

MATH 417 Homework 4

Due: Note that this is problems: Section 13.1: #3, #8, #11, Section 13.2: #1, #3, #6.

3. Analyze the following limits:

- a. $\lim_{(x,y) \rightarrow (0,0)} \frac{x+y}{x^2+y^2}$
- b. $\lim_{(x,y,z) \rightarrow (0,0,0)} \frac{x+y+z}{x^2+y^2+z^2}$
- c. $\lim_{(x,y) \rightarrow (0,0)} \frac{e^{x^2+y^2} - 1}{x^2+y^2}$

Solution:

a. This limit does not exist. To prove this consider the sequence $\{(\frac{1}{k}, -\frac{1}{k})\}_{k \geq 1}$. Observe that

$$\lim_{k \rightarrow \infty} (\frac{1}{k}, -\frac{1}{k}) = (0, 0).$$

and

$$\lim_{k \rightarrow \infty} \frac{\frac{1}{k} - \frac{1}{k}}{\frac{1}{k^2} + \frac{1}{k^2}} = 0.$$

On the other hand, the sequence $\{(\frac{1}{k}, 0)\}_{k \geq 1}$ also converges to $(0, 0)$ but

$$\lim_{k \rightarrow \infty} \frac{\frac{1}{k} + 0}{\frac{1}{k^2} + 0} = \lim_{k \rightarrow \infty} k = \infty.$$

Hence the limit cannot exist.

- b. This limit also does not exist and an argument similar to above works in this case as well, replacing our first sequence above by $\{(\frac{1}{k}, -\frac{1}{k}, 0)\}_{k \geq 1}$ and our second sequence by $\{(\frac{1}{k}, 0, 0)\}_{k \geq 1}$.
- c. This limit does exist and is equal to 1. To prove this, notice that via the series definition of the exponential function, we have

$$\begin{aligned} e^{x^2+y^2} - 1 &= \left(1 + (x^2 + y^2) + \frac{(x^2 + y^2)^2}{2!} + \frac{(x^2 + y^2)^3}{3!} + \dots\right) - 1 \\ &= (x^2 + y^2) + \frac{(x^2 + y^2)^2}{2!} + \frac{(x^2 + y^2)^3}{3!} + \dots \end{aligned}$$

Thus

$$\begin{aligned} \frac{e^{x^2+y^2} - 1}{x^2 + y^2} &= \frac{(x^2 + y^2) + \frac{(x^2+y^2)^2}{2!} + \frac{(x^2+y^2)^3}{3!} + \dots}{x^2 + y^2} \\ &= 1 + \frac{(x^2 + y^2)}{2!} + \frac{(x^2 + y^2)^2}{3!} + \dots \end{aligned}$$

It is clear that each term of the sum is continuous everywhere and hence for any sequence $\{(x_k, y_k)\}_{k \geq 1}$ that converges to $(0, 0)$, each of these terms converges to 0, except for the first, which is identically 1. Some care is needed here since we haven't discussed how to pass a limit into an infinite series, but in this case your intuition is correct and this limit converges to 1.

6. Let A be a subset of \mathbb{R}^n and let \mathbf{x} be a point in \mathbb{R}^n . Show that \mathbf{x} is a limit point of A if and only if every open ball about \mathbf{x} contains a point of A that is not equal to \mathbf{x} .

Solution: We first prove that if \mathbf{x} is a limit point of A then every open ball about \mathbf{x} contains a point of A that is not equal to \mathbf{x} . Suppose for a contradiction that this is not true. Then there is $r > 0$ such that $B_r(\mathbf{x})$ contains no points from A other than \mathbf{x} . Since \mathbf{x} is a limit point, there is at least one sequence $\{\mathbf{x}_k\}_{k \geq 1}$ in A

that converges to \mathbf{x} and for which no terms are equal to \mathbf{x} . By the definition of convergence of a series, there is some K such that for all $k \geq K$, $\mathbf{x}_k \in B_r(\mathbf{x})$. But this is a contradiction since such \mathbf{x}_k belong to A and we had assumed that $B_r(\mathbf{x})$ had no other points not in A .

Next we prove that if every open ball around \mathbf{x} has a point from A in it not equal to \mathbf{x} , then it is a limit point. Then for each $k \geq 1$, the ball $B_{\frac{1}{k}}(\mathbf{x})$ contains a point in A , which we call \mathbf{x}_k , which is not equal to \mathbf{x} . It is easy to show that the sequence $\{\mathbf{x}_k\}_{k \geq 1}$ converges to \mathbf{x} , and is in A by construction. Hence \mathbf{x} is a limit point of A .

8. Let $A \subset \mathbb{R}^n$. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be the characteristic function of A . Let x_* be a point in \mathbb{R}^n . Show that $\lim_{\mathbf{x} \rightarrow x_*} f(\mathbf{x})$ exists if and only if x_* is either an interior point or an exterior point of the set A .

Solution: We proved in a previous homework assignment that f is only continuous on the interior of A and the exterior of A . It follows from the fact that f is continuous at such points that if x_* is either in the interior of A or the exterior of A then $\lim_{\mathbf{x} \rightarrow x_*} f(\mathbf{x})$ since for all sequences $\{\mathbf{x}_k\}_{k \geq 1}$, $\lim_{k \rightarrow \infty} f(\mathbf{x}_k) = f(x_*)$.

On the other hand, let x_* be in the boundary of A . Since x_* is in the boundary of A , then for any $k \in \mathbb{N}$ we can find $\mathbf{x}_k, \mathbf{x}'_k \in B_{1/k}(\mathbf{x}_*)$ such that $\mathbf{x}_k \in A$ and $\mathbf{x}'_k \notin A$. Then we have

$$\lim_{\mathbf{x}_k \rightarrow x_*} f(\mathbf{x}_k) = 1 \neq 0 = \lim_{\mathbf{x}'_k \rightarrow x_*} f(\mathbf{x}'_k).$$

Thus $\lim_{\mathbf{x} \rightarrow x_*} f(\mathbf{x})$ does not exist.

1. Calculate the first-order partial derivatives of the following functions:

- $f(x, y, z) = x + yz + xy + x \sin(xy)$ for $(x, y, z) \in \mathbb{R}^3$
- $f(x, y, z) = \sin(x^2 y^2)/(1 + x^2 + y^3)$ for $(x, y, z) \in \mathbb{R}^3$
- $f(x, y, z) = \sqrt{1 + \cos^2(xy)}$ for $(x, y, z) \in \mathbb{R}^3$.

Solution:

- a. The first-order partials are defined everywhere in \mathbb{R}^3 and are equal to:

- $\frac{\partial f}{\partial x}(x, y, z) = 1 + y + \sin(xy) + xy \cos(xy),$
- $\frac{\partial f}{\partial y}(x, y, z) = z + x + x^2 \sin(xy),$
- $\frac{\partial f}{\partial z}(x, y, z) = y.$

- b. For all points in \mathbb{R}^2 the partial derivatives are equal to

- $\frac{\partial f}{\partial x}(x, y) = \frac{2y^2 x \cos(x^2 y^2)(1 + x^2 + y^3) - 2x \sin(x^2 y^2)}{(1 + y^3 + x^2)^2},$
- $\frac{\partial f}{\partial y}(x, y) = \frac{2x^2 y \cos(x^2 y^2)(1 + x^2 + y^3) - 3y \sin(x^2 y^2)}{(1 + y^3 + x^2)^2}.$

- c. Since $1 + \cos^2(xy)$ is everywhere positive, then the partial derivatives are defined in all \mathbb{R}^2 as

- $\frac{\partial f}{\partial x}(x, y) = \frac{-2y \sin(xy)}{\sqrt{1 + \cos^2(xy)}},$
- $\frac{\partial f}{\partial y}(x, y) = \frac{-2x \sin(xy)}{\sqrt{1 + \cos^2(xy)}}.$

3. For the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined in Example 13.9, show that neither the function $\partial f / \partial x : \mathbb{R}^2 \rightarrow \mathbb{R}$ nor the function $\partial f / \partial y : \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous at the point $(0, 0)$.

Solution: To show that $\frac{\partial f}{\partial x}$ is not continuous at $(0, 0)$, consider the sequence $\{(0, \frac{1}{k})\}_{k \geq 0}$ which converges to $(0, 0)$. Then

$$\lim_{k \rightarrow \infty} \frac{y_k^3 - x_k^2 y_k}{(x_k^2 + y_k^2)^2} = \lim_{k \rightarrow \infty} k = \infty.$$

But $\frac{\partial f}{\partial x}(0, 0) = 0$. Hence $\frac{\partial f}{\partial x}$ is not continuous. To show that $\frac{\partial f}{\partial y}$ is not continuous at $(0, 0)$, similarly consider the sequence $\{(\frac{1}{k}, 0)\}_{k \geq 0}$ and follow the same argument.

6. Define

$$g(x, y) = \begin{cases} x^2 y^2 / (x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Prove that the function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ has first-order partial derivatives. Is the function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ continuously differentiable?

Solution: When $(x, y) \neq (0, 0)$, then there is a neighborhood such that $g(x, y)$ is the quotient of two polynomials and hence for such points

$$\frac{\partial g}{\partial x}(x, y) = \frac{2xy^4}{(x^2 + y^2)^2}$$

and

$$\frac{\partial g}{\partial y}(x, y) = \frac{2yx^4}{(x^2 + y^2)^2}.$$

On the other hand, we have

$$\frac{\partial g}{\partial x}(0, 0) = \lim_{t \rightarrow 0} \frac{\frac{t^2(0)^2}{t^2+0^2} - 0}{t} = 0,$$

and

$$\frac{\partial g}{\partial y}(0, 0) = \lim_{t \rightarrow 0} \frac{\frac{t^2(0)^2}{0^2+t^2} - 0}{t} = 0.$$

So g has first-order partial derivatives. Both of these functions are continuous. It is clear that they are continuous everywhere except possibly $(0, 0)$ since they are the quotient of two polynomials and the polynomials in the denominator only vanish at $(0, 0)$. To show that $\frac{\partial g}{\partial x}$ is continuous at $(0, 0)$ note that

$$|2xy^4| \leq |2x(y^4 + 2x^2y^2 + x^4)| = |2x(x^2 + y^2)^2|$$

and hence

$$\left| \frac{2xy^4}{(x^2 + y^2)^2} \right| \leq \left| \frac{2x(x^2 + y^2)^2}{(x^2 + y^2)^2} \right| = |2x|.$$

For any sequence $\{(x_k, y_k)\}_{k \geq 1}$ which converges to $(0, 0)$, then $\{x_k\}_{k \geq 1}$ converges to 0. Hence we have

$$\lim_{k \rightarrow \infty} \left| \frac{2x_k y_k^4}{(x_k^2 + y_k^2)^2} \right| \leq \lim_{k \rightarrow \infty} |2x_k| = 0.$$

Thus

$$\lim_{k \rightarrow \infty} \left| \frac{2x_k y_k^4}{(x_k^2 + y_k^2)^2} \right| = 0 = \frac{\partial g}{\partial x}(0, 0)$$

So $\frac{\partial g}{\partial x}$ is continuous at $(0, 0)$. An analogous argument works for $\frac{\partial g}{\partial y}$.