A surprising connection between Khovanov's Heisenberg category and the asymptotic representation theory of symmetric groups.

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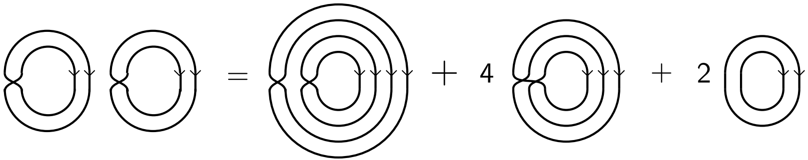
(Joint with Anthony Licata and Stuart Mitchell)

Fall Central AMS Sectional 2016: Combinatorial representation theory



Outline of the story

Khovanov's Heisenberg category $\mathcal{H}'$

Result: $\mathrm{End}_{\mathcal{H}'}(\mathbb{1}) \cong \Lambda^*$

Asymptotic rep. theory of symmetric groups

Shifted symmetric functions Λ^*

Khovanov's Heisenberg category $\mathcal{H}'$

Khovanov's Heisenberg category

Khovanov proposed a monoidal $\mathbb{C}$-linear category $\mathcal{H}'$, called *Khovanov's Heisenberg category* to categorify the Heisenberg algebra H.

$$\begin{tikzcd} \mathrm{Kar}(\mathcal{H}') \arrow[d, squiggly, "K_0"] \\ H \hookrightarrow K_0(\mathrm{Kar}(\mathcal{H}')) \end{tikzcd}$$

$$\left(\text{Conjecture: } H \cong K_0(\mathrm{Kar}(\mathcal{H}')) \right)$$

The idea

Classical: Heisenberg algebra H acts on direct sum of Grothendieck group of symmetric groups by linear operators from induction/restriction functors.

$$H \xrightarrow{\text{representation}} \bigoplus_{n\geq 0} K_0(\mathbb{C}[S_n]\text{-mod})$$

The idea

Classical: Heisenberg algebra H acts on direct sum of Grothendieck group of symmetric groups by linear operators from induction/restriction functors.

New: Khovanov's *Heisenberg category* $\mathcal{H}'$ acts on representation category of symmetric groups by induction/restriction functors themselves.

$$
\begin{tikzcd}
\mathrm{Kar}(\mathcal{H}') \arrow[r, "\text{categorical representation}"] \arrow[d, squiggly, "K_0"] & \bigoplus_{n\geq 0} \mathbb{C}[S_n]\text{-mod} \arrow[d, squiggly, "K_0"] \\
H \arrow[r, "\text{representation}"] & \bigoplus_{n\geq 0} K_0(\mathbb{C}[S_n]\text{-mod})
\end{tikzcd}
$$

Describing $\mathcal{H}'$

The category $\mathcal{H}'$:

Objects - Sequences of Q_+ and Q_-.

Morphisms - Oriented compact 1-manifolds immersed in the plane strip $\mathbb{R} \times [0,1]$.

You should think:

- Q_+ is 'like' $\mathrm{Ind}_{S_n}^{S_{n+1}}$,
- Q_- is 'like' $\mathrm{Res}_{S_{n-1}}^{S_n}$,
- Morphisms are 'like' natural transformations between compositions of these functors

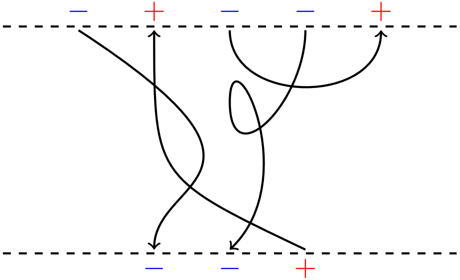
$$Q_+Q_-Q_-Q_+Q_- \quad \text{is 'like'} \quad \mathrm{Ind}_{S_{n-2}}^{S_{n-1}}\mathrm{Res}_{S_{n-2}}^{S_{n-1}}\mathrm{Res}_{S_{n-1}}^{S_n}\mathrm{Ind}_{S_{n-1}}^{S_n}\mathrm{Res}_{S_{n-1}}^{S_n}$$

Example:

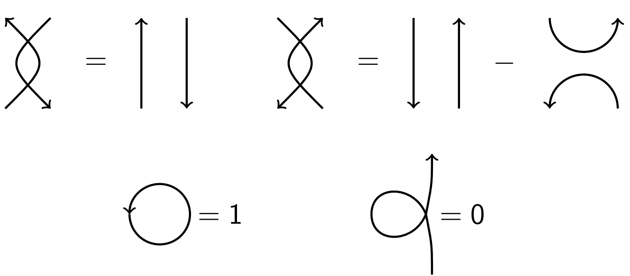
Objects Q_{-+--+}

$$\Big\uparrow \text{morphism}$$

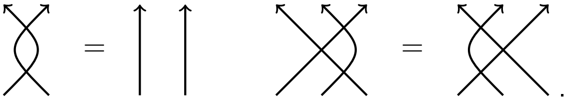
Object Q_{--+}



Local relations



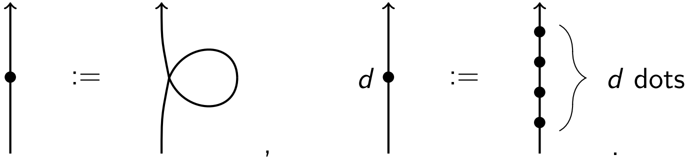
H relation
$Q_- Q_+ \cong Q_+ Q_- + 1$



S_n relations

Dots

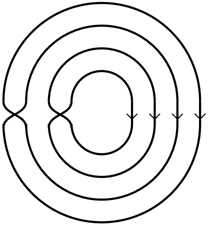
For simplification we write:



(As natural transformations correspond to multiplication by Jucys-Murphy element J_n).

Center of $\mathcal{H}'$

By definition the center of $\mathcal{H}'$ is $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ or the commutative algebra of all closed diagrams.



$\mathcal{H}'$ is **rich** in representation-theoretic data (contains all symmetric groups, affine degenerate Hecke algebras).

$$\Downarrow$$

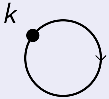
$\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ should be of combinatorial interest.

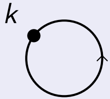
Center of $\mathcal{H}'$

Theorem (Khovanov)

$$\mathrm{End}_{\mathcal{H}'}(\mathbb{1}) \cong \mathbb{C}[c_0, c_1, \ldots] \cong \mathbb{C}[\tilde{c}_2, \tilde{c}_3, \ldots]$$

where

$c_k \longmapsto$ 

$\tilde{c}_k \longmapsto$ 

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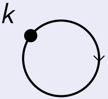
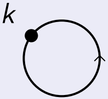
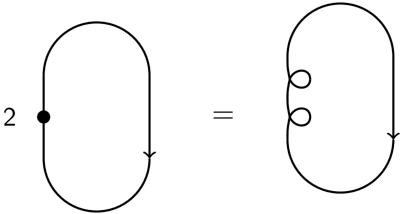
$c_k \longmapsto$ 

$\tilde{c}_k \longmapsto$ 

Reminder:



Theorem (K., Licata, Mitchell)

$\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ *is isomorphic as a* $\mathbb{C}$*-algebra to a deformation of the symmetric functions called the shifted symmetric functions* Λ^*,

$$\mathrm{End}_{\mathcal{H}'}(\mathbb{1}) \cong \Lambda^*.$$

The shifted symmetric functions Λ^*

Shifted symmetric functions

Call $f \in \mathbb{C}[x_1, \ldots, x_n]$ *shifted symmetric* if it is symmetric in the new variables

$$x_i' = x_i - i.$$

The *algebra of shifted symmetric functions* Λ^* has similar construction to classical symmetric functions Λ.

Λ	Λ^*
elements symmetric	elements shifted symmetric
graded by polynomial degree	filtered by polynomial degree

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Proposition (Okounkov-Olshanski)

$$gr(\Lambda^*) \cong \Lambda.$$

Shifted symmetric functions

Λ^* has many generators/bases analogous to Λ:

- $p^{\#}_{\lambda} = p_{\lambda} +$ l.o.t., *shifted power sums*
- $s^*_{\lambda} = s_{\lambda} +$ l.o.t., *shifted Schur functions*
- $e^*_k = e_k +$ l.o.t., *elementary shifted functions*
- $h^*_k = h_k +$ l.o.t., *homogeneous shifted functions*

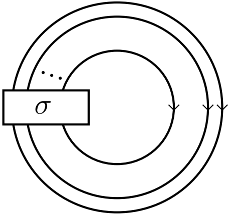
Λ^* as functions on $\mathbb{Y}$

Λ^* can also be realized as a subalgebra of functions on $\mathbb{Y}$, such that if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ and $f \in \Lambda^*$ then,

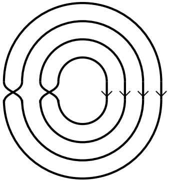
$$f(\lambda) := f(\lambda_1, \lambda_2, \ldots, \lambda_r, 0, 0, \ldots).$$

Dictionary between Λ^* and $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$

If $\sigma \in S_n$ of conjugacy class λ, then

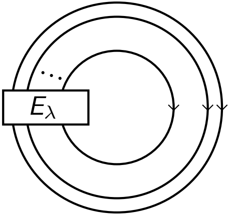
$p^{\#}_{\lambda} \longmapsto$ 

Example:

$p^{\#}_{(2,2)} \longmapsto$ 

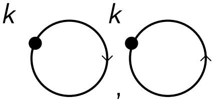
If E_λ is the Young idempotent of $\mathbb{C}[S_n]$ associated to λ then

$$s_\lambda^* \longmapsto \frac{1}{\dim(L^\lambda)}$$



Note $s_{(n)}^* = h_n^*$, $s_{(1^n)}^* = e_n^*$.

Remaining question:

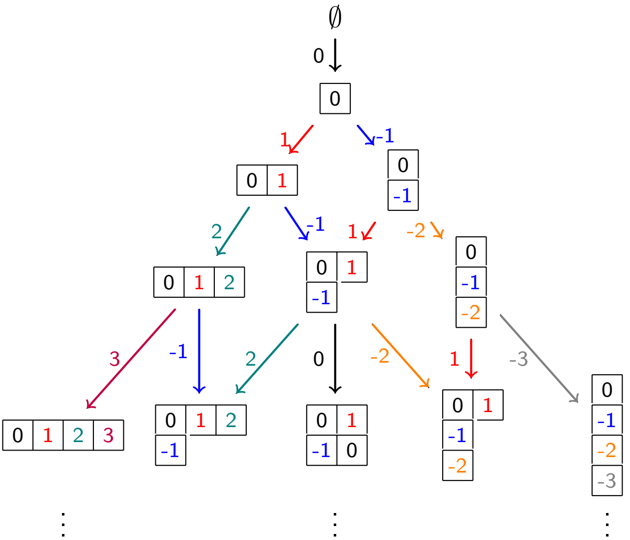


$$\xmapsto{\quad k \quad} \; ? \in \Lambda^*$$

For this we (surprisingly) need to turn to asymptotic representation theory.

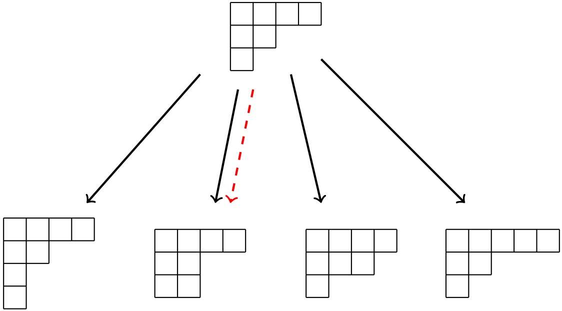
Moments of Kerov's transition measure $\widehat{m}_k$

Young's Lattice



Motivation for transition measure

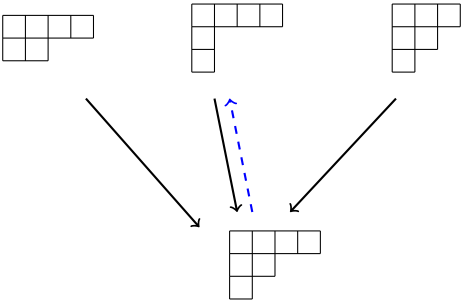
If start are partition $\lambda = (4,2,1)$ and move down one arrow, what is probability we land on partition $\mu = (4,2,2)$?



One choice is *transition probability*: $\frac{\dim(L^{(4,2,2)})}{|\mu| \dim(L^{\lambda})}$

Motivation for co-transition measure

Dually, if start are partition $\lambda = (4, 2, 1)$ and move **up** one arrow, what is probability we land on partition $\mu = (4, 1, 1)$?



Co-transition probability: $\frac{\dim(L^{(4,1,1)})}{\dim(L^{\lambda})}$

Moments of the transition/co-transition measure

To study these ideas, for each λ Kerov constructed probability measures on $\mathbb{R}$:

$$\widehat{\omega}_\lambda = \textit{ transition measure for } \lambda$$

$$\widehat{\omega}_\lambda = \textit{ co-transition measure for } \lambda$$

$\widehat{m}_k$, $\widecheck{m}_k$, and Λ^*

Set:

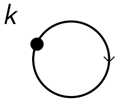
- kth moment of $\widehat{\omega}_\lambda = \widehat{m}_k(\lambda)$,
- kth moment of $\widecheck{\omega}_\lambda = \widecheck{m}_k(\lambda)$.

Can view $\widehat{m}_k$ and $\widecheck{m}_k$ as functions on $\mathbb{Y}$ by

$$\lambda \xmapsto{\widehat{m}_k} \widehat{m}_k(\lambda), \qquad \lambda \xmapsto{\widecheck{m}_k} \widecheck{m}_k(\lambda).$$

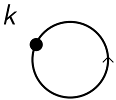
Then $\widehat{m}_k$ and $\widecheck{m}_k$ belong to Λ^*.

Then



$$\longmapsto \quad |\lambda|\breve{m}_k = \widehat{b}_k \in \Lambda^*$$

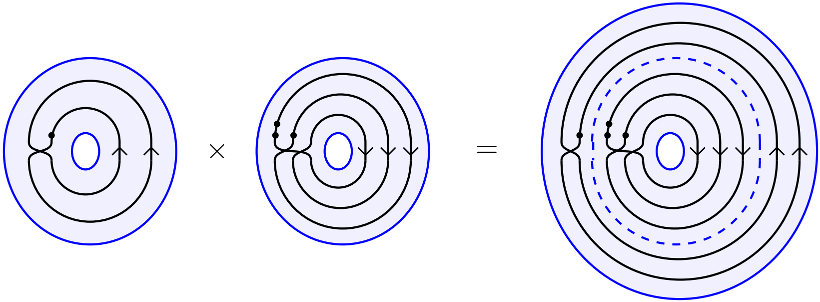
($\widehat{b}_k$ is a *Boolean* cumulant for $\widehat{m}_k$) and



$$\longmapsto \quad \widehat{m}_k \in \Lambda^*$$

$W_{1+\infty}$ and Λ^*

Another construction: The *trace* Tr (or zeroth Hochschild homology) of a Heisenberg category is noncommutative algebra of diagrams on an annulus.



Cautis-Lauda-Licata-Sussan showed $\mathrm{Tr}(\mathcal{H}') \cong W_{1+\infty}$ the vertex algebra from conformal field theory.

$W_{1+\infty}$ and Λ^*

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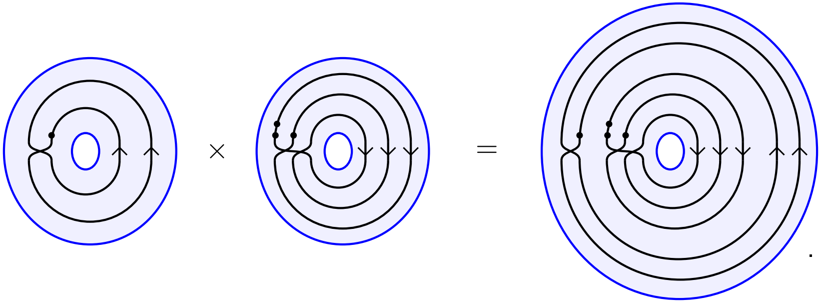
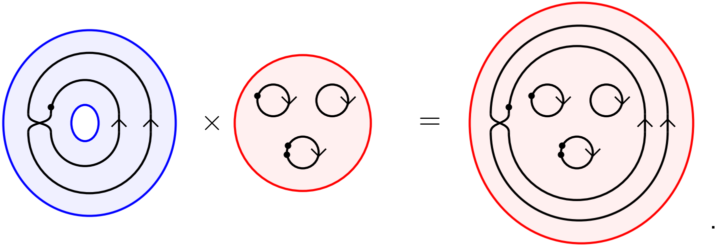


Cautis-Lauda-Licata-Sussan showed $\mathrm{Tr}(\mathcal{H}') \cong W_{1+\infty}$ the vertex algebra from conformal field theory.

$W_{1+\infty}$ and Λ^*

There is a natural action of $\mathrm{Tr}(\mathcal{H}')$ on $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ by placing a closed diagram from $\mathrm{End}_{\mathcal{H}'}(\mathbb{1})$ inside an annulus diagram from $\mathrm{Tr}(\mathcal{H}')$.



This gives purely planar realization of an action of $W_{1+\infty}$ on Λ^* which was first considered by Lascoux-Thibon.

Thank you.