Practice Exam 3

1. (a) Give an example of a 2×2 matrix which is NOT positive definite. Justify your answer.

   (b) Find the 3×3 symmetric matrix A associated with quadratic form defined by
$$\langle A\mathbf{x}, \mathbf{x} \rangle = 3x_1^2 + 2x_1x_2 - x_3^2.$$

2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function such that $\frac{\partial f}{\partial x_n}(\mathbf{x}) = c$ for all $\mathbf{x} \in \mathbb{R}^n$ and for some constant $c \in \mathbb{R}$. Show that we will not be able to find any extreme points using the second derivative test.

3. Determine whether the following functions $F : \mathbb{R}^3 \to \mathbb{R}^3$ are linear. When they are linear, find their corresponding matrix.

   (a) $F(x_1, x_2, x_3) = (x_1, x_1 - x_2, x_2 + x_3)$,

   (b) $F(x_1, x_2, x_3) = (x_1, x_2, x_2x_3)$,

   (c) $F(x_1, x_2, x_3) = (x_1, 0, 0)$,

   (d) $F(x_1, x_2, x_3) = (1, 0, 0)$,

   (e) $F(x_1, x_2, x_3) = (3x_1 + 2x_2, x_3, |x_2|)$,

   (f) $F(x_1, x_2, x_3) = (x_1 - x_2, x_1 + x_2, x_3)$.

4. Define $F(x, y) = (e^{xy} + 2x, y^2 + \sin(x - y))$ for $(x, y) \in \mathbb{R}^2$. Find the derivative matrix of the mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ at the points $(0, 0)$ and $(\pi, 0)$.

5. Suppose that $F : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable and that the derivative matrix $DF(\mathbf{x})$ has all entries equal to 0 for all $\mathbf{x} \in \mathbb{R}^2$. Prove that $F : \mathbb{R}^2 \to \mathbb{R}^2$ is constant, i.e. there is some $\mathbf{c} \in \mathbb{R}^2$ such that $F(\mathbf{x}) = \mathbf{c}$. (Feel free to assume anything from your previous calculus courses, but state what you are assuming.)