Research Statement

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1 Introduction

My research interests in mathematics focus on the connections between representation theory, combinatorics, and probability theory. I am particularly interested in the interaction between combinatorial representation theory and categorical representation theory. In this research statement I will describe some of my past and current projects which include:

- investigating a surprising connection between graphical Heisenberg categories on the one hand and symmetric functions and stochastic growth processes on the other.
- the study of crystal graphs associated to affine type quantum groups and their connections to the representation theory of cyclotomic quiver Hecke algebras and lattices of Young diagrams.

I am also doing work in geometric data analysis, machine learning, and compressive sensing. I briefly summarize some of these projects in Section 3.

2 Representation theory and combinatorics

2.1 The categorification program

Initiated by Crane and Frenkel [4], the idea underlying the categorification program is that many of the fundamental structures in algebra, geometry, and topology are the shadow of richer (often undiscovered) higher categories. Categorification, the process of revealing these higher analogues, then involves lifting set-theoretic constructions to category-theoretic constructions, roughly: replacing sets with categories, functions with functors, and equalities with isomorphisms. Though Khovanov's categorification of the Jones polynomial via Khovanov homology [15] may be the most famous example, far more familiar examples exist (for example, singular homology as a categorification of Betti numbers). Categorifying the fundamental algebras from representation theory has been a major, ongoing project in the field. Furthermore, theory developed in the course of this program has led to results of independent interest, such as the proof of one of the Kazhdan-Lusztig conjectures for all Coxeter types [5] via Soergel bimodules [35].

As consequence of this progress, a host of new categories have been discovered which sit "above" and "control" many of the classical objects in representation theory such as Hecke algebras, Heisenberg algebras, and quantum groups. These categories are rich in structure, and a major drive in my research has been using these categorical constructions to better understand the connections between representation theory and other areas of math, particularly combinatorics and probability theory. My first research project described below studies how categorifications of infinite dimensional Heisenberg algebras both provide graphical realizations of analogues of the symmetric functions and also naturally provide the framework for stochastic growth processes motivated by representation theory and combinatorics [22, 23]. In my second project, the categorification of quantum groups, via quiver Hecke algebras, is used to show how a fundamental isomorphism of combinatorially defined crystals can in fact be understood to be a consequence of induction/restriction functors from representation theory [24].

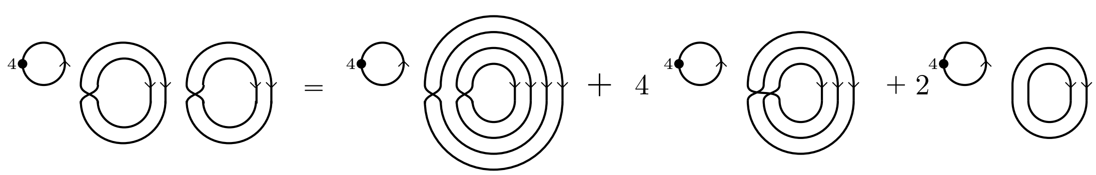
2.2 Heisenberg categories, symmetric functions, and growth processes

A *tower of algebras* $\{A_n\}_{n\geqslant 0}$ is a collection of nested algebras $A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$. The classic example is the symmetric group algebras $\{\mathbb{C}[S_n]\}_{n\geqslant 0}$. A result of Geissinger [7] showed that the direct sum of all Grothendieck groups of symmetric groups $\bigoplus_{n\geqslant 0} K_0(\mathbb{C}[S_n]\text{-mod})$ carries the structure of the Fock space representation of the infinite dimensional Heisenberg algebra $\mathfrak{h}$. In particular, the elements of $\mathfrak{h}$ act by linear operators coming from induction and restriction functors between symmetric group representations. This observation was the first hint of a deeper story in which a vastly richer *Heisenberg category* $\boldsymbol{\mathcal{H}}$, acts via genuine induction and restriction functors on the module categories $\bigoplus_{n\geqslant 0} A_n\text{-mod}$ for a tower of algebras $\{A_n\}_{n\geqslant 0}$.

$$\begin{tikzcd}
\boldsymbol{\mathcal{H}} \arrow[r, "\substack{\text{categorical}\\ \text{representation}}"] \arrow[d, rightsquigarrow, "K_0"] & \bigoplus_{n\geqslant 0} A_n\text{-mod} \arrow[d, rightsquigarrow, "K_0"] \\
\mathfrak{h} \arrow[r, "\text{representation}"] & \bigoplus_{n\geqslant 0} K_0(A_n\text{-mod})
\end{tikzcd}$$

The first Heisenberg category was defined by Khovanov [16] with $A_n = \mathbb{C}[S_n]$. Subsequently, Heisenberg categories associated to a range of towers of algebras have been discovered, including: Hecke algebras [27], Sergeev algebras [3], wreath product algebras [32], and degenerate cyclotomic affine Hecke algebras [28, 2]. In all these examples, $\boldsymbol{\mathcal{H}}$ can be defined diagrammatically, i.e. through an isotopy invariant calculus of planar diagrams with local relations and decorations.

All Heisenberg categories are additive monoidal categories and hence their center, the endomorphisms of the monoidal identity $Z(\boldsymbol{\mathcal{H}}) := \mathrm{End}_{\boldsymbol{\mathcal{H}}}(\mathbb{1})$, is a commutative algebra. In such diagrammatically defined categories, the center has a helpful diagrammatic definition as the algebra of all closed diagrams.

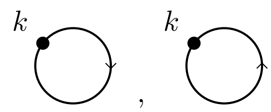


One of my ongoing projects is centered around studying the commutative algebra $Z(\boldsymbol{\mathcal{H}})$ corresponding to different towers of algebras. This began with a study of the center of Khovanov's Heisenberg category $Z(\boldsymbol{\mathcal{H}}_{\mathbb{C}[S_n]})$, which we proved is isomorphic to Λ^*, the algebra of shifted symmetric functions [22]. A deformation of the classical symmetric functions,

Λ^* was discovered by Kerov and Olshanksi [14] and further investigated by Okounkov and Olshanski [31] among many others. It has connections to Lie theory [31], Gromov-Witten invariants [30], and asymptotic representation theory [9].

As a consequence of our algebra isomorphism $Z(\mathcal{H}_{\mathbb{C}[S_n]}) \cong \Lambda^*$, we were able to describe an isotopy invariant graphical calculus for Λ^* using closed diagrams and a set of local relations. We showed how individual generators such as shifted Schur functions and properties of Λ^* such as involutions, have a natural graphical interpretation.

Surprisingly, under the isomorphism $Z(\mathcal{H}_{\mathbb{C}[S_n]}) \cong \Lambda^*$, the simplest generating diagrams (which are clockwise or counterclockwise oriented *bubbles* with k dots on them)



are shifted symmetric functions encoding moments for a family of probability measures, Kerov's *transition* and *co-transition measures*, parametrized by Young diagrams. This same probability measure was used as a tool to study the famous Plancherel growth process [12, 13]. This unexpected connection points to a deeper relationship between the combinatorics of categorical representation theory and the combinatorics of asymptotic representation theory. Exploring this connection more closely is one of the themes of my current work.

Furthermore, the results described above do not seem to be a symmetric group restricted phenomenon. In [23], we investigated the center of the twisted Heisenberg category $\mathcal{H}_{\mathbb{S}_n}$ of Cautis and Sussan [3] which is associated to the tower of Sergeev algebras $\{\mathbb{S}_n\}_{n\geqslant 0}$. While we showed that in this case $Z(\mathcal{H}_{\mathbb{S}_n})$ is isomorphic to the subalgebra of the symmetric functions generated by odd power sums $\mathbb{C}[p_1, p_3, \dots]$, similar connections to combinatorics and asymptotic representation theory hold. Notably, the bubble generators of $Z(\mathcal{H}_{\mathbb{S}_n})$ also correspond to functions encoding transition probabilities of a stochastic growth process (this time however on the graph of all strict Young diagrams).

There are a broad range of questions that deserve to be studied in this new field. Two that I am currently investigating are:

- *Centers of Heisenberg categories and new analogues of symmetric functions:* What new analogues of symmetric functions arise from centers of the more exotic Heisenberg categories (such as those associated with towers of degenerate cyclotomic affine Hecke algebras [28])? What combinatorial properties do they have?

- *Representation-theoretic growth processes and Heisenberg categories:* Do all Heisenberg categories define stochastic growth processes? Can we use this observation to define stochastic growth processes via functors and natural isomorphisms encoded in the graphical calculus of the associated Heisenberg category?

2.3 Crystal structure in representation categories

Crystals were first discovered by Kashiwara as combinatorial objects attached to certain quantum group $U_q(\mathfrak{g})$ representations that appear in the limit $q \to 0$ [11]. Since they can be described via colored directed graphs, crystals are substantially simpler than their corresponding representations, and in many cases allow us to translate questions in algebra to questions in combinatorics.

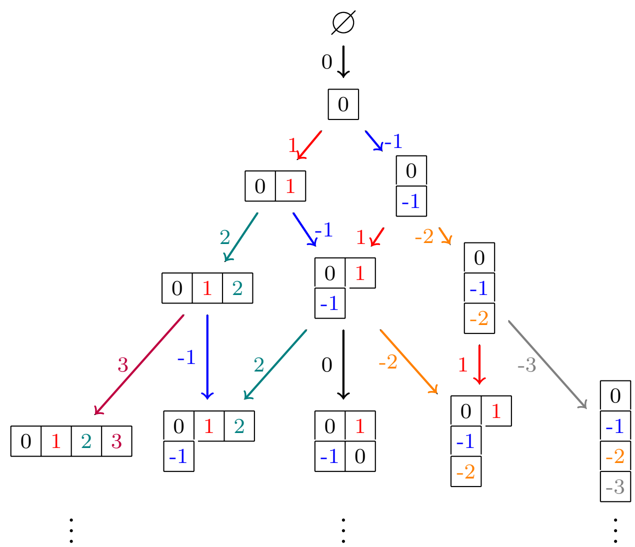


Figure 1: Young's lattice is a combinatorial model for the crystal corresponding to the highest weight representation V_{Λ_0} of $U_q(\mathfrak{sl}_\infty)$.

Crystals can also arise when a quantum group acts on a category via endofunctors. The first example of this was found when it was noticed that graphs defined via the branching rules of affine and cyclotomic Hecke algebras carry the structure of a crystal graph [25, 1, 8]. Only in hindsight was it realized that this is the result of a hidden categorical action of a quantum group. Subsequently many other representation categories with categorical action of a quantum group have been shown to carry a crystal structure. Examples include the category of finite-dimensional quiver Hecke algebra representations [26] and category $\mathcal{O}$ for the cyclotomic rDAHA [34].

categories	simple object in $\boldsymbol{\mathcal{C}}$	induction functor F_i between simple objects in $\boldsymbol{\mathcal{C}}$
	$\Big\downarrow K_0$	$\Big\downarrow K_0$
crystals	node of crystal	Kashiwara operator $\widetilde{f}_i$ (i-colored edge)

When a representation category carries a crystal structure one can try to understand how the combinatorics of the crystal lift to the representation theory. Such crystal-theoretic considerations allowed Lauda-Vazirani to classify all simple quiver Hecke algebra representations [26].

2.4 Results on crystal structure in categories

Khovanov and Lauda [17], [18] and independently Rouquier [33] invented *quiver Hecke algebras* R (also known as *Khovanov-Lauda-Rouquier (KLR) algebras*) to categorify the upper half $U_q^+(\mathfrak{g})$ of the quantum group for any fixed symmetrizable Kac-Moody algebra $\mathfrak{g}$.

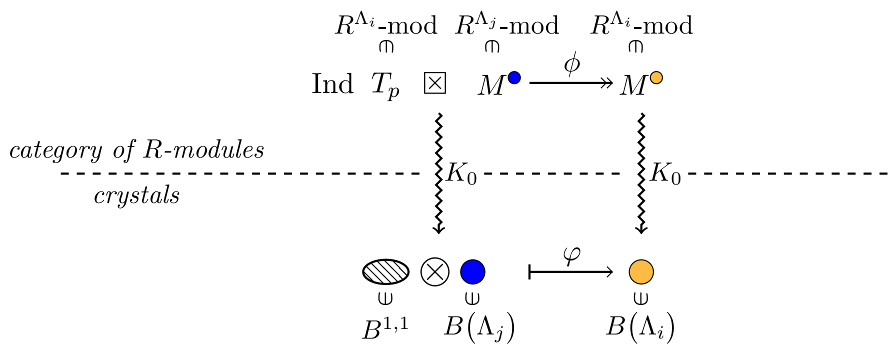
For each integral dominant weight Λ, the algebra R has a finite-dimensional quotient called a *cyclotomic quiver Hecke algebra* R^Λ. Lauda-Vazirani showed that the collection of simple R^{Λ}-modules carries the structure of $B(\Lambda)$ which is the crystal of the corresponding highest weight representation of $U_q(\mathfrak{g})$. Here nodes are the simple R^{Λ}-modules and Kashiwara operators $\widetilde{f}_i$ are given by refined induction functors between the simple modules [26].

When $\mathfrak{g}$ is of classical affine type, one class of crystals not previously appearing in the context of quiver Hecke algebras is the *Kirillov-Reshetikhin (KR) crystals* [29]. These are denoted by $B^{r,s}$ where $r \in I\backslash\{0\}$ for Dynkin indexing set I and $s \in \mathbb{Z}_{>0}$. A notable property of $B^{1,1}$ in particular is that for certain choices of fundamental weight Λ_j there is an isomorphism of crystals

$$\varphi : B^{1,1} \otimes B(\Lambda_j) \xrightarrow{\sim} B(\Lambda_i), \tag{1}$$

for some $i \in I$ [10]. This isomorphism is particularly exciting since it allows one to study the (infinite) crystal $B(\Lambda_i)$ through tensor products of the finite crystals $B^{1,1}$.

Given that R^{Λ_i}-mod (respectively R^{Λ_j}-mod) carries the structure of $B(\Lambda_i)$ (resp. $B(\Lambda_j)$), it is natural to ask whether (1) actually reflects representation-theoretic structure in R^{Λ_i}-mod. Vazirani and I showed that this is the case in [24]. Recall that each simple R^{Λ_i}-module M° corresponds to a node $\circ$ in $B(\Lambda_i)$, and similarly each simple R^{Λ_j}-module $M^{\bullet}$ corresponds to a node $\bullet$ in $B(\Lambda_j)$. We show that each assignment of the crystal isomorphism (1) is actually the shadow of a canonically defined surjective R^{Λ_i}-module homomorphism ϕ.



The simple R^{Λ_i}-module T_p here belongs to a family of "trivial" R^{Λ_i}-modules parametrized by paths p in the crystal $B^{1,1}$. Most importantly, the above correspondence intertwines the action of the Kashiwara operators $\widetilde{f}_i$ of the crystal graph with the action of their categorical analogue, the refined induction functors.

3 Other work: geometric data analysis, machine learning, and compressive sensing

As data is generated at a faster and faster rate, tools for extracting information from large, high-dimensional data sets have become increasingly valuable across science, engineering, and industry. Mathematicians have a prominent role to play in helping to design such algorithms. In this section I will briefly summarize some of my work in this field.

- *Secant-based dimensionality reduction*: The secant set S of a data set $D \subset \mathbb{R}^n$ encodes information about the spatial relationships between points in D and consequently is an important object of study when attempting to find projections of D that preserve D's structure.

In [19] we proposed an algorithm called the *secant avoidance projection (SAP) algorithm* to find projections which preserve the secant set of a data set. This iterative algorithm is designed for use on a GPU architecture with fast convergence for even very high-dimensional data sets. One issue that arose with this algorithm was that when $|D|$ is very large, it can be difficult to store all secants in S. We developed the *hierarchical secant avoidance projection (HSAP) algorithm* for such situations [21]. HSAP utilizes the intrinsic structure of D to inform subsampling and approximation of secants to vastly reduce the number of secants which need to be stored.

Finally, part of the output of both of these algorithms is the length of the projected secant least well preserved by the final output projection. By computing these lengths for all projection dimensions we arrive at a statistic for the data set D which we call the *κ-profile*. When the data sits on an m-manifold in $\mathbb{R}^n$ for $m < n$, the κ-profile can be related to m via Whitney's embedding theorem. We show in [20] that the κ-profile captures fundamental structure in D. In particular, for a data set $D(t)$ which changes with respect to a time parameter t, changes in the state of $D(t)$ over time are reflected by changes in $\kappa(t)$. As an example of this, we showed that the κ-profile for a weather data set taken in the Atlantic Ocean changed dramatically when a storm past through the region.

- *Searching for pure signals in hyperspectral imagery via the geometry of the Grassmannian*: Unlike RGB images, which only sample from three different spectral bands of light (red, green, and blue), hyperspectral imagery can sometimes sample more than 200. While hyperspectral images thus have strong discriminatory power, it can be difficult to extract information from them because they are often very high dimensional. One particularly important problem is understanding how to isolate the "most pure" signals from a hyperspectral image. Geometrically, this corresponds to picking out those spatial locations in the image whose spectral curve sits on the convex hull formed by all spectral curves in the image. In [6] we show how if one considers local patches rather than individual pixels, this endmember extraction problem is most naturally realized on a Grassmann manifold. We propose an algorithm to solve the problem in this setting.

- *Compressive sensing algorithm design for a single-pixel camera*[1]: Over the course of the last year, I have been involved with two projects with an industry partner which involve developing compressive sensing (CS) algorithms for single pixel imaging devices. This has involved choosing a CS sampling framework which captures the structure of images and also fits hardware constraints, developing and testing CS reconstruction algorithms which can reconstruct images in real-time with significant device noise, and writing a software package that implements all these algorithms for use in the field.

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