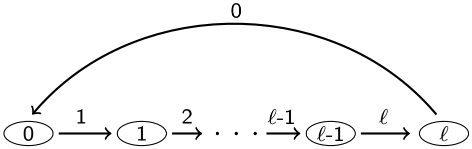
The Kirillov-Reshetikhin crystal $B^{1,1}$ and cyclotomic quiver Hecke algebras

Henry Kvinge, CSU

(Joint with Monica Vazirani)

CU algebraic Lie theory seminar



Reminder on crystals

Let $U_q(\mathfrak{g})$ be the quantum group associated to Kac-Moody algebra $\mathfrak{g}$ with Dynkin indexing set I.

A *crystal* is a combinatorial object that we can attach to certain $U_q(\mathfrak{g})$ representations V:

Reminder on crystals

Let $U_q(\mathfrak{g})$ be the quantum group associated to Kac-Moody algebra $\mathfrak{g}$ with Dynkin indexing set I.

A *crystal* is a combinatorial object that we can attach to certain $U_q(\mathfrak{g})$ representations V:

- Roughly corresponds to setting $q = 0$.

Reminder on crystals

Let $U_q(\mathfrak{g})$ be the quantum group associated to Kac-Moody algebra $\mathfrak{g}$ with Dynkin indexing set I.

A *crystal* is a combinatorial object that we can attach to certain $U_q(\mathfrak{g})$ representations V:

- Roughly corresponds to setting $q = 0$.
- For today, can think of crystals are I-colored, directed graphs.

Reminder on crystals

Let $U_q(\mathfrak{g})$ be the quantum group associated to Kac-Moody algebra $\mathfrak{g}$ with Dynkin indexing set I.

A *crystal* is a combinatorial object that we can attach to certain $U_q(\mathfrak{g})$ representations V:

- Roughly corresponds to setting $q = 0$.
- For today, can think of crystals are I-colored, directed graphs.

vertices	$\sim$	weight spaces of V
i-directed edges	$\sim$	action of $\widetilde{f}_i$ between weight spaces

Example

Representations of $U_q(\mathfrak{sl}_2)$ already look like directed graphs,

$$V(2),\ \dim(V(2)) = 3$$

$$\bullet \xrightarrow{f} \bullet \xrightarrow{f} \bullet$$

$$V(3),\ \dim(V(3)) = 4$$

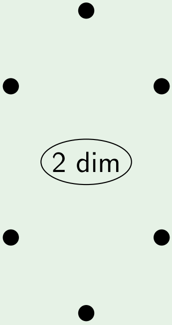
$$\bullet \xrightarrow{f} \bullet \xrightarrow{f} \bullet \xrightarrow{f} \bullet$$

$$V(4),\ \dim(V(4)) = 5$$

$$\bullet \xrightarrow{f} \bullet \xrightarrow{f} \bullet \xrightarrow{f} \bullet \xrightarrow{f} \bullet$$

Example

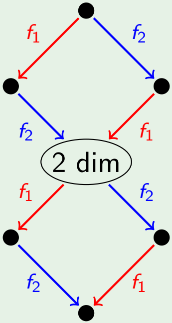
Adjoint representation V for $\mathfrak{sl}_3$:



- V has six 1-dimensional weight spaces, one 2-dimensional weight space.

Example

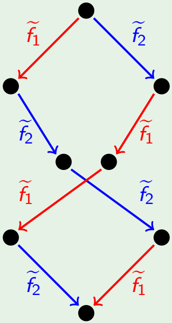
Adjoint representation V for $\mathfrak{sl}_3$:



- V has six 1-dimensional weight spaces, one 2-dimensional weight space.
- f_1 and f_2 map between weight spaces.

Example

Adjoint representation V for $\mathfrak{sl}_3$:



- V has six 1-dimensional weight spaces, one 2-dimensional weight space.
- f_1 and f_2 map between weight spaces.
- If we use $U_q(\mathfrak{sl}_3)$ and "rescale" operators f_i to $\widetilde{f}_i$, then "at $q = 0$" can find basis so that representation behaves like $\{1, 2\}$-colored graph.

Tensor Product of Crystals

Example

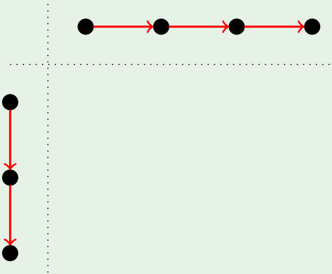
Crystals $B(2)$ and $B(3)$ associated to $\mathfrak{sl}_2$ representations $V(2)$ and $V(3)$

$$\underbrace{\bullet \to \bullet \to \bullet}_{B(2)} \qquad \text{and} \qquad \underbrace{\bullet \to \bullet \to \bullet \to \bullet}_{B(3)}$$

Tensor Product of Crystals

Example

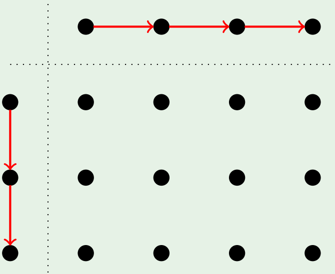
Then crystal $B(2) \otimes B(3)$ associated to $V(2) \otimes V(3)$ is



Tensor Product of Crystals

Example

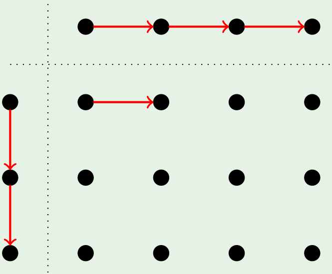
Then crystal $B(2) \otimes B(3)$ associated to $V(2) \otimes V(3)$ is



Tensor Product of Crystals

Example

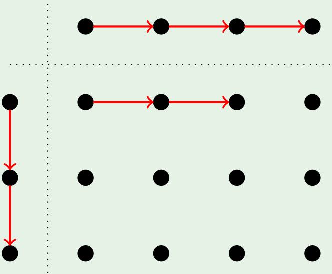
Then crystal $B(2) \otimes B(3)$ associated to $V(2) \otimes V(3)$ is



Tensor Product of Crystals

Example

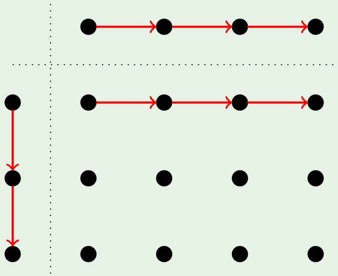
Then crystal $B(2) \otimes B(3)$ associated to $V(2) \otimes V(3)$ is



Tensor Product of Crystals

Example

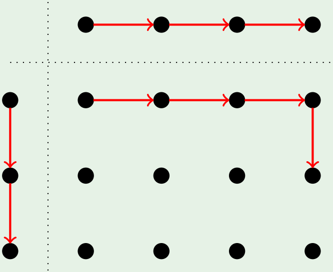
Then crystal $B(2) \otimes B(3)$ associated to $V(2) \otimes V(3)$ is



Tensor Product of Crystals

Example

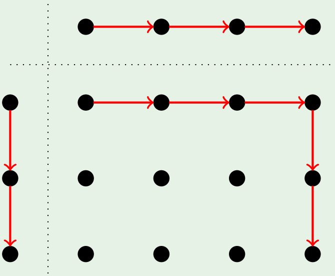
Then crystal $B(2) \otimes B(3)$ associated to $V(2) \otimes V(3)$ is



Tensor Product of Crystals

Example

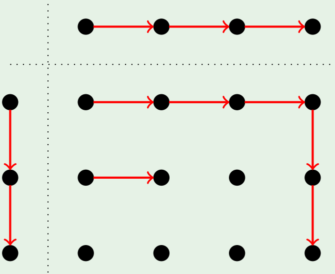
Then crystal $B(2) \otimes B(3)$ associated to $V(2) \otimes V(3)$ is



Tensor Product of Crystals

Example

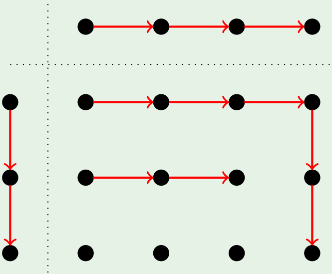
Then crystal $B(2) \otimes B(3)$ associated to $V(2) \otimes V(3)$ is



Tensor Product of Crystals

Example

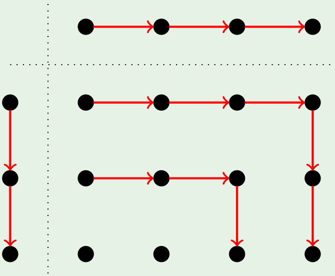
Then crystal $B(2) \otimes B(3)$ associated to $V(2) \otimes V(3)$ is



Tensor Product of Crystals

Example

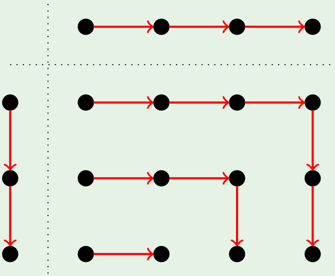
Then crystal $B(2) \otimes B(3)$ associated to $V(2) \otimes V(3)$ is



Tensor Product of Crystals

Example

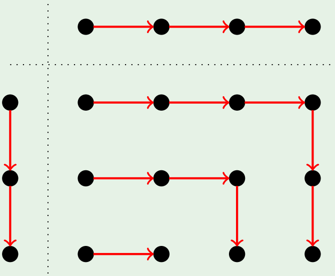
Then crystal $B(2) \otimes B(3)$ associated to $V(2) \otimes V(3)$ is



Tensor Product of Crystals

Example

Then crystal $B(2) \otimes B(3)$ associated to $V(2) \otimes V(3)$ is

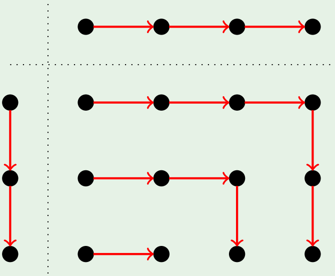


$$B(2) \otimes B(3) \cong B(1) \oplus B(3) \oplus B(5)$$

Tensor Product of Crystals

Example

Then crystal $B(2) \otimes B(3)$ associated to $V(2) \otimes V(3)$ is



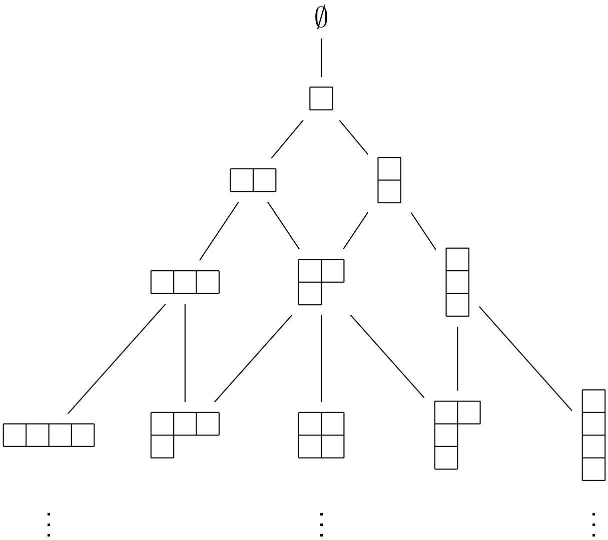
$$B(2) \otimes B(3) \cong B(1) \oplus B(3) \oplus B(5)$$

and

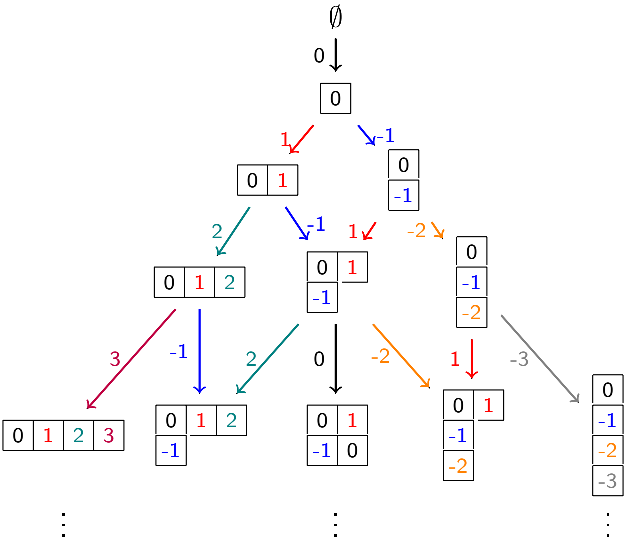
$$V(2) \otimes V(3) \cong V(1) \oplus V(3) \oplus V(5)$$

There are many useful combinatorial models for crystals...

Young's lattice of partitions



Young's lattice as a directed graph



Rich in connections to representation theory

Young's lattice as a directed graph:

Gives branching rule for symmetric groups:

- partitions in row n are simple $\mathcal{S}_n$ representations
- directed, i-colored edges correspond to induction (and restriction) functors

Rich in connections to representation theory

Young's lattice as a directed graph:

Gives branching rule for symmetric groups:

- partitions in row n are simple $\mathcal{S}_n$ representations
- directed, i-colored edges correspond to induction (and restriction) functors

Model for crystal $B(\Lambda_0)$ in type $\mathfrak{sl}_\infty$:

- partitions are nodes
- i-arrows correspond to $\widetilde{f}_i$ ($i \in I = \mathbb{Z}$)

Rich in connections to representation theory

Young's lattice as a directed graph:

Gives branching rule for symmetric groups:

- partitions in row n are simple $\mathcal{S}_n$ representations
- directed, i-colored edges correspond to induction (and restriction) functors

Model for crystal $B(\Lambda_0)$ in type $\mathfrak{sl}_\infty$:

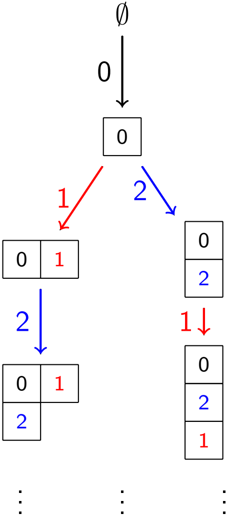
- partitions are nodes
- i-arrows correspond to $\widetilde{f}_i$ $(i \in I = \mathbb{Z})$

Can we say something similar for $\widehat{\mathfrak{sl}_{\ell+1}}$ (type $A_\ell^{(1)}$)?

Example: $B(\Lambda_0)$ in type $A_2^{(1)}$ ($\widehat{\mathfrak{sl}_3}$)

Similar model but...

- Nodes are now 3-restricted partitions
- Gives partial branching for:
  - Symmetric group algebras $\mathbb{F}_3\mathcal{S}_n$,
  - Cyclotomic Hecke algebras $H_n^{\Lambda_0}$,
  - Type $A_2^{(1)}$ cyclotomic KLR algebra R^{Λ_0}.



$B(\Lambda_0)$ for type $A_\ell^{(1)}$

Model of $B(\Lambda_0)$ has

- Nodes are now $(\ell+1)$-restricted partitions
- Gives partial branching for:

–Symmetric group algebras $\mathbb{F}_{\ell+1}\mathcal{S}_n$, (when $\ell+1$ prime)

–Cyclotomic Hecke algebras $H_n^{\Lambda_0}$, (with q an $\ell+1$ root of unity)

–Type $A_\ell^{(1)}$ cyclotomic KLR algebra R^{Λ_0}.

I will work today in language of cyclotomic Khovanov-Lauda-Rouquier (KLR) algebras (or cyclotomic quiver Hecke algebras).

...But results hold for both $H_n^{\Lambda_i}$ and $\mathbb{F}_{\ell+1}\mathcal{S}_n$.

Brief review of KLR algebras

Khovanov-Lauda and independently Rouquier invented associative, graded algebra R attached to any symmetrizable Cartan matrix A.

- R categorifies lower part of quantum group $U_q(\mathfrak{g})$.
- Twisted $\mathbb{Q}(q)$-bialgebra isomorphism,

$$U_q^-(\mathfrak{g}) \xrightarrow{\cong} \mathbb{Q}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} K_0(R\text{-}\mathrm{pmod})$$

  where R-pmod is category of finitely-generated, graded, projective R-modules.

Brief review of KLR algebras

For the remainder of talk:

- $A = [a_{ij}]$, Cartan matrix for classical affine type X_ℓ,

Brief review of KLR algebras

For the remainder of talk:

- $A = [a_{ij}]$, Cartan matrix for classical affine type X_ℓ,
- $I = \{0, 1, \ldots, \ell\}$ is Dynkin indexing set for A,

Brief review of KLR algebras

For the remainder of talk:

- $A = [a_{ij}]$, Cartan matrix for classical affine type X_ℓ,
- $I = \{0, 1, \ldots, \ell\}$ is Dynkin indexing set for A,
- $\{\alpha_i\}_{i \in I}$ are simple roots and positive root lattice is

$$Q^+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i,$$

Brief review of KLR algebras

- for $\nu \in Q^+$, $\nu = \sum_{i \in I} c_i \alpha_i$, set

$$\mathrm{ht}(\nu) = \sum_{i \in I} c_i, \tag{1}$$

Brief review of KLR algebras

- for $\nu \in Q^+$, $\nu = \sum_{i \in I} c_i \alpha_i$, set

$$\mathrm{ht}(\nu) = \sum_{i \in I} c_i, \tag{1}$$

- the set $\mathrm{Seq}(\nu)$ contains all ordered sequences of elements of I such that i appears c_i times.

Example

For $1, 2 \in I$, $\mathrm{ht}(\alpha_1 + 2\alpha_2) = 3$, and

$$\mathrm{Seq}(\alpha_1 + 2\alpha_2) = \{(122), (212), (221)\}.$$

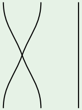
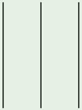
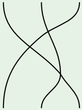
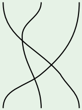
Brief review of KLR algebras

For $\nu \in Q^+$, algebra $R(\nu)$ can be presented by $\mathbb{C}$-linear combinations of braid-like planar diagrams with interacting strings:

,

Example

Some elements in $R(\alpha_1 + 2\alpha_2)$ are:

, 4 ,  $-$ 

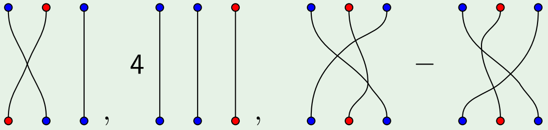
Brief review of KLR algebras

For $\nu \in Q^+$, algebra $R(\nu)$ can be presented by $\mathbb{C}$-linear combinations of braid-like planar diagrams with interacting strings:

- Between $\mathrm{ht}(\nu)$ points on top and $\mathrm{ht}(\nu)$ points on bottom,

Example

Some elements in $R(\alpha_1 + 2\alpha_2)$ are:



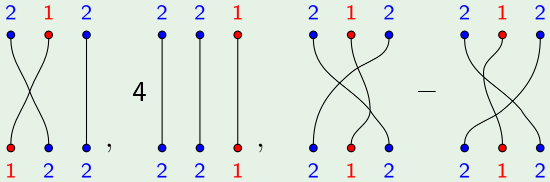
Brief review of KLR algebras

For $\nu \in Q^+$, algebra $R(\nu)$ can be presented by $\mathbb{C}$-linear combinations of braid-like planar diagrams with interacting strings:

- Between $\mathrm{ht}(\nu)$ points on top and $\mathrm{ht}(\nu)$ points on bottom, labelled by elements of $\mathrm{Seq}(\nu)$.

Example

Some elements in $R(\alpha_1 + 2\alpha_2)$ are:



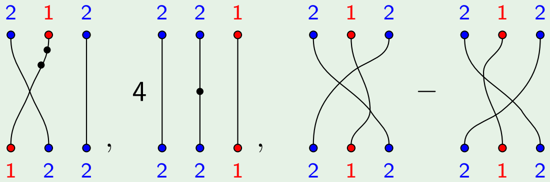
Brief review of KLR algebras

For $\nu \in Q^+$, algebra $R(\nu)$ can be presented by $\mathbb{C}$-linear combinations of braid-like planar diagrams with interacting strings:

- Between $\mathrm{ht}(\nu)$ points on top and $\mathrm{ht}(\nu)$ points on bottom, labelled by elements of $\mathrm{Seq}(\nu)$.
- Can add beads to strings.

Example

Some elements in $R(\alpha_1 + 2\alpha_2)$ are:



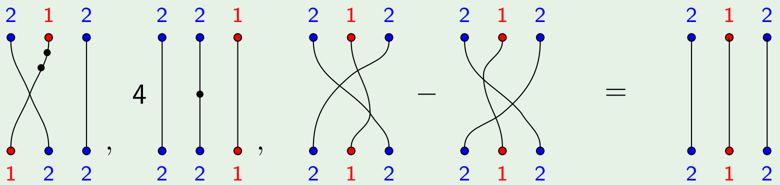
Brief review of KLR algebras

For $\nu \in Q^+$, algebra $R(\nu)$ can be presented by $\mathbb{C}$-linear combinations of braid-like planar diagrams with interacting strings:

- Between $\mathrm{ht}(\nu)$ points on top and $\mathrm{ht}(\nu)$ points on bottom, labelled by elements of $\mathrm{Seq}(\nu)$.
- Can add beads to strings.
- Modulo local relations.

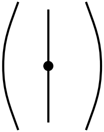
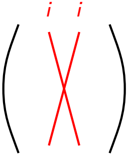
Example

Some elements in $R(\alpha_1 + 2\alpha_2)$ are:

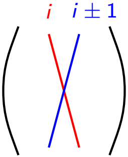


Brief review of KLR algebras

Grading is given by:

deg  $= 2$, deg 

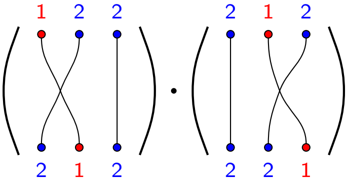
$= -2$,

deg 

$= 1$,

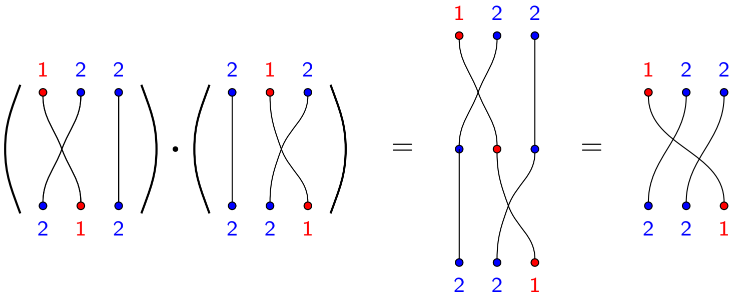
Brief review of KLR algebras

Multiplication is given my placing first diagram above second diagram,



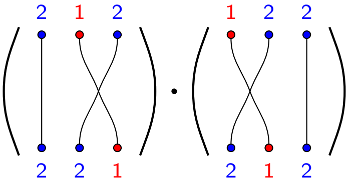
Brief review of KLR algebras

Multiplication is given my placing first diagram above second diagram,



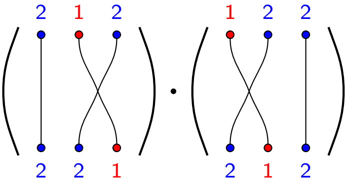
Brief review of KLR algebras

When labels of bottom of first diagram and top of second diagram do not agree, product is <u>zero</u>,



Brief review of KLR algebras

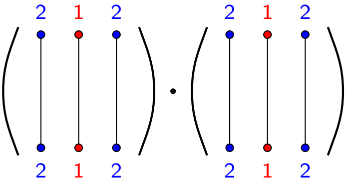
When labels of bottom of first diagram and top of second diagram do not agree, product is <u>zero</u>,



$$= \mathbf{0}$$

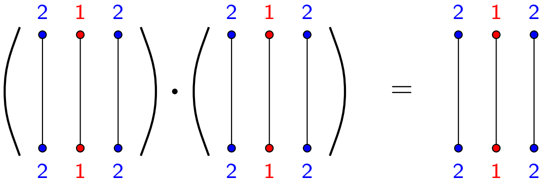
Brief review of KLR algebras

The elements of $R(\nu)$ in which no strings cross and which have no beads are idempotents



Brief review of KLR algebras

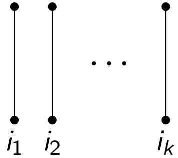
The elements of $R(\nu)$ in which no strings cross and which have no beads are idempotents



Brief review of KLR algebras

For $\underline{i} = (i_1, i_2, \ldots, i_k) \in \mathrm{Seq}(\nu)$ we write

$1_{\underline{i}} =$



Brief review of KLR algebras

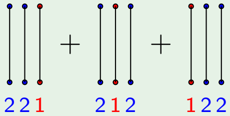
$R(\nu)$ has identity,

$$1 = \sum_{\underline{i} \in \mathrm{Seq}(\nu)} 1_{\underline{i}} \tag{2}$$

Example

For $\nu = \alpha_1 + 2\alpha_2$

$$1 = 1_{221} + 1_{212} + 1_{122} =$$



Brief review of KLR algebras

If M is an $R(\nu)$-module, we can decompose it into *weight spaces*,

$$M = \bigoplus_{\underline{i} \in \mathrm{Seq}(\nu)} 1_{\underline{i}} M$$

Brief review of KLR algebras

If M is an $R(\nu)$-module, we can decompose it into *weight spaces*,

$$M = \bigoplus_{\underline{i} \in \mathrm{Seq}(\nu)} 1_{\underline{i}} M$$

Because $R(\nu)$ is graded, we can take the *graded dimension* of each component, $\mathrm{gdim}(1_{\underline{i}} M) \in \mathbb{N}[q, q^{-1}]$. The *character* is defined as

$$\mathrm{Char}(M) = \sum_{\underline{i} \in \mathrm{Seq}(\nu)} \mathrm{gdim}(1_{\underline{i}} M)[\underline{i}].$$

Brief review of KLR algebras

Example

For type $A_2^{(1)}$, $1 \in I = \{0, 1, 2\}$, $R(2\alpha_1)$ has exactly one simple representation, $L(1^2)$,

$$\mathrm{Char}(L(1^2)) = (1 + q^{-2})[1\ 1] = q[2]_q![1\ 1].$$

Brief review of KLR algebras

Example

For type $A_2^{(1)}$, $1 \in I = \{0, 1, 2\}$, $R(2\alpha_1)$ has exactly one simple representation, $L(1^2)$,

$$\mathrm{Char}(L(1^2)) = (1 + q^{-2})[1\ 1] = q[2]_q![1\ 1].$$

$R(3\alpha_1)$ has exactly one simple representation, $L(1^3)$ with

$$\mathrm{Char}(L(1^3)) = (1 + q^{-2} + q^{-4})(1 + q^{-2})[1\ 1\ 1]$$

$$= q^3[3]_q![1\ 1\ 1]$$

Brief review of KLR algebras

Example

In type $A_2^{(1)}$, $\nu = \alpha_1 + 2\alpha_2$ there are 2 simple representations M_1 and M_2 with characters:

$$\mathrm{Char}(M_1) = (1 + q^{-2})[221] + [212]$$

$$\mathrm{Char}(M_2) = (1 + q^{-2})[122] + [212]$$

Brief review of KLR algebras

For each integral dominant weight Λ, R has a finite-dimensional quotient R^Λ, called the *cyclotomic KLR algebra*.

Brief review of KLR algebras

For each integral dominant weight Λ, R has a finite-dimensional quotient R^Λ, called the *cyclotomic KLR algebra*.

Two key points for this presentation:

- Simple R^Λ-modules carry structure of $B(\Lambda)$.
- Simple R-modules carry structure of $B(\infty)$.

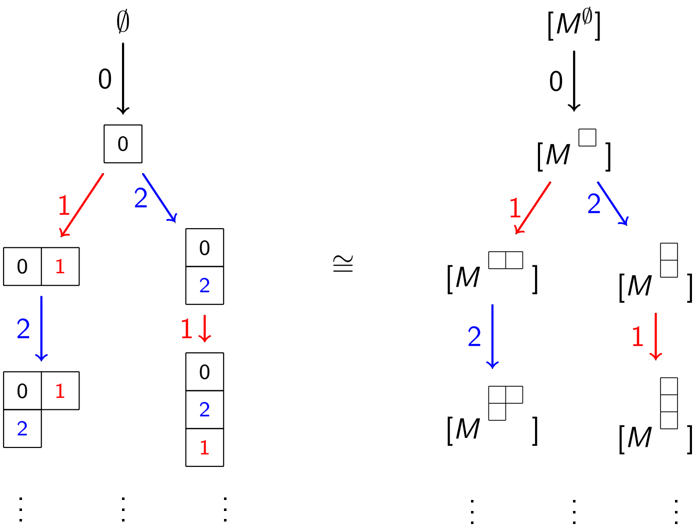
Brief review of KLR algebras

Crystal models:

	$B(\Lambda)$	$B(\infty)$
Nodes	simple R^{Λ}-modules	simple R-modules
Arrows, $\widetilde{f}_i$	refined induction functors	refined induction functors

For simple $M, N \in R\text{-}\mathrm{mod}$ (or $M, N \in R^{\Lambda}\text{-}\mathrm{mod}$),

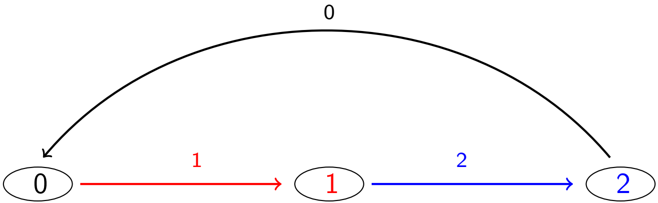
$$[M] \xrightarrow{i} [N] \qquad \Leftrightarrow \qquad \widetilde{f}_i M \cong N.$$


∅
0
0
1
2
0 1
0
2
2
1
0 1
2
0
2
1
≅
[M∅]
0
[M]
1
2
[M]
[M]
2
1
[M]
[M]

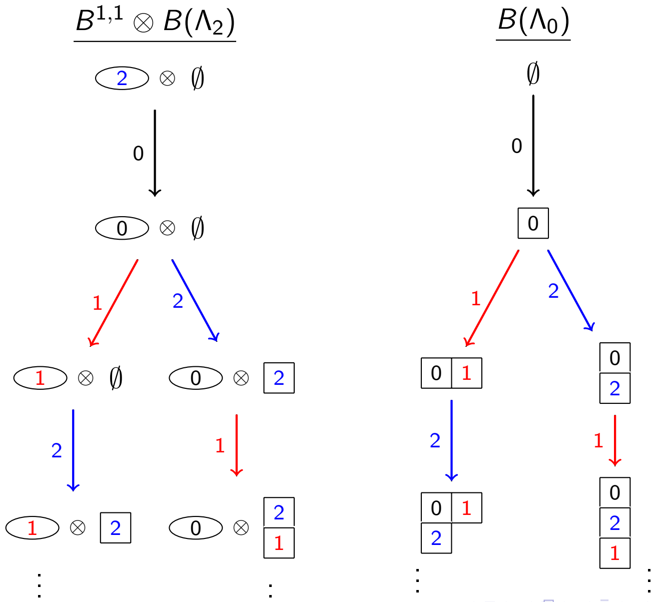
$$
\begin{tikzcd}[column sep=small]
 & {[M^{\emptyset}]} \arrow[d, "0"'] & \\
 & {[M^{\yng(1)}]} \arrow[dl, "1"'] \arrow[dr, "2"] & \\
{[M^{\yng(2)}]} \arrow[d, "2"'] & & {[M^{\yng(1,1)}]} \arrow[d, "1"'] \\
{[M^{\yng(2,1)}]} & & {[M^{\yng(1,1,1)}]} \\
\vdots & \vdots & \vdots
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
 & {[M^{\emptyset}]} \arrow[d, "0"'] & \\
 & {[\widetilde{f}_0 M^{\emptyset}]} \arrow[dl, "1"'] \arrow[dr, "2"] & \\
{[\widetilde{f}_1 \widetilde{f}_0 M^{\emptyset}]} \arrow[d, "2"'] & & {[\widetilde{f}_2 \widetilde{f}_0 M^{\emptyset}]} \arrow[d, "1"'] \\
{[\widetilde{f}_2 \widetilde{f}_1 \widetilde{f}_0 M^{\emptyset}]} & & {[\widetilde{f}_1 \widetilde{f}_2 \widetilde{f}_0 M^{\emptyset}]} \\
\vdots & \vdots & \vdots
\end{tikzcd}
$$

A crystal isomorphism

Kirillov-Reshetikhin crystal $B^{1,1}$ in type $A_2^{(1)}$



Compare


$B^{1,1} \otimes B(\Lambda_2)$
$B(\Lambda_0)$
2 ⊗ ∅
0
0 ⊗ ∅
1
2
1 ⊗ ∅
0 ⊗ 2
2
1
1 ⊗ 2
0 ⊗ 2 1
∅
0
0
1
2
0 1
0 2
2
1
0 1 2
0 2 1

- There is an isomorphism of crystals.

$$B^{1,1} \otimes B(\Lambda_2) \cong B(\Lambda_0)$$

- There is an isomorphism of crystals.

$$B^{1,1} \otimes B(\Lambda_2) \cong B(\Lambda_0)$$

- How to define this isomorphism in terms of Young diagram model for $B(\Lambda_0)$?

Notice,

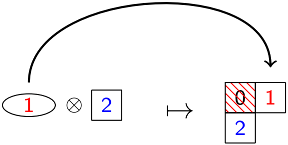
$$1 \otimes \boxed{2} \quad \mapsto \quad \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

- There is an isomorphism of crystals.

$$B^{1,1} \otimes B(\Lambda_2) \cong B(\Lambda_0)$$

- How to define this isomorphism in terms of Young diagram model for $B(\Lambda_0)$?

Notice,

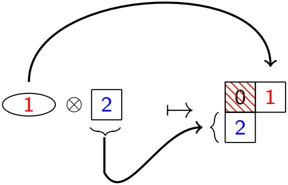


- There is an isomorphism of crystals.

$$B^{1,1} \otimes B(\Lambda_2) \cong B(\Lambda_0)$$

- How to define this isomorphism in terms of Young diagram model for $B(\Lambda_0)$?

Notice,

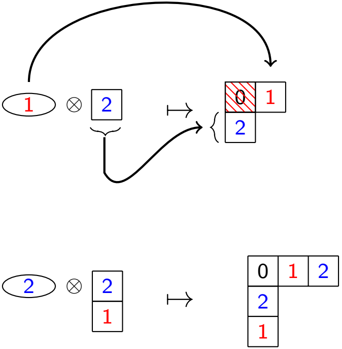


- There is an isomorphism of crystals.

$$B^{1,1} \otimes B(\Lambda_2) \cong B(\Lambda_0)$$

- How to define this isomorphism in terms of Young diagram model for $B(\Lambda_0)$?

Notice,

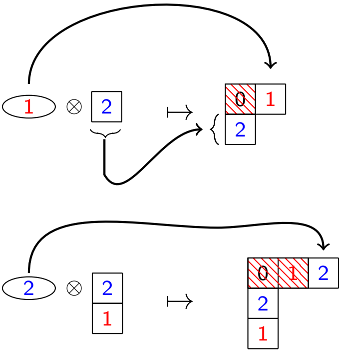


- There is an isomorphism of crystals.

$$B^{1,1} \otimes B(\Lambda_2) \cong B(\Lambda_0)$$

- How to define this isomorphism in terms of Young diagram model for $B(\Lambda_0)$?

Notice,

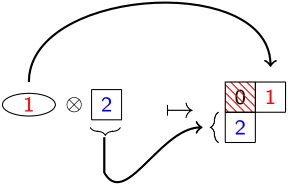


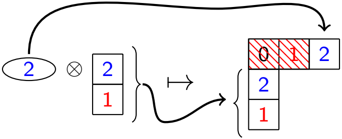
- There is an isomorphism of crystals.

$$B^{1,1} \otimes B(\Lambda_2) \cong B(\Lambda_0)$$

- How to define this isomorphism in terms of Young diagram model for $B(\Lambda_0)$?

Notice,



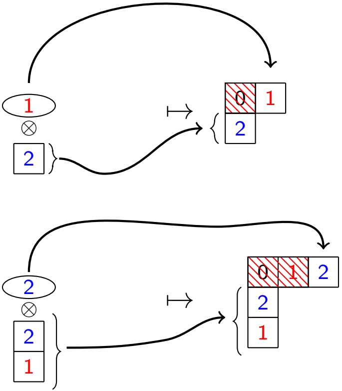


- There is an isomorphism of crystals.

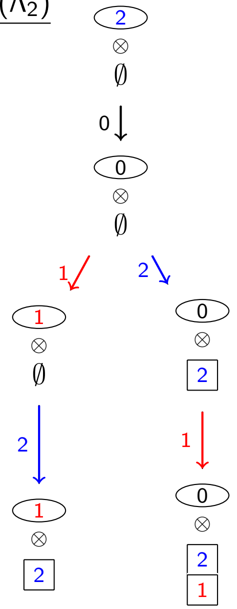
$$B^{1,1} \otimes B(\Lambda_2) \cong B(\Lambda_0)$$

- How to define this isomorphism in terms of Young diagram model for $B(\Lambda_0)$?

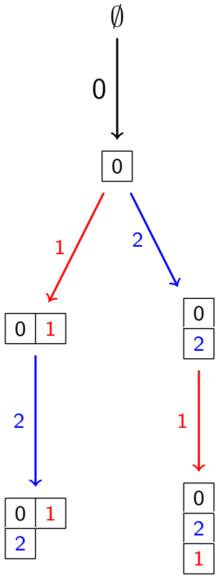
Notice,



$B^{1,1} \otimes B(\Lambda_2)$



$B(\Lambda_0)$



Because diagrams are $(\ell + 1)$-restricted, this map is a well-defined bijection.

More generally

- $B^{1,1}$ in type $A_\ell^{(1)}$ is an example of a perfect crystal of level 1 (also a Kirillov-Reshetikhin crystal).

$$
\begin{tikzcd}
\boxed{0} \arrow[r, "1"] & \boxed{1} \arrow[r, "2"] & \cdots \arrow[r, "\ell\text{-}1"] & \boxed{\ell\text{-}1} \arrow[r, "\ell"] & \boxed{\ell} \arrow[llll, bend right, "0"']
\end{tikzcd}
$$

- There is a crystal isomorphism

$$B^{1,1} \otimes B(\Lambda_{i-1}) \xrightarrow{\sim} B(\Lambda_i).$$

- $B(\Lambda_i)$ is complicated, but $B^{1,1}$ is easy to understand.
- Crystals behave nicely under tensor products. If we understand crystals B_1, B_2, it is easy to understand $B_1 \otimes B_2$.

- $B(\Lambda_i)$ is complicated, but $B^{1,1}$ is easy to understand.
- Crystals behave nicely under tensor products. If we understand crystals B_1, B_2, it is easy to understand $B_1 \otimes B_2$.

$$\Downarrow$$

Study $B(\Lambda_0)$ by iterating isomorphism:

$$B^{1,1} \otimes B(\Lambda_{i-1}) \cong B(\Lambda_i)$$

- $B(\Lambda_i)$ is complicated, but $B^{1,1}$ is easy to understand.
- Crystals behave nicely under tensor products. If we understand crystals B_1, B_2, it is easy to understand $B_1 \otimes B_2$.

$$\Downarrow$$

Study $B(\Lambda_0)$ by iterating isomorphism:

$$\begin{aligned} B^{1,1} \otimes B(\Lambda_{i-1}) &\cong B(\Lambda_i) \\ B^{1,1} \otimes B^{1,1} \otimes B(\Lambda_{i-2}) &\cong B(\Lambda_i) \end{aligned}$$

- $B(\Lambda_i)$ is complicated, but $B^{1,1}$ is easy to understand.
- Crystals behave nicely under tensor products. If we understand crystals B_1, B_2, it is easy to understand $B_1 \otimes B_2$.

$$\Downarrow$$

Study $B(\Lambda_0)$ by iterating isomorphism:

$$\begin{aligned} B^{1,1} \otimes B(\Lambda_{i-1}) &\cong B(\Lambda_i) \\ B^{1,1} \otimes B^{1,1} \otimes B(\Lambda_{i-2}) &\cong B(\Lambda_i) \\ B^{1,1} \otimes B^{1,1} \otimes B^{1,1} \otimes B(\Lambda_{i-3}) &\cong B(\Lambda_i) \\ &\vdots \end{aligned}$$

Main Question for this talk: Does the crystal isomorphism

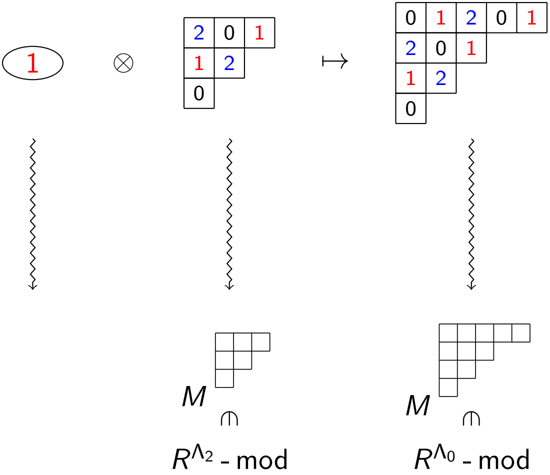
$$B^{1,1} \otimes B(\Lambda_{i-1}) \xrightarrow{\sim} B(\Lambda_i)$$

have a higher module-theoretic analogue for representation theory of KLR algebras?

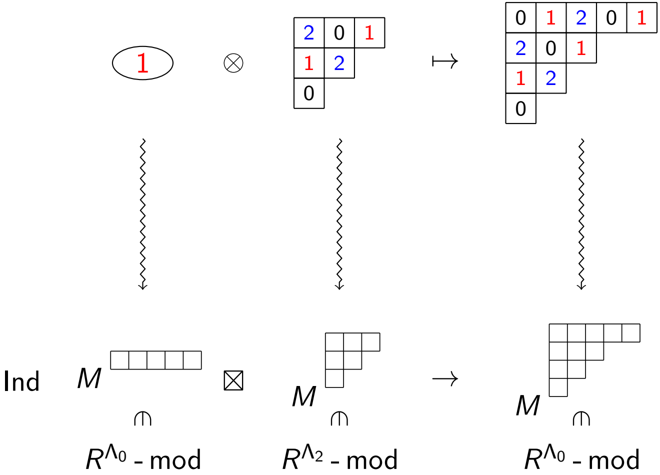
For case $A_2^{(1)}$, $\Lambda_i = \Lambda_0$, what is R-module analogue of this?...

$$1 \quad \otimes \quad \begin{array}{|c|c|c|} \hline 2 & 0 & 1 \\ \hline 1 & 2 \\ \cline{1-2} 0 \\ \cline{1-1} \end{array} \quad \mapsto \quad \begin{array}{|c|c|c|c|c|} \hline 0 & 1 & 2 & 0 & 1 \\ \hline 2 & 0 & 1 \\ \cline{1-3} 1 & 2 \\ \cline{1-2} 0 \\ \cline{1-1} \end{array}$$

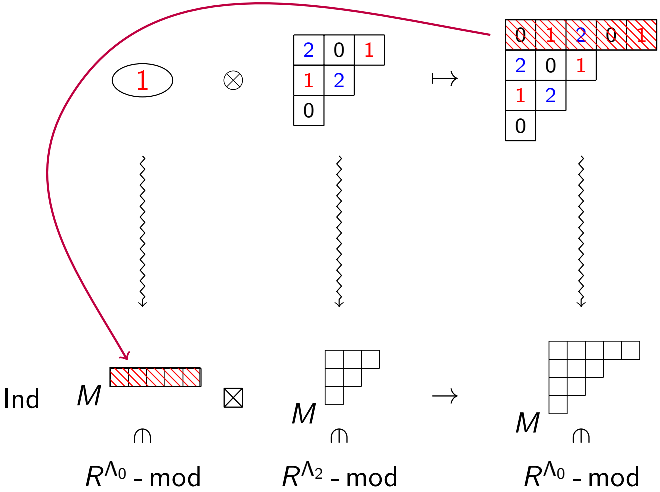
Should be the R^{Λ_0}-module homomorphism,



Should be the R^{Λ_0}-module homomorphism,



Should be the R^{Λ_0}-module homomorphism,



As with $\mathcal{S}_5$:

$$M^{\ydiagram{5}} \cong \text{1-dimensional ``trivial'' } R^{\Lambda_0}\text{-module}$$

and

$$\mathrm{Char}(M^{\ydiagram{5}}) = [0\ 1\ 2\ 0\ 1].$$

Easiest possible representation to work with!

But crystals are about much more than nodes.

Since

$$1 \quad \otimes \quad \begin{array}{|c|c|c|}\hline 2&0&1\\ \hline 1&2\\ \cline{1-2} 0\\ \cline{1-1}\end{array} \quad \mapsto \quad \begin{array}{|c|c|c|c|c|}\hline 0&1&2&0&1\\ \hline 2&0&1\\ \cline{1-3} 1&2\\ \cline{1-2} 0\\ \cline{1-1}\end{array}$$

tensor product rule for crystals gives

$$\tilde{f}_1\left(1 \quad \otimes \quad \begin{array}{|c|c|c|}\hline 2&0&1\\ \hline 1&2\\ \cline{1-2} 0\\ \cline{1-1}\end{array} \right) = 1 \quad \otimes \quad \tilde{f}_1\left(\begin{array}{|c|c|c|}\hline 2&0&1\\ \hline 1&2\\ \cline{1-2} 0\\ \cline{1-1}\end{array}\right) \mapsto \tilde{f}_1\left(\begin{array}{|c|c|c|c|c|}\hline 0&1&2&0&1\\ \hline 2&0&1\\ \cline{1-3} 1&2\\ \cline{1-2} 0\\ \cline{1-1}\end{array}\right)$$

With

$$1 \quad \otimes \quad \tilde{f}_1\left(\begin{ytableau} 2 & 0 & 1 \\ 1 & 2 \\ 0 \end{ytableau} \right) \quad \mapsto \quad \tilde{f}_1\left(\begin{ytableau} 0 & 1 & 2 & 0 & 1 \\ 2 & 0 & 1 \\ 1 & 2 \\ 0 \end{ytableau} \right)$$

should also have

$$\mathrm{Ind}\, M^{\ydiagram{5}} \boxtimes \tilde{f}_1\left(M^{\ydiagram{3,2,1}}\right) \quad \to \quad \tilde{f}_1\left(M^{\ydiagram{5,3,2,1}}\right)$$

Given $b_1 \in B(\Lambda_i)$, $b_2 \in B(\Lambda_{i-1})$ with

$$\textcircled{j} \boxtimes b_2 \mapsto b_1$$

and corresponding $M^{b_1} \in R^{\Lambda_i}\text{-mod}$, $M^{b_2} \in R^{\Lambda_{i-1}}\text{-mod}$, K.-Vazirani show:

Given $b_1 \in B(\Lambda_i)$, $b_2 \in B(\Lambda_{i-1})$ with

$$j \boxtimes b_2 \mapsto b_1$$

and corresponding $M^{b_1} \in R^{\Lambda_i}\text{-mod}$, $M^{b_2} \in R^{\Lambda_{i-1}}\text{-mod}$, K.-Vazirani show:

- $$\operatorname{Ind} T \boxtimes M^{b_2} \twoheadrightarrow M^{b_1}$$

  for appropriate "trivial" R^{Λ_i}-module T.

Given $b_1 \in B(\Lambda_i)$, $b_2 \in B(\Lambda_{i-1})$ with

$$j \boxtimes b_2 \mapsto b_1$$

and corresponding $M^{b_1} \in R^{\Lambda_i}\text{-mod}$, $M^{b_2} \in R^{\Lambda_{i-1}}\text{-mod}$, K.-Vazirani show:

- $$\operatorname{Ind} T \boxtimes M^{b_2} \twoheadrightarrow M^{b_1}$$

  for appropriate "trivial" R^{Λ_i}-module T.

- Action of $\widetilde{f}_i$ and $\widetilde{e}_i$ agree in module <u>and</u> crystal setting.

Given $b_1 \in B(\Lambda_i)$, $b_2 \in B(\Lambda_{i-1})$ with

$$j \boxtimes b_2 \mapsto b_1$$

and corresponding $M^{b_1} \in R^{\Lambda_i}\text{-mod}$, $M^{b_2} \in R^{\Lambda_{i-1}}\text{-mod}$, K.-Vazirani show:

- $$\operatorname{Ind} T \boxtimes M^{b_2} \twoheadrightarrow M^{b_1}$$

  for appropriate "trivial" R^{Λ_i}-module T.

- Action of $\widetilde{f}_i$ and $\widetilde{e}_i$ agree in module <u>and</u> crystal setting.

$B^{1,1} \otimes B(\Lambda_{i-1}) \cong B(\Lambda_i)$ **is the shadow of richer R-mod structure.**

Generalizing to other types

Question: How can we interpret nodes of $B^{1,1}$ without intuition from Young diagrams?

Is there another way to see that in $A_2^{(1)}$

$\textcolor{red}{1}$ corresponds to $M^{\square\square\square\square\square}$?

Is there another way to see that in $A_2^{(1)}$

$$\textcircled{1} \quad \text{corresponds to} \quad M^{\begin{array}{|c|c|c|c|c|}\hline &&&&\\\hline\end{array}} \; ?$$

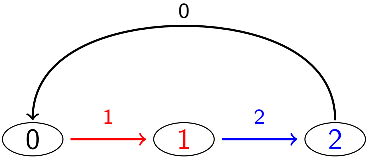
- $\begin{array}{|c|c|c|c|c|}\hline &&&&\\\hline\end{array}$ has residues $\begin{array}{|c|c|c|c|c|}\hline 0&1&2&0&1\\\hline\end{array}$

- the crystal $B^{1,1}$ has crystal graph

$$\begin{tikzcd} 0 \arrow[r, "1"] & 1 \arrow[r, "2"] & 2 \arrow[ll, bend right, "0"'] \end{tikzcd}$$

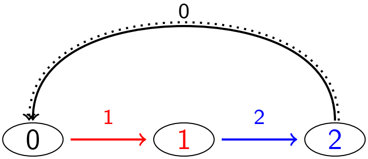
...Residues give directed walk in $B^{1,1}$

0	1	2	0	1



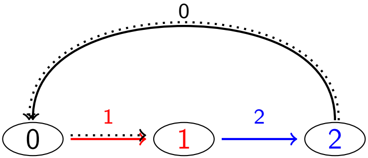
...Residues give directed walk in $B^{1,1}$

0	1	2	0	1

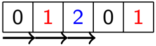


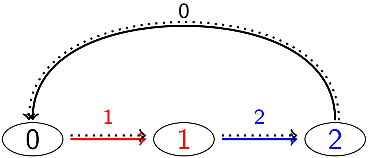
...Residues give directed walk in $B^{1,1}$

0	1	2	0	1



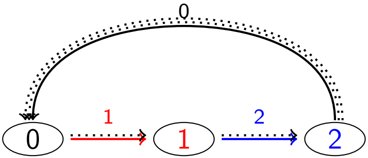
...Residues give directed walk in $B^{1,1}$



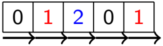


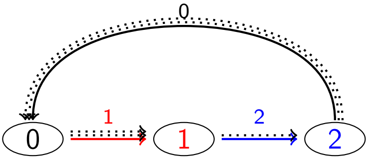
...Residues give directed walk in $B^{1,1}$

0	1	2	0	1



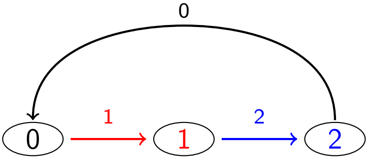
…Residues give directed walk in $B^{1,1}$





...Residues give directed walk in $B^{1,1}$

0	1	2	0	1

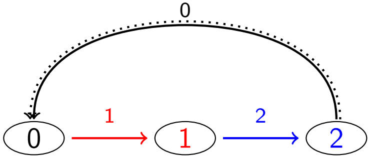


And walk describes how to build $M^{\square\square\square\square\square}$ with functors $\widetilde{f}_i$,

$$M^{\emptyset} \cong M^{\emptyset} =: \mathbb{1}$$

...Residues give directed walk in $B^{1,1}$

0	1	2	0	1

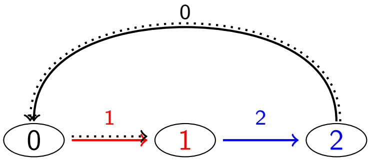


And walk describes how to build $M^{\square\square\square\square\square}$ with functors $\widetilde{f}_i$,

$$M^{\square} \cong \widetilde{f}_0 \mathbb{1}$$

...Residues give directed walk in $B^{1,1}$

0	1	2	0	1

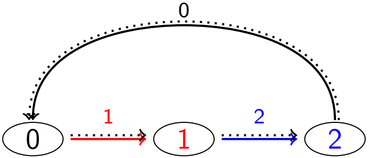


And walk describes how to build $M^{\square\square\square\square\square}$ with functors $\widetilde{f}_i$,

$$M^{\square\square} \cong \widetilde{f}_1 \widetilde{f}_0 \mathbb{1}$$

...Residues give directed walk in $B^{1,1}$

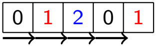
0	1	2	0	1

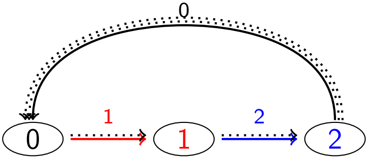


And walk describes how to build $M^{\square\square\square\square\square}$ with functors $\widetilde{f}_i$,

$$M^{\square\square\square} \cong \widetilde{f}_2 \widetilde{f}_1 \widetilde{f}_0 \mathbb{1}$$

...Residues give directed walk in $B^{1,1}$

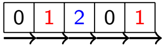


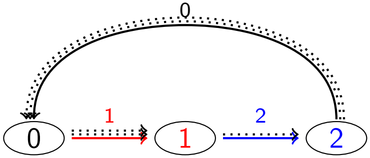


And walk describes how to build $M^{\square\square\square\square\square}$ with functors $\widetilde{f}_i$,

$$M^{\square\square\square\square} \cong \widetilde{f}_0 \widetilde{f}_2 \widetilde{f}_1 \widetilde{f}_0 \mathbb{1}$$

...Residues give directed walk in $B^{1,1}$

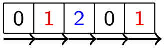


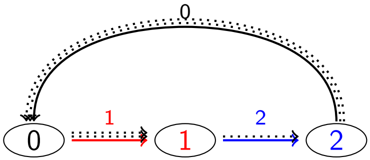


And walk describes how to build $M^{\square\square\square\square\square}$ with functors $\widetilde{f}_i$,

$$M^{\square\square\square\square\square} \cong \widetilde{f}_1 \widetilde{f}_0 \widetilde{f}_2 \widetilde{f}_1 \widetilde{f}_0 \mathbb{1}$$

...Residues give directed walk in $B^{1,1}$





And walk describes how to build $M^{\square\square\square\square\square}$ with functors $\widetilde{f}_i$,

$$M^{\square\square\square\square\square} \cong \widetilde{f}_{1}\widetilde{f}_{0}\widetilde{f}_{2}\widetilde{f}_{1}\widetilde{f}_{0}\,\mathbb{1}$$

Recall:

$$\mathrm{Char}(M^{\square\square\square\square\square}) = [0\ 1\ 2\ 0\ 1].$$

Using functors $\widetilde{f}_i$, we can build an R-module from any walk in $B^{1,1}$.

For a directed walk p in $B^{1,1}$ of length k which traverses edges colored

$$i_1, i_2, \ldots, i_k$$

set

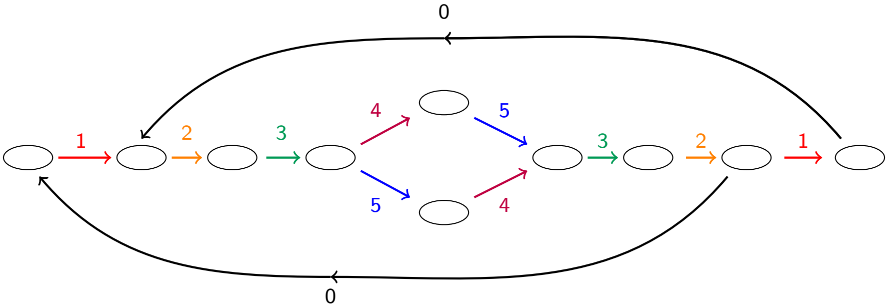
$$T_{p;k} := \widetilde{f}_{i_k} \ldots \widetilde{f}_{i_2} \widetilde{f}_{i_1} \mathbb{1}$$

$$\uparrow \tag{3}$$

Analogue for “trivial” modules in other types.

$B^{1,1}$ in type $D_5^{(1)}$

Directed walk

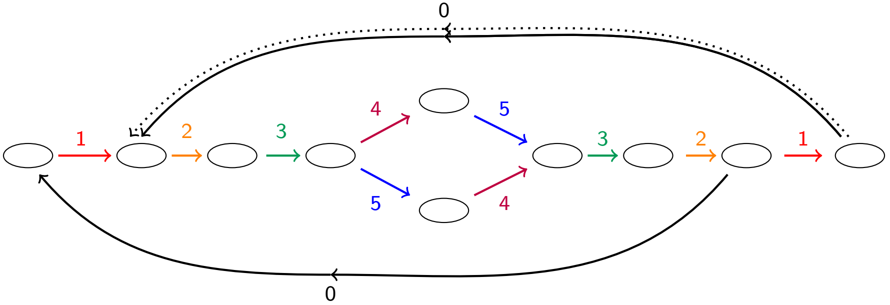


This corresponds to R-module

$$\mathbb{1}$$

$B^{1,1}$ in type $D_5^{(1)}$

Directed walk

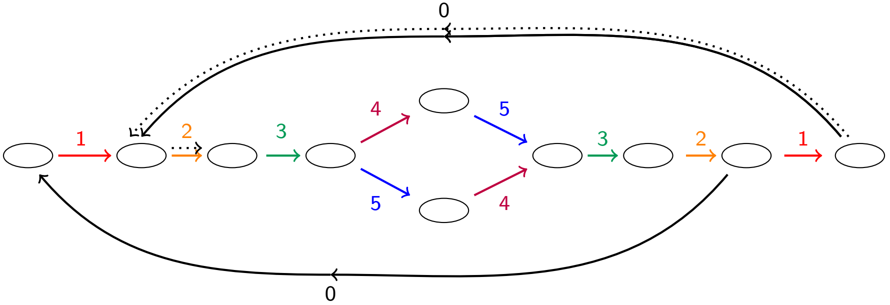


This corresponds to R-module

$$\widetilde{f}_0\, \mathbb{1}$$

$B^{1,1}$ in type $D_5^{(1)}$

Directed walk

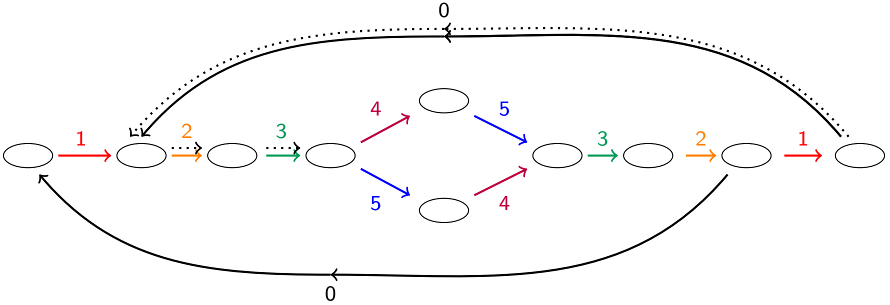


This corresponds to R-module

$$\widetilde{f}_2 \widetilde{f}_0 \mathbb{1}$$

$B^{1,1}$ in type $D_5^{(1)}$

Directed walk

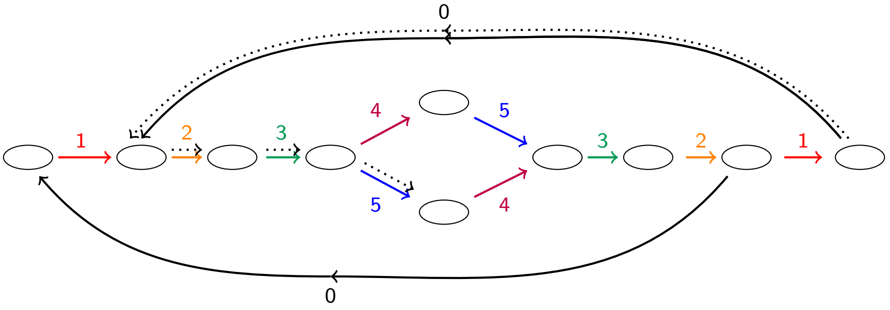


This corresponds to R-module

$$\widetilde{f_3}\,\widetilde{f_2}\,\widetilde{f_0}\,\mathbb{1}$$

$B^{1,1}$ in type $D_5^{(1)}$

Directed walk p

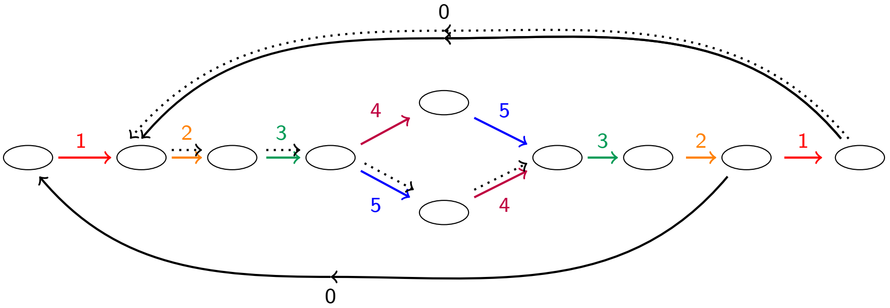


This corresponds to R-module

$$T_{p;4} = \widetilde{f_5}\,\widetilde{f_3}\,\widetilde{f_2}\,\widetilde{f_0}\,\mathbb{1}$$

$B^{1,1}$ in type $D_5^{(1)}$

Directed walk p

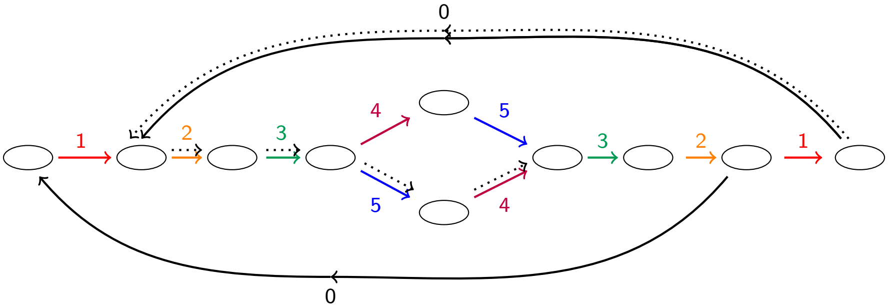


This corresponds to R-module

$$T_{p;4} = \widetilde{f_4}\,\widetilde{f_5}\,\widetilde{f_3}\,\widetilde{f_2}\,\widetilde{f_0}\,\mathbb{1},$$

$B^{1,1}$ in type $D_5^{(1)}$

Directed walk p

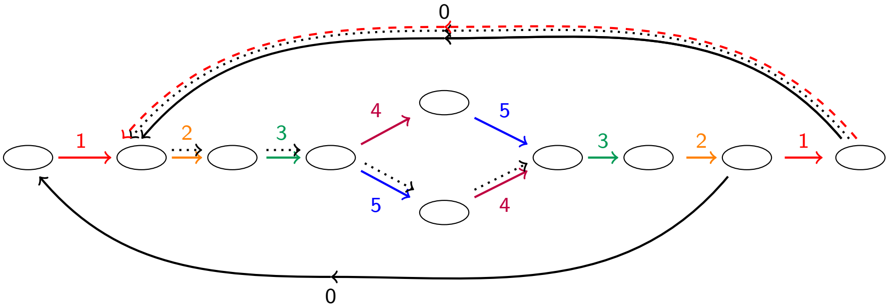


This corresponds to R-module

$$T_{p;5} = \widetilde{f}_4 \widetilde{f}_5 \widetilde{f}_3 \widetilde{f}_2 \widetilde{f}_0 \mathbb{1}, \quad \mathrm{Char}(T_{p;5}) = [0\ 2\ 3\ 5\ 4] + [0\ 2\ 3\ 4\ 5]$$

$B^{1,1}$ in type $D_5^{(1)}$

Directed walk p

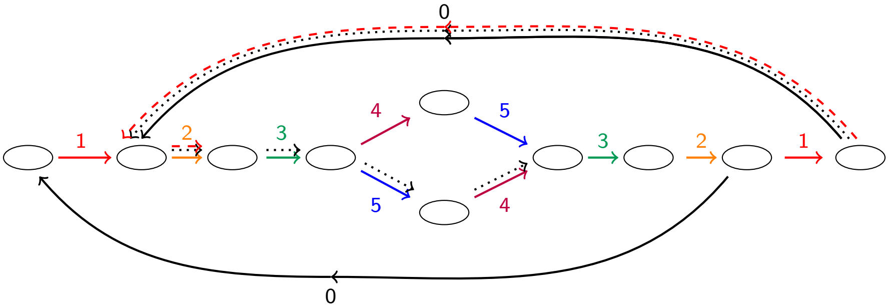


This corresponds to R-module

$$T_{p;5} = \widetilde{f_4}\widetilde{f_5}\widetilde{f_3}\widetilde{f_2}\widetilde{f_0}\,\mathbb{1}, \quad \mathrm{Char}(T_{p;5}) = [0\ 2\ 3\ 5\ 4] + [0\ 2\ 3\ 4\ 5]$$

$B^{1,1}$ in type $D_5^{(1)}$

Directed walk p

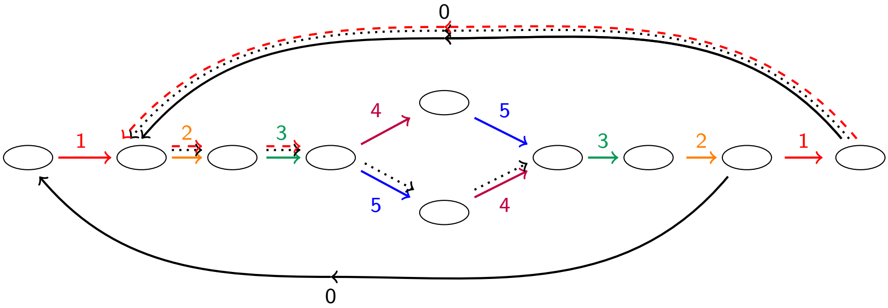


This corresponds to R-module

$$T_{p;5} = \widetilde{f_4}\widetilde{f_5}\widetilde{f_3}\widetilde{f_2}\widetilde{f_0}\,\mathbb{1}, \quad \mathrm{Char}(T_{p;5}) = [0\ 2\ 3\ 5\ 4] + [0\ 2\ 3\ 4\ 5]$$

$B^{1,1}$ in type $D_5^{(1)}$

Directed walk p

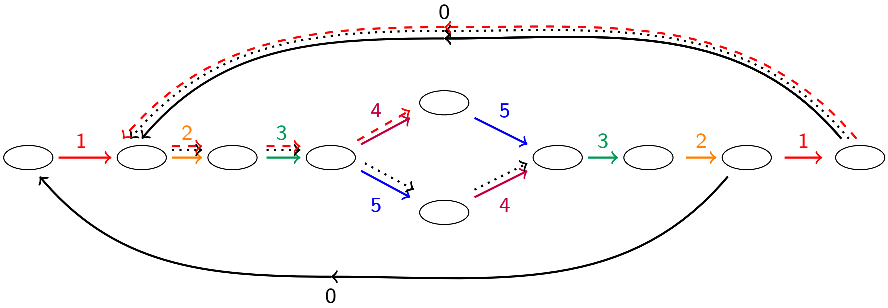


This corresponds to R-module

$$T_{p;5} = \widetilde{f_4}\widetilde{f_5}\widetilde{f_3}\widetilde{f_2}\widetilde{f_0}\,\mathbb{1}, \quad \mathrm{Char}(T_{p;5}) = [0\ 2\ 3\ 5\ 4] + [0\ 2\ 3\ 4\ 5]$$

$B^{1,1}$ in type $D_5^{(1)}$

Directed walk p

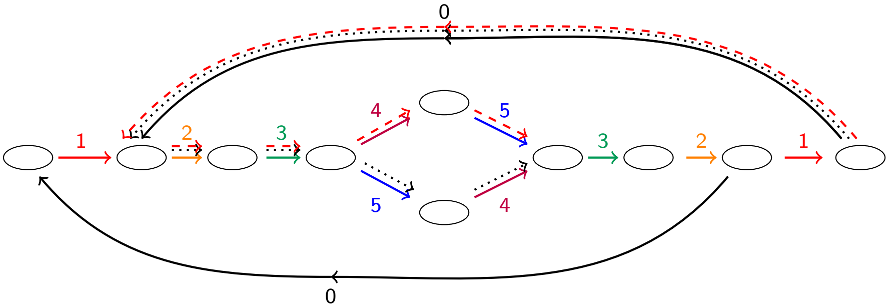


This corresponds to R-module

$$T_{p;5} = \widetilde{f_4}\,\widetilde{f_5}\,\widetilde{f_3}\,\widetilde{f_2}\,\widetilde{f_0}\,\mathbb{1}, \quad \mathrm{Char}(T_{p;5}) = [0\ 2\ 3\ 5\ 4] + [0\ 2\ 3\ 4\ 5]$$

$B^{1,1}$ in type $D_5^{(1)}$

Directed walk p



This corresponds to R-module

$$T_{p;5} = \widetilde{f_4}\,\widetilde{f_5}\,\widetilde{f_3}\,\widetilde{f_2}\,\widetilde{f_0}\,\mathbb{1}, \quad \mathrm{Char}(T_{p;5}) = [0\ 2\ 3\ 5\ 4] + [0\ 2\ 3\ 4\ 5]$$

Type X_ℓ

Question: Is there a module-theoretic interpretation of the crystal isomorphism

$$B^{1,1} \otimes B(\Lambda_{\sigma(i)}) \xrightarrow{\cong} B(\Lambda_i)$$

in other classical affine types X_ℓ? (when $B^{1,1}$ is perfect and Λ_i and $\Lambda_{\sigma(i)}$ is level 1).

Story is exactly same to type $A_\ell^{(1)}$ case, but "trivial" modules replaced by $T_{p;k}$.

If,

$$c \otimes b_1 \mapsto b_2$$

where $c \in B^{1,1}$, $b_1 \in B(\Lambda_{\sigma(i)})$, and $b_2 \in B(\Lambda_i)$, we showed:

Story is exactly same to type $A_\ell^{(1)}$ case, but "trivial" modules replaced by $T_{p;k}$.

If,

$$c \otimes b_1 \mapsto b_2$$

where $c \in B^{1,1}$, $b_1 \in B(\Lambda_{\sigma(i)})$, and $b_2 \in B(\Lambda_i)$, we showed:

- $$\operatorname{Ind} T_{p;k} \boxtimes M^{b_1} \twoheadrightarrow M^{b_2}.$$

  for appropriate directed walk p in $B^{1,1}$.
- Action of $\widetilde{e}_i$ and $\widetilde{f}_i$ agree in both crystal <u>and</u> module settings.

What <u>differs</u> in other classical types?

What <u>differs</u> in other classical types?

More complex $B^{1,1}$ results in more complex $T_{p;k}$. Two new subgraphs appear in $B^{1,1}$:

What <u>differs</u> in other classical types?

More complex $B^{1,1}$ results in more complex $T_{p;k}$. Two new subgraphs appear in $B^{1,1}$:

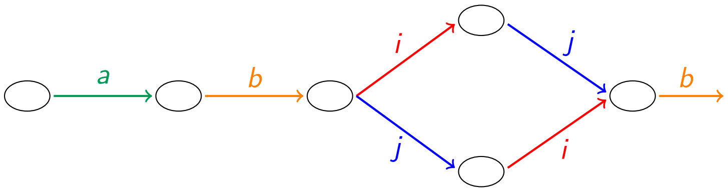
- Type $\mathcal{D}$ structures,

What <u>differs</u> in other classical types?

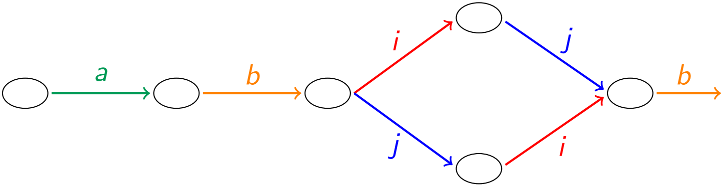
More complex $B^{1,1}$ results in more complex $T_{p;k}$. Two new subgraphs appear in $B^{1,1}$:

- Type $\mathcal{D}$ structures,
- Type $\mathcal{B}$ structures,

Type $\mathcal{D}$ structure

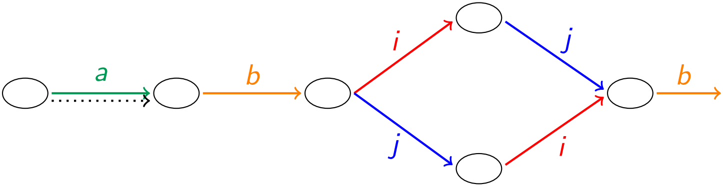


Type $\mathcal{D}$ structure



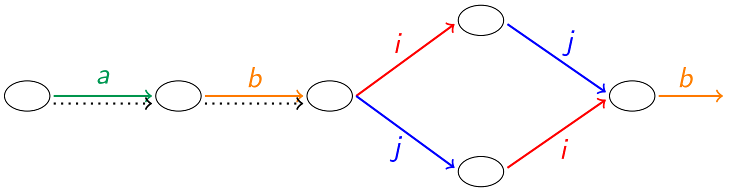
$\mathbb{1}$

Type $\mathcal{D}$ structure



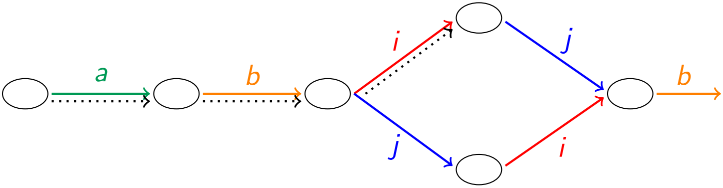
$$\mathrm{Char}(\widetilde{f_a}\,\mathbb{1}) = [a]$$

Type $\mathcal{D}$ structure



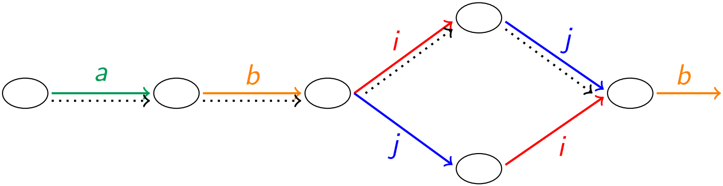
$$\mathrm{Char}(\widetilde{f_b}\,\widetilde{f_a}\,\mathbb{1}) = [a\ b]$$

Type $\mathcal{D}$ structure



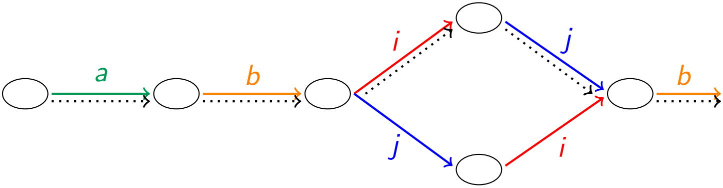
$$\mathrm{Char}(\widetilde{f_i}\widetilde{f_b}\widetilde{f_a}\,\mathbb{1}) = [a\ b\ i]$$

Type $\mathcal{D}$ structure



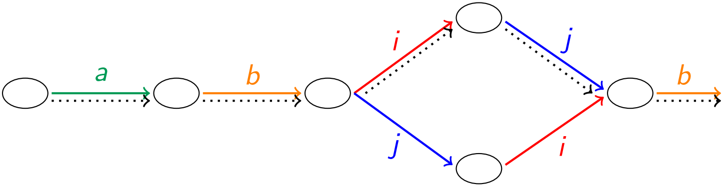
$$\mathrm{Char}(\widetilde{f_j}\widetilde{f_i}\widetilde{f_b}\widetilde{f_a}\,\mathbb{1}) = [a\ b\ i\ j] + [a\ b\ j\ i]$$

Type $\mathcal{D}$ structure



$$\mathrm{Char}(\widetilde{f_b}\widetilde{f_j}\widetilde{f_i}\widetilde{f_b}\widetilde{f_a}\,\mathbb{1}) = [a\ b\ i\ j\ b] + [a\ b\ j\ i\ b]$$

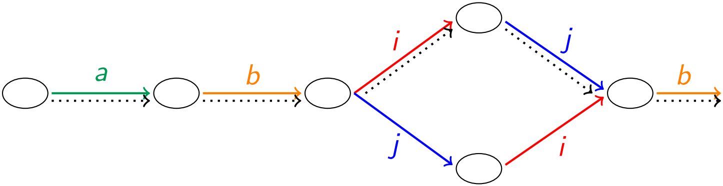
Type $\mathcal{D}$ structure



$$\mathrm{Char}(\widetilde{f_b}\widetilde{f_j}\widetilde{f_i}\widetilde{f_b}\widetilde{f_a}\,\mathbb{1}) = [a\ b\ i\ j\ b] + [a\ b\ j\ i\ b]$$

- Bifurcations double dimension.

Type $\mathcal{D}$ structure



$$\mathrm{Char}(\widetilde{f}_b\widetilde{f}_j\widetilde{f}_i\widetilde{f}_b\widetilde{f}_a\,\mathbb{1}) = [a\ b\ i\ j\ b] + [a\ b\ j\ i\ b]$$

- Bifurcations double dimension.
- Module does not see difference between two paths around bifurcation.

Type $\mathcal{B}$ structure

$\mathbb{1}$

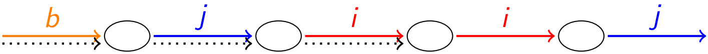
Type $\mathcal{B}$ structure

$$\mathrm{Char}(\widetilde{f_b}\,\mathbb{1}) = [b]$$

Type $\mathcal{B}$ structure

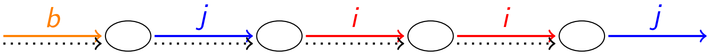
$$\mathrm{Char}(\widetilde{f_j}\widetilde{f_b}\,\mathbb{1}) = [b\ j]$$

Type $\mathcal{B}$ structure



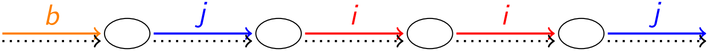
$$\mathrm{Char}(\widetilde{f_i}\widetilde{f_j}\widetilde{f_b}\,\mathbb{1}) = [b\ j\ i]$$

Type $\mathcal{B}$ structure



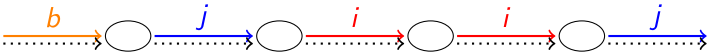
$$\mathrm{Char}(\widetilde{f_i}\widetilde{f_i}\widetilde{f_j}\widetilde{f_b}\,\mathbb{1}) = (1+q^{-2})[b\ j\ i\ i]$$

Type $\mathcal{B}$ structure



$$\mathrm{Char}(\widetilde{f}_j\widetilde{f}_i\widetilde{f}_i\widetilde{f}_j\widetilde{f}_b\,\mathbb{1}) = (1+q^{-2})[b\ j\ i\ i\ j]$$

Type $\mathcal{B}$ structure



$$\mathrm{Char}(\widetilde{f_j}\widetilde{f_i}\widetilde{f_i}\widetilde{f_j}\widetilde{f_b}\,\mathbb{1}) = (1+q^{-2})[b\ j\ i\ i\ j]$$

Traveling over adjacent i-arrows, multiply character by $[2] = (q^{-1}+q)$.

Iterating the construction

Recall, we can iterate:

$$
\begin{aligned}
B^{1,1} \otimes B(\Lambda_{i-1}) &\cong B(\Lambda_i) \\
B^{1,1} \otimes B^{1,1} \otimes B(\Lambda_{i-2}) &\cong B(\Lambda_i) \\
B^{1,1} \otimes B^{1,1} \otimes B^{1,1} \otimes B(\Lambda_{i-3}) &\cong B(\Lambda_i) \\
&\vdots
\end{aligned}
$$

Iterating the construction

Recall, we can iterate:

$$\begin{aligned} B^{1,1} \otimes B(\Lambda_{i-1}) &\cong B(\Lambda_i) \\ B^{1,1} \otimes B^{1,1} \otimes B(\Lambda_{i-2}) &\cong B(\Lambda_i) \\ B^{1,1} \otimes B^{1,1} \otimes B^{1,1} \otimes B(\Lambda_{i-3}) &\cong B(\Lambda_i) \\ &\vdots \end{aligned}$$

Similarly for simple $M \in R^{\Lambda_i}\text{-mod}$

Iterating the construction

Recall, we can iterate:

$$B^{1,1} \otimes B(\Lambda_{i-1}) \cong B(\Lambda_i)$$
$$B^{1,1} \otimes B^{1,1} \otimes B(\Lambda_{i-2}) \cong B(\Lambda_i)$$
$$B^{1,1} \otimes B^{1,1} \otimes B^{1,1} \otimes B(\Lambda_{i-3}) \cong B(\Lambda_i)$$
$$\vdots$$

Similarly for simple $M \in R^{\Lambda_i}\text{-mod}$

$$\operatorname{Ind} T_{p_1;k_1} \boxtimes M_1 \twoheadrightarrow M$$

Iterating the construction

Recall, we can iterate:

$$B^{1,1} \otimes B(\Lambda_{i-1}) \cong B(\Lambda_i)$$
$$B^{1,1} \otimes B^{1,1} \otimes B(\Lambda_{i-2}) \cong B(\Lambda_i)$$
$$B^{1,1} \otimes B^{1,1} \otimes B^{1,1} \otimes B(\Lambda_{i-3}) \cong B(\Lambda_i)$$
$$\vdots$$

Similarly for simple $M \in R^{\Lambda_i}\text{-mod}$

$$\mathrm{Ind}\; T_{p_1;k_1} \boxtimes M_1 \twoheadrightarrow M$$
$$\mathrm{Ind}\; T_{p_1;k_1} \boxtimes T_{p_2;k_2} \boxtimes M_2 \twoheadrightarrow M$$

Iterating the construction

Recall, we can iterate:

$$B^{1,1} \otimes B(\Lambda_{i-1}) \cong B(\Lambda_i)$$
$$B^{1,1} \otimes B^{1,1} \otimes B(\Lambda_{i-2}) \cong B(\Lambda_i)$$
$$B^{1,1} \otimes B^{1,1} \otimes B^{1,1} \otimes B(\Lambda_{i-3}) \cong B(\Lambda_i)$$
$$\vdots$$

Similarly for simple $M \in R^{\Lambda_i}\text{-mod}$

$$\mathrm{Ind}\ T_{p_1;k_1} \boxtimes M_1 \twoheadrightarrow M$$
$$\mathrm{Ind}\ T_{p_1;k_1} \boxtimes T_{p_2;k_2} \boxtimes M_2 \twoheadrightarrow M$$
$$\mathrm{Ind}\ T_{p_1;k_1} \boxtimes T_{p_2;k_2} \boxtimes T_{p_3;k_3} \boxtimes M_3 \twoheadrightarrow M$$
$$\vdots$$

Iterating the construction

In KLR case, process must terminate and we get decomposition,

$$\mathrm{Ind}\ T_{p_1;k_1} \boxtimes T_{p_2;k_2} \boxtimes \cdots \boxtimes T_{p_r;k_r} \twoheadrightarrow M$$

In type $A_\ell^{(1)}$ this decomposition is similar to those for Specht modules.

Iterating the construction

In KLR case, process must terminate and we get decomposition,

$$\mathrm{Ind}\ T_{p_1;k_1} \boxtimes T_{p_2;k_2} \boxtimes \cdots \boxtimes T_{p_r;k_r} \twoheadrightarrow M$$

In type $A_\ell^{(1)}$ this decomposition is similar to those for Specht modules.

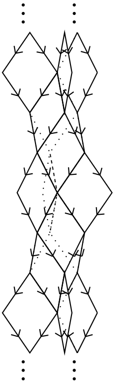
For other types, this seems to be new.

Work in progress

- Would like to generalize to other Kirillov-Reshetikhin crystals $B^{r,s}$

Work in progress

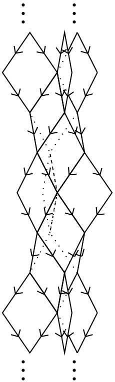
Similar arguments appear to work for $B^{r,1}$ in type $A_\ell^{(1)}$.



$B^{3,1}$ in type $A_5^{(1)}$

Work in progress

Similar arguments appear to work for $B^{r,1}$ in type $A_\ell^{(1)}$.



$B^{3,1}$ in type $A_5^{(1)}$

Key: "Trivial modules" $T_{p;k}$ arising from $B^{r,1}$ are homogeneous.

Work in progress

Homogeneous R-modules for simply-laced type fully classified by Kleshchev-Ram.

Work in progress

Homogeneous R-modules for simply-laced type fully classified by Kleshchev-Ram.

When $s > 1$ in $B^{r,s} \implies$, type $A_\ell^{(1)}$, $T_{p;k}$ are in general not homogeneous.

...New methods will be needed.

Future directions

- Can R representation theory provide new models for KR crystals?

Future directions

- Can R representation theory provide new models for KR crystals?

Example

Kleshchev and Ram have a beautiful combinatorial model for homogeneous R-modules for simply-laced type. Can we use this to construct a new combinatorial model for KR crystals of simply-laced type?

Future directions

- Can R representation theory provide new models for KR crystals?

Example

Kleshchev and Ram have a beautiful combinatorial model for homogeneous R-modules for simply-laced type. Can we use this to construct a new combinatorial model for KR crystals of simply-laced type?

- Is there any relationship between $T_{p;k}$ and cuspidal representations?

Thank you.