1

$$\because f\text{是定义在}E^n\text{上的凸函数，当}k=2\text{ 时，有}$$

$$\forall x^{(1)},x^{(2)}\in E^n$$
$$f\left(\lambda_1x^{(1)}+\lambda_2x^{(2)}\right)\le\lambda_1f\left(x^{(1)}\right)+\lambda_2f\left(x^{(2)}\right),\lambda_1+\lambda_2=1,\lambda\ge0$$

$$\because E^n\text{是个凸集，}\ \therefore\lambda_1x^{(1)}+\lambda_2x^{(2)}\in E^n$$

$$\text{当}k=3\text{ 时，令}\lambda_1x^{(1)}+\lambda_2x^{(2)}=x^{(12)},\lambda_3\ge0,\text{则}$$

$$\begin{aligned}f\left(\lambda_1x^{(1)}+\lambda_2x^{(2)}+\lambda_3x^{(3)}\right)&=f\left(x^{(12)}+\lambda_3x^{(3)}\right)\le f\left(x^{(12)}\right)+\lambda_3f\left(x^{(3)}\right)\\&=f\left(\lambda_1x^{(1)}+\lambda_2x^{(2)}\right)+\lambda_3f\left(x^{(3)}\right)\le\lambda_1f\left(x^{(1)}\right)+\lambda_2f\left(x^{(2)}\right)+\lambda_3f(x^{(3)}))\end{aligned}$$

$$\text{将此推导运用}k-2\text{ 次，即得}$$

$$f\left(\lambda_1x^{(1)}+\cdots+\lambda_kx^{(k)}\right)\le\lambda_1f\left(x^{(1)}\right)+\cdots+\lambda_kf\left(x^{(k)}\right),\ \lambda_1+\lambda_2+\cdots+\lambda_k=1,\lambda\ge0$$

2

$$\text{容易看出}S\text{是 4 个半空间的交集，}\ \because\text{ 半空间是凸集，凸集}\cap\text{凸集}=\text{凸集}$$

$$\therefore S\text{是凸集}$$

$$\because\begin{bmatrix}\dfrac{\partial^2f}{\partial x_1^2}&\dfrac{\partial^2f}{\partial x_1\partial x_2}\\\dfrac{\partial^2f}{\partial x_2\partial x_1}&\dfrac{\partial^2f}{\partial^2x_2}\end{bmatrix}=\begin{bmatrix}8x_2-24x_1^2&8x_1\\8x_1&-4\end{bmatrix},\text{且二阶主子式为}-4<0$$

$$\therefore Hessian\text{矩阵不是半正定的，所以}f(x_1,x_2)\text{不是}S\text{上的凸函数}$$

3

$$\text{易知}f(x)\text{在}R^2\text{上二阶可微，且}x_1\text{和}x_2\text{对调函数表达式不变，具有轮换对称性}$$

$$f'=0\Rightarrow\begin{cases}x_1=1\\x_2=1\end{cases}\text{或}\begin{cases}x_1=-1\\x_2=-1\end{cases}$$

$$\text{当}\begin{cases}x_1=1\\x_2=1\end{cases}\text{时，令}f''_{x_1x_1}=A,f''_{x_1x_2}=B,f''_{x_2x_2}=C\text{，易知}A=C$$

$\because AC-B^2>0, A<0 \therefore$ *此点是极大值点*

当$\begin{cases} x_1=-1 \\ x_2=-1 \end{cases}$*时，同上，* $\because AC-B^2>0, A>0$

$\therefore$ *此点是极小值点，极小值为*$-\dfrac{1}{3}$