MATH 369 Homework 10

Due: Tuesday April 23th, in class.

1. Find a basis for the null space and row space of

   (a) $A = \begin{pmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{pmatrix}$

   (b) $A = \begin{pmatrix} 2 & 0 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{pmatrix}$

   Solution:

   (a) The reduced row echelon form of this matrix is

   $$\begin{pmatrix} 1 & 0 & -16 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{pmatrix}.$$

   Then the row space of A has basis:

   $$\begin{pmatrix} 1 & 0 & -16 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & -19 \end{pmatrix}.$$

   On the other hand, by parametrizing all the solutions to $A\mathbf{x} = \mathbf{0}$, we can calculate that one null space basis for A is

   $$\begin{pmatrix} 16 \\ 19 \\ 1 \end{pmatrix}.$$

   (b) The reduced row echelon form of this matrix is

   $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   It follows then that a basis for the row space is

   $$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}.$$

   On the other hand, since A is invertible, the null space is trivial ($\{\mathbf{0}\}$).

2. Find a basis for the row and column space of the matrix:

   $$A = \begin{pmatrix} -1 & -4 & -7 & -3 \\ 2 & 0 & 2 & -2 \\ 2 & 3 & -4 & 1 \end{pmatrix}.$$

   Solution: Putting A into reduced row echelon form we have

   $$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

   Therefore a basis for the row space is given by

   $$\begin{pmatrix} 1 & 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}.$$

On the other hand, by noting the position of the leading 1's in the reduced row echelon form of the matrix, we get that a basis for the column space is

$$\begin{pmatrix}-1\\2\\2\end{pmatrix}, \quad \begin{pmatrix}-4\\0\\3\end{pmatrix}, \quad \begin{pmatrix}-7\\2\\-4\end{pmatrix}.$$

3. Find a subset of the vectors

$$\mathbf{v}_1 = \begin{pmatrix}1\\0\\1\\1\end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix}-3\\3\\7\\1\end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix}-1\\3\\9\\3\end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix}-5\\3\\5\\-1\end{pmatrix},$$

which form a basis for the space $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$.

Solution: The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are the columns of the matrix

$$A = \begin{pmatrix}1 & -3 & -1 & -5\\0 & 3 & 3 & 3\\1 & 7 & 9 & 5\\1 & 1 & 3 & -1\end{pmatrix}$$

When we transform this into reduced row echelon form this becomes

$$A = \begin{pmatrix}1 & 0 & 2 & -2\\0 & 1 & 1 & 1\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix}.$$

It follows then that a basis for $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ is given by

$$\mathbf{v}_1 = \begin{pmatrix}1\\0\\1\\1\end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix}-3\\3\\7\\1\end{pmatrix}.$$

4. Find a 3×3 matrix whose null space is:

   (a) a point,

   (b) a line,

   (c) a plane,

   (d) all of $\mathbb{R}^3$.

   Solution:

   (a) Any 3×3 matrix that is invertible will have a point as its null space. Therefore, an example would be

   $$A = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}.$$

   (b) Any 3×3 matrix whose null space is a line must have a 1-dimensional null space. One example of this would be

   $$A = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix}.$$

   (c) Any 3×3 matrix that has a plane as its null space must have a 2-dimensional null space. One example of this would be

   $$A = \begin{pmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix}.$$

(d) Any 3×3 matrix that has all of $\mathbb{R}^3$ as its null space must send every vector in $\mathbb{R}^3$ to the zero vector. Hence the only option is

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

5. Let A be a 5×7 matrix with rank 4.

(a) What is the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$? Explain.

(b) Does $A\mathbf{x} = \mathbf{b}$ have a solution for all vectors $\mathbf{b}$ in $\mathbb{R}^5$? Explain.

Solution:

(a) The dimension of the solution space of $A\mathbf{x} = \mathbf{0}$ is the nullity. Since A has 7 columns and rank 4 then the nullity must be 3.

(b) No. The column space has dimension 4 but the dimension of $\mathbb{R}^5$ is 5. It follows that there will be linear systems that will be inconsistent.