Chapter 1: Manifolds

1

In the first part, verify that the conditions required for metric is satisfied.

In the second part, one has to show that $B_{\bar{d}}(x, \varepsilon') \subset B_d(x, \epsilon) \subset B_{\bar{d}}(x, \varepsilon)$.

2

This is trivial.

3

(a). Basically, you have to make use of the local properties of Euclidean spaces.

(b). For (b), proceed in the following manner: Take $x_0 \in X$, let the set A be all points y in X such that there exists a path from x_0 to y. Show that this set is both open and closed; since the space is connected, $A = X$.

(c). Basically, one has to proceed in a similar fashion. But the issue is that the line connecting the limit point x and a point y in its neighborhood need not form an arc (the issue is that this need not result in a one-one path.)

To resolve this, observe that a continuous image of $[0, 1]$ will be compact. Since the space is locally metrizable, if x is a limit point of A, there is a sequence of points $\{x_n\}$ that converges to x in A. Choose a point x_n in some metrizable neighborhood of x, now the infimum of distances from x to this compact set will be realized at a point y in the range. Join y and x and remove the remainder of the path from y to x_n to obtain an arc to x, i.e., $x \in A$.

Similarly, one can prove openness of A, and from the fact that A is connected, we see that $A = X$.

4

(a). Topologist's Sine curve.

(b). Trivial.

(c). Define the relation $\sim$ on X by $x \sim y$ if there is an connected subset of X that contains x and y. The equivalence classes of X under $\sim$ are called the connected components of X.

It is easy to see that connected components are indeed connected, (show that C is the union of connected sets containing at least a point in common.)

If all the connected components are open, then the space is locally connected, since for every point x in X, C_x be the connected component to which x belongs. Since C_x is open, this is the neighborhood that we are looking.

Suppose the space is locally connected. Let C_x be a connected component, and pick $x \in C_x$. There exists an open connected neighborhood U. Since U is connected, it has to lie entirely in C_x, and hence C_x is open.

(d). Trivial

(e). Follows from 4.

5

(a). This is trivial.

(b). Follows from the fact that for $n \neq m$, $\mathbb{R}^n$ is not homeomorphism to $\mathbb{R}^m$.

6

(a). It is easy to see that open subsets of an n manifold is an n manifold.

Suppose M' be an n sub-manifold of M. If M' is not open, then for some $x \in M'$, every neighborhood of x contains a point outside of M', but there exists a neighborhood of M' that is homeomorphic to $\mathbb{R}^n$ and hence this neighborhood is open in M, a contradiction to the fact that M' is not open.

(b). Let x be a point of M which has a neighborhood of dimension n. Define A to be all points of M that has dimension equal to n. It is enough to show that A is both open and closed.

A is open: If $y \in A$, then y has a neighborhood homeomorphic to $\mathbb{R}^n$; clearly all points in these neighborhood lies in A, i.e., A is open.

A is closed: the space is locally metrizable, let y be a limit point of A, $\{y_n\}$ be a sequence of points in A that converges to y, if y has a neighborhood of dimension m where $m \neq n$, we have a contradiction. Since for large enough n, x_n has a neighborhood homeomorphic to $\mathbb{R}^n$ and also another neighborhood homeomorphic to $\mathbb{R}^m$ (I'm taking for faith that this can't happen.)

On second thought, we don't have to summon the locally-metrizable property.

7

(a). An easy application of the intermediate value theorem.

(b). I'm assuming that by an interval, the author is referring to an open interval, i.e., sets of the form (a, b).

This is trivial from 1.

(c). I'm assuming that by "f is homeomorphism", the author is referring to the fact that f is a homeomorphism between I and $f(I)$.

This is trivial from 2.

TODO 8

(a). It is easy to see that two components cannot be bounded at the same time, for, if they are bounded (call the components B and C), then $\mathbb{R} = A \cap B \cap C$ is also bounded, a contradiction.

So it is enough to show that one component is bounded. WLOG, assume that 0 lies "inside" the A. Let 0 belong to the component B. Since A is compact, A is bounded and thus the "inside" of A is bounded.

(b).

TODO 9

(a). Pick a point in $\mathbb{R}$, then $\mathbb{R} - \{x\}$ is disconnected, while $\mathbb{R}^n - \{x\}$, where $n > 1$, is connected.

(b).

TODO 11

The manifold M is σ compact, let $M = \cup M_i$, where M_i is compact.

Recall that compact metrizable spaces are first countable. Basically one has to take the countable union of all these countable sets to get a countable base for M.

Again, compact metrizable spaces are also

12

(a). $f(x, y) = \frac{-2x}{y-1}$.

(b). $f(x_1, x_2, \cdots, x_n) = (\frac{-2x_1}{x_n - 1}, \frac{2x_1}{x_n - 1}, \cdots, \frac{-2x_{n-1}}{x_n - 1})$.

13

(a). One has to show that the two definitions (the original definition, and the definition in which open sets need not contain the antipodal point), will give rise to the same topology.

Let us denote the set $\{-p \colon p \in V\}$ by $-V$. Since $\phi(p) = -p$ is a homeomorphism from S^1 onto S^1, V is homeomorphic to $-V$. In the second definition, observe that $f(V \cap (-V)) = f(V)$.

(b). In case of the Möbius strip, the issue is that the new definition will produce sets that are not open as per the original definition. For example $V = [0, 1/3) \times (-1, 0)$ is open in $[0, 1] \times (-1, 1)$; the new definition asserts that $f(V) = [[0, 1/2) \times (-1, 0)]$ is open in the Möbius strip, but it isn't open as per the original definition.

The two mappings differ in the symmetry of the map f.

TODO 14

(a).

(b). This follows from the definition of $\mathbb{P}^2$.

15

(a). Imagine S^1 lying inside $\mathbb{C}$, consider the map $f: S^1 \to S^1$ defined by $f(z) = z^2$. It is easy to see that this map is a quotient map since saturated open sets are mapped to open sets of S^1. (I think that saturated sets are of the form $V = V \cap (-V)$.)

Then it can be seen that the quotient space under this map is $\mathbb{P}^1$. There is a natural map $g: \mathbb{P}^1 \to S^1$ such that $g(p(x)) = f(x)$ where $p: S^1 \to \mathbb{P}^1$ is the identification map. Recall that g is a homeomorphism if and only if f is a quotient map.

(b). Let us imagine things in the following manner. S^n is a subspace of $\mathbb{R}^{n+1}$ defined by $\|\mathbf{x}\| = 1$, i.e., all points $\mathbf{x}$ such that $x_1^2 + \cdots + x_{n+1}^2 = 1$.

One can imagine S^{n-1} as a subspace of $\mathbb{R}^n$ in the following manner: all points $\mathbf{x}$ such that $x_1^2 + \cdots + x_n^2 = 1$ and $x_{n+1} = 0$.

Consider $D^n = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| < 1\}$, define $\varphi((x_1, \cdots, x_n)) = [(x_1, \cdots, x_n, x_{n+1})]$ where $[p] = \{p, -p\}$ and x_{n+1} is the positive real number that satisfies $x_{n+1}^2 = 1 - (x_1^2 + x_2^2 + \cdots + x_n^2)$. This map is the required homeomorphism.

18

(a). Consider $\mathbb{R}$, remove a one-point set. $\mathbb{R} - \{x\}$ is not connected and since compact sets are bounded, removing a compact set from this is not going to make this connected.

On the other hand, for any compact set C in $\mathbb{R}^n$ will be inside the set $[-M, M]^n$ and $\mathbb{R}^n - [-M, M]^n$ is connected

(b). Consider $C = S^{n-1}$ in $\mathbb{R}^n - \{0\}$. This is compact relative to $\mathbb{R}^n - \{0\}$ (since it is compact relative to $\mathbb{R}^n$.) $R^n - \{0\} - C$ has two connected components, one bounded and the other unbounded. In order to make $\mathbb{R}^n - \{0\} - K$ connected (where K is a compact set), one has to, at least, remove the bounded component completely.

Imagine the closed set $Y = [a, 0) \cup (0, b]$ for sufficiently small a and b such that Y lies entirely in K. Y is closed, since $Y = [a, b] \cap (\mathbb{R}^n - \{0\})$ and being a closed subset of K, it is compact. This is a contradiction.

19

(a). Given any compact subset C of $\mathbb{R}$, pick a closed interval that contains C, say K, then $\mathbb{R} - K$ has two components. To satisfy the property that $\varepsilon(K) \subset \varepsilon(C)$, $\varepsilon(C)$ has to be the component of $\mathbb{R} - C$ that contains one of the components of $\mathbb{R} - K$. Thus there are two ways of doing this, i.e., $\mathbb{R}$ has two ends.

(b). We repeat the same trick. Given any compact subset C of $\mathbb{R}^n$, $n > 1$, pick a closed n cell that contains C, say K, then $\mathbb{R} - K$ has exactly one component. We are forced to that components of $\mathbb{R} - C$ that contains $\mathbb{R} - K$, i.e., $\mathbb{R}^n$ has one end.

Suppose X "has one end", then for repeat the above step to see that X is one ended.

Suppose X is one ended. Suppose for a compact set C, $X - C$ has more than one component and for every compact set K such that $C \subset K$, $X - K$ has more than one component, then clearly, we have more than one choice for $\varepsilon(C)$, a contradiction.

(c). Refer to Theorem 29.1 in Topology, Munkres.

$\mathbb{R} \cup \varepsilon(R) = [0, 1]$ (Two point compactification of $\mathbb{R}$.)

$\mathbb{R}^n \cup \varepsilon(\mathbb{R}^n) = S^n$ (One point compactification of $\mathbb{R}^n$.)

TODO 25

(a). This is obvious from the fact that $\mathbb{R}^n$ can be "embedded" in $\mathbb{H}^n$ (that is, there is a continuous map from $\mathbb{R}^n$ to $\mathbb{H}^n$ that is homeomorphic to the image.)

(b). There are points on manifolds with boundary that do not have neighborhoods homeomorphic to $\mathbb{R}^n$. We call the collection of all these points as ∂M or the boundary of the points.

It is easy to show that ∂M is a closed subset of M, for if x is a limit point of ∂M and if x has a neighborhood that is homeomorphic to $\mathbb{R}^n$, then this neighborhood also contains a point from ∂M, which is a contradiction.

Consider the space $M - \partial M$ with the subspace topology. It is easy to see that all points of these space have a neighborhood homeomorphic to $\mathbb{R}^n$ (n may vary), and hence a manifold without boundary by definition.

Consider the space ∂M. If p is a point on ∂M and V be a neighborhood that is homeomorphic to $\mathbb{H}^n$ ($\mathbb{H}^n$ can be thought of as $\mathbb{R}^{n-1} \times [0, \infty)$.) Let $\phi \colon V \to \mathbb{H}^n$ be such a diffeomorphism, then we claim that p is a point of ∂M if and only if for $\phi(x) = 0$.

From the above claim, it is easy to see that ∂M, with the subspace topology is a manifold.

(c).

Chapter 2: Differentiable Structures

1

(a). Clearly, the symmetric and reflexive properties are satisfied, but there are issues with transitivity.

Consider the following maps:

$a\colon A \to \mathbb{R}^n$
$b\colon B \to \mathbb{R}^n$
$c\colon C \to \mathbb{R}^n$

The C^∞ related only means that the composition of one map and the inverse of the other map is C^∞ on the region were their domain overlaps. It could be true that $a \circ c^{-1}$ need not be C^∞ on $A \cap C$. An explicit example should be easy to construct.

(b). This has to do with the fact that $\mathscr{A}$ is an Atlas, and the newly added maps are all C^∞ related to **all** maps in the Atlas $\mathscr{A}$.

Let $\mathscr{A}'$ be the maximal atlas corresponding to $\mathscr{A}$. We have to prove the following: (x, X) and (y, Y) be two coordinates then x and y are C^∞ related.

When x or y belongs to $\mathscr{A}$, this is trivial. It is enough to prove for x and y not in $\mathscr{A}$.

WLOG, assume that there is a chart $(u, U) \in \mathscr{A}$ such that $A \in X \cap Y$ (in the general case, we have to repeat the following procedure for every charts.)

Thus $y \circ (x^{-1} = (y \circ u^{-1}) \circ (u \circ x^{-1})$ and hence is C^∞; similarly $x \circ y^{-1}$ is also C^∞.

TODO 2

3

(a). This is trivial from the fact that differentiability implies continuity and chain rule. (I'm assuming that these C^∞ functions are from $\mathbb{R}^n$ to $\mathbb{R}^m$.)

(b). The only if part can be easily seen from the fact that

$g \circ f \circ x^{-1} = (g \circ y^{-1}) \circ (y \circ f \circ x^{-1})$.

where $(x, A), (y, B)$ are charts of M and N respectively.

To show the only if part, choose $g = y^i$ and $g \circ f = f^i$ is C^∞ map from M into $\mathbb{R}$ for all $i = 1, 2, \cdots, m$ (here m is the dimension of N.) Which is equivalent to saying that the map f is C^∞. (See result (4) in page 31.)

4

There are infinite number of non-distinct ones. For example the ones that are generated by monomials of odd degree will be distinct. (I'm not sure if there is a better answer to this.)

TODO 5

(a). To say that an Atlas $\mathscr{A}$ is a maximal means the following: if h is a homeomorphism and such that it is C^∞ related to all elements of $\mathscr{A}$, then $h \in \mathscr{A}$.

Assume that $\mathscr{A}$ is maximal. Define $\mathscr{A}'$ to be the collection of all (x, U) in $\mathscr{A}$ with $U \subset N$. Suppose that h is a homeomorphism of an open subset of N into $\mathbb{R}^m$ that is C^∞ related to all elements of N, then clearly it can be thought of as a homeomorphism of an open subset of M into $\mathbb{R}^m$ and this map is C^∞ related to all elements of $\mathscr{A}$, i.e., $h \in \mathscr{A}$ which implies that $h \in \mathscr{A}'$.

(b). A subset U of N is open in N if and only if there exists an open set in M such that $U = N \cap M$. Clearly, the set of all $(x|_{V\cap N}, V \cap N)$ contains $\mathscr{A}'$; this collection is an Atlas. The equivalence follows from the fact that $\mathscr{A}'$ is maximal.

(c). To prove that $i \colon N \to M$ is C^∞, we need to show the following:

For $(y, Y) \in \mathscr{A}'$ and $(x, X) \in \mathscr{A}$, the map $x \circ i \circ y^{-1}$ is C^∞ (for all such choices of x and y.)

Since $(y, Y) = (z|_{V\cap N}, V \cap N)$ for some $(z, V) \in \mathscr{A}$, and $x \circ i = x|_{N\cap A}$, we see that $x \circ i \circ y^{-1} = x|_{N\cap A} \circ z^{-1}|_{N\cap V}$ and the RHS is, by definition of $\mathscr{A}$, C^∞.

6

Can be verified easily. (Refer to problem 1.12 for explicit form of P_1)

7

(a). This was motivated by example (2), page 33. In $\mathbb{R}^n$, one can construct a C^∞ function that joins any two points and such that the derivative at both the end points equal to the zero vector (2 provides a geometric picture of this.) With this in mind, we can proceed.

Consider points p and q on a C^∞ manifold $(M, \mathscr{A})$. Consider the arc from p to q (refer problem 1.3 (c) for existence of such an arc.) Now consider all charts (x, U) such that the intersection of the arc and the set U is non-empty. By compactness of the arc, one can choose finitely many such charts that covers the arc. In the range of these charts form the C^∞ function, paste them.

(b). One can adapt the above construction to form a one-one map that is C^∞.

TODO 8

9

Let S^n be covered using the standard atlas containing $2n$ homeomorphism, say $\mathscr{A}$; define $\mathscr{A}'$ to be the maximal atlas of the aforementioned atlas.

Observe that every chart in $(x, U) \in \mathscr{A}$ satisfies are such that the the map $g\colon S^n \to P^n$ has an inverse when restricted on U. Thus define $\mathscr{B}$ by taking the collection of all sets $x \circ g^{-1}$; clearly this is an atlas of $\mathbb{P}$. Let us call the maximal atlas of $\mathscr{B}$ as $\mathscr{B}'$. This is how we define C^∞ structure on $\mathbb{P}$. (I'm assuming that this is the differentiable structure that Spivak is talking about.)

One can show that if x is an element of $\mathscr{A}'$ if and only if there is an element z in $\mathscr{B}'$ such that $x = z \circ g$. The theorem easily follows from this remark.

The rank of f and $f \circ g$ are same.

10

(a). Obvious. At every point p, the function is C^∞, i.e., f is C^∞.

(b).

11

One can write down $D_j g$ as the limit of $(g(x + h_j(0, 0 \cdots, 0, 1)) - f(x))/h_j$ as $h_j \to 0$. This limit exists, hence the left and right limits exists and agrees with each other. It is easy to see that, in both cases (g replaced by h) the right limit is the same (as g agrees with f) and hence $D_j g = D_j h$.

TODO 12

The natural way to define a C^∞ structure on ∂M is the following: let $(M, \mathscr{A})$ be a C^∞ structure on M, then define $\mathscr{B}$ by taking the collection of all charts $(x|_{\partial M}, U \cap \partial M)$, where (x, U) is a chart of M.

Clearly, $\mathscr{B}$ is an atlas. It can be easily shown that this is also a maximal atlas. It can be shown that under this structure the inclusion map is C^∞, homeomorphic to its image, and hence an imbedding.

We need to prove the uniqueness of this structure .

16

Note that the function $p(x)/exp(-x)$ where p is a polynomial has a limit 0 as $x \to \infty$.

19

Follows from the hint.

TODO 33

(a). I'm guessing that the formula for determinant and the equation $\mathrm{Det}(A)$ will lead to an equation of the form $f(\mathbf{x}) = 1$, where $\mathbf{x}$ is a n^2 dimensional vector and f is a continuous function. The closed-ness of $\mathrm{SL}(n, \mathbb{R})$ follows from the fact that, as a subset of $\mathrm{GL}(n, \mathbb{R})$, it is an inverse of a closed set. (Note that the text mentions that $\mathrm{GL}(n, \mathbb{R})$ is an open subset of the n^2 dimensional Euclidean space; this can be shown with the help of a similar argument.)

(b).

Chapter 3: The Tangent Bundle

1

(a). Suppose there is a metric on M such that each U_i becomes open and that each x_i is homeomorphic to its image, then one can see that the topology on M is the following:

(a) If U is an open subset of $\mathbb{R}$, them U is open in M.

(b) If U is an open (as a subset of $\mathbb{R}$) neighborhood of 0, then $(U - \{0\}) \cup \{*\}$ is an open set in M.

Notice that the space is not Hasdorff (the condition is violated by the pair 0, $*$) Since metrizable spaces are Hausdorff, M is not metrizable.