1

将此非线性规划问题化为标准型，得

$$\min -b^T x, x \in R^n$$
$$s.t.\ 1 - x^T x \geq 0$$

$$KKT条件为\begin{cases} -b + wx = 0 \\ w(1 - x^T x) = 0 \\ w \geq 0 \end{cases}$$

$$得KKT点为 x = \frac{b}{||b||}，且此规划为凸规划，\therefore 此点满足最优性充分条件$$

2

经过观察可知，f是线性函数，可行域是半空间和圆的交集，是凸集，∴ 是个凸规划

∴ 必存在最优解，且在边界达到，将原问题化为

$$\min c^T x$$
$$s.t. Ax = 0$$
$$-x^T x + r^2 = 0$$

$$KKT条件\begin{cases} c - A^T v + 2v_{m+1} x = 0 \\ Ax = 0 \\ -x^T x + r^2 = 0 \end{cases}$$

$$解得 f_{min} = -r\sqrt{c^T(c - A^T v)}$$

$$最优解 x = \frac{r^2}{f_{min}}(c - A^T v)(f_{min} \neq 0)$$

$$当 c = A^T v 时，f_{min} = 0, 此时最优解不唯一$$

3

先求梯度

$$\nabla f(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \nabla g(x) = \begin{bmatrix} -2x_1 \\ -2(x_2 - 4) \end{bmatrix}, \nabla h(x) = \begin{bmatrix} 2(x_1 - 2) \\ 2(x_2 - 3) \end{bmatrix}$$

$$Lagrange函数 L(x, w, v) = x_2 - w(-x_1^2 - (x_2 - 4)^2 + 16) - v((x_1 - 2)^2 + (x_2 - 3)^2 - 13)$$

$$\nabla_x^2 L(x,w,v)=\begin{bmatrix}2(w-v) & 0\\ 0 & 2(w-v)\end{bmatrix}$$

对$x^{(1)}=\begin{bmatrix}0\\0\end{bmatrix}$来说，两个约束都是active constraint

$$KKT条件为\begin{cases}4v=0\\1-8w+6v=0\\w\ge 0\end{cases}$$

$$解得w=\frac{1}{8}>0, v=0, 因此满足一阶必要条件$$

$$令\begin{cases}\nabla g\left(x^{(1)}\right)^T d=0\\ \nabla h\left(x^{(1)}\right)^T d=0\end{cases}，解得d=0$$

$$即方向集G=\left\{d\middle|d\ne 0, \nabla g\left(x^{(1)}\right)^T d=0, \nabla h\left(x^{(1)}\right)^T d=0\right\}=\emptyset, \therefore x^{(1)}是局部最优解$$

对$x^{(2)}=\left(\frac{16}{5},\frac{32}{5}\right)^T$，两个约束都是active constraint

$$\nabla f\left(x^{(2)}\right)=(0,1)^T, \nabla g\left(x^{(2)}\right)=\left(-\frac{32}{5},-\frac{24}{5}\right)^T, \nabla h\left(x^{(2)}\right)=\left(\frac{12}{5},\frac{34}{5}\right)^T$$

$$KKT条件\begin{cases}\frac{32}{5}w-\frac{12}{5}v=0\\1+\frac{24}{5}w-\frac{34}{5}v=0\\w\ge 0\end{cases}的解为w=\frac{3}{40}>0, v=\frac{1}{5}, \therefore x^{(2)}是KKT点$$

$$方向集\begin{cases}\nabla g\left(x^{(2)}\right)^T d=0\\ \nabla h\left(x^{(2)}\right)^T d=0\end{cases}，解得d=0, \therefore x^{(2)}是局部最优解$$

对$x^{(3)}=\left(2,3+\sqrt{13}\right)^T$，只有等式约束是active constraint

$$KKT条件\ 1-2\sqrt{13}v=0\Rightarrow v=\frac{\sqrt{13}}{26}$$

$$方向集G=\{d|d=\begin{bmatrix}d_1\\0\end{bmatrix}, d_1\ne 0\}$$

$$Lagrange函数的Hessian矩阵\nabla_x^2 L\left(x^{(3)},w,v\right)=\begin{bmatrix}-2v & 0\\0 & -2v\end{bmatrix}=\begin{bmatrix}-\frac{1}{\sqrt{13}} & 0\\0 & -\frac{1}{\sqrt{13}}\end{bmatrix}$$

$$d^T\nabla_x^2 L\left(x^{(3)},w,v\right)d=(d_1,0)\begin{bmatrix}-\frac{1}{\sqrt{13}} & 0\\0 & -\frac{1}{\sqrt{13}}\end{bmatrix}\begin{bmatrix}d_1\\0\end{bmatrix}=-\frac{1}{\sqrt{13}}d_1^2<0,$$

$\therefore x^{(3)}$不满足二阶条件，不是局部最优解

4

$$令目标函数 = f(x), 等式约束 = h(x)$$

$$\nabla f(x) = \begin{bmatrix} x_1 - 1 \\ x_2 \end{bmatrix}, \nabla h(x) = \begin{bmatrix} -1 \\ 2\beta x_2 \end{bmatrix}$$

$$Lagrange函数为L(x,v) = \frac{1}{2}((x_1-1)^2 + x_2^2) - v(-x_1 + \beta x_2^2)$$

$$\nabla_x^2 L(x,v) = \begin{bmatrix} 1 & 0 \\ 0 & 1-2\beta v \end{bmatrix}$$

$$KKT条件\begin{cases} x_1 - 1 + v = 0 \\ x_2 - 2\beta v x_2 = 0 \end{cases}, 将x = (0,0)^T, 得v = 1$$

$$方向集G = \left\{ d \middle| d = \begin{bmatrix} 0 \\ d_2 \end{bmatrix} \right\}$$

$$d^T \nabla_x^2 L(x,v) d = (1-2\beta) d_2^2 > 0 \Rightarrow \beta < \frac{1}{2}$$

$$当\beta = \frac{1}{2}时，原问题 = \min \frac{1}{2}(x_1^2 + 1)$$

$$易知当x_1 = 0 时，原问题取得极小值. 综上，当\beta \le \frac{1}{2}时，\bar{x}是最优解$$

5

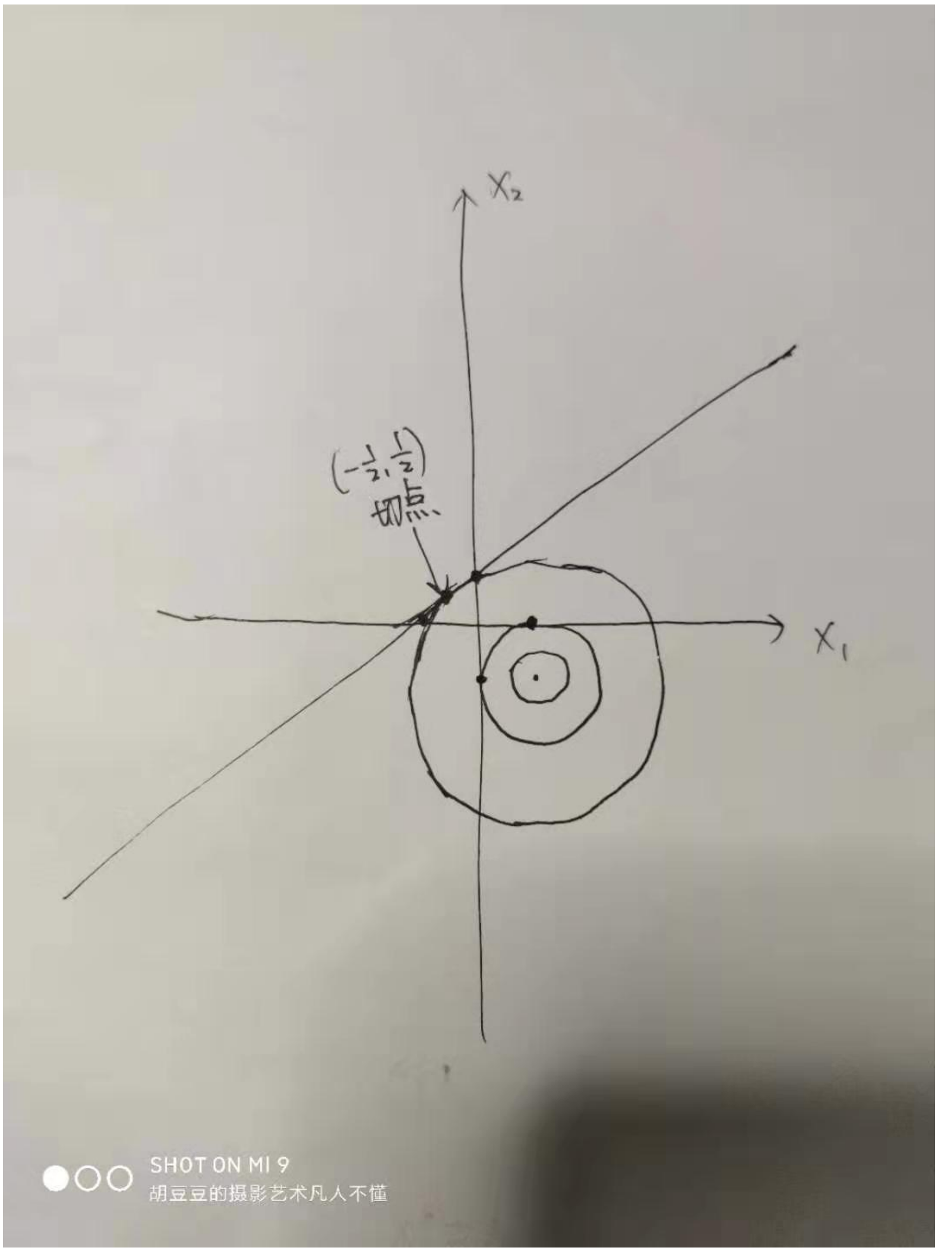
(1)

$$最优性条件法：令f(x) = 目标函数，g(x) = 不等式约束$$

$$\therefore \nabla f(x) = \begin{bmatrix} 2(x_1 - 1) \\ 2(x_2 + 1) \end{bmatrix}, \nabla g(x) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$KKT条件\begin{cases} 2(x_1 - 1) + w = 0 \\ 2(x_2 + 1) - w = 0 \\ w(-x_1 + x_2 - 1) = 0 \\ w \ge 0 \\ -x_1 + x_2 - 1 \ge 0 \end{cases}, 解得x = \left(-\frac{1}{2}, \frac{1}{2}\right)^T, w = 3, f_{min} = \frac{9}{2}$$

$$图解法：$$



如图，切点为最优解$x=\left(-\frac{1}{2},\frac{1}{2}\right)^T$，最优值为$\frac{9}{2}$

(2)

$$Lagrange函数L(x,w)=(x_1-1)^2+(x_2+1)^2-w(-x_1+x_2-1)$$

$$对偶问题的目标函数\theta(w)=\inf\{(x_1-1)^2+(x_2+1)^2-w(-x_1+x_2-1)|x\in R^2\}$$

$$=\inf\{x_1^2-2x_1+wx_1\}+\inf\{x_2^2+2x_2-wx_2\}+w+2$$

$$令w\geq 0$$

$$则\inf\{x_1^2-2x_1+wx_1\}=-\frac{1}{4}(w^2-4w+4)=\inf\{x_2^2+2x_2-wx_2\}$$

$$\theta(w) = -\frac{1}{2}w^2 + 3w, \text{对偶问题为}$$

$$max - \frac{1}{2}w^2 + 3w$$

$$s.t.\ w \geq 0$$