# RBE 502 HW 1 - Problem 2

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In this problem we look at a system of a 1 dimensional minecart on a frictionless path. Here, the equation for the system is represented as

$$m\dot{v} = u + \phi(t)$$

where $u$ is our control system, set to:

$$u = k_p(s - v) + k_d(0 - \dot{v}) = k_p s - k_p v - k_d \dot{v}$$

which plugged into our systems equation:

$$(m + k_d)\dot{v} + k_p v = k_p s + \phi(t)$$

We will attempt to find the upperbound for $\mid v \mid$ as $t \to \infty$ . To do this, first we shall rewrite our equation in a form of $\frac{dy}{dt} + ay = g(t)$.

$$\dot{v} + \frac{k_p v}{m + k_d} = \frac{k_p s + \phi(t)}{m + k_d}$$

...so our resulting function is

$$v = e^{-\frac{k_p}{m+kd}t} \int e^{\frac{k_p}{m+k_d}t} (\frac{k_p s + \phi(t)}{m + k_d})\, dt + c_1 e^{-\frac{k_p}{m+kd}t}$$

...and if we expand our integral into two separate integrals:

$$v = e^{-\frac{k_p}{m+kd}t} \int \frac{e^{\frac{k_p}{m+k_d}t} s k_p}{m + k_d} dt + e^{-\frac{k_p}{m+kd}t} \int \frac{e^{\frac{k_p}{m+k_d}t} \phi(t)}{m + k_d}\, dt + c_1 e^{-\frac{k_p}{m+kd}t}$$

...where we can easily take care of one of the integrals:

$$v = e^{-\frac{k_p}{m+kd}t} e^{\frac{k_p}{m+k_d}t} s + e^{-\frac{k_p}{m+kd}t} \int \frac{e^{\frac{k_p}{m+k_d}t} \phi(t)}{m + k_d}\, dt + c_1 e^{-\frac{k_p}{m+kd}t}$$

...which simplifies down to:

$$v = s + e^{-\frac{k_p}{m+kd}t} \int \frac{e^{\frac{k_p}{m+k_d}t} \phi(t)}{m + k_d}\, dt + c_1 e^{-\frac{k_p}{m+kd}t}$$

From here, we can utilize integration by parts for our remaining integral. For our purposes:

$$\int uv\,dt = u\int v\,dt - \int(\dot{u}\int v\,dt)\,dt$$

...where $u = \frac{\phi(t)}{m+k_d}$, $\dot{u} = \frac{\dot{\phi}(t)}{m+k_d}$, $v = e^{\frac{k_p}{m+k_d}t}$, and $\int v = \frac{e^{\frac{k_p}{m+k_d}t}(m+k_d)}{k_p}$, thus making our second integral:

$$\frac{\phi(t)}{m+k_d}\frac{e^{\frac{k_p}{m+k_d}t}(m+k_d)}{k_p} - \int\frac{\dot{\phi}(t)}{m+k_d}\frac{e^{\frac{k_p}{m+k_d}t}(m+k_d)}{k_p}$$

which simplifies to

$$\frac{\phi(t)e^{\frac{k_p}{m+k_d}t}}{kp} - \int\frac{\dot{\phi}(t)e^{\frac{k_p}{m+k_d}t}}{k_p}$$

When we plug this back into our prior equation:

$$v = s + e^{-\frac{k_p}{m+kd}t}(\frac{\phi(t)e^{\frac{k_p}{m+k_d}t}}{kp} - \int\frac{\dot{\phi}(t)e^{\frac{k_p}{m+k_d}t}}{k_p}) + c_1e^{-\frac{k_p}{m+kd}t}$$

which simplifies further to:

$$v = s + \frac{\phi(t)}{kp} - e^{-\frac{k_p}{m+kd}t}\int\frac{\dot{\phi}(t)e^{\frac{k_p}{m+k_d}t}}{k_p} + c_1e^{-\frac{k_p}{m+kd}t}$$

We take the limit of our resulting system equation as $t \to \infty$:

$$\lim_{t\to\infty} v(t) = s + \frac{\phi(t)}{k_p} - e^{-\infty}\int\frac{e^{\infty}\dot{\phi}(t)}{m+k_d}\,dt + c_1e^{-\infty}$$

...and since $e^{-\infty} = 0$:

$$\lim_{t\to\infty} v(t) = s + \frac{\phi(t)}{k_p}$$

Since $\phi(t) < a$ as per our description, we can thus say that the upperbound velocity of the given problem is defined by

$$v(t) < s + \frac{a}{k_p}$$