Dynamics of k-core percolation

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In many network applications nodes are stable provided they have at least k neighbors, and a network of k-stable nodes is called a k-core. The vulnerability to random attack is characterized by the size of culling avalanches which occur after a randomly chosen k-core node is removed. Simulations of lattices in two, three and four dimensions, as well as small world networks, indicate that power-law avalanches occur in first order k-core systems, while truncated avalanches are characteristic of second order cases.

The k-core of a network is the set of nodes and edges which remain after all nodes which have less than k neighbors, and the edges attached to them, have been culled or removed. Percolation of k-cores has applications ranging from magnetism (where it was invented and called bootstrap percolation)[1, 2] and rigidity percolation[3, 4], to social networks[5] and protein networks[6], as well as the jamming transition[7]. The behavior on Bethe lattices is particulary interesting due to the mixed nature of the transition where a first order jump in the order parameter is co-existent with a square root singularity[1, 3]. This behavior has recently been associated with the jamming transition in granular media [7] and has been analysed using a $1/d$ expansion, which confirmed that it persists in finite dimensions[8], and by large scale simulations, which suggest that in four dimensions the transition is not mixed[9]. In general, k-core percolation in finite dimensions is quite different than that occuring on random graphs, with metastability being a key new feature, so that for $k > z/2$ hypercubic lattices have thresholds approaching one in the large lattice limit[10, 11, 12, 13, 14], whereas on Bethe lattices the k-core threshold occurs at values less than one for all $k \leq z - 1$[1]. Moreover on random graphs k-core percolation is first order for all $k > 2$[1] whereas on triangular lattices and cubic lattices the transition is second order and in the connectivity percolation university class for $k = 3$[15, 16, 17, 18, 19, 20].

The vulnerability of networks to random node removal or attack, has a variety physical applications, including: radiation damage; species extinctions in biological networks; and random outages or cascading failures in communication or transportation networks[21]. There are also relations to avalanches in materials problems, including fracture and earthquakes[22, 23], and Barkhausen noise in magnetic materials[22, 24, 25]. In fact bootstrap percolation is related to the weak disorder limit of one of the key models of disordered magnets, the random field Ising model (RFIM)[26].

In an earlier paper[27], we defined elementary culling avalanches (ECA) in k-core or bootstrap problems to be

avalanches induced by random removal of one node on a stable k-core, which is a dynamical procedure analogous to that used in crackling noise simulations in random field systems[22]. We showed that on triangular and cubic lattices the ECA distribution provides a sensitive indicator of the nature of the bootstrap transition, even in the slowly convergent second order case $k = 3$ [15, 16, 17, 18, 19, 20] and the metastable $k = 4$ case[10, 11, 12, 13, 14].

Here we show that in systems ranging from small world networks to regular lattices, including hypercubic lattices in four dimensions, the dynamics of k-core percolation falls into two classes: (i) Second order systems which have truncated avalanche distributions where the largest avalanche grows very weakly with system size and; (ii) First order systems where avalanches are power law distributed and the largest avalanche is of order the system size, producing a first order jump in the infinite cluster probability at percolation. In the case of Bethe lattices, power law avalanches in first order cases can be demonstrated explicitly[28] (see below).

Culling cascades for a variety of lattices in finite dimensions are presented in Figs. 1 and 2a. These distributions are found by starting with a lattice with all sites present and then repeating two steps: (i) Randomly remove a site and all edges connected to it, (ii) Recursively cull all unstable sites until a stable k-core is achieved. The number of sites removed during step (ii) constitutes an elementary culling avalanche(ECA). The distribution of such avalanches, beginning with an undiluted lattice and continuing until the lattice is empty yields the cummulative culling avalanche distribution, $C(a)$. We also monitor the largest avalanche that occurs for a given lattice, a_{max}, which is the maximum cascade over the whole avalanche trace. The finite size scaling behavior of the maximum cascade is presented in the insets to Figs. 1 and 2a.

In Fig. 1 we present avalanche distributions for $k = 3, 4, 5$, on eight-coordinated lattices in three (Fig. 1a) and four dimensions (Fig. 1b). These lattices where chosen to demonstrate the effect of spatial dimension on the nature of the k-core transition. It is clear from this data that the $k = 4$ transition changes from second order to first order as we go from three to four dimensions as noted
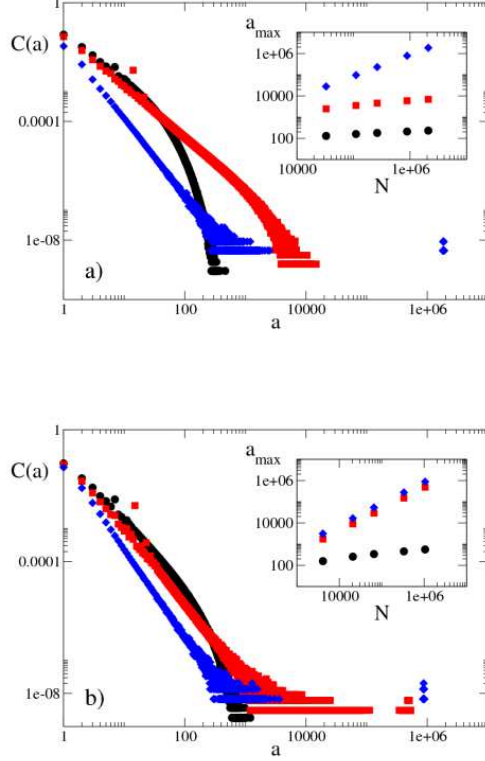
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FIG. 1: (Color online) Effect of lattice dimension (d), at fixed lattice co-ordination (8), on the avalanche distribution (ECA), $C(a)$, for the cases $k = 3$(black circles), $k = 4$(red squares) and $k = 5$ (blue diamonds). a) Body-centered cubic lattices ($d = 3$) and, b) Simple cubic lattices ($d = 4$). The 4-core transition changes from second order for $d = 3$ to first order for $d = 4$ [9], as is clearly seen in the avalanche distributions and in the finite size scaling of the largest avalanche, a_{max} (Insets). In a) 1000 samples were used, and in the main figure the lattice size $L = 128$. In b) 1000 samples were used, and in the main figure $L = 32$.

recently[9], however the $k = 3$ transition is second order in both three and four dimensions. In contrast on $z = 8$ Bethe lattices and random graphs, k-core percolation is first order for all $k > 2$. Clearly dimension plays an important role on the nature of the k-core transition, moreover whenever the transition is first order we see the same characteristic behavior: power law avalanches and a largest avalanche of order the sample size[27].

We investigated a second method for changing the nature of k-core transitions, namely by changing the range of the interactions. To do this, we created small world networks by starting with an undiluted triangular lattice and randomly removing a fraction f of bonds from it. These bonds are then randomly put back into the lattice, but they may connect any pair of sites at arbitrary distances, with the constraint that all sites remain six-coordinated. In this way we create six co-ordinated small
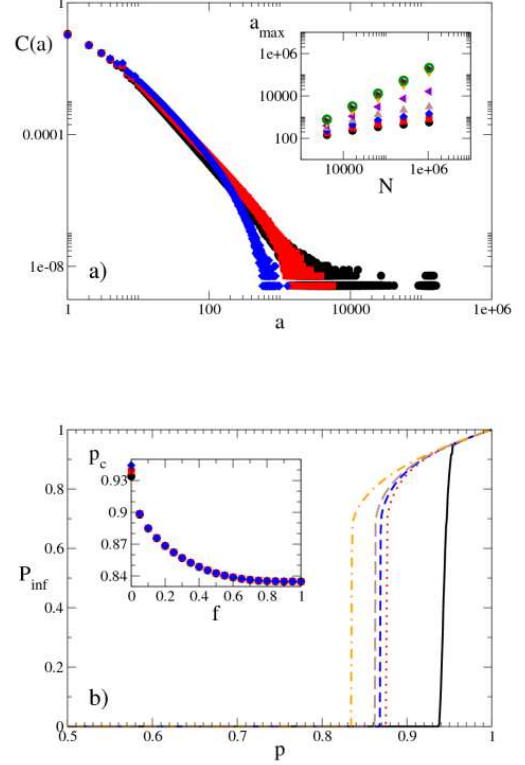

FIG. 2: (Color online) k-core percolation on six co-ordinated small world networks, with fraction f of small world connections. a) Avalanche distributions, $C(a)$, for 3-core cases with $f = 0.15$ (blue diamonds), $f = 0.20$ (red squares) and $f = 0.25$ (black circles). Inset: The largest avalanche, a_{max}, starting at the bottom, for $f = 0.15, 0.17, ..., 0.25$. b) The infinite 4-core probability, P_{inf} versus site probabiltiy p for small world fractions (from right to left) $f = 0, 0.15, 0.20, 0.25, 1$. Inset: the bootstrap percolation threshold, $p_c(f)$. There is a singular behavior in $p_c(f)$ as $f \to 0$ (see text). In the main figures of a) and b) and in the inset to a) from 1000 samples for each f, with $L = 1028$. In the inset to b) from 100 samples for $L = 256$(black circles), $L = 512$ (red squares), $L = 1024$(blue diamonds).

world networks with fraction f of long range connections. When f is zero, we have a regular triangular lattice, while when $f = 1$, we generate a random regular graph with co-ordination six everywhere. By varying f, we monitor the effect of long-range connections on the nature of the k-core transition, which also measures the vulnerability of the small world networks to random attack[21]. For $k = 3$, we find that, as illustrated in Fig. 2a, the transition changes from second order to first order when the fraction of small world connections reaches $f_c = 0.23(2)$. For $f > f_c$ there are power law avalanches and an extensive largest avalanche, while for $f < f_c$, the avalanche distribution is truncated, with no large avalanches.

The 4-core transition on 6-coordinated networks is first order for all f, however on triangular lattices it is

metastable so that for large lattices the threshold approaches one very slowly, while on random graphs there is a finite threshold $p_c < 1$. Nevertheless we find power law avalanches and an extensive largest avalanche in all of these cases. However, the behavior of the k-core threshold as a function of the fraction of small world connections is interesting and is presented in Fig. 2b. A notable feature of this data is that small world connections quickly stabilize the metastable 4-core of triangular lattices and, as is emphasized in the inset to Fig. 2b, there is a singular behavior in $p_c(f)$ as $f \to 0$. Presumably, this non-perturbative behavior is related to the stabilization of large holes in the triangular lattice by the addition of long-range loops characteristic of small world networks, and it may be possible to characterize this behavior using rigorous mathematical methods [10, 11, 12, 13].

We also observed power law avalanches and an extensive largest avalanche in the limit $f = 1$ which is the random graph limit. In this limit power law avalanches can be demonstrated explicitly using Bethe lattices, as we now show. Consider site diluted, z-coodinated Bethe lattices where a fraction p of the nodes are randomly removed, and let T be the branch probability that a node of the Bethe lattice is part of a k-core. The branch probability, T, is found by solving the self-consistent equation,

$$T = p \sum_{n=k}^{z-1} A_n^{z-1} T^n; \quad with \quad A_l^r = \binom{r}{l}(1-T)^{r-l} \quad (1)$$

which is well known from the original bootstrap paper[1] and from applications to rigidity percolation[3]. The branch probabilities are used to find the probability of a site being on the k-core,

$$X = p \sum_{n=k}^{z} A_n^z T^n. \quad (2)$$

We build on the results (1) and (2) to find the culling avalanche distributions for a branch, $B(a)$, and for the Bethe lattice, $P(a)$. $P(a)$ is the differential probability and is related to the cummulative probability, $C(a)$ through $C(a) = \int P(a, p)dp$.

As noted previously[28, 29], sites which have exactly k neighbors play a special role in culling avalanches on Bethe lattices. Following their terminology, we define the corona as a connected cluster of sites each of which is part of the k-core and has exactly k neighbors. With this definition, removal of a site of the k-core leads to a culling avalanche in the corona surrounding that site. The probability of finding a corona of size, a, surrounding a randomly chosen starting site on the Bethe lattice is given by,

$$P(a) = p \sum_{n=k}^{z} A_n^z \prod_{l=1}^{n} \sum_{a_l} B(a_l)\delta(S_n). \quad (3)$$

where $S_n = \sum_{l=1}^{n} a_l - a + 1$. This expression has the same form as Eq. (2), but with the term T^n in that
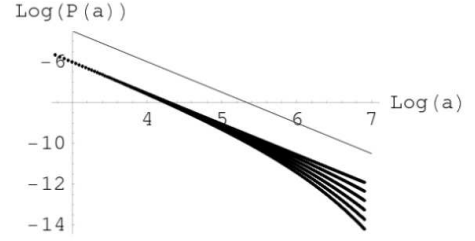

FIG. 3: A log-log plot of culling avalanches, $P(a)$ for 3-core percolation on a z=4 Bethe lattice, found from $P^*(y)$ (see text after Eq. (8) using an exact Mathematica algorithm. The dark lines are for 5 values of the site concentration, from the top starting with $p = 8/9 = p_c$ and then moving above p_c in increments of $\delta p = 10^{-4}$. The continuous line of slope 3/2, vertically shifted from the data, is included to demonstrate that $P(a) \sim 1/a^{3/2}$ at the 3-core threshold, p_c.

equation replaced by the product in Eq. (3). This term is a product over the probabilities of finding corona on each of the branches connected to the starting site, and the delta function ensures that the ECA's on these corona sum to the total avalanche size a. From this equation it is evident that the normalizations $\sum P(a) = X$ and $\sum B(a) = T$ hold. A similar reasoning is applied to the branch probabilities, yielding the extension of Eq. (1) to,

$$\frac{B(a)}{p} = \sum_{n=k}^{z-1} A_n^{z-1} T^n \delta(a) + A_{k-1}^{z-1} \prod_{l=1}^{k-1} \sum_{a_l} B(a_l)\delta(S_{k-1}). \quad (4)$$

There are two terms in this expression, as only corona sites on a branch initiate a culling avalanche on the branch. All other k-core sites terminate the ECA and contribute to the prefactor of $\delta(a)$ in this equation. This delta function is not present in Eq. (3) as the starting site there is removed by hand to initiate an ECA so it can have any co-ordination. Defining the generating functions, $B(y) = \sum B(a)y^a$, and $P(y) = \sum P(a)y^a$, and applying them to Eq. (3) yields,

$$P(y) = py \sum_{n=k}^{z} A_n^z B(y)^n. \quad (5)$$

Similarly, Eq. (4) becomes,

$$B(y) = p \sum_{n=k}^{z-1} A_n^{z-1} T^n + py A_{k-1}^{z-1} B(y)^{k-1}. \quad (6)$$

The probabilities $B(a)$ or $P(a)$ are extracted from the generating functions $B(y)$ or $P(y)$, by differentiation or by contour integration, whichever is more convenient.

The cases $k = 1, 2$ are the standard connectivity percolation and percolative backbone problems respectively, so the simplest non-trivial k-core problem on random graphs is $k = 3$, $z = 4$. The formalism outlined above can be carried through explicitly in this case and has a behavior

characteristic of all cases $k \geq 3$ [28, 29]. For this case, $B(y)$ obeys,

$$B(y) = pT^3 + 3py(1 - T)B(y)^2, \qquad (7)$$

where $T = 3/4 + 9(\delta p)^{\frac{1}{2}}/8\sqrt{2}$, which follows from Eq. (1) and $\delta p = p - p_c$. This illustrates the characteristic feature of a jump discontinuity and a continuous square root singularity, as noted before[1, 3]. Solving Eq. (7) and selecting the physical root, we find an explicit expression for the avalanche generating function,

$$B^*(y) = \frac{1 - [1 - 12p^2T^3(1 - T)y]^{\frac{1}{2}}}{6p(1 - T)y}. \qquad (8)$$

and finally, from Eq. (4), we find the generating function for Bethe lattices, $P^*(y) = py[4(1 - T)B^*(y)^3 + B^*(y)^4]$. We wrote a Mathematica program which finds the exact culling avalanche distribution from $P^*(y)$, and the results are presented in Fig. 3, which demonstrates the $3/2$ power law behavior at threshold characteristic of mean field avalanches. Asymptotic analysis near criticality is carried out by noting that the normalization condition $B^*(y = 1) = T$, enables an expansion of B^* near $y = 1$ and $p = p_c$. The leading term in the expansion of B^* is then, $B^* \approx 1 - [1 - y(1 + 9\delta p/2)]^{1/2}$, from which we find that near the threshold, p_c, the tail of the avalanche distribution obeys $P(a) \sim [1 - D(p - p_c)]^a/a^{3/2}$,

where $D = 9/2$ and $p_c = 8/9$ for 3-core percolation on 4-coordinated Bethe lattices. In experiments as well as computer simulations, it is more convenient to measure the integrated probability distribution of avalanches $C(a)$ and in the simulation results presented above, we integrate over the complete avalanche trace. The probability of large avalanches is significant in a small region of width $\frac{1}{\delta p}$ near threshold. Integration of $P(a)$ in this regime yields, $C(a) \sim a^{-\frac{5}{2}}$, for $a \to \infty$ which is observed in our simulations on random graphs, and is typical of avalanches occuring in infinite range models and Bethe lattices in a wide variety of non-equilibrium systems [22, 23, 29, 30, 31].

In all cases we studied, power law avalanches are associated with first order k-core transitions, both in metastable and regular first order cases. This is not without precident as in random field Ising hysteresis a first order jump occurs in the hysteresis loop despite the fact that power law avalanches occur [22, 24]. However in brittle fracture problems in finite dimensions, the avalanches are truncated when the loading sharing is short ranged[32, 33]. Presumably this is due to the greater amplification of stress on survivors in fracture problems with short range load sharing. Nevertheless, k-core dynamics capture the essential physics of instabilities in a broad class of complex network problems.

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