MATH 417 Homework 5

Note that this is problems: Section 13.3: #4, #6,#11, Section 14.1: #11, #15.

4. Suppose that the function $f : \mathbb{R}^n \to \mathbb{R}$ has first-order partial derivatives and that the point $\mathbf{x} \in \mathbb{R}^n$ is a local minimizer of f, meaning that there is $r > 0$ such that

$$f(\mathbf{x} + \mathbf{h}) \geqslant f(\mathbf{x})$$

if $dist(\mathbf{x} + \mathbf{h}, \mathbf{x}) < r$. Prove that $\nabla f(\mathbf{x}) = 0$.

Solution: For any $1 \leqslant i \leqslant n$, define $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(t) = f(\mathbf{x} + t\mathbf{e}_i).$$

Then it follows that for any $-r < t < r$, $\phi(t) \geqslant \phi(0)$ since $f(\mathbf{x} + t\mathbf{e}_i) \geqslant f(\mathbf{x})$, so $t = 0$ is a local minimum of $\phi(t)$. Furthermore, since f has first-order partial derivatives then

$$\phi'(t) = \frac{\partial f}{\partial x_i}(\mathbf{x} + t\mathbf{e}_i)$$

exists and it follows from single variable calculus that

$$\frac{\partial f}{\partial x_i}(\mathbf{x}) = \phi'(0) = 0.$$

Since this is true for all $1 \leqslant i \leqslant n$, the result follows.

6. Define $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x, y, z) = xyz + x^2 + y^2.$$

The MVT implies that there is a $0 < \theta < 1$ such that

$$f(1,1,1) - f(0,0,0) = \frac{\partial f}{\partial x}(\theta,\theta,\theta) + \frac{\partial f}{\partial y}(\theta,\theta,\theta) + \frac{\partial f}{\partial z}(\theta,\theta,\theta).$$

Find such a value θ.

Solution: We can take partial derivatives of f to obtain

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y,z) &= yz + 2x, \\
\frac{\partial f}{\partial y}(x,y,z) &= xz + 2y, \\
\frac{\partial f}{\partial z}(x,y,z) &= xy.
\end{aligned}$$

At (θ,θ,θ) these are

$$\begin{aligned}
\frac{\partial f}{\partial x}(\theta,\theta,\theta) &= \theta^2 + 2\theta, \\
\frac{\partial f}{\partial y}(\theta,\theta,\theta) &= \theta^2 + 2\theta, \\
\frac{\partial f}{\partial z}(\theta,\theta,\theta) &= \theta^2.
\end{aligned}$$

So what we need to solve is

$$3 = f(1,1,1) - f(0,0,0) = 3\theta^2 + 4\theta.$$

Applying the quadratic formula to this gives $\theta = -\frac{2}{3} - \frac{\sqrt{13}}{3}$ and $\theta = -\frac{2}{3} + \frac{\sqrt{13}}{3}$. Only the second of these is between 0 and 1, which gives us our answer.

11. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable. Define

$$K = \{ \mathbf{x} \in \mathbb{R}^n \mid ||\mathbf{x}|| \leqslant 1 \}.$$

Solution:

a. Prove that there is a point $\mathbf{x} \in K$ such that f obtains its minimum value.
b. Assume that if $\mathbf{p} \in \mathbb{R}^n$ and $||\mathbf{p}|| = 1$ then $\langle \nabla f(\mathbf{p}), \mathbf{p} \rangle > 0$. Show that the minimizer $\mathbf{x}$ in part a. has norm less than 1.

Solution:

a. First observe that by Theorem 13.20, since f is continuously differentiable, then f is continuous. Next, K is sequentially compact since it is closed and bounded. It then follows from the Extreme Value Theorem that f obtains a minimum and maximum value in K.
b. Next assume that if $\mathbf{p} \in \mathbb{R}^n$ and $||\mathbf{p}|| = 1$ then $\langle \nabla f(\mathbf{p}), \mathbf{p} \rangle > 0$. Assume for a contradiction that f obtains its minimum value on the boundary of K (i.e. at a point $\mathbf{p}$ such that $||\mathbf{p}|| = 1$). Then if we shrink $\mathbf{p}$ to $\frac{(k-1)\mathbf{p}}{k}$, then since $\frac{(k-1)\mathbf{p}}{k}$ lies in K, then

$$f\Big(\frac{(k-1)\mathbf{p}}{k}\Big) - f(\mathbf{p}) \geqslant 0.$$

Write $\mathbf{x} = \mathbf{p}$ and $\mathbf{h}_k = \frac{-\mathbf{p}}{k}$. Then the MVT tells us that there is $0 < \theta_k < 1$ such that

$$0 \leqslant f\Big(\frac{(k-1)\mathbf{p}}{k}\Big) - f(\mathbf{p}) = \langle \nabla f\Big(\frac{(k-\theta_k)\mathbf{p}}{k}\Big), \frac{-\mathbf{p}}{k} \rangle.$$

Multiplying by $-(k-\theta_k)$ then gives

$$0 \geqslant \langle \nabla f\Big(\frac{(k-\theta_k)\mathbf{p}}{k}\Big), \frac{(k-\theta_k)\mathbf{p}}{k} \rangle.$$

Since the scalar product and ∇f are both continuous, we can take the limit as $k \to \infty$, to get

$$0 \geqslant \langle \nabla f(\mathbf{p}), \mathbf{p} \rangle.$$

But this is a contradiction to our assumptions.

1. Define $f(x, y) = e^{2x+4y+1}$ for $(x, y) \in \mathbb{R}^2$. Find the equation for the tangent line of the graph of f at $(0, 0, e)$.

Solution: We can calculate that

$$\frac{\partial f}{\partial x}(x, y) = 2e^{2x+4y+1},$$
$$\frac{\partial f}{\partial x}(x, y) = 4e^{2x+4y+1}.$$

So,

$$f(0, 0) = e$$
$$\frac{\partial f}{\partial x}(0, 0) = 2e,$$
$$\frac{\partial f}{\partial x}(0, 0) = 4e.$$

Hence we get that the equation for the tangent plane is

$$\psi(x, y) = e + 2e(x) + 4e(y).$$

3. Let a, b, and c be positive numbers. The set of points $(x, y, z) \in \mathbb{R}^3$ such that

$$(x/a)^2 + (y/b)^2 - (z/c)^2 = 1$$

is called a hyperboloid. Find the equation of the tangent plane to this hyperboloid at (x_0, y_0, z_0) for z_0 positive.

Solution: Solving for z we get that this surface is the graph of the function

$$f(x, y) = z = \pm c\sqrt{(x/a)^2 + (y/b)^2 + 1}.$$

We want z positive. Since we know c is positive, we pick the positive branch above. The partial derivatives with respect to x and y are then

$$\begin{aligned}
\frac{\partial f}{\partial x}(x, y) &= \frac{cx}{a\sqrt{(x/a)^2 + (y/b)^2 + 1}}, \\
\frac{\partial f}{\partial y}(x, y) &= \frac{cy}{b\sqrt{(x/a)^2 + (y/b)^2 + 1}}.
\end{aligned}$$

Hence the tangent plane is given by

$$\psi(x, y) = c\sqrt{(x_0/a)^2 + (y_0/b)^2 + 1} + \frac{cx_0(x - x_0)}{a\sqrt{(x_0/a)^2 + (y_0/b)^2 + 1}} + \frac{cy_0(y - y_0)}{b\sqrt{(x_0/a)^2 + (y_0/b)^2 + 1}}.$$