STA202 – OPPORTUNITY THEORY

NILAI HARAPAN

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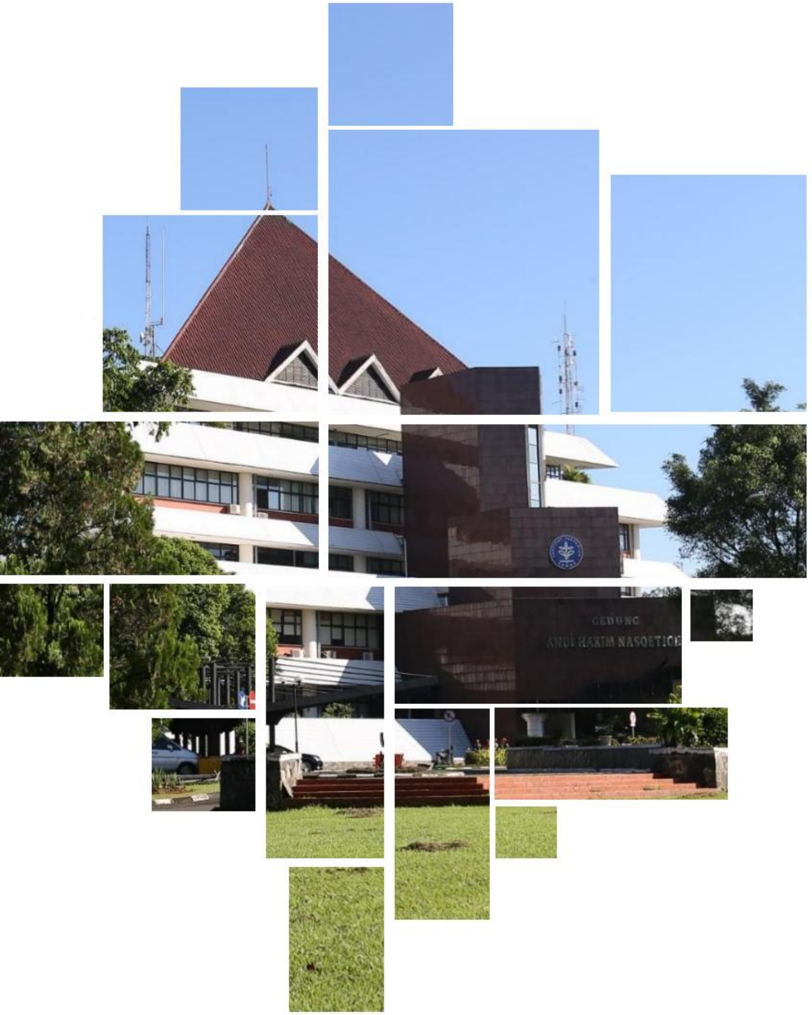

Program Studi Sarjana Statistika dan Sains Data
Departemen Statistika - FMIPA

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Outline

1. Expected Value
2. Variety
3. Expected Value of a Function of a Random Variable
4. The nature of the value of hope and variety
5. Moment
6. Moment Generating Function

REFERENCE:

1. Ross SM. 2010. A first course in probability. 8th ed. New Jersey: Pearson Prentice Hall.
2. Wackerly DD, Mendenhall W, Scheaffer RL. 2008. Mathematical Statistics with Applications. Seventh Edition. California: Thomson Learning, Inc

1. VALUE OF EXPECTATIONS

One of the concepts that is often used in statistical theory is the expected value of a random variable.

Expected value can be seen as the balance point of a probability function in a real line (abscis of random variable value), and is generally the called average.

Definition:

Consider a random variable X which has the probability mass (density) function ( ).

The mean or expected value of X is defined as:

$$= \ (\ ) \ \underset{ÿ}{=} \ (=)$$

if pa discrete

and

$$= \ (\ ) \ \overset{ÿ}{\underset{ÿ}{=}} \ (\ )$$

if pa is continuous

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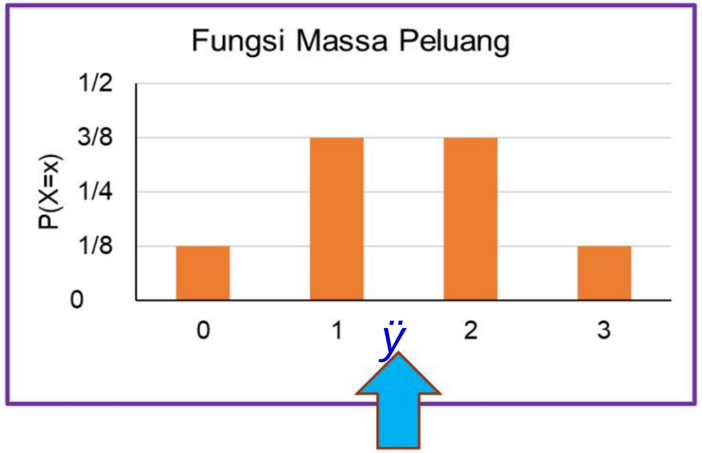
Illustration: Expected Value on discrete



center of mass

Pay attention to Illustration-fmp

x	0	1	2	3
P(X=x)	1/8	3/8	3/8	1/8

ÿ Expected value of random variable X:

$$(\ ) = = 0 \times \left(\frac{1}{8}\right) + \left(1 \times \frac{3}{8}\right) + \left(2 \times \frac{3}{8}\right) + \left(3 \times \frac{1}{8}\right) = \frac{3}{2}$$

Illustration: Expected Value pa continuous

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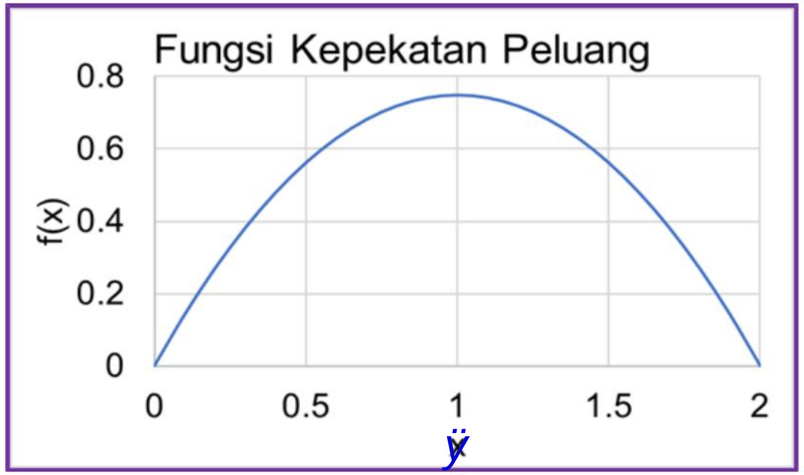
Pay attention to Illustration-fkp

The random variable has fkp:

$$(\ )\quad \frac{}{8}(4\ 2\qquad ^{2})\quad ,\quad 0<<2$$

$$=3\qquad 0\qquad ,\ \text{other}$$



center of mass

- Expected value of random variable X:

$$(\ )\quad 0\ddot{y} = = \overset{2}{0} \times \frac{3}{8}(4\ 2\qquad ^{2}) \qquad \overset{2}{} \frac{3}{8}(4\quad 2\ 2\qquad ^{3})$$

$$= \quad (\ )\quad \frac{3}{8}\left(\frac{4}{3}\quad 3\quad \frac{1}{2}\quad 4\right)\Big|_{=0}^{=2}$$

$$= \quad (\ )\quad \frac{3}{8}\left[\left(\frac{32}{3}\ 8\ (0\right)0) = \qquad\right] \quad \frac{3}{8}\left[\left(\frac{8}{3}\right)\right] = 1$$

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2. VARIETY

ÿ Variety of discrete random variable X

$$E(X) \quad 2 \quad (x) \; P(x) \quad 2 \quad x$$

ÿ Variable continuous random variable X

$$E(X) \quad ÿÿÿ \quad 2 \quad ÿ \; ÿ \; (x) \; f(x) dx \quad 2 \quad 3 \quad -ÿ$$

ÿ Alternative formulas in

calculation of the variance of a distribution

$$E(X) \; (ÿ) \quad 2 \quad 2 \quad 2$$

ÿ Standard deviation of discrete/continuous random variables

$$ÿ \; ÿ \; ÿ \quad \sqrt{\;\;2}$$

Illustration: Discrete pa Variety

Watch Illustration-fmp

The probability mass function X is:

x	0	1	2	3
P(X=x)	1/8	3/8	3/8	1/8

ÿ = $(\ \ ) = \frac{3}{2}$

ÿ $(\ \ ^2) = \ 3 = 0 \ \ ^2 \ (\ \ )$

$$= 0^2\left(\frac{1}{8}\right) + 1^2\left(\frac{3}{8}\right) + 2^2\left(\frac{3}{8}\right) + 3^2\left(\frac{1}{8}\right) = \frac{24}{8} = 3$$

then _ $^2(\ \ ) = (\ \ ^2) \ \ ^2 = 3 \quad \left(\frac{3}{2}\right)^2 = \frac{3}{4}$

Standard deviation: = $\frac{1}{2}\sqrt{3}$

Illustration: Continuous pa variety

Suppose the random variable Y spreads according to the following probability density function:

$$(\ ) = 3 \quad \frac{}{64}\, 2\, y\, (4\ y \text{ for } 0\ 4) \qquad 0 \quad \text{for others}$$

$$ÿ = \quad (\ ) = 0^{4}\ \frac{3}{64}\, 3\, y\, (4\ y \quad)$$

$$= \frac{3}{64}\,{}_{0}^{4}\,(4\ y^{3}\ 4\ y \quad) \qquad = \frac{3}{64}\left(y\ Ÿ^{4ÿ}\ \frac{1}{5}\ {}^{5}\right)_{ÿ\,0}^{4} = 2.4$$

$$ÿ \quad (\ ^{2}) = 0^{4}\ \frac{3}{64}\, 4\, y\, (4\ y \quad)$$

$$= \frac{3}{64}\,\overline{\ }_{0}^{4}\,(4\ y^{4}\ 5\ y \quad) \qquad = \frac{3}{64}\left(\frac{4}{5}\ 5\ y \quad \frac{1}{6}\, y^{6}\right)_{ÿ\,0}^{4} = 6.4$$

$$ÿ \quad {}^{2}(\ ) = \quad (\ ^{2}) \quad {}^{2}\ 2 = 6.4\ 2.4 \quad = 0.64$$

3. EXPECTATION VALUE OF RANDOM VARIABLE FUNCTIONS

If (X) is a function of the random variable X, then the expected value of (X) is:

(X) = E[(X)] = ÿ ( ) (=) if X is discrete

ÿ ÿ ÿ ( ) ( ) ÿ if X is continuous

if the sum or integral exists.

Illustration: expected value of discrete pa function

- Suppose the random variable X represents the number of cars washed between 16:00 and 17:00 at a car wash having the following probability mass function:

x	4	5	6	7	8	9
$P(X = x)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$

  - The random variable g(X) = 2X-1 represents the car wash fee
  - the average car wash revenue for that hour is:

$$[\ (\ ) = [2\ 1\ ] = \sum_{=4}^{9} [2\ 1\ (\ )\ ]$$

$$= (7)(1/12 + (9)(1/12) + 11(1/4) + 13(1/4) + 15(1/6) + 17(1/6) = 12\tfrac{2}{3}$$

Illustration: the expected value of the continuous pa function

Suppose the random variable X spreads according to the following probability density function :

$$(\ ) = \frac{1}{3} \quad {}^{2} \text{ for } 1 \ 2$$
$$0 \text{ for other}$$

Expected value $(\ ) = 4 + 3$:

$$[4 + 3] = \int_{1}^{2} (4+3)\left(\frac{1}{3} \ {}^{2}\right)$$

$$= \frac{3}{-} \int_{1}^{2} (4 \ {}^{3} + 3 \ {}^{2}) = 8$$

4. NATURE OF HOPE VALUE AND VARIETY

Suppose a and b are constants and the random variable X has a mean x and a variance 2 **a)** x

E(a) a and Var(a) 0

b) E(bX)ÿ bÿX and Var(bX) b 2 2 X

c) Let Y = a + bX o The mean of the random variable Y is:

Y E(a bX) a bÿX

o The variance of the random variable Y is:

Var(a bX)ÿ b 2 2 2 X

o So that the standard deviation of Y is:

Y bÿX

Illustration: The Nature of Expectation Value and Discrete Variety of Pa

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Notice the Discrete Illustrations:

- From the illustration it is known: = $(\ ) = \frac{3}{2}$ and $^{2}(\ ) = \frac{3}{4}$
- Determine the expected value and variance $(\ )$ of X = 6 + 4

Solution:

ÿ $[\ (\ )] = 6 [+ 4 = 6 + 4] = 6 \quad [\ ] \quad \left(\frac{3}{2}\right) + 4 = 5$

ÿ $^{2}[\ (\ )] = \ ^{2}[6 + 4 = 36] \quad ^{2}[\ ] + 0 = 36 \left(\frac{3}{4}\right) = 27$

Illustration: The Nature of the Value of Hope and the Variety of Continuous Pa

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- Pay attention to Continuous Illustrations:
- From the illustration it is known: Y= 2.4 and $^2(\ ) = 0.64$
- Determine the expected value and variance of $(\ ) = 4 + 3$

Solution:

ÿ $[\ (\ )] = 4 + [3 = 4 + 3] = 4\ 2.4 [+ 3] = 12.6 \quad (\quad)$

ÿ $^2[\ (\ )] = \ ^2[4 + 3 =]16 \quad ^2[\ ] + 0 = 16\ (0.64) = 3.84$

5. MOMENT

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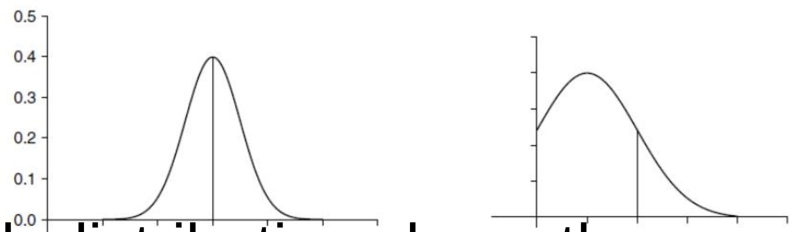
- Although the mean *and* standard deviation are *descriptive* measures of the location of the center and the distribution or dispersion of a probability function ( ), they do not provide a unique characteristic of a distribution.

- Two distributions that have the same mean and variance, but have different shapes, as shown in the following figure:



- Both distributions have the same mean and variance = 1, = 1 2

- To get a good approximation to a probability distribution a higher degree moment is needed.

The th moment of the random variable is defined as:

ÿ = [ ] for = 1,2, 3, ,

The center moment of the random variable X defined as:

= [( )] for = 2, 3, 4,

Moment (cont.)

Third moment standardized to mean

$$_3 = \frac{E[(X \quad)^3]}{^3} = \frac{^3}{^{3ÿ2}_2}$$

is called *the* slope of the distribution

Slope is used to measure the symmetry of a distribution function to the average.

A distribution is said to be symmetric if the right and left sides of the center are the same.

If _ $_3$ = 0 then the distribution is symmetric about the mean

if right $_3$ > 0 then the distribution has a long tail at the tail

if < 0 then $_3$ the distribution has a long tail to the left of the tail of the distribution.

Normal distribution has zero slope coefficient.

Moment (cont.)

Fourth moment standardized to mean

$$_{4} = \frac{E[(X \quad)^{4}]}{^{4}} = \frac{^{4}}{_{2}^{2}}$$

This is called kurtosis of the distribution

Kurtosis measures the taper or slope of a distribution compared to a distribution normal.

Kurtosis is measured by the size of the tail of a distribution.

Positive kurtosis indicates the distribution has few observations in the tail scatter

Negative kurtosis indicates that there are many observations in the tail of the distribution.

Distributions with relatively long tails are called leptokurtic, and vice versa for short tail called platokurtic .

A distribution with the same kurtosis as a normal distribution is called mesokurtic . The normal distribution has a kurtosis value $_4$ = 3.

Illustration : Skewness

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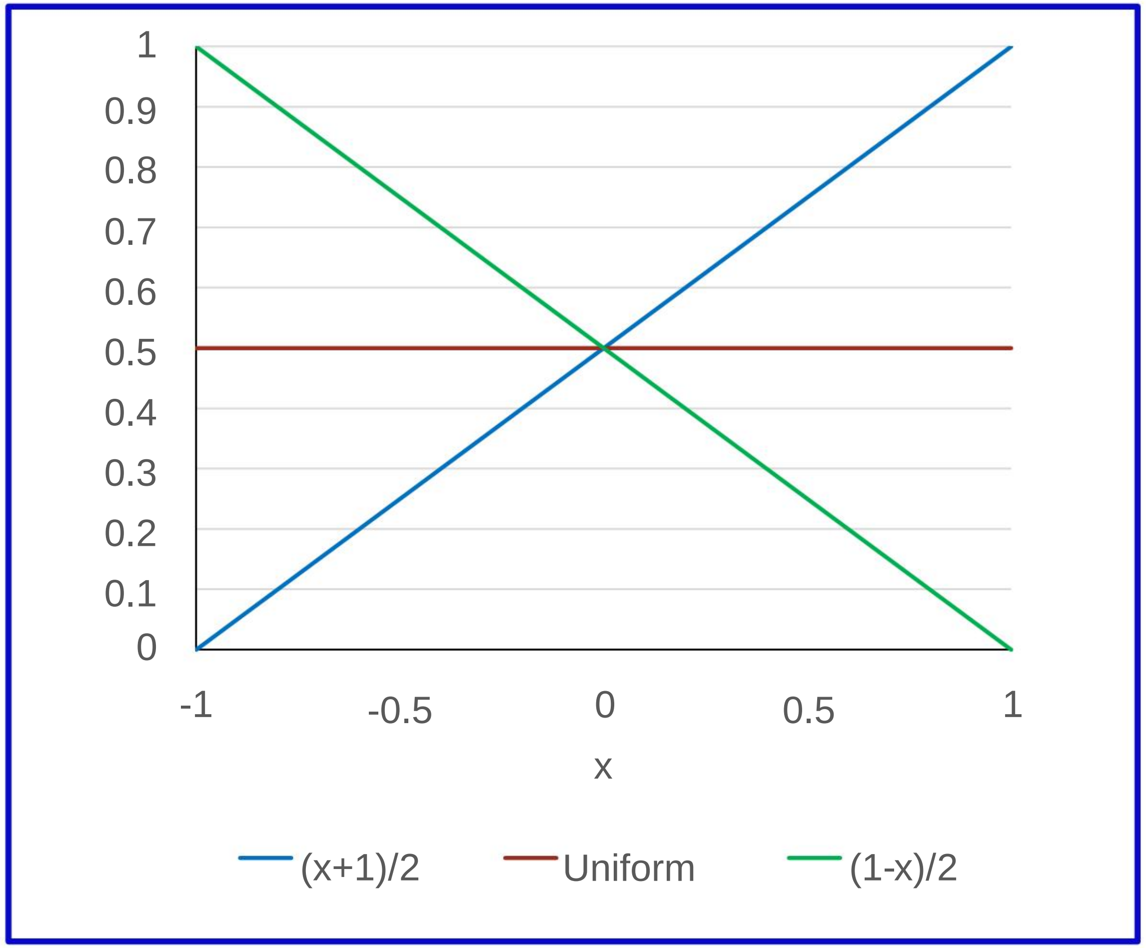
Suppose the random variable X has a probability density function:

ÿ ( ) = (1 /2 for 1 < < 1, zero for others

ÿ ( ) = 1/2 for 1 < < 1, zero for others

ÿ ( ) = 1(/2 for) 1 < < 1, zero for others



Determine the value of the *skewness* of the distribution

the !!!

Illustration: Taper *(Kurtosis)*

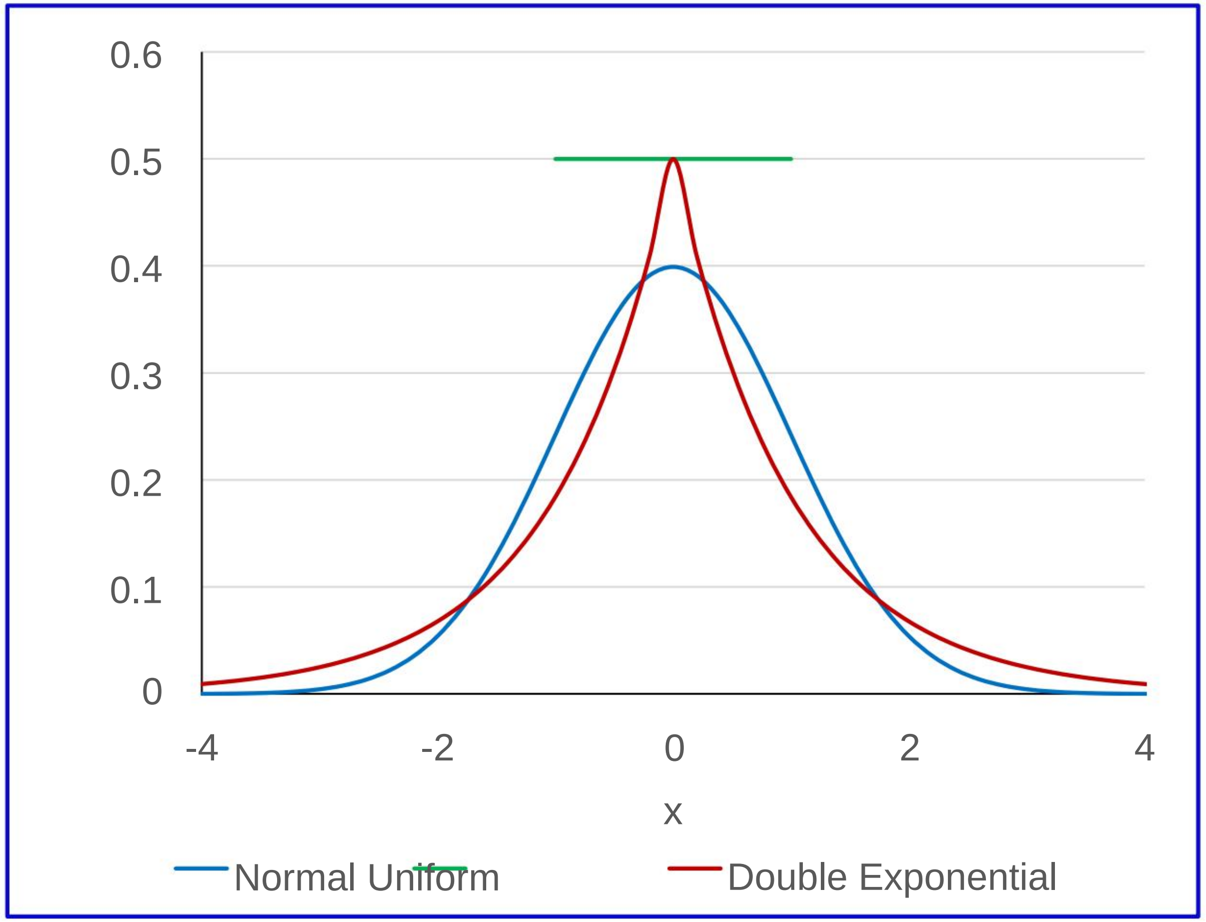
Suppose the random variable X has a probability density function:

ÿ ( ) = $\frac{1}{\sqrt{2}}$ 2 2for < < ,

zero for others

ÿ ( ) = 1/2 for 1 < < 1, zero for others

ÿ ( ) = $\frac{1}{2}$ for ÿ ÿ < < , zero for other



Determine the value of the kurtosis distribution the !!!

6. MOMENT GENERATING FUNCTION

Suppose X is a random variable, there is is a positive integer so that the expected value of E [X] exists for < < .

The moment generating function of the random variable X is defined as:

$$M_X(\ ) = E[\ ^X] = \begin{cases} \ddot{y} \quad ^X \ (\ ) \text{ when X is discrete} \\ \ddot{y} \ \ddot{y} \\ \quad ^X \ (\ ) \qquad \text{if X is continuous} \\ \ddot{y}_{\ddot{y}} \end{cases}$$

o $\ ^X = 1 + \ + \frac{(\ )^2}{2!} + \frac{(\ )^3}{3!} + \ + \frac{(\ )}{!} + $

o $M_X(\ ) = E[\ ^X] = E\left[1 + X + \frac{(\ X)^2}{2!} + \frac{(\ X)^3}{3!} + \ + \frac{(\ X)}{!} + \right]$

$$= 1 + E[X] + \frac{^2}{2!}E[X^2] + \frac{^3}{3!}E[X^3] + \ + \frac{}{!}E[X^n] +$$

Moment Generating Function (cont.)

$$2\ M(X) = E[X] + \frac{2}{2!}E[X] + \frac{3}{3!}3E[X] + + \frac{n}{!}E[X^n] +$$

o The first derivative of $M(X)$ against is obtained

$$\frac{MX(\ )}{} = MX^{ÿ}(\ ) = E[X] + E[X^2] + \frac{2}{2}E[X^3] + + \frac{(\ 1\ )}{(\ 1!\ )}E[X^n] +$$

Evaluate the derivative at value = 0, all terms except EX are zero. Up to MX [ ]

$$^{ÿ}(0) = EX[\ ]$$

In the same way, the second derivative of MX will be obtained MX $^{ÿ}(0) = E[X^2]$

If you continue until the th derivative on MX you will get:

$$\frac{(\ )}{MX\ \_}\Big|_{=0} = MX^{(\ )}\ 0 (=)^{ÿ}$$ for = 1,2,

Illustration: Discrete Moment Generating Function

Let X be a binomial spread with the following probability mass function:

$$(\ ) \ = \ (\ ) \quad (1 \ ) \text{ for } = 0.1,2, \qquad ,$$

The moment generating function is as follows:

$$MX = E \ (\ ) \ [\ X] = \sum_{=0} (\ ) \ (1 \ )$$

$$= \sum_{=0} (\ )(\ \ ) \ (1 \ ) \qquad = [+ 1 \ (\ \ )]$$

for < <

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Illustration: Discrete Moment Generating Function (cont.)

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- The first and second derivatives of ( ) MX against are obtained:
  - $M_X'(\ ) = (\quad)[\quad + 1(\quad)]^{1}$
  - $M_X''(\ ) = 1(\quad)(\quad)^2[\quad + 1(\quad)]^{2} + (\quad)[\quad + 1(\quad)]^{1}$
- Then:
  - $E[X] = M_X'(0) =$
  - $E[X^2] = M_X''(0) = 1 + (\quad)$
  - Variety:

$$^2 = E[X^2] - (E[X])^2 = 1 + (\quad) \quad (\quad)^2 = (1\ )$$

Illustration: Continuous Moment Generating Function

Let X have a standard normal distribution with the probability density function as following:

$$(\ ) = \frac{1}{\sqrt{2}} \cdot -\frac{1}{2} \ ^2 \ ; < <$$

The moment generating function is:

$$MX = E \ (\ ) \ [\ ^{X}] = \int_{ÿ}^{ÿ} x \frac{1}{2\sqrt{\ }} \cdot -\frac{1}{2} \ ^2 = \int_{ÿ}^{ÿ} \frac{1}{2\sqrt{\ }} \cdot \frac{1}{2}(\ 2ÿ2 \ )$$

$$= \int_{ÿ}^{ÿ} \frac{1}{\sqrt{\ }} \cdot \frac{1}{2}(\ 2ÿ2 + \ \ 2) + \frac{2}{\ } \ 2 = \ \frac{2}{\ } \int_{ÿ}^{ÿ} \frac{1}{\sqrt{\ }} \cdot \frac{1}{2}(\ \cdot \ )^2 = \ \frac{2}{2}$$

The Nature of the Moment Generating Function

1. $M_{X+a}(t) = e^{at} M_X(t)$
2. $M_{aX}(t) = M_X(at)$
3. If X_1, with , X is a stochastic independent random variable their respective moment generating functions are M_{X_1} , $(\,)$, M_X $(\,)$ and $Y = X_1 + + X$ then $M_Y(\,) = M_{X_1}(\,) M_X(\,)$

Illustration: The Nature of the Moment Generating Function

ÿ Determine the moment generating function ~ (, ^{2})

ÿ Let ~ 0.1 ( ), then Fpm standard normal distribution: ( ) = $\frac{2}{2}$

ÿ Then Fpm for = + is:

( ) = + ( ) = E((+)) = ( )

= ( ) ($\frac{2\ 2}{2}$ =) $\frac{2\ 2}{2}$ +

ÿ So that = + ~ (, ^{2})

QUESTION

4.38. If $E[X] = 1$ and $\text{Var}(X) = 5$, find

(a) $E[(2 + X)^2]$;

(b) $\text{Var}(4 + 3X)$.

5.7. The density function of X is given by

$$f(x) = \begin{cases} a + bx^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

If $E[X] = \frac{3}{5}$, find a and b.

5.4. The probability density function of X, the lifetime of a certain type of electronic device (measured in hours), is given by

$$f(x) = \begin{cases} \dfrac{10}{x^2} & x > 10 \\ 0 & x \le 10 \end{cases}$$

(a) Find $P\{X > 20\}$.

(b) What is the cumulative distribution function of X?

(c) What is the probability that, of 6 such types of devices, at least 3 will function for at least 15 hours? What assumptions are you making?

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QUESTION

6.4 **Basketball shots** To win a basketball game, two competitors play three rounds of one three-point shot each. The series ends if one of them scores in a round but the other misses his shot or if both get the same result in each of the three rounds. Assume competitors A and B have 30% and 20% of successful attempts, respectively, in three-point shots and that the outcomes of the shots are independent events.

a. Verify the probability that the series ends in the second round is 23.56%. (*Hint:* Sketch a tree diagram and write out the sample space of all possible sequences of wins and losses in the three rounds of the series, find the probability for each sequence and then add up those for which the series ends within the second round).

b. Find the probability distribution of X = number of rounds played to end the series.

c. Find the expected number of rounds to be played in the series.

6.10 TRY **Ideal number of children** Let X denote the response of a randomly selected person to the question, "What is the ideal number of children for a family to have?" The probability distribution of X in the United States is approximately as shown in the table, according to the gender of the person asked the question.

Probability Distribution of X = Ideal Number of Children

x	P(x) Females	P(x) Males
0	0.01	0.02
1	0.03	0.03
2	0.55	0.60
3	0.31	0.28
4	0.11	0.08

Note that the probabilities do not sum to exactly 1 due to rounding error.

a. Show that the means are similar, 2.50 for females and 2.39 for males.

b. The standard deviation for the females is 0.770 and 0.758 for the males. Explain why a practical implication of the values for the standard deviations is that males hold slightly more consistent views than females about the ideal family size.

QUESTION

3.153 Find the distributions of the random variables that have each of the following moment-generating functions:

a $m(t) = [(1/3)e^t + (2/3)]^5$.

b $m(t) = \dfrac{e^t}{2 - e^t}$.

c $m(t) = e^{2(e^t - 1)}$.

3.156 Suppose that Y is a random variable with moment-generating function $m(t)$.

a What is $m(0)$?

b If $W = 3Y$, show that the moment-generating function of W is $m(3t)$.

c If $X = Y - 2$, show that the moment-generating function of X is $e^{-2t}m(t)$.


thank you