MATH 417 Homework 2

Due: Friday, August 31, in class.
Note that this is problems: Chapter 11.2: #3, #6,#7,#8,#10.

3. Show that an open ball in $\mathbb{R}^n$ is bounded.

   Solution: Let $\mathbf{u}$ be a point in $\mathbb{R}^n$ and consider the open ball $B_r(\mathbf{u})$ for $r > 0$. We will show that $B_r(\mathbf{u})$ is bounded. Set $M = ||\mathbf{u}||$. We claim that for all $\mathbf{v} \in B_r(\mathbf{u})$, $||\mathbf{v}|| < M + r$. To prove this, note that by the triangle inequality:
$$dist(\mathbf{0}, \mathbf{v}) \leqslant dist(\mathbf{0}, \mathbf{u}) + dist(\mathbf{u}, \mathbf{v}) < M + r.$$
   It follows that $B_r(\mathbf{u})$ is bounded.

6. Let A be a subset of $\mathbb{R}^n$ and let the function $f : A \to \mathbb{R}$ be continuous.

   (a) If A is bounded, is $f(A)$ bounded?

   (b) If A is closed, is $f(A)$ closed?

   Solution:

   (a) Not necessarily. Consider the following counterexample. We know that the set $A = (0, 1)$ is bounded in $\mathbb{R}$ and the function $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = 1/x$ is continuous. However $f(A) = (1, \infty)$ which is definitely not bounded.

   (b) Not necessarily. Consider the following counterexample. We know that $\mathbb{R}$ is closed in $\mathbb{R}$ and that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \arctan(x)$ is continuous on $\mathbb{R}$. Then it can be checked that $f(\mathbb{R}) = (-\pi/2, \pi/2)$, which is not closed.

7. Suppose that the function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and that $f(\mathbf{u}) \geqslant ||\mathbf{u}||$ for every point in $\mathbf{u}$ in $\mathbb{R}^n$. Prove that $f^{-1}([0, 1])$ is sequentially compact.

   Solution: We will show that $f^{-1}([0, 1])$ is closed and bounded. It will then follow from Theorem 11.18 that $f^{-1}([0, 1])$ is sequentially compact. First note that because $[0, 1]$ is closed in $\mathbb{R}$ and f is continuous, then $f^{-1}([0, 1])$ must be closed, this is Theorem 11.12. It remains to show that $f^{-1}([0, 1])$ is bounded.

   We show that for all $\mathbf{u} \in f^{-1}([0, 1])$, $||\mathbf{u}|| \leqslant 1$. Because $\mathbf{u} \in f^{-1}([0, 1])$, there is $x \in [0, 1]$ such that $f(\mathbf{u}) \in [0, 1]$. Hence we have
$$||\mathbf{u}|| \leqslant f(\mathbf{u}) < 1.$$
   It follows that $f^{-1}([0, 1])$ is bounded and therefore $f^{-1}([0, 1])$ is sequentially compact.

8. Let A and B be sequentially compact subsets of $\mathbb{R}$. Define $K = \{(x, y) \in \mathbb{R}^2 \mid x \in A, y \in B\}$. Prove that K is sequentially compact.

   Solution: Suppose that $\{(x_k, y_k)\}_{k \geqslant 0}$ is a sequence in K, so that $\{x_k\}_{k \geqslant 0}$ and $\{y_k\}_{k \geqslant 0}$ are sequences in A and B respectively. Since A is sequentially compact, $\{x_k\}_{k \geqslant 0}$ has a subsequence $\{x_{k_j}\}_{j \geqslant 0}$ that converges to a value $x \in A$. We can use $\{x_{k_j}\}_{j \geqslant 0}$ to define a subsequence of $\{y_k\}_{k \geqslant 0}$ in B given by $\{y_{k_j}\}_{j \geqslant 0}$. Since B is sequentially continuous, $\{y_{k_j}\}_{j \geqslant 0}$ also has a convergent subsequence $\{y_{k_{j_i}}\}_{i \geqslant 0}$ which converges to some y in B. This defines a subsequence of $\{x_{k_j}\}_{j \geqslant 0}$, $\{x_{k_{j_i}}\}_{i \geqslant 0}$, that also converges to x since $\{x_{k_j}\}_{j \geqslant 0}$ converges to x. Then $\{(x_{k_{j_i}}, y_{k_{j_i}})\}_{i \geqslant 0}$ is a subsequence of $\{(x_k, y_k)\}_{k \geqslant 0}$ in K and by the componentwise convergence criterion for sequences, this subsequence converges to $(x, y) \in K$. Since this argument applies to all sequences in K, it follows that K is sequentially compact.

10. A mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *Lipschitz* if there is a number C
$$dist(F(\mathbf{u}), F(\mathbf{v})) \leqslant C dist(\mathbf{u}, \mathbf{v})$$
    for all points $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$. Show that a Lipschitz mapping is uniformly continuous.

Solution: Pick a point $\mathbf{u} \in \mathbb{R}^n$. For any $\epsilon > 0$, set $\delta = \frac{\epsilon}{C}$. Then because F is Lipschitz continuous, we have that for all $\mathbf{v}$ such that

$$dist(\mathbf{u}, \mathbf{v}) < \delta$$

we have

$$dist(F(\mathbf{u}), F(\mathbf{v})) \leqslant C dist(\mathbf{u}, \mathbf{v}) < C\delta = \epsilon.$$

Since our choice of δ does not depend upon $\mathbf{u}$, then F is uniformly continuous.