On spirale

June 20, 2017

Abstract

1 Dynamical model & objectives

We consider the following model:

$$\dot{d}(t) = -v(t)cos(\alpha(t))$$

$$\dot{\alpha}(t) = -\omega(t) + \frac{v(t)}{d(t)} sin(\alpha(t))$$

two states to be controlled $d(t), \alpha(t)$.
full measurements;

The objective: to follow a spiral path.

For the moment, the objective is to follow :

$$\dot{d}^\star(t) = -v^\star cos(\alpha^\star)$$

where $v^\star, \alpha^\star$ are given reals.

I have got a problem : $d(t)$ should be different from zero. Practical stability?

2 error dynamics

Let consider the following reference trajectory:

$$\begin{cases} \dot{d}^\star(t) = -v^\star cos(\alpha^\star) \\ \dot{\alpha}^\star = 0 \end{cases}$$

we define the errors :

$$\begin{cases} e_d(t) = d(t) - d^\star(t) \\ e_\alpha(t) = \alpha(t) - \alpha^\star \end{cases}$$

Its dynamics are given by:

$$\begin{cases} \dot{e}_d(t) = v^\star cos(\alpha^\star) - v(t)cos(\alpha(t)) \\ \dot{e}_\alpha(t) = -\omega(t) + \frac{v(t)}{d(t)} sin(\alpha(t)) \end{cases}$$

$$\begin{cases} \dot{e}_d(t) = v^\star cos(\alpha^\star) - v(t)cos(e_\alpha(t) + \alpha^\star) \\ \dot{e}_\alpha(t) = -\omega(t) + \frac{v(t)}{e_d(t)+d^\star(t)} sin(e_\alpha(t) + \alpha^\star) \end{cases}$$

The objective is to find control functions such that the origin of the system defined with (e_d, e_α) is GAS.

3 First control

Let choose a Lyapunov function:

$$V(e) = \frac{1}{2}e_d(t)^2 + \frac{1}{2}e_\alpha^2$$

$$\dot{V}(e) = e_d(v^\star cos(\alpha^\star) - v(t)cos(e_\alpha(t)+\alpha^\star)) + e_\alpha(-\omega(t) + \frac{v(t)}{e_d(t)+d^\star(t)}sin(e_\alpha(t)+\alpha^\star))$$

Let choose :

$$v(t) = \frac{cos(\alpha^\star)}{cos(e_\alpha+\alpha^\star)}v^\star + \frac{1}{cos(e_\alpha+\alpha^\star)}\lambda_d e_d$$
$$\omega(t) = \frac{v(t)}{e_d(t)+d^\star(t)}sin(e_\alpha(t)+\alpha^\star) + \lambda_\alpha e_\alpha$$

We obtain therefore:

$$\dot{V}(e) = -\lambda_d e_d^2 - \lambda_\alpha e_\alpha^2$$

It should work.

4 Second control

Based on exact state linearisation with an extra constraint : $v(t) = v^\star$ (dunno why ! ;))

$$\ddot{e}_d(t) = v^\star sin(e_\alpha + \alpha^\star)\dot{e}_\alpha(t)$$

$$\ddot{e}_d(t) = v^\star sin(e_\alpha + \alpha^\star)(-\omega(t) + \frac{v(t)}{e_d(t)+d^\star(t)}sin(e_\alpha(t)+\alpha^\star))$$

Let choose the control

$$\omega(t) = \frac{v(t)}{e_d(t)+d^\star(t)}sin(e_\alpha(t)+\alpha^\star) + \frac{1}{v^\star sin(e_\alpha+\alpha^\star)}(\lambda_1 e_d(t) + \lambda_2\dot{e}_d(t))$$

to obtain

$$\ddot{e}_d(t) + \lambda_2\dot{e}_d(t) + \lambda_1 e_d(t) = 0.$$

Let consider the following state :

$$z(t) = \begin{bmatrix} e_d(t) \\ \dot{e}_d(t) \end{bmatrix}$$

The system in closed loop is defined by

$$\dot{z}(t) = \begin{bmatrix} 0 & 1 \\ -\lambda_1 & -\lambda_2 \end{bmatrix} z(t)$$

Asymptotic Stability of $z(t)$ is classically derived from the choice of coefficients λ_1, λ_2. Furthermore, it implies also the asymptotic stability of e_α also. Indeed,

$$\dot{e}_d(t) = v^\star(cos(\alpha^\star) - cos(e_\alpha(t) + \alpha^\star))$$

We get then

$$Arcos(-\frac{\dot{e}_d(t)}{v^\star} + cos(\alpha^\star)) - \alpha^\star = e_\alpha(t)$$

It will defined a diffeomorphism if $| -\frac{\dot{e}_d(t)}{v^\star} + cos(\alpha^\star)| \leq 1$, meaning that

$$v^\star(-1 + cos(\alpha^\star)) \leq \dot{e}_d(t) \leq v^\star(1 + cos(\alpha^\star))$$

The diffeomorphism is defined by :

$$z(t) = \phi(x) = \begin{bmatrix} e_d(t) \\ v^\star(cos(\alpha^\star) - cos(e_\alpha(t) + \alpha^\star)) \end{bmatrix}$$

5 Third control

Idea : convergence in finite time toward the spiral.

Just of have the principle :

Let consider the differential equation $\dot{y}(t) = -\sqrt{y}$, with the initial condition $y(0)$ for $t = 0$. 0 is n equilibrium point.

The solution is this equation is :

$$y(t) = (-t + t_0 + \sqrt{y(t_0)})^2$$

$y(t) = 0$ for all $t \geq t_0 + \sqrt{t_0}$

Convergence in finite time. Application:

Consider the following system:

$$\dot{y}(t) = \beta y(t) + u(t)$$

We consider the following Lyapunov functional:

$$V = \frac{1}{2}y^2(t)$$

$$\dot{V}(t) = y(t)\dot{y}(t) = y(t)(\beta y(t) + u(t))$$

Let choose $u(t) = -\beta y(t) - \lambda sign(y(t))$ We obtain

$$\dot{V}(t) = -\beta|y(t)| = -\beta\sqrt{2V(t)}$$

We prove therefore the convergence in finite time of $V(t)$ and also $y(t)$.

This sort of equation will be used in order to design some controllers allowing the convergence in finite time toward the origin. Hence, recovering the first type of controller:

5.1 first controller modified

Let choose a Lyapunov function:

$$V(e) = \frac{1}{2}e_d(t)^2 + \frac{1}{2}e_\alpha^2$$

$$\dot{V}(e) = e_d(v^\star cos(\alpha^\star) - v(t)cos(e_\alpha(t)+\alpha^\star)) + e_\alpha(-\omega(t) + \frac{v(t)}{e_d(t)+d^\star(t)} sin(e_\alpha(t)+\alpha^\star))$$

Let choose :

$$\begin{aligned} v(t) &= \frac{cos(\alpha^\star)}{cos(e_\alpha+\alpha^\star)}v^\star + \frac{1}{cos(e_\alpha+\alpha^\star)}\lambda_d sign(e_d) \\ \omega(t) &= \frac{v(t)}{e_d(t)+d^\star(t)} sin(e_\alpha(t)+\alpha^\star) + \lambda_\alpha sign(e_\alpha) \end{aligned}$$

We obtain therefore:

$$\dot{V}(e) = -\lambda_d|e_d| - \lambda_\alpha|e_\alpha|$$

We therefore obtain:

$$\dot{V}(e) \le -2max(\lambda_d, \lambda_\alpha)\sqrt{V}$$

proving the stability in finite time.

6 Idea for the next future

For the first controller, we may look for a more general quadratic function of the form :

$$V = \begin{bmatrix} e_d(t) \\ e_\alpha(t) \end{bmatrix}^T P \begin{bmatrix} e_d(t) \\ e_\alpha(t) \end{bmatrix},$$

with $P \in \mathcal{S}_2^+$