RBE502 - Homework Set 4

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Introduction

In this homework set we are looking at the dynamic system of a quadcopter.

The quadrotor operates with a body frame located at its center of mass (in the center of the quadcopter, equidistant and within the same plane as its rotors) $\boldsymbol{c}$. We have a fixed reference frame $\boldsymbol{e}$ serving as our intertial frame.

The external forces and moments on the system are represented by $\boldsymbol{r}$ and $\boldsymbol{n}$, where $\boldsymbol{r} = r_1 c_1 + r_2 c_2 + r_3 c_3$ and $\boldsymbol{n} = n_1 c_1 + n_2 c_2 + n_3 c_3$ directly applied to the center of mass. We are assuming that the torque of the rotor is proportionaly related to the input thrust via the constant $\sigma > 0$, for $\tau_i = \sigma u_i$. We wll be utilizing $\boldsymbol{I}$ as our inertial matrix, where:

$$\boldsymbol{I} = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \tag{1}$$

We have also had established for us that there exists a rotation matrix - $R_{C/E}$ - which rotates from frame C to frame E. This rotaton matrix is the result of a Euler angle $z-y-x$ rotation along ϕ, θ, and ψ, respectively. This rotation matrix is such that:

$$R_{C/E} = \begin{bmatrix} \cos(\psi)\cos(\theta) & \cos(\psi)\sin(\phi)\sin(\theta) - \cos(\phi)\sin(\psi) & \sin(\phi)\sin(\psi) + \cos(\phi)\cos(\psi)\sin(\theta) \\ \cos(\theta)\sin(\psi) & \cos(\phi)\cos(\psi) + \sin(\phi)\sin(\psi)\sin(\theta) & \cos(\phi)\sin(\psi)\sin(\theta) - \cos(\psi)\sin(\phi) \\ -\sin(\theta) & \cos(\theta)\sin(\phi) & \cos(\phi)\cos(\theta) \end{bmatrix} \tag{2}$$

System Analysis

For our problem set, we are proivded that the equations that descirbe our system can be derived via:

$$\dot{x} = \boldsymbol{v} \tag{3}$$

$$\dot{\alpha} = T^{-1}\omega \tag{4}$$

$$\dot{v} = -g\boldsymbol{e}_3 + \frac{1}{m} R_{C/E}(u_1 + u2 + u3 + u4)\boldsymbol{c_3} + R_{C/E}\boldsymbol{r} \tag{5}$$

$$\dot{\omega} = \boldsymbol{I}^{-1}((u_2 - u_4)lc_1 + (u_3 - u_1)lc_2 + (u_1 - u_2 + u_3 - u_4)\sigma c_3 + \boldsymbol{n} - \boldsymbol{\omega} \times \boldsymbol{I}\boldsymbol{\omega}) \tag{6}$$

$$T^{-1} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \frac{\sin\phi}{\cos\theta} & \frac{\cos\phi}{\theta} \end{bmatrix} \tag{7}$$

When we expand upon these functions, we find ourselves with a several matricies that make up a system of equations.

$$\dot{\boldsymbol{x}} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{8}$$

$$\dot{\boldsymbol{\alpha}} = \begin{bmatrix} \omega_1 + \omega_3\, \cos(\phi)\, \tan(\theta) + \omega_2\, \sin(\phi)\, \tan(\theta) \\ \omega_2\, \cos(\phi) - \omega_3\, \sin(\phi) \\ \frac{\omega_3\, \cos(\phi)}{\cos(\theta)} + \frac{\omega_2\, \sin(\phi)}{\cos(\theta)} \end{bmatrix} \tag{9}$$

$$\dot{\boldsymbol{v}} = \begin{bmatrix} \frac{(\sin(\phi)\, \sin(\psi)+\cos(\phi)\, \cos(\psi)\, \sin(\theta))\, (u_1+u_2+u_3+u_4)}{m} \\ -\frac{(\cos(\psi)\, \sin(\phi)-\cos(\phi)\, \sin(\psi)\, \sin(\theta))\, (u_1+u_2+u_3+u_4)}{m} \\ \frac{\cos(\phi)\, \cos(\theta)\, (u_1+u_2+u_3+u_4)}{m} - g \end{bmatrix} \tag{10}$$

$$\dot{\boldsymbol{\omega}} = \begin{bmatrix} \frac{l\, (u_2-u_4)+\mathrm{Iy}\, \omega_2\, \omega_3-\mathrm{Iz}\, \omega_2\, \omega_3}{\mathrm{Ix}} \\ -\frac{l\, (u_1-u_3)+\mathrm{Ix}\, \omega_1\, \omega_3-\mathrm{Iz}\, \omega_1\, \omega_3}{\mathrm{Iy}} \\ \frac{\sigma\, (u_1-u_2+u_3-u_4)+\mathrm{Ix}\, \omega_1\, \omega_2-\mathrm{Iy}\, \omega_1\, \omega_2}{\mathrm{Iz}} \end{bmatrix} \tag{11}$$

Now that we have these system equations, we can append them to form our $\dot{\boldsymbol{z}}$.

$$\dot{\boldsymbol{z}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{\phi}_1 \\ \dot{\theta}_2 \\ \dot{\psi}_3 \\ \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \\ \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \omega_1 + \omega_3\, \cos(\phi)\, \tan(\theta) + \omega_2\, \sin(\phi)\, \tan(\theta) \\ \omega_2\, \cos(\phi) - \omega_3\, \sin(\phi) \\ \frac{\omega_3\, \cos(\phi)}{\cos(\theta)} + \frac{\omega_2\, \sin(\phi)}{\cos(\theta)} \\ \frac{(\sin(\phi)\, \sin(\psi)+\cos(\phi)\, \cos(\psi)\, \sin(\theta))\, (u_1+u_2+u_3+u_4)}{m} \\ -\frac{(\cos(\psi)\, \sin(\phi)-\cos(\phi)\, \sin(\psi)\, \sin(\theta))\, (u_1+u_2+u_3+u_4)}{m} \\ \frac{\cos(\phi)\, \cos(\theta)\, (u_1+u_2+u_3+u_4)}{m} - g \\ \frac{l\, (u_2-u_4)+\mathrm{Iy}\, \omega_2\, \omega_3-\mathrm{Iz}\, \omega_2\, \omega_3}{\mathrm{Ix}} \\ -\frac{l\, (u_1-u_3)+\mathrm{Ix}\, \omega_1\, \omega_3-\mathrm{Iz}\, \omega_1\, \omega_3}{\mathrm{Iy}} \\ \frac{\sigma\, (u_1-u_2+u_3-u_4)+\mathrm{Ix}\, \omega_1\, \omega_2-\mathrm{Iy}\, \omega_1\, \omega_2}{\mathrm{Iz}} \end{bmatrix} \tag{12}$$

Problem 1 - Part A

In this part of the problem, we are considering $\boldsymbol{z}_0 = \begin{bmatrix} x_1 & x_2 & x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$ and $\boldsymbol{u}_0 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T \frac{mg}{4}$. We wish to show that $(z, u) = (z_0, u_0)$ is an equilibrium point of this quadrotor system. Taking these values as defined, and plugging them into our $\dot{\boldsymbol{z}}$ calculates to the simple output of:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \\ \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \\ \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{13}$$

Since we see in the above equations that in this position with the targeted input, we have a system in a stable equilibrium. We can claim as such as there are no non-zero velocities or accelerations at this point.

At any given location in space with no external forces, the drone can hover when perfectly level - ie $\phi = \theta = \psi = 0$ - with no acceleration and orientation change if the resulting thrust is $\frac{mg}{4}$ - or, each motor providing resulting thrust of one fourth the vehicle's weight to cancel the force of gravity.

Problem 1 - Part B

We now wish to find the linear approximation of the quadrotor system using the vacinity of the aforementioned equilibrium point $(\boldsymbol{z_0}, \boldsymbol{u}_0)$.

To do this, we will be denoting z_d as our desired z, and u_d as our *desired* u. We will note then that $e := z_d - z$ and $w := u_d - u$. Using Taylor Serie's approximation we can then create a linear approximation.

$$f(z,u) \approx f(z_d, u_d) + \frac{\partial f}{\partial z}\bigg|_{(z,u)=(z_d,u_d)} (z - z_d) + \frac{\partial f}{\partial u}\bigg|_{(z,u)=(z_d,u_d)} (u - u_d) \tag{14}$$

If we let our linearized approximation have $\boldsymbol{A} := \frac{\partial f}{\partial z}\Big|_{(z,u)=(z_d,u_d)}$ and $\boldsymbol{B} := \frac{\partial f}{\partial u}\Big|_{(z,u)=(z_d,u_d)}$, we can then write an equation for $\dot{e} = \boldsymbol{A}e + \boldsymbol{B}w$. To create our $\boldsymbol{A}$ and $\boldsymbol{B}$, we utilized the Jacobian of our $\dot{z}$, solved prior, to z and u. Thus $\boldsymbol{A}$ is a $12x12$ matrix and $\boldsymbol{B}$ is a $12x4$ matrix. If we utilize the equilibrium point from Part A, we can then plug in z_0 and u_0, defined earlier. Thus:

$$\boldsymbol{A} = \begin{bmatrix} \frac{\partial \dot{z}_1}{\partial x_1} & \frac{\partial \dot{z}_1}{\partial x_2} & \cdots & \frac{\partial \dot{z}_1}{\partial \omega_2} & \frac{\partial \dot{z}_1}{\partial \omega_3} \\ \vdots & & \ddots & & \vdots \\ \frac{\partial \dot{z}_{12}}{\partial x_1} & \frac{\partial \dot{z}_{12}}{\partial x_2} & \cdots & \frac{\partial \dot{z}_{12}}{\partial \omega_2} & \frac{\partial \dot{z}_{12}}{\partial \omega_3} \end{bmatrix} \tag{15}$$

$$\boldsymbol{B} = \begin{bmatrix} \frac{\partial \dot{z}_1}{\partial u_1} & \frac{\partial \dot{z}_1}{\partial u_2} & \frac{\partial \dot{z}_0}{\partial u_3} & \frac{\partial \dot{z}_0}{\partial u_4} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \dot{z}_{12}}{\partial u_1} & \frac{\partial \dot{z}_{12}}{\partial u_2} & \frac{\partial \dot{z}_{12}}{\partial u_3} & \frac{\partial \dot{z}_{12}}{\partial u_4} \end{bmatrix} \tag{16}$$

The $\boldsymbol{A}$ matrix is large, but can be simplified in writing by noting that, for this system it can be divided into four quadrants. Two exist as 0_6, which represents a $6x6$ all-zero matrix. Another, I_6, represents a $6x6$ identity matrix. Finally, our bottom corner which we'll label as A_{21}, is a matrix we'll specifically define below.

$$A_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 & g & 0 \\ 0 & 0 & 0 & -g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{17}$$

$$\boldsymbol{A} = \begin{bmatrix} 0_6 & I_6 \\ A_{21} & 0_6 \end{bmatrix} \tag{18}$$

Our $\boldsymbol{B}$ matrix is defined below:

$$\boldsymbol{B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{m} & \frac{1}{m} & \frac{1}{m} & \frac{1}{m} \\ 0 & \frac{l}{\mathrm{Ix}} & 0 & -\frac{l}{\mathrm{Ix}} \\ -\frac{l}{\mathrm{Iy}} & 0 & \frac{l}{\mathrm{Iy}} & 0 \\ \frac{\sigma}{\mathrm{Iz}} & -\frac{\sigma}{\mathrm{Iz}} & \frac{\sigma}{\mathrm{Iz}} & -\frac{\sigma}{\mathrm{Iz}} \end{bmatrix} \tag{19}$$

We thus have a linearized model for the system with $\dot{e} = \boldsymbol{A}e + \boldsymbol{B}w$.

Problem 1 - Part C

Now we are tasked with determining if our linear system is controllable. To do this, we must check to see if matrix $\boldsymbol{C}$ has full rank. We define $\boldsymbol{C}$ as:

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{A}^0\boldsymbol{B} & \boldsymbol{A}^1\boldsymbol{B} * \boldsymbol{A}^2\boldsymbol{B} & \dots & \boldsymbol{A}^{n-1}\boldsymbol{B} \end{bmatrix} \tag{20}$$

...with $n = 12$ for each value within our $\dot{z}$. This results in a $12x48$ matrix that is far too large to output here. When we look at the rank of this matrix, we see that it is full ranked - $rank(\boldsymbol{C}) = 12$. This tells us that the linearized system is indeed controllable.

Problem 2

In this problem, we let $\dot{x} = \boldsymbol{A}x$, for $x \,\epsilon\, \mathbb{R}$ and $\boldsymbol{A} \,\epsilon\, \mathbb{R}^{nxn}$. We assume $\boldsymbol{A}$ has distinct real eigenvalues $\lambda_i < 0$ for $i = 1, 2, \dots, n$. We wish to prove that $x = 0$ is the only equilibrium point of the system, and that it is also asymptotically stable.

If $\boldsymbol{A}$ has real eigenvalues, we can thus present them as such (shrunk for room):

$$\begin{bmatrix} \dot{\boldsymbol{x}}_1 \\ \dot{\boldsymbol{x}}_2 \\ \dot{\boldsymbol{x}}_3 \\ \vdots \\ \dot{\boldsymbol{x}}_{n-2} \\ \dot{\boldsymbol{x}}_{n-1} \\ \dot{\boldsymbol{x}}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_{n-2} & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & \lambda_{n-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix} \tag{21}$$

From this, we can determine that $\dot{x}_i = \lambda_i(0)$, such that $\dot{x}_i = 0$ for each x_i. We then look at the solution of this system:

$$e^{\boldsymbol{A}t} := \sum_{k=0}^{\infty} \frac{\boldsymbol{A}^k t^k}{k!} \tag{22}$$

which we can write as...

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + c_3 e^{\lambda_3 t} + \dots + c_n e^{\lambda_n t} \tag{23}$$

Since we stated earlier that we assumed that each lambda has a negative real parts in its eigenvalues ($\lambda_i < 0$), this results in:

$$x(t) = c_1 e^{-\infty} + c_2 e^{-\infty} + c_3 e^{-\infty} + \dots + c_n e^{-\infty} \tag{24}$$

...which would push all values to 0. This means that $||x(t)|| \to 0$ as $t \to \infty$.