MATH 417 Homework 1

Due: Friday, August 31, in class.
Note that this is problems: Section 10.1: #5, #8, #13; Section 10.2 #3, 8a; Section 10.3 #9, 12.

10.1

5. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Prove that

$$\langle \mathbf{u}, \mathbf{v} \rangle = \frac{||\mathbf{u}+\mathbf{v}||^{\mathbf{2}} - ||\mathbf{u}-\mathbf{v}||^{\mathbf{2}}}{\mathbf{4}}.$$

Proof: Using the definition of the norm on $\mathbb{R}^n$ we have

$$||\mathbf{u}+\mathbf{v}||^{\mathbf{2}} - ||\mathbf{u}-\mathbf{v}||^{\mathbf{2}} = \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v} \rangle - \langle \mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v} \rangle.$$

By the linearity and symmetry of the scalar product it follows that

$$\begin{aligned} \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v} \rangle - \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v} \rangle &= \big(\langle \mathbf{u}, \mathbf{u} \rangle + \mathbf{2}\langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle\big) - \big(\langle \mathbf{u}, \mathbf{u} \rangle - \mathbf{2}\langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle\big) \\ &= 4\langle \mathbf{u}, \mathbf{v} \rangle. \end{aligned}$$

Dividing both sides by 4 gives the desired equality.

8. Let $\mathbf{u} = (\mathbf{a}, \mathbf{b})$ and $\mathbf{v} = (\mathbf{c}, \mathbf{d})$ be nonzero points in the plane $\mathbb{R}^2$ and let θ be the radian measure of the angle with vertex at $\mathbf{0}$ formed by $\mathbf{0}$, $\mathbf{u}$, and $\mathbf{v}$.

   (a) Prove that
   $$||\mathbf{u}||^2||\mathbf{v}||^2 - (\langle \mathbf{u}, \mathbf{v} \rangle)^2 = (||\mathbf{u}||\ ||\mathbf{v}|| \sin(\theta))^2.$$

   (b) Express the left-hand side of the above equation in components to obtain
   $$|ad - bc| = |\ ||\mathbf{u}||\ ||\mathbf{v}|| \sin(\theta)|.$$

   (c) Use (b) to verify that $|ad - bc|/2$ is the area of the triangle with vertices $\mathbf{0}$, $\mathbf{u}$, and $\mathbf{v}$ and that, as a consequence, $|ad - bc|$ is the area of the parallelogram with vertices $\mathbf{0}$, $\mathbf{u}$, $\mathbf{u}+\mathbf{v}$, and $\mathbf{v}$.

Solution:

(a) Recall the identity $\cos(\theta)^2 = 1 - \sin(\theta)^2$. From Proposition 10.3 in the textbook
$$\langle \mathbf{u}, \mathbf{v} \rangle = ||\mathbf{u}||\ ||\mathbf{v}|| \cos(\theta).$$
Squaring both sides gives,
$$\langle \mathbf{u}, \mathbf{v} \rangle^2 = ||\mathbf{u}||^2\ ||\mathbf{v}||^2 \cos(\theta)^2 = ||\mathbf{u}||^2\ ||\mathbf{v}||^2 (1 - \sin(\theta)^2).$$
Subtracting $||\mathbf{u}||^2\ ||\mathbf{v}||^2$ from both sides and multiplying both sides by (-1) then gives the result.

(b) In components, the left side gives
$$(a^2 + b^2)(c^2 + d^2) - (ac + bd)^2 = a^2d^2 + b^2c^2 - 2acbd = (ad - bc)^2.$$
Taking square roots of both sides then gives the result.

(c) Recall that the formula for the area of a parallelogram of height h and base length b is bh. The parallelogram with vertices $\mathbf{0}$, $\mathbf{u}$, $\mathbf{u}+\mathbf{v}$, and $\mathbf{v}$ can be described as having base length $||\mathbf{u}||$ and height $||\mathbf{v}|| \sin(\theta)$ (or equivalently, having base length $||\mathbf{v}||$ and height $||\mathbf{u}|| \sin(\theta)$). Hence by (b), the area of this parallelogram is
$$bh = \Big| ||\mathbf{u}||\ ||\mathbf{v}|| \sin(\theta) \Big| = |ad - bc|.$$

13. Given two continuous functions $f : [0,1] \to \mathbb{R}$ and $g : [0,1] \to \mathbb{R}$, we define the scalar product of f and g, denoted by $\langle f, g\rangle$, by the formula

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx.$$

   (a) Verify that this scalar product has the properties of the scalar product in $\mathbb{R}^n$, i.e. for $f, g, h : [0,1] \to \mathbb{R}$ as above and $\alpha, \beta \in \mathbb{R}$,

      i. $\langle f, g\rangle = \langle g, f\rangle$,
      ii. $\langle \alpha f + \beta g, h\rangle = \alpha\langle f, h\rangle + \beta\langle g, h\rangle$.

   (b) Follow the proof of the Cauchy-Schwarz inequality for points in $\mathbb{R}^n$ to prove that

$$\left|\int_0^1 f(x)g(x)dx\right| \leqslant \sqrt{\int_0^1 [f(x)]^2dx}\sqrt{\int_0^1 [g(x)]^2dx}.$$

Solution:

   (a) The first property follows from the fact that for any functions $f, g : [0,1] \to \mathbb{R}$, $f(x)g(x) = g(x)f(x)$ for all $x \in [0,1]$ and hence

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx = \int_0^1 g(x)f(x)dx = \langle g, f\rangle.$$

   The second property follows from the fact that the integral is linear, that is,

$$\langle \alpha f + \beta g, h\rangle = \int_0^1 (\alpha f(x) + \beta g(x))h(x)dx$$

$$\int_0^1 \alpha f(x)h(x) + \beta g(x)h(x)dx = \alpha\int_0^1 f(x)h(x)dx + \beta\int_0^1 g(x)h(x)dx = \alpha\langle f, h\rangle + \beta\langle g, h\rangle.$$

   (b) If any of the three integrals is zero then both sides of the inequality are zero and we are done (showing this takes quite a bit of machinery that we don't yet have, specifically *measure theory*). So assume all the integrals are non-zero, in particular, set

$$c = \int_0^1 f(x)g(x)dx \quad \text{and} \quad d = \int_0^1 g(x)^2dx.$$

   These integrals are finite since $[0,1]$ is compact and we are assuming f and g are continuous. Define a function $h : [0,1] \to \mathbb{R}$ by $h = f - \frac{c}{d}g$. This function is continuous. One can also check that

$$\int_0^1 g(x)h(x)dx = 0$$

   by direct calculation from the definition of h. Since $h(x)^2$ is a positive function, we can use the properties of this scalar product in (a) to show that

$$0 \leqslant \int_0^1 h(x)^2 = \int_0^1 f(x)(f(x) - \frac{c}{d}g(x))dx = \int_0^1 f(x)^2dx - \frac{c}{d}\int_0^1 f(x)g(x)dx.$$

   The inequality then follows from recalling the definition of c and d, rearranging terms, and taking a square root.

10.2

3. Suppose that $\{\mathbf{u}_k\}$ is a sequence of points in $\mathbb{R}^n$ that converges to the point $\mathbf{u}$. Prove that the sequence of real numbers $\{||\mathbf{u}_k||\}$ converges to $||\mathbf{u}||$.

Solution: We first show that

$$\Big|\, ||\mathbf{u}_k|| - ||\mathbf{u}|| \,\Big| \leqslant ||\mathbf{u}_k - \mathbf{u}||.$$

For some $k \geqslant 0$, assume that $||\mathbf{u}|| \geqslant ||\mathbf{u}_k||$ (the case where $||\mathbf{u}|| \leqslant ||\mathbf{u}_k||$ is completely analogous). By the triangle inequality, we have

$$||\mathbf{u}|| = ||\mathbf{u}_k + (\mathbf{u} - \mathbf{u}_k)|| \leqslant ||\mathbf{u}_k|| + ||\mathbf{u} - \mathbf{u}_k||.$$

Subtracting $||\mathbf{u}_k||$ from both sides gives

$$||\mathbf{u}|| - ||\mathbf{u}_k|| \leqslant ||\mathbf{u} - \mathbf{u}_k||.$$

Since $||\mathbf{u}|| \geqslant ||\mathbf{u}_k||$ then we can take the absolute value of both sides to get

$$\Big| \; ||\mathbf{u}_k|| - ||\mathbf{u}|| \; \Big| \leqslant ||\mathbf{u}_k - \mathbf{u}||.$$

Now suppose that $\{\mathbf{u}_k\}$ converges to $\mathbf{u}$. Then for any $\epsilon > 0$ there is K such that for all $k \geqslant K$,

$$dist(\mathbf{u}_k, \mathbf{u}) < \epsilon.$$

But by the reverse triangle equality above, we have that for all $k \geqslant K$,

$$dist(||\mathbf{u}_k||, ||\mathbf{u}||) = \Big| \; ||\mathbf{u}_k|| - ||\mathbf{u}|| \; \Big| \leqslant ||\mathbf{u}_k - \mathbf{u}|| = dist(\mathbf{u}_k, \mathbf{u}) < \epsilon$$

which means that the sequence of real numbers $\{||\mathbf{u}_k||\}$ converges to $||\mathbf{u}||$.

8.a A sequence of points $\{\mathbf{u}_k\}$ in $\mathbb{R}^n$ is said to be a *Cauchy sequence* provided that for each positive ϵ there is an index K such that

$$dist(\mathbf{u}_k, \mathbf{u}_l) < \epsilon \qquad \text{if } k \geqslant K \text{ and } l \geqslant K.$$

Prove that $\{\mathbf{u}_k\}$ is a Cauchy sequence if and only if each component sequence is a Cauchy sequence.

Solution: Suppose that $\{\mathbf{u}_k\}$ is Cauchy so that for any $\epsilon > 0$ there is K such for all $k, \ell \geqslant K$,

$$dist(\mathbf{u}_k, \mathbf{u}_\ell) < \epsilon.$$

By the properties of the component projection

$$|p_i(\mathbf{u}_k) - p_i(\mathbf{u}_\ell)| = |p_i(\mathbf{u}_k - \mathbf{u}_\ell)| \leqslant ||\mathbf{u}_k - \mathbf{u}_\ell|| = dist(\mathbf{u}_k, \mathbf{u}_\ell) < \epsilon.$$

Since we can do this for any $\epsilon > 0$ and any $1 \leqslant i \leqslant n$, it follows that each component sequence is Cauchy.

Now assume that each component sequence is Cauchy. Then for a value $\epsilon' = \frac{\epsilon}{\sqrt{n}} > 0$ and $1 \leqslant i \leqslant n$ there is K_i such that for $k, \ell \geqslant K_i$,

$$|p_i(\mathbf{u}_k) - p_i(\mathbf{u}_k)| < \epsilon'.$$

Note that by picking $K := \max_{1 \leqslant i \leqslant n} K_i$ we get for all $k, \ell \geqslant K$

$$|p_1(\mathbf{u}_k) - p_1(\mathbf{u}_\ell)| + \cdots + |p_n(\mathbf{u}_k) - p_n(\mathbf{u}_\ell)| < (\epsilon')^2 + \cdots + (\epsilon')^2 = n(\epsilon')^2.$$

But

$$||\mathbf{u}_k - \mathbf{u}_\ell||^2 = |p_1(\mathbf{u}_k) - p_1(\mathbf{u}_\ell)| + \cdots + |p_n(\mathbf{u}_k) - p_n(\mathbf{u}_\ell)|$$

and

$$n(\epsilon')^2 = \epsilon^2.$$

So we have that for all $k, \ell \geqslant K$,

$$||\mathbf{u}_k - \mathbf{u}_\ell||^2 < \epsilon^2$$

and hence

$$||\mathbf{u}_k - \mathbf{u}_\ell|| < \epsilon.$$

10.3

1. Let A and B be subsets of $\mathbb{R}^n$ with $A \subseteq B$.
   (a) Prove that $intA \subseteq intB$.
   (b) Is it necessarily true that $bdA \subseteq bdB$?

Solution:

(a) Supposed that $\mathbf{u} \in intA$. Then by definition there exists some $r > 0$ such that $B_r(\mathbf{u}) \subseteq A$. But since $A \subseteq B$, then $B_r(\mathbf{u}) \subseteq B$ and hence $\mathbf{u}$ is an interior point of B. It follows that $intA \subseteq intB$.

(b) No, it is not necessarily true. Consider the case where $A = \{0\} \subset \mathbb{R}$ and $B = [-1, 1] \subset \mathbb{R}$. Then $A \subset B$ but $bdA = \{0\}$ and $bdB = \{-1, 1\}$.

2. For a subset A of $\mathbb{R}^n$, the *closure* of A, denoted by clA, is defined by

$$clA = intA \cup bdA.$$

Prove that $A \subseteq clA$ and that $A = clA$ if and only if A is closed in $\mathbb{R}^n$.

Solution: We first prove that $A \subseteq clA$. Suppose that $\mathbf{u} \in A$. Then for any ball $B_r(\mathbf{u})$ about $\mathbf{u}$, $B_r(\mathbf{u})$ always contains at least one point of A, namely $\mathbf{u}$. Then there are two cases: either we can find a sufficiently small $r > 0$ such that $B_r(\mathbf{u})$ is contained in A or we cannot find such an r. In the first case $\mathbf{u} \in intA$ and in the second $\mathbf{u} \in bdA$. So $\mathbf{u} \in clA$ and $A \subseteq clA$.

Next we show that $A = clA$ if and only if A is closed in $\mathbb{R}^n$. First consider the case where $A = clA$. It follows then that $bdA \subseteq A$. Thus by Proposition 10.9, A is closed.

On the other hand if A is closed then by Proposition 10.9 $bdA \subseteq A$. Since $intA \subseteq A$, then $clA \subseteq A$ and thus by the first part of this problem $A = clA$.