MATH 369 Homework 4

Due: Thursday February 28, in class.

1. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix}.$$

Find a parametrization of all vectors $\mathbf{b}$ such that there is an $\mathbf{x}$ for which $A\mathbf{x} = \mathbf{b}$.

Solution: We do row reductions to get

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 3 & 4 & 5 & b_2 \\ 6 & 7 & 8 & b_3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & -2 & -4 & b_2 - 3b_1 \\ 0 & 0 & 0 & b_3 + \frac{3}{2}b_1 - \frac{5}{2}b_2 \end{array}\right).$$

For this system to be consistent, we must have

$$b_3 + \frac{3}{2}b_1 - \frac{5}{2}b_2 = 0 \implies b_3 = -\frac{3}{2}b_1 + \frac{5}{2}b_2.$$

The full set of $\mathbf{b}$ can be parametrized by

$$\mathbf{b} = \begin{pmatrix} t \\ r \\ -\frac{3}{2}t + \frac{5}{2}r \end{pmatrix}.$$

Note that here, t and r can be any real numbers.

2. Say whether each of the maps below is linear. Justify your reasoning using the definition of a linear map.

(a) The map $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x + y \\ y - z + 1 \\ x + 5y + z \end{pmatrix}.$$

(b) The transformation $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ which takes a vector in $\mathbb{R}^2$ and rotates it θ degrees counterclockwise about the origin.

(c) The transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ which sends

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x + y \\ 2x^2 - y \end{pmatrix}.$$

Solution:

(a) This is not a linear map. Observe for example that it does not respect scalar multiplication:

$$2T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 14 \end{pmatrix} \neq \begin{pmatrix} 6 \\ 1 \\ 14 \end{pmatrix} = T\left(2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right).$$

(b) This is a linear map. If we think of scalar multiplication as the process of extending or shrinking a vector (while keeping its direction constant), then whether we first shrink/extend a vector (multiply by a scalar) and then rotate it by θ or rotate it by θ and then shrink/extend it, we get the same result.

It is also not too hard to see that it doesn't matter whether we rotate vectors and then add them, or add them and then rotate them, we get the same answer either way.

(c) This is not a linear transformation. Again, we have

$$2T\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}6\\2\end{pmatrix} \neq \begin{pmatrix}6\\6\end{pmatrix} = T\left(2\begin{pmatrix}1\\1\\1\end{pmatrix}\right).$$

3. Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation and that we know that

$$T(\mathbf{e}_1) = \begin{pmatrix}1\\2\end{pmatrix}, \qquad T(\mathbf{e}_2) = \begin{pmatrix}1\\-1\end{pmatrix}, \qquad T(\mathbf{e}_3) = \begin{pmatrix}2\\1\end{pmatrix}.$$

Calculate the value of $T(\mathbf{v})$ where

$$\mathbf{v} = \begin{pmatrix}5\\-2\\4\end{pmatrix}.$$

Solution: We write

$$\begin{pmatrix}5\\-2\\4\end{pmatrix} = 5\begin{pmatrix}1\\0\\0\end{pmatrix} - 2\begin{pmatrix}0\\1\\0\end{pmatrix} + 4\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Hence, using the linearity of T, we have

$$\begin{aligned} T\begin{pmatrix}5\\-2\\4\end{pmatrix} &= 5T\begin{pmatrix}1\\0\\0\end{pmatrix} - 2T\begin{pmatrix}0\\1\\0\end{pmatrix} + 4T\begin{pmatrix}0\\0\\1\end{pmatrix} \\ &= 5\begin{pmatrix}1\\2\end{pmatrix} - 2\begin{pmatrix}1\\-1\end{pmatrix} + 4\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}11\\16\end{pmatrix} \end{aligned}$$