Practice Exam 1

1. (a) Give a definition of a closed set.

   (b) Give at least two equivalent definitions of a continuous function.

   Solution: See textbook or lecture notes.

2. Let $\mathbf{u}$ and $\mathbf{v}$ be two points in $\mathbb{R}^n$. Let $R, r > 0$. Show that if $dist(\mathbf{u}, \mathbf{v}) = R + r$, then the open balls $B_R(\mathbf{u})$ and $B_r(\mathbf{v})$ are disjoint (i.e. $B_R(\mathbf{u}) \cap B_r(\mathbf{v}) = \varnothing$).

   Solution: To show that these sets are disjoint we show that if $\mathbf{w} \in B_R(\mathbf{u})$ then $\mathbf{w} \notin B_r(\mathbf{v})$. Since $\mathbf{w} \in B_R(\mathbf{u})$ we must have $dist(\mathbf{w}, \mathbf{u}) < R$. Then by the reverse triangle inequality

   $$\Big|R + r - ||\mathbf{w} - \mathbf{u}||\Big| = \Big|||\mathbf{u} - \mathbf{v}|| - ||\mathbf{w} - \mathbf{u}||\Big| \leqslant ||\mathbf{w} - \mathbf{v}||.$$

   Because $dist(\mathbf{w}, \mathbf{u}) = ||\mathbf{w} - \mathbf{u}|| < R$ we must have that

   $$r < (R - ||\mathbf{w} - \mathbf{u}||) + r = \Big|R + r - ||\mathbf{w} - \mathbf{u}||\Big| \leqslant ||\mathbf{w} - \mathbf{v}||.$$

   Thus $dist(\mathbf{w}, \mathbf{v}) > r$ and hence $\mathbf{w} \notin B_r(\mathbf{v})$.

   Showing that $w \in B_r(\mathbf{v})$ implies $w \notin B_R(\mathbf{u})$ is completely equivalent. This implies that $B_R(\mathbf{u}) \cap B_r(\mathbf{v}) = \varnothing$.

3. Prove that the set $A = \{\, (x, y, z) \in \mathbb{R}^3 \mid x^2 + xy + z^3 \neq 0 \,\} \subseteq \mathbb{R}^3$ is open in $\mathbb{R}^3$.

   Solution: First observe that the function $f : \mathbb{R}^3 \to \mathbb{R}$ is continuous since it is a polynomial in the projection component functions $f = p_1^2 + p_1 p_2 + p_3^3$, and p_1, p_2, p_3 are continuous. Note that $\mathbb{R}^3 \backslash A = f^{-1}(\{0\})$. The intervals $(-\infty, 0)$ and $(0, \infty)$ are open and therefore their union $(-\infty, 0) \cup (0, \infty)$ is open (this is a basic property of open sets). Since f is continuous, $f^{-1}((-\infty, 0) \cup (0, \infty))$ is open. But

   $$f^{-1}((-\infty, 0) \cup (0, \infty)) = A.$$

4. Let A be an unbounded set. Show that there is a sequence $\{\mathbf{u}_k\}_{k \geqslant 1}$ in A such that no subsequence of $\{\mathbf{u}_k\}_{k \geqslant 1}$ converges.

   Solution: Because A is unbounded, for any $M > 0$, we can find $\mathbf{u} \in A$ such that $||\mathbf{u}|| \geqslant M$. Thus for each $k \in \mathbb{Z}_{\geqslant 0}$, there is $\mathbf{u}_k$ such that $||\mathbf{u}_k|| \geqslant k$. In this way we construct a sequence $\{\mathbf{u}_k\}_{k \geqslant 0}$. We show that $\{\mathbf{u}_k\}_{k \geqslant 0}$ has no subsequences that converge. Suppose for a contradiction that there is a subsequence $\{\mathbf{u}_{k_j}\}_{j \geqslant 0}$ of $\{\mathbf{u}_k\}_{k \geqslant 0}$ that converges to $\mathbf{u}$. Set $\ell = ||\mathbf{u}||$. We know that the norm function is continuous so it must be the case that $\lim_{j \to \infty} ||\mathbf{u}_{k_j}|| = ||\mathbf{u}||$. However, since

   $$k_j \leqslant ||\mathbf{u}_{k_j}||$$

   then

   $$\ell < \infty = \lim_{j \to \infty} k_j \leqslant \lim_{j \to \infty} ||\mathbf{u}_{k_j}||$$

   a contradiction.

5. Let $f : \mathbb{R}^n \to \mathbb{R}$ be the function defined so that

   $$f(x_1, x_2, \ldots, x_n) = \begin{cases} 1 & \text{if all } x_i \text{ are rational for } 1 \leqslant i \leqslant n \\ 0 & \text{otherwise.} \end{cases}$$

   Prove that f is not continuous at any point in $\mathbb{R}^n$.

   Solution: We first show that f is not continuous at all points $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ such that all x_i are rational for $1 \leqslant i \leqslant n$. Assume for a contradiction that f is continuous at such a point $\mathbf{x}$. Then for $\epsilon = \frac{1}{2}$ we can find $\delta > 0$ so that for all $\mathbf{v} \in B_r(\mathbf{u})$,

   $$|f(\mathbf{v}) - f(\mathbf{u})| < \frac{1}{2}.$$

But for any $\delta > 0$ we can find an irrational number $y_1 \in \mathbb{R}$ such that $|y_1 - x_1| < \delta$. Set

$$\mathbf{y} = (y_1, x_2, \ldots, x_n).$$

Then

$$||\mathbf{y} - \mathbf{x}|| = ||(y_1 - x_1, 0, \ldots, 0)|| = |y_1 - x_1| < \delta$$

so $\mathbf{y} \in B_r(\mathbf{u})$ but $|f(\mathbf{y}) - f(\mathbf{x})| = |1 - 0| = 1 > \frac{1}{2}$ a contradiction. So f is not continuous at $\mathbf{x} = (x_1, \ldots, x_n)$ such that x_i are rational.

The case of $\mathbf{x} = (x_1, \ldots, x_n)$ where at least one x_i is irrational is completely analogous.

6. Show that the function $f : (0, 1] \to \mathbb{R}$ defined by $f(x) = \ln(x^2)$ is continuous but not uniformly continuous. (You can assume that $\ln(x)$ is continuous on $(0, \infty)$.

   Solution: To prove that f is continuous, note that $g : (0, 1] \to \mathbb{R}$ defined by $g(x) = x^2$ is continuous on $(0, 1]$ and $g((0, 1]) = (0, \infty)$. We also know that $h : (0, \infty) \to \mathbb{R}$ defined by $h(x) = \ln(x)$ is continuous on $(0, \infty)$. Since the domain of h and image of g match up ($g((0, 1]) \subseteq (0, \infty)$), then $h \circ g = f$ is continuous.

   Consider the sequences $u_k = 1/k$ and $v_k = 1/k^2$. Then

$$\lim_{k\to\infty} dist(u_k, v_k) = \lim_{k\to\infty} \left| \frac{1}{k} - \frac{1}{k^2} \right| = \lim_{k\to\infty} \left| \frac{k-1}{k^2} \right| = 0.$$

   On the other hand

$$\lim_{k\to\infty} dist(f(u_k), f(v_k)) = \lim_{k\to\infty} \left| \ln\left(\frac{1}{k^2}\right) - \ln\left(\frac{1}{k^4}\right) \right| = \lim_{k\to\infty} \left| \ln\left(\frac{k^2-1}{k^4}\right) \right| = \infty.$$

   This is a contradiction. So f is not uniformly continuous.