Chapter 1: Manifolds

1

In the first part, verify that the conditions required for metric is satisfied.

In the second part, one has to show that $B_{\bar{d}}(x, \varepsilon') \subset B_d x, \epsilon \subset B_{\bar{d}} x, \varepsilon$.

2

This is trivial.

3

Basically, you have to make use of the local properties of the Euclidean space.

For (b), proceed in the following manner: Take $x_0 \in X$, let the set A be all points y in X such that there exists a path from x_0 to y. Show that this set is both open and closed; since the space is connected, $A = X$.

For (c), I'm guessing that we have to follow in a similar fashion. But here, to show that A is both open and closed looks difficult.

4

1. Topologist's Sine curve.
2. Trivial.
3. Define the relation $\sim$ on X by $x \sim y$ if there is an connect subset of X that contains x and y. The equivalence classes of X under $\sim$ are called the connected components of X.

   It is easy to see that connected components are indeed connected, (show that C is the union of connected sets containing at least a point in common.)

   If all the connected components are open, then the space is locally connected, since for every point x in X, C_x be the connected component to which x belongs. Since C_x is open, this is the neighborhood that we are looking.

   Suppose the space is locally connected. Let C_x be a connected component, and pick $x \in C_x$. There exists an open connected neighborhood U. Since U is connected, it has to lie entirely in C_x, and hence C_x is open.
4. Trivial
5. Follows from 4.

5

1. This is trivial.
2. Follows from the fact that for $n \neq m$, $\mathbb{R}^n$ is not homeomorphism to $\mathbb{R}^m$.

6

1. It is easy to see that open subsets of an n manifold is an n manifold.

   Suppose M' be an n sub-manifold of M. If M' is not open, then for some $x \in M'$, every neighborhood of x contains a point outside of M', but there exists a neighborhood of M' that is homeomorphic to $\mathbb{R}^n$ and hence this neighborhood is open in M, a contradiction to the fact that M' is not open.

2. Let x be a point of M which has a neighborhood of dimension n. Define A to be all points of M that has dimension equal to n. It is enough to show that A is both open and closed.

   A is open: If $y \in A$, then y has a neighborhood homeomorphic to $\mathbb{R}^n$; clearly all points in these neighborhood lies in A, i.e., A is open.

   A is closed: the space is locally metrizable, let y be a limit point of A, $\{y_n\}$ be a sequence of points in A that converges to y, if y has a neighborhood of dimension m where $m \neq n$, we have a contradiction. Since for large enough n, x_n has a neighborhood homeomorphic to $\mathbb{R}^n$ and also another neighborhood homeomorphic to $\mathbb{R}^m$ (I'm taking for faith that this can't happen.) On second thought, we don't have to summon the locally-metrizable property.

7

1. An easy application of the intermediate value theorem.
2. I'm assuming that by an interval, the author is referring to an open interval, i.e., sets of the form (a, b).

   This is trivial from 1.

3. I'm assuming that by "f is homeomorphism", the author is referring to the fact that f is a homeomorphism between I and $f(I)$.

   This is trivial from 2.