MATH 369 Homework 11

Due: Thursday May 9th, in class.

This homework is a little longer because it will be worth more points, but you will also have more time to do it.

On this homework, you do not need to show your work for row reduction calculations or matrix inverse calculations. If you choose to use software however, remember that you will need to be able to do calculations by hand on the final (the problems will be of manageable size).

1. Let B and B' be the bases

$$B = \left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$$

and

$$B' = \left\{ \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \right\}.$$

(a) Find the transition matrix $P_{\text{stan}\to B}$ from the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to B.

(b) Find the transition matrix $P_{B\to B'}$ from B to B'.

(c) Use (a) to find

$$[\mathbf{w}]_B$$

where in terms of the standard basis

$$\mathbf{w} = \begin{pmatrix} -5 \\ 8 \\ -5 \end{pmatrix}$$

(d) Use (b) and your previous answer to find $[\mathbf{w}]_{B'}$.

(e) Make a conjecture how we should interpret $P_{B\to B'}P_{\text{stan}\to B}$.

Solution:

(a) There are quite a few ways to solve this. One way is to note that

$$P_{B\to\text{stan}} = \begin{pmatrix} 2 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$$

Hence

$$P_{\text{stan}\to B} = (P_{B\to\text{stan}})^{-1} = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} & -\frac{5}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \\ -1 & 0 & 2 \end{pmatrix}.$$

(b) We have

$$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} + 5\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix},$$

$$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix} - 3\begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix},$$

$$\begin{pmatrix}1\\2\\1\end{pmatrix} = \frac{5}{2}\begin{pmatrix}3\\1\\-5\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\1\\-3\end{pmatrix} + 6\begin{pmatrix}-1\\0\\2\end{pmatrix}.$$

Hence

$$P_{B\to B'} = \begin{pmatrix} 3 & 2 & \frac{5}{2} \\ -2 & -3 & -\frac{1}{2} \\ 5 & 1 & 6 \end{pmatrix}.$$

(c) We have

$$[\mathbf{w}]_B = P_{\text{stan}\to B}\mathbf{w} = \begin{pmatrix}9\\-9\\-5\end{pmatrix}_B.$$

(d) We have

$$[\mathbf{w}]_{B'} = P_{B\to B'}[\mathbf{w}]_B = \begin{pmatrix} 3 & 2 & \frac{5}{2} \\ -2 & -3 & -\frac{1}{2} \\ 5 & 1 & 6 \end{pmatrix}\begin{pmatrix}9\\-9\\-5\end{pmatrix}_B = \begin{pmatrix}-\frac{7}{2}\\ \frac{23}{2} \\ 6\end{pmatrix}_{B'}.$$

(e) $P_{B\to B'}P_{\text{stan}\to B} = P_{\text{stan}\to B'}$.

2. Let

$$A = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

(a) Compute

$$\begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}.$$

What do you notice?

(b) Write A in terms of the basis

$$B = \left\{ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix} \right\}.$$

(c) If you were working with matrix A which basis would you rather work with: the standard basis or B? (There is no right answer here, the question is just supposed to make you think.)

Solution:

(a) We have

$$\begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}.$$

and

$$\begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}2\\-2\end{pmatrix}.$$

(b) We have that

$$P_{B\to\text{stan}} = \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$$

so that

$$P_{\text{stan}\to B} = (P_{B\to\text{stan}})^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

Then written in terms of B,

$$P_{\text{stan}\to B} A P_{B\to\text{stan}} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

(c) Since A is a diagonal matrix in B, then this would probably be an easier basis to work with A in.

3. Let A be the matrix

$$A = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}.$$

(a) What is the characteristic equation for A?

(b) What are the eigenvalues of A?

(c) What are the eigenvectors of A?

Solution:

(a) The characteristic equation of A is

$$(1-\lambda)(1-\lambda) = 0.$$

(b) The only eigenvalue of this matrix is 1.

(c) There is only a 1-dimensional eigenspace for eigenvalue 1. It is spanned by

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

4. Let

$$A = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

(a) One eigenvalue is 3 with corresponding eigenvector

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

What are the other two eigenvalue/eigenvector pairs.

(b) What is $A^{10}\mathbf{v}_3$? (This should take very little calculation.)

Solutions:

(a) The characteristic equation of this matrix is

$$-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0.$$

Since we know that 3 is an eigenvalue, then we can divide this by $(\lambda - 3)$,

$$\frac{-\lambda^3 + 6\lambda^2 - 11\lambda + 6}{\lambda - 3} = -\lambda^2 + 3\lambda - 2 = (2-\lambda)(-1+\lambda).$$

Therefore the other eigenvalues are 2 and 1. We will find eigenvectors for each of these eigenvalues:

- To find the eigenvalue for $\lambda = 2$ we need to calculate the null space of

$$A - 2I_3 = \begin{pmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{pmatrix}.$$

The reduced row echelon form of this matrix with a right column equal to zero (since we are calculating the null space) is

$$\begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence a basis for the null space of the matrix $A - 2I_2$ is

$$\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}.$$

We can check that this is exactly an eigenvector of A with eigenvalue 2.

- To find the eigenvalue for $\lambda = 1$ we need to calculate the null space of

$$A - I_3 = \begin{pmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}.$$

The reduced row echelon form of this matrix with a right column equal to zero (since we are calculating the null space) is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence a basis for the null space of the matrix $A - I_3$ is

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

We can check that this is exactly an eigenvector of A with eigenvalue 1.

(b) We have $A\mathbf{v}_3 = 3^{10}\mathbf{v}_3 = 59049\mathbf{v}_3$.

5. Suppose that A is an $n \times n$ matrix, and $\mathbf{v}_1$ and $\mathbf{v}_2$ are both eigenvectors with eigenvalue 3 so that

$$A\mathbf{v}_1 = 3\mathbf{v}_1 \qquad \text{and} \qquad A\mathbf{v}_2 = 3\mathbf{v}_2.$$

(a) Show that $\mathbf{v}_1 + \mathbf{v}_2$ is an eigenvector with eigenvalue 3.
(b) Show that $10\mathbf{v}_1$ is an eigenvector with eigenvalue 3.
(c) Conclude the space of all eigenvectors of A with eigenvalue 3 is a subspace of $\mathbb{R}^n$.

Solution:

(a) We have

$$A(\mathbf{v}_1 + \mathbf{v}_2) = A\mathbf{v}_1 + A\mathbf{v}_2 = 3\mathbf{v}_1 + 3\mathbf{v}_2 = 3(\mathbf{v}_1 + \mathbf{v}_2),$$

so $\mathbf{v}_1 + \mathbf{v}_2$ is an eigenvector with eigenvalue 3.

(b) We have

$$A(10\mathbf{v}_1) = 10A\mathbf{v}_1 = 10(3\mathbf{v}_1) = 3(10\mathbf{v}_1).$$

(c) Recall that for a set of vectors to be a subspace, two properties must be true:
- If $\mathbf{v}_1$ and $\mathbf{v}_2$ are both in the set of vectors, then $\mathbf{v}_1 + \mathbf{v}_2$ is also in the set.
- If $\mathbf{v}_1$ is in the set, then for any scalar k, $k\mathbf{v}_1$ is also in the set.

But our work above shows that both of these properties are true for the set of all eigenvectors of A with eigenvalue 3.

6. Consider the matrix

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

R rotates a vector $\mathbf{v} \in \mathbb{R}^2$ by $90°$. Does R have any eigenvalues/eigenvectors? If not, give a rough geometric explanations about why this might be.

Solution: The characteristic equation of this matrix is

$$\lambda^2 + 1 = 0.$$

This polynomial has no real roots so this matrix has no real eigenvalues.

Note that for any nonzero vector $\mathbf{v}$ in $\mathbb{R}^2$, rotation by $90°$ will change the direction of $\mathbf{v}$ in such a way that $R\mathbf{v}$ is not a scalar multiple of $\mathbf{v}$ (this would not be true if R was rotation by $180°$). Therefore it makes sense that R does not have any eigenvalues/eigenvectors.