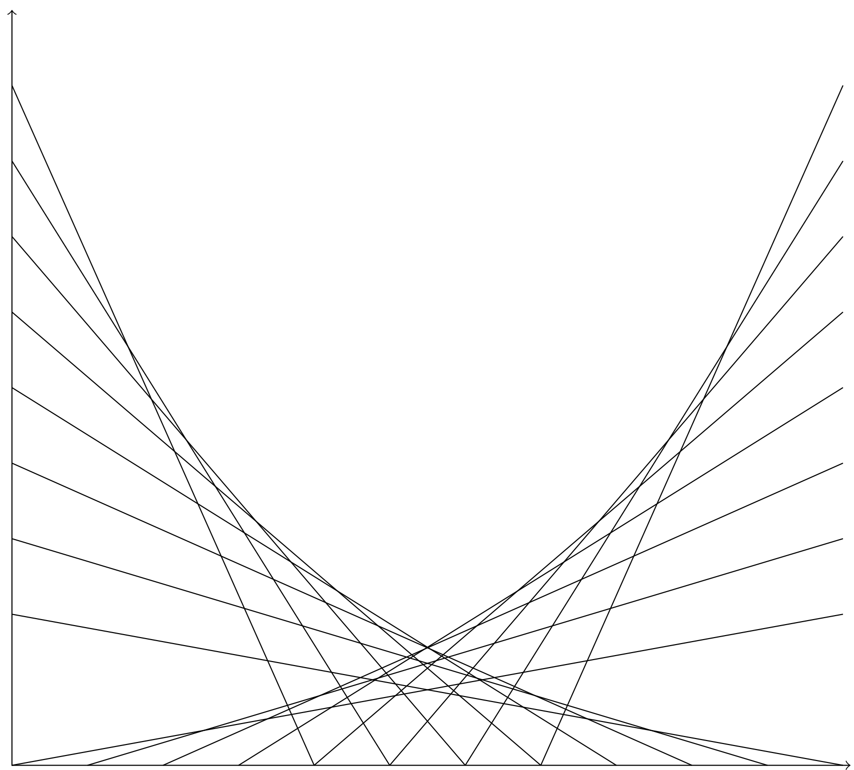
Optimization Handbook



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Benders decomposition

1 Introduction

Benders decomposition is a solution method for solving certain large-scale optimization problems. It is particularly suited for problems in which a set of variables are said to be *complicating* in the sense that fixing them to a given value makes the problem easy. Briefly, the Benders decomposition approach seperates an original problem into several decision stages. A first-stage *master* problem is solved using only a subset of variables, then, the values of the remaining variables are determined by a so-called *subproblem* depending on the first-stage variables. If the master problem's optimal solution yields an infeasible subproblem, a *feasibility cut* is added to the master problem, which is then re-solved. Due to the structure of the reformulation, the Benders algorithm starts with a *restricted master problem* where only a subset of constraints are considered while the others are iteratively added.

This technique was first introduced in [1] and has since been generalized to non-linear mixed integer problems.

2 Formal derivation

Consider the following problem :

$$\text{minimize } c^T x + f(y) \tag{8.1}$$
$$\text{s.t. } Ax + g(y) = b \tag{8.2}$$
$$y \in Y, x \geq 0 \tag{8.3}$$

where variable y is a *complicating constraint*. Note that it may be complicating due to the form of f or g but also by our ability to enforce the constraint $y \in Y$. We assume that fixing y to a given value $\hat{y}$ turns our problem into an easy-to-solve problem.

We can notice that our problem is equivalent to the following one :

$$\min \left\{ f(y) + \min \left\{ c^T x : Ax = b - g(y) \right\} : y \in Y \right\}$$

Let us denote by $q(y)$ the value of the minimization problem over x : $q(y) = \min\{c^T x : Ax = b - g(y)\}$. By duality, the following holds

$$q(y) = \max\{(b - g(y))^T \pi : A^T \pi \leq c\}$$

Note that the feasibility space of the dual does not depend on the values of y and let us apply the decomposition theorem for polyhedra 3 on it :

$$\{A^T \pi \leq c\} = \left\{ \sum_i u^i \alpha_i + \sum_j v^j \beta_j \right\}$$
$$\sum_i \alpha_i = 1 \quad \text{and} \quad \alpha, \beta \geq 0$$

where $\{u^i\}_i$ denotes the extreme points of $\{x|Ax \leq c\}$ and $\{v^j\}_j$ denotes the extreme rays of the polyhedral cone $\{x|Ax \leq 0\}$. Intuitively, the convex combination of the extreme points of $\{x|Ax \leq c\}$ defines the optimal solutions (since we know that there exists at least one optimal solution corresponding to an extreme point of the considered polyhedron) while the conical combination of extreme rays of $\{x|Ax \leq 0\}$

defines the feasibility region. The following theorem will allow us to reformulate our problem :

Theorem 1. *Let ($\mathcal{P}$) be the following problem :*

$$\max\{c^T x : Ax \leq b, x \geq 0\} \qquad (\mathcal{P})$$

Then, ($\mathcal{P}$) is upper bounded if and only if

$$c^T v^j \leq 0 \quad \forall j = 1, ..., J$$

where $\{v^j\}_{j=1,...,J}$ *denotes the set of extreme rays of* $\{x|Ax \leq 0\}$.

Proof. $\Rightarrow$: By contradiction, let us suppose that ($\mathcal{P}$) is upper bounded and that there exists k such that $c^T v^k > 0$. Let us consider a feasible solution to ($\mathcal{P}$) denoted by $u = z + t$ where z is a convex combination of the extreme points of $\{x|Ax \leq b\}$ and t an element of the conical combinations of the extreme rays of $\{x|Ax \leq 0\}$ (see theorem 3). Let $\lambda \in \mathbb{R}_+$. Since v^k is in the conical polyhedra $\{x|Ax \leq 0\}$, it holds that $\lambda A v^k \leq 0$. Moreover, we have $At \leq 0$. Hence, $A(t + \lambda v^k) \leq 0$. Now, since $v^k \geq 0$, we have $\lambda v^k \geq 0$ and since $t \geq 0$ it holds that $t + \lambda v^k \geq 0$. This shows that, for any λ, $z + \lambda v^k$ is a feasible solution for problem ($\mathcal{P}$). However, its associated objective value is given by $c^T u + \lambda c^T v^k \to +\infty$ when $\lambda \to +\infty$ since, by assumption, $c^T v^k > 0$. This contradicts the fact that ($\mathcal{P}$) is upper bounded.
$\Leftarrow$: Let us consider a solution $u = z + t$ then $c^T u = c^T z + \sum_j c^T v^j \beta_j \leq c^T z$ since $c^T v^j \leq 0, \forall j = 1, ..., J$. Problem ($\mathcal{P}$) is therefore upper bounded by $\sum_i \alpha_i \max_i\{c^T u^i\}$. $\square$

Theorem 1 can be intuitively understood by interpreting $c^T v^j$ as $\langle c, v^j \rangle$ (scalar product). The necessary and sufficiant condition that $\langle c, v^j \rangle \leq 0$ simply expresses that the the hyperplane defining the objective function must be *oriented* (i.e., increasing) in the direction of the origin of the cone formed by the extreme rays of the polyhedron. It is depicted in figure 8.1.

If we apply theorem 1 to our specific problem, we find that a necessary and sufficient condition for $q(y)$ to be upper bounded is that $(b - g(y))^T v^j \leq 0, \forall j = 1, ..., J$ where v^j denotes an extreme ray of the polyhedron $\{x|Ax \leq c\}$ and J a list for their indices.

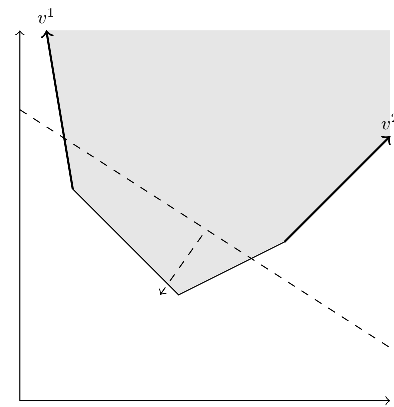

Figure 8.1: Illustration of theorem 1

Moreover, we know that there exists at least one optimal solution realised in an extreme point of the feasible region. Therefore, by calling $u^i, i = 1, ..., I$ the extreme points of the feasible region, the problem of finding the value of $q(y)$ can be reformulated as

$$q(y) = \min\{q : q \geq (b - g(y))^T u^i \quad \forall i = 1, ..., I\}$$

Assuming that this problem is bounded. We finally can write our original problem as the following

$$\begin{aligned} \text{minimize} \;\; & f(y) + q && (8.4)\\ \text{s.t.} \;\; & y \in Y && (8.5)\\ & (b - g(y))^T u^i \leq q \quad \forall i = 1, ..., I && (8.6)\\ & (b - g(y))^T v^j \leq 0 \quad \forall j = 1, ..., J && (8.7) \end{aligned}$$

where constraints 8.6 are called *optimality cuts/constraints* since they define the extreme points of the feasible region and constraints 8.7 are called *feasibility cuts/constraints* since they enforce that the problem is bounded.

3 Algorithm

3.1 Pseudo code

It is clear that our final formulation implies an exponential number of constraints since polyhedra typically have an exponential number of extreme points and extreme rays. The idea of the Benders Decomposition Algorithm is to work with a relaxation of the problem where only a limited number of constraints are considered. The algorithm then tries to reach the optimality of the original problem by a *clever* choice of constraints to be added iteratively. The algorithm is presented in 1.

3.2 Generating a feasibility cut

In **Step 2** of algorithm 1, it is asked to find an extreme ray such that $(b-g(y))^T v^j > 0$, i.e., a direction in which the problem is unbouned. A way to do that is to use the Simplex Tableau. Indeed, if a problem is unbounded, this implies that there exists a variable whose reduced cost is positive (i.e., ready to enter the basis) while the associated column is composed of postive terms (i.e., no constraint bounds its value). The associated column is in fact an extreme ray of the polyhedron.

Algorithm 1 Benders Decomposition Algorithm

Step 0 : Find an extreme point of $\{x|Ax \leq c, x \geq 0\}$ (e.g., via the phase 1 of the Simplex)
$1 \to p, 0 \to k$

Step 1 : Solve relaxed problem with only p optimality constraints and k feasibility constraints :

$$\begin{aligned} \text{minimize } & f(y) + q \\ \text{s.t. } & y \in Y \\ & (b - g(y))^T u^i \leq q \quad \forall i = 1, ..., p \\ & (b - g(y))^T v^j \leq 0 \quad \forall j = 1, ..., k \end{aligned}$$

Let $\bar{y}, \bar{q}$ be the optimal solution thus obtained.

Step 2 : Check the feasible of $(\bar{y}, \bar{q})$ for the original problem by solving

$$q(\bar{y}) = \max\{(b - g(y))^T \pi : A^T \pi \leq c, \pi \geq 0\}$$

Then,

If $q(\bar{y}) = +\infty$:
The problem is unbounded and, from theorem 1, one can find an extreme ray v^j such that $(b - g(y))^T v^j > 0$.
Add feasibility cut to the relaxed problem.
Increment k.
Got to **Step 1**.

Else :
Let $\bar{\pi}$ be the optimal solution of cost $q(\bar{y})$.

If $\bar{q} < q(\bar{y})$:
Add optimality cut $(b - g(y))^T \bar{\pi} \leq q$
Increment p.
Got to **Step 1**.

Else :
The solution is optimal.

4 Stabilisation methods

4.1 Bundle methods

4.2 Proximal methods

5 Generalization

In this section, we take interest in a generalization of the Benders decomposition applicable for the following problem

$$\begin{aligned} \text{minimize } & f(x, y) && (8.8) \\ \text{s.t. } & g(x, y) \leq 0 && (8.9) \\ & x \in X && (8.10) \\ & y \in Y && (8.11) \end{aligned}$$

under the following hypothesis :

(i) X is a convex set

(ii) $f(x, y)$ and $g(x, y)$ are convex-in-x over X

Again, we can write the *projection* on the y-variable space, thus obtaining :

$$\min\{\inf\{f(x, y) : g(x, y) \leq 0, x \in X\} : y \in Y\}$$

and let us denote by $h(y)$ the minimization problem over x, i.e.,

$$h(y) = \inf\{f(x, y) : g(x, y) \leq 0, x \in X\}$$

Chapter 9

Dantzig-Wolfe decomposition

Part II

Non-linear Optimization

Appendices

Appendix A

Convex sets

1 Polyhedra

1.1 Definitions

Combinations and hulls

First, the following definitions will allow us to introduce some geometrical notions :

Definition 1 (Convex combination). *Let $x_1, ..., x_n$ be a finite set of vectors in a real vector space, a convex combination of these vectors is a vector of the form*

$$\sum_{i=1}^{n} \alpha_i x_i \qquad \textit{with} \sum_{i=1}^{k} \alpha_i = 1, \alpha \geq 0$$

Definition 2 (Convex hull).

$$\textit{conv}(X) = \left\{ \sum_{i=1}^{n} \alpha_i x_i \middle| x_i \in X, \sum_{i=1}^{n} \alpha_i = 1, \alpha \geq 0 \right\}$$

Definition 3 (Conical combination). *Let $x_1, ..., x_n$ be a finite set of vectors in a real vector space, a conical combination of these vectors is a vector of the form*

$$\sum_{i=1}^{k} \alpha_i x_i \qquad \textit{with } \alpha \geq 0$$

Definition 4 (Conical hull).

$$\textit{cone}(X) = \left\{ \sum_{i=1}^{n} \alpha_i x_i \middle| x_i \in X, \alpha_i \geq 0 \right\}$$

Polyhedra, polytopes and polyhedral cones

Definition 5 (Polyhedron). *(plural : polyhedra)*

$$P = \{x \in \mathbb{R}^n | Ax \leq b\}$$

Definition 6 (Polytope). *A polytope is the convex set a finite number of points. Alternatively,*

$$P = \{x \in \mathbb{R}^n | Ax \leq b, u^- \leq x \leq u^+\}$$

Definition 7 (Polyhedral cone).

$$P = \{x \in \mathbb{R}^n | Ax \leq 0, x \geq 0\}$$

Observation 1 (Important). *Note that no real consensus has been reach on the definitions of polyhedra and polytopes. Sometimes, polyhedra corresponds to three dimensional solids with polygonial faces while polytopes denote the extension of a polyhedra to higher dimensions.*

It comes from these definitions that every polytopes and every polyhedral cones are polyhedra. These geometrical objects are depicted in figure A.1.

1.2 Theorems

We first recall the following definition in order to properly enounce some important theorems.

Definition 8 (Minkowski sum). *Let A and B be two vector spaces, the Minkowki sum is defined as*

$$A + B = \{a + b | a \in A, b \in B\}$$

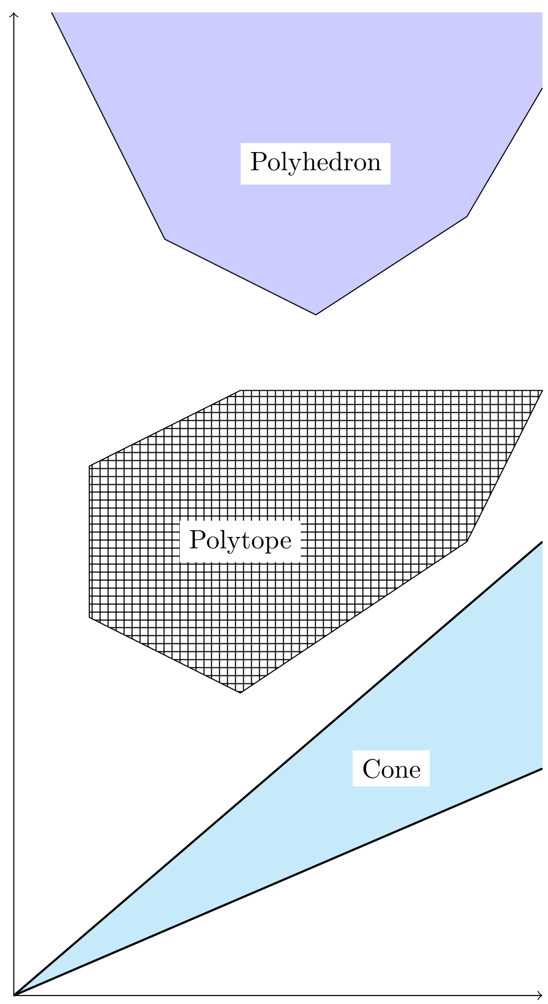

Figure A.1: A polyhdron, a polytope and a polyhedral cone depicted in 2D

Theorem 2 (Affine Minkowski-Weyl)**.** *Let there be a polyhedron defined by a set of inequalities,* $P = \{x \in \mathbb{R}^n | Ax \leq b\}$*. There exists vectors* $x_1, ..., x_q \in \mathbb{R}^{,}n$ *and* $y_1, ..., y_r \in \mathbb{R}^n$ *such that*

$$P = cone(x_1, ..., x_q) + conv(y_1, ..., y_r)$$

Theorem 3 (Decomposition theorem for polyhedra)**.** *A set* P *of vectors in a Euclidean space is a polyhedron if and only if it is the Minkowki sum of a polytope* Q *and a polyhedral cone* C*.*

$$P = Q + C$$

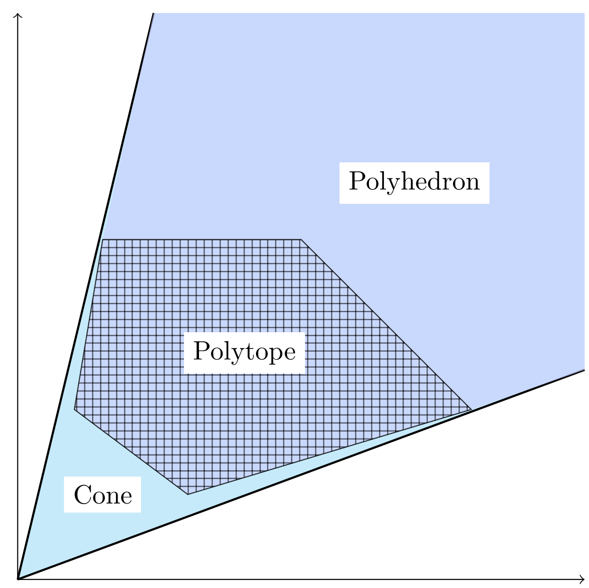

Figure A.2: Illustration of theorem 3

This decomposition theorem is illustrated in figure A.2. Intuitively, the polytope defines the *lower* shape of the polyhedron and the polyhedral cone defines the *queue* of the polyhedron.

Bibliography

[1] J. F. Benders. Partitioning procedures for solving mixed-variables programming problems. *Numerische Mathematik*, 4(1):238–252, Dec 1962.

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