COHERENT SYSTEMS OF PROBABILITY MEASURES ON GRAPHS FOR REPRESENTATIONS OF FREE FROBENIUS TOWERS

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ABSTRACT. First formally defined by Borodin and Olshanski, a coherent system on a graded graph is a sequence of probability measures which respect the action of certain down/up transition functions between graded components. In one common example of such a construction, each measure is the Plancherel measure for the symmetric group S_n and the down transition function is induced from the inclusions $S_n \hookrightarrow S_{n+1}$.

In this paper we generalize the above framework to the case where $\{A_n\}_{n \geqslant 0}$ is any free Frobenius tower and A_n is no longer assumed to be semisimple. In particular, we describe two coherent systems on graded graphs defined by the representation theory of $\{A_n\}_{n \geqslant 0}$ and connect one of these systems to a family of central elements of $\{A_n\}_{n \geqslant 0}$. When the algebras $\{A_n\}_{n \geqslant 0}$ are not semisimple, the resulting coherent systems reflect the duality between simple A_n-modules and indecomposable projective A_n-modules.

CONTENTS

1. INTRODUCTION

Study of the asymptotic representation theory of symmetric groups has uncovered deep results through the synthesis of techniques from probability theory, combinatorics, and algebra. One idea frequently used both implicitly and explicitly in this field is that of a coherent system. Coherent systems were first formally defined in [4] where they are a key part of the framework used to construct infinite-diffusion processes. Given a graded graph with vertex set $V = \bigcup_{n \geqslant 0} V_n$, a coherent system consists of a sequence of

probability measures $\{M_n\}_{n\geqslant 0}$ on the graded components of V, which are consistent with the action of a down transition function $p_\downarrow$ between components.

A basic example of this construction occurs when Pl_n is taken to be the Plancherel measure on $\mathbb{Y}_n$. The sequence $\{Pl_n\}_{n\geqslant 0}$ is coherent with respect to down and up transition functions each controlled by the number of standard Young tableau of appropriate shape. On the other hand, this coherent system can also be described in terms of the representation theory of the tower of symmetric group algebras $\{\mathbb{C}[S_n]\}_{n\geqslant 0}$. From this point of view, Pl_n is defined via the decomposition of the left regular representation of $\mathbb{C}[S_n]$ into a direct sum of simple $\mathbb{C}[S_n]$ representations, and $p_\downarrow$ and $p_\uparrow$ are defined via induction and restriction functors.

One of the main goals of this paper is to understand what assumptions are necessary to generalize the above construction to an arbitrary tower of algebras $\{A_n\}_{n\geqslant 0}$. We show that when for all $n \geqslant 0$:

(1) the induction and restriction functors between A_n representations and A_{n+1} representations are biadjoint (so that A_{n+1} is a Frobenius extension of A_n),
(2) A_{n+1} is a free (A_n, A_n)-bimodule,

then there a two coherent systems that generalize the Plancherel system corresponding to $\{\mathbb{C}[S_n]\}_{n\geqslant 0}$. The observation that there are two such systems to choose from reflects the fact that when A_n is not semisimple, there arises a distinction between simple A_n-modules and indecomposable projective A_n-modules. In particular, in one coherent system the down transition functions are controlled by the dimensions of simple A_n-modules and the up transition functions are controlled by the dimensions of indecomposable projective A_n-modules, while in the other coherent system the reverse is true.

That the existence of coherent systems should depend on categorical conditions is not surprising given the recent appearance of coherent systems within categorical representation theory. It was shown in [16] for example, that certain moments associated to down/up transition functions for the tower $\{\mathbb{C}[S_n]\}_{n\geqslant 0}$ appear naturally within the center of Khovanov's Heisenberg category [14], a monoidal category which conjecturally categorifies the infinite dimensional Heisenberg algebra. A similar phenomenon [17] was observed for down/up transition functions associated to towers of Sergeev algebras, which appear in the twisted Heisenberg category of Cautis and Sussan [7].

In the classical case of $\{\mathbb{C}[S_n]\}_{n\geqslant 0}$, Biane observed that data related to $p_\downarrow$ is encoded by elements of $\{Z(\mathbb{C}[S_n])\}_{n\geqslant 0}$ [2]. This allows for certain questions related to asymptotic processes to be translated into algebraic/-combinatorial questions. We show that when each A_n in the free Frobenius tower $\{A_n\}_{n\geqslant 0}$ is further assumed to be a symmetric Frobenius algebra, then there are families of elements in $\{Z(A_n)\}_{n\geqslant 0}$ which encode data not only for

$p_\downarrow$, but also $p_\uparrow$. We plan to investigate these elements and their possible connection to symmetric functions in greater detail in a later paper.

One of the broader goals of this work is to pave the way for studies of the asymptotic representation theory of more exotic towers of algebras. In particular, because of their connections to Lie theory and geometry, we think that understanding the asymptotic representation theory of the wreath product algebras attached to a Frobenius algebra F would be particularly interesting (see Example 2.1.3). Degenerate cyclotomic Hecke algebras are another natural generalization of symmetric groups (see Example 2.1.4). This family of non-semisimple towers of algebras has deep connections to Lie theory [15, Part 1]. Furthermore, as their representation theory is indexed by certain multipartitions, we hypothesize that using the coherent systems described in this paper, an entire new family of limit shapes could be constructed.

The paper is structured as follows: in Section 2 we review Frobenius extensions and Frobenius towers. In Section 3 we describe the representation theory of Frobenius towers particularly in relation to elements from the centralizers $\{Z(A_{n+1}, A_n)\}_{n \geqslant 0}$. In Section 4 we define a pair of coherent systems on certain graphs attached to free Frobenius towers and connect one of these to a family of central elements. Finally in Section 4.1 we discuss future directions of research.

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2. Frobenius extensions and Frobenius towers

In this section we review some of the basic properties of Frobenius extensions and Frobenius towers. All the algebras that we introduce are assumed to be finite-dimensional $\mathbb{C}$-algebras unless otherwise stated. We point the reader toward [12] for a detailed introduction to Frobenius extensions as well as a more diverse set of examples.

Recall that a tower of $\mathbb{C}$-algebras $\{A_n\}_{n \geqslant 0}$ is a nested sequence of algebras

$$A_0 \subset A_1 \subset A_2 \subset \dots \tag{1}$$

In this paper we will always assume that $A_0 = \mathbb{C}$ which adds nice properties to the tower. Due to the inclusions in (1), for any $0 \leqslant k_1, k_2 < n$, A_n is an (A_{k_1}, A_{k_2})-bimodule, with the action of A_{k_1} given by the natural left-multiplication against A_n and the action of A_{k_2} given by the natural right-multiplication against A_n. When considering A_n as a bimodule we will sometimes use the following notation:

- To denote A_n as a (A_{k_1}, A_{k_2})-bimodule we write $_{k_1}(A_n)_{k_2}$.
- To denote A_n as a (A_{k_1}, A_n)-bimodule we write $_{k_1}(A_n)$.

- To denote A_n as a (A_n, A_{k_2})-bimodule we write $(A_n)_{k_2}$.
- To denote A_n as a (A_n, A_n)-bimodule we write A_n.

This bimodule structure defines a restriction functor $\mathrm{Res}^{A_n}_{A_k} : A_n\text{-}\mathrm{Mod} \to$ $A_k\text{-}\mathrm{Mod}$ and an induction functor $\mathrm{Ind}^{A_n}_{A_k} : A_k\text{-}\mathrm{Mod} \to A_n\text{-}\mathrm{Mod}$ for each $0 \leqslant k < n$, where $A_k\text{-}\mathrm{Mod}$ (resp. $A_n\text{-}\mathrm{Mod}$) is the category of finite-dimensional A_k-modules (resp. A_n-modules). Specifically for $N \in A_n\text{-}\mathrm{Mod}$ and $K \in A_k\text{-}\mathrm{Mod}$,

$$N \xrightarrow{\mathrm{Res}^{A_n}_{A_k}} {}_k(A_n) \otimes_{A_n} N. \tag{2}$$

and

$$K \xrightarrow{\mathrm{Ind}^{A_n}_{A_k}} (A_n)_k \otimes_{A_k} K. \tag{3}$$

Henceforth for brevity we write

$$\mathrm{Ind}^n_k := \mathrm{Ind}^{A_n}_{A_k} \qquad \text{and} \qquad \mathrm{Res}^n_k := \mathrm{Res}^{A_n}_{A_k}$$

when the tower of algebras is understood from the context. Via (2) we can identify Res^n_k with the (A_k, A_n)-bimodule ${}_k(A_n)$ and via (3) we can identify Ind^n_k with the (A_n, A_k)-bimodule $(A_n)_k$. This description allows us to translate natural transformations between compositions of the functors Res^n_k and Ind^n_k into bimodule homomorphisms between tensor products of ${}_k(A_n)$ and $(A_n)_k$.

Example 2.1.   (1) The tower of symmetric group algebras $\{\mathbb{C}[S_n]\}_{n \geqslant 0}$ is our motivating example for this paper. Recall that S_n is generated by Coxeter generators $s_1, s_2, \ldots, s_{n-1}$ subject to relations

$$s_i^2 = 1, \qquad\qquad\qquad\qquad 1 \leqslant i \leqslant n - 1, \tag{4}$$

$$s_i s_j = s_j s_i, \qquad\qquad |i - j| > 1, \ \ 1 \leqslant i, j \leqslant n - 1, \tag{5}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \qquad\qquad\qquad 1 \leqslant i \leqslant n - 2. \tag{6}$$

We will always assume that the inclusion $S_k \hookrightarrow S_n$ is the standard one with S_k mapping to the subgroup of S_n generated by $s_1, s_2, \ldots, s_{k-1}$. By linear extension this defines an inclusion $\mathbb{C}[S_k] \hookrightarrow \mathbb{C}[S_n]$.

(2) Recall that the Clifford superalgebra Cl_n is the unital associative algebra with n generators $c_1, c_2, \ldots, c_n$ such that for $1 \leqslant i, j \leqslant n$:

$$c_i^2 = 1 \qquad \text{and} \qquad c_i c_j = -c_j c_i \text{ for } i \neq j.$$

The superalgebra structure is defined by setting each generator c_i to be an odd element. The *Sergeev superalgebra* is defined as $\mathbb{S}_n := Cl_n \rtimes \mathbb{C}[S_n]$, where the action of S_n is given by permuting the indices of $c_1, c_2, \ldots, c_n$ so that:

$$s_i c_i = c_{i+1} s_i, \qquad\qquad 1 \leqslant i \leqslant n - 1,$$
$$s_i c_{i+1} = c_i s_i, \qquad\qquad 1 \leqslant i \leqslant n - 1,$$
$$s_i c_j = c_j s_i \qquad\qquad j \neq i, \ i + 1.$$

The Sergeev algebras form a tower $\{\mathbb{S}_n\}_{n \geqslant 0}$ via the embedding

$$\mathbb{S}_n \hookrightarrow \mathbb{S}_{n+1}$$

which sends $c_i \mapsto c_i$ for $1 \leqslant i \leqslant n$ and $s_i \mapsto s_i$ for $1 \leqslant i \leqslant n-1$.

One reason that Sergeev algebras are interesting is that studying simple representations of $\{\mathbb{S}_n\}_{n \geqslant 0}$ is equivalent to studying projective representations of $\{S_n\}_{n \geqslant 0}$. See [15] and [24] for detailed studies of the algebras $\{\mathbb{S}_n\}_{n \geqslant 0}$, their representation theory, and connections to combinatorics and Lie theory. $\{\mathbb{S}\}_{n \geqslant 0}$ is another tower of algebras whose asymptotic representation theory has been well-studied, see for example [9], [10], [3], [21].

(3) Let F be a Frobenius graded superalgebra (that is, F is $\mathbb{Z} \times \mathbb{Z}_2$ graded). The symmetric group S_n acts on $F^{\otimes n}$ by superpermutations. More precisely, for homogeneous $f_1, f_2, \ldots, f_n \in F$ and $1 \leqslant i \leqslant n-1$,

$$s_i \cdot (f_1 \otimes \cdots \otimes f_i \otimes f_{i+1} \otimes \cdots \otimes f_n) \qquad (7)$$

$$= (-1)^{\bar{f}_i \overline{f_{i+1}}}(f_1 \otimes \cdots \otimes f_{i+1} \otimes f_i \otimes \cdots \otimes f_n)$$

where $\bar{f}_i$ and $\overline{f_{i+1}}$ denote the $\mathbb{Z}_2$-degree of f_i and f_{i+1} respectively.

The wreath product algebras $\{F^{\otimes n} \rtimes S_n\}_{n \geqslant 0}$ induced from (7) form a tower via the embedding

$$F^{\otimes n} \rtimes S_n \hookrightarrow F^{\otimes(n+1)} \rtimes S_{n+1}$$

which for $f_1, \ldots, f_n \in F$ and $\omega \in S_n$ sends

$$(f_1 \otimes \cdots \otimes f_n)\omega \mapsto ((f_1 \otimes \cdots \otimes f_n \otimes 1)\omega.$$

$F^{\otimes n} \rtimes S_n$ also inherits a $(\mathbb{Z} \times \mathbb{Z}_2)$-grading from F by setting S_n to sit in degree $(0,0)$.

In fact, Example 2.1.2 is a special case of $F^{\otimes n} \rtimes S_n$ with $F = Cl_1$. For a description of the representation theory of $F^{\otimes n} \rtimes S_n$ in the non-super, ungraded setting see [19, Theorem A.5], while for the super, graded case see [22, Section 4].

(4) The *degenerate affine Hecke algebra* H_n is generated by elements $s_1, s_2, \ldots s_{n-1}$ and $x_1, x_2, \ldots, x_n$, such that $s_1, \ldots, s_{n-1}$ satisfy relations (4)-(6), $x_1, \ldots, x_n$ commute, and:

$$s_j x_i = x_i s_j, \qquad\qquad i \neq j, j+1,$$
$$s_i x_i = x_{i+1} s_i - 1, \qquad\qquad 1 \leqslant i \leqslant n-1.$$

Thus as a $\mathbb{C}$-vector space

$$H_n \cong \mathbb{C}[x_1, \ldots, x_n] \otimes \mathbb{C}[S_n].$$

Let P^+ be the dominant weight lattice for $\mathfrak{g} = \mathfrak{sl}_\infty$ and let I be the associated Dynkin indexing set. Also let $\lambda = \sum_{i \in I} \lambda_i \omega_i \in P^+$ where $\lambda_i \in \mathbb{Z}_{\geqslant 0}$ and ω_i is the ith fundamental weight. The integer

$d = \sum_{i \in I} \lambda_i$ is called the *level* of λ. Let I^λ be the two-sided ideal of H_n^λ generated by the element

$$\prod_{i \in I} (x_1 - i)^{\lambda_i}.$$

The quotient algebra $H_n^\lambda = H_n / I^\lambda$ is called the *degenerate cyclotomic Hecke algebra* associated to λ. By abuse of notation we write $x_i, s_i \in H_n^\lambda$ for the images of $x_i, s_i \in H_n$ in this quotient. H_n^λ is finite dimensional, with the set

$$\{x_1^{t_1} x_2^{t_2} \ldots x_n^{t_n} \sigma \mid t_1, t_2, \ldots, t_n < d, \ \sigma \in S_n\}$$

being a basis [15, Theorem 3.2.2], so that $\dim(H_n^\lambda) = d^n n!$. The algebras $\{H_n^\lambda\}_{n \geqslant 0}$ generalize symmetric group algebras because when $\lambda = \omega_0$, $H_n^\lambda \cong \mathbb{C}[S_n]$.

The collection $\{H_n^\lambda\}_{n \geqslant 0}$ forms a tower via the embeddings

$$H_n^\lambda \hookrightarrow H_{n+1}^\lambda$$

which send $x_i \mapsto x_i$ for $1 \leqslant i \leqslant n$ and $s_i \mapsto s_i$ for $1 \leqslant i \leqslant n-1$. The tower $\{H_n^\lambda\}_{n \geqslant 0}$ has a rich representation theory which is described in detail in [15, Part I]. In particular it has a crystal structure isomorphic to the highest weight crystal $B(\lambda)$ in affine type A. The algebras H_n^λ will be one of our prime examples of a tower of non-semisimple algebras.

In all the examples above, we follow the usual convention that $A_0 = \mathbb{C}$.

Remark 2.2. It is worth noting that all the examples in Example 2.1 are particular instances of cyclotomic wreath product algebras as defined in [23]. While we could have thus presented a single unified example, we felt it was more appropriate to partition our examples into digestible chunks as above for the benefit of the reader that may not be familiar with cyclotomic wreath product algebras.

For $0 \leqslant k < n$ it is always the case that the functor Ind_k^n is left adjoint to Res_k^n. This is known as *Frobenius reciprocity*. When Res_k^n is also left adjoint to Ind_k^n, so that $(\mathrm{Ind}_k^n, \mathrm{Res}_k^n)$ are a pair of biadjoint functors, then we say that A_n is a *Frobenius extension* of A_k. When in addition $_k(A_n)_k$ is a free (A_n, A_n)-bimodule, then we say that A_n is a *free Frobenius extension* of A_k. If A_{n+1} is a Frobenius extension of A_n for all $n \geqslant 0$ in the tower $\{A_n\}_{n \geqslant 0}$, then we say that $\{A_n\}_{n \geqslant 0}$ is a *Frobenius tower*. When A_{n+1} is a free Frobenius extension of A_n for all $n \geqslant 0$, then we say that $\{A_n\}_{n \geqslant 0}$ is a *free Frobenius tower*.

Theorem 2.3. [1, Corollary 1.2] The algebra A_{n+1} is a free Frobenius extension of A_n if and only if there is a (A_n, A_n)-bimodule homomorphism

$$E_{n+1,n} : \ _n(A_{n+1})_n \to A_n,$$

a finite set of elements $B_{n+1,n} \subset A_{n+1}$, and a vector space isomorphism $^\vee : A_{n+1} \to A_{n+1}$ such that for any $a \in A_{n+1}$,

$$a = \sum_{b \in B_{n+1,n}} E_{n+1,n}(ab^\vee)b = \sum_{b \in B_{n+1,n}} b^\vee E_{n+1,n}(ba). \tag{8}$$

We call $B_{n+1,n}$ and $B_{n+1,n}^\vee$ a *dual basis pair*, $E_{n+1,n}$ a *Frobenius homomorphism*, and the triple $(E_{n+1,n}, B, B^\vee)$ a *Frobenius system*.

Corollary 2.4. When A_{n+1} is a free Frobenius extension of A_n then $B_{n+1,n}$ is a basis for A_{n+1} as a left A_n-module, $B_{n+1,n}^\vee$ is a basis for A_{n+1} as a right A_n-module, and

$$E_{n+1,n}(b(b')^\vee) = \delta_{b,b'} \qquad \text{for any } b, b' \in B_{n+1,n}.$$

Example 2.5. All the towers listed in Example 2.1 are examples of free Frobenius towers. We describe their Frobenius homomorphisms and examples of dual bases below.

(1) For $A_{n+1} = \mathbb{C}[S_{n+1}]$ and $A_n = \mathbb{C}[S_n]$, $E_{n+1,n} : \mathbb{C}[S_{n+1}] \twoheadrightarrow \mathbb{C}[S_n]$ is defined on group elements so that for $g \in S_{n+1}$

$$E_{n+1,n}(g) = \begin{cases} g & \text{if } g \in S_n \\ 0 & \text{otherwise.} \end{cases}$$

Alternatively, since as an $(\mathbb{C}[S_n], \mathbb{C}[S_n])$-bimodule $\mathbb{C}[S_{n+1}]$ decomposes as

$$_n(\mathbb{C}[S_{n+1}])_n \cong \mathbb{C}[S_n] \oplus \left(\mathbb{C}[S_n] \otimes_{\mathbb{C}[S_{n-1}]} \mathbb{C}[S_n] \right), \tag{9}$$

then $E_{n+1,n}$ can also be viewed as projection onto the first summand on the right side of (9).

One choice for $B_{n+1,n}$ is the set of right coset representatives of S_n in S_{n+1}. Then $B_{n+1,n}^\vee$ is the set of left coset representatives of S_n in S_{n+1}. In particular, one dual basis pair is

$$B_{n+1,n} = \{s_n s_{n-1} \dots s_i \mid 1 \leqslant i \leqslant n+1\}$$

and

$$B_{n+1,n}^\vee = \{s_i \dots s_{n-1} s_n \mid 1 \leqslant i \leqslant n+1\}$$

(here, the convention is that when $i = n+1$, the corresponding element is the identity). Note that in this case $b \overset{\vee}{\mapsto} b^\vee$ sends an element of $B_{n+1,n}$ to its inverse.

(2) Let F be a Frobenius graded superalgebra with trace map $\text{tr} : F \to \mathbb{C}$ of degree $(-\delta, \tau) \in \mathbb{Z} \times \mathbb{Z}_2$. Let B_F and $\widehat{B}_F$ be homogeneous dual bases of F with respect to tr.

If $\{A_n\}_{n \geqslant 0} = \{F^{\otimes(n)} \rtimes S_n\}_{n \geqslant 0}$, then from [22, Proposition 3.4], as an (A_n, A_n)-bimodule A_{n+1} decomposes as

$$_n(A_{n+1})_n \cong \left(\bigoplus_{b \in B_F} (1 \otimes \cdots \otimes 1 \otimes b) A_n \right) \oplus \left(A_n \otimes_{A_{n-1}} A_n \right)$$

and
$$(1 \otimes \cdots \otimes 1 \otimes b)A_n \cong \Pi^{\bar{b}}A_n,$$

where $\Pi : A_n\text{-}\mathrm{Mod} \to A_n\text{-}\mathrm{Mod}$ is the parity shift functor (recall that $\bar{b} \in \mathbb{Z}_2$ is the parity of b in the $\mathbb{Z}_2$-grading). Then $E_{n+1,n} : A_{n+1} \to A_n$ corresponds to the (A_n, A_n)-bimodule homomorphism that first projects

$$A_{n+1} \twoheadrightarrow \bigoplus_{b \in B_F} (1 \otimes \cdots \otimes 1 \otimes b)A_n$$

followed by an (A_n, A_n)-bimodule homomorphism

$$\bigoplus_{b \in B_F} (1 \otimes \cdots \otimes 1 \otimes b)A_n \twoheadrightarrow A_n.$$

From [22, Proposition 3.4], for $f_1, \ldots, f_{n+1} \in F$ and $\sigma \in S_{n+1}$,

$$E_{n+1,n}\Big((f_1 \otimes \cdots \otimes f_{n+1})\sigma\Big)$$
$$= \begin{cases} (-1)^{\tau(\overline{f_1} + \cdots + \overline{f_n})} \operatorname{tr}(f_{n+1})(f_1 \otimes \cdots \otimes f_n)\sigma & \text{if } \sigma \in S_n \\ 0 & \text{otherwise.} \end{cases}$$

Corresponding dual bases are

$$B_{n+1,n} = \{(1^{\otimes n} \otimes b)s_n \ldots s_i \mid b \in B_F,\ 1 \leqslant i \leqslant n + 1\}$$

and

$$B_{n+1,n}^{\vee} = \{s_i \ldots s_n(1^{\otimes n} \otimes b) \mid b \in \widehat{B_F},\ 1 \leqslant i \leqslant n + 1\}.$$

(3) For $\lambda \in P^+$ with λ of level d, by [15, Lemma 7.6.1], as an $(H_n^\lambda, H_n^\lambda)$-bimodule, H_{n+1}^λ decomposes as

$$_n(H_{n+1}^\lambda)_n \cong \Big(\bigoplus_{j=0}^{d-1} H_n^\lambda x_{n+1}^j\Big) \oplus (H_n^\lambda \otimes_{H_{n-1}^\lambda} H_n^\lambda)$$

where $H_n^\lambda x_{n+1}^j \cong H_n^\lambda$. Then $E_{n+1,n} : H_{n+1}^\lambda \twoheadrightarrow H_n^\lambda$ corresponds to projection to the summand $H_n^\lambda x_{n+1}^{d-1}$. Recall that in the case where $\lambda = \omega_0$ then $H_n^\lambda \cong \mathbb{C}[S_n]$, and $E_{n+1,n}$ is the Frobenius homomorphism described in Example 2.5.1.

Define

$$y_{n,k} := \sum_{t=k}^{d-1} (-1)^{d-1-t} x_n^{t-k} \det\Big(E_{n,n-1}(x_n^{d+j-i})\Big)_{i,j=1,\ldots,d-1-t}$$

where the determinant in this expression is 1 when $t = d - 1$. By [20, Proposition 5.11], a pair of dual bases with respect to $E_{n+1,n}$ is

$$B_{n+1,n} = \{\, s_n \ldots s_i y_{i,a} \mid i = 1, \ldots, n+1,\ a = 0, \ldots, d-1 \,\}$$

and

$$B_{n+1,n}^{\vee} = \{\, x_i^a s_i \ldots s_n \mid i = 1, \ldots, n+1,\ a = 0, \ldots, d-1 \,\}.$$

While we only required that the algebra A_{n+1} immediately above A_n be a Frobenius extension of A_n, this assumption is in fact enough to ensure that for any $0 \leqslant k < n+1$, A_{n+1} is a Frobenius extension of A_k.

Proposition 2.6. [12, Section 1.3] Let $\{A_n\}_{n \geqslant 0}$ be a Frobenius tower. Then for any $0 \leqslant k < n$, A_n is a Frobenius extension of A_k with Frobenius homomorphism $E_{n,k} := E_{k+1,k} \circ \cdots \circ E_{n,n-1}$, and dual basis pair

$$B_{n,k} := \{ b^{(k+1)} \ldots b^{(n-1)} b^{(n)} \mid b^{(i)} \in B_{i,i-1} \}$$

and

$$B_{n,k}^\vee := \{ (b^{(n)})^\vee (b^{(n-1)})^\vee \ldots (b^{(k+1)})^\vee \mid b^{(i)} \in B_{i,i-1} \}.$$

Remark 2.7. As a consequence of the assumption that $A_0 = \mathbb{C}$, each A_n in the Frobenius tower $\{A_n\}_{n \geqslant 0}$ is actually a Frobenius algebra with trace $E_{n,0} : A_n \to \mathbb{C}$.

When for all $a, a' \in A_n$,

$$E_{n+1,k}(aa') = E_{n+1,k}(a'a)$$

then we say that A_{n+1} is a *symmetric* Frobenius extension of A_k. Note that when this property is true for $E_{n,0}$, then A_n is a *symmetric Frobenius algebra* (also known as a *symmetric algebra*). We will generally assume that each A_n in our tower $\{A_n\}_{n \geqslant 0}$ is symmetric as a Frobenius algebra (this is true for all towers in Example 2.5), though we do not assume that $E_{n,k}$ is symmetric for $k > 0$ (this is not true for $\{H_n^\lambda\}_{n \geqslant 0}$ in Example 2.1.4).

The Frobenius homomorphism $E_{n,k}$ actually gives the co-unit of the adjunction of the functor Res_k^n and its right adjoint Ind_k^n. That is, the natural transformation $\epsilon_{n,k} : \mathrm{Res}_k^n \mathrm{Ind}_k^n \to \mathbb{1}$ can be translated to an (A_k, A_k)-bimodule homomorphism from ${}_k(A_n) \otimes_{A_n} (A_n)_k \cong {}_k(A_n)_k$ to A_k. This homomorphism is $E_{n,k}$.

In the language of bimodules, the unit $\eta_{n,k} : \mathbb{1} \to \mathrm{Ind}_k^n \mathrm{Res}_k^n$ of the adjunction with Ind_k^n the right adjoint of Res_k^n defines an (A_n, A_n)-bimodule homomorphism $H_{n,k} : A_n \to (A_n)_k \otimes_{A_k} {}_k(A_n)$. For $a \in A_n$, $H_{n,k}$ sends

$$a \xmapsto{H_{n,k}} \sum_{b \in B_{n,k}} ab^\vee \otimes b.$$

For any $c \in A_n$, c defines an (A_n, A_n)-bimodule homomorphism

$$T_{c,n,k} : (A_n)_k \otimes_{A_k} {}_k(A_n) \to (A_n)_k \otimes_{A_k} {}_k(A_n)$$

via the assignment such that for $a \otimes a' \in (A_n)_k \otimes_{A_k} {}_k(A_n)$,

$$a \otimes a' \xmapsto{T_{c,n,k}} ac \otimes a'.$$

Finally, there is an (A_n, A_n)-bimodule homomorphism $M_{n,k} : (A_n)_k \otimes_{A_k} {}_k(A_n) \to A_n$, known as the *multiplication map* which maps $a \otimes a' \in (A_n)_k \otimes_{A_k} {}_k(A_n)$ to

$$a \otimes a' \xmapsto{M_{n,k}} aa'$$

(in fact this is the co-unit of the adjunction of Ind_k^n with Res_k^n as its right adjoint).

We then define $N_{n,k} : A_n \to A_n$ so that for $c \in A_n$,

$$c \mapsto (M_{n,k} \circ T_{c,n,k} \circ H_{n,k})(1) = \sum_{b \in B_{n,k}} b^\vee cb.$$

Remark 2.8. When $\{A_n\}_{n \geqslant 0}$ is the tower of Iwahori-Hecke algebras, the map $N_{n,k}$ is sometimes referred to as the *relative norm* [11].

2.1. **The centers of** $\{A_n\}_{n \geqslant 0}$**.** We write $Z(A_n)$ for the center of A_n and $Z(A_n, A_k)$ for the centralizer of A_k in A_n. That is

$$Z(A_n, A_k) := \{\, a \in A_n \mid aa' = a'a, \ \text{for all } a' \in A_k\}.$$

Proposition 2.9. For all $0 \leqslant k < n$:
  (1) $E_{n,k} : A_n \to A_k$ maps $Z(A_n, A_k)$ into $Z(A_k)$.
  (2) $N_{n,k} : A_n \to A_n$ maps A_n into $Z(A_n)$.

Proof. Part (1) follows from the fact that $E_{n,k}$ is a (A_k, A_k)-bimodule homomorphism. Let $a \in Z(A_n, A_k)$. Then for any $a' \in A_k$,

$$a'E_{n,k}(a) = E_{n,k}(a'a) = E_{n,k}(aa') = E_{n,k}(a)a'.$$

Part (2) follows from the observation that for any $a \in A_n$,

$$N_{n,k}(a) = \sum_{b \in B_{n,k}} b^\vee ab = (M_{n,k} \circ T_{a,n,k} \circ H_{n,k})(1).$$

Since each map $M_{n,k}$, $T_{a,n,k}$, and $H_{n,k}$ is an (A_n, A_n)-bimodule homomorphism, then for any $a' \in A_n$,

$$a'(M_{n,k} \circ T_{a,n,k} \circ H_{n,k})(1) = (M_{n,k} \circ T_{a,n,k} \circ H_{n,k})(a')$$
$$= (M_{n,k} \circ T_{a,n,k} \circ H_{n,k})(1)a'.$$

$\square$

For $0 \leqslant k < n$, the elements $N_{n,k}(1)$ will play a central role in this paper. We therefore set

$$C_{n,k} := N_{n,k}(1) = \sum_{b \in B_{n,k}} b^\vee b.$$

Note that for all $0 \leqslant k < n$, $C_{n,k}$ belongs to $Z(A_n)$. In [12], the element $C_{n,k}$ is referred to as the $E_{n,k}$-*index* of the Frobenius extension A_n of A_k. When $k = 0$ and $E_{n,0} : A_n \to \mathbb{C}$ is symmetric, $C_{n,0}$ is referred to as the *central Casimir element* in [5].

Lemma 2.10. [5, Proposition 3.5] When A_n is a symmetric Frobenius algebra with respect to $E_{n,0}$ then

$$C_{n,0} = \sum_{b \in B_{n,0}} b^\vee b = \sum_{b \in B_{n,0}} bb^\vee.$$

In general,

$$\sum_{b \in B_{n,k}} b^\vee b \neq \sum_{b \in B_{n,k}} b b^\vee \tag{10}$$

(see Example 2.11.2 below). Because the element on the right side of (10) will appear again later, we denote it by

$$C_{n,k}^\iota := \sum_{b \in B_{n,k}} b b^\vee.$$

Example 2.11.   (1) When $A_n = \mathbb{C}[S_n]$ and $A_k = \mathbb{C}[S_k]$ then

$$C_{n,k}^\iota = C_{n,k} = \sum_{g \in S_n/S_k} g g^{-1} = \left| S_n/S_k \right| = (n)(n-1) \ldots (k+1).$$

(2) Let $\lambda = \omega_i + \omega_j$ for $i, j \in \mathbb{Z}$ so that the level of λ is $d = 2$. Then with the help of Lemma 5.12 in [20] a routine calculation shows:

$$C_{3,2} = 2(x_1 + x_2 + x_3) - 3(i + j) \in H_3^\lambda.$$

On the other hand

$$C_{3,2}^\iota = 2(s_2 s_1 x_1 s_1 s_2 + s_2 x_2 s_2 + x_3) - 3(i + j)$$

$$= (6x_3 - 4x_3 s_2 - 2 s_2 s_1 s_2) - 3(i + j) \in H_3^\lambda.$$

It is not hard to show that in this case $C_{3,2}^\iota$ is not even in $Z(H_3^\lambda)$.

3. The representation theory of Frobenius towers

We begin by setting some notation related to the representation theory of $\{A_n\}_{\geqslant 0}$ and recalling some fundamental facts. Much of this section is modeled after the representation theory for degenerate cyclotomic Hecke algebras (Example 2.1.4) in [15, Part I]. In this section we assume that $\{A_n\}_{\geqslant 0}$ is a Frobenius tower. We denote the indexing set of isomorphism classes of simple A_n-representations by Γ_n (note that this is a finite set since A_n is assumed to be finite-dimensional) and set

$$\Gamma := \bigcup_{n \geqslant 0} \Gamma_n.$$

For $\lambda \in \Gamma_n$ let L^λ be the simple A_n-module corresponding to λ. Each L^λ has a unique projective cover which we denote by P^λ. The set $\{P^\lambda\}_{\lambda \in \Gamma_n}$ gives a complete list of isomorphism classes of indecomposable projective A_n-modules. Recall from Section 2 that we write A_n-Mod for the category of finite-dimensional A_n-modules. We furthermore set A_n-Pmod to be the category of finite-dimensional projective A_n-modules.

We denote by $G_0(A_n\text{-}\mathrm{Mod})$ the triangulated Grothendieck group of finite dimensional A_n-modules. That is, $G_0(A_n\text{-}\mathrm{Mod})$ is the abelian group generated by the set $\{[M] \mid M \in A_n\text{-}\mathrm{Mod}\}$ subject to the relations such that for $M, N, K \in A_n\text{-}\mathrm{Mod}$,

$$[M] + [N] = [K]$$

if there is a short exact sequence

$$0 \to M \to K \to N \to 0.$$

$G_0(A_n\text{-}\mathrm{Mod})$ is a free abelian group with basis $\{[L^\lambda] \mid \lambda \in \Gamma_n\}$. We denote by $K_0(A_n\text{-}\mathrm{Pmod})$ the split Grothendieck group of finite dimensional projective A_n-modules, so that $K_0(A_n\text{-}\mathrm{Pmod})$ is the abelian group generated by $\{[P] \mid P \in A_n\text{-}\mathrm{Pmod}\}$ with the property that for $P, P', P'' \in A_n\text{-}\mathrm{Pmod}$,

$$[P] + [P'] = [P'']$$

if and only if

$$P \oplus P' \cong P''.$$

$K_0(A_n\text{-}\mathrm{Pmod})$ is a free abelian group with basis $\{[P^\lambda] \mid \lambda \in \Gamma_n\}$. We set

$$G_0(A) = \bigoplus_{n \geqslant 0} G_0(A_n\text{-}\mathrm{Mod}) \qquad \text{and} \qquad K_0(A) = \bigoplus_{n \geqslant 0} K_0(A_n\text{-}\mathrm{Pmod}).$$

Note that since both Ind_n^{n+1} and Res_n^{n+1} are exact functors that send projectives to projectives, they descend to linear operators on $G_0(A)$ and $K_0(A)$. By abuse of notation we denote both the genuine functors Ind_n^{n+1} and Res_n^{n+1} and their corresponding linear operators on $G_0(A)$ and $K_0(A)$ using the same notation.

There is a form $\langle \cdot, \cdot \rangle : K_0(A) \times G_0(A) \to \mathbb{Z}_{\geqslant 0}$ such that for $m, n \geqslant 0$ and $P \in A_m\text{-}\mathrm{Pmod}$, $L \in A_n\text{-}\mathrm{Mod}$ then

$$\langle [P], [L] \rangle = \dim(\mathrm{Hom}_{A_n}(P, L)) \qquad \text{if } m = n$$

and $\langle [P], [L] \rangle = 0$ otherwise. In fact, reducing to the basis for $K_0(A)$, $\{[P^\lambda]\}_{\lambda \in \Gamma}$, and the basis for $G_0(A)$, $\{[L^\lambda]\}_{\lambda \in \Gamma}$, $\langle \cdot, \cdot \rangle$ can be defined by the property that for $\lambda, \eta \in \Gamma$

$$\langle [P^\lambda], [L^\eta] \rangle = \delta_{\lambda, \eta}. \tag{11}$$

The equality in (11) relies on basic properties of projective covers.

Since $\{A_n\}_{n \geqslant 0}$ is a Frobenius tower, Ind_n^{n+1} and Res_n^{n+1} are biadjoint and hence for any $\mu \in \Gamma_n$ and $\lambda \in \Gamma_{n+1}$

$$\langle \mathrm{Ind}_n^{n+1}[P^\mu], [L^\lambda] \rangle = \langle [P^\mu], \mathrm{Res}_n^{n+1}[L^\lambda] \rangle \tag{12}$$

and

$$\langle \mathrm{Res}_n^{n+1}[P^\lambda], [L^\mu] \rangle = \langle [P^\lambda], \mathrm{Ind}_n^{n+1}[L^\mu] \rangle.$$

We define the following elements:

- $\kappa(\mu, \lambda)$ for $\mu \in \Gamma_n$ and $\lambda \in \Gamma_{n+1}$ such that for $[P^\mu] \in K_0(A_n\text{-}\mathrm{Pmod})$,

$$\mathrm{Ind}_n^{n+1}[P^\mu] = \sum_{\lambda \in \Gamma_{n+1}} \kappa(\mu, \lambda)[P^\lambda]. \tag{13}$$

- $\kappa^*(\mu, \lambda)$ for $\mu \in \Gamma_n$ and $\lambda \in \Gamma_{n+1}$ such that for $[L^\mu] \in G_0(A_n\text{-}\mathrm{Mod})$,

$$\mathrm{Ind}_n^{n+1}[L^\mu] = \sum_{\lambda \in \Gamma_{n+1}} \kappa^*(\mu, \lambda)[L^\lambda]. \tag{14}$$

- $\bar{\kappa}(\lambda, \mu)$ for $\mu \in \Gamma_n$ and $\lambda \in \Gamma_{n+1}$ such that for $[P^\lambda] \in K_0(A_{n+1})$,

$$\operatorname{Res}_n^{n+1}[P^\lambda] = \sum_{\mu \in \Gamma_n} \bar{\kappa}(\lambda, \mu)[P^\mu]. \qquad (15)$$

- $\bar{\kappa}^*(\lambda, \mu)$ for $\mu \in \Gamma_n$ and $\lambda \in \Gamma_{n+1}$ such that for $[L^\lambda] \in G_0(A_{n+1})$,

$$\operatorname{Res}_n^{n+1}[L^\lambda] = \sum_{\mu \in \Gamma_n} \bar{\kappa}^*(\lambda, \mu)[L^\mu]. \qquad (16)$$

Remark 3.1. Note that it follows from the definition of $K_0(\cdot)$ that (13) and (15) can be lifted to the level of modules

$$\operatorname{Ind}_n^{n+1} P^\mu \cong \bigoplus_{\lambda \in \Gamma_{n+1}} (P^\lambda)^{\oplus \kappa(\mu, \lambda)}$$

and

$$\operatorname{Res}_n^{n+1} P^\lambda \cong \bigoplus_{\mu \in \Gamma_n} (P^\mu)^{\oplus \bar{\kappa}(\lambda, \mu)}.$$

Of course, a similar decomposition does not occur in general for $\operatorname{Ind}_n^{n+1} L^\mu$ and $\operatorname{Res}_n^{n+1} L^\lambda$ when the algebras $\{A_n\}_{n \geqslant 0}$ are not semisimple. In this case (14) and (16) can be interpreted at the level of modules via composition series.

The following proposition is essentially a generalization of Corollary 9.3.2 in [15].

Proposition 3.2. For all $n \geqslant 0$ and $\mu \in \Gamma_n$ and $\lambda \in \Gamma_{n+1}$,
- $\kappa(\mu, \lambda) = \bar{\kappa}^*(\lambda, \mu)$,
- $\kappa^*(\mu, \lambda) = \bar{\kappa}(\lambda, \mu)$.

Proof. It follows from (11) and (13) that

$$\langle \operatorname{Ind}_n^{n+1}[P^\mu], [L^\lambda] \rangle = \kappa(\mu, \lambda)$$

while from (11) and (16)

$$\langle [P^\mu], \operatorname{Res}_n^{n+1}[L^\lambda] \rangle = \bar{\kappa}^*(\lambda, \mu).$$

From (12) we then get that $\kappa(\mu, \lambda) = \bar{\kappa}^*(\lambda, \mu)$. The proof that $\kappa^*(\mu, \lambda) = \bar{\kappa}(\lambda, \mu)$ is analogous. $\qquad \square$

Given Proposition 3.2, we henceforth use the notation $\kappa(\mu, \lambda)$ and $\kappa^*(\mu, \lambda)$ exclusively. We extend κ and κ^* to all of $\Gamma \times \Gamma$ by setting

$$\kappa(\mu, \lambda) = \kappa^*(\mu, \lambda) = 0$$

when there is no $n \geqslant 0$ such that $|\mu| = n$ and $|\lambda| = n + 1$.

By a classical result, as a left A_n-module A_n can be decomposed into a direct sum of indecomposable projective A_n-modules so that

$$A_n \cong \bigoplus_{\mu \in \Gamma_n} (P^\mu)^{\dim(L^\mu)}. \qquad (17)$$

This is a generalization of the well-known decomposition of the left regular representation of a finite group into a direct sum of simple representations.

We will use the notation $\{e_{\mu,1}, e_{\mu,2}, \ldots, e_{\mu,\dim(L^\mu)}\}$ for a complete set of central, orthogonal idempotents corresponding to the summands $(P^\mu)^{\oplus \dim(L^\mu)}$.

3.1. Elements of $Z(A_{n+1}, A_n)$ and representations of A_n.

Suppose that $x \in Z(A_{n+1}, A_n)$, and that $M \in A_{n+1}\text{-Mod}$. Then there is an induced action of x on $\mathrm{Res}_n^{n+1}(M)$. Indeed, $x \in \mathrm{End}_{A_n}(\mathrm{Res}_n^{n+1} M)$ since for any $m \in \mathrm{Res}_n^{n+1}(M)$ and $a \in A_n$,

$$a(xm) = x(am).$$

Let $\lambda \in \Gamma_{n+1}$ and recall the decomposition

$$\mathrm{Res}_n^{n+1} P^\lambda \cong \bigoplus_{\mu \in \Gamma_n} (P^\mu)^{\oplus \kappa^*(\mu, \lambda)}. \tag{18}$$

The action of x is stable on each summand in this decomposition since x necessarily commutes with the idempotent in A_n correspond to this summand. It follows that $x \in \mathrm{End}_{A_n}(P^\mu)$ for each $\mu \in \Gamma_n$, and this action depends on $\lambda \in \Gamma_{n+1}$ where $P^\mu \subseteq \mathrm{Res}_n^{n+1}(P^\lambda)$.

Recall that for $\alpha \in \mathbb{C}$, the generalized α-eigenspace of $x \in \mathrm{End}_\mathbb{C}(M)$ in M is defined to be

$$M[\alpha] := \{\, m \in M \mid (x - \alpha)^k m = 0 \ \text{ for } k >> 0\}.$$

Since $(x - a)$ commutes with A_n it follows that as an A_n-module

$$M \cong \bigoplus_{\alpha \in \mathbb{C}} M[\alpha].$$

In the remainder of this section we will assume that $\{x_n\}_{n \geqslant 1}$ is a sequence of elements such that for all $n \geqslant 0$,

(a) $x_{n+1} \in Z(A_{n+1}, A_n)$,
(b) x_{n+1} has constant generalized eigenvalue on $(P^\mu)^{\oplus \kappa^*(\mu, \lambda)} \subseteq \mathrm{Res}_n^{n+1} P^\lambda$ for each $\mu \in \Gamma_n$, $\lambda \in \Gamma_{n+1}$.

Under this assumption, to each $\lambda \in \Gamma_{n+1}$ and $\mu \in \Gamma_n$ we can associate $\alpha_\mu^\lambda \in \mathbb{C}$ which is the generalized eigenvalue of x_{n+1} on $(P^\mu)^{\oplus \kappa^*(\mu, \lambda)}$ in $\mathrm{Res}_n^{n+1} P^\lambda$.

Example 3.3. The model for the sequence $\{x_n\}_{n \geqslant 0}$ described above is the collection of Jucys-Murphy elements in $\{\mathbb{C}[S_n]\}_{n \geqslant 0}$ and related algebras. For reference we list some of these elements below:

(1) The classical Jucys-Murphy elements $\{J_n\}_{n \geqslant 0}$ in $\{\mathbb{C}[S_n]\}_{n \geqslant 0}$, are defined by $J_1 := 0$ and for $n > 1$,

$$J_n := (1, n) + (2, n) + \cdots + (n - 1, n) \in \mathbb{C}[S_n].$$

$\{J_n\}_{n \geqslant 0}$ satisfy assumptions (a) and (b).

(2) The tower of Sergeev algebras $\{\mathbb{S}_n\}_{n \geqslant 0}$ has its own version of Jucys-Murphy elements which by abuse of notation we also denote by $\{J_n\}_{n \geqslant 0}$. These are defined so that $J_1 := 0$ and

$$J_n := \sum_{i=1}^{n-1} (1 + c_i c_n)(i, n).$$

(3) Just as Sergeev algebras are just a specific example of Frobenius wreath product algebras $\{F^{\otimes n} \otimes S_n\}_{n \geqslant 0}$, the Jucys-Murphy elements of $\{\mathbb{S}_n\}_{n \geqslant 0}$ are just a specific case of Jucys-Murphy elements in $\{F^{\otimes n} \otimes S_n\}_{n \geqslant 0}$. These are defined so that $J_1 := 0$ and

$$J_n = \sum_{i=1}^{n-1} t_{i,n}(1,n)$$

where

$$t_{i,n} = \sum_{b \in B_F} (1^{\otimes i-1} \otimes b \otimes 1^{\otimes n-i-1} \otimes b^{\vee}).$$

(4) Finally, the image of $\{x_n\}_{n \geqslant 1}$ in the quotient algebras $\{H_n^{\lambda}\}_{n \geqslant 0}$ satisfy assumptions (a) and (b).

For a representation $N \in A_n\text{-Mod}$ write $\mathrm{Tr}_N : A_n \to \mathbb{C}$ for the usual trace function on images of A_n in $\mathrm{End}_{\mathbb{C}}(N)$. The normalized trace

$$\widetilde{\mathrm{Tr}}_N := \frac{\mathrm{Tr}_N}{\dim(N)}$$

is a familiar tool in asymptotic representation theory. In what follows however, we will need to work with the Frobenius homomorphism $E_{n,0} : A_n \to \mathbb{C}$ (rather than trace functions) and associate it to a specific projective representation P. We do this by pre-composing with multiplication by the corresponding idempotent e_P. That is, we consider the map $E_{n,0}(e_P \cdot) : A_n \to \mathbb{C}$. In order to arrive at the correct analogue of normalized trace in this setting we rescale e_P to

$$\widetilde{e}_P = \frac{e_P}{\dim(P)}.$$

Note that $\widetilde{e}_P$ is not an idempotent unless P is 1-dimensional.

Although it is not an element of A_n, for any $\mu \in \Gamma_n$ we can calculate $\mathrm{Tr}_{P^{\mu}}(x_{n+1})$ after specifying that $P^{\mu} \subseteq \mathrm{Res}_n^{n+1} P^{\lambda}$ for some $\lambda \in \Gamma_{n+1}$. Then for $k \geqslant 0$ and $e_{\mu} \in Z(A_n)$ the central idempotent corresponding to some $P^{\mu} \subseteq \mathrm{Res}_n^{n+1} P^{\lambda}$,

$$\mathrm{Tr}_{A_n}(e_{\mu} x_{n+1}^k) = \mathrm{Tr}_{P^{\mu}}(x_{n+1}^k) = \dim(P^{\mu})(\alpha_{\mu}^{\lambda})^k$$

and

$$\mathrm{Tr}_{A_n}(\widetilde{e}_{\mu} x_{n+1}^k) = (\alpha_{\mu}^{\lambda})^k.$$

Proposition 3.4 gives a relationship between the trace on A_{n+1} with respect to left multiplication against A_{n+1}, $E_{n+1,0}$, and $C_{n+1,0}$.

Proposition 3.4. [5, Corollary 3.10] Let A_{n+1} be a symmetric Frobenius algebra with respect to $E_{n+1,0} : A_{n+1} \to \mathbb{C}$ and $\mathrm{Tr}_{A_{n+1}} : A_{n+1} \to \mathbb{C}$ be the trace on A_{n+1} with respect to A_{n+1} as a left A_{n+1}-module. Then for $a \in A_{n+1}$,

$$\mathrm{Tr}_{A_{n+1}}(a) = E_{n+1,0}(C_{n+1,0}a).$$

Corollary 3.5. [5, Corollary 3.11] For $\lambda \in \Gamma_{n+1}$, let $e_\lambda \in Z(A_{n+1})$ be a central idempotent corresponding to P^λ. For $a \in A_{n+1}$,

$$\mathrm{Tr}_{P^\lambda}(a) = E_{n+1,0}(C_{n+1,0}e_\lambda a)$$

and

$$\widetilde{\mathrm{Tr}}_{P^\lambda}(a) = E_{n+1,0}(C_{n+1,0}\widetilde{e}_\lambda a).$$

Corollary 3.6. Let $\lambda \in \Gamma_{n+1}$ and let $e_{\lambda,i}$ and $e_{\lambda,j}$ both be central idempotents of A_{n+1} such that as A_{n+1}-modules $e_{\lambda,i}A_{n+1} \cong e_{\lambda,j}A_{n+1} \cong P^\lambda$. Then

$$E_{n+1,0}(C_{n+1,0}e_{\lambda,i}a) = E_{n+1,0}(C_{n+1,0}e_{\lambda,j}a).$$

Proposition 3.4 has an analogue for Frobenius towers where $A_0 = \mathbb{C}$.

Proposition 3.7. Let $\{A_n\}_{n\geqslant 0}$ be a Frobenius tower with $A_0 = \mathbb{C}$ and assume A_n and A_{n+1} are symmetric Frobenius algebras with respect to $E_{n,0}$ and $E_{n+1,0}$ respectively. Let $\mathrm{Tr}_{A_{n+1}} : A_{n+1} \to \mathbb{C}$ be the trace on A_{n+1} with respect to A_{n+1} as a left A_{n+1}-module and $\mathrm{Tr}_{A_n} : A_n \to \mathbb{C}$ be the trace on A_n with respect to A_n as a left A_n-module. Then for $a \in Z(A_{n+1}, A_n)$

$$\mathrm{Tr}_{A_{n+1}}(a) = \mathrm{Tr}_{A_n}(E_{n+1,n}(C^\iota_{n+1,n}a)).$$

Proof. Recall that

$$B = \{b^{(n)}b^{(n+1)} \mid b^{(n)} \in B_{n,0},\ b^{(n+1)} \in B_{n+1,n}\}$$

and

$$B^\vee = \{(b^{(n+1)})^\vee(b^{(n)})^\vee \mid b^{(n+1)} \in B_{n+1,n},\ b^{(n)} \in B_{n,0}\}$$

are dual bases for A_{n+1} with respect to $E_{n+1,0} = E_{n,0} \circ E_{n+1,n}$. Proposition 3.4 implies that for $a \in Z(A_{n+1}, A_n)$ we have

$$\mathrm{Tr}_{A_{n+1}}(a) = E_{n+1,0}(C_{n+1,0}a))$$

$$= \sum_{\substack{b^{(n+1)}\in B_{n+1,n},\\ b^{(n)}\in B_{n,0}}} E_{n+1,0}((b^{(n+1)})^\vee(b^{(n)})^\vee b^{(n)}b^{(n+1)}a).$$

By Lemma 2.10 this is equal to

$$\sum_{\substack{b^{(n+1)}\in B_{n+1,n},\\ b^{(n)}\in B_{n,0}}} E_{n+1,0}(b^{(n)}b^{(n+1)}(b^{(n+1)})^\vee(b^{(n)})^\vee a).$$

as $E_{n+1,0}$ is assumed to be symmetric. Recalling that $E_{n+1,n}$ is an (A_n, A_n)-bimodule homomorphism and $a \in Z(A_{n+1}, A_n)$ commutes with $(b)^\vee \in A_n$ we get

$$\sum_{\substack{b^{(n+1)}\in B_{n+1,n},\\ b^{(n)}\in B_{n,0}}} E_{n,0}(E_{n+1,n}(b^{(n)}b^{(n+1)}(b^{(n+1)})^\vee(b^{(n)})^\vee a))$$

$$= \sum_{b^{(n)}\in B_{n,0}} E_{n,0}(b^{(n)}E_{n+1,n}(C^\iota_{n+1,n}a)(b^{(n)})^\vee). \tag{19}$$

Since $E_{n,0}$ is symmetric, (19) is equivalent to

$$\sum_{b^{(n)}\in B_{n,0}} E_{n,0}\Big((b^{(n)})^{\vee}b^{(n)}E_{n+1,n}\big(C^{\iota}_{n+1,n}a\big)\Big).$$

$$= E_{n,0}\Big(C_{n,0}E_{n+1,n}\big(C^{\iota}_{n+1,n}a\big)\Big).$$

Applying Proposition 3.4 then gives the desired result. $\qquad\square$

The following theorem will allow us to connect the down/up transition functions which will be introduced in Section 4 with the centers $\{Z(A_n)\}_{n\geqslant 0}$.

Theorem 3.8. Let $\mu \in \Gamma_n$ and let $e_\mu \in Z(A_n)$ be a central idempotent corresponding to P^μ. Assume that x_n and x_{n+1} satisfy assumptions (a) and (b).

(1) For all $k \geqslant 0$, $E_{n+1,n}(C_{n+1,0}x^k_{n+1}) \in Z(A_n)$ and

$$E_{n,0}\Big(\widetilde{e}_\mu E_{n+1,n}(C_{n+1,0}x^k_{n+1})\Big) = \sum_{\lambda\in\Gamma_{n+1}} \frac{\kappa^*(\mu,\lambda)\dim(L^\lambda)}{\dim(L^\mu)}(\alpha^\lambda_\mu)^k.$$

(2) For all $k \geqslant 0$, $N^n_0(x^k_n) \in Z(A_n)$ and

$$E_{n,0}\Big(\widetilde{e}_\mu N^n_0(x^k_n)\Big) = \sum_{\eta\in\Gamma_{n-1}} \frac{\kappa^*(\eta,\mu)\dim(P^\eta)}{\dim(P^\mu)}(\alpha^\mu_\eta)^k.$$

Proof. (1) $C_{n+1,0} \in Z(A_{n+1}, A_n)$ by Proposition 2.9.2. Since x_{n+1} is also in $Z(A_{n+1}, A_n)$ by assumption, then $C_{n+1,0}x^k_{n+1} \in Z(A_{n+1}, A_n)$. It then follows from Proposition 2.9.1 that $E_{n+1,n}(C_{n+1,0}x^k_{n+1}) \in Z(A_n)$.

Recall that we write

$$\mathrm{Res}^{n+1}_n P^\lambda = \bigoplus_{\gamma\in\Gamma_n} (P^\gamma)^{\oplus\kappa^*(\gamma,\lambda)}.$$

Hence the trace of the action of x^k_{n+1} on $\mathrm{Res}^{n+1}_n P^\lambda$ is precisely,

$$\mathrm{Tr}_{\mathrm{Res}^{n+1}_n P^\lambda}(x^k_{n+1}) = \sum_{\gamma\in\Gamma_n} \kappa^*(\gamma,\lambda)\dim(P^\gamma)(\alpha^\lambda_\gamma)^k \qquad (20)$$

Let $e_{\lambda,\mu}$ be a central idempotent corresponding to $(P^\mu)^{\oplus\kappa^*(\lambda,\mu)} \subseteq \mathrm{Res}^{n+1}_n P^\lambda$. Then

$$\mathrm{Tr}_{\mathrm{Res}^{n+1}_n P^\lambda}(e_{\lambda,\mu}x^k_{n+1}) = \kappa^*(\lambda,\mu)\dim(P^\mu)(\alpha^\lambda_\mu)^k.$$

Because of the decomposition (17) if we let f_μ be the idempotent corresponding to $(P^\mu)^{\dim(L^\mu)} \subseteq A_n$,

$$f_\mu := \sum_{i=1}^{\dim(L^\mu)} e_{\mu,i}$$

and if we set

$$\widehat{f}_\mu := \frac{f_\mu}{\dim(P^\mu)} = \sum_{i=1}^{\dim(L^\mu)} \widetilde{e}_{\mu,i},$$

then applying (20) to decomposition (17) we have

$$\mathrm{Tr}_{A_{n+1}}(f_\mu x_{n+1}^k) = \sum_{\lambda \in \Gamma_n} \kappa^*(\mu, \lambda) \dim(L^\lambda) \dim(P^\mu)(\alpha_\mu^\lambda)^k$$

and

$$\mathrm{Tr}_{A_{n+1}}(\widehat{f}_\mu x_{n+1}^k) = \sum_{\lambda \in \Gamma_n} \kappa^*(\mu, \lambda) \dim(L^\lambda)(\alpha_\mu^\lambda)^k. \qquad (21)$$

Next by Proposition 3.4

$$\mathrm{Tr}_{A_{n+1}}(\widehat{f}_\mu x_{n+1}^k) = E_{n+1,0}(C_{n+1,0}\widehat{f}_\mu x_{n+1}^k) \qquad (22)$$

$$= E_{n,0}\Big(E_{n+1,n}(C_{n+1,0}\widehat{f}_\mu x_{n+1}^k) \Big)$$

$$= E_{n,0}\Big(\widehat{f}_\mu E_{n+1,n}(C_{n+1,0}x_{n+1}^k) \Big)$$

$$= \dim(L^\mu) E_{n,0}\Big(\widetilde{e}_\mu E_{n+1,n}(C_{n+1,0}x_{n+1}^k) \Big)$$

where the third equality follows from the fact that $C_{n+1,0} \in Z(A_{n+1})$ and the last equality follows from Corollary 3.6. Combining (21) and (22) then gives the desired result.

(2) It follows from Proposition 2.9.2 that $N_0^n(x_n^k) \in Z(A_n)$.

Because we are assuming that $E_{n,0}$ is symmetric and because $\widetilde{e}_\mu \in Z(A_n)$,

$$E_{n,0}(\widetilde{e}_\mu N_0^n(x_n^k)) = E_{n,0}\Big(\widetilde{e}_\mu \sum_{b \in B_{n,0}} b^\vee x_n^k b \Big)$$

$$= E_{n,0}\Big(\widetilde{e}_\mu \sum_{b \in B_{n,0}} bb^\vee x_n^k \Big).$$

Then by Lemma 2.10 and Proposition 3.4 we have,

$$E_{n,0}\Big(\widetilde{e}_\mu \sum_{b \in B_{n,0}} bb^\vee x_n^k \Big) = E_{n,0}\Big(\widetilde{e}_\mu C_{n,0}x_n^k \Big)$$

$$= \frac{1}{\dim(P^\mu)} \mathrm{Tr}_{P^\mu}(x_n^k).$$

Because P^μ decomposes as

$$\mathrm{Res}_{n-1}^n P^\mu = \bigoplus_{\eta \in \Gamma_{n-1}} (P^\eta)^{\oplus \kappa^*(\eta, \mu)}$$

and x_n has constant generalized eigenvalue α_η^μ on each P^η, then

$$\frac{1}{\dim(P^\mu)} \mathrm{Tr}_{P^\mu}(x_n^k) = \sum_{\eta \in \Gamma_{n-1}} \frac{\kappa^*(\eta, \mu) \dim(P^\eta)}{\dim(P^\mu)}(\alpha_\eta^\mu)^k.$$

$\square$

4. Two coherent systems on free Frobenius towers

Recall that a graded graph is a triple $G = (V, \rho, E)$ where V is a discrete set of vertices, $\rho : V \to \mathbb{Z}$ is a rank function, and E is a multiset of edges such that $(x, y) \in E$ for $x, y \in V$ implies that $\rho(y) = \rho(x) + 1$ (see [8] for more details). Observe that ρ induces a decomposition of V into graded components $V = \bigcup_{k \in \mathbb{Z}} V_k$. We begin this section by defining two graded graphs, G_Γ and G_Γ^* which will be central to this paper.

Definition 4.1.
  (1) Let G_Γ be the graded graph such that:
      (a) the vertex set of G_Γ is equal to Γ, and the nth graded component is Γ_n,
      (b) the vertices $\mu \in \Gamma_n$ and $\lambda \in \Gamma_{n+1}$ only have an edge between them if $\kappa(\mu, \lambda) > 0$.
  (2) Let G_Γ^* be the graded graph such that:
      (a) the vertex set of G_Γ is equal to Γ, and the nth graded component is Γ_n,
      (b) the vertices $\mu \in \Gamma_n$ and $\lambda \in \Gamma_{n+1}$ only have an edge between them if $\kappa^*(\mu, \lambda) > 0$.

The coherent system formalism was first defined in [4, Section 1] in the context of graded sets. Our usage of this concept in terms of graded graphs is closer to the set-up in [21, Section 2.2], but the distinction between all these cases is essentially superficial.

We begin by reviewing some definitions. For a graded graph G with vertex set V and graded components $V_0, V_1, V_2, \ldots$ a *down transition function* $p_\downarrow : V \times V \to [0, 1]$ is a function that satisfies the following criteria.

  - For $(\lambda, \mu) \in V \times V$, $p_\downarrow(\lambda, \mu) > 0$ if and only if $\lambda \in V_{n+1}$ and $\mu \in V_n$ for some $n \in \mathbb{Z}$ and λ and μ are connected by an edge.
  - For fixed $\lambda \in V_{n+1}$,

$$\sum_{\mu \in V_n} p^\downarrow(\lambda, \mu) = 1.$$

Let $\{M_n\}_{n \geqslant 0}$ be a collection of probability measures such that M_n is a probability measure on V_n. The measures $\{M_n\}_{n \geqslant 0}$ are said to be *coherent* with respect to down transition function $p_\downarrow$ if for any $\mu \in V_n$,

$$\sum_{\lambda \in V_{n+1}} M_{n+1}(\lambda) p_\downarrow(\lambda, \mu) = M_n(\mu).$$

In this paper we assume that for all $n \geqslant 0$ and $\mu \in V_n$, $M_n(\mu) > 0$. When this assumption is satisfied, then down transition function $p_\downarrow : V \times V \to [0, 1]$ and coherent measures $\{M_n\}_{n \geqslant 0}$ together define an *up-transition function*

$p_\uparrow : V \times V \to [0,1]$ such that when $\mu \in V_n$ and $\lambda \in V_{n+1}$,

$$p_\uparrow(\mu, \lambda) = \frac{M_{n+1}(\lambda)}{M_n(\mu)} p_\downarrow(\lambda, \mu)$$

and otherwise

$$p_\uparrow(\mu, \lambda) = 0.$$

By construction the up-transition function $p_\uparrow$ satisfies

$$\sum_{\lambda \in V_{n+1}} M_n(\mu) p_\uparrow(\mu, \lambda) = M_{n+1}(\lambda).$$

The triple $(p_\downarrow, \{M_n\}_{n \geqslant 0}, p_\uparrow)$ is called a *coherent system*.

Recall that $\{A_n\}_{n \geqslant 0}$ is a free Frobenius tower and the simple representations of A_n are indexed by the set Γ_n. We introduce the probability measure $Pl_n : \Gamma_n \to [0,1]$ defined such that for $\mu \in \Gamma_n$,

$$Pl_n(\mu) = \frac{\dim(L^\mu)\dim(P^\mu)}{\dim(A_n)}.$$

That this is a probability measure follows from (17). The measures $\{Pl_n\}_{n \geqslant 0}$ are one generalization of Plancherel measure to the non-semisimple setting.

For $\mu \in \Gamma_n$ and $\eta \in \Gamma_{n-1}$, set

$$p_\downarrow(\mu, \eta) = \frac{\kappa(\eta, \mu)\dim(L^\eta)}{\dim(L^\mu)}$$

and

$$p_\downarrow^*(\mu, \eta) = \frac{\kappa^*(\eta, \mu)\dim(P^\eta)}{\dim(P^\mu)}.$$

Proposition 4.2. The measures $\{Pl_n\}_{n \geqslant 0}$ are coherent with respect to $p_\downarrow$ on G_Γ and $p_\downarrow^*$ on G_{Γ^*}.

Proof. For $\mu \in \Gamma_n$ we first observe that since A_{n+1} is assumed to be a free right A_n-module, as a right A_n-module A_{n+1} must have rank $\dim(A_{n+1})/\dim(A_n)$ and

$$\dim\left(\mathrm{Ind}_n^{n+1} P^\mu\right) = \frac{\dim(A_{n+1})\dim(P^\mu)}{\dim(A_n)}. \tag{23}$$

On the other hand using the notation of (13),

$$\dim\left(\mathrm{Ind}_n^{n+1} P^\mu\right) = \sum_{\lambda \in \Gamma_{n+1}} \kappa(\mu, \lambda)\dim(P^\lambda). \tag{24}$$

Then using the equivalence between (23) and (24) we have

$$\sum_{\lambda \in \Gamma_{n+1}} M_{n+1}(\lambda) p_\downarrow(\lambda, \mu) = \sum_{\lambda \in \Gamma_{n+1}} \frac{\kappa(\mu, \lambda)\dim(L^\mu)\dim(P^\lambda)}{\dim(A_{n+1})}$$

$$= \frac{\dim(P^\mu)\dim(L^\mu)}{\dim(A_n)} = M_n(\mu).$$

The proof that $\{Pl_n\}_{n \geqslant 0}$ is coherent with respect to $p_\downarrow^*$ is analogous. $\square$

Observe that it follows by definition that the up-transition function associated to $(p_\downarrow, \{Pl_n\}_{n \geqslant 0})$ is defined so that for $\mu \in \Gamma_n$ and $\lambda \in \Gamma_{n+1}$,

$$p_\uparrow(\mu, \lambda) = \frac{\dim(A_n)}{\dim(A_{n+1})} \frac{\kappa(\mu, \lambda) \dim(P^\lambda)}{\dim(P^\mu)}$$

while the up-transition function associated to $(p_\downarrow^*, \{Pl_n\}_{n \geqslant 0})$ is

$$p_\uparrow^*(\mu, \lambda) = \frac{\dim(A_n)}{\dim(A_{n+1})} \frac{\kappa^*(\mu, \lambda) \dim(L^\lambda)}{\dim(L^\mu)}.$$

Remark 4.3. Observe that in the coherent system $(p_\downarrow, \{Pl_n\}_{n \geqslant 0}, p_\uparrow)$ on G_Γ, the dimension of simple representations control the down transition probabilities while the dimension of indecomposable projective representations control the up probabilities. For the coherent system $(p_\downarrow^*, \{Pl_n\}_{n \geqslant 0}, p_\uparrow^*)$ on $G_{\Gamma*}$ the reverse is true. When all $\{A_n\}_{n \geqslant 0}$ are semisimple then the distinction between simple A_n-modules and indecomposable projective A_n-modules disappears and hence $(p_\downarrow, \{Pl_n\}_{n \geqslant 0}, p_\uparrow)$ and $(p_\downarrow^*, \{Pl_n\}_{n \geqslant 0}, p_\uparrow^*)$ (as well as the underlying graphs G_Γ and $G_{\Gamma*}$) are identical.

Because of assumptions (a)-(b), the elements $\{x_n\}_{n \geqslant 0}$ define coordinates for elements of Γ. Specifically, to $\mu \in \Gamma_n$ we associate a finite collection of elements from $\mathbb{C}$,

$$\{\alpha_\eta^\mu \mid \eta \in \Gamma_{n-1}, \kappa(\eta, \mu) > 0\} \bigsqcup \{\alpha_\mu^\lambda \mid \lambda \in \Gamma_{n+1}, \kappa(\mu, \lambda) > 0\}.$$

We get another set of coordinates for μ by substituting $\kappa^*(\cdot, \cdot)$ for $\kappa(\cdot, \cdot)$ above. In specific examples it is often the case that the sets above take entries in $\mathbb{Z}$. This is true for Examples 3.3.1-3.3.2 and 3.3.4. When it is the case that the coordinates for Γ take values in $\mathbb{R}$ then to each $\mu \in \Gamma$ we can associate two probability measures $\nu_{\uparrow,\mu}$ and $\nu_{\downarrow,\mu}$ on $\mathbb{R}$:

- The transition measure is

$$\nu_{\uparrow,\mu} = \sum_{\lambda \in \Gamma_{n+1}} p_\uparrow(\mu, \lambda) \delta_{\alpha_\mu^\lambda}.$$

- The co-transition measure is

$$\nu_{\downarrow,\mu} = \sum_{\eta \in \Gamma_{n-1}} p_\downarrow(\mu, \eta) \delta_{\alpha_\eta^\mu}.$$

Where for $a \in \mathbb{R}$, δ_a is the Dirac delta measure on $\mathbb{R}$. We can analogously define $\nu_{\uparrow,\mu}^*$ and $\nu_{\downarrow,\mu}^*$ associated to $p_\uparrow^*(\mu, \lambda)$ and $p_\downarrow^*(\mu, \eta)$ respectively.

For $k \geqslant 0$, we can take the kth moments with respect to $\nu_{\uparrow,\mu}$, $\nu_{\downarrow,\mu}$, $\nu_{\uparrow,\mu}^*$, and $\nu_{\downarrow,\mu}^*$. We denote these as $m_{\uparrow,k}(\mu)$, $m_{\downarrow,k}(\mu)$, $m_{\uparrow,k}^*(\mu)$, and $m_{\downarrow,k}^*(\mu)$ respectively. They will be:

$$m_{\uparrow,k}(\mu) := \sum_{\lambda \in \Gamma_{n+1}} p_\uparrow(\mu, \lambda)(\alpha_\mu^\lambda)^k, \tag{25}$$

$$m_{\downarrow,k}(\mu) := \sum_{\eta \in \Gamma_{n-1}} p_\downarrow(\mu, \eta)(\alpha_\eta^\mu)^k, \tag{26}$$

$$m^*_{\uparrow,k}(\mu) := \sum_{\lambda \in \Gamma_{n+1}} p^*_{\uparrow}(\mu,\lambda)(\alpha^{\lambda}_{\mu})^k, \qquad\qquad (27)$$

$$m^*_{\downarrow,k}(\mu) := \sum_{\eta \in \Gamma_{n-1}} p^*_{\downarrow}(\mu,\eta)(\alpha^{\mu}_{\eta})^k. \qquad\qquad (28)$$

As suggested by the notation, we may regard $m_{\uparrow,k}$ as a function from $\Gamma \to \mathbb{R}$ via the association:

$$\mu \xmapsto{\;m_{\uparrow,k}\;} m_{\uparrow,k}(\mu).$$

We can view $m_{\downarrow,k}$, $m^*_{\uparrow,k}$, and $m^*_{\downarrow,k}$ analogously.

One may ask which pair of coherent systems $(p_{\downarrow}, \{Pl_n\}_{n\geqslant 0}, p_{\uparrow})$ or $(p^*_{\downarrow}, \{Pl_n\}_{n\geqslant 0}, p^*_{\uparrow})$ is more natural from an algebraic perspective. The next theorem shows that at least the moment functions $m^*_{\uparrow,k}$, $m^*_{\downarrow,k}$, associated with coherent system $(p^*_{\downarrow}, \{Pl_n\}_{n\geqslant 0}, p^*_{\uparrow})$ are captured by elements in $\{Z(A_n)\}_{n\geqslant 0}$.

Theorem 4.4. Let $\{A_n\}_{n\geqslant 0}$ be a free Frobenius tower and assume that $E_{n,0}$ is symmetric for all $n \geqslant 0$. Let $\{x_n\}_{n\geqslant 0}$ be elements satisfying assumptions (a) and (b), and let $\mu \in \Gamma$ with e_{μ} an idempotent corresponding to P^{μ}. Then

(1) $E_{n,0}\Big(\widetilde{e}_{\mu}E_{n+1,n}(C_{n+1,0}x^k_{n+1})\Big) = m^*_{\uparrow,k}(\mu).$

(2) $E_{n,0}\Big(\widetilde{e}_{\mu,i}N^n_0(x^k_n)\Big) = \dfrac{\dim(A_n)}{\dim(A_{n-1})}m^*_{\downarrow,k}(\mu).$

Proof. This follows from Theorem 3.8 and definitions (27)-(28).   □

Specific examples of the functions $m^*_{\uparrow,k}$ and $m^*_{\downarrow,k}$ have appeared before in the asymptotic representation theory literature. To our knowledge however, this is the first time that they have been written down for a general Frobenius tower, especially when the tower is not assumed to be semisimple. In [13] where $A_n = \mathbb{C}[S_n]$ and Γ is the set of all Young diagrams, $\{m^*_{k,\uparrow}(\mu)\}_{k\geqslant 0}$, $\{m^*_{k,\downarrow}(\mu)\}_{k\geqslant 0}$ were studied in relation to asymptotic properties of the Plancherel measure on symmetric groups. In this setting the coordinates of μ are Kerov's famous interlacing coordinates and Theorem 4.4.1 is an observation by Biane in [2]. Lassalle proved that $\{m^*_{k,\downarrow}\}_{k\geqslant 0}$ (properly scaled) and $\{m^*_{k,\uparrow}\}_{k\geqslant 0}$ are each sets of generators of the shifted symmetric functions [18, Theorem 6.4].

A similar story was discovered for the tower $\{\mathbb{S}_n\}_{n\geqslant 0}$ in [21], where Petrov's $\mathbf{g}^{\uparrow}_k$ and $\mathbf{g}^{\downarrow}_{k+1}$ are identical to our $m^*_{2k,\uparrow}$ and $m^*_{2k,\downarrow}$. It was also shown in [21] that in this case $\{m^*_{2k,\uparrow}\}_{k\geqslant 0}$ and $\{m^*_{2k,\downarrow}\}_{k\geqslant 0}$ are each sets of generators for the subalgebra of the symmetric functions generated by odd power sum functions. The connection between $\{Z(\mathbb{S}_n)\}_{n\geqslant 0}$ and $\{m^*_{2k,\uparrow}\}_{k\geqslant 0}$, $\{m^*_{2k,\downarrow}\}_{k\geqslant 0}$ was observed in [17, Theorem 2.17].

More recently, the functions $\{m^*_{k,\uparrow}\}_{k\geqslant 0}$ and $\{m^*_{k,\downarrow}\}_{k\geqslant 0}$ have appeared in the context of categorical representation theory. In [16], the versions associated to the tower $\{\mathbb{C}[S_n]\}_{n\geqslant 0}$ (i.e. moments of Kerov's transition and co-transition measure) were shown to arise naturally as elements of the center of Khovanov's Heisenberg category [14, Theorem 5.5]. In [17, Theorem

5.5], $m_{2k,\downarrow}^*$ and $m_{2k,\uparrow}^*$ corresponding to $\{\mathbb{S}_n\}_{n\geqslant 0}$ appeared as elements of the center of the twisted Heisenberg category of Cautis and Sussan [6].

4.1. Future directions.
Below we suggest a number of directions for future research.

(1) **New families of limit shapes:** The framework described above should make it possible to define new families of limit shapes for certain free Frobenius towers. For example, the simple representations of H_n^λ are parametrized by proper subsets of multipartitions, which "grow" from specific integer locations on $\mathbb{Z}$ determined by λ. This growth process should include structure not seen in the classical case of symmetric groups.

(2) **Deeper connections to Heisenberg categories:** As was noted above, the functions $\{m_{k,\uparrow}^*\}_{k\geqslant 0}$ and $\{m_{k,\downarrow}^*\}_{k\geqslant 0}$ appear at least twice in categorifications of Heisenberg algebras. This observation is interesting, but what would be more interesting is to understand how far this connection extends and whether Heisenberg categories can shed any new light on problems in asymptotic representation theory. In particular, it seems possible that the diagrammatics of Heisenberg categories might offer a new avenue for studying questions in asymptotic representation theory.

(3) **Connections to symmetric functions:** In two of the examples that have been extensively studied ($\{\mathbb{C}[S_n]\}_{n\geqslant 0}$ and $\{\mathbb{S}_n\}_{n\geqslant 0}$), the corresponding functions $\{m_{k,\uparrow}^*\}_{k\geqslant 0}$ and $\{m_{k,\downarrow}^*\}_{k\geqslant 0}$ can be identified with certain sets of generators for algebras related to symmetric functions. There is evidence that similar results may hold more broadly (for example, when $\{A_n\}_{n\geqslant 0}$ is a tower of degenerate cyclotomic Hecke algebras). It would be interesting to better understand the properties (especially combinatorial) of the commutative algebras generated by $\{m_{k,\uparrow}^*\}_{k\geqslant 0}$ and $\{m_{k,\downarrow}^*\}_{k\geqslant 0}$ respectively.

References

1. Allen D. Bell and Rolf Farnsteiner, *On the theory of Frobenius extensions and its application to Lie superalgebras*, Trans. Amer. Math. Soc. **335** (1993), no. 1, 407–424.

2. Philippe Biane, *Representations of symmetric groups and free probability*, Adv. Math. **138** (1998), no. 1, 126–181.

3. Alexei Borodin, *Multiplicative central measures on the Schur graph*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **240** (1997), no. Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 2, 44–52, 290–291.

4. Alexei Borodin and Grigori Olshanski, *Infinite-dimensional diffusions as limits of random walks on partitions*, Probab. Theory Related Fields **144** (2009), no. 1-2, 281–318.

5. Michel Broué, *Higman's criterion revisited*, Michigan Math. J. **58** (2009), no. 1, 125–179.

6. Sabin Cautis, Anthony Licata, and Joshua Sussan, *Braid group actions via categorified Heisenberg complexes*, Compos. Math. **150** (2014), no. 1, 105–142.

7. Sabin Cautis and Joshua Sussan, *On a categorical Boson-Fermion correspondence*, Comm. Math. Phys. **336** (2015), no. 2, 649–669.

8. Sergey Fomin, *Duality of graded graphs*, J. Algebraic Combin. **3** (1994), no. 4, 357–404.

9. Vladimir Ivanov, *The Gaussian limit for projective characters of large symmetric groups*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **283** (2001), no. Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 6, 73–97, 259.

10. _____, *Plancherel measure on shifted Young diagrams*, Representation theory, dynamical systems, and asymptotic combinatorics, Amer. Math. Soc. Transl. Ser. 2, vol. 217, Amer. Math. Soc., Providence, RI, 2006, pp. 73–86.

11. Lenny K. Jones, *Centers of generic Hecke algebras*, Trans. Amer. Math. Soc. **317** (1990), no. 1, 361–392.

12. Lars Kadison, *New examples of Frobenius extensions*, University Lecture Series, vol. 14, American Mathematical Society, Providence, RI, 1999.

13. Sergei Kerov, *Transition probabilities of continual Young diagrams and the Markov moment problem*, Funktsional. Anal. i Prilozhen. **27** (1993), no. 2, 32–49, 96.

14. Mikhail Khovanov, *Heisenberg algebra and a graphical calculus*, Fund. Math. **225** (2014), no. 1, 169–210.

15. Alexander Kleshchev, *Linear and projective representations of symmetric groups*, Cambridge Tracts in Mathematics, vol. 163, Cambridge University Press, Cambridge, 2005.

16. Henry Kvinge, Anthony M. Licata, and Stuart Mitchell, *Khovanov's Heisenberg category, moments in free probability, and shifted symmetric functions*, to appear in Algebraic Combinatorics, (arXiv:1610.04571) (2018).

17. Henry Kvinge, Can Ozan Oğuz, and Michael Reeks, *The center of the twisted Heisenberg category, factorial Schur Q-functions, and transition functions on the Schur graph*, To appear in the Journal of Algebraic Combinatorics, (arXiv:1712.09626) (2017).

18. Michel Lassalle, *Jack polynomials and free cumulants*, Adv. Math. **222** (2009), no. 6, 2227–2269.

19. Ian G. Macdonald, *Symmetric functions and Hall polynomials*, second ed., Oxford Classic Texts in the Physical Sciences, The Clarendon Press, Oxford University Press, New York, 2015.

20. Marco Mackaay and Alistair Savage, *Degenerate cyclotomic hecke algebras and higher level heisenberg categorification*, Journal of Algebra **505** (2018), 150 – 193.

21. Leonid Petrov, *Random walks on strict partitions*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **373** (2009), no. Teoriya Predstavleniĭ, Dinamicheskie Sistemy, Kombinatornye Metody. XVII, 226–272, 351.

22. Daniele Rosso and Alistair Savage, *A general approach to Heisenberg categorification via wreath product algebras*, Math. Z. **286** (2017), no. 1-2, 603–655.

23. Alistair Savage, *Affine wreath product algebras*, International Mathematics Research Notices (2018), rny092.

24. Jinkui Wan and Weiqiang Wang, *Lectures on spin representation theory of symmetric groups*, Proceedings for Taipei winter school 2010, Bulletin of Institute of Mathematics Academia Sinica **7** (2012), 91–164.