# MATH 369 Homework 3

**Due:** Thursday February 14, in class.

1. Find the value of $a$ that solves the matrix multiplication equation.

$$\begin{pmatrix} a & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & -3 \end{pmatrix} \begin{pmatrix} a \\ 1 \\ 1 \end{pmatrix} = 0.$$

   **Solution:** Direct computation gives that the above matrix product is equal to

$$a^2 + 2a + 1 = 0.$$

   Therefore the only value for $a$ that satisfies the equation is $a = -1$.

2. (a) If $A$ and $B$ are $n \times n$ matrices and $AB = \mathbf{0}$, is it always true that either $A = \mathbf{0}$ or $B = \mathbf{0}$? If it is always true, justify your answer. If not, give an example of a case where this is not true.

   (b) Find a $2 \times 2$ matrix $B$ with $B \neq I_2$ and $B \neq \mathbf{0}$ so that

$$B^2 = B.$$

   **Solution:**

   (a) This is not true. Consider the example

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

   (b) One example is

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

3. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ c & 4 \end{pmatrix}.$$

   (a) For which values of $c$ does $A$ have an inverse?

   (b) For what values of $c$ does the linear system

$$\begin{array}{rcrcl} x & + & 2y & = & 0 \\ cx & + & 4y & = & 0 \end{array}$$

   have infinitely many solutions?

   **Solution:**

   (a) The matrix $A$ has an inverse if and only if $1(4) - 2c \neq 0$. This is true when $c \neq 2$.

   (b) You can check that there are infinite solutions when $c = 2$. One method of doing this is to do Gaussian elimination with the goal of eliminating a row.

4. An $n \times n$ matrix is called an *idempotent* if $A^2 = A$. Show that if $A$ is an idempotent then $(I - A)$ is also an idempotent.

   **Solution:** If $A$ is an idempotent, then $A^2 = A$. Then

$$(I - A)^2 = (I - A)(I - A) = I^2 - I \cdot A - A \cdot I + A^2 = I - A - A + A = I - A.$$

5. Let $A$ and $B$ be $n \times n$ matrices. It is generally NOT true that $(A+B)^{-1} = A^{-1} + B^{-1}$. Show this by finding two $2 \times 2$ matrices $A$, $B$ which are each invertible and for which

$$(A+B)^{-1} \neq A^{-1} + B^{-1}.$$

**Solution:** Almost any choice will work. A very simple example is

$$A = B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

From class we know

$$A^{-1} = B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We have

$$A^{-1} + B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

But

$$(A+B)^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

6. Find the inverse $A^{-1}$ to the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 4 & 0 \\ 0 & 1 & 6 \end{pmatrix}.$$

**Solution:** There are a number of methods that can be used to solve this. Your end solution should be

$$A^{-1} = \frac{1}{22} \begin{pmatrix} 24 & 2 & -8 \\ 6 & 6 & -2 \\ -1 & -1 & 4 \end{pmatrix}.$$