Category Theory

A Brief Introduction to Category Theory

Introduction

Category theory formalizes mathematical structures and their concepts in terms of labeled directed graph called category. It has has been used to formalize concepts of other high-level abstractions such as sets, rings, and groups. Informally category theory can be thought as is a general theory of functions.

Category Theory

A category **S** consists of the following data:

- Objects. These are referred to by generic symbols like A, B, C,...
- Arrows. These are referred to by generic symbols like f, g, h,...

Where an object is not an arrow and an arrow is not an object. A category **S** is also required to satisfy the following axioms:

Composition Law

Given two arbitrary arrows

$$A \xrightarrow{f} B_1 \quad \text{and} \quad B_2 \xrightarrow{g} C$$

we can form the composition

$$A \xrightarrow{f * g} C$$

g following *f* if and only if $B_1 = B_2$, e.g. in the case where we have

$$A \xrightarrow{f} B \xrightarrow{g} C$$

Associative Law

Composition of arbitrary arrows

$$A \xrightarrow{f} B \quad , \quad B \xrightarrow{g} C$$

and

$$C \xrightarrow{h} D$$

is associative, i.e, the following relation holds:

$$h*(g*f) = (h*g)*f$$

Identity Law

For arbitrary objects, the arrows

$$A \xrightarrow{1_A} A \quad , \quad B \xrightarrow{1_B} B$$

and

$$A \xrightarrow{f} B$$

must obey the following equations;

$$f*1_A = 1_B*f = f$$

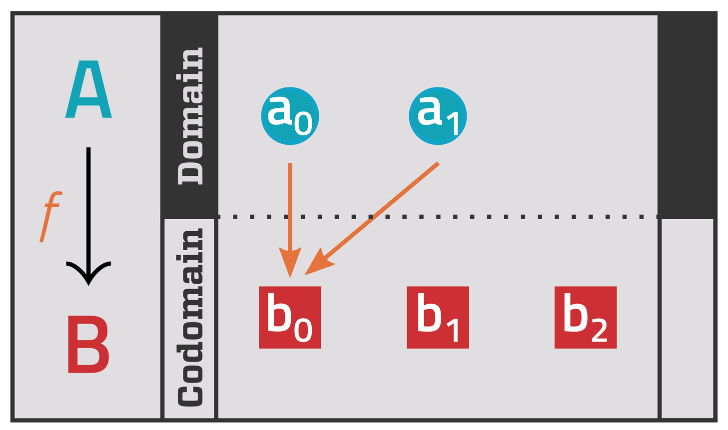
External vs. Internal Diagram

The representations

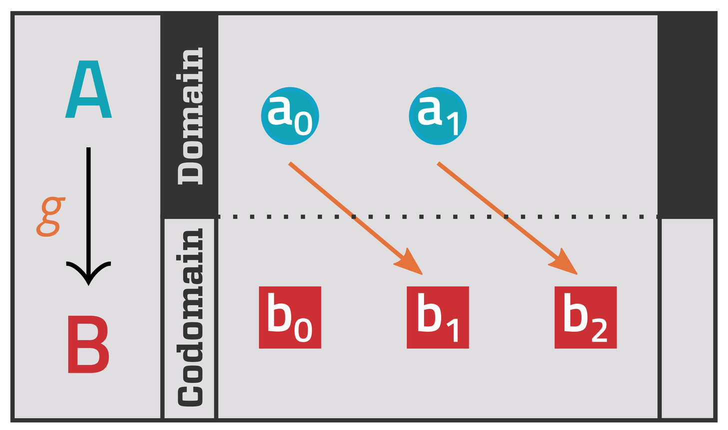
$$A \xrightarrow{f} B \quad \text{and} \quad A \xrightarrow{g} B$$

are called External Diagrams of the maps *f* and *g*. They show the Domain and Codomain but gives no insight to the "internal structure" of the maps.

We can also represent maps as an Internal Diagram or Map-Graph:



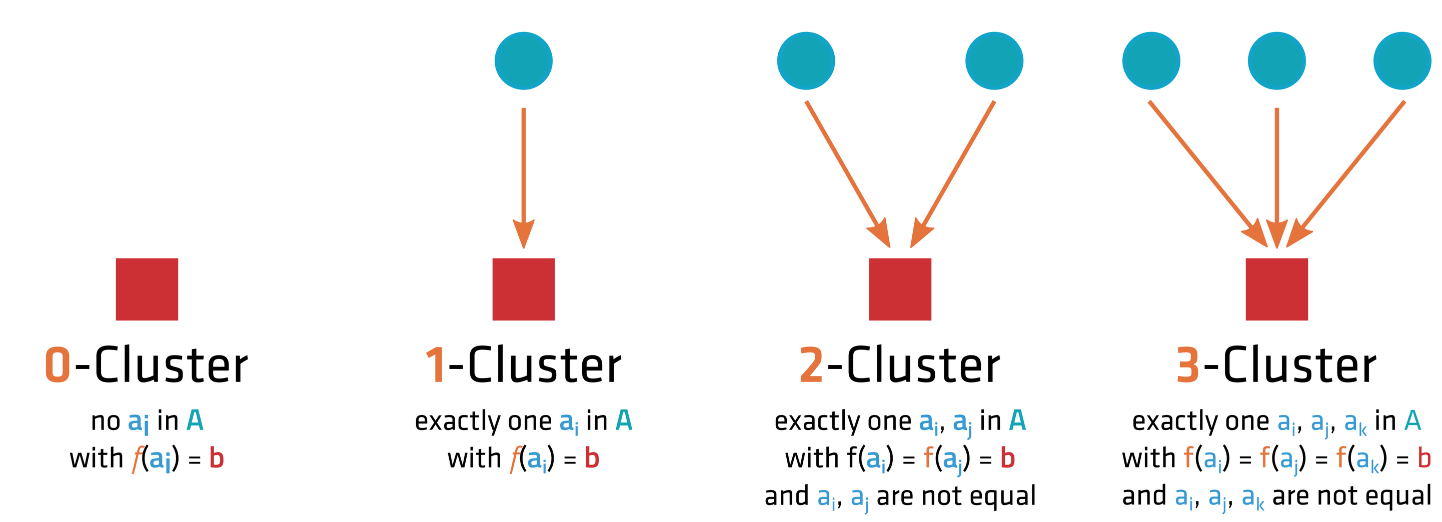
$$f(a_0) = f(a_1) = b_0$$



$$f(a_0) = b_1,\ f(a_1) = b_2$$

Components of a Map

The number of components in a mapping *f* from A to B is determined by the size of B; the types of components are determined by the size of A and the nature of *f*



Composition

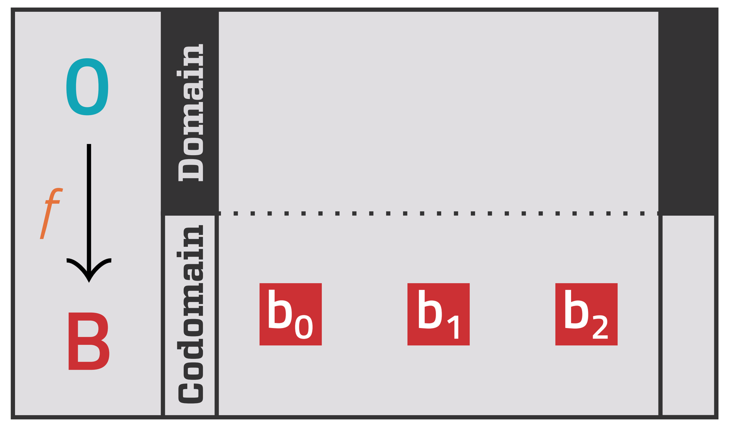
From the law of composition we know that whenever we have a partial triangle (left), we can complete it (right).

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] & & C \end{tikzcd} \quad \xRightarrow{h = f*g} \quad \begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & & C \end{tikzcd}$$

By doing this we construct a Commutative Diagram, where all paths between two arbitrary objects can be interpreted as the same arrow. Note that in a composition clusters can only increase or stay the same size, they cannot decrease in size.

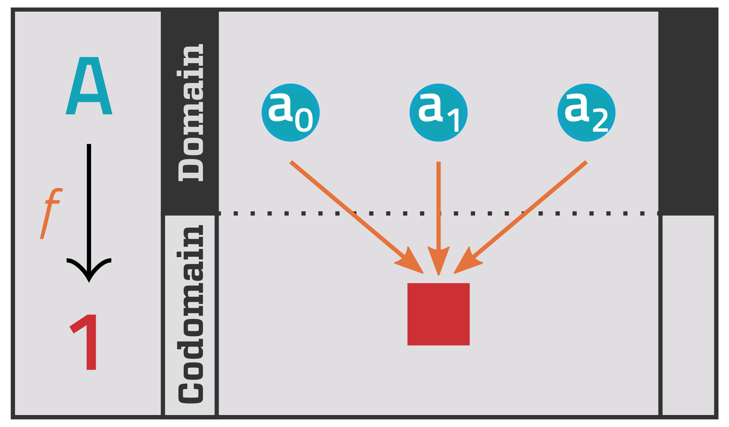
Initial Object

In **S**, a object **0** is said to be an Initial Object of **S**, if for all sets **X**, there is a unique map **0** to **X**.



Terminal Object

In **S**, a set **1** is said to be an Terminal object of **S**, if for all sets **X**, there is a unique map **X** to **1**.



Duality

To find the dual of a category **S** we do the following

- Interchange each occurrence of "source" in S with "target".
- Interchange the order of composing morphisms. That is, replace each occurrence of f of g with g of f.

The dual of a category is often called the the opposite category as the dual views a mapping from the codomain's view. A example of the dual is seen below.

$$A \xrightarrow{f} B \qquad A \xleftarrow{f^{op}} B$$

Conclusion

Category theory provides a very strong tool set to understanding function and other high-level abstractions. It is often considered a strong alternative to set theory for formalizing concepts. While it it is not formal taught at undergraduate level, many of its concepts are seen through out other topics.

References

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