MATH 417 Homework 5

Note that this is problems: Section 13.2: #4, #5,#12, Section 13.3: #1, #3.

4. Suppose that the function $g : \mathbb{R}^2 \to \mathbb{R}$ has the property that

$$|g(x,y)| \leqslant x^2 + y^2 \quad \text{for all } (x,y) \in \mathbb{R}^2.$$

Prove that $g : \mathbb{R}^2 \to \mathbb{R}$ has partial derivatives with respect to both x and y at the point $(0,0)$.

Solution: First note that since

$$|g(0,0)| \leqslant 0^2 + 0^2 = 0$$

then $g(0,0) = 0$. Then we can compute that

$$\left|\frac{\partial f}{\partial x}(0,0)\right| = \lim_{t\to 0}\left|\frac{g(t,0) - g(0,0)}{t}\right|$$

$$= \lim_{t\to 0}\left|\frac{g(t,0)}{t}\right|.$$

(Note that here we use the fact that the absolute value function is continuous). We also have that $|g(t,0)| \leqslant t^2$, so

$$\lim_{t\to 0}\left|\frac{g(t,0)}{t}\right| \leqslant \lim_{t\to 0}|t| = 0.$$

It follows that $\frac{\partial f}{\partial x}(0,0)$ exists and $\frac{\partial f}{\partial x}(0,0) = 0$. A completely analogous argument shows that $\frac{\partial f}{\partial y}(0,0)$ exists and $\frac{\partial f}{\partial x}(0,0) = 0$.

5. Suppose that the function $f : \mathbb{R}^2 \to \mathbb{R}$ has first-order partial derivatives and that

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial y}(x,y) = 0 \quad \text{for all } (x,y) \in \mathbb{R}^2.$$

Prove that the function $f : \mathbb{R}^2 \to \mathbb{R}$ is constant, that is, that there is some number c such that

$$f(x,y) = c \quad \text{for all } (x,y) \in \mathbb{R}^2.$$

Solution: We assume here the fact that for a differentionable function $g : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = 0$ for all $x \in \mathbb{R}$, it must be the case that $f(x) = c$ for some $c \in \mathbb{R}$. Now return to $f : \mathbb{R}^2 \to \mathbb{R}$. First, for fixed $a \in \mathbb{R}$, define $g_{1,a} : \mathbb{R} \to \mathbb{R}$ so that

$$g_{1,a}(x) := f(x,a).$$

It is clear that $g'_{1,a}(x) = \frac{\partial f}{\partial x}(x,a)$. Similarly for fixed $b \in \mathbb{R}$, define $g_{2,b} : \mathbb{R} \to \mathbb{R}$ so that

$$g_{2,b}(y) := f(b,y).$$

Again we have $g'_{2,b}(x) = \frac{\partial f}{\partial y}(b,y)$. Then by assumption we have that for all $a,b \in \mathbb{R}$,

$$g'_{1,a}(x) = g'_{2,b}(y) = 0.$$

Hence,

$$g_{1,a}(x) = c_{1,a} \qquad \text{and} \qquad g_{2,b}(y) = c_{2,b}.$$

We aim to show that for some $c \in \mathbb{R}$, $c_{1,a} = c_{2,b} = c$ for all $a,b \in \mathbb{R}$. Observe that because $g_{1,a}(x) = f(x,a)$, and $g_{2,b}(y) = f(b,y)$ then

$$c_{2,b} = g_{2,b}(a) = f(b,a) = g_{1,a}(b) = c_{1,a}.$$

Now for arbitrary $(\alpha,\beta) \in \mathbb{R}^2$ we have that $f(\alpha,\beta) = f(\alpha,b) = c$. So $f(x,y) = c$ for all $(x,y) \in \mathbb{R}^2$.

12. See book description.

Solution:

a. Suppose that there are two potential functions $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$ for ϕ and ψ. Then we have that

$$\frac{\partial}{\partial x}(f_1 - f_2) = \phi - \phi = 0.$$

and

$$\frac{\partial}{\partial y}(f_1 - f_2) = \psi - \psi = 0.$$

Hence by the previous problem we have that

$$f_1(x, y) - f_2(x, y) = c,$$

which implies that f_1 and f_2 only differ by a constant.

b. Since ϕ and ψ are continuously differentiable, then f has continuous second-order partials and by Theorem 13.10

$$\frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}.$$

Hence

$$\frac{\partial \psi}{\partial x}(x, y) = \frac{\partial \phi}{\partial y}(x, y).$$

1. For each of the following functions, find the derivative vector $\nabla f(\mathbf{x})$ for those points $\mathbf{x} \in \mathbb{R}^2$ where it is defined:

a. $f(\mathbf{x}) = e^{||\mathbf{x}||^2}$.

b. $f(x, y) = \frac{\sin(xy)}{\sqrt{x^2+y^2+1}}$

c. $f(\mathbf{x}) = \frac{1}{||\mathbf{x}||^2}$.

Solution:

a. ∇f is defined for all $\mathbf{x} \in \mathbb{R}^2$, with

$$\nabla f(x, y) = (2xe^{||\mathbf{x}||^2}, 2ye^{||\mathbf{x}||^2}).$$

b. ∇f is defined for all $\mathbf{x} \in \mathbb{R}^2$ with

$$\begin{aligned}&\nabla f(x, y) = \\ &(y\cos(xy)(x^2 + y^2 + 1)^{-1/2} - x\sin(xy)(x^2 + y^2 + 1)^{-3/2}, \\ &x\cos(xy)(x^2 + y^2 + 1)^{-1/2} - y\sin(xy)(x^2 + y^2 + 1)^{-3/2}).\end{aligned}$$

c. ∇f is defined everywhere except for $(0, 0)$. For $(x, y) \neq (0, 0)$ we have

$$\nabla f(x, y) = \Big(\frac{-2x}{(x^2 + y^2)^2}, \frac{-2y}{(x^2 + y^2)^2}\Big).$$

3. Suppose that the functions $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuously differentiable. Find a formula for $\nabla(g \circ f)(\mathbf{x})$ in terms of $\nabla f(\mathbf{x})$ and $g'(f(\mathbf{x}))$.

Solution: By definition we have

$$\nabla(g \circ f)(\mathbf{x}) = (\frac{\partial(g \circ f)}{\partial x_1}(\mathbf{x}), \frac{\partial(g \circ f)}{\partial x_2}(\mathbf{x}), \dots, \frac{\partial(g \circ f)}{\partial x_n}(\mathbf{x})).$$

Now we know that by the chain rule so that for $1 \leqslant i \leqslant n$,

$$\frac{\partial(g \circ f)}{\partial x_i}(\mathbf{x}) = g'(f(\mathbf{x}))\frac{\partial f}{\partial x_i}.$$

Thus we have

$$\nabla(g \circ f)(\mathbf{x}) = \nabla f(\mathbf{x})g'(f(\mathbf{x})).$$