Practice Exam 2

1. (a) Give an example of a subset A of $\mathbb{R}$ and a point x in A that is not a limit point of the set A.

   (b) Given a function $f : \mathbb{R}^n \to \mathbb{R}$ with first-order partial derivatives and $\mathbf{p} \in \mathbb{R}^n$ with $||\mathbf{p}|| = 1$, what is a formula for the directional derivative $\frac{\partial f}{\partial \mathbf{p}}(\mathbf{x})$ at point $\mathbf{x} \in \mathbb{R}^n$?

2. Prove that the function

$$g(x, y) = \begin{cases} \frac{x^2 y^4}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{otherwise} \end{cases}$$

   has first-order partial derivatives. Is g continuously differentiable?

3. Prove that

$$\lim_{(x,y)\to(0,0)} \frac{(1 + 2x + y^2)^{3/2} - 1 - 3x}{\sqrt{x^2 + y^2}} = 0.$$

   (Hint: you should be able to prove this without actually having to compute the limit.)

4. Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(x, y, z) = \cos(xy^2) + e^{z^2+x}.$$

   What is the direction of fastest increase of f at the point $(\pi/2, 1, 0)$? What is a direction (from $(x, y, z) = (\pi/2, 1, 0)$) in which f is not increasing?

5. Let A be a subset of $\mathbb{R}^n$ and let the point x_* in $\mathbb{R}^n$ be a limit point of A. Suppose that the function $g : A \to \mathbb{R}$ is bounded; that is, there is a number c such that

$$|g(\mathbf{x})| \leqslant c \qquad \text{for all } \mathbf{x} \in A.$$

   Prove that if $\lim_{\mathbf{x}\to x_*} f(\mathbf{x}) = 0$, then $\lim_{\mathbf{x}\to x_*}[g(\mathbf{x})f(\mathbf{x})] = 0$.