MATH 417 Homework 2

Note that this is problems: Chapter 11.1: #3, #4,#6,#10,#11.

3. Fix a point $\mathbf{v}$ in $\mathbb{R}^n$ and define the function $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(\mathbf{u}) = \langle \mathbf{u}, \mathbf{v} \rangle \qquad \text{for } \mathbf{u} \in \mathbb{R}^n.$$

Prove that the function f is continuous.

Solution: Set $\mathbf{v} = (v_1, \ldots, v_n)$. By Proposition 11.1 we know that the projection functions $p_1, \ldots, p_n : \mathbb{R}^n \to \mathbb{R}$ are continuous. By Theorem 11.3 then,

$$v_1 p_1 + v_2 p_2 + \cdots + v_n p_n = \langle \cdot, \mathbf{v} \rangle = f(\cdot)$$

is continous.

4. Suppose that the function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and that $f(\mathbf{u}) > 0$ if the point $\mathbf{u} \in \mathbb{R}^n$ has at least one rational component. Prove that $f(\mathbf{u}) \geqslant 0$ for all points $\mathbf{u} \in \mathbb{R}^n$.

Solution: Choose any point $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$. From our knowledge of $\mathbb{R}$, we know that for all $1 \leqslant i \leqslant n$ we can find a sequence $\{u_{i,k}\}_{k \geqslant 1}$ that converges to u_i where all $u_{i,k}$ are rational (regardless of whether u_i is rational or irrational). Define $\mathbf{u}_k := (u_{1,k}, u_{2,k}, \ldots, u_{n,k})$. By the componentwise convergence criterion $\lim_{k\to\infty} \mathbf{u}_k = \mathbf{u}$. Hence, because f is continous $\lim_{k\to\infty} f(\mathbf{u}_k) = \mathbf{u}$. But since all components of each term of $\mathbf{u}_k$ are rational, $f(\mathbf{u}_k) > 0$. Taking the limit of both sides we get

$$f(\mathbf{u}) = \lim_{k\to\infty} f(\mathbf{u}) \geqslant 0.$$

6. Suppose that the functions $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are both continuous. Prove that the set

$$\mathcal{O} = \{\mathbf{u} \in \mathbb{R}^n \mid f(\mathbf{u}) = g(\mathbf{u}) = 0\}$$

is closed in $\mathbb{R}^n$.

Solution: We will show that $\mathcal{O}$ is closed by showing that if $\{\mathbf{u}_k\}_{k \geqslant 0}$ is a sequence in $\mathcal{O}$ that converges to a value $\mathbf{u}$, then $\mathbf{u} \in \mathcal{O}$. So suppose that $\{\mathbf{u}_k\}_{k \geqslant 1}$ is a sequence in $\mathcal{O}$ that converges to $\mathbf{u}$. Thus for each $\mathbf{u}_k$, $f(\mathbf{u}_k) = g(\mathbf{u}_k) = 0$. Since f and g are continuous, it then follows that

$$f(\mathbf{u}) = \lim_{k\to\infty} f(\mathbf{u}_k) = \lim_{k\to\infty} 0 = 0.$$

and

$$g(\mathbf{u}) = \lim_{k\to\infty} g(\mathbf{u}_k) = \lim_{k\to\infty} 0 = 0.$$

It then follows that $\mathbf{u} \in \mathcal{O}$ and hence $\mathcal{O}$ is closed.

10. Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ is continuous. Suppose that $\mathbf{u}$ is a point in $\mathcal{O}$ at which $f(\mathbf{u}) > 0$. Prove that there is an open ball $\mathcal{B}$ about $\mathbf{u}$ such that $f(\mathbf{v}) > f(\mathbf{u})/2$ for all $\mathbf{v} \in \mathcal{B}$.

Solution: Set $\epsilon = \frac{f(\mathbf{u})}{2}$. Since f is continuous at $\mathbf{u}$ we have that there is a $\delta > 0$ such that for all $dist(\mathbf{v}, \mathbf{u}) = ||\mathbf{v} - \mathbf{u}|| < \delta$ (note that because $\mathcal{O}$ is open, we are guaranteed that by picking δ sufficiently small, all points satisfying this criteria are actually in $\mathcal{O}$),

$$dist(f(\mathbf{v}), f(\mathbf{u})) = |f(\mathbf{v}) - f(\mathbf{u})| < \frac{f(\mathbf{u})}{2}.$$

This is equivalent to

$$-\frac{f(\mathbf{u})}{2} < f(\mathbf{v}) - f(\mathbf{u}) < \frac{f(\mathbf{u})}{2}$$

which gives us

$$\frac{f(\mathbf{u})}{2} < f(\mathbf{v}) < \frac{3f(\mathbf{u})}{2}.$$

Since this is true for all $\mathbf{v}$ such that $dist(\mathbf{v}, \mathbf{u}) < \delta$, then this is true for all points in the ball $B_\delta(\mathbf{u})$.

11. Let A be a subset of $\mathbb{R}^n$. The *characteristic function* on the set A is the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(\mathbf{u}) = \begin{cases} 1 & \text{if } \mathbf{u} \in A \\ 0 & \text{if } \mathbf{u} \notin A. \end{cases}$$

Prove that this characteristic function is continuous at each interior point of A and at each exterior point of A but fails to be continuous at each boundary point of A.

Solution: Recall that we can decompose $\mathbb{R}^n$ into the disjoint set.

$$\mathbb{R}^n = intA \cup bdA \cup extA.$$

- Suppose first that $\mathbf{u}$ is a point in $intA$. Then there is a ball $B_r(\mathbf{u})$ for $r > 0$ such that $B_r(\mathbf{u}) \subset A$. Now for any $\epsilon > 0$, choose $\delta = r$. Then

$$0 = |1 - 1| = |f(\mathbf{v}) - f(\mathbf{u})| = dist(f(\mathbf{v}), f(\mathbf{u})) < \epsilon$$

  whenever $\mathbf{v} \in B_r(\mathbf{u})$. So f is continuous at $\mathbf{u}$.

- Next suppose that $\mathbf{u}$ is a point in $extA$. Then there is a ball $B_r(\mathbf{u})$ for $r > 0$ such that $B_r(\mathbf{u}) \subset \mathbb{R}^n \backslash A$. Now for any $\epsilon > 0$, choose $\delta = r$. Then

$$0 = |0 - 0| = |f(\mathbf{v}) - f(\mathbf{u})| = dist(f(\mathbf{v}), f(\mathbf{u})) < \epsilon$$

  whenever $\mathbf{v} \in B_r(\mathbf{u})$. So f is continuous at $\mathbf{u}$.

- Finally suppose that $\mathbf{u}$ is a point in bdA. Assume that in fact $\mathbf{u} \in A$. Since all open balls centered at $\mathbf{u}$ contain an element not in A, by considering the sequence of balls $B_1(\mathbf{u})$, $B_{\frac{1}{2}}(\mathbf{u})$, $B_{\frac{1}{3}}(\mathbf{u}), \ldots,$ we can construct a sequence of points $\{\mathbf{u}_k\}$ converging to $\mathbf{u}$ but in $\mathbb{R}^n \backslash A$. Then

$$\lim_{k \to \infty} f(\mathbf{u}_k) = \lim_{k \to \infty} 0 = 0 \neq 1 = f(\mathbf{u}).$$

  So f is not continuous at $\mathbf{u}$.

  The case where $\mathbf{u} \in bdA$ but $\mathbf{u} \in \mathbb{R}^n \backslash A$ is completely analogous.