ENB 439 Tutorial - Dead reckoning

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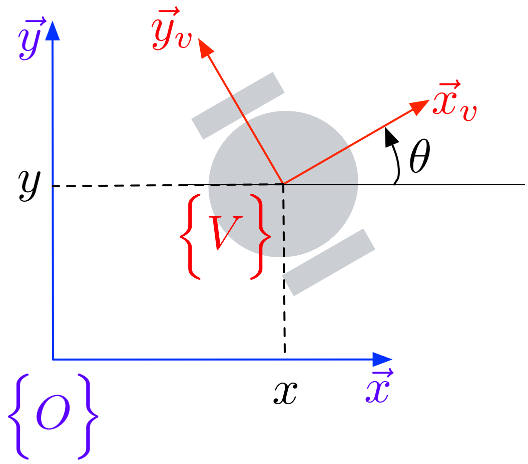

Figure 1: Differential model

We consider a differential non-holonomic robot whose model is presented in figure 1. Its configuration corresponds to the coordinates x and y of the point V in the world frame and to its orientation θ with respect to $\vec{x}$. We note $\chi(t) = [x(t),\ y(t),\ \theta(t)]^T$. The robot control inputs correspond to $u(t) = [v(t),\ \omega(t)]^T$, where $v(t)$ is the linear velocity along $\vec{x}_v$ and $\omega(t)$ is the angular velocity about $\vec{z}_v$.
Now, we introduce the robot kinematic model :

$$\dot{x}(t) = v(t)\cos(\theta(t)) \tag{1a}$$

$$\dot{y}(t) = v(t)\sin(\theta(t)) \tag{1b}$$

$$\dot{\theta}(t) = \omega(t) \tag{1c}$$

We consider that the robot is controlled using a computer. Thus the control inputs $u(t)$ are sent to the robot with an interval T_s which corresponds to the sampling time. This means that the control input $u(t_k)$ applied between instants t_k and t_{k+1}, where $t_{k+1} = t_k + T_s$, is constant. For $t \in [t_k, t_{k+1}]$, we can rewrite 1a, 1b and 1c as follows :

$$\dot{x}(t) = \upsilon(t_k)\cos(\theta(t)) \tag{2a}$$

$$\dot{y}(t) = \upsilon(t_k)\sin(\theta(t)) \tag{2b}$$

$$\dot{\theta}(t) = \omega(t_k) \tag{2c}$$

where $\upsilon(t_k)$ and $\omega(t_k)$ are constant.
Our goal is to solve (2a), (2b) and (2c) between t_k and t_{k+1}. Thus we can obtain the robot configuration $\chi(t_{k+1})$ knowing $\chi(t_k)$ and $u(t_k)$.

- First, we consider the robot orientation. To achieve this aim, we integrate 2c:

$$\begin{aligned} \theta(t) &= \int_{t_k}^{t} \dot{\theta}(t)dt \\ &= \int_{t_k}^{t} \omega(t_k)dt \\ &= [\omega(t_k)t]_{t_k}^{t} \\ &= \omega(t_k)t - \omega(t_k)t_k + cst \end{aligned} \tag{3}$$

  For $t = t_k$, equation (3) becomes :

$$\begin{aligned} \theta(t_k) &= \omega(t_k)t_k - \omega(t_k)t_k + cst \\ &= cst \end{aligned} \tag{4}$$

  Using (3) and (4), we obtain :

$$\theta(t) = \omega(t_k)t - \omega(t_k)t_k + \theta(t_k) \tag{5}$$

  Finally, we considere $t = t_{k+1}$ and $T_s = t_{k+1} - t_k$, which leads to

$$\begin{aligned} \theta(t_{k+1}) &= \omega(t_k)t_{k+1} - \omega(t_k)t_k + \theta(t_k) \\ &= \omega(t_k)T_s + \theta(t_k) \end{aligned} \tag{6}$$

Now, we consider equations (2a) and (2b). As $\theta(t)$ is not supposed to be constant between t_k and t_{k+1}, we have to consider two cases. The first one is $\omega(t_k) = 0$ and the second is $\omega(t_k) \neq 0$.

- First we consider $\omega(t_k) = 0$, and then $\theta(t) = \theta(t_k)$ between t_k and t_{k+1}. This leads to :

$$\begin{aligned} x(t) &= \int_{t_k}^{t} \dot{x}(t)dt \\ &= \int_{t_k}^{t} \upsilon(t_k)\cos(\theta(t_k))dt \\ &= [\upsilon(t_k)\cos(\theta(t_k))t]_{t_k}^{t} \\ &= \upsilon(t_k)\cos(\theta(t_k))t - \upsilon(t_k)\cos(\theta(t_k))t_k + cst \end{aligned} \tag{7}$$

For $t = t_k$, equation (7) becomes :

$$\begin{aligned} x(t_k) &= \upsilon(t_k)\cos(\theta(t_k))t_k - \upsilon(t_k)\cos(\theta(t_k))t_k + cst \\ &= cst \end{aligned} \tag{8}$$

Considering $t = t_{k+1}$, $T_s = t_{k+1} - t_k$ and using (7) and (8), we finally obtain :

$$\begin{aligned} x(t_{k+1}) &= \upsilon(t_k)\cos(\theta(t_k))t_{k+1} - \upsilon(t_k)\cos(\theta(t_k))t_k + x(t_k) \\ &= \upsilon(t_k)\cos(\theta(t_k))T_s + x(t_k) \end{aligned} \tag{9}$$

The same reasoning is done for $y(t_{k+1})$ which leads to :

$$y(t_{k+1}) = \upsilon(t_k)\sin(\theta(t_k))T_s + y(t_k) \tag{10}$$

- We now consider $\omega(t_k) \neq 0$. This leads to :

$$\begin{aligned} x(t) &= \int_{t_k}^{t} \dot{x}(t)dt \\ &= \int_{t_k}^{t} \upsilon(t_k)\cos(\theta(t))dt \end{aligned} \tag{11}$$

We inject (5) into (11), which leads to :

$$x(t) = \int_{t_k}^{t} \upsilon(t_k) \cos\big(\omega(t_k)t - \omega(t_k)t_k + \theta(t_k)\big) dt \tag{12}$$

To integrate this last equation, we propose to integrate by substitution. We recall hereafter the principle.

$$\int_a^b f(g(t))g'(t)dt = \int_{g(a)}^{g(b)} f(z)dz \tag{13}$$

To use (13), we propose to define :

$$\begin{gathered} g(t) = \omega(t_k)t - \omega(t_k)t_k + \theta(t_k) \\ g'(t) = \omega(t_k) \\ f(t) = \tfrac{\upsilon(t_k)}{\omega(t_k)} \cos(g(t)) \end{gathered} \tag{14}$$

Thus, we can rewrite (12) as :

$$\begin{aligned} x(t) &= \textstyle\int_a^b f(g(t))g'(t)dt \\ &= \textstyle\int_{t_k}^{t} \frac{\upsilon(t_k)}{\omega(t_k)} \cos\big(\omega(t_k)t - \omega(t_k)t_k + \theta(t_k)\big)\omega(t_k)dt \\ &= \textstyle\int_{g(t_k)}^{g(t)} \frac{\upsilon(t_k)}{\omega(t_k)} \cos(z)dz \\ &= \textstyle\int_{\theta(t_k)}^{\omega(t_k)t-\omega(t_k)t_k+\theta(t_k)} \frac{\upsilon(t_k)}{\omega(t_k)} \cos(z)dz \\ &= \Big[\tfrac{\upsilon(t_k)}{\omega(t_k)} \sin(z)\Big]_{\theta(t_k)}^{\omega(t_k)t-\omega(t_k)t_k+\theta(t_k)} \\ &= \tfrac{\upsilon(t_k)}{\omega(t_k)}\Big(\sin\big(\omega(t_k)t - \omega(t_k)t_k + \theta(t_k)\big) - \sin\big(\theta(t_k)\big)\Big) + cst \end{aligned} \tag{15}$$

For $t = t_k$, equation (15) becomes :

$$\begin{aligned} x(t_k) &= \tfrac{\upsilon(t_k)}{\omega(t_k)}\Big(\sin\big(\omega(t_k)t_k - \omega(t_k)t_k + \theta(t_k)\big) - \sin\big(\theta(t_k)\big)\Big) + cst \\ &= \tfrac{\upsilon(t_k)}{\omega(t_k)}\Big(\sin\big(\theta(t_k)\big) - \sin\big(\theta(t_k)\big)\Big) + cst \\ &= cst \end{aligned} \tag{16}$$

Considering $t = t_{k+1}$, $T_s = t_{k+1} - t_k$ and using (15) and (16), we finally obtain:

$$\begin{aligned} x(t_{k+1}) &= \tfrac{\upsilon(t_k)}{\omega(t_k)}\Big(\sin\big(\omega(t_k)t_{k+1} - \omega(t_k)t_k + \theta(t_k)\big) - \sin\big(\theta(t_k)\big)\Big) + x(t_k) \\ &= \tfrac{\upsilon(t_k)}{\omega(t_k)}\Big(\sin\big(\omega(t_k)T_s + \theta(t_k)\big) - \sin\big(\theta(t_k)\big)\Big) + x(t_k) \end{aligned} \tag{17}$$

Following the same reasoning, we obtain :

$$y(t_{k+1}) = -\tfrac{\upsilon(t_k)}{\omega(t_k)}\Big(\cos\big(\omega(t_k)T_s + \theta(t_k)\big) - \cos\big(\theta(t_k)\big)\Big) + y(t_k) \tag{18}$$

Finally, we have computed the solutions to (1a), (1b) and (1c), which allows us to calculate $\chi(t_{k+1})$ knowing $\chi(t_k)$ and $u(t_k)$:

1. if $\omega(t_k) = 0$

$$\begin{cases} x(t_{k+1}) = \upsilon(t_k)\cos(\theta(t_k))T_s + x(t_k) \\ y(t_{k+1}) = \upsilon(t_k)\sin(\theta(t_k))T_s + y(t_k) \\ \theta(t_{k+1}) = \omega(t_k)T_s + \theta(t_k) \end{cases} \tag{19}$$

2. if $\omega(t_k) \neq 0$

$$\begin{cases} x(t_{k+1}) = x(t_k) + \tfrac{\upsilon(t_k)}{\omega(t_k)}\Big(\sin\big(\omega(t_k)T_s + \theta(t_k)\big) - \sin\big(\theta(t_k)\big)\Big) \\ y(t_{k+1}) = y(t_k) - \tfrac{\upsilon(t_k)}{\omega(t_k)}\Big(\cos\big(\omega(t_k)T_s + \theta(t_k)\big) - \cos\big(\theta(t_k)\big)\Big) \\ \theta(t_{k+1}) = \omega(t_k)T_s + \theta(t_k) \end{cases} \tag{20}$$

Questions

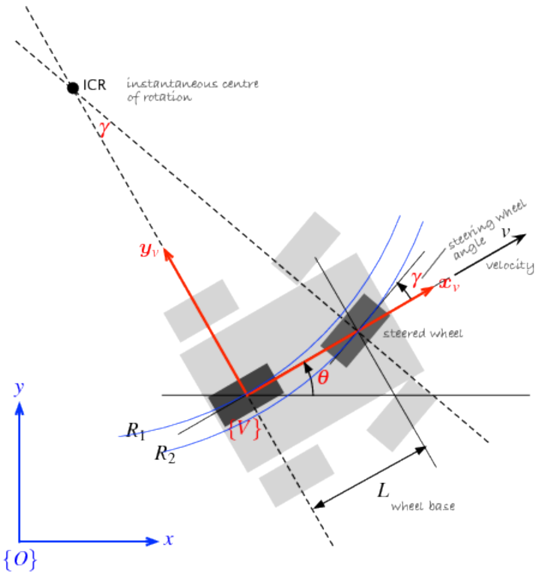

Figure 2: Car-like model

We now consider a car like robot whose model is presented in figure 2. Its configuration is defined by $\chi(t) = [x(t),\ y(t),\ \theta(t),\ \gamma(t)]^T$. Its kinematic is given by:

$$\begin{cases} \dot{x}(t) = v(t)\cos(\theta(t)) \\ \dot{y}(t) = v(t)\sin(\theta(t)) \\ \dot{\theta}(t) = \frac{v(t)}{L}\tan\gamma(t) \\ \dot{\gamma}(t) = \omega(t) \end{cases} \tag{21}$$

1. For $t \in [t_k, t_{k+1}]$, we suppose $\gamma(t) = \frac{1}{2}\omega(t_k)(t - t_k) + \gamma(t_k)$. Using this assumption and the calculus made for the differential model, compute $\chi(t_{k+1})$ knowing $\chi(t_k)$ and $u(t_k)$. Implement the solution with Matlab.

2. We consider Euler's scheme :

$$\dot{\chi}(t_k) = \frac{\chi(t_{k+1}) - \chi(t_k)}{T_s} \tag{22}$$

   Using (22), compute $\chi(t_{k+1})$ knowing $\chi(t_k)$ and $u(t_k)$. Implement the solution with Matlab.

3. To integrate system, a function called *ode45* is provided. Use this function to compute the robot configuration evolution.