MATH 369 Homework 6

Due: Thursday March 14, in class.

1. Choose t so that the vectors:

$$\mathbf{v} = \begin{pmatrix} 10 \\ -2 \\ -1 \\ 3 \\ t \end{pmatrix} \qquad \text{and} \qquad \mathbf{w} = \begin{pmatrix} 2 \\ -3 \\ 4 \\ 5 \\ 1 \end{pmatrix}$$

are orthogonal.

Solution: For $\mathbf{v}$ and $\mathbf{w}$ to be orthogonal we must have

$$0 = \mathbf{v} \cdot \mathbf{w} = 20 + 6 - 4 + 15 + t,$$

so we get $t = -37$.

2. Suppose that $\mathbf{v}$ is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$ in $\mathbb{R}^n$. Show that $\mathbf{v}$ is orthogonal to $k_1\mathbf{u}_1 + k_2\mathbf{u}_2$ for any $k_1, k_2 \in \mathbb{R}$.

Solution: If $\mathbf{v}$ is orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$ then we have

$$\mathbf{v} \cdot \mathbf{u}_1 = 0 \qquad \text{and} \qquad \mathbf{v} \cdot \mathbf{u}_2 = 0.$$

Thus, by the properties of the dot product we have

$$\begin{aligned} \mathbf{v} \cdot (k_1\mathbf{u}_1 + k_2\mathbf{u}_2) &= \mathbf{v} \cdot (k_1\mathbf{u}_1) + \mathbf{v} \cdot (k_2\mathbf{u}_2) \\ &= k_1(\mathbf{v} \cdot \mathbf{u}_1) + k_2(\mathbf{v} \cdot \mathbf{u}_2) = 0 + 0 = 0. \end{aligned}$$

3. Decompose the vector

$$\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

into a sum $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1$ has the same direction as

$$\mathbf{z} = \begin{pmatrix} 4 \\ -4 \\ 2 \\ -2 \end{pmatrix}$$

and $\mathbf{w}_2$ is orthogonal to $\mathbf{z}$.

Solution: To find $\mathbf{w}_1$ we compute the projection of $\mathbf{u}$ onto $\mathbf{z}$ to get

$$\mathbf{w}_1 = \left(\frac{\mathbf{u} \cdot \mathbf{z}}{||\mathbf{z}||^2}\right)\mathbf{z} = \frac{2}{40}\begin{pmatrix} 4 \\ -4 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{8}{40} \\ -\frac{8}{40} \\ \frac{4}{40} \\ -\frac{4}{40} \end{pmatrix}.$$

The part orthogonal to $\mathbf{z}$ is then

$$\mathbf{u} - \mathbf{w}_1 = \begin{pmatrix} \frac{72}{40} \\ \\ \frac{48}{40} \\ \\ \frac{36}{40} \\ \\ \frac{84}{40} \end{pmatrix}.$$

4. For each of the sets V described below with specified addition and scalar multiplication operation, state whether V is a vector space or not. If it is not a vector space, explain at least one axiom that it violates. If it is a vector space, justify this by showing that the 10 axioms all hold.

(i)
- The set: V is the set of polynomials with real coefficients.
- Addition operation: the standard addition of polynomials.
- Scalar multiplication: the standard multiplication of a polynomial by a real number.

(ii)
- The set: V is the set of vectors in $\mathbb{R}^2$ taking the form

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \text{where} \quad x_2 > 0.$$

- Addition operation: the standard addition of vectors in $\mathbb{R}^2$.
- Scalar multiplication: the standard multiplication of vectors in $\mathbb{R}^2$ by a scalar.

(iii)
- The set: V is the set of vectors in $\mathbb{R}^2$ taking the form

$$\mathbf{x} = \begin{pmatrix} 2t \\ t \end{pmatrix}.$$

- Addition operation: the standard addition of vectors in $\mathbb{R}^2$.
- Scalar multiplication: the standard multiplication of vectors in $\mathbb{R}^2$ by a scalar.

(iv)
- The set: V is all 2×2 matrices.
- Addition operation: the standard addition of matrices of size 2×2.
- Scalar multiplication: the standard multiplication of 2×2 matrices by a scalar.

(v)
- The set: V is all vectors v in $\mathbb{R}^3$ such that $||v|| = 1$ (that is, all points on the unit-sphere).
- Addition operation: the standard addition of vectors in $\mathbb{R}^3$.
- Scalar multiplication: the standard multiplication of vectors in $\mathbb{R}^3$ by a scalar.

Solution:

(i) The set of polynomials with real coefficients is a vector space. We check the following. We let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

and

$$g(x) = b_k x^k + b_{k-1} x^{k-1} + \cdots + b_1 x + b_0$$

be some arbitrary polynomials for $a_n, \ldots, a_0, b_k, \ldots, b_0$ real numbers and $n, k \geqslant 0$ integers. We will assume that $k < n$.

(1) For any two polynomials

$$\begin{aligned} f(x) + g(x) &= (a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0) + (b_k x^k + b_{k-1} x^{k-1} + \cdots + b_1 x + b_0) \\ &= a_n x^n + \cdots + (a_k + b_k) x^k + \cdots + (a_1 + b_1) x + (a_0 + b_0). \end{aligned}$$

It is clear that this last term is still a polynomial. So the first axiom holds.

(2) Commutativity holds because we have

$$f(x)+g(x) = (a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0) + (b_kx^k + b_{k-1}x^{k-1} + \cdots + b_1x + b_0)$$
$$= a_nx^n + \cdots + (a_k + b_k)x^k + \cdots + (a_1 + b_1)x + (a_0 + b_0)$$
$$= a_nx^n + \cdots + (b_k + a_k)x^k + \cdots + (b_1 + a_1)x + (b_0 + a_0)$$
$$= (b_kx^k + b_{k-1}x^{k-1} + \cdots + b_1x + b_0) + (a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0) = g(x) + f(x).$$

(3) Associativity holds using an almost identical argument to above (basically it holds because it holds for the real numbers).

(4) There is a zero vector among the polynomials, the constant polynomial $z(x) = 0$. Indeed, an easy calculation shows that

$$z(x) + f(x) = 0 + (a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0) = (a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0) = f(x).$$

(5) The negative vector corresponding to polynomial

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

is just

$$-f(x) = -a_nx^n - a_{n-1}x^{n-1} - \cdots - a_1x - a_0.$$

Indeed,

$$f(x) + (-f(x)) = (a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0) + (-a_nx^n - a_{n-1}x^{n-1} - \cdots - a_1x - a_0) = 0$$

and we saw that $z(x) = 0$ is the zero vector for this space.

(6) For any scalar λ and an arbitrary polynomial $f(x)$ we have

$$\lambda f(x) = \lambda(a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0)$$
$$= \lambda a_nx^n + \lambda a_{n-1}x^{n-1} + \cdots + \lambda a_1x + \lambda a_0.$$

But since $\lambda a_n, \ldots, \lambda a_0$ are just real numbers, this is again a polynomial. So this axiom holds.

(7) Scalar multiplication distributes with respect to vector addition. We have

$$\lambda(f(x) + g(x)) = \lambda\Big((a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0) + (b_kx^k + b_{k-1}x^{k-1} + \cdots + b_1x + b_0)\Big)$$
$$= \lambda(a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0) + \lambda(b_kx^k + b_{k-1}x^{k-1} + \cdots + b_1x + b_0)$$
$$= \lambda f(x) + \lambda g(x).$$

(8) Scalar multiplication distributes with respect to scalar addition. For another scalar μ we have

$$(\lambda + \mu)f(x) = (\lambda + \mu)(a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0)$$
$$\lambda(a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0) + \mu(a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0)$$
$$\lambda f(x) + \mu f(x).$$

(9) Scalar multiplication behaves nicely with respect to multiplication of scalars. We have

$$\lambda(\mu f(x)) = \lambda(\mu a_nx^n + \mu a_{n-1}x^{n-1} + \cdots + \mu a_1x + \mu a_0)$$
$$= \lambda\mu a_nx^n + \lambda\mu a_{n-1}x^{n-1} + \cdots + \lambda\mu a_1x + \lambda\mu a_0$$
$$= \lambda\mu(a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0) = (\lambda\mu)f(x).$$

(10) Finally, scalar multiplication by the number 1 fixes a vector.

$$1 \cdot f(x) = 1(a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = f(x).$$

(ii) This is not a vector space. To see this, note that the vector

$$\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is in V, but

$$(-1)\mathbf{v} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

is not in V because the second entry is less than 0. Hence axiom (6) is violated.

(iii) This set is a vector space. We note that because V is a subset of $\mathbb{R}^2$ with the same vector addition and scalar multiplication operations, we can use the fact that $\mathbb{R}^2$ is a vector space to show that some of the axioms also hold for V.

(1) Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in V so that for some real numbers t, r we have

$$\mathbf{u} = \begin{pmatrix} 2t \\ t \end{pmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{pmatrix} 2r \\ r \end{pmatrix}.$$

Then we can write

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} 2t \\ t \end{pmatrix} + \begin{pmatrix} 2r \\ r \end{pmatrix} = \begin{pmatrix} 2(t+r) \\ (t+r) \end{pmatrix}$$

and this vector must be in V because $(t+r)$ is just another real number like t and r.

(2) Commutativity of vector addition holds because this holds for all vectors in $\mathbb{R}^2$.

(3) Associativity of vector addition holds because this holds for all vectors in $\mathbb{R}^2$.

(4) We already know the vector that plays the role of the zero vector in $\mathbb{R}^2$ is

$$\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By choosing $t = 0$ we get that

$$\begin{pmatrix} 2t \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is in V.

(5) Since all the elements of V are also in $\mathbb{R}^2$, then we know that the additive inverse of

$$\begin{pmatrix} 2t \\ t \end{pmatrix}$$

is just

$$\begin{pmatrix} -2t \\ -t \end{pmatrix} = \begin{pmatrix} 2(-t) \\ (-t) \end{pmatrix}.$$

It follows that for any vector in V, the inverse is also in V because we can just choose parameter $(-t)$ instead of t.

(6) If

$$\mathbf{u} = \begin{pmatrix} 2t \\ t \end{pmatrix}$$

is in V then for scalar λ,

$$\lambda\mathbf{u} = \begin{pmatrix} \lambda 2t \\ \lambda t \end{pmatrix} = \begin{pmatrix} 2(\lambda t) \\ (\lambda t) \end{pmatrix}$$

is in V by choosing λt instead of t.

The rest of the properties follow directly from the fact that vectors in V are also in $\mathbb{R}^2$ with the same vector addition and scalar multiplication actions and axioms (7)-(10) then hold because $\mathbb{R}^2$ is a vector space.

(iv) The set of all 2×2 matrices with real entries is a vector space. We check the following. We let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and

$$B = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

be some arbitrary 2×2 matrices.

(1) It is easy to check that

$$A + B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} a+x & b+y \\ c+z & d+w \end{pmatrix}$$

is also a matrix.

(2) Commutativity

$$A + B = B + A$$

holds by Theorem 1.4.1(a) in the textbook.

(3) Let C be another 2×2 matrix. Associativity

$$A + (B + C) = (A + B) + C$$

holds by Theorem 1.4.1(b) in the textbook.

(4) The 2×2 zero matrix $\mathbf{0}$ has the role of zero vector since

$$A + \mathbf{0} = A$$

for any 2×2 matrix A by Theorem 1.4.2(a).

(5) Any 2×2 matrix A has an additive inverse given by $(-1)A = -A$. Indeed, by Theorem 1.4.2(c)

$$A + (-A) = \mathbf{0}.$$

(6) For any scalar λ and an 2×2 matrix A we have

$$\lambda A = \lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}.$$

This last term is also a 2×2 matrix, so this axiom is satisfied.

(7) Scalar multiplication distributes with respect to vector addition. This follows from Theorem 1.4.1(h).

(8) Scalar multiplication distributes with respect to scalar addition. This follows from Theorem 1.4.1(j).

(9) Scalar multiplication behaves nicely with respect to multiplication of scalars. This follows from Theorem 1.4.1(l).

(10) Finally, scalar multiplication by the number 1 fixes a matrix.

$$1 \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} (1)a & (1)b \\ (1)c & (1)d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(v) This is not a vector space. To see this note that the vectors

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

are both in V since

$$||\mathbf{u}|| = \sqrt{1^2 + 0^2 + 0^2} = 1$$

and

$$||\mathbf{v}|| = \sqrt{0^2 + 1^2 + 0^2} = 1.$$

On the other hand, $\mathbf{u} + \mathbf{v}$ is NOT in V since

$$||\mathbf{u} + \mathbf{v}|| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}.$$

Thus axiom (1) is violated.