Practice Exam 3

1. True or false: If A is a 10×15 matrix and the row space of A has dimension 7, then the nullity of A is 3.

   Solution: False. Since the row space of A has dimension 7, then the column space of A also has dimension 7. That is, the rank of A is 7. Since the nullity is the number of columns minus the rank of A, then the nullity is 8.

2. Consider the set of vectors:

$$\mathbf{v}_1 = \begin{pmatrix} -3 \\ -9 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} -7 \\ -8 \\ -2 \end{pmatrix}.$$

   (a) Find a basis for $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ as the row space of a matrix.

   (b) Find a **subset** of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ that forms a basis for $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$.

   (c) Are the bases you found in (a) and (b) bases for $\mathbb{R}^3$. Why or why not?

   Solution:

   (a) Take A to be the matrix

$$A = \begin{pmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 4 & -1 & 2 \\ -7 & -8 & -2 \end{pmatrix}$$

   then the reduced row echelon form of the matrix is

$$\begin{pmatrix} 1 & 0 & \frac{6}{13} \\ 0 & 1 & -\frac{2}{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

   It follows that a basis for the row space of A is

$$\begin{pmatrix} 1 \\ 0 \\ \frac{6}{13} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -\frac{2}{13} \end{pmatrix}.$$

   (b) Take A to be the matrix

$$A = \begin{pmatrix} -3 & 1 & 4 & -7 \\ -9 & 3 & -1 & 8 \\ 0 & 0 & 2 & -2 \end{pmatrix}.$$

   then the reduced row echelon form of the matrix is

$$\begin{pmatrix} 1 & -\frac{1}{3} & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

   So a subset of the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ which give a basis for the span is $\mathbf{v}_1$ and $\mathbf{v}_3$.

(c) The bases for $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ are not bases for $\mathbb{R}^3$ since all bases for $\mathbb{R}^3$ have 3 elements and bases for $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ have only two elements ($\mathbb{R}^3$ is 3-dimensional while $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ is 2-dimensional).

3. Let A be the matrix

$$A = \begin{pmatrix} 1 & 2 & 5 & 3 \\ 2 & 0 & 2 & 2 \\ 3 & 1 & 5 & 6 \end{pmatrix}$$

(a) What is a basis for the row space of A?

(b) What is a basis for the column space of A?

(c) What is a basis for the null space of A?

(d) What is the dimension for the null space of A^T?

(e) What is the rank and nullity of A?

Solution: Put into reduced row echelon form, A becomes

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

From this we can read off the various bases:

(a) A basis for the row space is

$$\begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) A basis for the column space is

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}.$$

(c) A basis for the null space of A is

$$\begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix}.$$

(d) Since the reduced row echelon form of A^T is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

then the null space of A^T has dimension 0.

(e) The rank of A is 3 and the nullity of A is 1.

4. Suppose that A is the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & -3 & 6 \\ 0 & 1 & -1 \end{pmatrix}.$$

(a) Check that $\det(A) \neq 0$.

(b) Explain why this means that the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_1 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \mathbf{v}_1 = \begin{pmatrix} 0 \\ 6 \\ -1 \end{pmatrix}$$

are linearly independent.

(c) Explain why this means that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ span $\mathbb{R}^3$.

(d) Does $\det(A) \neq 0$ also imply that the row vectors of A are a basis for $\mathbb{R}^3$.

Solution:

(a) The determinant of A is -1.

(b) Because the determinant is not zero, A is invertible. This means that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. This is equivalent to saying that the only way to get $\mathbf{0}$ from a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ is with all coefficients equal to 0.

(c) Because the determinant is not zero, A is invertible. This means that for any $\mathbf{b}$ in $\mathbb{R}^3$, the equation $A\mathbf{x} = \mathbf{b}$ is consistent. But this is equivalent to saying that the columns of A span $\mathbb{R}^3$.

(d) Yes, the rows of A span $\mathbb{R}^3$ because the rows of A are the columns of A^T and we know that $\det(A^T) = \det(A) \neq 0$.

5. Let A be an $m \times n$ matrix, and $\mathbf{b}$ a vector in $\mathbb{R}^m$. Suppose that $\mathbf{v}_1$ and $\mathbf{v}_2$ are both solutions to the equation $A\mathbf{v} = \mathbf{b}$. What fundamental matrix space of A does $\mathbf{v}_1 - \mathbf{v}_2$ belong to? Explain.

Solution: The difference $\mathbf{v}_1 - \mathbf{v}_2$ belongs to the null space of A because

$$A(\mathbf{v}_1 - \mathbf{v}_2) = A\mathbf{v}_1 - A\mathbf{v}_2 = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$