
$$|2xy^4| \leqslant |2x(y^4 + 2x^2y^2 + x^4)| = |2x(x^2+y^2)^2|$$
   and hence
$$\Big|\frac{2xy^4}{(x^2+y^2)^2}\Big| \leqslant \Big|\frac{2x(x^2+y^2)^2}{(x^2+y^2)^2}\Big| = |2x|.$$
   For any sequence $\{(x_k,y_k)\}_{k\geqslant 1}$ which converges to $(0,0)$, then $\{x_k\}_{k\geqslant 1}$ converges to 0. Hence we have
$$\lim_{k\to\infty} \Big|\frac{2x_ky_k^4}{(x_k^2+y_k^2)^2}\Big| \leqslant \lim_{k\to\infty} |2x_k| = 0.$$
   Thus
$$\lim_{k\to\infty} \Big|\frac{2x_ky_k^4}{(x_k^2+y_k^2)^2}\Big| = 0 = \frac{\partial g}{\partial x}(0,0)$$
   So $\frac{\partial g}{\partial x}$ is continuous at $(0,0)$. An analogous argument works for $\frac{\partial g}{\partial y}$.