MATH 369 Homework 8

Due: Tuesday April 16th, in class.

1. Find the coordinates/coefficients of

$$\mathbf{w} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

in terms of the basis

$$\mathbf{u}_1 = \begin{pmatrix} 3 \\ 8 \end{pmatrix} \qquad \text{and} \qquad \mathbf{u}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Solution: This problem amounts to expressing the vector

$$\begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

as a linear combination of the vectors

$$\mathbf{u}_1 = \begin{pmatrix} 3 \\ 8 \end{pmatrix} \qquad \text{and} \qquad \mathbf{u}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This is equivalent to solving the equation

$$\begin{pmatrix} 3 & 1 \\ 8 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}.$$

A solution to this is

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -\frac{6}{5} \\ \\ \frac{28}{5} \end{pmatrix}$$

that is

$$\begin{pmatrix} 2 \\ -4 \end{pmatrix} = -\frac{6}{5} \begin{pmatrix} 3 \\ 8 \end{pmatrix} + \frac{28}{5} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The coordinates/coefficients are $\Big(-\frac{6}{5}, \frac{28}{5} \Big)$.

2. #21 in Section 4.4.

Solution:

(a) The elements of $\{T_A(\mathbf{e}_1), T_A(\mathbf{e}_2), T_A(\mathbf{e}_2)\}$ are

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}.$$

We can use a determinant to check that these are linearly independent.

(b) The elements of $\{T_A(\mathbf{e}_1), T_A(\mathbf{e}_2), T_A(\mathbf{e}_2)\}$ are

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

We can use a determinant to check that these are linearly dependent.

3. #14 in Section 4.5.

   Solution: In order to solve this problem, we need to show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ spans the same space as $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. We can do this by noting that we can write each element of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ as a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and vice versa. Indeed

$$\begin{aligned}\mathbf{v}_1 &= \mathbf{u}_1,\\ \mathbf{v}_2 &= \mathbf{u}_2 - \mathbf{u}_1,\\ \mathbf{v}_3 &= \mathbf{u}_3 - \mathbf{u}_2 - \mathbf{u}_1\end{aligned}$$

   and

$$\begin{aligned}\mathbf{u}_1 &= \mathbf{v}_1,\\ \mathbf{u}_2 &= \mathbf{v}_1 + \mathbf{v}_2,\\ \mathbf{u}_3 &= \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3.\end{aligned}$$

   Then by Theorem 4.5.4, since V is 3-dimensional ($\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ is a basis) and $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ span V, then $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ must be a basis for V.

4. #17 in Section 4.5.

   Solution: This set spans a subspace that is either 2 or 3-dimensional. It is at most 3-dimensional because all these vectors live in $\mathbb{R}^3$ which is 3-dimensional, and at least 2-dimensional because two of the vectors are linearly independent ($\mathbf{v}_1$ and $\mathbf{v}_2$ for example). But we can show that $\mathbf{v}_3$ and $\mathbf{v}_4$ are linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_4 = \mathbf{v}_1 - \mathbf{v}_2.$$

   Hence $\mathbf{v}_1$ and $\mathbf{v}_2$ are not only linearly independent, but they also span the space. Thus they are a basis.

5. #3 in Section 4.7.

   Solution:

   (a) This is not in the column space.

   (b) This is in the column space with:

$$\begin{pmatrix}1\\9\\1\end{pmatrix} - 3\begin{pmatrix}-1\\3\\1\end{pmatrix} + \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}5\\1\\-1\end{pmatrix}.$$