MATH 417 Homework 8

Note that this is problems: Section 14.3 #1,2,8; Section 15.1 #1,4,6.

1. Analyze the local extrema of the following functions:

   (a) $f(x, y) = e^{x^2-4y+y^2}$ for $(x, y) \in \mathbb{R}^2$,

   (b) $g(x, y, z) = e^{x^2-4y+y^2} + z^2$ for $(x, y, z) \in \mathbb{R}^3$,

   (c) $f(x, y) = (x^2 + y^2)e^{x^2+y^2}$ for $(x, y) \in \mathbb{R}^2$,

   (d) $f(x, y) = x^3y^2(6 - x - y)$ for $(x, y) \in \mathbb{R}^2$,

   Solution:

   (a) We have
   $$\nabla f(x, y) = (2xe^{x^2-4y+y^2}, (-4 + 2y)e^{x^2-4y+y^2}).$$
   This is equal to $(0, 0)$ when $(x, y) = (0, 2)$. At this point the Hessian is
   $$\nabla^2 f(0, 2) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$
   This matrix is positive definite and hence $(0, 2)$ is a local minimizer of f.

   (b) We have
   $$\nabla g(x, y, z) = (2xe^{x^2-4y+y^2}, (-4 + 2y)e^{x^2-4y+y^2}, 2z).$$
   This is zero when $(x, y, z) = (0, 2, 0)$. At $(0, 2, 0)$ the Hessian is
   $$\nabla^2 g(0, 2, 0) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$
   This matrix is positive definite and hence $(0, 2, 0)$ is a local minimizer of g.

   (c) We have
   $$\nabla f(x, y) = \left(2x(x^2 + y^2 + 1)e^{x^2+y^2}, 2y(x^2 + y^2 + 1)e^{x^2+y^2}\right).$$
   This is equal to $(0, 0)$ when $(x, y) = (0, 0)$. At $(0, 0)$ the Hessian is
   $$\nabla^2 f(0, 0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$
   So this point is a local minimizer of f.

   (d) We have
   $$\nabla f(x, y) = \left(x^2y^2(18 - 4x - 3y), x^3y(12 - 2x - 3y)\right).$$
   This is equal to $(0, 0)$ when $(x, y) = (0, 0)$ or $(x, y) = (3, 2)$. At $(0, 0)$ the Hessian is
   $$\nabla^2 f(0, 0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$
   Therefore we cannot say if this point is a maximizer or minimizer of f. At $(3, 2)$ the Hessian is
   $$\nabla^2 f(0, 0) = \begin{bmatrix} -504 & -108 \\ -108 & -162 \end{bmatrix}.$$
   We notice that the top left element is negative and the determinant of this matrix is positive, then this point is a strict maximizer of f.

Suppose that

2. Find necessary and sufficient conditions for a 2×2 symmetric matrix to be negative definite. Use this information to state and prove a sufficient condition for a point to be a local maximizer for a function of two variables.

   Solution: A 2×2 matrix
   $$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$
   is negative definite if $a < 0$ and $ac - b^2 > 0$. The argument is completely analogous to Proposition 14.15. The function $Q : \mathbb{R}^2 \to \mathbb{R}^2$ associated with A is
   $$Q(x, y) = ax^2 + 2bxy + cy^2 \qquad \text{for } (x, y) \in \mathbb{R}^2.$$
   For points (x, y) with $y \neq 0$, set $t = x/y$ and $p(t) = at^2 + 2bt + c$. Observe that
   $$Q(x, y) = y^2(a(x/y)^2 + 2b(x/y) + c) = y^2 p(t).$$
   The polynomial $p(t)$ is negative for all t if and only if $a < 0$ and $ac - b^2 > 0$. If $y = 0$ then $Q(x, 0) = ax^2 < 0$ if and only if $a < 0$.

   Applying this to Theorem 14.22 we get that $\mathbf{x} \in \mathbb{R}^2$ is a strict local maximizer for $f : \mathbb{R}^2 \to \mathbb{R}$ if

   (a) $\frac{\partial f}{\partial x}(\mathbf{x}) = 0$ and $\frac{\partial f}{\partial y}(\mathbf{x}) = 0$

   (b) $\frac{\partial^2 f}{\partial x^2}(\mathbf{x}) < 0$ and
   $$\frac{\partial^2 f}{\partial x^2}(\mathbf{x})\frac{\partial^2 f}{\partial y^2}(\mathbf{x}) - \left(\frac{\partial^2 f}{\partial x \partial y}(\mathbf{x})\right)^2 > 0.$$

8. Suppose that the function $f : \mathbb{R}^n \to \mathbb{R}$ has continuous second-order partial derivatives. Let $\mathbf{x}$ be a point in $\mathbb{R}^n$ at which $\nabla f(\mathbf{x}) = 0$ and such that all entries of the Hessian matrix $\nabla^2 f(\mathbf{x})$ are also 0. By giving specific examples, show that it is possible for the point $\mathbf{x}$ to be a local maximum, a local minimum, or neither.

   Solution: Consider the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by
   $$f(x_1, \dots, x_n) = c$$
   for all $(x_1, \dots, x_n) \in \mathbb{R}^n$ with $c \in \mathbb{R}$ some constant. Then every point $\mathbf{x}$ is both a local maximizer and a local minimizer (though not strict). Furthermore, it is easy to check that $\nabla f(\mathbf{x}) = \mathbf{0}$ and $\nabla^2 f(\mathbf{x})$ is the matrix of zeros.

   Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x^3 + y^3$. The point $(x, y) = (0, 0)$ is not a maximizer or minimizer for f. At this point you can check that $\nabla f(0, 0) = 0$ and $\nabla^2 f$ is the matrix of zeros.

1. Which of the following mappings $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ is linear.

   (a) $\mathbf{F}(x, y) = (-y, e^x)$ for $(x, y) \in \mathbb{R}^2$,

   (b) $\mathbf{F}(x, y) = (x - y^2, 2y)$ for $(x, y) \in \mathbb{R}^2$,

   (c) $\mathbf{F}(x, y) = 17(x, y)$ for $(x, y) \in \mathbb{R}^2$,

   Solution: (a) and (b) are not linear. (c) is linear.

4. Show that there is no linear mapping $\mathbf{T} : \mathbb{R}^2 \to \mathbb{R}^2$.

   Solution: We use proof by contradiction. Suppose that $\mathbf{T}$ is linear. Then
   $$(0, 1) = \mathbf{T}(-2, -2) = -2\mathbf{T}(1, 1) = -2(4, 0) = (-8, 0),$$
   a contradiction. Hence $\mathbf{T}$ cannot be linear.

6. For a point (x, y) in the plane $\mathbb{R}^2$, define $\mathbf{T}(x, y)$ to be the point on the line $\ell = \{(x, y) \in \mathbb{R}^2 \mid y = 2x\}$ that is closest to (x, y). Show that the mapping $\mathbf{T} : \mathbb{R}^2 \to \mathbb{R}^2$ is linear and find the 3×3 matrix associated to this mapping.

Solution: Given a point $(x_0, y_0) \in \mathbb{R}^2$. The shortest line from (x_0, y_0) to ℓ will intersect ℓ at the point on ℓ that is closest to (x_0, y_0) and this line will necessarily be orthogonal to ℓ. Hence the line should have slope $-\frac{1}{2}$ and must pass through the point (x_0, y_0). The equation such a line is

$$y - y_0 = -\frac{1}{2}(x - x_0)$$

that is

$$y = -\frac{1}{2}(x - x_0) + y_0.$$

We want the intersection of this line with ℓ. This gives the two equations

$$x = \frac{2}{5}x_0 + \frac{2}{5}y_0 \quad \text{and} \quad y = \frac{2}{5}x_0 + \frac{4}{5}y_0.$$

Hence, given a point (x_0, y_0), projection onto the closest point on ℓ is given by

$$\begin{bmatrix} \frac{2}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

It follows from this observation that the mapping is linear.