MATH 369 Homework 11

Due: Friday May 10th, via email, under my office door, or during class on May 9th.

This homework is a little longer because it will be worth more points, but you will also have more time to do it.

On this homework, you do not need to show your work for row reduction calculations or matrix inverse calculations. If you choose to use software however, remember that you will need to be able to do calculations by hand on the final (the problems will be of manageable size).

1. Let B and B' be the bases
$$B = \left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$$
and
$$B' = \left\{ \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \right\}.$$
   (a) Find the transition matrix $P_{\text{stan}\to B}$ from the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to B.
   (b) Find the transition matrix $P_{B\to B'}$ from B to B'.
   (c) Use (a) to find
   $$[\mathbf{w}]_B$$
   where in terms of the standard basis
   $$\mathbf{w} = \begin{pmatrix} -5 \\ 8 \\ -5 \end{pmatrix}$$
   (d) Use (b) and your previous answer to find $[\mathbf{w}]_{B'}$.
   (e) Make a conjecture about how we should interpret $P_{B\to B'}P_{\text{stan}\to B}$.

2. Let
$$A = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$$
   (a) Compute
   $$\begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$
   What do you notice?
   (b) Write A in terms of the basis
   $$B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$
   (c) If you were working with matrix A which basis would you rather work with: the standard basis or B? (There is no right answer here, the question is just supposed to make you think.)

3. Let A be the matrix
$$A = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}.$$
   (a) What is the characteristic polynomial for A?
   (b) What are the eigenvalues of A?

(c) What are the eigenvectors of A?

4. Let

$$A = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

(a) One eigenvalue is 3 with corresponding eigenvector

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

What are the other two eigenvalue/eigenvector pairs?

(b) What is $A^{12}\mathbf{v}_3$?

5. Suppose that A is an $n \times n$ matrix, and $\mathbf{v}_1$ and $\mathbf{v}_2$ are both eigenvectors with eigenvalue 3 so that

$$A\mathbf{v}_1 = 3\mathbf{v}_1 \qquad \text{and} \qquad A\mathbf{v}_2 = 3\mathbf{v}_2.$$

(a) Show that $\mathbf{v}_1 + \mathbf{v}_2$ is an eigenvector of A with eigenvalue 3.

(b) Show that $10\mathbf{v}_1$ is an eigenvector with eigenvalue 3.

(c) Use these observations to show that the space of all eigenvectors of A with eigenvalue 3 is a subspace of $\mathbb{R}^n$.

6. Consider the matrix

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Recall that R rotates a vector $\mathbf{v} \in \mathbb{R}^2$ by $90°$. Does R have any eigenvalues/eigenvectors? If not, give a rough geometric explanation for why this might be.