

Efficient Robbins-Monro Procedure for Binary Data

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SUMMARY

The Robbins-Monro procedure does not perform well in the estimation of extreme quantiles, because the procedure is implemented using asymptotic results, which are not suitable for binary data. Here we propose a modification of the Robbins-Monro procedure and derive the optimal procedure for binary data under some reasonable approximations. The improvement obtained by using the optimal procedure for the estimation of extreme quantiles is substantial.

Some key words: Quantile estimation; Sequential design; Stochastic approximation.

1. INTRODUCTION

In many applications, interest centres on finding the threshold of a variable that will cause certain amount of successes, or failures, in the output. For example, an explosive designer may be interested in finding the level of shock necessary to make 99.99% of the explosives fire (Neyer, 1994); see Joseph and Wu (2002) for another application. The problem can be formally stated as follows. Let Y be a binary response with probability of success $M(x)$, where x is the variable whose threshold is of interest. The objective is to find the value θ of x for which $M(\theta) = \alpha$, for given α . Usually $M(x)$ is a distribution function obtained by assuming some distribution for a latent variable underlying the binary response, so that θ can be regarded as the α -quantile of this distribution. The function $M(x)$ is unknown to the experimenter, but the experimenter can observe Y at different values of x in order to find θ . The objective is to devise an experimental strategy that will help one to estimate θ with minimum number of observations and with great accuracy. The experiment can be performed by using a sequential design in which the x 's are chosen sequentially and are allowed to depend on the data already observed .

One sequential design strategy known as stochastic approximation is to choose x_1, x_2, \dots such that $x_n \rightarrow \theta$ in probability. Robbins & Monro(1951) proposed the procedure

$$x_{n+1} = x_n - a_n(y_n - \alpha), \tag{1}$$

where y_n is the binary response observed at x_n and $\{a_n\}$ is a pre-specified sequence of positive constants. Robbins & Monro showed that $x_n \rightarrow \theta$ in probability if

$$\sum_{n=1}^{\infty} a_n = \infty, \text{ and } \sum_{n=1}^{\infty} a_n^2 < \infty. \tag{2}$$

Robbins & Monro recommended the simple but non-optimal choice $a_n = c/n$ for some constant c . Based on the results of Chung (1954), Hodges & Lehmann (1956) and Sacks (1958), the procedure is fully asymptotically efficient with $a_n = \{n\dot{M}(\theta)\}^{-1}$ under certain conditions, where \dot{M} denotes the first derivative of M ; for use in practical implementations see for example Wetherill and Glazebrook (1986). The optimal choice of a_n in small samples has not been investigated, although in most experiments this is the most interesting case.

Wetherill (1963) showed through simulation that the Robbins-Monro procedure works quite well for $\alpha = 0.5$, corresponding to LD50, but performs very poorly for extreme quantiles, even with a number of modifications. Cochran & Davis (1965) and Young & Easterling (1994) obtained similar conclusions based on extensive simulation results.

Although the Robbins-Monro procedure is not restricted to binary data, binary data possess two properties that can be exploited to improve the procedure: the variance of Y is a function of x given by $M(x)\{1 - M(x)\}$, and $M(x)$ is a distribution function. In this article we make use of these properties to produce optimal Robbins-Monro procedures for binary data. We also propose a slight modification to the Robbins-Monro procedure to get better convergence properties.

Several methods for quantile estimation based on binary data are proposed in the literature; see for example Wu (1985), McLeish & Tosh (1990), Kalish (1990), Neyer (1994) and Sitter & Wu (1999) among many others. The objective of this study is to find the optimal sequential procedure within the class of the Robbins-Monro type procedures, not to find the best overall method.

2. OPTIMAL ROBBINS-MONRO PROCEDURE

In most applications, the distribution function $M(x)$ is from a location family with location parameter θ . Therefore hereafter we denote $M(x)$ by $M(x - \theta)$. Thus $M(0) = \alpha \in (0, 1)$ is specified. We also assume that $\dot{M}(0) > 0$ is known. The experimenter starts the experiment at some value x_1 , which is believed to be close to θ based on some prior knowledge. Therefore we may choose a prior distribution for θ with $E(\Theta) = x_1$ and $\text{var}(\Theta) = \tau_1^2 < \infty$, where τ_1 represents the initial uncertainty of θ with respect to x_1 . Let $Z_n = x_n - \Theta$. Note that, although x_1 is a fixed quantity, x_2, \dots, x_n are random because of their dependence on past data. Consider a modified Robbins-Monro process given by

$$x_{n+1} = x_n - a_n(y_n - b_n).$$

For binary data, $\{b_n\}$ is a sequence of constants in $(0, 1)$. They need not be equal to α , but $\{b_n\}$ is expected to get close to α as n gets larger. We will see that, if we use a b_n different from α , the performance of the Robbins-Monro procedure can be greatly improved. Thus our model is

$$y_n | Z_n \sim \text{Ber}\{M(Z_n)\},$$

$$Z_{n+1} = Z_n - a_n(y_n - b_n),$$

$$E(Z_1) = 0 \text{ and } E(Z_1^2) = \tau_1^2.$$

The objective is to find sequences $\{a_n\}$ and $\{b_n\}$ such that $Z_n \rightarrow 0$ in probability at the fastest rate. First we investigate the conditions under which the desired convergence can be obtained.

Suppose the sequence $\{b_n\}$ satisfies the condition

$$\sum_{n=2}^{\infty} a_n |b_n - \alpha| \sum_{j=1}^{n-1} a_j < \infty. \quad (3)$$

Then we have the following convergence result whose proof closely follows that of Robbins & Monro (1951). The above condition together with (2) ensures that b_n converges to α . Moreover, because $\sum_{j=1}^{n-1} a_j$ increases with n , the convergence of b_n to α should be fast enough for (3) to hold. All proofs are given in the Appendix.

THEOREM 1. *If (2) and (3) hold, then $Z_n \rightarrow 0$, in probability, as $n \rightarrow \infty$.*

There are infinitely many sequences for $\{a_n\}$ and $\{b_n\}$ satisfying the conditions of the Theorem 1. We are seeking the particular sequence that gives the best convergence properties. We propose to choose a_n and b_n such that $E(Z_{n+1}^2)$ is a minimum subject to the condition that $E(Z_{n+1}) = 0$. Similar ideas of choosing two sequences to minimise the conditional mean squared error in sequential designs was employed by Hu (1997, 1998) in a Bayesian framework.

We have that

$$E(Z_{n+1}) = E(Z_n) - a_n[E\{M(Z_n)\} - b_n] = 0.$$

Since the sequences a_1, \dots, a_{n-1} and b_1, \dots, b_{n-1} are obtained such that $E(Z_2) = \dots = E(Z_n) = 0$, we obtain

$$b_n = E\{M(Z_n)\}. \quad (4)$$

Let $\tau_n^2 = E(Z_n^2)$. From (A1), we have

$$\tau_{n+1}^2 = \tau_n^2 - 2a_n E[Z_n \{M(Z_n) - b_n\}] + a_n^2 \left([E\{M(Z_n)\} - b_n]^2 + E\{M(Z_n)\}[1 - E\{M(Z_n)\}] \right). \quad (5)$$

Minimising τ_{n+1}^2 with respect to a_n and using (4), we obtain

$$a_n = \frac{E\{Z_n M(Z_n)\}}{E\{M(Z_n)\}[1 - E\{M(Z_n)\}]}. \quad (6)$$

Unfortunately, the optimal sequences depend on the function M , which is unknown to the experimenter. Therefore the best we can do is to choose a function G that can closely approximate the true function M and derive the sequences. The resulting Robbins-Monro procedure is optimal only when $M = G$ and approximately optimal otherwise. It can be considered as an efficient procedure as long as the deviation of G from the true function is not severe. The choice of $a_n = \{\dot{M}(0)\}^{-1}$ is based on a linear approximation of M around 0, which is not good for a distribution function particularly in the tail areas.

Consider an approximation for $M(z)$ given by,

$$G(z) = \Phi\{\Phi^{-1}(\alpha) + \beta z\}, \quad (7)$$

where $\beta = \dot{M}(0)/\phi\{\Phi^{-1}(\alpha)\}$, Φ is the standard normal distribution function, and ϕ is its density function. Now a_n and b_n can be obtained from (4) and (6) using $G(z)$ instead of $M(z)$. We also need the distribution of Z_n to evaluate the expectations in (4) and (6). It is easy to show that the density function of Z_{n+1} is

$$f_{Z_{n+1}}(z) = \{1 - G(z - a_n b_n)\} f_{Z_n}(z - a_n b_n) + G\{z + a_n(1 - b_n)\} f_{Z_n}\{z + a_n(1 - b_n)\}.$$

The $f_{Z_n}(z)$ can be recursively computed starting with $f_{Z_1}(z)$. Let $Z_1 \sim N(0, \tau_1^2)$, so that $f_{Z_1}(z) = 1/\tau_1 \phi(z/\tau_1)$. Clearly it is difficult to compute the expectations in (4) and (6) with this exact distribution, and therefore we resort to approximation. It is quite natural to approximate the distribution of Z_n by $N(0, \tau_n^2)$ as the first two moments of the two

distributions match exactly. It turns out that this is a very good approximation, as verified by plotting these functions for various values of α , τ_1 , and n . Using this approximation we obtain

$$\begin{aligned} b_n &= E\{G(Z_n)\} = \int_{-\infty}^{\infty} \Phi\{\Phi^{-1}(\alpha) + \beta z\} \frac{1}{\tau_n} \phi(z/\tau_n) dz = \Phi\left\{\frac{\Phi^{-1}(\alpha)}{(1 + \beta^2 \tau_n^2)^{\frac{1}{2}}}\right\}, \\ a_n &= \frac{E\{Z_n G(Z_n)\}}{E\{G(Z_n)\}[1 - E\{G(Z_n)\}]} = \frac{1}{b_n(1 - b_n)} \int_{-\infty}^{\infty} z \Phi\{\Phi^{-1}(\alpha) + \beta z\} \frac{1}{\tau_n} \phi(z/\tau_n) dz \\ &= \frac{1}{b_n(1 - b_n)} \frac{\beta \tau_n^2}{(1 + \beta^2 \tau_n^2)^{\frac{1}{2}}} \phi\left\{\frac{\Phi^{-1}(\alpha)}{(1 + \beta^2 \tau_n^2)^{\frac{1}{2}}}\right\}. \end{aligned}$$

Substituting in (5) we obtain $\tau_{n+1}^2 = \tau_n^2 - b_n(1 - b_n)a_n^2$, which shows that there is an improvement by moving from x_n to x_{n+1} . Let $c_n = \beta a_n b_n(1 - b_n)$ and $\nu_n = \beta^2 \tau_n^2$. Then the optimal Robbins-Monro procedure can be written as

$$x_{n+1} = x_n - \frac{c_n}{\beta b_n(1 - b_n)}(y_n - b_n), \quad (8)$$

where

$$c_n = \frac{\nu_n}{(1 + \nu_n)^{\frac{1}{2}}} \phi\left\{\frac{\Phi^{-1}(\alpha)}{(1 + \nu_n)^{\frac{1}{2}}}\right\}, \quad b_n = \Phi\left\{\frac{\Phi^{-1}(\alpha)}{(1 + \nu_n)^{\frac{1}{2}}}\right\} \quad \text{and} \quad \nu_{n+1} = \nu_n - \frac{c_n^2}{b_n(1 - b_n)}, \quad (9)$$

with $\nu_1 = \beta^2 \tau_1^2$. Note that c_n and b_n are sequences that can be specified before the experiment. They can be easily computed once $\nu_1 \in (0, \infty)$ is specified. The procedure can work only with a finite value of ν_1 , which implies that a noninformative prior for θ cannot be used. Therefore the Bayesian formulation of the problem with a proper prior was very crucial in the development of the above procedure. After n experiments the best estimate of θ is x_{n+1} . We can also obtain a $(1 - \gamma)$ credible interval for θ as $x_{n+1} \pm \Phi^{-1}(\gamma/2)\tau_{n+1}$, where $\tau_{n+1} = \nu_{n+1}^{\frac{1}{2}}/\beta$.

PROPOSITION 1. *As $n \rightarrow \infty$, $\nu_n \rightarrow 0$, $b_n \rightarrow \alpha$ and $c_n \rightarrow 0$.*

By Proposition 1 we have that $\tau_n^2 \rightarrow 0$ and therefore $Z_n \rightarrow 0$ in probability, if M is equal to the normal distribution function given in (7). We can expect the result to hold even when the true model is not normal. Fortunately this hypothesis is true as stated in the following proposition.

PROPOSITION 2. *For the procedure in (8), $Z_n \rightarrow 0$ in probability, as $n \rightarrow \infty$*

It is clear from (9) that b_n always lies between α and $1/2$. The consequence of this is important. Even if we are interested in extreme quantiles, for small n , the optimal Robbins-Monro procedure is operated as though we are interested in a quantile between α and LD50. The search is moved closer to α as n increases since $b_n \rightarrow \alpha$. This is markedly different from the ordinary Robbins-Monro procedure in (1). Wetherill (1963) noted that the Robbins-Monro procedure performs miserably for the estimation of small or large quantiles. This is mainly because of the unequal up and down movements of the procedure. Since we use b_n instead of α this imbalance is somewhat mitigated. Consider an example of Wetherill. Suppose $\alpha = 0.25$ and the true model is a logistic function given by $M(z) = \{1 + 3 \exp(-z)\}^{-1}$. Suppose that the experiment started exactly at θ . If we observe $y_1 = 1$, which can happen with probability 0.25, then $x_2 = \theta - \{1/\dot{M}(0)\}(1 - 0.25) = \theta - 4$. Thus x_2 is lower than θ . The search will move up if we observe $y = 0$. Suppose we observed $y_2 = \dots = y_n = 0$. Then

$$x_{n+1} = \theta - 4 + \frac{0.25}{0.25 \times 0.75} \left(\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right),$$

which crosses θ from below at $n = 31$. In other words, it will take at least 30 steps to get back to the true value θ . Now consider the optimal Robbins-Monro procedure in (8). Here

$\beta = \dot{M}(0)/\phi\{\Phi(0.25)\} = 0.59$. We select τ_1 such that x_2 is exactly the same as that in the Robbins-Monro procedure. This value is 4.475. Now $x_7 = \theta - 4 + 1.579 + 0.973 + 0.655 + 0.478 + 0.370 = \theta + 0.055$. Thus the sequence crosses θ from below in just 5 steps as opposed to the 30 steps taken by the Robbins-Monro procedure.

Note that we require the value of $\dot{M}(0)$ to compute β . In practice, it is unlikely that the experimenter will know the exact value of $\dot{M}(0)$ and the procedure has to be implemented using a guessed value of $\dot{M}(0)$. It is clear from the proof of Proposition 2 that the procedure will work irrespective of the value of β , but the convergence can slow down if the guessed value is far away from the true value. If a good guess cannot be made for $\dot{M}(0)$, one can try to estimate it adaptively from the data by fitting a parametric model such as the one in (7). For the Robbins-Monro procedure such an adaptive procedure under some truncation rule gives the same asymptotic performance as that of the original procedure; see Lai & Robbins (1979) and Wu (1985) for more details.

3. SIMULATIONS

We now compare the performance of the optimal Robbins-Monro procedure in (8) with the Robbins-Monro procedure in (1) through simulations. For the Robbins-Monro procedure a_n is taken as $\{n\dot{M}(0)\}^{-1}$, which is the standard choice. The following six models are selected for the simulation study.

$$\text{Normal distribution: } M(z) = \Phi\{\Phi^{-1}(\alpha) + z\}.$$

$$\text{Uniform distribution: } M(z) = \begin{cases} 0 & , z < -3\alpha \\ \alpha + z/3 & , -3\alpha \leq z \leq 3(1 - \alpha) \\ 1 & , z > 3(1 - \alpha). \end{cases}$$

$$\text{Logistic distribution: } M(z) = (1 + \frac{1 - \alpha}{\alpha} e^{-z})^{-1}.$$

$$\text{Extreme value distribution: } M(z) = 1 - \exp\{\log(1 - \alpha) e^z\}.$$

$$\text{Skewed logistic distribution: } M(z) = (1 + \frac{1 - \sqrt{\alpha}}{\sqrt{\alpha}} e^{-z})^{-2}.$$

$$\text{Cauchy distribution: } M(z) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1}\{z + \tan(\pi\alpha - \pi/2)\}.$$

Let $\theta = 0$. We choose 20 samples to estimate θ . Thus the best estimate of θ is x_{21} . We let $\tau_1 = 1$ be the initial uncertainty, which means that, with probability about 95%; θ is within ± 2 of the starting value x_1 . The starting value is randomly generated from $N(0, \tau_1^2)$ and is used for both procedures. Then at different values of α between 0.1 and 0.9, 1000 simulations were performed. The mean squared error of x_{21} is calculated and is plotted in Fig.1. We see that the optimal Robbins-Monro procedure performs uniformly better than the Robbins-Monro procedure, with great improvement for the extreme quantiles. The performance of the two procedures around LD50 is comparable, which agrees with the findings of Wetherill (1963). Fig. 2 shows the estimated density curves for x_{21} for different α values with the logistic distribution as the true model. It clearly shows how the performance of the Robbins-Monro procedure deteriorates as we move towards the extreme quantiles. The entire process is repeated with $\tau_1 = 2$, with essentially the same conclusions.

This simulation study clearly demonstrate the superior performance of the optimal Robbins-Monro procedure over the Robbins-Monro procedure. The optimal Robbins-Monro proce-

cedure did a good job even for the estimation of extreme quantiles. The procedure appears to be robust against the model assumptions as it performed well for a wide range of distributions.

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APPENDIX

Proofs

Proof of Theorem 1: We have that

$$E(Z_{n+1}^2|Z_n) = Z_n^2 - 2a_n Z_n \{M(Z_n) - b_n\} + a_n^2 E\{(y_n - b_n)^2|Z_n\}.$$

Let $\tau_n^2 = E(Z_n^2)$, $d_n = E[Z_n \{M(Z_n) - b_n\}]$ and $e_n = E\{(y_n - b_n)^2\}$. Then

$$\tau_{n+1}^2 = \tau_n^2 - 2a_n d_n + a_n^2 e_n, \tag{A1}$$

which gives

$$\tau_{n+1}^2 = \tau_1^2 - 2 \sum_{j=1}^n a_j d_j + \sum_{j=1}^n a_j^2 e_j. \tag{A2}$$

Since $b_n \in (0, 1)$ and $y_n = 0$ or 1 , we have that $0 < e_n < 1$ for all n . Thus $0 < \sum_{j=1}^n a_j^2 e_j < \sum_{j=1}^n a_j^2$. Therefore by condition (2) the positive-term series $\sum_{j=1}^n a_j^2 e_j$ converges.

We have that

$$\sum_{j=1}^n a_j d_j = \sum_{j=1}^n a_j E[Z_j \{M(Z_j) - b_j\}] = \sum_{j=1}^n a_j E[Z_j \{M(Z_j) - \alpha\}] + \sum_{j=1}^n a_j (\alpha - b_j) E(Z_j).$$

Since M is a distribution function and $\dot{M}(0) > 0$, there exist δ and δ' such that $0 < \delta \leq M(z) - \alpha/z \leq \delta' < \infty$ for all z . This implies that $E[Z_j\{M(Z_j) - \alpha\}] \geq \delta E(Z_j^2) = \delta\tau_j^2$. Thus

$$\sum_{j=1}^n a_j d_j \geq \delta \sum_{j=1}^n a_j \tau_j^2 + \sum_{j=1}^n a_j (\alpha - b_j) E(Z_j).$$

Consider the magnitude of the second term on the right-hand side:

$$\left| \sum_{j=1}^n a_j (\alpha - b_j) E(Z_j) \right| \leq \sum_{j=1}^n a_j |\alpha - b_j| |E(Z_j)| < \sum_{j=2}^n a_j |\alpha - b_j| \sum_{i=1}^{j-1} a_i,$$

because $Z_n = Z_1 - \sum_{j=1}^{n-1} a_j (y_j - b_j) \Rightarrow |E(Z_n)| < \sum_{j=1}^{n-1} a_j$. Also, since $\tau_{n+1}^2 \geq 0$ for all n ,

from () we obtain $\sum_{j=1}^n a_j d_j \leq (\tau_1^2 + \sum_{j=1}^n a_j^2 e_j)/2$. Thus

$$\begin{aligned} \delta \sum_{j=1}^n a_j \tau_j^2 &\leq \sum_{j=1}^n a_j d_j - \sum_{j=1}^n a_j (\alpha - b_j) E(Z_j) \\ &< \frac{1}{2} (\tau_1^2 + \sum_{j=1}^n a_j^2 e_j) + \sum_{j=2}^n a_j |\alpha - b_j| \sum_{i=1}^{j-1} a_i. \end{aligned}$$

The right-hand side converges by condition (3) and therefore the positive-term series $\sum_{j=1}^n a_j \tau_j^2$

converges. Now

$$\begin{aligned} \sum_{j=1}^n |a_j d_j| &\leq \sum_{j=1}^n a_j |E[Z_j\{M(Z_j) - \alpha\}]| + \sum_{j=1}^n a_j |\alpha - b_j| |E(Z_j)| \\ &< \delta' \sum_{j=1}^n a_j \tau_j^2 + \sum_{j=2}^n a_j |\alpha - b_j| \sum_{i=1}^{j-1} a_i, \end{aligned}$$

Since the right-hand side converges, the series $\sum_{j=1}^n a_j d_j$ converges absolutely and hence the

series converges. Thus, from (A2), $\lim \tau_n^2$ exists and is equal to $\tau_1^2 - 2 \sum_{j=1}^{\infty} a_j d_j + \sum_{j=1}^{\infty} a_j^2 e_j$.

However, since $0 \leq \sum_{n=1}^{\infty} a_n \tau_n^2 < \infty$ and $\sum_{n=1}^{\infty} a_n = \infty$, this limit must be 0 and therefore

$Z_n \rightarrow 0$ in probability. \diamond

Proof of Proposition 1: From (9), we have

$$\nu_{n+1} = \nu_n - \frac{\nu_n^2}{1 + \nu_n} I\left\{\frac{\Phi^{-1}(\alpha)}{(1 + \nu_n)^{\frac{1}{2}}}\right\}, \quad (\text{A3})$$

where $I(u) = \phi^2(u)/[\Phi(u)\{1 - \Phi(u)\}]$ is the Fisher information of u in binary data with probability of success equal to $\Phi(u)$. It is well known that $0 < I(u) \leq 2/\pi$. Thus $\nu_{n+1} < \nu_n$ and $\nu_{n+1} \geq \nu_n(1 - 2/\pi) \geq \nu_1(1 - 2/\pi)^n > 0$ for all n . Hence the sequence $\{\nu_n\}$ converges. Let $h(\nu_n)$ be the right-hand side of (A3). Then $\lim \nu_n$ satisfies the equation $\nu = h(\nu)$, for which 0 is a unique solution. Thus, from (9), we obtain $b_n \rightarrow \alpha$ and $c_n \rightarrow 0$. \diamond

Proof of Proposition 2: Let $n^* = 1/I\{\Phi^{-1}(\alpha)\}$ and $\nu^* = 1/I^2\{\Phi^{-1}(\alpha)\}$. We have that $\dot{h}(0) = 1$. Since $\dot{h}(\nu)$ is a continuous function in ν , there exists a $\bar{\nu} \in (0, \nu^*]$ such that $\dot{h}(\nu) > 0$ for all $\nu \leq \bar{\nu}$. Also, since $\nu_n \rightarrow 0$, there exists an \bar{n} such that $\nu_n < \bar{\nu}$ for all $n \geq \bar{n}$. Let $\tilde{n} = \max\{\lceil n^* \rceil, \bar{n}, \lceil \nu^*/\bar{\nu} \rceil\}$ and $\nu' = \tilde{n}\bar{\nu}$, where $\lceil x \rceil$ is the smallest integer greater than or equal to x . Thus $\nu_{\tilde{n}} \leq \nu_{\bar{n}} < \bar{\nu} = \nu'/\tilde{n}$. Suppose that $\nu_n \leq \nu'/n$ for some $n \geq \tilde{n}$. Since $\nu' \geq \nu^* > 1$, we have for $n \geq \tilde{n}$ that

$$I\left\{\frac{\Phi^{-1}(\alpha)}{(1 + \nu'/n)^{\frac{1}{2}}}\right\} \geq I\{\Phi^{-1}(\alpha)\} = \frac{n^* + \nu^*}{(n^* + 1)\nu^*} \geq \frac{n + \nu^*}{(n + 1)\nu^*} \geq \frac{n + \nu'}{(n + 1)\nu'}.$$

Since $\nu_n \leq \nu'/n \leq \nu'/\tilde{n} = \bar{\nu}$ and $h(\nu)$ is increasing in ν for all $\nu \leq \bar{\nu}$, from (A3) we obtain

$$\nu_{n+1} \leq \frac{\nu'}{n} \left[1 - \frac{\nu'}{n + \nu'} I\left\{\frac{\Phi^{-1}(\alpha)}{(1 + \nu'/n)^{\frac{1}{2}}}\right\}\right] \leq \frac{\nu'}{n + 1}.$$

Then by mathematical induction $\nu_n \leq \nu'/n$ for all $n \geq \tilde{n}$.

Let $\nu'' = \min\{1, \tilde{n}\nu_{\tilde{n}}\}$. Then $\nu_{\tilde{n}} \geq \nu''/\tilde{n}$. Suppose $\nu_n \geq \nu''/n$ for some $n \geq \tilde{n}$. Since $\nu'' \leq 1$, we have that $I\{\Phi^{-1}(\alpha)/(1 + \nu''/n)^{\frac{1}{2}}\} < 1 \leq (n + \nu'')/((n + 1)\nu'')$. Since $\nu''/n \leq \nu_n \leq$

$\nu'/n \leq \bar{\nu}$, from (A3) we obtain

$$\nu_{n+1} \geq \frac{\nu''}{n} \left[1 - \frac{\nu''}{n + \nu''} I \left\{ \frac{\Phi^{-1}(\alpha)}{(1 + \nu''/n)^{\frac{1}{2}}} \right\} \right] \geq \frac{\nu''}{n+1}.$$

Then by mathematical induction $\nu_n \geq \nu''/n$ for all $n \geq \tilde{n}$.

We have for all $n \geq \tilde{n}$ that

$$\begin{aligned} a_n &= \frac{\nu_n}{\beta(1 + \nu_n)^{\frac{1}{2}}} \frac{\phi\{\Phi^{-1}(\alpha)/(1 + \nu_n)^{\frac{1}{2}}\}}{\Phi\{\Phi^{-1}(\alpha)/(1 + \nu_n)^{\frac{1}{2}}\}[1 - \Phi\{\Phi^{-1}(\alpha)/(1 + \nu_n)^{\frac{1}{2}}\}]} \\ &\leq \frac{\nu'}{\beta(n^2 + n\nu')^{\frac{1}{2}}} \frac{1/\sqrt{(2\pi)}}{\alpha(1 - \alpha)} \leq \frac{\nu'}{\sqrt{(2\pi)\alpha(1 - \alpha)\beta n}}, \\ \text{and } a_n &\geq \frac{\nu''}{\beta(n^2 + n\nu'')^{\frac{1}{2}}} \frac{\phi\{\Phi^{-1}(\alpha)\}}{1/2(1 - 1/2)} \geq \frac{4\nu''\phi(\Phi^{-1}(\alpha))}{\beta(n+1)}. \end{aligned}$$

Thus $\{a_n\}$ satisfies (2). Also $\sum_{j=1}^{n-1} a_j = O(\log n)$ as $n \rightarrow \infty$.

Using Taylor series expansion we obtain

$$b_n = \Phi\left\{\frac{\Phi^{-1}(\alpha)}{(1 + \nu_n)^{\frac{1}{2}}}\right\} = \alpha - \nu_n \frac{\Phi^{-1}(\alpha)}{2} \phi\{\Phi^{-1}(\alpha)\} + O(\nu_n^2).$$

Thus $|b_n - \alpha| = O(1/n)$, which implies that $a_n|b_n - \alpha| \sum_{j=1}^{n-1} a_j = O(\log n/n^2)$ and therefore

(3) holds. Thus, by Theorem 1, $Z_n \rightarrow 0$, in probability. \diamond

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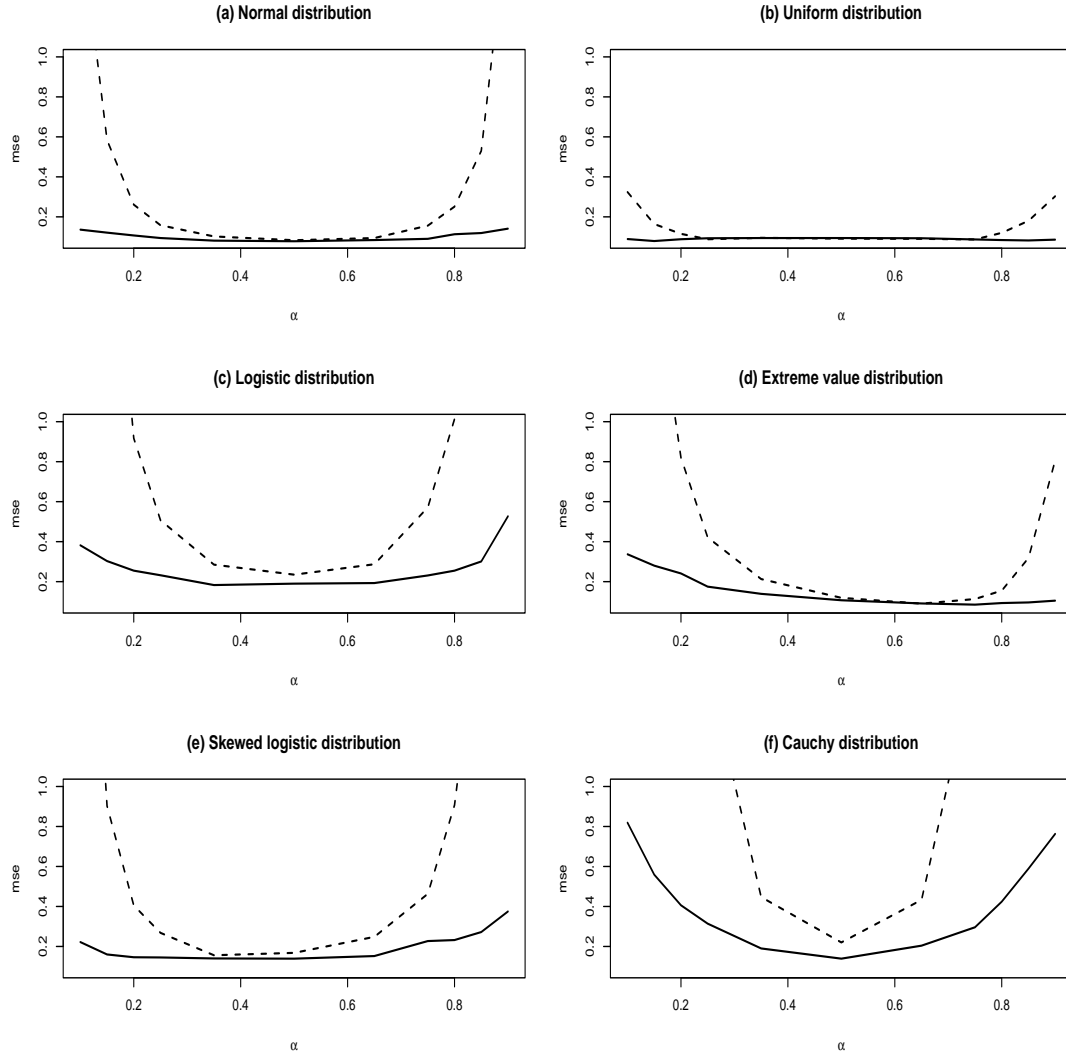


Figure 1: Simulation study. Mean squared error of estimator of θ against α , for the six models, using Robbins-Monro, dashed, and optimal Robbins-Monro, solid.

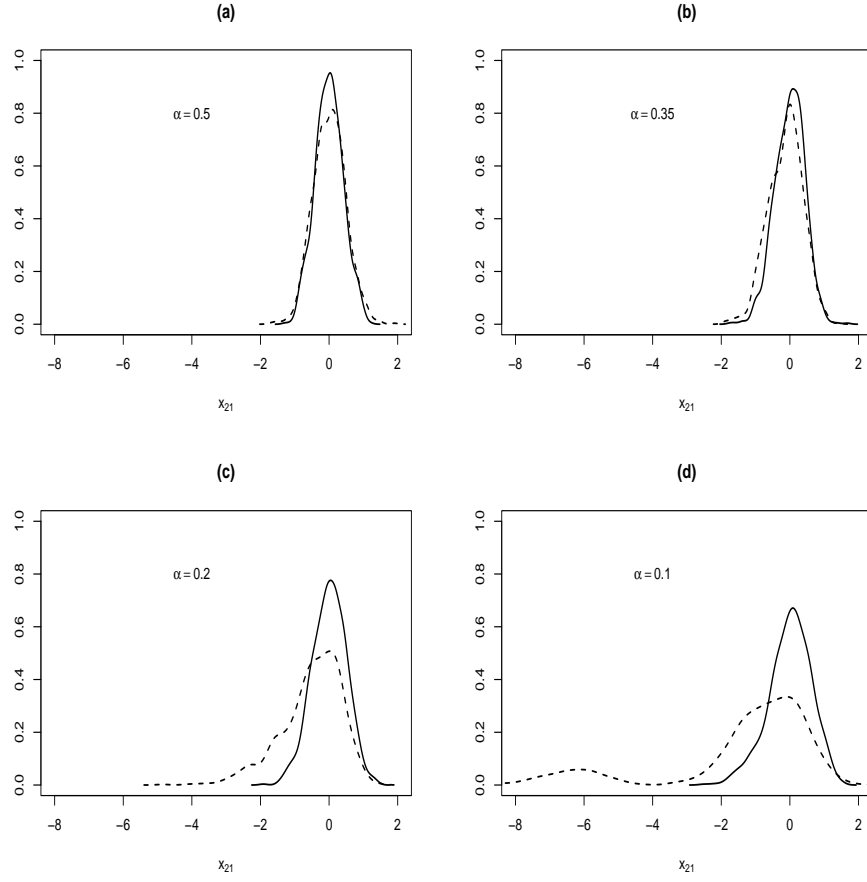


Figure 2: Simulation study. Density curves of the estimates of θ for different α with $M =$ logistic distribution and $\theta = 0$, using Robbins-Monro, dashed, and optimal Robbins-Monro, solid. (a) $\alpha = 0.5$, (b) $\alpha = 0.35$, (c) $\alpha = 0.2$, (d) $\alpha = 0.1$.