MATH 369 Homework 5

Due: Thursday March 7, in class.

1. (a) Calculate the determinants of the matrices:

$$A = \begin{pmatrix} 1 & 4 & -1 \\ 2 & 0 & -1 \\ 7 & -10 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

   (b) Determine whether the matrices above are invertible.

   Solultion:

   (a) We have

$$\det(A) = 1(0-10) - 4(4+7) - 1(-20) = -34$$

   and

$$\det(B) = 1(48-45) - 2(36-42) + 3(32-35) = 0.$$

   (b) It follows from what we learned in class that since A has non-zero determinant, it is invertible, and since B has zero determinant, then it is non-invertible.

2. Let $n = 1,000,000$. Calculate $\det(I_n)$. Justify your answer using our rule for calculating determinants from class (or whatever your favorite method is).

   Solultion: Notice that when $n = 1$, then I_n is just 1 and $\det(I_n) = 1$. Now suppose that we know that $\det(I_{n-1}) = 1$. Then using our formula from class, we can write $\det(I_n)$ as a sum of elements from the first row of I_n times their corresponding cofactors.

$$\det(I_n) = a_{1,1}C_{1,1} + a_{1,2}C_{1,2} + \cdots + a_{1,n}C_{1,n}.$$

   But all $a_{1,1}, a_{1,2}, \ldots, a_{1,n}$ are zero except for $a_{1,1} = 1$. Hence we have

$$\det(I_n) = C_{1,1} = \det(I_{n-1}) = 1.$$

   Since we already know that $\det(I_1) = 1$, what we showed above implies that $\det(I_2) = 1$, which implies that $\det(I_3) = 1$, etc. In this way we can show that for any n (including $n = 1,000,000$), $\det(I_n) = 1$.

3. Find the values of λ such that

$$A = \begin{pmatrix} \lambda - 4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda - 1 \end{pmatrix}$$

   has $\det(A) = 0$.

   Solultion: We calculate

$$\det(A) = (\lambda - 4)(\lambda^2 - \lambda - 6).$$

   Since we want to find those λ for which $\det(A) = 0$. We need to find the roots of the above polynomial. Note that

$$\det(A) = (\lambda - 4)(\lambda^2 - \lambda - 6) = (\lambda - 4)(\lambda - 3)(\lambda + 2).$$

   Hence $\det(A) = 0$ when $\lambda = 4, 3, -2$.

4. For which values of θ is

$$A = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   invertible?

   Solultion: The determinant of A is

$$\det(A) = 1(\cos^2(\theta) + \sin^2(\theta)) = 1,$$

   so this matrix is invertible for all θ.

5. (a) Calculate the norm of the vector

$$\mathbf{v} = \begin{pmatrix} 6 \\ -1 \\ 2 \\ 0 \end{pmatrix}.$$

(b) Find a unit vector that points in the same direction as $\mathbf{v}$.

Solultion:

(a) The norm of this vector is

$$||\mathbf{v}|| = \sqrt{6^2 + (-1)^2 + 2^2 + 0^2} = \sqrt{41}.$$

(b) The unit vector that points in the same direction of $\mathbf{v}$ is

$$\frac{1}{||\mathbf{v}||}\mathbf{v} = \begin{pmatrix} \frac{6}{\sqrt{41}} \\ \frac{-1}{\sqrt{41}} \\ \frac{2}{\sqrt{41}} \\ 0 \end{pmatrix}.$$

6. Give a rough explanation of what the matrix

$$A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

does to a vector

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

when we multiply $A\mathbf{x}$. It may be helpful to choose a value of θ such as $\frac{\pi}{4}$ and look where A sends some easy vectors, such as $\mathbf{e}_1$ and $\mathbf{e}_2$. (Hint: this question relates to Homework 4).

Solultion: The matrix A takes a vector $\mathbf{x}$ in $\mathbb{R}^2$ and rotates it by θ radians counterclockwise.