Practice Exam 1

1. Consider the linear system:

$$\begin{array}{rcrcrcl} x & + & 3y & - & z & = & 1 \\ 4x & + & 2y & + & z & = & 2 \\ 5x & - & y & - & z & = & 3. \end{array}$$

This system has exactly one solution. Find it in two ways:

(a) By putting the corresponding matrix into reduced row echelon form.

(b) By finding the inverse of the corresponding coefficient matrix.

Solution:

(a) If we do Gauss-Jordan elimination we get

$$\left(\begin{array}{ccc|c} 1 & 3 & -1 & 1 \\ 4 & 2 & 1 & 2 \\ 5 & -1 & -1 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{11}{20} \\ 0 & 1 & 0 & \frac{1}{20} \\ 0 & 0 & 1 & -\frac{6}{20} \end{array}\right).$$

This of course tells us that $x = \frac{11}{20}$, $y = \frac{1}{20}$, $z = -\frac{6}{20}$.

(b) To solve using inverses we simply invert

$$A = \begin{pmatrix} 1 & 3 & -1 \\ 4 & 2 & 1 \\ 5 & -1 & -1 \end{pmatrix}$$

to get

$$A^{-1} = \frac{1}{40}\begin{pmatrix} -1 & 4 & 5 \\ 9 & 4 & -5 \\ -14 & 16 & -10 \end{pmatrix}.$$

Multiplying

$$\frac{1}{40}\begin{pmatrix} -1 & 4 & 5 \\ 9 & 4 & -5 \\ -14 & 16 & -10 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{11}{20} \\ \frac{1}{20} \\ -\frac{6}{20} \end{pmatrix}$$

which agrees with the solution that we found above.

2. (a) Find a value for a such that the augmented matrix

$$\begin{array}{rcrcrcl} 3x & + & 2y & + & z & = & 0 \\ 6x & + & y & + & 4z & = & 0 \\ 9x & + & 5y & + & az & = & 0 \end{array}$$

has exactly one free variable when put into reduced row echelon form.

(b) For the value of a you found above, write out parametric equations that give all solutions to the system.

Solution:

(a) Leaving a as an unknown we can start the process of Gauss-Jordan elimination to get

$$\left(\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 6 & 1 & 4 & 0 \\ 9 & 5 & a & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 0 & -3 & 2 & 0 \\ 9 & 5 & a & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 0 & -3 & 2 & 0 \\ 0 & -1 & a-3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 0 & -3 & 2 & 0 \\ 0 & 0 & a-\frac{11}{3} & 0 \end{array}\right).$$

For the system to have a free variable, we clearly need the last row to be zero. This will be the case when $a = \frac{11}{3}$.

(b) Setting $a = \frac{11}{3}$ we can put the corresponding augmented matrix in reduced row echelon form

$$\left(\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 6 & 1 & 4 & 0 \\ 9 & 5 & a & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & \frac{7}{9} & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

It follows that a parametrization of all solutions to the system is given by $x = -\frac{7t}{9}$, $y = \frac{2t}{3}$, $z = t$.

3. Solve the matrix equation

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -9 \\ -21 \end{pmatrix}$$

for x and y. Use any method that you like.

Solution: There are a few equivalent ways to do this. One is to calculate

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 1 & -4 \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ 3 & -12 \end{pmatrix}.$$

The inverse of this matrix is

$$\frac{1}{3} \begin{pmatrix} -12 & 5 \\ -3 & 1 \end{pmatrix}.$$

Hence the solution is

$$\frac{1}{3} \begin{pmatrix} -12 & 5 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} -9 \\ -21 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

4. Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 0 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 2 & 2 \\ 0 & -1 \end{pmatrix} \quad C = \begin{pmatrix} -1 & 4 \\ 0 & -2 \end{pmatrix}.$$

Calculate:

(a) $AB^T + C$ if it is defined. If it is not, state why.

(b) $tr(BA^T + C)$ if it is defined. If it is not, state why.

Solution:

(a) This is not defined. A is a 3×2 matrix and B^T is a 2×3 matrix. Therefore AB^T is a 3×3 matrix, which cannot be added to the 2×2 matrix C.

(b) This is not defined for the same reason.

5. A matrix A is called *orthogonal* if $A^{-1} = A^T$. Show that the matrix

$$A = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{pmatrix}$$

is orthogonal.

Solution: To show that A is orthogonal, we just need to show that A^T is the inverse of A. This can be done directly:

$$AA^T = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By a theorem from class, we do not need to also check $A^T A = I_3$.

6. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}.$$

Find

$$\mathbf{u} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \quad \mathbf{u}' = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

such that

$$A\mathbf{u} = A\mathbf{u}' = \mathbf{0}$$

and $\mathbf{u} \neq \mathbf{u}'$.

Solution: The point is that we showed in class that A was singular (not invertible). Therefore we should not be surprised that there are more than one $\mathbf{u}$ such that $A\mathbf{u} = \mathbf{0}$. One way to find some of these would be to put A into reduced row echelon form

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

Therefore all solutions to $A\mathbf{u} = \mathbf{0}$ take the form

$$\mathbf{u} = \begin{pmatrix} t \\ -t \end{pmatrix}.$$

Two choices are

$$\mathbf{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \mathbf{u} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}.$$