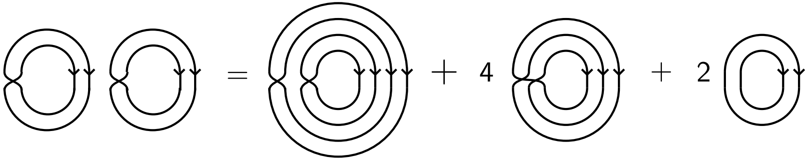
Khovanov's Heisenberg category, moments in free probability, and shifted symmetric functions.

Henry Kvinge, Colorado State

(Joint with Anthony Licata and Stuart Mitchell)

University of Virginia Algebra Seminar

September 27, 2017



Outline of the story

① Khovanov's Heisenberg category $\mathcal{H}'$

Result: $Z(\mathcal{H}') \cong \Lambda^*$

③ Asymptotic rep. theory of symmetric groups

② Shifted symmetric functions Λ^*

Khovanov's Heisenberg category $\mathcal{H}'$

Khovanov's Heisenberg category

Khovanov proposed a monoidal $\mathbb{C}$-linear category $\mathcal{H}'$, called *Khovanov's Heisenberg category* to categorify the Heisenberg algebra H.

$$\begin{tikzcd} \mathcal{H}' \arrow[d, rightsquigarrow, "K_0"] \\ H \hookrightarrow K_0(\mathcal{H}') \end{tikzcd}$$

category-theoretic structures

set-theoretic structures

Reminder: H is generated by $p_n, q_n \in \mathbb{Z}_{\geq 0}$ subject to the relations

$$p_n q_m = q_m p_n + \delta_{n,m} 1, \; p_n p_m = p_m p_n, \; q_n q_m = q_m q_n.$$

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$$
\begin{tikzcd}
 & \bigoplus_{n\geq 0} \mathbb{C}[S_n]\text{-mod} \arrow[d, squiggly, "K_0"] \\
H \arrow[r, "\substack{\text{Fock space}\\ \text{representation}}"] & \bigoplus_{n\geq 0} K_0(\mathbb{C}[S_n]\text{-mod}) \cong \Lambda
\end{tikzcd}
$$

The idea

Classical: Heisenberg algebra H acts on symmetric functions, (direct sum of Grothendieck group of symmetric groups by linear operators coming from induction/restriction functors).
New: Khovanov's *Heisenberg category* $\mathcal{H}'$ acts on representation category of symmetric groups by induction/restriction functors themselves.

$$
\begin{array}{ccc}
\mathrm{Kar}(\mathcal{H}') & \xrightarrow{\text{categorical representation}} & \bigoplus_{n\geq 0} \mathbb{C}[S_n]\text{-mod} \\
\Big\downarrow K_0 & & \Big\downarrow K_0 \\
H & \xrightarrow{\text{Fock space representation}} & \bigoplus_{n\geq 0} K_0(\mathbb{C}[S_n]\text{-mod}) \cong \Lambda
\end{array}
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--- ? ----------------------

Describing $\mathcal{H}'$

The category $\mathcal{H}'$:

Objects - Sequences of Q_+ and Q_-.

Morphisms - Oriented compact 1-manifolds immersed in the plane strip $\mathbb{R} \times [0,1]$. Up to isotopy.

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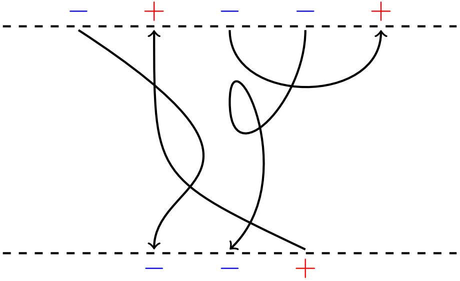
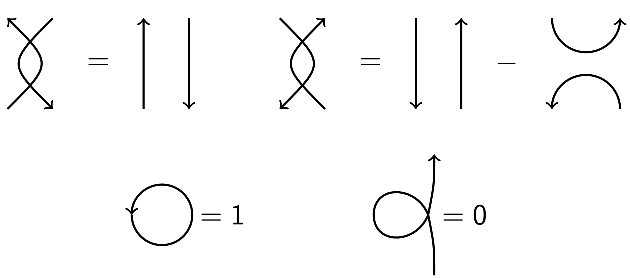
Example:

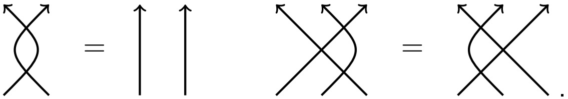
$$\begin{array}{c} \text{Object } Q_- Q_+ Q_- Q_- Q_+ \\ \Big\uparrow \text{morphism} \\ \text{Object } Q_- Q_- Q_+ \end{array}$$



Local relations



Motivation:
H relation
$Q_- Q_+ \cong Q_+ Q_- \oplus 1$



Motivation:
S_n relations

You should think:

- Q_+ is 'like' $\mathrm{Ind}_{S_n}^{S_{n+1}}$,
- Q_- is 'like' $\mathrm{Res}_{S_{n-1}}^{S_n}$,

$$Q_+ Q_- Q_- Q_+ Q_- \quad \text{is 'like'} \quad \mathrm{Ind}_{S_{n-2}}^{S_{n-1}} \mathrm{Res}_{S_{n-2}}^{S_{n-1}} \mathrm{Res}_{S_{n-1}}^{S_n} \mathrm{Ind}_{S_{n-1}}^{S_n} \mathrm{Res}_{S_{n-1}}^{S_n}$$

- Morphisms are 'like' natural transformations between compositions of these functors

Example:

Mackey decomposition for symmetric groups:

$$\mathrm{Ind}_{S_{n-1}}^{S_n} \circ \mathrm{Res}_{S_{n-1}}^{S_n} \oplus \mathrm{Id}_n \cong \mathrm{Res}_{S_n}^{S_{n+1}} \circ \mathrm{Ind}_{S_n}^{S_{n+1}} .$$

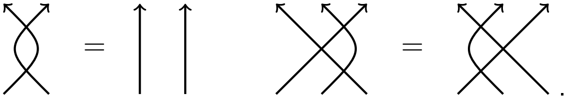
One can show that:

$$Q_+ Q_- \quad \oplus \mathbb{1} \quad \cong Q_- Q_+.$$

Heisenberg relation:

$$q_1 p_1 \quad +1 \quad = p_1 q_1.$$

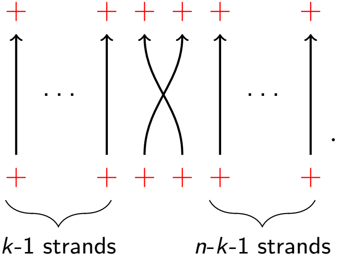
Relations



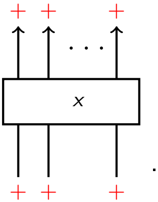
mean there are homomorphisms

$$\mathbb{C}[S_n] \to \mathrm{Hom}_{\mathcal{H}'}(\underbrace{Q_+ \dots Q_+}_{n}, \underbrace{Q_+ \dots Q_+}_{n})$$

$$\begin{array}{c}(k, k+1) = s_k \\ \pitchfork \\ S_n\end{array} \longmapsto$$



For $x \in \mathbb{C}[S_n]$, set

$x \quad \mapsto$ 

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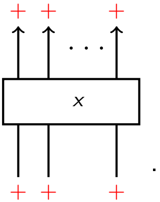
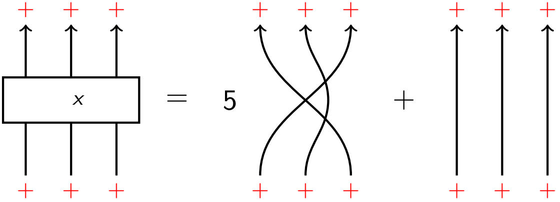
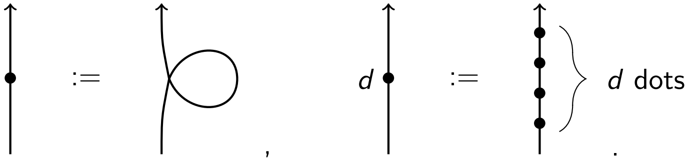
$x \quad \mapsto$ 

Example: In $\mathbb{C}[S_3]$, if $x = 5(1,3) + 1$ then



Dots

For simplification we write:

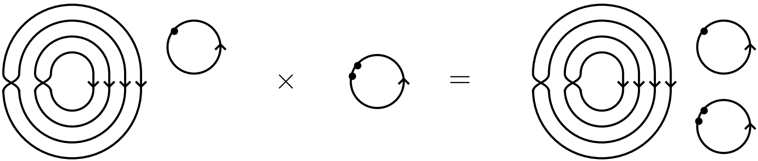


Acts on $\bigoplus_{n\geq 0} \mathbb{C}[S_n]$ - mod as multiplication by Jucys-Murphy elements.

$$J_0 := 0, \quad J_k := (1,k) + (2,k) + \cdots + (k-1,k).$$

Center of $\mathcal{H}'$

By definition the center $Z(\mathcal{H}')$ of $\mathcal{H}'$ is graphically the commutative $\mathbb{C}$-algebra of all closed diagrams.



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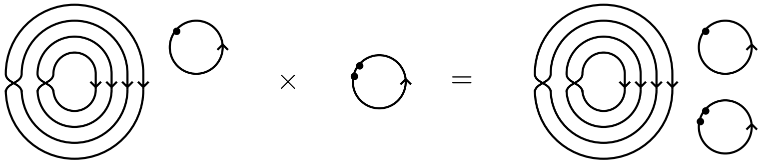


$\mathcal{H}'$ is **rich** in representation-theoretic data (morphism spaces contain all symmetric groups, affine degenerate Hecke algebras).

$$\Downarrow$$

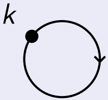
$Z(\mathcal{H}')$ should contain interesting information.

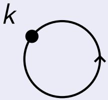
Center of $\mathcal{H}'$

Theorem (Khovanov)

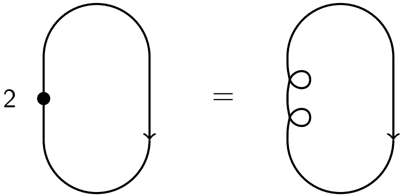
$$Z(\mathcal{H}') \cong \mathbb{C}[c_0, c_1, \dots] \cong \mathbb{C}[\tilde{c}_2, \tilde{c}_3, \dots]$$

where

$c_k \longmapsto$ 

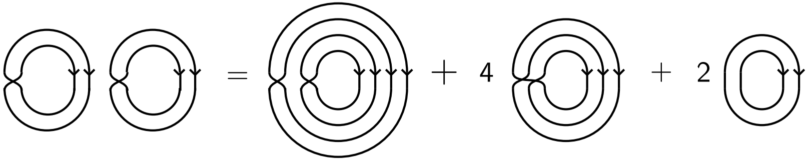
$\tilde{c}_k \longmapsto$ 

Reminder: A dot labelled with a k is k right-twisted curls.



Question: Is known combinatorics/structure encoded by the diagrammatics of $Z(\mathcal{H}')$?

Example: Is there a combinatorial interpretation of relations such as



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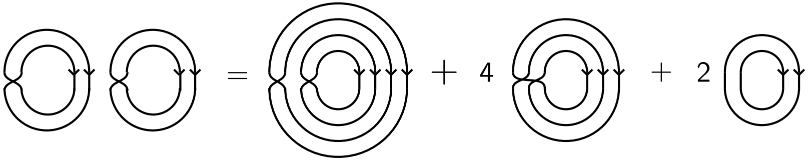

$$f_{(2)} \quad f_{(2)} \quad = \quad f_{(2,2)} \quad + \quad 4f_{(3)} \quad + \quad 2f_{(1,1)}$$

Guess: Corresponds to mulitiplication of some symmetric functions or analogues of symmetric functions, $\{f_\lambda\}$?

The shifted symmetric functions Λ^*

Shifted symmetric functions

Call $f \in \mathbb{C}[x_1, \ldots, x_n]$ *shifted symmetric* if it is symmetric in the new variables

$$x_i' = x_i - i.$$

Example: $x_1x_2 + x_2x_3 + x_1x_3 + x_2 + 2x_3 \quad \leftarrow$ shifted symmetric

$$\Downarrow \qquad x_1' = x_1 - 1, \;\; x_2' = x_2 - 2, \;\; x_3' = x_3 - 3$$

$(x_1'x_2' + x_2'x_3' + x_1'x_3') + 5(x_1' + x_2' + x_3') + 19 \quad \leftarrow$ symmetric

Shifted symmetric functions

The *algebra of shifted symmetric functions* Λ^* has similar construction to classical symmetric functions Λ.

Λ	Λ^*
$\Lambda = \varprojlim \Lambda_n$	$\Lambda^* = \varprojlim \Lambda^*_n$
elements symmetric	elements shifted symmetric
graded by polynomial degree	filtered by polynomial degree

Shifted symmetric functions

Λ^* has many generators/bases analogous to Λ:

- $p^{\#}_\lambda = p_\lambda +$ l.o.t., *shifted power sums*
- $s^*_\lambda = s_\lambda +$ l.o.t., *shifted Schur functions*
- $e^*_k = e_k +$ l.o.t., *elementary shifted functions*
- $h^*_k = h_k +$ l.o.t., *homogeneous shifted functions*

Shifted symmetric functions

Proposition (Okounkov-Olshanski)

$$gr(\Lambda^*) \cong \Lambda.$$

Shifted symmetric functions

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Example:

$$e_2^*(x_1, x_2, x_3) = \underbrace{\boxed{x_1 x_2 + x_2 x_3 + x_1 x_3}}_{\text{elementary symmetric polynomial } e_2} + x_2 + 2x_3$$

Λ^* as functions on Young diagrams $\mathcal{P}$

Λ^* can also be realized as a subalgebra of functions on Young diagrams $\mathcal{P}$.

Given $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ and $f \in \Lambda^*$ then,

$$f(\lambda) := f(\lambda_1, \lambda_2, \ldots, \lambda_r, 0, 0, \ldots).$$

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Example: $\lambda = (4, 2, 1)$,

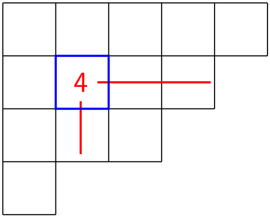
$e_2^*(x_1, x_2, x_3) = x_1 x_2 + x_2 x_3 + x_1 x_3 + x_2 + 2x_3$

$e_2^*(\lambda) = e_2^*(4, 2, 1) = 4 \cdot 2 + 2 \cdot 1 + 4 \cdot 1 + 2 + 2 \cdot 1$

Shifted Schur functions

Recall hook length

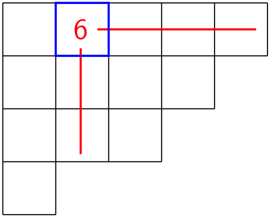
hook length($\square$) $= 4$



Shifted Schur functions

Recall hook length

hook length($\square$) = 6



Shifted Schur functions

Theorem: [Okounkov] If $|\lambda| \leq |\mu|$ then

$$s^*_\mu(\lambda) = \begin{cases} \prod_{\square \in \mu} (\text{hook length } \square) & \lambda = \mu \\ 0 & \text{otherwise} \end{cases}$$

Labeled hook lengths

8	6	5	3	1
6	4	3	1	
4	2	1		
1				

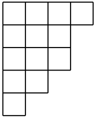
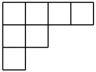
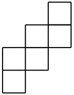
$s^*_{(5,4,3,1)}\Big($  $\Big) = 8 \cdot 6 \cdot 4 \cdot 1 \cdot 6 \cdot 4 \cdot 2 \cdot 5 \cdot 3 \cdot 1 \cdot 3 \cdot 1 \cdot 1$

Corollary: $\dim(L^\lambda) = \frac{n!}{s^*_\lambda(\lambda)}$

Shifted symmetric functions

Let λ/μ be a skew Young diagram, $\lambda \vdash n$, $\mu \vdash k$,

Example:

$\lambda =$  , $\mu =$  , $\lambda/\mu =$ 

[Okounkov-Olshanski]

$$\#\text{ standard tableaux of shape } \lambda/\mu = \frac{\dim(L^{\lambda})s^{*}_{\mu}(\lambda)}{n(n-1)\dots(n-k+1)}$$

Example: [Okounkov-Olshanski]

For $\mu \vdash k$, $\lambda \vdash n$,

$$p^{\#}_{\mu}(\lambda) = \begin{cases} \frac{(n \downharpoonleft k)}{\dim L^{\lambda}} \chi^{\lambda}(\mu, 1^{n-k}) & k \leq n \\ 0 & \text{otherwise.} \end{cases}$$

where

$$(n \downharpoonleft k) = n(n-1)\dots(n-k+1),$$

$$\dim(L^{\lambda}) = \text{dim. of simple } S_n\text{-rep } L^{\lambda}$$

$\chi^{\lambda}(\mu, 1^{n-k}) =$ value of character corresponding to simple representation L^{λ} on element of cycle type $(\mu, 1^{n-k})$.

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But $\quad p^{\#}_{\mu_1} p^{\#}_{\mu_2} \dots p^{\#}_{\mu_r} = p^{\#}_{(\mu_1, \mu_2, \dots, \mu_r)} +$ l.o.t.

Theorem (K., Licata, Mitchell)

$Z(\mathcal{H}')$ is isomorphic as a $\mathbb{C}$-algebra to the shifted symmetric functions Λ^*,

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To establish the isomorphism, we use the fact that for all $n \geq 0$ there are surjective homomorphisms.

$$\begin{tikzcd}
\Lambda^* \arrow[dr, two heads, "\text{Ivanov-Kerov}"'] & & Z(\mathcal{H}') \arrow[dl, two heads] \\
& Z(\mathbb{C}[S_n]) &
\end{tikzcd}$$

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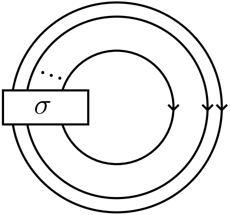
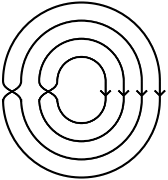
\end{tikzcd}$$

Dictionary between Λ^* and $Z(\mathcal{H}')$

If $\sigma \in S_n$ of conjugacy class λ, then

$p_\lambda^\# \longmapsto$ 

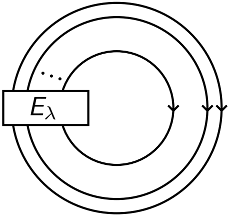
Example:

$p_{(2,2)}^\# \longmapsto$ 

Dictionary between Λ^* and $Z(\mathcal{H}')$

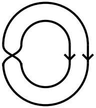
If E_λ is the central idempotent of $\mathbb{C}[S_n]$ associated to L^λ then

$$s^*_\lambda \longmapsto \frac{1}{\dim(L^\lambda)}$$

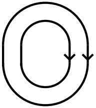


Example: $L^{(2)}$ is trivial representation for $\mathbb{C}[S_2]$ with idempotent $E_{(2)} = \frac{1}{2}(s_1 + 1)$ so

$$s^*_{(2)} \longmapsto \frac{1}{2}$$



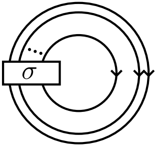
$$+ \ \frac{1}{2}$$



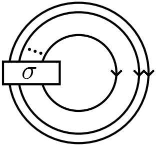
Dictionary between Λ^* and $Z(\mathcal{H}')$

Because $h_n^* = s_{(n)}^*$, $e_n^* = s_{(1^n)}^*$,

$$e_n^* \longmapsto \frac{1}{n!} \sum_{\sigma \in S_n}$$



$$h_n^* \longrightarrow \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{\mathrm{sgn}(\sigma)}$$



Involutions on Λ^*

Okounkov and Olshanski identified an involution $\xi : \Lambda^* \to \Lambda^*$ defined so that if $f \in \Lambda^*$:

$$\xi(f)(\lambda) := f(\lambda').$$

In particular

$$s^*_\lambda \longmapsto s^*_{\lambda'}$$

$$p^\#_k \longmapsto (-1)^k p^\#_k$$

$$e^*_k \longmapsto h^*_k$$

$$h^*_k \longmapsto e^*_k$$

Involutions on Λ^*

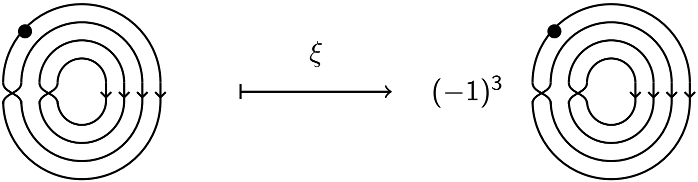
What is the diagrammatic interpretation?

For diagram $D \in Z(\mathcal{H}')$,

$$\xi(D) = (-1)^{c(D)} D$$

$c(D) := \#\{\text{dots and crossings in } D\}$

Example:



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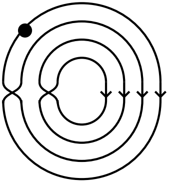
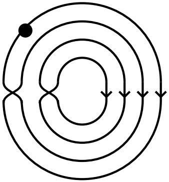
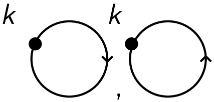


$$\xmapsto{\xi} \quad -$$



Remaining question:

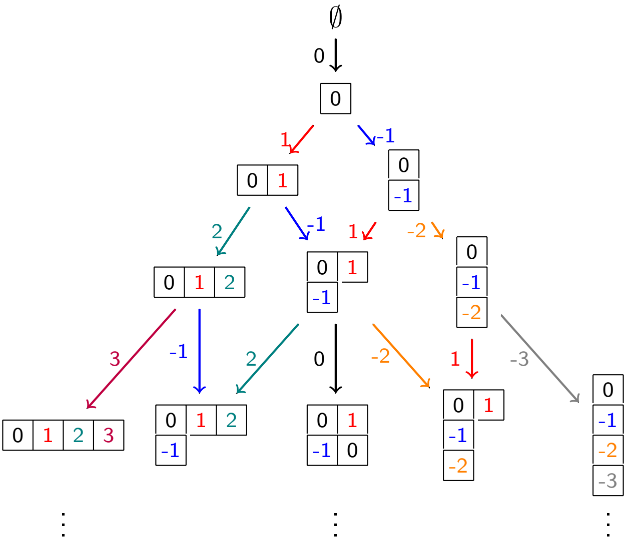


$$\longmapsto \quad ? \in \Lambda^*$$

For this we need to turn to asymptotic representation theory.

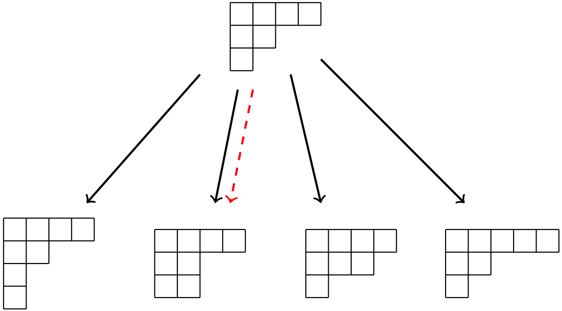
Moments of Kerov's transition measure $\widehat{m}_k$

Young's Lattice



Motivation for transition measure

Assign probability of moving from $\lambda = (4,2,1)$ to $\mu = (4,2,2)$?



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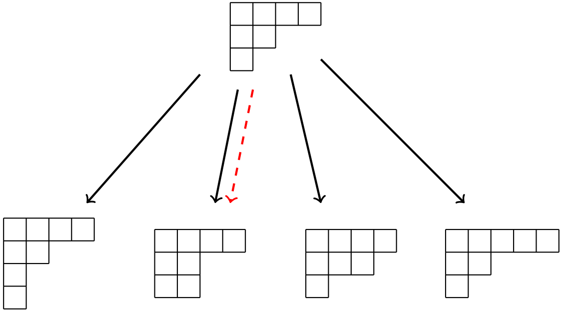

One choice is *transition probability*: $\widehat{q}_\lambda((4,2,2)) = \dfrac{\dim(L^{(4,2,2)})}{|\mu| \dim(L^\lambda)}$

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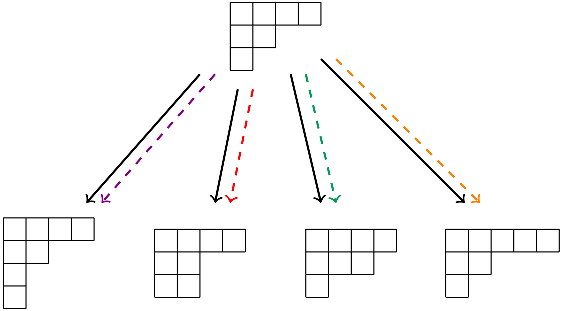
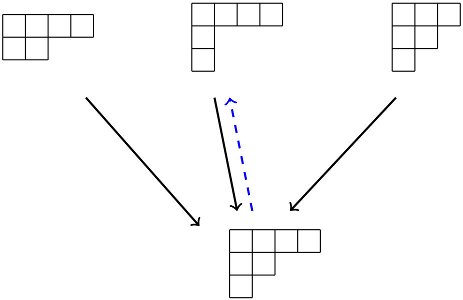


One choice is *transition probability*: $\widehat{q}_\lambda((4,2,2)) = \frac{\dim(L^{(4,2,2)})}{|\mu| \dim(L^\lambda)}$

$$\frac{\dim(L^{(4,2,1,1)})}{|\mu| \dim(L^\lambda)} + \frac{\dim(L^{(4,2,2)})}{|\mu| \dim(L^\lambda)} + \frac{\dim(L^{(4,3,1)})}{|\mu| \dim(L^\lambda)} + \frac{\dim(L^{(5,2,1)})}{|\mu| \dim(L^\lambda)} = 1$$

Motivation for co-transition measure

Dually, assign probability of moving from $\lambda = (4,2,1)$ to $\mu = (4,1,1)$?



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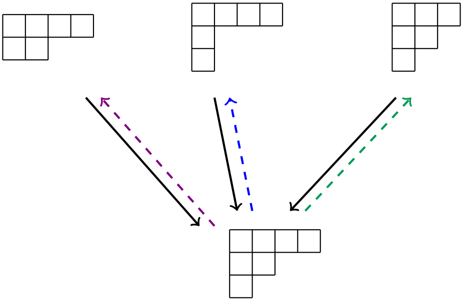


Co-transition probability: $\check{q}_\lambda((4,1,1)) = \frac{\dim(L^{(4,1,1)})}{\dim(L^\lambda)}$

$$\frac{\dim(L^{(4,2)})}{\dim(L^\lambda)} + \frac{\dim(L^{(4,1,1)})}{\dim(L^\lambda)} + \frac{\dim(L^{(3,2,1)})}{\dim(L^\lambda)} = 1$$

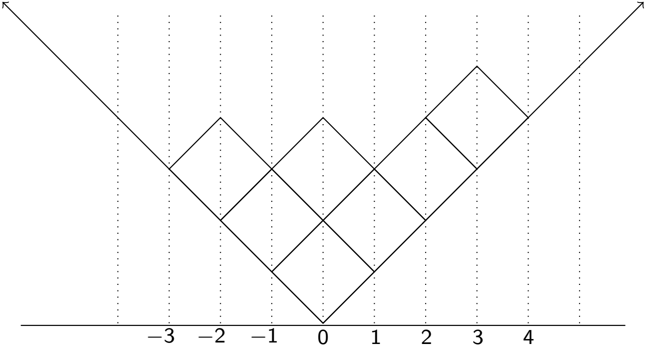
The transition/co-transition measure

For each λ, Kerov constructed two probability measures on $\mathbb{R}$ based on the transition/co-transition measure for λ and the interlacing coordinates of λ:

$$\widehat{\omega}_\lambda = \textit{ transition measure for } \lambda = \sum_{\lambda \nearrow \mu} \widehat{q}_\lambda(\mu) \delta_{\mathsf{content}(\mu/\lambda)}$$

$$\check{\omega}_\lambda = \textit{ co-transition measure for } \lambda = \sum_{\mu \nearrow \lambda} \check{q}_\lambda(\mu) \delta_{\mathsf{content}(\lambda/\mu)}$$

Example: $\lambda = (4, 2, 1)$.



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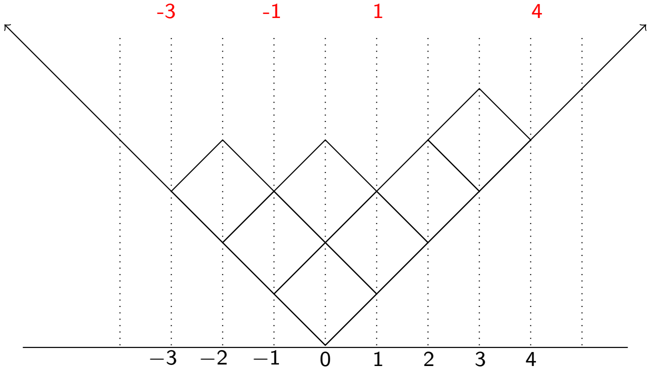

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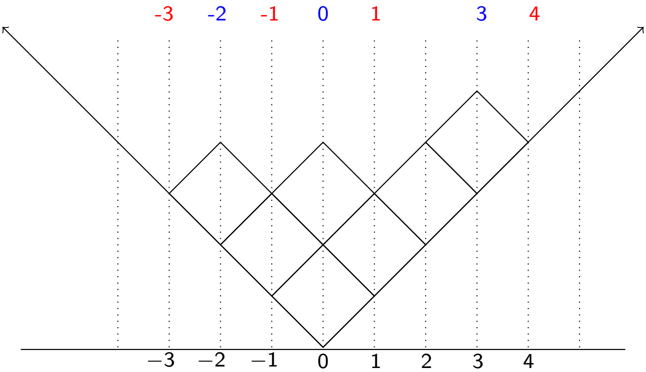
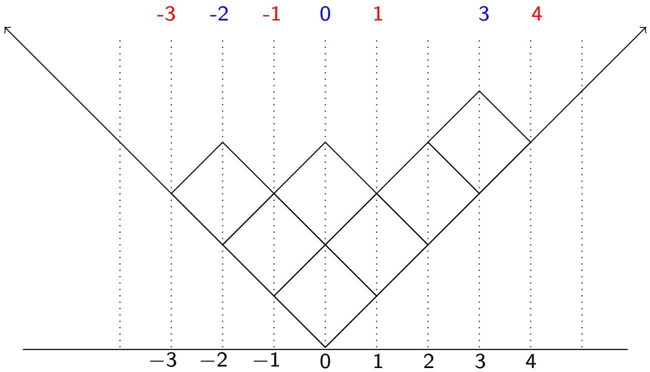


Example: $\lambda = (4, 2, 1)$.

$$\widehat{\omega}_\lambda = \quad \widehat{q}_\lambda((4,2,1,1))\delta_{-3} + \widehat{q}_\lambda((4,2,2))\delta_{-1} + \widehat{q}_\lambda((4,3,1))\delta_1 + \widehat{q}_\lambda((5,2,1))\delta_4$$



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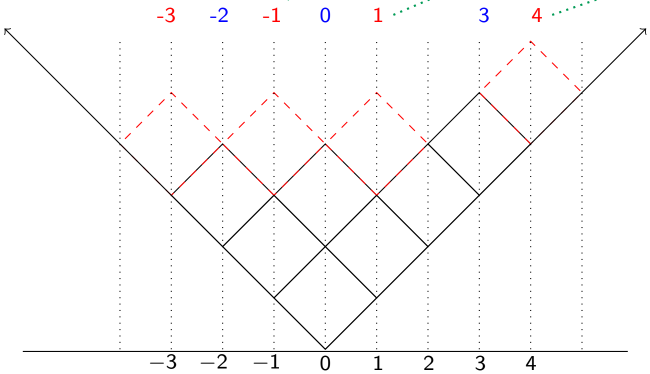
Contents of cells we can add



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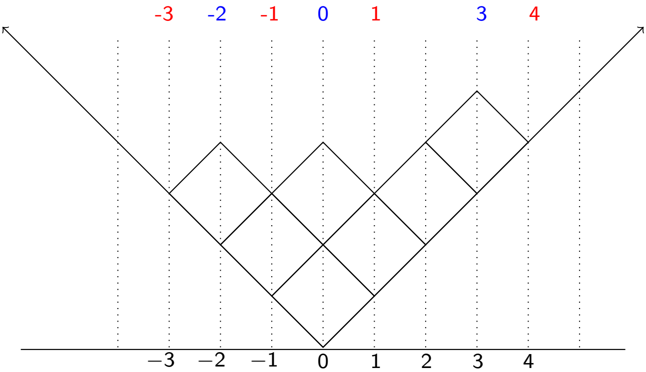
$$\breve{\omega}_\lambda = \breve{q}_\lambda((4,2))\delta_{-2} + \breve{q}_\lambda((4,1,1))\delta_0 + \breve{q}_\lambda((3,2,1))\delta_3$$



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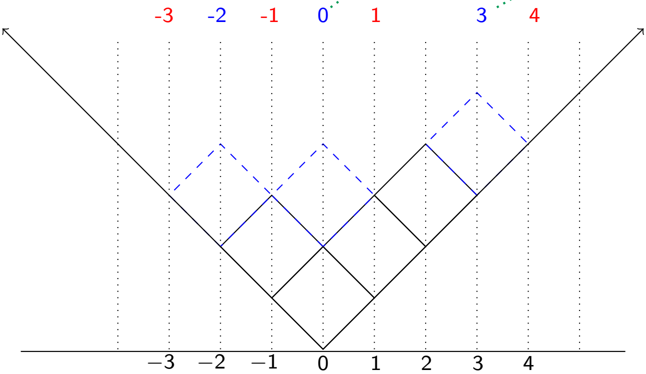
Contents of cells we can remove



Moments and Boolean cumulants

Can consider moments and cumulants for $\widehat{\omega}_\lambda$ and $\breve{\omega}_\lambda$.

- $\widehat{m}_k(\lambda) = k$th moment of $\widehat{\omega}_\lambda$,
- $\breve{m}_k(\lambda) = k$th moment of $\breve{\omega}_\lambda$.
- $\widehat{b}_k(\lambda) = k$th Boolean cumulant of $\widehat{\omega}_\lambda$.

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$$\lambda \overset{\widehat{m}_k}{\longmapsto} \widehat{m}_k(\lambda), \qquad \lambda \overset{\widecheck{m}_k}{\longmapsto} \widecheck{m}_k(\lambda), \qquad \lambda \overset{\widehat{b}_k}{\longmapsto} \widehat{b}_k(\lambda).$$

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[Lassalle] $\widehat{m}_k$, $p_1^{\#}\widecheck{m}_k$, $\widehat{b}_k$ belong to Λ^*.

An algebraic approach to $\widehat{m}_k$, $\check{m}_k$

From Biane's study of the asymptotic representation theory of symmetric groups and free probability:

- $\widehat{m}_k(\lambda) = \chi^\lambda(\mathrm{pr}_n(J_{n+1}^k))$
- $p_1^{\#}(\lambda)\check{m}_k(\lambda) = \chi^\lambda\Big(\sum_{i=1}^n s_i \dots s_{n-1} J_n^k s_{n-1} \dots s_i \Big)$

where J_{n+1} = Jucys-Murphy element $= \displaystyle\sum_{i=1}^{n+1} (i, n+1)$

$$\mathrm{pr}_n(\sigma) = \begin{cases} \sigma & \text{if } \sigma(n+1) = n+1 \\ 0 & \text{otherwise} \end{cases}$$

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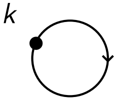
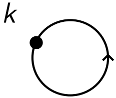
We use this interpretation to connect $\widehat{m}_k$, $\check{m}_k$ and $Z(\mathcal{H}')$.

Then



$$\longmapsto \qquad p_1^{\#}\check{m}_k = \widehat{b}_k \in \Lambda^*$$



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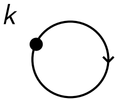
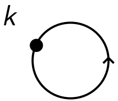


$$\longmapsto \quad \boxed{p_1^{\#}\check{m}_k = \widehat{b}_k} \in \Lambda^*$$

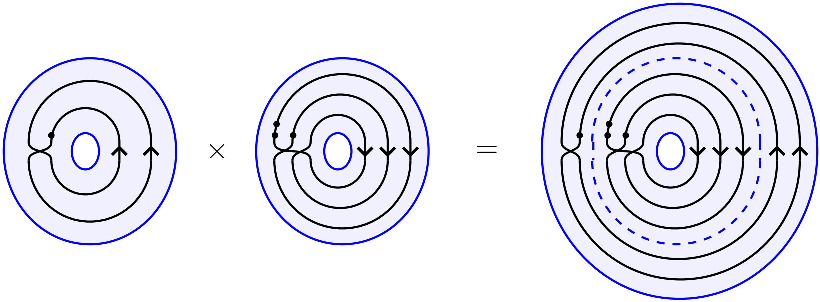


$$\longmapsto \quad \widehat{m}_k \in \Lambda^*$$

Question: Can any of the results of Kerov, Olshanski, Okounkov, Vershik, etc. be obtained purely diagrammatically?

$W_{1+\infty}$ and Λ^*

Another construction: The *trace* (or zeroth Hochschild homology) of the Heisenberg category $\mathrm{Tr}(\mathcal{H}')$ is a noncommutative algebra of diagrams on an annulus.



Cautis-Lauda-Licata-Sussan showed $\mathrm{Tr}(\mathcal{H}') \cong W_{1+\infty}$ the vertex algebra from conformal field theory.

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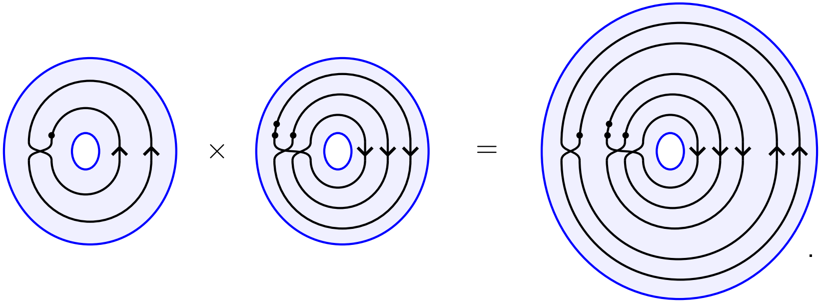

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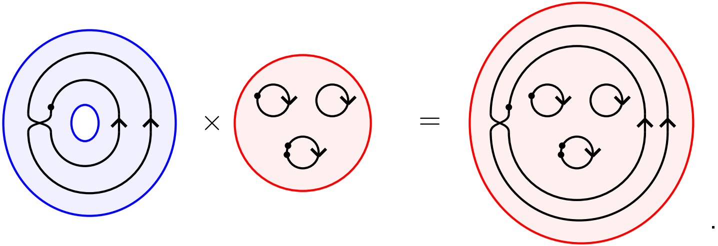
$W_{1+\infty}$ and Λ^*

There is a natural action of $\mathrm{Tr}(\mathcal{H}')$ on $Z(\mathcal{H}')$ by placing a closed diagram from $Z(\mathcal{H}')$ inside an annulus diagram from $\mathrm{Tr}(\mathcal{H}')$.



This gives purely planar realization of an action of $W_{1+\infty}$ on Λ^* which was first considered by Lascoux-Thibon.

Thank you.