Optimization Handbook

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Benders decomposition

1 Introduction

Benders decomposition is a solution method for solving certain large-scale optimization problems. It is particularly suited for problems in which a set of variables are said to be *complicating* in the sense that fixing them to a given value makes the problem easy. Briefly, the Benders decomposition approach seperates an original problem into several decision stages. A first-stage *master* problem is solved using only a subset of variables, then, the values of the remaining variables are determined by a so-called *subproblem* depending on the first-stage variables. If the master problem's optimal solution yields an infeasible subproblem, a *feasibility cut* is added to the master problem, which is then re-solved. Due to the structure of the reformulation, the Benders algorithm starts with a *restricted master problem* where only a subset of constraints are considered while the others are iteratively added.

This technique was first introduced in [1] and has since been generalized to non-linear mixed integer problems.

2 Formal derivation

Consider the following problem :

$$\begin{aligned} \text{minimize } & c^T x + f(y) && (8.1)\\ \text{s.t. } & Ax + g(y) = b && (8.2)\\ & y \in Y, x \geq 0 && (8.3) \end{aligned}$$

where variable y is a *complicating constraint*. Note that it may be complicating due to the form of f or g but also by our ability to enforce the constraint $y \in Y$. We assume that fixing y to a given value $\hat{y}$ turns our problem into an easy-to-solve problem.

We can notice that our problem is equivalent to the following one :

$$\min \left\{ f(y) + \min \left\{ c^T x : Ax = b - g(y) \right\} : y \in Y \right\}$$

Let us denote by $q(y)$ the value of the minimization problem over x : $q(y) = \min\{c^T x : Ax = b - g(y)\}$. By duality, the following holds

$$q(y) = \max\{(b - g(y))^T \pi : A^T \pi \leq c\}$$

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Appendices

Appendix A

Convexity

Definition 1 (Minkowski sum). *Let A and B be two vector spaces, the Minkowki sum is defined as*

$$A + B = \{a + b | a \in A, b \in B\}$$

Definition 2 (Convex combination). *Let $x_1, ..., x_n$ be a finite set of vectors in a real vector space, a convex combination of these vectors is a vector of the form*

$$\sum_{i=1}^{n} \alpha_i x_i \qquad \text{with} \ \sum_{i=1}^{k} \alpha_i = 1, \alpha \geq 0$$

Definition 3 (Convex hull).

$$conv(X) = \left\{ \sum_{i=1}^{n} \alpha_i x_i \middle| x_i \in X, \sum_{i=1}^{n} \alpha_i = 1, \alpha \geq 0 \right\}$$

Definition 4 (Conical combination). *Let $x_1, ..., x_n$ be a finite set of vectors in a real vector space, a conical combination of these vectors is a vector of the form*

$$\sum_{i=1}^{k} \alpha_i x_i \qquad \text{with} \ \alpha \geq 0$$

Definition 5 (Conical hull).

$$cone(X) = \left\{ \sum_{i=1}^{n} \alpha_i x_i \middle| x_i \in X, \alpha_i \geq 0 \right\}$$

Theorem 1 (Affine Minkowski-Weyl). *Let there be a polytope defined by a set of inequalities, $P = \{x \in \mathbb{R}^n | Ax \leq b\}$. There exists vectors $x_1, ..., x_q \in \mathbb{R}^{\cdot}n$ and $y_1, ..., y_r \in \mathbb{R}^n$ such that*

$$P = cone(x_1, ..., x_q) + conv(y_1, ..., y_r)$$

Observation 1. *In the affine Minkowski-Weyl theorem, the smallest set of vectors $x_1, ..., x_q$ is the set of extreme rays of P and the smallest set of vectors $y_1, ..., y_r$ is the set of extreme points of P.*

Bibliography

[1] J. F. Benders. Partitioning procedures for solving mixed-variables programming problems. *Numerische Mathematik*, 4(1):238–252, Dec 1962.