MATH 369 Homework 2

Due: Thursday February 7, in class.

1. Write a short justification for each of your answers below:

   (a) If A is an augmented matrix for a system with three unknowns and five equations, what is the maximum number of leading 1's in its reduced row echelon form?

   (b) If B is an augmented matrix for a system with five unknowns and three equations, what is the maximum possible number of free parameters in the system?

   (c) If C is an augmented matrix for a system with three unknowns and five equations, then what is the minimum possible number of rows of zeros in any row echelon form of C?

   Solution:

   (a) We can have at most 1 leading one in each row and column (we ignore the column corresponding to the constants in the system). Since the number of unknowns is less than the number of equations, this is the limiting factor in determining how many leading 1's we can have. In this case then we can have at most 3 leading 1's.

   (b) Note that if all equations have coefficients of zero, then every unknown is free (this corresponds to the augmented matrix with all zero rows). If for some reason we do not allow this case, then thinking back to #3 in the previous homework, we see that we could have that in reduced row echelon form we have one non-zero row. In this case we have one leading 1 and 4 free variables.

   (c) Since each row of a system put into reduced row echelon form either has a leading 1 or is a zero row then the minimum number of possible rows of zeros is equivalent to the maximum number leading 1's. We already decided that this was 3 in (a). Hence the minimum number of rows of zeros is 2.

2. Consider the matrices

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -2 & 3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 \\ 3 & -4 \\ -4 & -2 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 \\ -3 & 4 \\ 2 & 0 \end{pmatrix}$$

   For each of the expressions below either compute the result or state that it is not defined.

   (a) $A + B$,

   (b) $B + C$,

   (c) AB,

   (d) BC,

   (e) $AC + B$.

   Solution:

   (a) This isn't defined since A and B don't have the same size.

   (b) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ -2 & -2 \end{pmatrix}$

   (c) $\begin{pmatrix} -7 & -13 \\ 7 & -16 \end{pmatrix}$

   (d) This isn't defined since the inner dimensions of B and C don't match.

   (e) This isn't defined since AC will have size 2×2 whereas B has size 3×2.

3. (a) Give an example of a 2×2 matrix A and a 2×2 matrix B such that $AB \neq BA$.

(b) Give an example of a 2×2 matrix A and a 2×2 matrix B such that $AB = BA$.

Solution:

(a) One example is given by

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

We have

$$AB = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

but

$$BA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

(b) One example is

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We have

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

4. Let

$$A = \begin{pmatrix} -1 & 7 \\ 7 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 6 & -2 \\ -2 & -\frac{1}{2} \end{pmatrix}.$$

(a) Show that $A^T = A$ and $B^T = B$.

(b) Give a general condition that tells whether a 2×2 matrix

$$C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has the property that $C^T = C$.

Solution:

(a) You should check this but it is routine.

(b) Note that taking the transpose of a 2×2 matrix only switches element $(2,1)$ and $(1,2)$. Hence the matrix C is invariant with respect to the transpose if and only if $c = b$.