Practice Exam 1

1. (a) Give at least two definitions of a closed set in $\mathbb{R}^n$.

   (b) Give at least two equivalent definitions of a continuous function $f : A \to \mathbb{R}^m$ for $A \subseteq \mathbb{R}^n$.

2. Let $\mathbf{u}$ and $\mathbf{v}$ be two points in $\mathbb{R}^n$. Let $R, r > 0$. Show that if $dist(\mathbf{u}, \mathbf{v}) = R + r$, then the open balls $B_R(\mathbf{u})$ and $B_r(\mathbf{v})$ are disjoint (i.e. $B_R(\mathbf{u}) \cap B_r(\mathbf{v}) = \varnothing$).

3. Prove that the set $A = \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + xy + z^3 \neq 0 \} \subseteq \mathbb{R}^3$ is open in $\mathbb{R}^3$.

4. Let A be an unbounded set. Show that there is a sequence $\{\mathbf{u}_k\}_{k \geqslant 1}$ in A such that no subsequence of $\{\mathbf{u}_k\}_{k \geqslant 1}$ converges.

5. Let $f : \mathbb{R}^n \to \mathbb{R}$ be the function defined so that

$$f(x_1, x_2, \dots, x_n) = \begin{cases} 1 & \text{if all } x_i \text{ are rational for } 1 \leqslant i \leqslant n \\ 0 & \text{otherwise.} \end{cases}$$

   Prove that f is not continuous at any point in $\mathbb{R}^n$.

6. Show that the function $f : (0, 1] \to \mathbb{R}$ defined by $f(x) = \ln(x^2)$ is continuous but not uniformly continuous. (You can assume that $\ln(x)$ is continuous on $(0, \infty)$).