

# Research Statement

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## 1 Introduction

My research interests in mathematics focus on the connections between representation theory, combinatorics, and probability theory. I am particularly interested in the interaction between combinatorial representation theory and categorical representation theory. In this research statement I will describe some of my past and current projects which include:

- investigating a surprising connection between graphical Heisenberg categories on the one hand and symmetric functions and stochastic growth processes on the other.
- the study of crystal graphs associated to affine type quantum groups and their connections to the representation theory of cyclotomic quiver Hecke algebras and lattices of Young diagrams.

I am also doing work in geometric data analysis, machine learning, and compressive sensing. I briefly summarize some of my work in these fields in Section 3.

## 2 Representation theory and combinatorics

### 2.1 The categorification program

Initiated by Crane and Frenkel [4], the idea underlying the categorification program is that many of the fundamental structures in algebra, geometry, and topology are the shadow of richer (often undiscovered) higher categories. Categorification, the process of revealing these higher analogues, then involves lifting set-theoretic constructions to category-theoretic constructions, roughly: replacing sets with categories, functions with functors, and equalities with isomorphisms. Though Khovanov’s categorification of the Jones polynomial via Khovanov homology [14] may be the most famous example, far more familiar examples exist (for example, singular homology as a categorification of Betti numbers). Categorifying the fundamental algebras from representation theory has been a major, ongoing project in the field. Furthermore, theory developed in the course of this program has led to results of independent interest, such as the proof of the Kazhdan-Lusztig conjecture for all Coxeter types [5] via Soergel bimodules [34].

As consequence of this progress, a host of new categories have been discovered which sit “above” and “control” many of the classical objects in representation theory such as Hecke algebras, Heisenberg algebras, and quantum groups. These categories are rich in structure, and a major drive in my research has been using these categorical constructions to better understand the connections between representation theory and other areas of math, particularly combinatorics and probability theory. My first research project described below studies how categorifications of infinite dimensional Heisenberg algebras both provide graphical realizations of analogues of the symmetric functions and also naturally provide the framework for

stochastic growth processes motivated by representation theory and combinatorics [21, 22]. In the second project, the categorification of quantum groups, via quiver Hecke algebras, is used to show how a fundamental isomorphism of combinatorially defined crystals can in fact be understood to be a consequence of induction/restriction functors from representation theory [23].

## 2.2 Heisenberg categories, symmetric functions, and growth processes

A *tower of algebras*  $\{A_n\}_{n \geq 0}$  is a collection of nested algebras  $A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$ . The classic example is the symmetric group algebras  $\{\mathbb{C}[S_n]\}_{n \geq 0}$ . A result of Geissinger [7] showed that the direct sum of all Grothendieck groups of symmetric groups  $\bigoplus_{n \geq 0} K_0(\mathbb{C}[S_n]\text{-mod})$  carries the structure of the Fock space representation of the infinite dimensional Heisenberg algebra  $\mathfrak{h}$ . In particular, the elements of  $\mathfrak{h}$  act by linear operators coming from induction and restriction functors between symmetric group representations. This observation was the first hint of a deeper story in which a vastly richer *Heisenberg category*  $\mathcal{H}$ , acts via genuine induction and restriction functors on the modules categories  $\bigoplus_{n \geq 0} A_n\text{-mod}$  for a tower of algebras  $\{A_n\}_{n \geq 0}$ .

$$\begin{array}{ccc}
 \mathcal{H} & \xrightarrow{\text{categorical representation}} & \bigoplus_{n \geq 0} A_n\text{-mod} \\
 \vdots \downarrow K_0 & & \downarrow K_0 \vdots \\
 \mathfrak{h} & \xrightarrow{\text{representation}} & \bigoplus_{n \geq 0} K_0(A_n\text{-mod})
 \end{array}$$

The first Heisenberg category was defined by Khovanov [15] with  $A_n = \mathbb{C}[S_n]$ . Subsequently, Heisenberg categories associated to a range of towers of algebras have been discovered, including: Hecke algebras [26], Sergeev algebras [3], wreath product algebras [31], and degenerate cyclotomic affine Hecke algebras [27, 2]. In all these examples,  $\mathcal{H}$  can be defined diagrammatically, i.e. through an isotopy invariant calculus of planar diagrams with local relations and decorations.

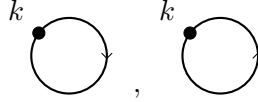
All Heisenberg categories are additive monoidal categories and hence their center, the endomorphisms of the monoidal identity  $Z(\mathcal{H}) := \text{End}_{\mathcal{H}}(1)$ , is a commutative algebra. In such diagrammatically defined categories, the center has a helpful diagrammatic definition as the algebra of all closed diagrams.

One of my ongoing projects is centered around studying the commutative algebra  $Z(\mathcal{H})$  corresponding to different towers of algebras. I started by studying the center of Khovanov's Heisenberg category  $Z(\mathcal{H}_{\mathbb{C}[S_n]})$ , and proved that it is isomorphic to  $\Lambda^*$ , the algebra of shifted

symmetric functions [21]. A deformation of the classical symmetric functions,  $\Lambda^*$  was discovered by Kerov and Olshanski [13] and further investigated by Okounkov [29] and Okounkov and Olshanski [30] among many others.

As a consequence of the algebra isomorphism  $Z(\mathcal{H}_{\mathbb{C}[S_n]}) \cong \Lambda^*$ , we were able to describe an isotopy invariant graphical calculus for  $\Lambda^*$  using closed diagrams and a set of local relations. We showed how individual generators such as shifted Schur functions and properties of  $\Lambda^*$  such as involutions emerge in a natural way from the graphical calculus.

Surprisingly, under the isomorphism  $Z(\mathcal{H}_{\mathbb{C}[S_n]}) \cong \Lambda^*$ , the simplest generating diagrams which are clockwise or counterclockwise oriented *bubbles* with  $k$  dots on them



are shifted symmetric functions encoding moments for a family of probability measures, Kerov's *transition* and *co-transition measures*, parametrized by Young diagrams. This same probability measure was used as a tool to study the famous Plancherel growth process [11, 12]. This unexpected connection points to a deeper relationship between the combinatorics of categorical representation theory and the combinatorics of asymptotic representation theory. Exploring this connection more closely is one of the themes of my current work.

Furthermore, the results described above do not seem to be a symmetric group restricted phenomenon. In [22], we investigated the center of the twisted Heisenberg category  $\mathcal{H}_{S_n}$  of Cautis and Sussan [3] which is associated to the tower of Sergeev algebras  $\{S_n\}_{n \geq 0}$ . While we showed that in this case  $Z(\mathcal{H}_{S_n})$  is isomorphic to the subalgebra of the symmetric functions generated by odd power sums  $\mathbb{C}[p_1, p_3, \dots]$ , similar connections to combinatorics and asymptotic representation theory hold. Notably, the bubble generators of  $Z(\mathcal{H}_{S_n})$  also correspond to functions encoding transition probabilities of a stochastic growth process (this time however on the graph of all strict Young diagrams).

There are a broad range of questions that deserve to be studied in this new field. Two that I am currently investigating are:

- *Centers of Heisenberg categories and new analogues of symmetric functions:* What new analogues of symmetric functions arise from centers of the more exotic Heisenberg categories (such as those associated with towers of degenerate cyclotomic affine Hecke algebras [27])? What combinatorial properties do they have?
- *Representation-theoretic growth processes and Heisenberg categories:* Do all Heisenberg categories define stochastic growth process? Can we in define stochastic growth processes via functors and natural isomorphisms encoded in the graphical calculus of the category?

### 2.3 Crystal structure in representation categories

Crystals were first discovered by Kashiwara as combinatorial objects attached to certain quantum group  $U_q(\mathfrak{g})$  representations that appear in the limit  $q \rightarrow 0$  [10]. Since they can be described via colored directed graphs, crystals are substantially simpler than their corresponding representations, and in many cases allow us to translate questions in algebra to questions in combinatorics.

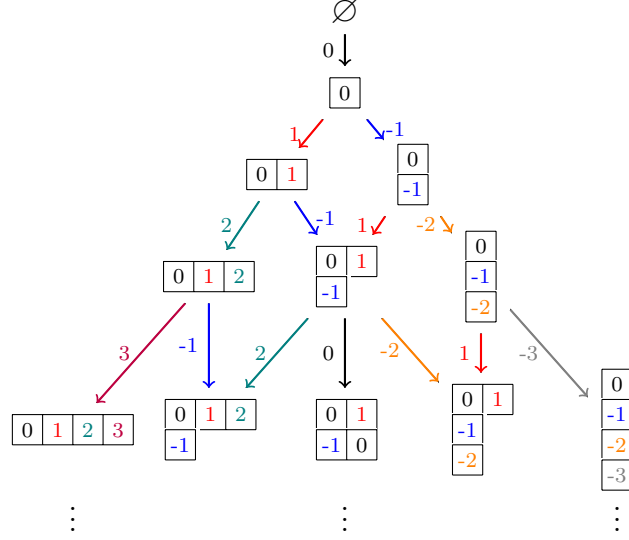


Figure 1: Young's lattice is a combinatorial model for the crystal corresponding to the highest weight representation  $V_{\Lambda_0}$  of  $U_q(\mathfrak{sl}_\infty)$ .

Crystals can also arise when a quantum group acts on a category via endofunctors. The first example of this was found when it was noticed that graphs defined via the branching rules of affine and cyclotomic Hecke algebras carry the structure of a crystal graph [24, 1, 8]. Only in hindsight was it realized that this is the result of a hidden categorical action of a quantum group. Subsequently many other representation categories with categorical action of a quantum group have been shown to carry a crystal structure. Examples include the category of finite-dimensional quiver Hecke algebra representations [25] and category  $\mathcal{O}$  for the cyclotomic rDAHA [33].

$$\begin{array}{ccccc}
 & \text{simple object in } \mathcal{C} & & \text{induction functor } F_i \text{ between simple objects in } \mathcal{C} & \\
 \text{categories} & \downarrow K_0 & \text{-----} & \downarrow K_0 & \\
 \text{crystals} & \text{node of crystal} & & \text{Kashiwara operator } \tilde{f}_i \text{ (} i\text{-colored edge)} & 
 \end{array}$$

When a representation category carries a crystal structure one can try to understand how the combinatorics of the crystal lift to the representation theory. Such crystal-theoretic considerations allowed Lauda-Vazirani to classify all simple quiver Hecke algebra representations [25].

## 2.4 Results on crystal structure in categories

Khovanov and Lauda [16], [17] and independently Rouquier [32] invented *quiver Hecke algebras*  $R$  (also known as *Khovanov-Lauda-Rouquier (KLR) algebras*) to categorify the upper half  $U_q^+(\mathfrak{g})$  of the quantum group for any fixed symmetrizable Kac-Moody algebra  $\mathfrak{g}$ .

For each integral dominant weight  $\Lambda$ , the algebra  $R$  has a finite-dimensional quotient called a *cyclotomic quiver Hecke algebra*  $R^\Lambda$ . Lauda-Vazirani showed that the collection of

simple  $R^\Lambda$ -modules carries the structure of  $B(\Lambda)$  which is the crystal of the corresponding highest weight representation of  $U_q(\mathfrak{g})$ . Here nodes are the simple  $R^\Lambda$ -modules and Kashiwara operators  $\tilde{f}_i$  are given by refined induction functors between the simple modules [25].

When  $\mathfrak{g}$  is of classical affine type, one class of crystals not previously appearing in the context of quiver Hecke algebras is the *Kirillov-Reshetikhin (KR) crystals* [28]. These are denoted by  $B^{r,s}$  where  $r \in I \setminus \{0\}$  for Dynkin indexing set  $I$  and  $s \in \mathbb{Z}_{>0}$ . A notable property of  $B^{1,1}$  in particular is that for certain choices of fundamental weight  $\Lambda_j$  there is an isomorphism of crystals

$$\varphi : B^{1,1} \otimes B(\Lambda_j) \xrightarrow{\sim} B(\Lambda_i), \quad (1)$$

for some  $i \in I$  [9]. This isomorphism is particularly exciting since it allows one to study the (infinite) crystal  $B(\Lambda_i)$  through tensor products of the finite crystals  $B^{1,1}$ .

Given that  $R^{\Lambda_i}\text{-mod}$  (respectively  $R^{\Lambda_j}\text{-mod}$ ) carries the structure of  $B(\Lambda_i)$  (resp.  $B(\Lambda_j)$ ), it is natural to ask whether (1) actually reflects representation-theoretic structure in  $R^{\Lambda_i}\text{-mod}$ . Vazirani and I showed that this is the case in [23]. Recall that each simple  $R^{\Lambda_i}$ -module  $M^\bullet$  corresponds to a node  $\bullet$  in  $B(\Lambda_i)$ , and similarly each simple  $R^{\Lambda_j}$ -module  $M^\bullet$  corresponds to a node  $\bullet$  in  $B(\Lambda_j)$ . We show that each assignment of the crystal isomorphism (1) is actually the shadow of a canonically defined surjective  $R^{\Lambda_i}$ -module homomorphism  $\phi$ .

$$\begin{array}{c}
 \begin{array}{ccccc}
 R^{\Lambda_i}\text{-mod} & R^{\Lambda_j}\text{-mod} & & R^{\Lambda_i}\text{-mod} \\
 \cap & \cap & & \cap \\
 \text{Ind } T_p \otimes M^\bullet & \xrightarrow{\phi} & M^\bullet
 \end{array} \\
 \begin{array}{c}
 \text{category of } R\text{-modules} \\
 \hline
 \text{crystals}
 \end{array} \\
 \begin{array}{ccc}
 \downarrow K_0 & & \downarrow K_0 \\
 \begin{array}{ccc}
 \text{shaded circle} \otimes \text{blue circle} & \xrightarrow{\varphi} & \text{orange circle} \\
 \downarrow \psi & & \downarrow \psi \\
 B^{1,1} \otimes B(\Lambda_j) & & B(\Lambda_i)
 \end{array}
 \end{array}
 \end{array}$$

The simple  $R^{\Lambda_i}$ -module  $T_p$  here belongs to a family of “trivial”  $R^{\Lambda_i}$ -modules parametrized by paths  $p$  in the crystal  $B^{1,1}$ . Most importantly, the above correspondence intertwines the action of the Kashiwara operators  $\tilde{f}_i$  of the crystal graph with the action of their categorical analogue, the refined induction functors.

### 3 Other work: geometry, data analysis, and compressive sensing

As data is generated at a faster and faster rate, tools for extracting information from large, high-dimensional data sets have become increasingly valuable across science, engineering, and industry. Mathematicians have a prominent role to play in helping to design such algorithms. In this section I will briefly summarize some of my work in this field.

- *Secant-based dimensionality reduction*: The secant set  $S$  of a data set  $D \subset \mathbb{R}^n$  encodes information on the spatial relationships between points in  $D$  and consequently is an important object of study when attempting to find projections of  $D$  that preserve distances between points in  $D$ .

In [18] we proposed an algorithm called the *secant avoidance projection (SAP) algorithm* to find projections which preserve the secant set of a data set. This iterative algorithm is designed for use on a GPU architecture with fast convergence for even very high-dimensional data sets. One issue that arose with this algorithm was that when  $|D|$  is very large, it can be difficult to store all secants in  $S$ . We developed the *hierarchical secant avoidance projection (HSAP) algorithm* for such situations [20]. HSAP utilizes the intrinsic structure of  $D$  to inform subsampling and approximation of secants to vastly reduce the number of secants which need to be stored.

Finally, part of the output of both of these algorithms is the length of the projected secant least well preserved by the final output projection. By computing these lengths for all projection dimensions we arrive at a statistic for the data set  $D$  which we call the  $\kappa$ -profile. When the data sits on an  $m$ -manifold in  $\mathbb{R}^n$  for  $m < n$ , the  $\kappa$ -profile can be related to  $m$  via Whitney’s embedding theorem. We show in [19] that while the  $\kappa$ -profile is completely general, it captures fundamental structure in  $D$ . In particular, for a data set  $D(t)$  which changes with respect to a time parameter  $t$ , changes in the state of  $D(t)$  over time are reflected by changes in  $\kappa(t)$ . As an example of this, we showed that the  $\kappa$ -profile for a weather data set taken in the Atlantic changed dramatically when a storm past through the region.

- *Searching for pure signals in hyperspectral imagery via the geometry of the Grassmannian*: Unlike RGB images, which only sample from three different spectral bands of light (red, green, and blue), hyperspectral imagery can sample more than 200. While hyperspectral images thus have strong discriminatory power, it can be difficult to extract information from them because they are often very high dimensional. One particularly important problem is understanding how to isolate the “most pure” signals from a hyperspectral image. Geometrically, this corresponds to picking out those spatial locations in the image whose spectral curve sits on the convex hull formed by all spectral curves in the image. In [6] we show how if one considers local patches rather than individual pixels this endmember extraction problem is most naturally realized on a Grassmann manifold. We propose an algorithm to solve the problem in this setting.
- *Compressive sensing algorithm design for a single-pixel camera*<sup>1</sup>: Over the course of the last year, I have been involved with two projects with an industry partner which involve developing compressive sensing (CS) algorithms for single pixel imaging devices. This has involved choosing a CS sampling framework which captures the structure of images and also fits hardware constraints, developing and testing CS reconstruction algorithms which can reconstruct images in real-time with significant device noise, and writing a software package that implements all these algorithms for use in the field.

## References

- [1] Susumu Ariki, *On the decomposition numbers of the Hecke algebra of  $G(m, 1, n)$* , J. Math. Kyoto Univ. **36** (1996), no. 4, 789–808.
- [2] J. Brundan, *On the definition of Heisenberg category*, arxiv:1709.06589 (2017).

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<sup>1</sup>Work described in this section is ITAR-protected and hence some details have been omitted.

- [3] Sabin Cautis and Joshua Sussan, *On a categorical Boson-Fermion correspondence*, Comm. Math. Phys. **336** (2015), no. 2, 649–669.
- [4] Louis Crane and Igor B. Frenkel, *Four-dimensional topological quantum field theory, Hopf categories, and the canonical bases*, J. Math. Phys. **35** (1994), no. 10, 5136–5154, Topology and physics.
- [5] Ben Elias and Geordie Williamson, *The Hodge theory of Soergel bimodules*, Ann. of Math. (2) **180** (2014), no. 3, 1089–1136.
- [6] Elin Farnell, Henry Kvinge, Michael Kirby, and Chris Peterson, *Endmember extraction on the Grassmannian*, 2018 IEEE Data Science Workshop (DSW), June 2018, pp. 71–75.
- [7] Ladnor Geissinger, *Hopf algebras of symmetric functions and class functions*, Combinatoire et représentation du groupe symétrique (Actes Table Ronde C.N.R.S., Univ. Louis-Pasteur Strasbourg, Strasbourg, 1976), Springer, Berlin, 1977, pp. 168–181. Lecture Notes in Math., Vol. 579.
- [8] Ian Grojnowski, *Affine  $sl_p$  controls the representation theory of the symmetric group and related Hecke algebras*, arXiv:9907129 (1999).
- [9] Seok-Jin Kang, Masaki Kashiwara, Kailash C. Misra, Tetsuji Miwa, Toshiki Nakashima, and Atsushi Nakayashiki, *Perfect crystals of quantum affine Lie algebras*, Duke Math. J. **68** (1992), no. 3, 499–607.
- [10] Masaki Kashiwara, *On crystal bases of the  $Q$ -analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), no. 2, 465–516.
- [11] Sergei Kerov, *Transition probabilities of continual Young diagrams and the Markov moment problem*, Funktsional. Anal. i Prilozhen. **27** (1993), no. 2, 32–49, 96.
- [12] ———, *A differential model for the growth of Young diagrams*, Proceedings of the St. Petersburg Mathematical Society, Vol. IV, Amer. Math. Soc. Transl. Ser. 2, vol. 188, Amer. Math. Soc., Providence, RI, 1999, pp. 111–130. MR 1732430
- [13] Sergei Kerov and Grigori Olshanski, *Polynomial functions on the set of Young diagrams*, C. R. Acad. Sci. Paris Sér. I Math. **319** (1994), no. 2, 121–126.
- [14] Mikhail Khovanov, *A categorification of the Jones polynomial*, Duke Math. J. **101** (2000), no. 3, 359–426. MR 1740682
- [15] ———, *Heisenberg algebra and a graphical calculus*, Fund. Math. **225** (2014), no. 1, 169–210.
- [16] Mikhail Khovanov and Aaron D. Lauda, *A diagrammatic approach to categorification of quantum groups. I*, Represent. Theory **13** (2009), 309–347.
- [17] ———, *A diagrammatic approach to categorification of quantum groups II*, Trans. Amer. Math. Soc. **363** (2011), no. 5, 2685–2700.

- [18] Henry Kvinge, Elin Farnell, Michael Kirby, and Chris Peterson, *A GPU-oriented algorithm design for secant-based dimensionality reduction*, 2018 17th International Symposium on Parallel and Distributed Computing (ISPD), June 2018, pp. 69–76.
- [19] ———, *Monitoring the shape of weather, soundscapes, and dynamical systems: a new statistic for dimension-driven data analysis on large data sets*, Submitted to IEEE Big-Data 2018, August 2018.
- [20] Henry Kvinge, Elin Farnell, Michael Kirby, and Chris Peterson, *Too many secants: a hierarchical approach to secant-based dimensionality reduction on large data sets*, The 2018 IEEE High Performance Extreme Computing Conference, IEEE, 2018, Accepted.
- [21] Henry Kvinge, Anthony M. Licata, and Stuart Mitchell, *Khovanov’s Heisenberg category, moments in free probability, and shifted symmetric functions*, arXiv:1610.04571 (2016), To appear in Algebraic Combinatorics.
- [22] Henry Kvinge, Can Ozan Oğuz, and Michael Reeks, *The center of the twisted Heisenberg category, factorial Schur  $Q$ -functions, and transition functions on the Schur graph*, ArXiv e-prints (2017).
- [23] Henry Kvinge and Monica Vazirani, *A combinatorial categorification of the tensor product of the Kirillov-Reshetikhin crystal  $B^{1,1}$  and a fundamental crystal*, Algebras and Representation Theory (2017).
- [24] Alain Lascoux, Bernard Leclerc, and Jean-Yves Thibon, *Hecke algebras at roots of unity and crystal bases of quantum affine algebras*, Comm. Math. Phys. **181** (1996), no. 1, 205–263.
- [25] Aaron D. Lauda and Monica Vazirani, *Crystals from categorified quantum groups*, Adv. Math. **228** (2011), no. 2, 803–861.
- [26] Anthony Licata and Alistair Savage, *Hecke algebras, finite general linear groups, and Heisenberg categorification*, Quantum Topol. **4** (2013), no. 2, 125–185.
- [27] Marco Mackaay and Alistair Savage, *Degenerate cyclotomic Hecke algebras and higher level Heisenberg categorification*, J. Algebra **505** (2018), 150–193.
- [28] Masato Okado and Anne Schilling, *Existence of Kirillov-Reshetikhin crystals for nonexceptional types*, Represent. Theory **12** (2008), 186–207.
- [29] Andrei Okounkov, *Quantum immanants and higher Capelli identities*, Transform. Groups **1** (1996), no. 1-2, 99–126.
- [30] Andrei Okounkov and Grigori Olshanski, *Shifted Schur functions*, Algebra i Analiz **9** (1997), no. 2, 73–146.
- [31] Daniele Rosso and Alistair Savage, *A general approach to Heisenberg categorification via wreath product algebras*, Math. Z. **286** (2017), no. 1-2, 603–655. MR 3648512
- [32] Raphaël Rouquier, *2-Kac-Moody algebras*, arXiv:0812.5023 (2008).



- [33] Peng Shan, *Crystals of Fock spaces and cyclotomic rational double affine Hecke algebras*, Ann. Sci. Éc. Norm. Supér. (4) **44** (2011), no. 1, 147–182.
- [34] Wolfgang Soergel, *Kategorie  $\mathcal{O}$ , perverse Garben und Moduln über den Koinvarianten zur Weylgruppe*, J. Amer. Math. Soc. **3** (1990), no. 2, 421–445. MR 1029692