MATH 417 Homework 5

Note that this is problems: Section 14.2 #1,3,5,7,10.

1. Define

$$f(x,y) = e^{xy} + x^2 + 2xy \qquad \text{for } (x,y) \in \mathbb{R}^2$$

   (a) Define $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(t) = f(2t, 3t)$ for $t \in \mathbb{R}$. Calculate $\phi''(0)$ directly.

   (b) Find the Hessian matrix of the function $f : \mathbb{R}^2 \to \mathbb{R}$ at the point $(0,0)$ and use formula (14.11) to calculate

$$\phi''(0) = \frac{d^2}{dt^2}[f(2t,3t)]\Big|_{t=0}$$

Solution:

   (a) A quick calculation shows that

$$\phi(t) = e^{6t^2} + 16t^2.$$

   Hence $\phi''(t) = 12e^{6t^2} + 144t^2e^{6t^2} + 32$, and in particular $\phi''(0) = 44$.

   (b) We calculate

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y) &= ye^{xy} + 2x + 2y, \\
\frac{\partial f}{\partial y}(x,y) &= xe^{xy} + 2x, \\
\frac{\partial^2 f}{\partial x^2}(x,y) &= y^2e^{xy} + 2, \\
\frac{\partial^2 f}{\partial x \partial y}(x,y) &= e^{xy} + xye^{xy} + 2, \\
\frac{\partial^2 f}{\partial y^2}(x,y) &= x^2e^{xy},
\end{aligned}$$

   We observe that the value $\mathbf{h}$ associated to $\phi(t)$ is $\mathbf{h} = (2,3)$ while $\mathbf{x} = (0,0)$. Since we are interested in $t = 0$, we then should calculate:

$$\left\langle \begin{bmatrix} 2 & 3 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\rangle = 44,$$

   which agrees with what we have above.

3. Suppose that the function $f : \mathbb{R}^2 \to \mathbb{R}$ has continuous second-partial derivatives, and at the origin $(0,0)$ suppose that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

   Let $\mathbf{h}$ be a nonzero point in the plane $\mathbb{R}^2$ and suppose that

$$\langle \nabla^2 f(0,0)\mathbf{h}, \mathbf{h} \rangle > 0.$$

   Use the single-variable theory to prove that there is some positive number r such that

$$f(t\mathbf{h}) > f(0,0) \qquad \text{if } 0 < |t| < r.$$

Solution: Define $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(t) := f(t\mathbf{h}).$$

In particular, $\phi(0) = f(0,0)$. It follows from Theorem 14.12 that

$$\phi'(t) = \langle \nabla f(t\mathbf{h}), \mathbf{h} \rangle.$$

Since $\nabla f(0,0) = (0,0)$ then $\phi'(0) = 0$, so $t = 0$ is a critical point of ϕ. It also follows from Theorem 14.12 that

$$\phi''(t) = \langle \nabla^2 f(t\mathbf{h})\mathbf{h}, \mathbf{h} \rangle.$$

By assumption then $\phi''(0) > 0$. It follows from the second derivative test then that $t = 0$ is a local minimizer of ϕ. Thus there is some $r > 0$ such that for all $t \in B_r(0)$,

$$\phi(t) > \phi(0).$$

Translating this from ϕ to f then gives the result.

5. Let a, b, and c be real numbers with $a \neq 0$, and define $p(t) = at^2 + 2bt + c$.

   (a) Show that $p(t) > 0$ for every number t if and only if $a > 0$ and $ac - b^2 > 0$.

   (b) Show that $p(t) < 0$ for every number t if and only if $a < 0$ and $ac - b^2 > 0$.

   Solution: Set

   $$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

   and observe that if $\mathbf{x} = (t, 1)$ then $\langle A\mathbf{x}, \mathbf{x} \rangle = p(t)$. Hence $p(t) > 0$ if A is positive definite, which is equivalent to the conditions $a > 0$ and $ac - b^2 > 0$. This shows one direction. To show that $p(t) > 0$ implies that $a > 0$ and $ac - b^2 > 0$, suppose for a contradiction that either $a < 0$ (since we assume $a \neq 0$) or $ac - b^2 \leqslant 0$. If $a < 0$ then for sufficiently large t, $p(t) < 0$, a contradiction. If $ac - b^2 < 0$, then $p(t)$ has real roots so that there are $t \in \mathbb{R}$ such that $p(t) = 0$, a contradiction. Hence we must have $a > 0$ and $ac - b^2 > 0$.

   The second part is completely analogous.

7. For each of the following quadratic functions, find a 2×2 matrix with which it is associated.

   (a) $h(x, y) = x^2 - y^2$,

   (b) $g(x, y) = x^2 + 8xy + y^2$.

   Solution:

   (a) Choose
   $$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

   (b) Choose
   $$A = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$$

10. Define the function $Q : \mathbb{R} \to \mathbb{R}$ by $Q(x) = x^4$. Observe that

    $$Q(x) > 0 \quad \text{for all } x \neq 0.$$

    Show that there is no positive number c such that

    $$Q(x) \geqslant cx^2 \quad \text{for all } x \neq 0.$$

    Explain why this does not contradict Proposition 14.16.

    Solution: It is clear that if $x \neq 0$ then $x^4 > 0$, and thus the same is true for $Q(x)$. Suppose that there were such a number $c > 0$. Then we would have

    $$x^4 \geqslant cx^2$$

    for all $x \neq 0$. Hence

    $$x^2 \geqslant c > 0.$$

    But setting $x = \frac{\sqrt{c}}{2}$ then leads to a contradiction. This example does not contradict Proposition 14.16 since Q as defined above is not actually a quadratic form.