MATH 417 Homework 9

Note that this is problems: Section 15.2 #2,4,7

2. Define
$$F(x,y,z) = (xyz, x^2 + xz, 1 + 3x)$$
for all $(x,y,z) \in \mathbb{R}^3$. Find the derivative matrix of the mapping $F : \mathbb{R}^3 \to \mathbb{R}^3$ at the points $(1,2,3)$, $(0,1,0)$, and $(-1,4,0)$.

   Solution: The gradients of the component functions are
$$\nabla F_1(x,y,z) = (yz, xz, xy),$$
$$\nabla F_2(x,y,z) = (2x + z, 0, x),$$
$$\nabla F_3(x,y,z) = (3,0,0).$$
Thus, the derivative matrix at $(1,2,3)$ is
$$DF(1,2,3) = \begin{bmatrix} 6 & 3 & 2 \\ 5 & 0 & 1 \\ 3 & 0 & 0 \end{bmatrix},$$
the derivative matrix at $(1,2,3)$ is
$$DF(0,1,0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix},$$
and the derivative matrix at $(-1,4,0)$ is
$$DF(1,2,3) = \begin{bmatrix} 0 & 0 & -4 \\ -2 & 0 & -1 \\ 3 & 0 & 0 \end{bmatrix}.$$

4. Suppose that A is an $m \times n$ matrix. Define the mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ by
$$F(\mathbf{x}) = A\mathbf{x},$$
for $\mathbf{x} \in \mathbb{R}^n$. Prove that $DF(\mathbf{x}) = A$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

   Solution: Write
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}.$$
Then
$$F(\mathbf{x}) = F(x_1, x_2, \dots, x_n) = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n \end{bmatrix}.$$
Thus the ith component function is
$$F_i(x_1, \dots, x_n) = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \cdots + a_{in}x_n.$$
The gradient of this function is
$$\nabla F_i(x_1, \dots, x_n) = (a_{i1}, a_{i2}, a_{i3}, \dots, a_{in}).$$

Thus

$$DF(\mathbf{x}) = \begin{bmatrix} \nabla F_1(x_1, \ldots, x_n) \\ \nabla F_2(x_1, \ldots, x_n) \\ \vdots \\ \nabla F_m(x_1, \ldots, x_n) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{bmatrix}.$$

7. Give a proof of the First-Order Approximation Theorem based on the Mean Value Theorem.

Solution: We assume that $\mathcal{O} \subseteq \mathbb{R}^n$ is open and that $F : \mathcal{O} \to \mathbb{R}^m$ is continuously differentiable. Let $\mathbf{x} \in \mathcal{O}$. Since $\mathcal{O}$ is open, we can find $r > 0$ such that $B_r(\mathbf{x}) \subseteq \mathcal{O}$. We consider the limit

$$\lim_{\mathbf{h} \to 0} \frac{||F(\mathbf{x} + \mathbf{h}) - [F(\mathbf{x}) + DF(\mathbf{x})\mathbf{h}]||}{||\mathbf{h}||}. \tag{1}$$

For $\mathbf{h}$ sufficiently close to 0, $\mathbf{x} + \mathbf{h} \subseteq B_r(\mathbf{x})$ and hence the line segment between $\mathbf{x}$ and $\mathbf{x} + \mathbf{h}$ is also in $B_r(\mathbf{x})$ since this set is convex. Thus for each such sufficiently small $\mathbf{h}$ we can write the limit (1) as

$$\lim_{\mathbf{h} \to 0} \frac{||A\mathbf{h} - DF(\mathbf{x})\mathbf{h}||}{||\mathbf{h}||},$$

where the ith column A is equal to $\nabla F_i(\mathbf{x} + \theta_i \mathbf{h})$, and each $\theta_i \in (0, 1)$. By the generalized Cauchy-Schwarz inequality we have

$$\lim_{\mathbf{h} \to 0} \frac{||A\mathbf{h} - DF(\mathbf{x})\mathbf{h}||}{||\mathbf{h}||} = \lim_{\mathbf{h} \to 0} \frac{||A - DF(\mathbf{x})||||\mathbf{h}||}{||\mathbf{h}||} = \lim_{\mathbf{h} \to 0} ||A - DF(\mathbf{x})||.$$

Because each ∇F_i is assumed to be continuous then as $\mathbf{h} \to 0$, $\nabla F_i(\mathbf{x} + \theta_i \mathbf{h}) \to \nabla F_i(\mathbf{x})$, and consequently $A \to DF(\mathbf{x})$, so

$$\lim_{\mathbf{h} \to 0} ||A - DF(\mathbf{x})|| = 0.$$