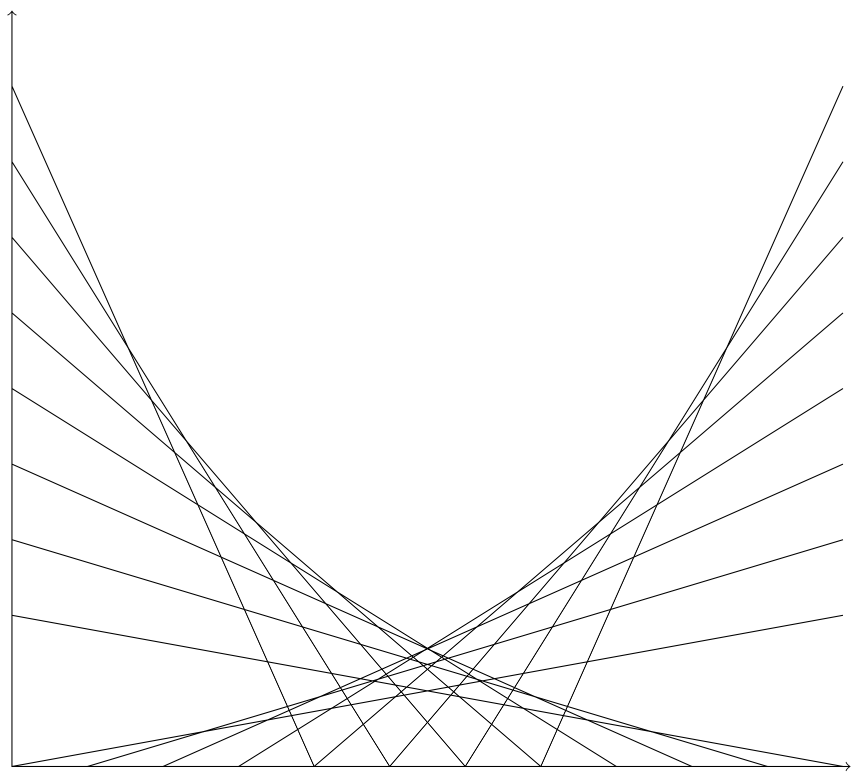
Optimization Handbook



Henri Lefebvre

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Part I

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Chapter 2

Introduction to linear programming

1 Problems description

1.1 Definitions

Linear programming (LP) takes interests in problems in which both the objective function and the constraints are linear with respect to the decision variables. The feasible region is therefore composed of linear inequalities or linear equalities. For the sake of generality, we often consider a problem of minimizing a linear objective function subject to equality constraints. Such a problem is said to be in *standard form*. Note that the standard form is only used as a way to present a homogeneous framework for theoretical work. In the following section, we show how any formulated linear program can be reduced to a problem written in standard form.

Some examples of problems which can be formulated in a linear fashion are presented in section 3. Yet, we give here a very simple example for the sake of understanding. This problem will also allow us to have a first geometrical interpretation of some mathematical objects considered in linear programming.

Let us consider a production plant where two kinds of items are to be produced. Both items are made of two raw matrials RM_1 and RM_2. Items of type A need 10 units of RM_1 and 4 units of RM_2 while items of type B need 2 units of RM_1 and 4 units of RM_2 to be produced. The problem is to decide how many items of type A and of type B should be produced in the plant, knowing that the raw materials are of fixed amount. Let us assume that we have 50 units of raw material RM_1 and 60 units of RM_2. A so-called *feasible solution* is a decision which satisfies the capacity constraints of the raw materials (i.e., we do not produce more items than we have raw material to do so). A feasible solution is said to be *optimal* if it minimizes or maximizes a certain quantity. Here, we will consider the minimization of the cost. Let us assume that items of type A cost 5 euros to be produced and that items B cost 4 euros. The problem can be stated as :

$$\begin{aligned} \text{minimize } & 3x + 4y \\ \text{s.t. } & 10x + 4y \leq 50 \\ & 2x + 3y \leq 30 \\ & x \geq 0, y \geq 0 \end{aligned}$$

Here, one may argue that the decisions variables x and y need to take integer values. This in fact depends on the context and application of the problem. We will see however that integer linear programming (ILP) problems as well as mixed integer linear programming (MILP) where we have both continuous and integer decision variables are much harder to solve. Yet we will consider quite efficient algorithms to solve such kinds of problems, among which : the branch-and-bound approach and the cutting planes algorithm.

Since our problem is a two-dimensional problem, in that sense that we have two decision real (or integer) variables to decide, one can plot the *feasible region* of the problem. The feasible region denots the set of feasible solutions. Figure 2.1 depicts the feasible region of our problem.

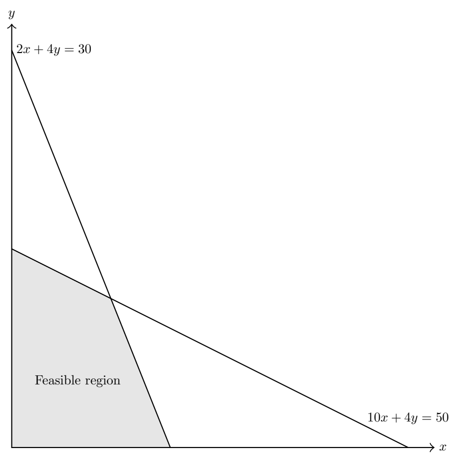

Figure 2.1: 2D representation of a feasible region

One should indeed see that above this feasible region stands a plane (or hyperplane in higher dimensions) defining the objective function. Since the objective function is a place, it is clear that the optimal solution can be found in one of the extreme points of the feasible region. This property, in fact, holds in general and is called the *Fundamental theorem of linear programming* and will be formally introduced in section 2.

1.2 Standard form

Formal definition

Definition 1 (Standard form)**.** *A linear programming problem is said to be in standard form if it is written as*

$$\begin{aligned} \textit{minimize } & c^T x \\ \textit{s.t. } & Ax = b \\ & x \geq 0 \end{aligned}$$

Examples

In this section, we show how we can turn linear problems which are not originally in the standard form to a problem written in standard form.

Negative variables If a given problem uses a negative variable, say $x \leq 0$. It suffices to consider the opposite decision variable $\hat{x} = -x$. The problem is now in standard form.

Real variables If a given problem uses a free variable (i.e., a variable which can be positive or negative) $x \in \mathbb{R}$. We introduce two positive decision variables $x^+ \geq 0$ and $x^- \geq 0$ and write x as $x^+ - x^- \in \mathbb{R}$. The problem is now in standard form.

Inequality constraints If a given problem defines the feasible region with inequality constraint, so-called *slack* variables can be introduced. Considering an inequality constraint $\sum_j a_j x_j \leq b$, we introduce variable $s \geq 0$ so that $\sum_j a_j x_j + s = b$. We do not restrict the values of s (other than by its sign) and do not associate any cost in the objective function. Therefore, the value of s in the optimal solution corresponds to the $b - \sum_j a_j x_j$, hence the name of slack variables. The obtained problem is now in standard form.

2 Fundamental Theorem of LP

The fundamental theorem of linear programming as been intuitively introduced in the previous section. We enounce it here formally and prove its validity and geometrical interpretation. First, we introduce the following assumption which will hold in general in the subsequent theorems :

Assumption 1 (Full rank assumption)**.** *Considering the feasible region $\{x|Ax = b, x \geq 0\}$ where A is a $m \times n$ matrix. We assume that rank$(A) = n$*

The fundamental theorem of LP characterizes the optimal solutions of a given problem. In that sense, it reduces the search space for optimality. To properly enouce the theorem, we need to introduce the concept of *basic* solutions.

Consider a feasible region defined as

$$\{x \in \mathbb{R} | Ax = b, x \geq 0\}$$

where A is a $m \times n$-matrix of full rank (see assumption 1). If we select n linearly independent columns from A, then the set of columns represent a basis of $\mathbb{R}^n$. Let us denote by B the matrix corresponding to that basis. Since B is non-singular, the following system can be solved uniquely :

$$Bx_B = b$$

where x_B is an m-dimensional vector. Then, clearly, the vector x defined as $x = [x_B, \mathbf{0}]$ belongs to the feasible region since $Ax = A[x_B, \mathbf{0}] = Ax_B + A\mathbf{0} = b$. This consideration leads to the following definition :

Definition 2 (Basic solution)**.** *Considering a feasible region $\{x \in \mathbb{R} | Ax = n, x \geq 0\}$ where A is a matrix of full rank. A vector x is said to be basic if and only if there exists a basis B of $\mathbb{R}^n$ composed of n columns of A and $x = [B^{-1}b, \mathbf{0}]$. If, moreover, it is positive element-wise then it is said to be a feasible basic solution.*

We can now enounce the theorem :

Theorem 1 (Fundamental theorem of linear programming)**.** *Consider the following LP problem in standard form :*

$$\begin{aligned} \textit{minimize } & c^T x \\ \textit{s.t. } & Ax = b \\ & x \geq 0 \end{aligned}$$

Then, under the full rank assumption (1),

(i) if there is a feasible solution, there is a basic feasible solution

(ii) if there is an optimal feasible solution, there is an optimal basic feasible solution.

Proof.

(i) Let x be a feasible solution and let us denote by $a_1, ..., a_n$ the columns of A. Since it is feasible, the following holds :

$$a_1 x_1 + ... + a_n x_n = b$$

Assume now that exactely p variables x_i are greater than zero. For the sake of exposure, we'll assume that they correspond to the p first variables. That is :

$$a_1 x_1 + ... + a_p x_p = b$$

Then two different situations may occur :

$(a_1, ..., a_p)$ **are linearly independent** : then clearly we have $p \leq m$. If $p = m$, then x is a basic feasible solution and the proof is complete. If $p < m$ then since A is of full rank, one can find $m - p$ columns of A which forms a basis when added to the p columns $a_1, ..., a_p$. By setting $x_i = 0$ for all $i > p$, we obtain a (degenerate) basic feasible solution.

$(a_1, ..., a_p)$ **are linearly dependent** : Then there exists coefficients $\lambda_i, ..., \lambda_p$ all non-zero, at least one of which can be assumed to be positive, such that

$$a_1 \lambda_1 + ... + a_p \lambda_p = 0$$

Let $\varepsilon \in \mathbb{R}$, Since $Ax = b$ holds, it also holds that

$$(x_1 - \varepsilon\lambda_1)a_1 + ... + (x_p - \varepsilon\lambda_p)a_p = b \quad (2.1)$$

For $\varepsilon = 0$, this reduces to the original feasible solution. As ε increases, the different components may increase, decrease or remain constant depending on the sign of λ_i. Since we assumed that at least one λ_i is positive, then at least one component will decrease as ε increases. We increase ε to the first point where one or more components become zero. That is, we choose

$$\varepsilon = \min\{x_i / \lambda_i : \lambda_i > 0\}$$

For this specific value, the solution built in 2.1 is feasible and has at most $p-1$ positive variables. Repeating this process as much as necessary, we can eliminate positive variables untill we obtain a feasible solution with corresponding columns which are linearly dependent.

(ii) Let us consider an optimal solution x^* and, as in proof of (i), let us suppose that there are exactely p positive variables. Again, two cases may occure as to whether the selected columns of A are linearly dependent or not. If the columns are independent, the proof goes as for (i). If they are dependent, we still can apply the idea of the proof of (ii) yet we need to check that the objective value of $x^* - \varepsilon\lambda$ remains optimal. Note that the objective value of the constructed solution is

$$c^T x^* + \varepsilon c^T \lambda$$

By contradiction, suppose that $c^T\lambda \neq 0$. Then one can find a sufficiently small value for ε so that $c^T x^* + \varepsilon c^T \lambda < c^T x^*$ which contradicts the optimality of x^*. Hence, $c^T y = 0$ and the obtained solution is optimal.

□

Before stating an equivalence result previously mentioned between basic variables and extreme points of polyhedra, let us take a small detour with a geometrical example of the idea of the proof of theorem 1, point *(i)*. For that purpose, let us consider the following feasible region :

$$K = \left\{ x \in \mathbb{R}^3_+ \middle| \begin{array}{ll} x_1 + x_2 + x_3 & = 1 \\ x_2 + x_3 & = \frac{2}{3} \end{array} \right\}$$

K is depicted in figure 2.2. As well, a (non-basic) feasible point is depicted with coordinates $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Indeed, one can easily check that

$$\frac{1}{3}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{3}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{3}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\\frac{2}{3}\end{pmatrix}$$

Yet, clearly, this collection of vectors are linear dependent, in particular we have :

$$1\begin{pmatrix}1\\1\end{pmatrix} + 0\begin{pmatrix}1\\0\end{pmatrix} + (-1)\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

From the two above equations, we obtain by multiplying the second one by a scalar $\varepsilon > 0$ and subtracting the first one :

$$\left(\frac{1}{3} - \varepsilon\right)\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{3}\begin{pmatrix}1\\0\end{pmatrix} + \left(\frac{1}{3} + \varepsilon\right)\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\\frac{2}{3}\end{pmatrix}$$

If we choose $\varepsilon = \frac{1}{3}$ (i.e., $\min\{x_i/\lambda_i : \lambda_i > 0\}$), we find that $x = (0, \frac{2}{3}, \frac{1}{3})$ is also a feasible solution and it has one coefficient set to zero. Since we have two constraints ($m = 2$), this solution is a basic solution (also depicted in figure 2.2).

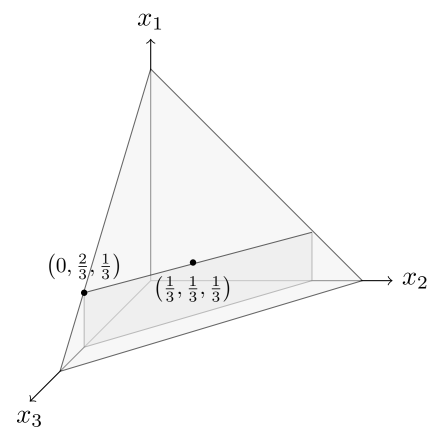

Figure 2.2: 3D representation of feasible region K

We can see that our basic feasible solution is indeed an extreme point of the polytope K. The following theorem states that this remark holds in general.

Theorem 2 (Equivalence between basic solutions and extreme points)**.** *Considering a matrix A of full rank (see assumption 1), a vector x is an extreme point of the polyhedron $\{x|Ax = b, x \geq 0\}$ if and only if it is a basic feasible solution.*

Proof.

$\Rightarrow$: Let x be a basic feasible solution for $\{x|Ax = b, x \leq 0\}$ where A is a $m \times n$ matrix. We have

$$a_1x_1 + ... + a_mx_m = b$$

By contradiction, let us suppose that there exists two different points $y, z \in \{x|Ax = b, x \geq 0\}$ such that x is a convex combination of these points, i.e., :

$$x = \alpha y + (1 - \alpha)z \quad 0 < \alpha < 1$$

Since $x \geq 0$ and α is positive, it holds that the last $n - m$ components of y and z are also equal to zero. Thus :

$$y_1a_1 + ... + y_ma_m = b$$
$$z_1a_1 + ... + z_ma_m = b$$

Yet since $a_1, ..., a_m$ are linearly independent, it follows that $x = y = z$ which contradicts that $y \neq z$. Hence, x is an extreme point.

$\Leftarrow$: Let x be an extreme point of $\{x|Ax = b, x \geq 0\}$ and let us assume that the nonzero components of x are the first k components, i.e.,

$$a_1x_1 + ... + a_kx_k = b$$

Let us show that $a_1, ..., a_k$ are linearly independent (i.e., that x is a basic feasible solution). By contradiction, suppose that $a_1, ..., a_k$ are linearly dependent. Then, there exists a vector $\lambda \geq 0$ with at least one non-zero coefficient such that

$$\lambda_1a_1 + ... + \lambda_ka_k = 0$$

Since $x \geq 0$, one can find a value for ε such that

$$x + \varepsilon\lambda \geq 0 \quad x - \varepsilon\lambda \geq 0$$

Yet, if this were the case, it would holds that $x = \frac{1}{2}(x + \varepsilon\lambda) + \frac{1}{2}(x - \varepsilon\lambda)$ which is a convex combination of two distinct feasible points. This contradicts the fact that x is an exterme point. Hence, x is a basic feasible solution.

□

One final remark should be made regarding the definition of a basic solution. Indeed, in general, the coefficients of a basic solution may not be all nonzero. We therfore give the following definition :

Definition 3 (Degenerate solution)**.** *A basic solution is said to be degenerate if and only if one or more of the basic variables value are zero.*

Note that, in presence of degeneracy, ambiguity arises since one could interchange freely the zero-valuated basic variables with non-basic variables.

Observation 1. *The fundamental theorem of linear programming reduces the search space for optimality to the set of extreme points of the feasible region. Yet, note that for a problem with n variables and m constraints, we have at most*

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

extreme points (or equivalently, basic solutions).

3 Examples

3.1 Minimum cost flow problem

3.2 Support Vector Machine

3.3 Knapsack problem

Chapter 3

The Simplex algorithm

In chapter 2, we showed with the fundamental theorem of linear programming that the search space for optimality was reduced indeed to the set of extreme points of the feasible region, or equivalently of basic feasible solutions for the problem. The idea of the Simplex algorithm is to go from one extreme point to another in such a way that the objective function is always improved (i.e., decreases in case of a minimization). By assumption, the considered polyhedron is lower bounded (upper bounded in case of maximization) and therefore, the algorithm reaches the optimal point. The method can be thought of as a travel from an original extreme point to the one which maximizes the objective function.

The name Simplex comes from the name of the geometrical generalization of triangles to higher dimensions. Its name comes from the idea that it is the "simplest" closed geometrical object in n dimension.

The first section derives the algorithm formally. Then we explain in more details the geometrical interpretation of the Simplex algorithm. Since the algorithm starts with an initial basic feasible solution, the next section will explain how such a point can be found by using the same Simplex algorithm. In the three last sections, we give the pseudo code of the Revised Simplex algorithm written in matrix form and introduce two variants of the Simplex : (1) the bounded simplex which deals with bounded variables and (2) the transportation Simplex which is used for transportation problems.

1 Formal derivation

1.1 Assumptions

For the sake of demonstrations, we introduce the following assumption :

Assumption 2 (Nondegeneracy assumption)**.** *Every basic feasible solution is a nondegenerate basic feasible solution.*

1.2 Pivoting and Gauss reduction

In this section, we explain how one can move from one basic solution to another with a standard operation from linear algebra called pivoting. This operation is used in Gauss method for solving a system of linear equations. Consider the following set of equations :

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1\\
&a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2\\
&\vdots \qquad\qquad\qquad \ddots \qquad \vdots\\
&a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m
\end{aligned}$$

Of course, it is well known that if $m < n$ and if the equations are not redondant (i.e., they are linearly independent) there is not a unique solution. Yet, following the princple of pivoting from the Gauss elimination technique (see appendix **??**) one can turn this system in a so-called *canonical form* expressed as

$$\begin{array}{cccccccccccc}
x_1 & & & & & + & \bar{a}_{1(m+1)}x_{m+1} & + & ... & + & \bar{a}_{1n}x_n & = & \bar{b}_1 \\
& x_2 & & & & + & \bar{a}_{2(m+1)}x_{m+1} & + & ... & + & \bar{a}_{2n}x_n & = & \bar{b}_2 \\
& & x_3 & & & + & \bar{a}_{3(m+1)}x_{m+1} & + & ... & + & \bar{a}_{3n}x_n & = & \bar{b}_3 \\
& & & \ddots & & & \vdots & & \ddots & & & & \vdots \\
& & & & x_m & + & \bar{a}_{m(m+1)}x_{m+1} & + & ... & + & \bar{a}_{mn}x_n & = & \bar{b}_m
\end{array}$$

which we often write, for the sake of synthesis, in a so-called *tableau* :

$$\begin{array}{ccccccccc}
x_1 & x_2 & x_3 & ... & x_m & x_{m+1} & ... & x_n & \\
1 & 0 & 0 & 0 & 0 & \bar{a}_{1(m+1)} & ... & \bar{a}_{1n} & \bar{b}_1 \\
0 & 1 & 0 & 0 & 0 & \bar{a}_{2(m+1)} & ... & \bar{a}_{2n} & \bar{b}_2 \\
0 & 0 & 1 & 0 & 0 & \bar{a}_{3(m+1)} & ... & \bar{a}_{3n} & \bar{b}_3 \\
& & & \ddots & & \vdots & \ddots & & \vdots \\
0 & 0 & 0 & 0 & 1 & \bar{a}_{m(m+1)} & ... & \bar{a}_{mn} & \bar{b}_m
\end{array}$$

It is clear that if one posesses a system of equations written in cannonical form, then the solution given by $x = (\bar{b}_1, \bar{b}_2, ..., \bar{b}_m, 0, 0, ..., 0)$ is a basic solution. The idea of the Simplex algorithm is, in fact, to pivot from one canonical form to another. The question however is how to select the variable which will enter the basis and which one will leave the basis in order to increase a given objective function. This is deatailed in the following sub-sections.

1.3 Vector leaving the basis

In this section, we show how one can decide which variable should leave the basis. In fact, in the previous section, we showed that the pivot operation allows us to move from a basic solution to another, however, such a move does not guarantee the feasibility of the obtained basic solution. In other words, it is not established that the pivot operation will keep the positivity of the variables. We present here a sufficient condition for the pivot operation to keep the feasibility property when moving from one basic solution to another.

Let x be a basic solution. We have

$$a_1x_1 + a_2x_2 + ... + a_mx_m = b$$

where $x_i > 0, \forall i = 1...m$ (nondegeneracy assumption). And suppose that we want to bring x_q in the basis. The question is how to choose which variable has to leave the basis in order to keep feasibility of the new basic solution. For that purpose, let us write a_q in terms of the current basis :

$$a_q = \lambda_1a_1 + \lambda_2a_2 + ... + \lambda_ma_m$$

From the two above equalities, we derive the following :

$$(x_1 - \varepsilon\lambda_1)a_1 + (x_2 - \varepsilon\lambda_2)a_2 + ... + (x_m - \varepsilon\lambda_m)x_m + \varepsilon a_q = b$$

for any $\varepsilon > 0$, which is a linear combination of at most $m+1$ vectors. Setting $\varepsilon = 0$ yields the current basic feasible solution. As ε increases, the coefficient of a_q increases. Yet, it yields a non basic variable, in general. The coefficients of the other vectors may increase or decrease with ε depending on the original coefficients (i.e., if we have $\lambda_i > 0$). Therefore, by taking the first value of ε which makes vanishing such a vector, we ensure the feasibility of the obtained new basic solution. Formally, we choose ε such that :

$$\varepsilon = \min\{x_i/\lambda_i : \lambda_i > 0\}$$

If the minimum is achieved by more than one variable, the new basic feasible solution is degenerate. If, however, none of the λ_i are positive, this means that all the coefficients increase as ε increases, without restriction, while keeping feasibility. This corresponds to a case where the polyhedron is unbounded.

1.4 Vector entering the basis

In the previous section, we have shown how to choose which variable had to leave in order to keep feasibility when we want to insert a given variable in the basis. In that sense, it allows us to travel from one basic solution to another while keeping feasibility. In this section, we show how to choose which variable should enter the basis in order to increase a given objective function. For that purpose, consider the following objective function :

$$c^Tx = c_1x_1 + c_2x_2 + ... + c_nx_n$$

And consider a feasible basic solution $\hat{x} = [\hat{x}_B, 0]$. Its objective value is given by

$$c_1\hat{x}_1 + c_2\hat{x}_2 + ... + c_m\hat{x}_m$$

The key idea here, is to express the objective function value of a general solution x in terms of the objective value of the basic solution $\hat{x}$. This can be achieved by solving the following system for $x_{m+1}, ..., x_n$:

$$
\begin{aligned}
x_1 &= \bar{b}_1 - \sum_{j=m+1}^{p} \bar{a}_{1j} x_j \\
x_2 &= \bar{b}_2 - \sum_{j=m+1}^{p} \bar{a}_{2j} x_j \\
&\vdots \qquad \vdots \\
x_m &= \bar{b}_m - \sum_{j=m+1}^{p} \bar{a}_{mj} x_j
\end{aligned}
$$

Doing so, we obtain that

$$c^T x = \underbrace{\sum_{j=1}^{m} c_j x_j}_{c^T \hat{x}_B} + \sum_{j=m+1}^{n} (c_j - z_j) x_j$$

where

$$z_j = \sum_{i=1}^{m} \bar{a}_{ij} c_i$$

This result gives us a condition for a vector to benifically enter the basis. Indeed, if, for a given variable j, we have $c_j - z_j < 0$ then it means that increasing the value of x_j from zero to a positive value will decrease the objective function. Hence, going from the solution $\hat{x}$ to another solution which includes $x_j > 0$ will yield a lower value of the objective.

We now can state the two following theorems :

Theorem 3. *Given a non-degenerate basic feasible solution with corresponding objective value z_0. If there exist a column such that $c_j - z_j < 0$, then there is a feasible solution with objective value $z < z_0$. If the column $\bar{a}_j$ can be substituted for some vector in the original basis to yield a feasible basic solution, then this solution will have an objective value $z < z_0$. If, however, $\bar{a}_j$ cannot be substituted to yield a basic feasible solution, the problem is unbounded and the optimal solution tends to minus infinity.*

Theorem 4 (Optimality condition)**.** *If, for some basic feasible solution, $c_j - z_j \geq 0$ for all j, then the solution is optimal.*

We introduce the standard notation $r_j = c_j - z_j$. These coefficients are called *reduced cost* or *relative costs* since they measure the cost of a variable respectively to a given basis. We can interpret these numbers as the gain we would obtain to use a real variable x_j instead of the linear combination giving $x_j = \sum_{j=1}^{m} \bar{a}_j x_j$. Another interpretation is to see the reduced cost as the amount by which the objective cost would have to decrease (for minimization problems) in order to make the entrance of a column profitable.

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Chapter 9

Benders decomposition

1 Introduction

Benders decomposition is a solution method for solving certain large-scale optimization problems. It is particularly suited for problems in which a set of variables are said to be *complicating* in the sense that fixing them to a given value makes the problem easy. Briefly, the Benders decomposition approach seperates an original problem into several decision stages. A first-stage *master* problem is solved using only a subset of variables, then, the values of the remaining variables are determined by a so-called *subproblem* depending on the first-stage variables. If the master problem's optimal solution yields an infeasible subproblem, a *feasibility cut* is added to the master problem, which is then re-solved. Due to the structure of the reformulation, the Benders algorithm starts with a *restricted master problem* where only a subset of constraints are considered while the others are iteratively added.

This technique was first introduced in [1] and has since been generalized to non-linear mixed integer problems.

2 Formal derivation

Consider the following problem :

$$\text{minimize } c^T x + f(y) \tag{9.1}$$

$$\text{s.t. } Ax + g(y) = b \tag{9.2}$$

$$y \in Y, x \geq 0 \tag{9.3}$$

where variable y is a *complicating constraint.* Note that it may be complicating due to the form of f or g but also by our ability to enforce the constraint $y \in Y$. We assume that fixing y to a given value $\hat{y}$ turns our problem into an easy-to-solve problem.

We can notice that our problem is equivalent to the following one :

$$\min \left\{ f(y) + \min \left\{ c^T x : Ax = b - g(y) \right\} : y \in Y \right\}$$

Let us denote by $q(y)$ the value of the minimization problem over x : $q(y) = \min\{c^T x : Ax = b - g(y)\}$. By duality, the following holds

$$q(y) = \max\{(b - g(y))^T \pi : A^T \pi \leq c\}$$

Note that the feasibility space of the dual does not depend on the values of y and let us apply the decomposition theorem for polyhedra 7 on it :

$$\{A^T \pi \leq c\} = \left\{ \sum_i u^i \alpha_i + \sum_j v^j \beta_j \right\}$$

$$\sum_i \alpha_i = 1 \quad \text{and} \quad \alpha, \beta \geq 0$$

where $\{u^i\}_i$ denotes the extreme points of $\{x|Ax \leq c\}$ and $\{v^j\}_j$ denotes the extreme rays of the polyhedral cone $\{x|Ax \leq 0\}$. Intuitively, the convex combination of the extreme points of $\{x|Ax \leq c\}$ defines the optimal solutions (since we know that there exists at least one optimal solution corresponding to an extreme point of the considered polyhedron) while the conical combination of extreme rays of $\{x|Ax \leq 0\}$ defines the feasibility region. The following theorem will allow us to reformulate our problem :

Theorem 5. *Let ($\mathcal{P}$) be the following problem :*

$$\max\{c^T x : Ax \leq b, x \geq 0\} \qquad (\mathcal{P})$$

Then, ($\mathcal{P}$) is upper bounded if and only if

$$c^T v^j \leq 0 \quad \forall j = 1, ..., J$$

where $\{v^j\}_{j=1,...,J}$ denotes the set of extreme rays of $\{x|Ax \leq 0\}$.

Proof. $\Rightarrow$: By contradiction, let us suppose that ($\mathcal{P}$) is upper bounded and that there exists k such that $c^T v^k > 0$. Let us consider a feasible solution to ($\mathcal{P}$) denoted by $u = z + t$ where z is a convex combination of the extreme points of $\{x|Ax \leq b\}$ and t an element of the conical combinations of the extreme rays of $\{x|Ax \leq 0\}$ (see theorem 7). Let $\lambda \in \mathbb{R}_+$. Since v^k is in the conical polyhedra $\{x|Ax \leq 0\}$, it holds that $\lambda A v^k \leq 0$. Moreover, we have $At \leq 0$. Hence, $A(t + \lambda v^k) \leq 0$. Now, since $v^k \geq 0$, we have $\lambda v^k \geq 0$ and since $t \geq 0$ it holds that $t + \lambda v^k \geq 0$. This shows that, for any λ, $z + \lambda v^k$ is a feasible solution for problem ($\mathcal{P}$). However, its associated objective value is given by $c^T u + \lambda c^T v^k \to +\infty$ when $\lambda \to +\infty$ since, by assumption, $c^T v^k > 0$. This contradicts the fact that ($\mathcal{P}$) is upper bounded.
$\Leftarrow$: Let us consider a solution $u = z + t$ then $c^T u = c^T z + \sum_j c^T v^j \beta_j \leq c^T z$ since $c^T v^j \leq 0, \forall j = 1, ..., J$. Problem ($\mathcal{P}$) is therefore upper bounded by $\sum_i \alpha_i \max_i\{c^T u^i\}$. □

Theorem 5 can be intuitively understood by interpreting $c^T v^j$ as $\langle c, v^j \rangle$ (scalar product). The necessary and sufficient condition that $\langle c, v^j \rangle \leq 0$ simply expresses that the the hyperplane defining the objective function must be *oriented* (i.e., increasing) in the direction of the origin of the cone formed by the extreme rays of the polyhedron. It is depicted in figure 9.1.

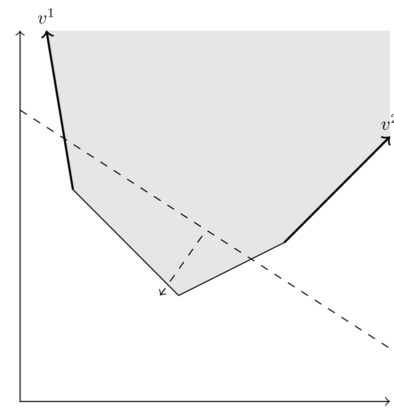

Figure 9.1: Illustration of theorem 5

If we apply theorem 5 to our specific problem, we find that a necessary and sufficient condition for $q(y)$ to be upper bounded is that $(b - g(y))^T v^j \leq 0, \forall j = 1, ..., J$ where v^j denotes an extreme ray of the polyhedron $\{x|Ax \leq c\}$ and J a list for their indices. Moreover, we know that there exists at least one optimal solution realised in an extreme point of the feasible region. Therefore, by calling $u^i, i = 1, ..., I$ the extreme points of the feasible region, the problem of finding the value of $q(y)$ can be reformulated as

$$q(y) = \min\{q : q \geq (b - g(y))^T u^i \quad \forall i = 1, ..., I\}$$

Assuming that this problem is bounded. We finally can write our original problem as the following

$$\begin{aligned} \text{minimize } & f(y) + q && (9.4)\\ \text{s.t. } & y \in Y && (9.5)\\ & (b - g(y))^T u^i \leq q \quad \forall i = 1, ..., I && (9.6)\\ & (b - g(y))^T v^j \leq 0 \quad \forall j = 1, ..., J && (9.7) \end{aligned}$$

where constraints 9.6 are called *optimality cuts/constraints* since they define the extreme points of the feasible region and constraints 9.7 are called *feasibility cuts/constraints* since they enforce that the problem is bounded.

3 Algorithm

3.1 Pseudo code

It is clear that our final formulation implies an exponential number of constraints since polyhedra typically have an exponential number of extreme points and extreme rays. The idea of the Benders Decomposition Algorithm is to work with a relaxation of the problem where only a limited number of constraints are considered. The algorithm then tries to reach the optimality of the original problem by a *clever* choice of constraints to be added iteratively. The algorithm is presented in 1.

3.2 Generating a feasibility cut

In **Step 2** of algorithm 1, it is asked to find an extreme ray such that $(b-g(y))^T v^j > 0$, i.e., a direction in which the problem is unbouned. A way to do that is to use the Simplex Tableau. Indeed, if a problem is unbounded, this implies that there exists a variable whose reduced cost is positive (i.e., ready to enter the basis) while the associated column is composed of postive terms (i.e., no constraint bounds its value). The associated column is in fact an extreme ray of the polyhedron.

4 Stabilisation methods

4.1 Bundle methods

4.2 Proximal methods

5 Generalization

In this section, we take interest in a generalization of the Benders decomposition applicable for the following problem

$$\text{minimize } f(x,y) \tag{9.8}$$
$$\text{s.t. } g(x,y) \le 0 \tag{9.9}$$
$$x \in X \tag{9.10}$$
$$y \in Y \tag{9.11}$$

under the following hypothesis :

(*i*) X is a convex set

Algorithm 1 Benders Decomposition Algorithm

Step 0 : Find an extreme point of $\{x|Ax \le c, x \ge 0\}$ (e.g., via the phase 1 of the Simplex)
$1 \to p, 0 \to k$

Step 1 : Solve relaxed problem with only p optimality constraints and k feasibility constraints :

$$\begin{aligned} \text{minimize } & f(y) + q \\ \text{s.t. } & y \in Y \\ & (b-g(y))^T u^i \le q \quad \forall i = 1, ..., p \\ & (b-g(y))^T v^j \le 0 \quad \forall j = 1, ..., k \end{aligned}$$

Let $\bar{y}, \bar{q}$ be the optimal solution thus obtained.

Step 2 : Check the feasible of $(\bar{y}, \bar{q})$ for the original problem by solving

$$q(\bar{y}) = \max\{(b-g(y))^T \pi : A^T \pi \le c, \pi \ge 0\}$$

Then,

If $q(\bar{y}) = +\infty$:
The problem is unbounded and, from theorem 5, one can find an extreme ray v^j such that $(b-g(y))^T v^j > 0$.
Add feasibility cut to the relaxed problem.
Increment k.
Got to **Step 1**.

Else :
Let $\bar{\pi}$ be the optimal solution of cost $q(\bar{y})$.

If $\bar{q} < q(\bar{y})$:
Add optimality cut $(b-g(y))^T \bar{\pi} \le q$
Increment p.
Got to **Step 1**.

Else :
The solution is optimal.

(ii) $f(x, y)$ and $g(x, y)$ are convex-in-x over X

Again, we can write the *projection* on the y-variable space, thus obtaining :

$$\min \{\inf\{f(x, y) : g(x, y) \leq 0, x \in X\} : y \in Y\}$$

and let us denote by $h(y)$ the minimization problem over x, i.e.,

$$h(y) = \inf\{f(x, y) : g(x, y) \leq 0, x \in X\}$$

Chapter 10

Dantzig-Wolfe decomposition

Chapter 11

Robust optimization

Chapter 12

Stochastic optimization

Part II

Non-linear Optimization

Chapter 13

Introduction

Chapter 14

Descent methods

Chapter 15

Newton methods

Chapter 16

Non-differentiable optimization

Appendices

Appendix A

Convex sets

1 Polyhedra

1.1 Definitions

Combinations and hulls

First, the following definitions will allow us to introduce some geometrical notions :

Definition 4 (Convex combination)**.** *Let $x_1, ..., x_n$ be a finite set of vectors in a real vector space, a convex combination of these vectors is a vector of the form*

$$\sum_{i=1}^{n} \alpha_i x_i \qquad \textit{with } \sum_{i=1}^{k} \alpha_i = 1, \alpha \geq 0$$

Definition 5 (Convex hull)**.**

$$conv(X) = \left\{ \sum_{i=1}^{n} \alpha_i x_i \middle| x_i \in X, \sum_{i=1}^{n} \alpha_i = 1, \alpha \geq 0 \right\}$$

Definition 6 (Conical combination)**.** *Let $x_1, ..., x_n$ be a finite set of vectors in a real vector space, a conical combination of these vectors is a vector of the form*

$$\sum_{i=1}^{k} \alpha_i x_i \qquad \textit{with } \alpha \geq 0$$

Definition 7 (Conical hull)**.**

$$cone(X) = \left\{ \sum_{i=1}^{n} \alpha_i x_i \middle| x_i \in X, \alpha_i \geq 0 \right\}$$

Polyhedra, polytopes and polyhedral cones

Definition 8 (Polyhedron)**.** *(plural : polyhedra)*

$$P = \{x \in \mathbb{R}^n | Ax \leq b\}$$

Definition 9 (Polytope)**.** *A polytope is the convex set a finite number of points. Alternatively,*

$$P = \{x \in \mathbb{R}^n | Ax \leq b, u^- \leq x \leq u^+\}$$

Definition 10 (Polyhedral cone)**.**

$$P = \{x \in \mathbb{R}^n | Ax \leq 0, x \geq 0\}$$

Observation 2 (Important)**.** *Note that no real consensus has been reach on the definitions of polyhedra and polytopes. Sometimes, polyhedra corresponds to three dimensional solids with polygonial faces while polytopes denote the extension of a polyhedra to higher dimensions.*

It comes from these definitions that every polytopes and every polyhedral cones are polyhedra. These geometrical objects are depicted in figure A.1.

1.2 Theorems

We first recall the following definition in order to properly enounce some important theorems.

Definition 11 (Minkowski sum)**.** *Let A and B be two vector spaces, the Minkowki sum is defined as*

$$A + B = \{a + b | a \in A, b \in B\}$$

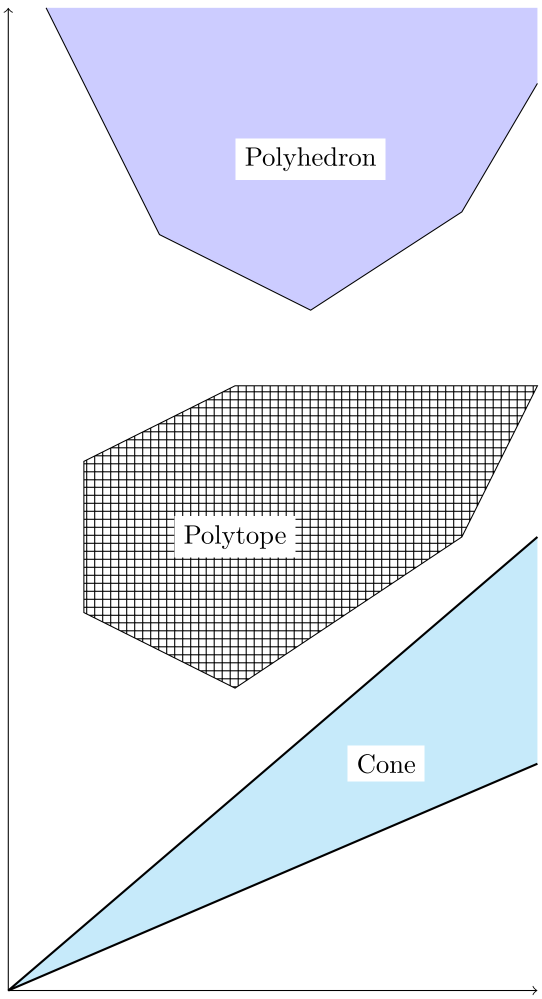

Figure A.1: A polyhdron, a polytope and a polyhedral cone depicted in 2D

Theorem 6 (Affine Minkowski-Weyl)**.** *Let there be a polyhedron defined by a set of inequalities,* $P = \{x \in \mathbb{R}^n | Ax \leq b\}$*. There exists vectors* $x_1, ..., x_q \in \mathbb{R}'n$ *and* $y_1, ..., y_r \in \mathbb{R}^n$ *such that*

$$P = cone(x_1, ..., x_q) + conv(y_1, ..., y_r)$$

Theorem 7 (Decomposition theorem for polyhedra)**.** *A set* P *of vectors in a Euclidean space is a polyhedron if and only if it is the Minkowki sum of a polytope* Q *and a polyhedral cone* C*.*

$$P = Q + C$$

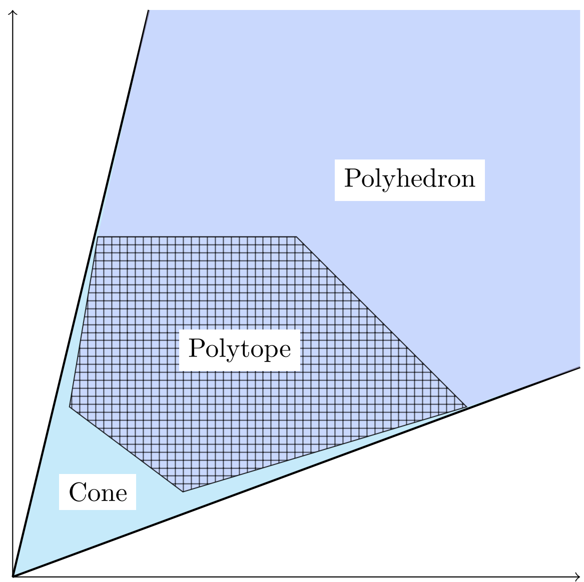

Figure A.2: Illustration of theorem 7

This decomposition theorem is illustrated in figure A.2. Intuitively, the polytope defines the *lower* shape of the polyhedron and the polyhedral cone defines the *queue* of the polyhedron.

Bibliography

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