MATH 417 Homework 5

Note that this is problems: Section 13.3: #4, #6,#11, Section 14.1: #11, #15.

4. Suppose that the function $f : \mathbb{R}^n \to \mathbb{R}$ has first-order partial derivatives and that the point $\mathbf{x} \in \mathbb{R}^n$ is a local minimizer of f, meaning that there is $r > 0$ such that

$$f(\mathbf{x}+\mathbf{h}) \geqslant f(\mathbf{x})$$

if $dist(\mathbf{x}+\mathbf{h}, \mathbf{x}) < r$. Prove that $\nabla f(\mathbf{x}) = 0$.

Solution: For any $1 \leqslant i \leqslant n$, define $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(t) = f(\mathbf{x} + t\mathbf{e}_i).$$

Then it follows that for any $-r < t < r$, $\phi(t) \geqslant \phi(0)$ since $f(\mathbf{x}+t\mathbf{e}_i) \geqslant f(\mathbf{x})$, so $t = 0$ is a local minimum of $\phi(t)$. Furthermore, since f has first-order partial derivatives then

$$\phi'(t) = \frac{\partial f}{\partial x_i}(\mathbf{x} + t\mathbf{e}_i)$$

exists and it follows from single variable calculus that

$$\frac{\partial f}{\partial x_i}(\mathbf{x}) = \phi'(0) = 0.$$

Since this is true for all $1 \leqslant i \leqslant n$, the result follows.

6. Define $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x, y, z) = xyz + x^2 + y^2.$$

The MVT implies that there is a $0 < \theta < 1$ such that

$$f(1, 1, 1) - f(0, 0, 0) = \frac{\partial f}{\partial x}(\theta, \theta, \theta) + \frac{\partial f}{\partial y}(\theta, \theta, \theta) + \frac{\partial f}{\partial z}(\theta, \theta, \theta).$$

Find such a value θ.

Solution: We can take partial derivatives of f to obtain

$$\begin{aligned}\frac{\partial f}{\partial x}(x, y, z) &= yz + 2x,\\ \frac{\partial f}{\partial y}(x, y, z) &= xz + 2y,\\ \frac{\partial f}{\partial z}(x, y, z) &= xy.\end{aligned}$$

At (θ, θ, θ) these are

$$\begin{aligned}\frac{\partial f}{\partial x}(\theta, \theta, \theta) &= \theta^2 + 2\theta,\\ \frac{\partial f}{\partial y}(\theta, \theta, \theta) &= \theta^2 + 2\theta,\\ \frac{\partial f}{\partial z}(\theta, \theta, \theta) &= \theta^2.\end{aligned}$$

So what we need to solve is

$$3 = f(1, 1, 1) - f(0, 0, 0) = 3\theta^2 + 4\theta.$$

Applying the quadratic formula to this gives $\theta = -\frac{2}{3} - \frac{\sqrt{13}}{3}$ and $\theta = -\frac{2}{3} + \frac{\sqrt{13}}{3}$. Only the second of these is between 0 and 1, which gives us our answer.

11. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable. Define

$$K = \{\ \mathbf{x} \in \mathbb{R}^n \mid ||\mathbf{x}|| \leqslant 1\ \}.$$

Solution:

a. Prove that there is a point $\mathbf{x} \in K$ such that f obtains its minimum value.
b. Assume that if $\mathbf{p} \in \mathbb{R}^n$ and $||\mathbf{p}|| = 1$ then $\langle \nabla f(\mathbf{p}), \mathbf{p} \rangle > 0$. Show that the minimizer $\mathbf{x}$ in part a. has norm less than 1.

Solution:

a. First observe that by Theorem 13.20, since f is continuously differentiable, then f is continuous. Next, K is sequentially compact since it is closed and bounded. It then follows from the Extreme Value Theorem that f obtains a minimum and maximum value in K.
b. Next assume that if $\mathbf{p} \in \mathbb{R}^n$ and $||\mathbf{p}|| = 1$ then $\langle \nabla f(\mathbf{p}), \mathbf{p} \rangle > 0$. Assume for a contradiction that f obtains its minimum value on the boundary of K (i.e. at a point $\mathbf{p}$ such that $||\mathbf{p}|| = 1$). Then if we shrink $\mathbf{p}$ to $\frac{(k-1)\mathbf{p}}{k}$, then since $\frac{(k-1)\mathbf{p}}{k}$ lies in K, then

$$f\Big(\frac{(k-1)\mathbf{p}}{k}\Big) - f(\mathbf{p}) \geqslant 0.$$

Write $\mathbf{x} = \mathbf{p}$ and $\mathbf{h}_k = \frac{-\mathbf{p}}{k}$. Then the MVT tells us that there is $0 < \theta_k < 1$ such that

$$0 \leqslant f\Big(\frac{(k-1)\mathbf{p}}{k}\Big) - f(\mathbf{p}) = \langle \nabla f\Big(\frac{(k-\theta_k)\mathbf{p}}{k}\Big), \frac{-\mathbf{p}}{k} \rangle.$$

Multiplying by $-(k-\theta_k)$ then gives

$$0 \geqslant \langle \nabla f\Big(\frac{(k-\theta_k)\mathbf{p}}{k}\Big), \frac{(k-\theta_k)\mathbf{p}}{k} \rangle.$$

Since the scalar product and ∇f are both continuous, we can take the limit as $k \to \infty$, to get

$$0 \geqslant \langle \nabla f(\mathbf{p}), \mathbf{p} \rangle.$$

But this is a contradiction to our assumptions.

11. Prove that

$$\lim_{(x,y)\to(0,0)} \frac{\sin(2x+2y) - 2x - 2y}{\sqrt{x^2+y^2}} = 0$$

Solution: We can calculate that for $f(x, y) = \sin(2x + 2y)$,

$$\frac{\partial f}{\partial x}(x, y) = 2\cos(2x + 2y),$$
$$\frac{\partial f}{\partial x}(x, y) = 2\cos(2x + 2y).$$

So,

$$f(0, 0) = 0$$
$$\frac{\partial f}{\partial x}(0, 0) = 2,$$
$$\frac{\partial f}{\partial x}(0, 0) = 2.$$

Hence we get that the first order approximation for f at $(x, y) = (0, 0)$ is

$$\psi(x, y) = 2x + 2y.$$

Since f is continuously differentiable on all of $\mathbb{R}^2$, result then follows from Theorem 14.2.

13. Suppose that the function $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous. Let a and b be any real numbers. Prove that

$$\lim_{(x,y)\to(0,0)} [f(x,y) - (f(0,0) + ax + by)] = 0.$$

Is it true that

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - (f(0,0) + ax + by)}{\sqrt{x^2 + y^2}} = 0?$$

Solution: Since f and $ax + by$ are continuous, it follows that

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$$

and

$$\lim_{(x,y)\to(0,0)} ax + by = 0.$$

The first result follows from these calculations.

The second limit is not always true in general. Consider the case where $f(x,y) = (a+1)x + by$. Then we have

$$\begin{aligned}
&\lim_{(x,y)\to(0,0)} \frac{f(x,y) - (f(0,0) + ax + by)}{\sqrt{x^2 + y^2}} \\
&= \lim_{(x,y)\to(0,0)} \frac{(a+1)x + by - ax - by}{\sqrt{x^2 + y^2}} \\
&= \lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2 + y^2}}.
\end{aligned}$$

Considering the sequence $(x_k, y_k) = (1/k, 0)$ we see that

$$= \lim_{k\to\infty} \frac{x_k}{\sqrt{x_k^2 + 0^2}} = 1.$$