ECE535: Digital Signal Processing

Project 1: Sample Rate Conversion

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1 Preliminaries

Part (a)

a) $s[n] = \dfrac{\sin A\pi(n-D)}{A\pi(n-D)}$ → undefinded at $n = D$

→ Use L'Hosiptal to find values at $n = D$

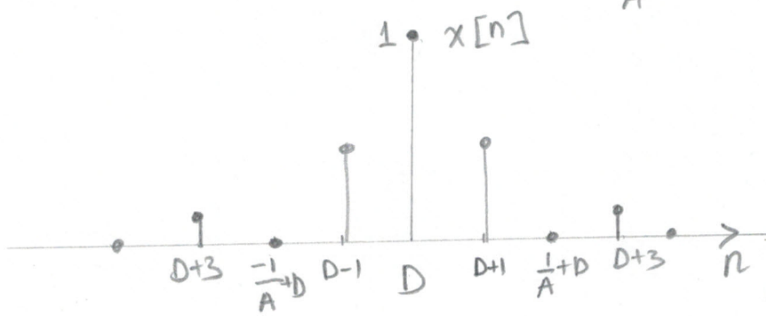
$$s[D] = \left.\frac{\frac{d}{dn}\sin A\pi(n-D)}{\frac{d}{dn}A\pi(n-D)}\right|_{n=D} = \frac{A\pi\cos A\pi(n-D)}{A\pi} = 1$$

$$\Rightarrow \underline{\underline{x[D] = \left(s[D]\right)^2 = 1^2 = 1}}.$$

Zero crossings: $A\pi(n-D) = k\pi$, k is integer $\neq 0$

$$n - D = \frac{k}{A}$$

$$n = \frac{k}{A} + D$$



Maximum value: 1 at point of sym: $n = D$

locations of zero crossings: $n = \dfrac{k}{A} + D$ where $k \in \mathbb{Z}$ but $k \neq 0$

Figure 1: Analyses to find maximum values and zero crossings

Part (b) and (c)

b) From Table 2.3 :

$$\frac{\sin(w_c n)}{\pi n} \xleftrightarrow{DTFT} X(e^{j\omega}) = \begin{cases} 1, & |\omega| < w_c \\ 0, & w_c < |\omega| \le \pi \end{cases}$$

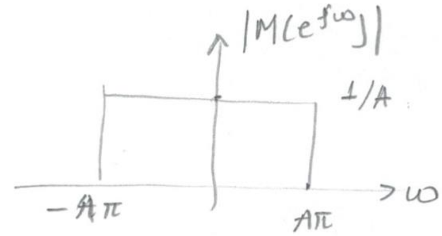
$$\frac{1}{A} \cdot \frac{\sin(A\pi n)}{A\pi n} \xleftrightarrow{DTFT} \begin{cases} 1/A, & |\omega| < A\pi \\ 0, & A\pi < |\omega| < \pi \end{cases}$$

$$\frac{\sin(A\pi(n-D))}{A\pi n} \xleftrightarrow{DTFT} \begin{cases} \frac{1}{A} \cdot e^{-j\omega D}, & |\omega| < A\pi \\ 0, & A\pi < |\omega| < \pi \end{cases} = M(e^{j\omega})$$

b) $$x[n] = \left[\frac{\sin(A\pi(n-D))}{A\pi(n-D)}\right]^2 \xleftrightarrow{DTFT} \frac{1}{2\pi} M(e^{j\omega}) * M(e^{j\omega})$$

Convolution expend the support to:

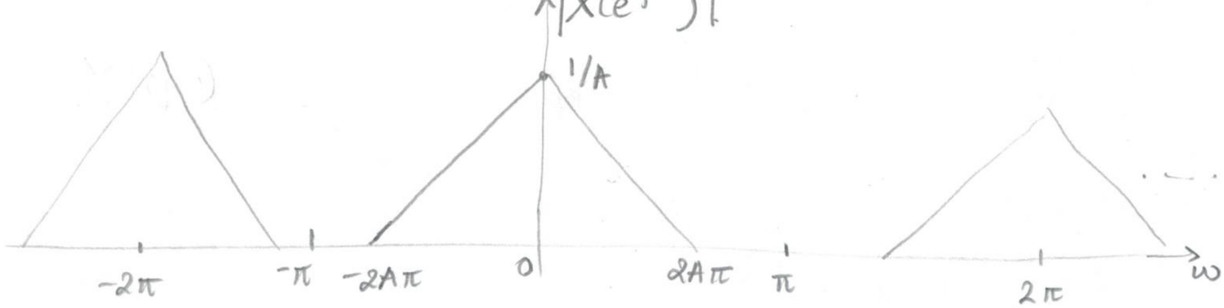
$-2A\pi < |\omega| < 2A\pi$



Maximum height:

$$\frac{1}{2\pi} \cdot \left(\frac{1}{A} \cdot \frac{1}{A}\right) \cdot 2A\pi = \frac{1}{A}$$

Convolution of two rectangular boxes is a triangle:

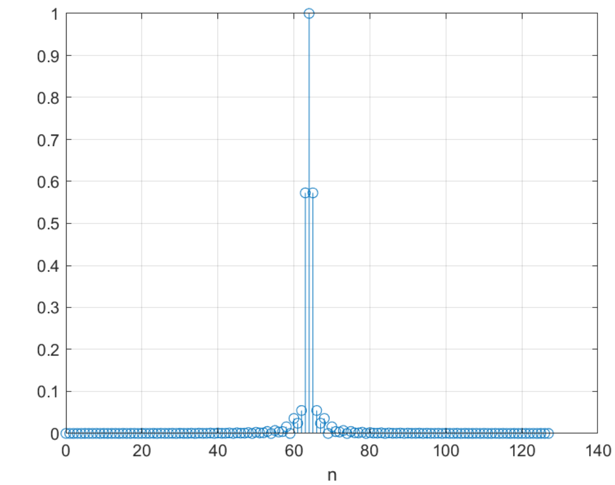


$$X(e^{j\omega}) = \begin{cases} \frac{-1}{2\pi A^2}\omega + \frac{1}{A}, & 0 < |\omega| < 2A\pi \\ \frac{1}{2\pi A^2}\omega + \frac{1}{A}, & -2A\pi < \omega \le 0 \\ 0, & o.w \end{cases}$$

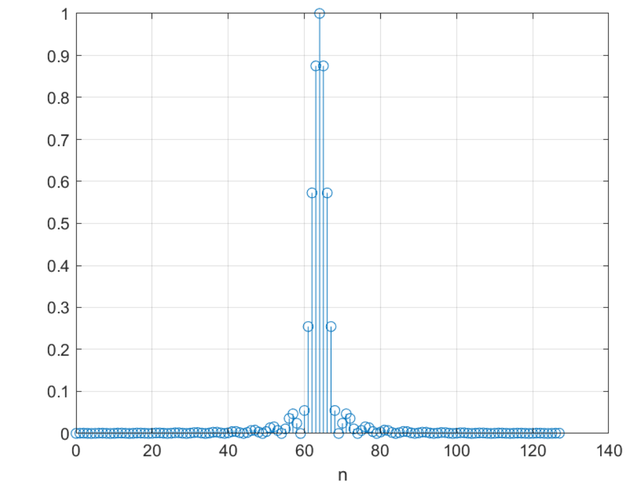
← Repeat every 2π

2 Upsampling

Part (a)



(a) 128 samples of $x_1[n] = sinc(0.4(n-64))^2$

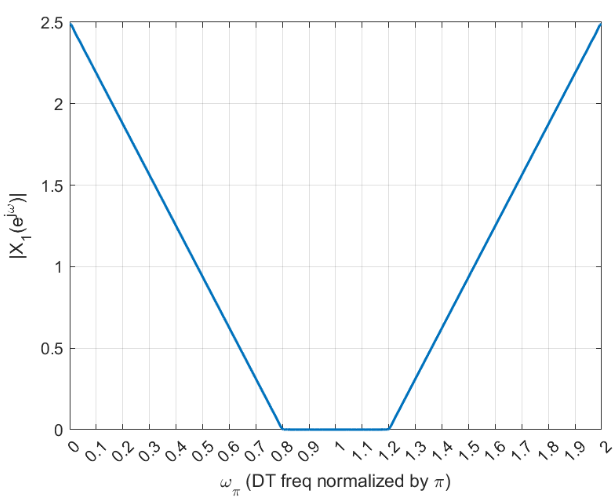


(b) 128 samples of $x_2[n] = sinc(0.2(n-64))^2$

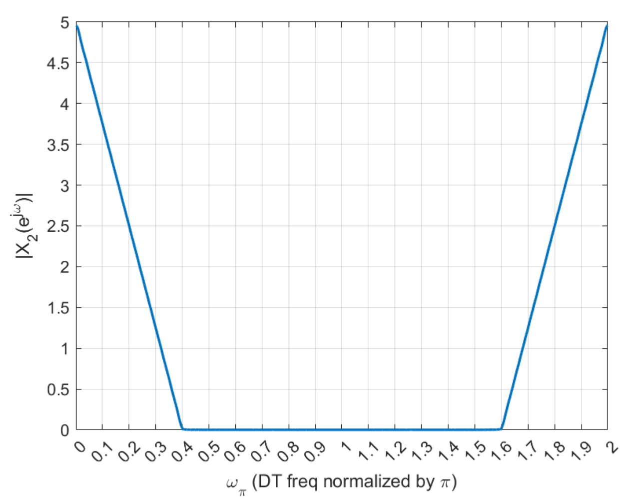
Figure 3: Verify symmetry, peak values, and zero crossings

In Fig. 3, both of the sinc's are symmetric about $D = 64$, and have peak values of 1's.

Part (b)



(a) Magnitudes of DFT of $x_1[n]$



(b) Magnitudes of DFT of $x_2[n]$

Figure 4: 2048 samples of DFT of $x_1[n]$ and $x_2[n]$.

The triangle shapes in Fig. 4 agree with the analytical results. The maximum height of $X_1(e^{j\omega})$ is $1/0.4 = 2.5$, and $X_1(e^{j\omega})$ is bandlimited to $2A\pi = 2(0.4)\pi = 0.8\pi$. Similarly, The maximum height of $X_2(e^{j\omega})$ is $1/0.2 = 5$, and $X_2(e^{j\omega})$ is bandlimited to $2A\pi = 2(0.2)\pi = 0.4\pi$.

Part (c)

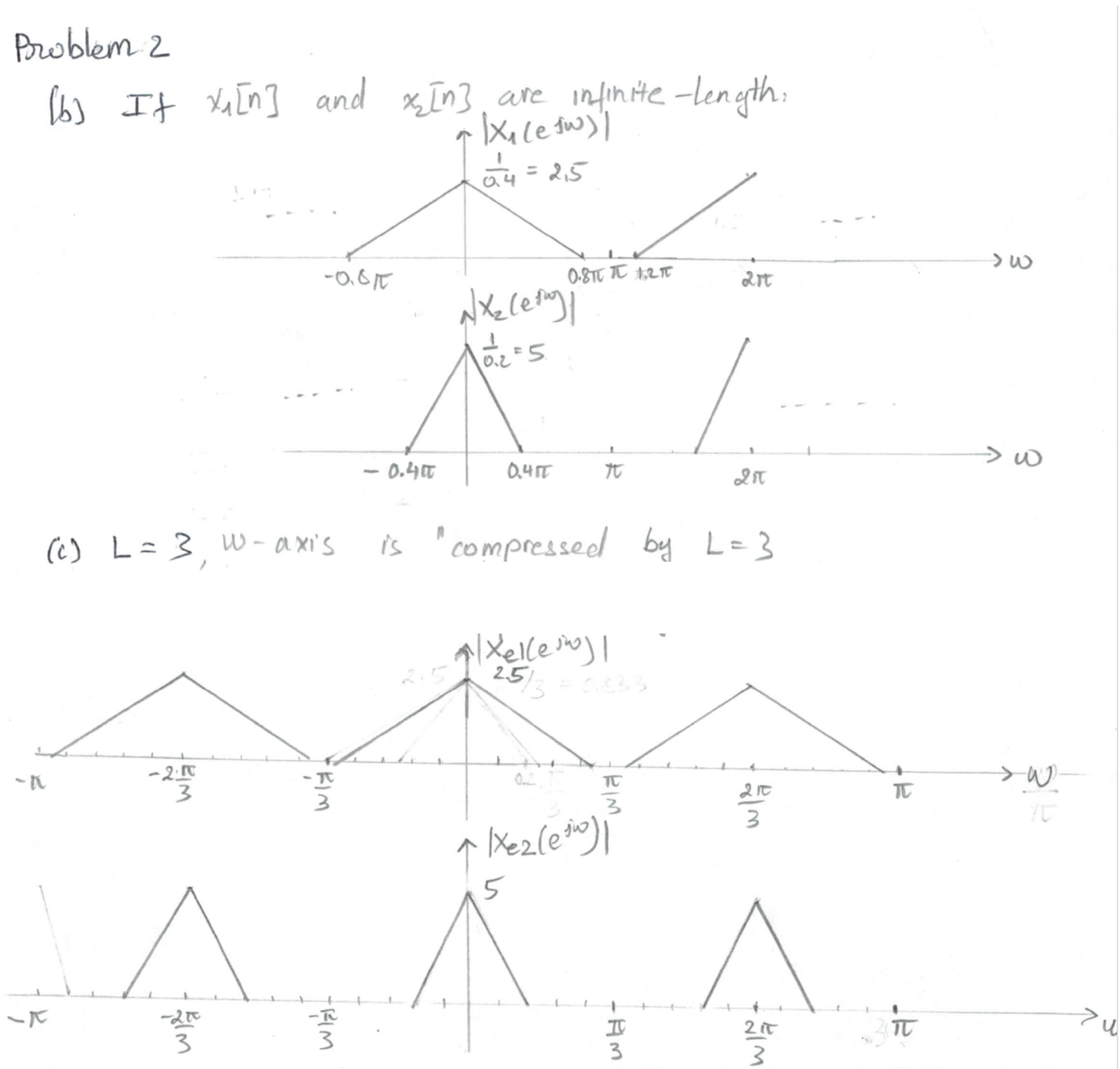
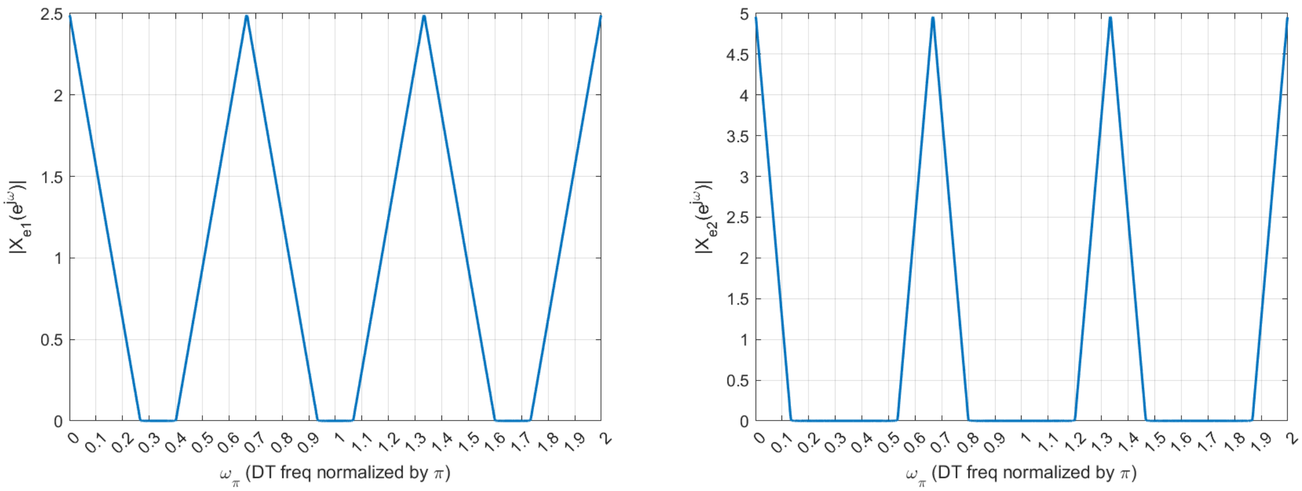


Figure 5: Sketches of the DTFT's of expanded signals

Fig. 5 presents the prediction of how upsampling affects the DTFT of $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$.

Based on knowledge of the upsampling operation, the frequency axis is scaled, or compressed, by $L = 3$. For $X_1(e^{j\omega})$, the highest frequency of 0.8π is scaled to $0.8\pi/3$. Similarly, for $X_2(e^{j\omega})$, the highest frequency of 0.4π is scaled to $0.4\pi/3$. In any period of $[0, 2\pi]$, three triangles are observed, or three copies of $X_1(e^{j\omega})$ or $X_2(e^{j\omega})$ are observed. The height of $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$ remains unchanged through upsampling.



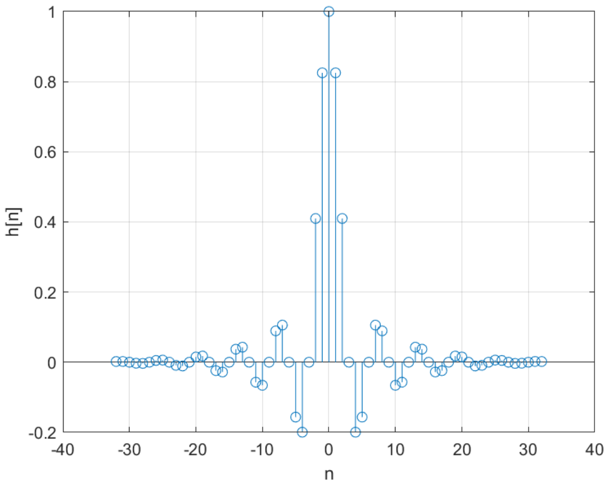
(a) Magnitudes of DTFT of $x_{e1}[n]$ (b) Magnitudes of DTFT of $x_{e2}[n]$

Figure 6: DTFT of expanded signals $x_{e1}[n]$ and $x_{e2}[n]$.

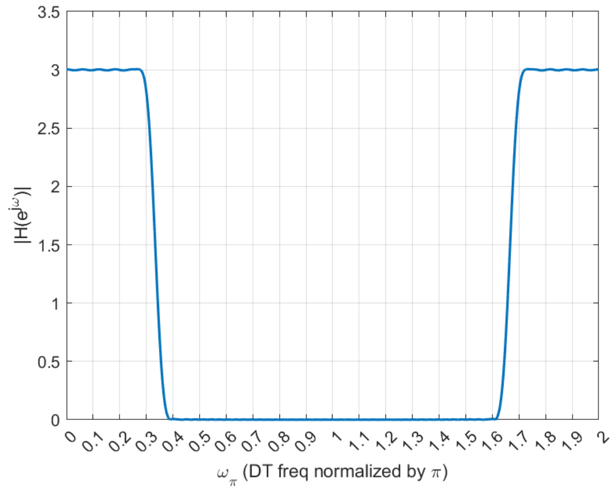
The MATLAB results in Fig. 6 confirms the prediction in paper pencil analysis in Fig. 5.

Part (d)

From plot (b) of Fig. 7, the frequency response of the LPF satisfy the requirements: cut off frequency $\omega_c = \pi/L = \pi/3$, and a gain of $L = 3$.



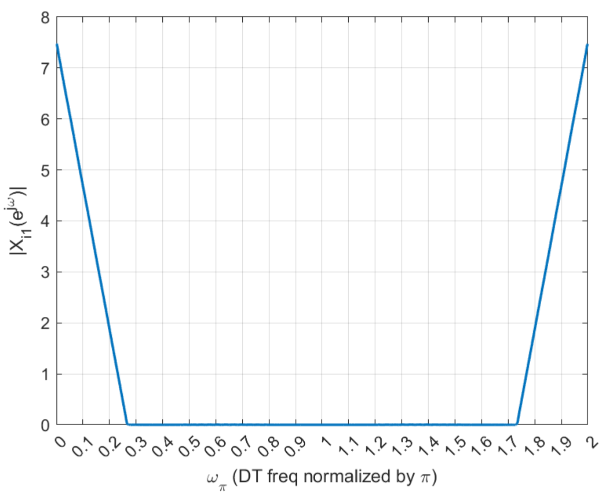
(a) Impulse Response of LPF for Upsampling



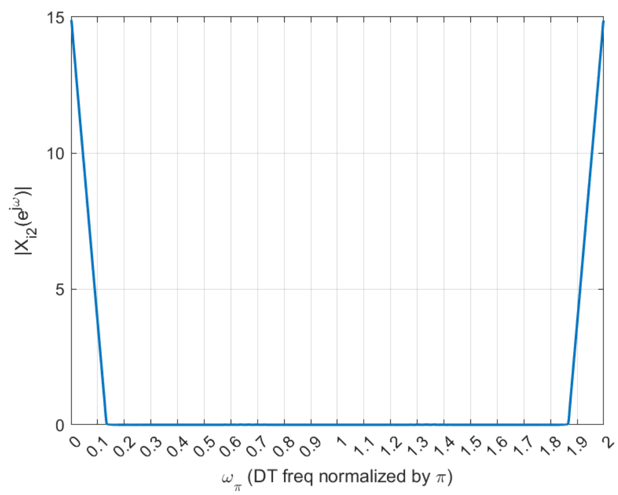
(b) Frequency Response of LPF for Upsampling

Figure 7: Low Pass Filter Design for Upsampling

Part (e)



(a) DFT of interpolated signal $x_{i1}[n]$



(b) DFT of interpolated signal $x_{i2}[n]$

Figure 8: Magnitude of DFT of interpolated signals

The effect of the LPF is observed by comparing Fig. 6 and Fig. 8. The LPF filters out all the copies from upsampling between $\omega_c < |\omega| < \pi$; that is $\frac{\pi}{3} < |\omega| < \pi$. After filtering, the magnitude of these transform are similar to as if the continuous time signals had been sampled 3 times faster, where the frequency axis is scaled by 3 and the magnitude spectra are periodic of 2π, and not periodic of $\frac{2\pi}{3}$. Since the LPF has a gain of $L = 3$, the maximum height of $X_{i1}(e^{j\omega})$= $2.5{\times}3 = 7.5$, and the maximum height of $X_{i2}(e^{j\omega})$= $5 \times 3 = 15$.

3 Downsampling

Part (a)

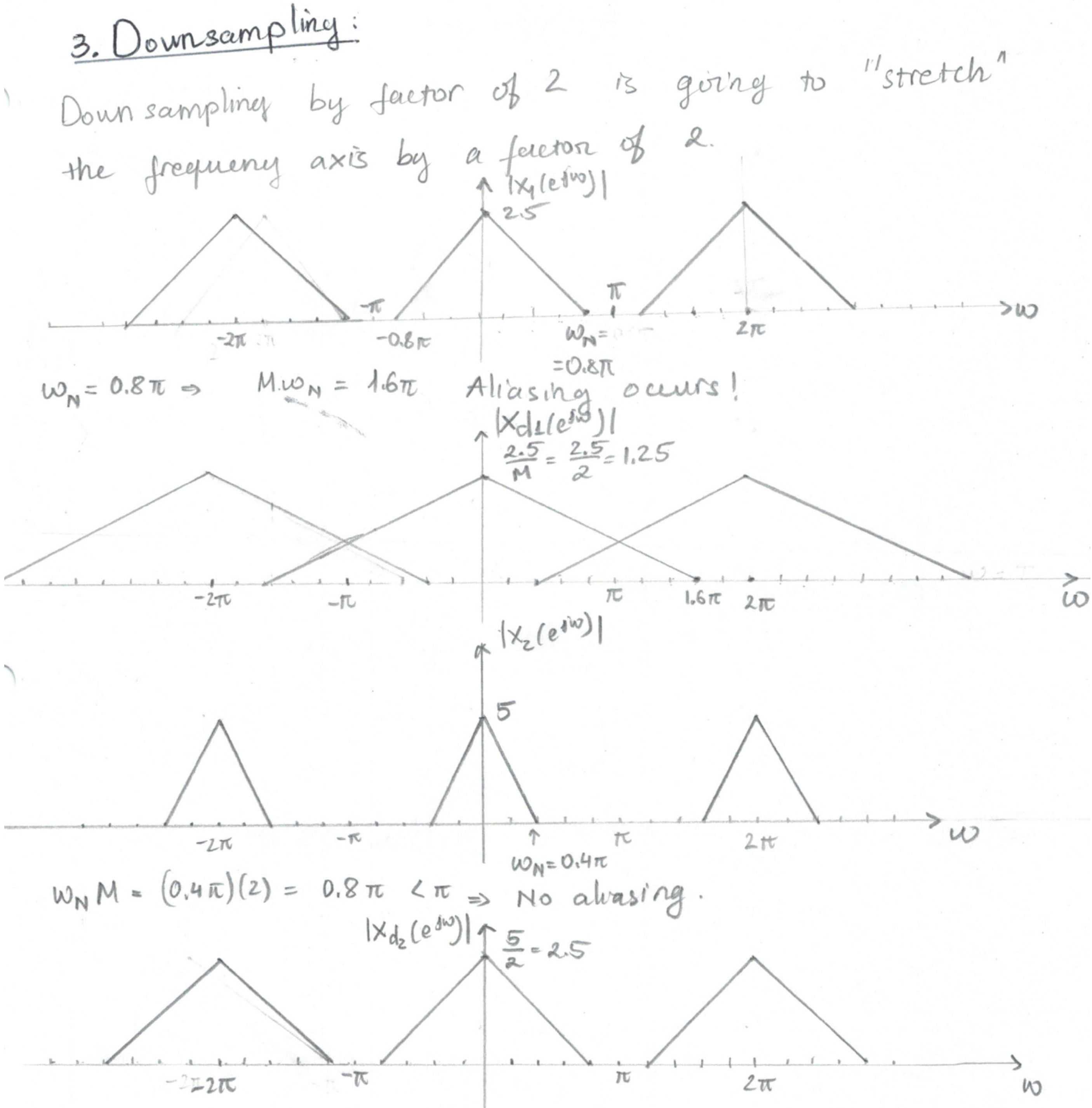
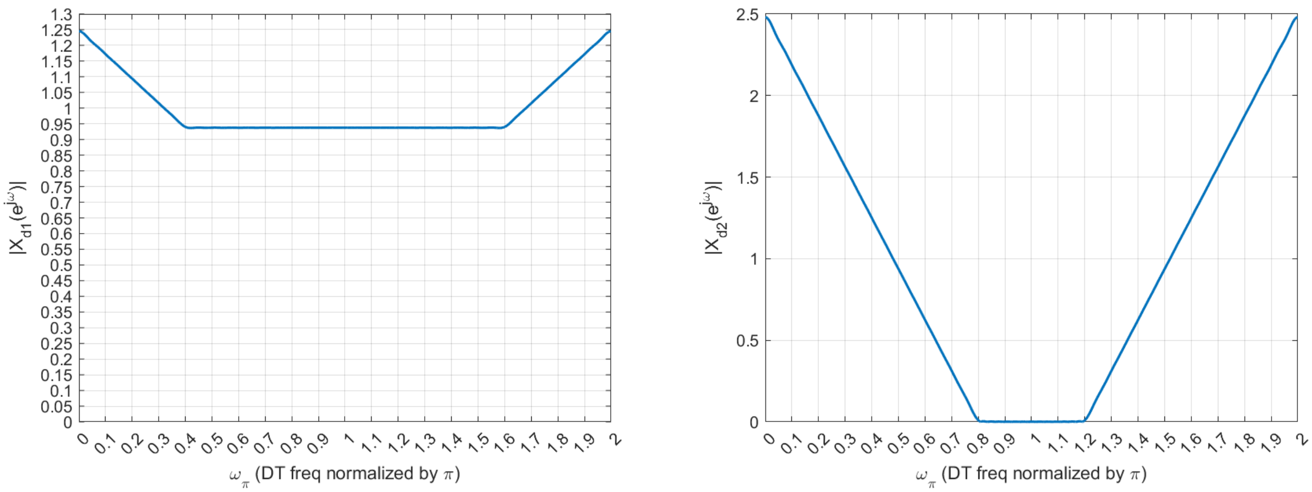


Figure 9: Analysis of downsampling and aliasing

For $X_1(e^{j\omega})$, since the maximum frequency is 0.8π, downsampling with $M = 2$ shall causes aliasing because $0.8\pi \times 2 = 1.6\pi > \pi$. Between the period of $[0, 2\pi]$, unaffected frequencies are $(0, 0.4\pi)$ and $(1.6\pi, 2\pi)$. Aliasing occurs between frequencies $(0.4\pi, 1.6\pi)$. However, $X_2(e^{j\omega})$ has the maximum frequency of 0.4π will not be aliased because $0.4\pi \times 2 = 0.8\pi < \pi$.

Part (b)



(a) Aliasing occurs with $X_{d1}(e^{j\omega})$

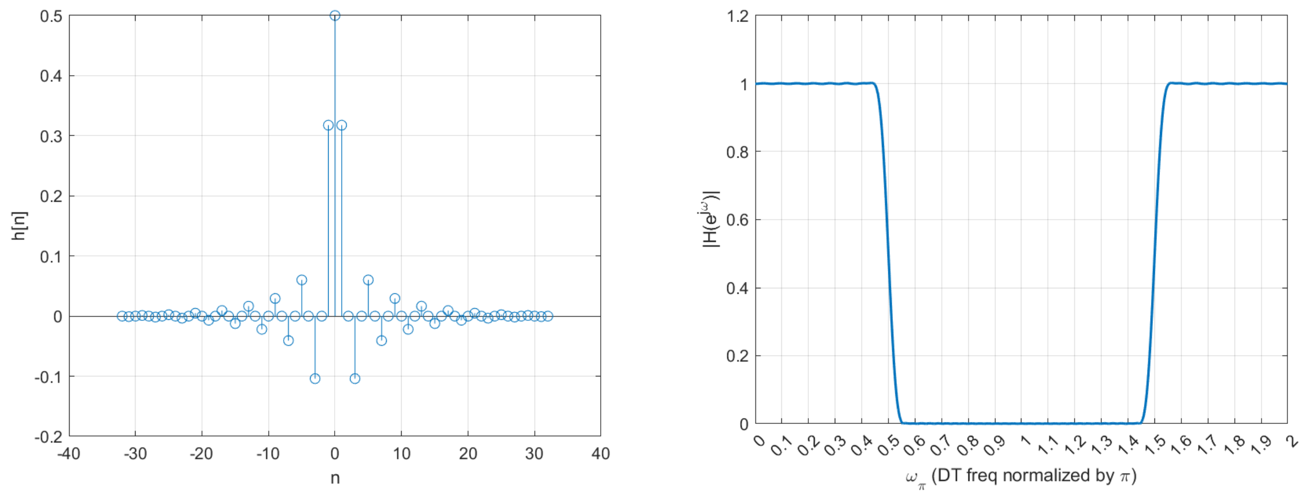
(b) $X_{d2}(e^{j\omega})$ has no alisaing

Figure 10: Magnitude of DTFT of downsampling with M=2

The MATLAB results in Fig. 10 agrees with the analytical results in Fig. 9. The aliasing frequencies for $X_{d1}(e^{j\omega})$ are verified to be $(0.4\pi, 1.6\pi)$, while frequencies $(0, 0.4\pi)$ and $(1.6\pi, 2\pi)$are unaffected. $X_{d2}(e^{j\omega})$ has no aliasing. Note that the heights are scaled by the $M = 2$ factor. The maximum height of $X_{d1}(e^{j\omega})$ is $2.5/2 = 1.25$. The maximum height of $X_{d2}(e^{j\omega})$ is $5/2 = 2.5$.

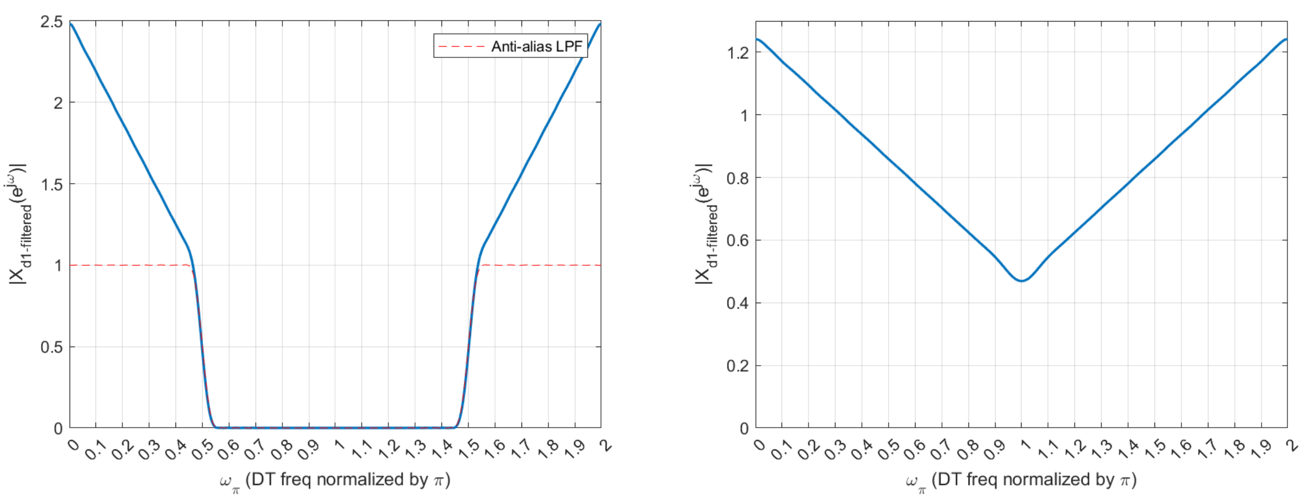
Part (c)

The LPF has a cut off frequency $\omega_c = \pi/M = \pi/2$, and a gain of 1.



(a) Impulse Response of LPF for Downsampling (b) Frequency Response of LPF for Downsampling

Figure 11: Filter design for prior-downsampling



(a) $x_{d1}[n]$ is anti-alias filtered prior to downsampling (b) Downsampling anti-alias filtered $x_{d1}[n]$

Figure 12: Implement anti-alias filter to signal $x_{d1}[n]$ prior to downsampling

4 Practical Implementation: Changing the sample rate of an audio signal

Part (a)

Since $\frac{11\,025\,\mathrm{Hz}}{8820\,\mathrm{Hz}} = 1.25$, we need to upsample by $L = 4$ and then downsample by $M = 5$.

Part (b)

We need to upsample first before downsampling. Since the downsampling factor is $M = 5$, on the period of $[-\pi, \pi]$ the audio signal must the bandlimited to $|\omega| < \pi/5 = 0.2\pi$ to avoid aliasing. Otherwise, the audio signal must be LPF with cutoff frequency $w_c = \pi/5$ prior to downsampling. This method is also not desirable because a good range of frequencies in the audio signal is sacrificed.

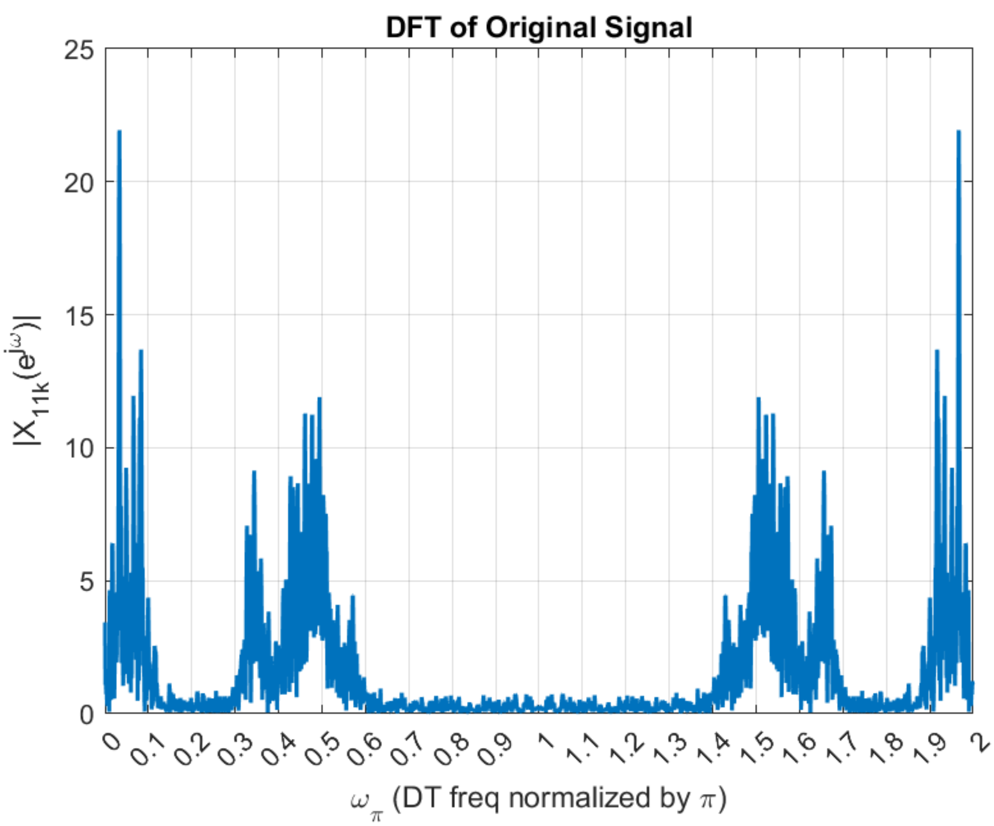


Figure 13: Magnitude of DFT of the original audio signal $X_{11k}(e^{j\omega})$

Fig. 13 shows that $X_{11k}(e^{j\omega})$ is not bandlimited to $\pi/5$. Therefore, directly downsampling with $M = 5$ would guarantee aliasing. Prior-downsample filtering is also undesirable because frequencies content between $(0.3\pi, 0.6\pi)$ and $(1.4\pi, 1.7\pi)$ will be lost. A more suitable plan of action is to upsampling first then downsampling. Upsamping with $L = 4$, and LPF the expanded signal, would guarantee the audio signal is bandlimited to $|\omega| < \pi/4$. Then, downsampling with the factor $M = 5$ can only cause aliasing if $X_{11k}(e^{j\omega})$ has frequencies contents near π. Note in Fig. 13 that frequencies contents on $(0.8\pi, \pi)$ are close to 0.

Part (c)

Only one LPF filter for interpolating the upsampled audio signal is needed.

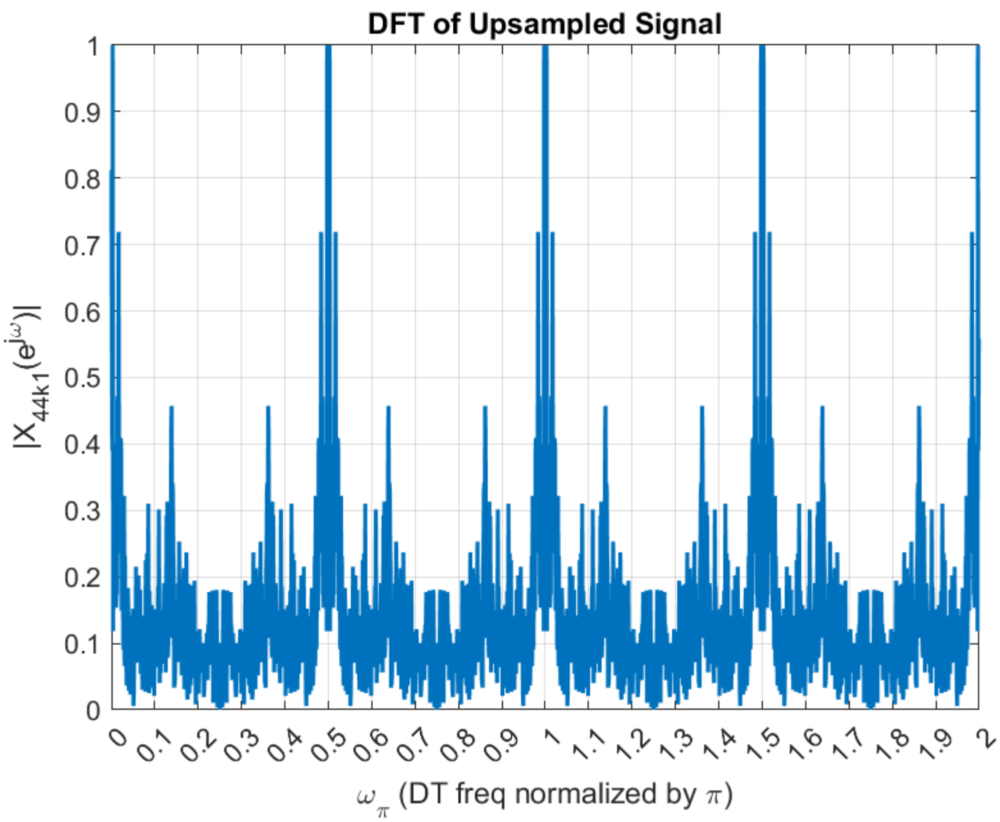
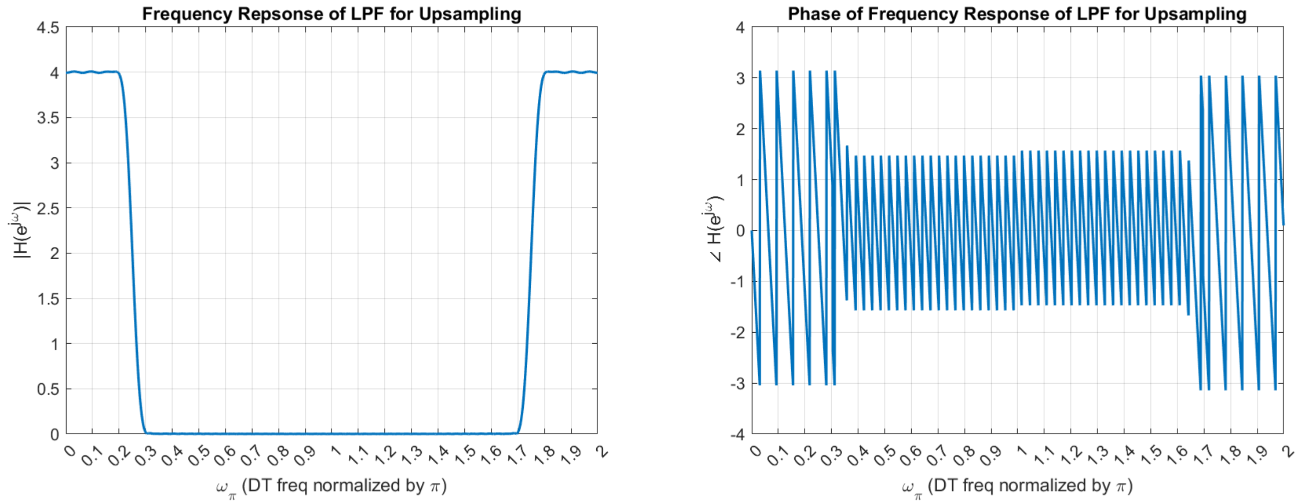


Figure 14: Upsampling the audio signal by $L = 4$ to obtain $X_{44k1}(e^{j\omega})$. Four "copies" of $X_{11k}(e^{j\omega})$ can now be observed on period $[0, 2\pi]$, which confirm the upsampling is carried out correctly.



(a) Magnitude of Frequency Response

(b) Phase of Frequency Response

Figure 15: Upsampling LPF design with a cutoff frequency of $w_c = \pi/4 = 0.25\pi$ and a gain $L = 4$.

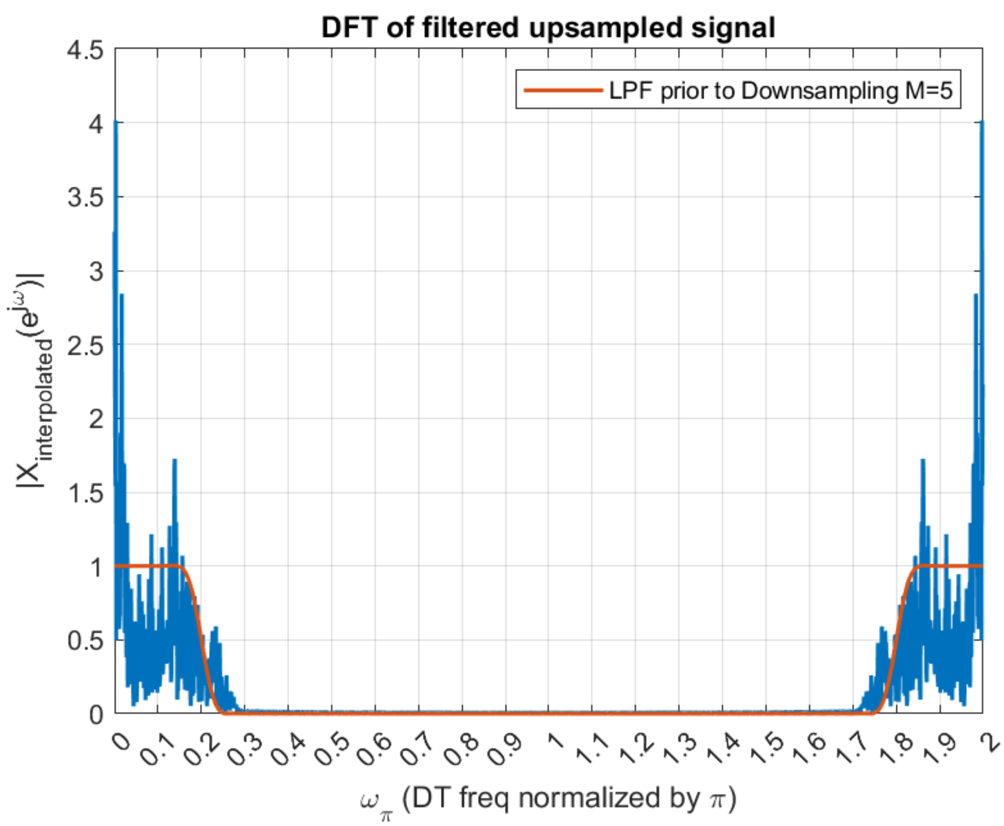


Figure 16: Interpolating expanded signal $X_{44k1}(e^{j\omega})$, using the upsampling LPF designed in Fig. 15. If a prior-downsampling LPF were to be applied, the results would not be changed significantly

Fig. 16 is the audio signal being upsampled by $M = 4$ and then filtered by a LPF with a cutoff frequency of $w_c = \pi/4 = 0.25\pi$ and a gain $L = 4$. The spectrum in Fig. 16 is now similar to as if the signal in Fig. 13 is sampled 4 times faster.

The next step is to downsample the interpolated upsampled signal. In Fig. 16, the red curve is the anti-aliasing filtered designed for downsampling with a factor of 5. Notice the frequency content that the LPF will filter out is not significant. For this specific application of the given audio signal, the anti-alias LPF is not necessary.

Another way to reason why the anti-alias LPF is not needed can be done by observing Fig. 13. If ω if the frequency axis of the original signal, then after both the up-and-down sampling process, the final frequency axis is scaled by $\frac{5}{4}\omega$. Therefore, any frequency contents beyond 0.8π in Fig. 13 will be aliased, because $\frac{5}{4}\omega > \pi$ if $\omega > \frac{4}{5}\pi$. In Fig. 13, the frequency contents on $(0.8\pi, 1.2\pi)$ are very close to 0. Hence, the results of of aliasing is not significant.

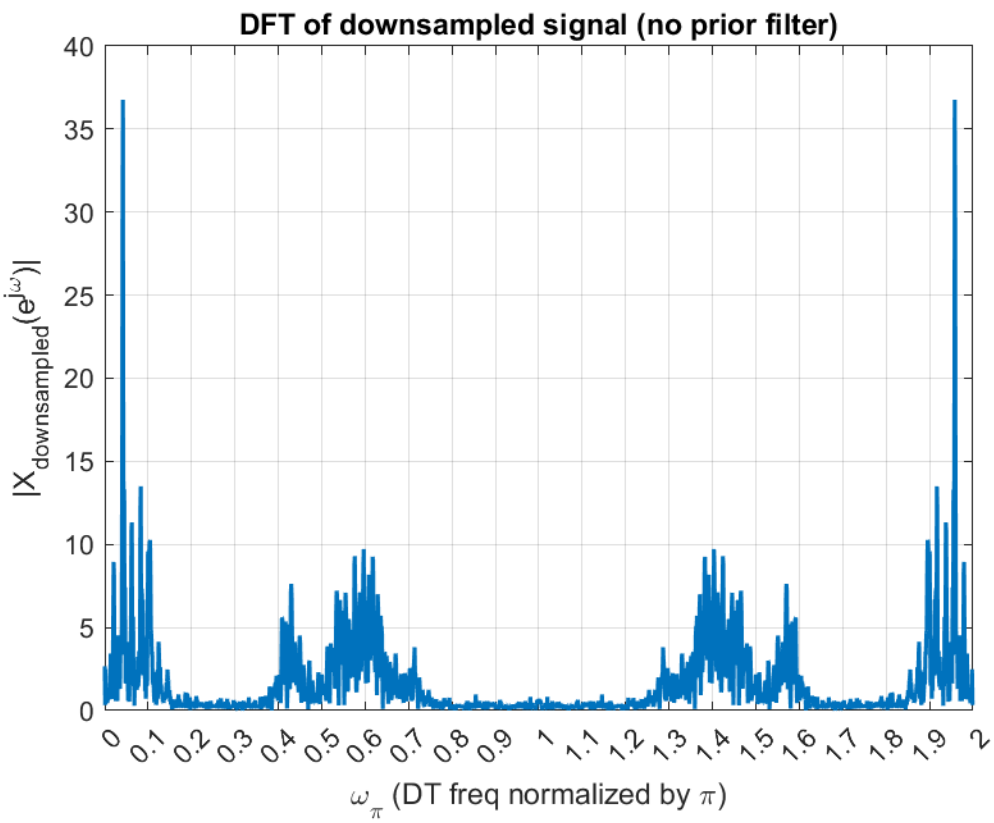


Figure 17: Downsampling the upsampled and interpolted signal. No prior-downsampling low pass filtering is needed.