Practice Final

1. For
$$A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$
   (a) Find the characteristic equation of A.
   (b) What are the eigenvalues of A?
   (c) Find eigenvectors for both eigenspaces.

   Solutions:

   (a) The characteristic equation is
$$\lambda^2 - 3\lambda - 4 = 0.$$
   (b) The eigenvalues are $\lambda = 4$ and $\lambda = -1$.
   (c) A basis for the eigenspace associated with $\lambda = 4$ is
$$\begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$
   A basis for the eigenspace associated with $\lambda = -1$ is
$$\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

2. The characteristic equation for
$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$
   is $\lambda^3 + 3\lambda + 2 = 0$. The solutions for this equation are $\lambda = 2, \lambda = -1, \lambda = -1$. Find three linearly independent eigenvectors of A (one for $\lambda = 2$ and two for $\lambda = -1$).

   Solutions: An eigenvector for A corresponding to $\lambda = 2$ is
$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$
   Two linearly independent eigenvectors associated with $\lambda = -1$ are
$$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

3. Let $P_{\text{stan}\to B}$ be the transition matrix from the standard basis to B. If
$$P_{\text{stan}\to B} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}$$
   then what is B (written in terms of the standard basis)?

   Solutions: We have
$$P_{B\to\text{stan}} = (P_{\text{stan}\to B})^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$
   Then from our formula for computing $P_{B\to\text{stan}}$ we must have:
$$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

4. Suppose that the characteristic equation for a square matrix A is

$$p(\lambda) = (\lambda - 1)(\lambda - 3)^2(\lambda - 4)^3.$$

   (a) What is the size of A?
   (b) Is A invertible?
   (c) How many different eigenspaces does A have?

   Solution:

   (a) Note that the total power on λ is 6, so A is a 6×6 matrix.
   (b) A is invertible since 0 is not a root of p.
   (c) There are 3 eigenspaces of A.

5. Let B be the basis

$$B = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

   Suppose that when A is written in terms of B, it takes the form

$$A = \begin{pmatrix} 4 & 1 & -3 \\ -1 & 8 & -3 \\ -1 & -1 & 6 \end{pmatrix}.$$

   What is A written in terms of the standard basis?

   Solution: We easily get that

$$P_{B \to \text{stan}} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

   Hence

$$P_{\text{stan} \to B} = (P_{B \to \text{stan}})^{-1} = \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{pmatrix}.$$

   Hence, written in the standard basis:

$$P_{B \to \text{stan}} A P_{\text{stan} \to B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Some book problems on older material: There are more problems that you will be able to do, so pick the problems related to material that you find challenging:

- Section 1.1: # 15, 16.
- Section 1.2: # 5-8, 15, 16, 27, 28.
- Section 1.3: # 5, 13, 25.
- Section 1.4: # 10, 15, 16, 25-28, 48.
- Section 1.5: # 13-15 (use any method you want to find the inverse).
- Section 1.8: # 21-24, 27, 28.
- Section 2.1: # 15-18, 21-24 (use any method you want), 38, 39.
- Section 2.3: # 7-10, 15-18, 34.
- Section 3.2: # 5, 6, 26.

- Section 3.3: # 15-20, 29, 34.
- Section 4.1: # 9-11.
- Section 4.2: # 4, 5, 11, 15, 19.
- Section 4.3: # 2, 7, 12, 14, 24.
- Section 4.4: # 11, 13, 20, 29.
- Section 4.5: # 15, 17, 18.
- Section 4.7: # 9-10, 14-17, 27.
- Section 4.8: # 3-5, 7, 18, 21, 29.