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$\hat{x}$ 为KKT点的必要条件是，存在乘子$w_i \geq 0 (i \in I)$和$v_j (j = 1, \ldots, l)$，使得

$$\nabla f(\hat{x}) - \sum_{i \in I} w_i \nabla g_i(\hat{x}) - \sum_{j=1}^{l} v_j \nabla h_j(\hat{x}) = 0$$

记$A_1 = \left[\nabla g_{i_1}(\hat{x}), \ldots, \nabla g_{i_k}(\hat{x})\right], w = (w_1, \ldots, w_k)^T, B = [\nabla h_1(\hat{x}), \ldots, \nabla h_l(\hat{x})], v = (v_1, \ldots, v_l)^T$

$$= p - q, p \geq 0, q \geq 0.$$

$$Lagrange函数可改写为(-A_1, -B_1, B) \begin{bmatrix} w \\ p \\ q \end{bmatrix} = -\nabla f(\hat{x}), \begin{bmatrix} w \\ p \\ q \end{bmatrix} \geq 0$$

根据$Farkas$引理，上式有解的充要条件是

$$\begin{bmatrix} -A_1^T \\ -B_1^T \\ B^T \end{bmatrix} d \leq 0, -\nabla f(\hat{x})^T d > 0 \text{ 无解}$$

$$即\begin{cases} \nabla f(\hat{x})^T d < 0, \\ A_1^T d \geq 0 \\ B^T d = 0 \end{cases} \text{ 无解。因此线性规划的最优值为 } 0$$