Chapter 1: Manifolds

1

In the first part, verify that the conditions required for metric is satisfied.

In the second part, one has to show that $B_{\bar{d}}(x, \varepsilon') \subset B_d(x, \epsilon) \subset B_{\bar{d}}(x, \varepsilon)$.

2

This is trivial.

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(a). Basically, you have to make use of the local properties of Euclidean spaces.

(b). For (b), proceed in the following manner: Take $x_0 \in X$, let the set A be all points y in X such that there exists a path from x_0 to y. Show that this set is both open and closed; since the space is connected, $A = X$.

(c). Basically, one has to proceed in a similar fashion. But the issue is that the line connecting the limit point x and a point y in its neighborhood need not form an arc (the issue is that this need not result in a one-one path.)

To resolve this, observe that a continuous image of $[0, 1]$ will be compact. Since the space is locally metrizable, if x is a limit point of A, there is a sequence of points $\{x_n\}$ that converges to x in A. Choose a point x_n in some metrizable neighborhood of x, now the infimum of distances from x to this compact set will be realized at a point y in the range. Join y and x and remove the remainder of the path from y to x_n to obtain an arc to x, i.e., $x \in A$.

Similarly, one can prove openness of A, and from the fact that A is connected, we see that $A = X$.

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(a). Topologist's Sine curve.

(b). Trivial.

(c). Define the relation $\sim$ on X by $x \sim y$ if there is an connected subset of X that contains x and y. The equivalence classes of X under $\sim$ are called the connected components of X.

It is easy to see that connected components are indeed connected, (show that C is the union of connected sets containing at least a point in common.)

If all the connected components are open, then the space is locally connected, since for every point x in X, C_x be the connected component to which x belongs. Since C_x is open, this is the neighborhood that we are looking.

Suppose the space is locally connected. Let C_x be a connected component, and pick $x \in C_x$. There exists an open connected neighborhood U. Since U is connected, it has to lie entirely in C_x, and hence C_x is open.

(d). Trivial

(e). Follows from 4.

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(a). This is trivial.

(b). Follows from the fact that for $n \neq m$, $\mathbb{R}^n$ is not homeomorphism to $\mathbb{R}^m$.

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(a). It is easy to see that open subsets of an n manifold is an n manifold.

Suppose M' be an n sub-manifold of M. If M' is not open, then for some $x \in M'$, every neighborhood of x contains a point outside of M', but there exists a neighborhood of M' that is homeomorphic to $\mathbb{R}^n$ and hence this neighborhood is open in M, a contradiction to the fact that M' is not open.

(b). Let x be a point of M which has a neighborhood of dimension n. Define A to be all points of M that has dimension equal to n. It is enough to show that A is both open and closed.

A is open: If $y \in A$, then y has a neighborhood homeomorphic to $\mathbb{R}^n$; clearly all points in these neighborhood lies in A, i.e., A is open.

A is closed: the space is locally metrizable, let y be a limit point of A, $\{y_n\}$ be a sequence of points in A that converges to y, if y has a neighborhood of dimension m where $m \neq n$, we have a contradiction. Since for large enough n, x_n has a neighborhood homeomorphic to $\mathbb{R}^n$ and also another neighborhood homeomorphic to $\mathbb{R}^m$ (I'm taking for faith that this can't happen.)

On second thought, we don't have to summon the locally-metrizable property.

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(a). An easy application of the intermediate value theorem.

(b). I'm assuming that by an interval, the author is referring to an open interval, i.e., sets of the form (a, b).

This is trivial from 1.

(c). I'm assuming that by "f is homeomorphism", the author is referring to the fact that f is a homeomorphism between I and $f(I)$.

This is trivial from 2.

TODO 8

(a). It is easy to see that two components cannot be bounded at the same time, for, if they are bounded (call the components B and C), then $\mathbb{R} = A \cap B \cap C$ is also bounded, a contradiction.

So it is enough to show that one component is bounded. WLOG, assume that 0 lies "inside" the A. Let 0 belong to the component B. Since A is compact, A is bounded and thus the "inside" of A is bounded.

(b).

TODO 9

(a). Pick a point in $\mathbb{R}$, then $\mathbb{R} - \{x\}$ is disconnected, while $\mathbb{R}^n - \{x\}$, where $n > 1$, is connected.

(b).

TODO 11

The manifold M is σ compact, let $M = \cup M_i$, where M_i is compact.

Recall that compact metrizable spaces are first countable. Basically one has to take the countable union of all these countable sets to get a countable base for M.

Again, compact metrizable spaces are also

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(a). $f(x, y) = \frac{-2x}{y-1}$.

(b). $f(x_1, x_2, \cdots, x_n) = (\frac{-2x_1}{x_n-1}, \frac{2x_1}{x_n-1}, \cdots, \frac{-2x_{n-1}}{x_n-1})$.

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(a). One has to show that the two definitions (the original definition, and the definition in which open sets need not contain the antipodal point), will give rise to the same topology.

Let us denote the set $\{-p : p \in V\}$ by $-V$. Since $\phi(p) = -p$ is a homeomorphism from S^1 onto S^1, V is homeomorphic to $-V$. In the second definition, observe that $f(V \cap (-V)) = f(V)$.

(b). In case of the Möbius strip, the issue is that the new definition will produce sets that are not open as per the original definition. For example $V = [0, 1/3) \times (-1, 0)$ is open in $[0, 1] \times (-1, 1)$; the new definition asserts that $f(V) = [[0, 1/2) \times (-1, 0)]$ is open in the Möbius strip, but it isn't open as per the original definition.

The two mappings differ in the symmetry of the map f.

TODO 14

(a).

(b). This follows from the definition of $\mathbb{P}^2$.

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(a). Imagine S^1 lying inside $\mathbb{C}$, consider the map $f\colon S^1 \to S^1$ defined by $f(z) = z^2$. It is easy to see that this map is a quotient map since saturated open sets are mapped to open sets of S^1. (I think that saturated sets are of the form $V = V \cap (-V)$.)

Then it can be seen that the quotient space under this map is $\mathbb{P}^1$. There is a natural map $g\colon \mathbb{P}^1 \to S^1$ such that $g(p(x)) = f(x)$ where $p\colon S^1 \to \mathbb{P}^1$ is the identification map. Recall that g is a homeomorphism if and only if f is a quotient map.

(b). Let us imagine things in the following manner. S^n is a subspace of $\mathbb{R}^{n+1}$ defined by $\|\mathbf{x}\| = 1$, i.e., all points $\mathbf{x}$ such that $x_1^2 + \cdots + x_{n+1}^2 = 1$.

One can imagine S^{n-1} as a subspace of $\mathbb{R}^n$ in the following manner: all points $\mathbf{x}$ such that $x_1^2 + \cdots + x_n^2 = 1$ and $x_{n+1} = 0$.

Consider $D^n = \{\mathbf{x} \in \mathbb{R}^n \colon \|\mathbf{x}\| < 1\}$, define $\varphi((x_1, \cdots, x_n)) = [(x_1, \cdots, x_n, x_{n+1})]$ where $[p] = \{p, -p\}$ and x_{n+1} is the positive real number that satisfies $x_{n+1}^2 = 1 - (x_1^2 + x_2^2 + \cdots + x_n^2)$. This map is the required homeomorphism.

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(a). Consider $\mathbb{R}$, remove a one-point set. $\mathbb{R} - \{x\}$ is not connected and since compact sets are bounded, removing a compact set from this is not going to make this connected.

On the other hand, for any compact set C in $\mathbb{R}^n$ will be inside the set $[-M, M]^n$ and $\mathbb{R}^n - [-M, M]^n$ is connected

(b). Consider $C = S^{n-1}$ in $\mathbb{R}^n - \{0\}$. This is compact relative to $\mathbb{R}^n - \{0\}$ (since it is compact relative to $\mathbb{R}^n$.) $R^n - \{0\} - C$ has two connected components, one bounded and the other unbounded. In order to make $\mathbb{R}^n - \{0\} - K$ connected (where K is a compact set), one has to, at least, remove the bounded component completely.

Imagine the closed set $Y = [a, 0) \cup (0, b]$ for sufficiently small a and b such that Y lies entirely in K. Y is closed, since $Y = [a, b] \cap (\mathbb{R}^n - \{0\})$ and being a closed subset of K, it is compact. This is a contradiction.

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(a). Given any compact subset C of $\mathbb{R}$, pick a closed interval that contains C, say K, then $\mathbb{R} - K$ has two components. To satisfy the property that $\varepsilon(K) \subset \varepsilon(C)$, $\varepsilon(C)$ has to be the component of $\mathbb{R} - C$ that contains one of the components of $\mathbb{R} - K$. Thus there are two ways of doing this, i.e., $\mathbb{R}$ has two ends.

(b). We repeat the same trick. Given any compact subset C of $\mathbb{R}^n$, $n > 1$, pick a closed n cell that contains C, say K, then $\mathbb{R} - K$ has exactly one component. We are forced to that components of $\mathbb{R} - C$ that contains $\mathbb{R} - K$, i.e., $\mathbb{R}^n$ has one end.

Suppose X "has one end", then for repeat the above step to see that X is one ended.

Suppose X is one ended. Suppose for a compact set C, $X - C$ has more than one component and for every compact set K such that $C \subset K$, $X - K$ has more than one component, then clearly, we have more than one choice for $\varepsilon(C)$, a contradiction.

(c). Refer to Theorem 29.1 in Topology, Munkres.

$\mathbb{R} \cap \varepsilon(R) = [0, 1]$ (Two point compactification of $\mathbb{R}$.)

$\mathbb{R}^n \cap \varepsilon(\mathbb{R}^n) = S^n$ (One point compactification of $\mathbb{R}^n$.)

Chapter 2: Differentiable Structures

1

(a). Clearly, the symmetric and reflexive properties are satisfied, but there are issues with transitivity.

Consider the following maps:

$a\colon A \to \mathbb{R}^n$
$b\colon B \to \mathbb{R}^n$
$c\colon C \to \mathbb{R}^n$

The C^∞ related only means that the composition of one map and the inverse of the other map is C^∞ on the region were their domain overlaps. It could be true that $a \circ c^{-1}$ need not be C^∞ on $A \cap C$. An explicit example should be easy to construct.

(b). This has to do with the fact that $\mathscr{A}$ is an Atlas, and the newly added maps are all C^∞ related to **all** maps in the Atlas $\mathscr{A}$.

Let $\mathscr{A}'$ be the maximal atlas corresponding to $\mathscr{A}$. We have to prove the following: (x, X) and (y, Y) be two coordinates then x and y are C^∞ related.

When x or y belongs to $\mathscr{A}$, this is trivial. It is enough to prove for x and y not in $\mathscr{A}$.

WLOG, assume that there is a chart $(u, U) \in \mathscr{A}$ such that $A \in X \cap Y$ (in the general case, we have to repeat the following procedure for every charts.)

Thus $y \circ (x^{-1} = (y \circ u^{-1}) \circ (u \circ x^{-1})$ and hence is C^∞; similarly $x \circ y^{-1}$ is also C^∞.

TODO 2

3

(a). This is trivial from the fact that differentiability implies continuity and chain rule. (I'm assuming that these C^∞ functions are from $\mathbb{R}^n$ to $\mathbb{R}^m$.)

(b). The only if part can be easily seen from the fact that

$g \circ f \circ x^{-1} = (g \circ y^{-1}) \circ (y \circ f \circ x^{-1})$.

where $(x, A), (y, B)$ are charts of M and N respectively.

To show the only if part, choose $g = y^i$ and $g \circ f = f^i$ is C^∞ map from M into $\mathbb{R}$ for all $i = 1, 2, \cdots, m$ (here m is the dimension of N.) Which is equivalent to saying that the map f is C^∞. (See result (4) in page 31.)