MATH 369 Homework 8

Due: Tuesday April 9th, in class.

1. Does the set of vectors
$$S = \left\{ \begin{pmatrix} 1 \\ 6 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} \right\}$$
have the same span as the set
$$S' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}?$$
Justify your answer.

   Solution: No. We already know that $span(S') = \mathbb{R}^3$. On the other hand, in order for $span(S) = \mathbb{R}^3$ we must have
$$\det \begin{pmatrix} 1 & 2 & -1 \\ 6 & 4 & 2 \\ 4 & -1 & 5 \end{pmatrix} \neq 0.$$
However, we can check that this determinant is actually equal to 0.

2. The solutions to the equation $A\mathbf{x} = \mathbf{0}$ where
$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$
form a subspace. Is this subspace the trivial subspace ($\{\mathbf{0}\}$) or is it larger?

   Solution: No, the solution space is not trivial. For example,
$$\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$
belongs to this subspace, and it is not the zero vector.

3. Determine whether the vectors are linearly independent or linearly dependent in $\mathbb{R}^3$:

   (a)
$$\begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}, \quad \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}.$$

   (b)
$$\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}, \quad \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 7 \\ 0 \\ -2 \end{pmatrix}.$$

   Solution:

   (a) This set is linearly independent. Indeed we can calculate that the determinant of the matrix
$$\begin{pmatrix} -3 & 5 & 1 \\ 0 & -1 & 1 \\ 4 & 2 & 3 \end{pmatrix}$$
is non-zero and hence the equation
$$\begin{pmatrix} -3 & 5 & 1 \\ 0 & -1 & 1 \\ 4 & 2 & 3 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

must have a unique solution. This is equivalent to saying that $k_1 = 0$, $k_2 = 0$, $k_3 = 0$ is the only coefficients that give

$$k_1 \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix} + k_2 \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} + k_3 \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

(b) This set must be linearly dependent by Theorem 4.3.3.

4. Remember that polynomials form a vector space. Show that the polynomials $\{2 - x + 4x^2, 3 + 6x + 2x^2, -5 - 20x + 2x^2\}$ are linearly dependent.

   Solution: We have

$$2(2 - x + 4x^2) + (-3)(3 + 6x + 2x^2) + (-1)(-5 - 20x + 2x^2) = 0.$$

5. Show that for any vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in a vector space V, the vectors $\mathbf{u} - \mathbf{v}$, $\mathbf{v} - \mathbf{w}$, and $\mathbf{w} - \mathbf{u}$ form a linearly dependent set.

   Solution: We have

$$(\mathbf{u} - \mathbf{v}) + (\mathbf{v} - \mathbf{w}) + (\mathbf{w} - \mathbf{u}) = \mathbf{0}.$$