Research Statement

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1 Introduction

My research interests in mathematics span the entire spectrum of application, from my theory-centered research in representation theory (Section 2), to my more applied work developing new secant-based dimensionality reduction algorithms for data science (Section 3), to my development and implementation of compressive sensing reconstruction algorithms for use in a real device being developed in a joint academia-industry partnership (Section 4).

Much of my research is also characterized by the way in which it both explores and utilizes connections between different domains of mathematics and beyond. For example, my research in representation theory, explores connections between this subject and combinatorics, category theory, and probability theory. On the other hand my research in data science focuses on utilizing geometric and topological frameworks to better handle, interpret, and analyze massive data sets. Finally, the projects I have worked on in compressive sensing utilize a GPU-framework to facilitate fast reconstruction of image. For me, the most exciting mathematics sits at the intersection of, and draws from, many disciplines both in and beyond math itself.

2 Representation theory

2.1 The categorification program

Initiated by Crane and Frenkel [9], the idea underlying the categorification program is that many of the fundamental structures in algebra, geometry, and topology are the shadow of richer (often undiscovered) higher categories. Categorification, the process of revealing these higher analogues, then involves lifting set-theoretic constructions to category-theoretic constructions, roughly: replacing sets with categories, functions with functors, and equalities with isomorphisms. Though Khovanov's categorification of the Jones polynomial via Khovanov homology [21] may be the most famous example, far more familiar examples exist (for example, singular homology is a categorification of Betti numbers). Categorifying the fundamental algebras from representation theory has been a major, ongoing project in the field. Milestones include the categorification of quantum groups [23, 31] and Reshetikhin-Turaev knot invariants [42]. Furthermore, theory developed in the course of this program has led to results of independent interest, such as the proof of the Kazhdan-Lusztig conjecture for all Coxeter types [10] via Soergel bimodules [41].

As consequence of this progress, a host of new categories have been discovered which sit "above" and "control" many of the classical objects in representation theory such as Hecke algebras, Heisenberg algebras, and quantum groups. These categories are rich in structure, and experience has shown that while features of an algebra might seem incidental at the set-theoretic level, at the categorical level they are often revealed to be basic consequences of structure in the higher category (for example, the canonical basis of quantum groups [23, 39]). A major drive in my research has been using these categorifications to better understand the connections between representation theory and other areas of math such as combinatorics and probability theory. In my first research project described below, categorifications of the infinite dimensional Heisenberg algebra both provide a graphical interpretation of analogues of the symmetric functions and also naturally provide the framework for stochastic growth processes motivated by representation theory [27, 28]. In the second project, the categorification of quantum groups in classical affine type is used to show how a basic crystal isomorphism can in fact be understood to be a consequence of the representation theory of a cyclotomic quiver Hecke algebras [29].

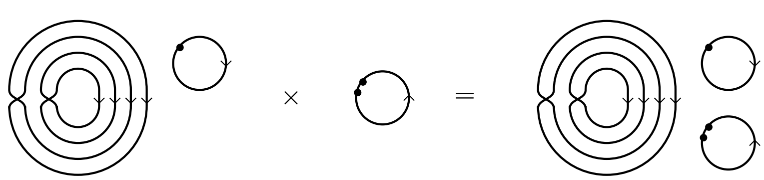
2.2 Heisenberg categories, symmetric functions, and growth processes

A *tower of algebras* $\{A_n\}_{n\geqslant 0}$ is a collection of nested algebras $A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$. The classic example is the symmetric group algebras $\{\mathbb{C}[S_n]\}_{n\geqslant 0}$. A result of Geissinger [13] showed that the direct sum of all Grothendieck groups of symmetric groups $\bigoplus_{n\geqslant 0} K_0(\mathbb{C}[S_n]\text{-mod})$ carries the structure of the Fock space representation of the infinite dimensional Heisenberg algebra $\mathfrak{h}$. In particular, the elements of $\mathfrak{h}$ act by linear operators coming from induction and restriction functors between symmetric group representations. This observation was the first hint of a deeper story in which a vastly richer *Heisenberg category* $\boldsymbol{\mathcal{H}}$, acts via genuine induction and restriction functors on the modules categories $\bigoplus_{n\geqslant 0} A_n$-mod for a tower of algebras $\{A_n\}_{n\geqslant 0}$.

$$\begin{tikzcd}
\boldsymbol{\mathcal{H}} \arrow[rr, "\substack{\text{categorical}\\ \text{representation}}"] \arrow[d, squiggly, "K_0"] & & \bigoplus_{n\geqslant 0} A_n\text{-mod} \arrow[d, squiggly, "K_0"] \\
\mathfrak{h} \arrow[rr, "\text{representation}"] & & \bigoplus_{n\geqslant 0} K_0(A_n\text{-mod})
\end{tikzcd}$$

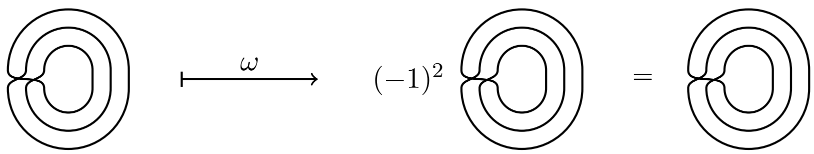
The first Heisenberg category was defined by Khovanov [22] and is associated to the $\{\mathbb{C}[S_n]\}_{n\geqslant 0}$. Subsequently, Heisenberg categories associated to a range of algebras have been discovered, including: Hecke algebras [33], Sergeev algebras [6], wreath product algebras [38], and degenerate cyclotomic affine Hecke algebras [34, 5]. In all these examples, $\boldsymbol{\mathcal{H}}$ can be defined diagrammatically, i.e. through a calculus of planar diagrams with local relations and decorations.

All Heisenberg categories are additive monoidal categories and hence their center, the endomorphisms of the monoidal identity $Z(\boldsymbol{\mathcal{H}}) := \mathrm{End}_{\boldsymbol{\mathcal{H}}}(\mathbb{1})$, is a commutative algebra. In such diagrammatically defined categories, the center has a helpful diagrammatic definition as the algebra of all closed diagrams.

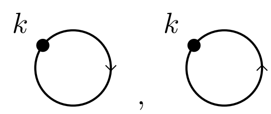


We studied Khovanov's Heisenberg category $\mathcal{H}_{\mathbb{C}[S_n]}$, associated to $\{\mathbb{C}[S_n]\}_{n\geqslant 0}$, and proved that $Z(\mathcal{H}_{\mathbb{C}[S_n]})$ is isomorphic to Λ^*, the algebra of shifted symmetric functions [27]. A deformation of the classical symmetric functions, Λ^* was discovered by Kerov and Olshanksi [20] and further investigated by Okounkov [36], Okounkov and Olshanski [37], and Ivanov and Olshanski [15] just to name a few.

As a consequence of the algebra isomorphism $Z(\mathcal{H}_{\mathbb{C}[S_n]}) \cong \Lambda^*$, we were able to describe a graphical calculus for Λ^* using closed diagrams and a set of local relations. We showed how individual generators such as shifted Schur functions and properties of Λ^* such as involutions emerge in a natural way from the graphical calculus. For example, the involution $\omega : \Lambda^* \to \Lambda^*$ which exchanges the shifted analogue of elementary symmetric functions with the shifted analogue of homogeneous symmetric functions corresponds to multiplying each collection of closed diagrams by $(-1)^c$ where c is the number of crossings in the diagram.



Surprisingly, under the isomorphism $Z(\mathcal{H}_{\mathbb{C}[S_n]}) \cong \Lambda^*$, the simplest diagrams corresponding to a clockwise or counterclockwise oriented *bubble* with k dots on it



is a shifted symmetric function encoding moments for a family of probability measures, Kerov's *transition* and *co-transition measures*, parametrized by Young diagrams. This same probability measure was used as a tool to study the famous Plancherel growth process [18, 19]. This unexpected phenomenon points to a deeper connection between categorical representation theory and asymptotic representation theory. Exploring this connection more closely is one of the themes of my current work.

Furthermore, the connection between $Z(\mathcal{H}_{\mathbb{C}[S_n]})$, symmetric functions, and asymptotic representation theory does not seems to be a coincidence. In [28], we investigated the center of the twisted Heisenberg category $\mathcal{H}_{\mathbb{S}_n}$ of Cautis and Sussan [6] which is associated to the tower of Sergeev algebras $\{\mathbb{S}_n\}_{n\geqslant 0}$. While we showed that in this case $Z(\mathcal{H}_{\mathbb{S}_n})$ is isomorphic to the subalgebra of the symmetric functions generated by odd power sums $\mathbb{C}[p_1, p_3, \dots]$, similar connections to combinatorics and asymptotic representation theory hold. Notably, the bubble generators of $Z(\mathcal{H}_{\mathbb{S}_n})$ also correspond to functions encoding transition probabilities of a stochastic growth process (this time however on the graph of all strict Young diagrams).

Two questions we are currently investigating are:

- *Centers of Heisenberg categories and new analogues of symmetric functions:* What new analogues of symmetric functions arise from centers of the more exotic Heisenberg categories (such as those associated with towers of degenerate cyclotomic affine Hecke algebras [34])? What combinatorial properties do they have?

- *Representation-theoretic growth processes and Heisenberg categories:* Do all Heisenberg categories define stochastic growth process? Can we in define stochastic growth processes via functors and natural isomorphisms?

2.3 Crystal structure in representation categories

Crystals were first discovered by Kashiwara as combinatorial objects attached to certain quantum group $U_q(\mathfrak{g})$ representations that appear in the limit $q \to 0$ [17]. Since they can be described via colored directed graphs, crystals are substantially simpler than their corresponding representations. They still however carry fundamental combinatorial information.

Crystals can also arise when a quantum group acts on a category via endofunctors. The first example of this was found when it was noticed that simple representations of affine and cyclotomic Hecke algebras carry the structure of a crystal graph [30, 1, 14]. Only in hindsight was it realized that this is the result of a categorical action of a quantum group. Subsequently many other representation categories with categorical action of a quantum group have been shown to carry a crystal structure. Examples include the category of finite-dimensional quiver Hecke algebra representations [32] and category $\mathcal{O}$ for the cyclotomic rDAHA [40].

categories	simple object in $\boldsymbol{\mathcal{C}}$	induction functor F_i between simple objects in $\boldsymbol{\mathcal{C}}$
	$\rightsquigarrow K_0$	$\rightsquigarrow K_0$
crystals	node of crystal	Kashiwara operator $\widetilde{f}_i$ (i-colored edge)

When a representation category carries a crystal structure one can try to understand how the combinatorics of the crystal lift to the representation theory. Such crystal-theoretic considerations allowed Lauda-Vazirani to classify all simple quiver Hecke algebra representations [32].

2.4 Results on crystal structure in categories

Khovanov and Lauda [23], [24] and independently Rouquier [39] invented *quiver Hecke algebras R* (also known as *Khovanov-Lauda-Rouquier (KLR) algebras*) to categorify the upper half $U_q^+(\mathfrak{g})$ of the quantum group for any fixed symmetrizable Kac-Moody algebra $\mathfrak{g}$.

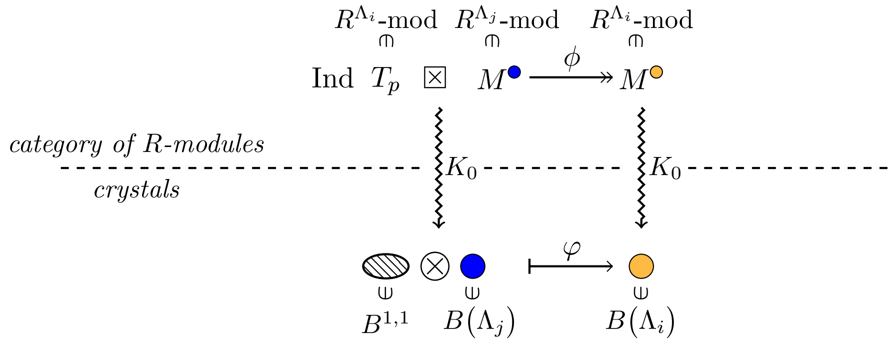
For each integral dominant weight Λ, the algebra R has a finite-dimensional quotient called a *cyclotomic quiver Hecke algebra R^Λ*. Lauda-Vazirani showed that the collection of simple R^Λ-modules carries the structure of $B(\Lambda)$ which is the crystal of the corresponding highest weight representation of $U_q(\mathfrak{g})$. Here nodes are the simple R^Λ-modules and Kashiwara operators $\widetilde{f}_i$ are given by refined induction functors between the simple modules [32].

When $\mathfrak{g}$ is of classical affine type, one class of crystals not previously appearing in the context of quiver Hecke algebras is the *Kirillov-Reshetikhin (KR) crystals* [35]. These are denoted by $B^{r,s}$ where $r \in I \backslash \{0\}$ for Dynkin indexing set I and $s \in \mathbb{Z}_{>0}$. A notable property of $B^{1,1}$ in particular is that for certain choices of fundamental weight Λ_j there is an isomorphism of crystals

$$\varphi : B^{1,1} \otimes B(\Lambda_j) \xrightarrow{\sim} B(\Lambda_i), \tag{1}$$

for some $i \in I$ [16]. This isomorphism is particularly exciting since it allows one to study the (infinite) crystal $B(\Lambda_i)$ through tensor products of the finite crystals $B^{1,1}$.

Given that R^{Λ_i}-mod (respectively R^{Λ_j}-mod) carries the structure of $B(\Lambda_i)$ (resp. $B(\Lambda_j)$), it is natural to ask whether (1) actually reflects representation-theoretic structure in R^{Λ_i}-mod. Vazirani and I showed that this is the case in [29]. Recall that each simple R^{Λ_i}-module M° corresponds to a node $\circ$ in $B(\Lambda_i)$, and similarly each simple R^{Λ_j}-module $M^{\bullet}$ corresponds to a node $\bullet$ in $B(\Lambda_j)$. We show that each assignment of the crystal isomorphism (1) is actually the shadow of a canonically defined surjective R^{Λ_i}-module homomorphism ϕ.



The simple R^{Λ_i}-module T_p here belongs to a family of "trivial" R^{Λ_i}-modules parametrized by paths p in the crystal $B^{1,1}$. Most importantly, the above correspondence intertwines the action of the Kashiwara operators $\widetilde{f}_i$ of the crystal graph with the action of their categorical analogue, the refined induction functors.

3 Geometry and data analysis

As data is generated at a faster and faster rate, tools for extracting information from large, high-dimensional data sets have become increasingly valuable across science, engineering, and industry. For data in low-dimensions we are taught in grade-school to plot the data and use our visual intuition to guide our understanding. When the data sits in high dimensions however, relying on our innate understanding of space becomes very limiting. On the other hand geometry and topology have sophisticated tools for understanding shape and space, honed over the last 200 years. My work in data science utilizes a geometric framework to improve data analytics.

3.1 Secant sets, dimensionality reduction, and Whitney's theorem

The secant set S of a data set $D \subset \mathbb{R}^n$ is the set of differences of all distinct points, i.e. $S := \{x - y \mid x, y \in D,\ x \neq y\}$. While S contains strictly less information than D, it does contain the relational information important to a good dimensionality reduction algorithm. More informally, any "good" reduction of the data should keep points that are far apart, far apart, i.e. the reduction should preserve secants. Furthermore, by measuring how well a projection preserves secants, we obtain a measure of how much information we have lost in the process of reducing the data.

Inspired by the constructive proof of Whitney's embedding theorem Broomhead and Kirby developed a family of algorithms designed to solve the optimization problem

$$P^* := \underset{P \in \mathrm{Proj}(n,k)}{\arg\max} \left(\min_{s \in \tilde{S}} ||Ps||_2 \right) \tag{2}$$

where $\tilde{S}$ is the set of unit secants of D and $\mathrm{Proj}(n,k)$ is the set of orthogonal projections from $\mathbb{R}^n$ to $\mathbb{R}^k$ [2, 3, 4]. One weakness of these algorithms is that they do not scale well to very large data sets. With this in mind we developed two new secant-based dimensionality reduction algorithms which also solve (2):

1. *The secant avoidance algorithm (SAP)* [25]: Given an initial projection $P^{(0)}$ of rank k, at each iteration i this algorithm shifts the k-dimensional subspace corresponding to the current projection $P^{(i)}$ toward the unit secant in $\tilde{S}$ least well preserved by $P^{(i)}$. The output is a projection of rank k which we take as a local solution to (2). In order to promote speed, the SAP algorithm was designed with GPU architecture in mind. Indeed, tests suggest that while the SAP algorithm is competitive with previous secant-based dimensionality reduction algorithms in terms of output, it is substantially faster when implemented on a GPU.

2. *The hierarchical secant avoidance algorithm (HSAP)* [26]: Because for a data set with T points, the number of secants grows as $O(T^2)$, for very large data sets storing secants becomes an issue. The HSAP algorithm addresses this problem by utilizing the essential ideas of the SAP algorithm, but rather than optimizing with respect to all unit secants, HSAP instead optimizes a mixture of linear approximations to secant sets for a given cluster of points and the secants between these clusters.

Another motivation for developing new secant-based dimensionality reduction techniques is their ability to help suggest the dimension of a data set. Along with a projection $P : \mathbb{R}^n \to \mathbb{R}^n$, output of the SAP and HSAP algorithms also includes $\kappa := \min_{s\in\tilde{S}} ||Ps||_2$, i.e. the length of the secant worst preserved by P. It follows that

$$\kappa||x-y||_{\ell_2} \leqslant ||P(x)-P(y)||_{\ell_2} \leqslant ||x-y||_{\ell_2}$$

for all points x, y in the data set. So when $\kappa > 0$, P is a bi-Lipschitz function. Besides giving the nice property that P has a well-conditioned inverse on the data set, this also means that P preserves the topological or Hausdorff dimension of D [11]. By studying the values of κ as we change the input k (the rank of the desired projection) and recalling Whitney's result that any smooth m-dimensional real manifold M can be embedded in $\mathbb{R}^{2m+1}$, we can make educated conjectures about the intrinsic dimension of the data we study. A toy example is shown in Figure 1 from [25].

We also have

My current projects in dimensionality reduction seek to answer questions including

- *Dimension of layers of a neural network:* It would be interesting to

3.2 Grassmann manifolds and hyperspectral imaging

It has become increasingly clear that better results can be achieved in data analytics when the data is realized in the "right" geometry. This often means using spaces which are not traditionally considered within classical data science such as Grassmann manifolds, flag manifolds, and Stiefel manifolds. The use of Grassmann manifolds is particularly useful when one

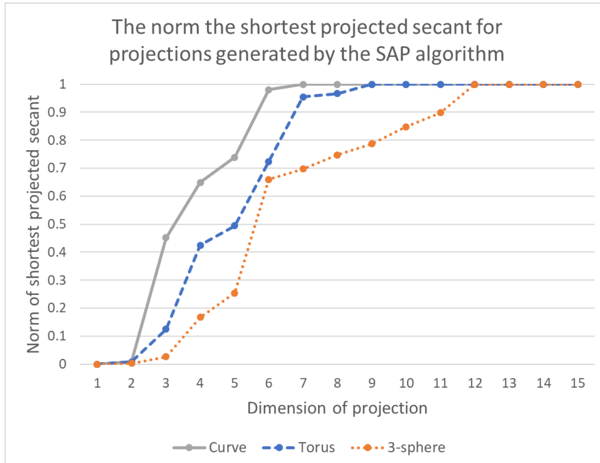
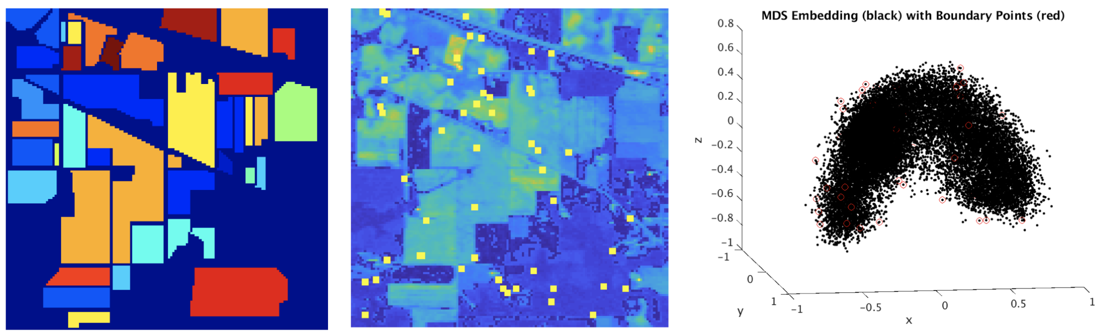


Figure 1: Values of κ as a function of projection rank k for points drawn from: a curve, a 2-dimensional torus, and the 3-sphere. All were embedded non-trivially into $\mathbb{R}^{15}$.

wants to capture variation of state in the data, for example, variations in the illumination of faces [7] or variations in signal for hyperspectral detection [8].

Given this, there is a significant need to develop non-Euclidean analogues of classical data science algorithms. In [12] we introduced an endmember extraction algorithm for points on a Grassmannian. Given a cluster of points $D \subset \mathbb{R}^n$, endmember extraction refers to the process of finding the subset D' of D containing of vertices of the convex hull of D. The elements of D', called *endmembers*, correspond to the "pure" signals of D. In hyperspectral imaging, where endmember extraction is commonly used, endmembers generally correspond to spectral signals of pure compounds, while the rest of D consists of mixtures of these pure, spectral signals.

While a single spectral signal of length n can be encoded as a point in $\mathbb{R}^n$, it often makes sense to instead combine variations from several signals within a single point. Imagine the spectral signal for "forest". This would in general not be a single spectral vector but the space spanned by a number of different signals corresponding to different tree species, bark and leaves, etc. Points then no longer correspond to vectors but rather to subspaces of $\mathbb{R}^n$ and hence points on the Grassmannian $G(n, k)$ (where k is the dimension of the subspace). One can use chordal distance on the Grassmannian to define endmembers in this context. Our Grassmannian endmember extraction algorithm provides a way to find the most pure space of signals (often taken to be a local patch from the image). In Figure 2 we show the result of running our algorithm on 3×3 patches from the Indian Pines hyperspectral image. The yellow squares in (b) correspond patches that the algorithm has determine correspond to endmembers.



(a) (b) (c)

Figure 2: (a) The class labels to the Indian Pines hyperspectral data cube. (b) A visualization of several bands of the Indian Pines data cube with points of interest identified by the Grassmannian endmember extraction algorithm superimposed. (c) The embedding of points from the Indian Pines hyperspectral data cube from $Gr(9, 200)$ to $\mathbb{R}^3$ with those points identified as endmembers via the algorithm circled with red.

4 Compressive sensing and hyperspectral imaging

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