CS534 - HW 1

Keith Chester

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Problem 1

In problem 1, we are tasked with creating a recursive and linear time agent for following propositonal logic statements. The work associated with this problem can be found in *problem1.py*.

Problem 2

In this problem we are exploring a first-order logical knowledge base and writing out logical expressions utilizing it. The knowledge base is represented as:

- CopyOf(d, a) - Predicate - Disk d is a copy of album a
- Owns(p, d) - Predicate - Person p owns disk d
- Sings(p, s, a) - Predicate - Album a inludes a recording of song s sung by person p
- Wrote(p, s) - Person p wrote song s

We are also injecting the following constants:

- $McCartney$ - a person
- $Gershwin$ - a person
- $BHoliday$ - a person
- Joe - a person
- $EleanorRigby$ - a song
- $TheManILove$ - a song
- $Revolver$ - an album

Within this, we express the following statements using first-order logic:

(a) $\text{Wrote}(Gershwin, TheManILove)$

(b) $\neg\,\text{Wrote}(Gershwin, EleanorRigby)$

(c) $\text{Wrote}(Gershwin, TheManILove) \vee \text{Wrote}(McCartney, TheManILove)$

(d) $\exists s\,\text{Wrote}(Joe, s)$

(e) $\exists d\,\text{Owns}(Joe, CopyOf(d, Revolver))$

(f) $\forall s\,\text{Sings}(McCartney, s, Revolver) \Rightarrow \text{Wrote}(McCartney, s)$

(g) $\forall s\,\text{Sings}(p, s, Revolver)\neg\,\text{Wrote}(Gershwin, s)$

(h) $\forall s\,\text{Wrote}(Gershwin, s) \Rightarrow \text{Sings}(p, s, a)$

(i) $\exists a \forall s\, \text{Sings}(p, s, a)\, \text{Wrote}(Joe, s)$

(j) $\exists d, a\, \text{Owns}(Joe, \text{CopyOf}(d, a)) \wedge \text{Sings}(BHoliday, s, a)$

(k) $\exists d_i\, \text{CopyOf}(d_i, a) \forall a\, \text{Sings}(McCartney, a, s) \wedge \forall\, \text{Owns}(Joe, d)$

(l) $\exists d\, \text{CopyOf}(d, a) \forall a\, \text{Sings}(BHoliday, s, a) \wedge \text{Owns}(Joe, d)$

Problem 3

In this queston, we are looking at the following table of three binary input atributes, and a singular binary output:

Example	A_1	A_2	A_3	Output y
x_1	1	0	0	0
x_2	1	0	1	0
x_3	0	1	0	0
x_4	1	1	1	1
x_5	1	1	0	1

a.

Using the Gini Index, we aim to create a decision tree for this data. For Gini Index, we are looking at the probability distribution of a particular variable being wrong when randomly chosen. From this we can begin to branch our tree into a set of decisions that lead us to a particular preditctive result.

$$Gini = 1 - \sum_{i=1}^{n} p_i^2 \tag{1}$$

...where p_i is the probability of a given object being classified into a particular class.

So, calculating the Gini index for our given data:
For A_1, we are look at the indexes of positive and negative outputs:

$$A_1 = 1, y = 1 \rightarrow \frac{2}{4} = 0.5 \tag{2}$$

$$A_1 = 1, y = 0 \rightarrow \frac{2}{4} = 0.5 \tag{3}$$

...for the Gini calculation of:

$$1 - (0.5^2 + 0.5^2) = 0.5 \tag{4}$$

$$A_1 = 0, y = 1 \rightarrow \frac{0}{1} = 0 \tag{5}$$

$$A_1 = 0, y = 0 \rightarrow \frac{1}{1} = 1 \tag{6}$$

...for the Gini calculation of:

$$1 - (0^2 + 1^2) = 0 \tag{7}$$

and the weighted sum of these Gini indexes to create the Gini index for A_1:

$$\sum p_i * g_i = (\frac{4}{5} * 0.5 + \frac{1}{5} * 0) = 0.4 \tag{8}$$

Now we continue this for A_2:

$$A_2 = 1, y = 1 \rightarrow \frac{2}{3} = 0.67 \tag{9}$$

$$A_2 = 1, y = 0 \rightarrow \frac{1}{3} = 0.33 \tag{10}$$

...for the Gini calculation of:

$$1 - (0.67^2 + 0.33^2) = 0.44 \tag{11}$$

$$A_2 = 0, y = 1 \rightarrow \frac{0}{2} = 0 \tag{12}$$

$$A_2 = 0, y = 0 \rightarrow \frac{2}{2} = 1 \tag{13}$$

...for the Gini calculation of:

$$1 - (0^2 + 1^2) = 0 \tag{14}$$

and the weighted sum of these Gini indexes to create the Gini index for A_2:

$$\sum p_i * g_i = (\frac{3}{5} * 0.44 + \frac{2}{5} * 0) = 0.27 \tag{15}$$

...and finally, following the process for A_3:

$$A_3 = 1, y = 1 \rightarrow \frac{1}{2} = 0.5 \tag{16}$$

$$A_3 = 1, y = 0 \rightarrow \frac{1}{2} = 0.5 \tag{17}$$

...for the Gini calculation of:

$$1 - (0.5^2 + 0.5^2) = 0.5 \tag{18}$$

$$A_3 = 0, y = 1 \rightarrow \frac{1}{3} = 0.33 \tag{19}$$

$$A_3 = 0, y = 0 \rightarrow \frac{2}{3} = 0.67 \tag{20}$$

...for the Gini calculation of:

$$1 - (0.33^2 + 0.67^2) = 0.44 \tag{21}$$

and the weighted sum of these Gini indexes to create the Gini index for A_3:

$$\sum p_i * g_i = (\frac{2}{5} * 0.5 + \frac{3}{5} * 0.44) = 0.27 \tag{22}$$

So the Gini indexes for each input is

Input	Gini index
A_1	0.4
A_2	0.27
A_3	0.27

Both A_3 and A_2 have the same Gini index, thus either can be utilized as the root node of our decision tree. For our purposes, we will choose A_3. From this we repeat the computations above for the positive and negative branches of our A_3 input.

Positive A_3:

Example	A_1	A_2	A_3	Output y
x_2	1	0	1	0
x_4	1	1	1	1

Gini Index of $A_1 = 1$ when $A_3 = 1$:

$$A_1 = 1, y = 1 \rightarrow \frac{1}{2} = 0.5 \tag{23}$$

$$A_1 = 1, y = 0 \rightarrow \frac{1}{2} = 0.5 \tag{24}$$

$$1 - (0.5^2 + 0.5^2) = 0.5 \tag{25}$$

Gini Index of $A_1 = 0$ when $A_3 = 1$ is not applicable, as there is no examples anymore of $A_1 = 0$. Thus our weighted Gini Index is still 0.5.

Gini Index of $A_2 = 1$ when $A_3 = 1$:

$$A_2 = 1, y = 1 \rightarrow \frac{1}{1} = 1.0 \tag{26}$$

$$A_2 = 1, y = 0 \rightarrow \frac{0}{1} = 0.0 \tag{27}$$

$$1 - (1.0^2 + 0.0^2) = 0.0 \tag{28}$$

Ginin Index of $A_2 = 0$ when $A_3 = 1$:

$$A_2 = 1, y = 1 \rightarrow \frac{0}{1} = 0.0 \tag{29}$$

$$A_2 = 1, y = 0 \rightarrow \frac{1}{1} = 1.0 \tag{30}$$

$$1 - (0^2 + 1.0^2) = 0.0 \tag{31}$$

...for a Gini index, when weighted, to still be 0.0 for A_2.

Input	Gini index
A_1	0.5
A_2	0.0

...so we see that we want to use A_2 for the next split as it has the lowest Gini index, and completes this branch of the tree as we have a perfect split.

Negative A_3:

Example	A_1	A_2	A_3	Output y
x_1	1	0	0	0
x_3	0	1	0	0
x_5	1	1	0	1

Gini Index of $A_1 = 1$ when $A_3 = 0$:

$$A_1 = 1, y = 1 \rightarrow \frac{1}{2} = 0.5 \tag{32}$$

$$A_1 = 1, y = 0 \rightarrow \frac{1}{2} = 0.5 \tag{33}$$

$$1 - (0.5^2 + 0.5^2) = 0.5 \tag{34}$$

Gini Index of $A_1 = 0$ when $A_3 = 0$:

$$A_1 = 1, y = 1 \rightarrow \frac{0}{1} = 0 \tag{35}$$

$$A_1 = 1, y = 0 \rightarrow \frac{1}{1} = 1.0 \tag{36}$$

$$1 - (1.0^2 + 0.0^2) = 0.0 \tag{37}$$

...for the weighted sum of a Gini index of A_1 when $A_3 = 0$:

$$\sum p_i * g_i = (\frac{2}{3} * 0.5 + \frac{1}{3} * 0) = 0.33 \tag{38}$$

Moving onto the Gini Index of A_2 when $A_3 = 0$, we first explore when $A_2 = 1$:

$$A_2 = 1, y = 1 \rightarrow \frac{1}{2} = 0.5 \tag{39}$$

$$A_2 = 1, y = 0 \rightarrow \frac{1}{2} = 0.5 \tag{40}$$

$$1 - (0.5^2 + 0.5^2) = 0.5 \tag{41}$$

Looking at $A_2 = 0$ when $A_3 = 0$:

$$A_2 = 1, y = 1 \rightarrow \frac{0}{1} = 0.0 \tag{42}$$

$$A_2 = 1, y = 0 \rightarrow \frac{1}{1} = 1.0 \tag{43}$$

$$1 - (1.0^2 + 0.0^2) = 0.0 \tag{44}$$

...and our weight index for A_2 when $A_3 = 0$ is:

$$\sum p_i * g_i = (\frac{2}{3} * 0.5 + \frac{1}{3} * 0) = 0.33 \tag{45}$$

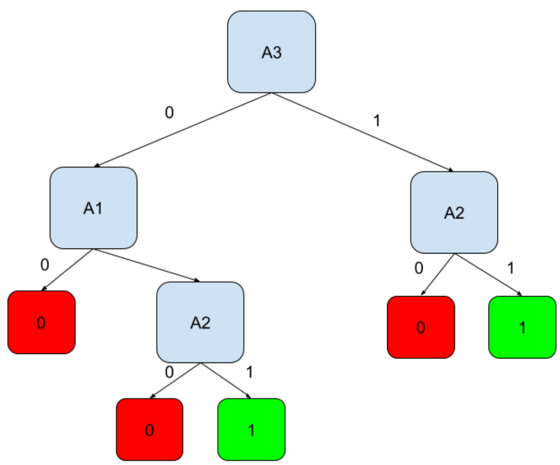
for our collected Gini indexs being:

Input	Gini index
A_1	0.33
A_2	0.33

...thus making it inconclusive which of A_1 or A_2 we could branch off of. We shall choose A_1 for our purposes. One more branch to go!

This next branch doesn't need much in the way of calculations; to save time, we can just observe that when $A_3 = 0$ and $A_1 = 1$, $A_2 = y$. When $A_3 = 0$ and $A_1 = 0$, $y = 0$.

Thus we exhaustively formed the following tree:



b.

Now we utilize information gain to create a decision tree for this data. Here we will be calculating the amount of entropy for each branch. The highest information gain is the columns with the least entropy. We calculate entropy via:

$$-\sum ci = 1p(x_i) \log_b p(x_i) \tag{46}$$

...then, when we look at the potential splitting of that column, we look at the information gain from the column via looking at the change in entropy and looking at the weighted sum of post-split entropy and comparing it back to our original entropy. We are looking for larger information gain.

$$\sum^{V \epsilon A} \frac{|T_V|}{T} * Entropy(T_V) \tag{47}$$

Looking at our table once again:

Example	A_1	A_2	A_3	Output y
x_1	1	0	0	0
x_2	1	0	1	0
x_3	0	1	0	0
x_4	1	1	1	1
x_5	1	1	0	1

...we can begin calculating the information gain for each node A_i.
For A_1:

$$-(\frac{4}{5}\log_2\frac{4}{5}) + -(\frac{1}{5}\log_2\frac{1}{5}) = 0.72 \tag{48}$$

...and then we look at the information gain for a potential split on A_1:

Example	A_1	A_2	A_3	Output y
x_1	1	0	0	0
x_2	1	0	1	0
x_4	1	1	1	1
x_5	1	1	0	1

$$-(\frac{2}{4}\log_2\frac{2}{4}) + -(\frac{2}{4}\log_2\frac{2}{4}) = 1 \tag{49}$$

Example	A_1	A_2	A_3	Output y
x_3	0	1	0	0

$$-(\frac{0}{1}\log_2\frac{0}{1}) + -(\frac{1}{1}\log_2\frac{1}{1}) = 0 \tag{50}$$

$$\frac{4}{5} * 1 + \frac{1}{5} * 0 = 0.8 \tag{51}$$

For A_2:

$$-(\frac{3}{5}\log_2\frac{3}{5}) + -(\frac{2/5}{\log_2}\frac{2}{5}) = 0.97 \tag{52}$$

...and then we look for the information gain if we split on A_2:

Example	A_1	A_2	A_3	Output y
x_3	0	1	0	0
x_4	1	1	1	1
x_5	1	1	0	1

$$-(\frac{2}{3}\log_2\frac{2}{3}) + -(\frac{1}{3}\log_2\frac{1}{3}) = 0.92 \tag{53}$$

Example	A_1	A_2	A_3	Output y
x_1	1	0	0	0
x_2	1	0	1	0

$$-(\frac{0}{2}\log_2\frac{0}{2}) + -(\frac{2}{2}\log_2\frac{2}{2}) = 0.0 \tag{54}$$

$$\frac{3}{5} * 0.92 + \frac{2}{5} * 0 = 0.55 \tag{55}$$

For A_3:

$$-(\frac{2}{5}\log_2\frac{2}{5}) + -(\frac{3/5}{\log_2}\frac{3}{5}) = 0.97 \tag{56}$$

...and now exploring the information gain if we split on A_3:

Example	A_1	A_2	A_3	Output y
x_2	1	0	1	0
x_4	1	1	1	1

$$-(\frac{1}{2}\log_2\frac{1}{2})+-(\frac{1}{2}\log_2\frac{1}{2})=1.0 \tag{57}$$

Example	A_1	A_2	A_3	Output y
x_1	1	0	0	0
x_3	0	1	0	0
x_5	1	1	0	1

$$-(\frac{1}{3}\log_2\frac{1}{3})+-(\frac{2}{3}\log_2\frac{2}{3})=0.92 \tag{58}$$

$$\frac{2}{5}*1.0+\frac{3}{5}*0.92=0.95 \tag{59}$$

...which leads us to this entropy for each A_i and the information gain if we split on each.

Input	Entropy	Split Entropy	Gain
A_1	0.72	0.80	-0.08
A_2	0.97	0.55	0.42
A_3	0.97	0.95	0.02

From this we can see we have the highest information gain if we split on A_2. We continue this process for each of the resulting branches from this split.

$A_2 = 1$

Example	A_1	A_2	A_3	Output y
x_3	0	1	0	0
x_4	1	1	1	1
x_5	1	1	0	1

A_1**:**

$$-(\frac{2}{3}\log_2\frac{2}{3})+-(\frac{1}{3}\log_2\frac{1}{3})=0.92 \tag{60}$$

...and now exploring the information gain if we split on A_1:

Example	A_1	A_2	A_3	Output y
x_4	1	1	1	1
x_5	1	1	0	1

$$-(\frac{2}{2}\log_2\frac{2}{2})+-(\frac{0}{2}\log_2\frac{0}{2})=0.0 \tag{61}$$

Example	A_1	A_2	A_3	Output y
x_3	0	1	0	0

$$-(\frac{0}{1}\log_2\frac{0}{1})+-(\frac{1}{1}\log_2\frac{1}{1})=0.0 \tag{62}$$

for an obvious weighted entropy of 0.0.

A_3**:**

$$-(\frac{1}{3}\log_2\frac{1}{3})+-(\frac{2}{3}\log_2\frac{2}{3})=0.92 \tag{63}$$

...and now exploring the information gain if we split on A_3:

Example	A_1	A_2	A_3	Output y
x_4	1	1	1	1

$$-(\frac{1}{1}\log_2\frac{1}{1}) + -(\frac{1}{1}\log_2\frac{0}{1}) = 0.0 \tag{64}$$

Example	A_1	A_2	A_3	Output y
x_3	0	1	0	0
x_5	1	1	0	1

$$-(\frac{1}{2}\log_2\frac{1}{2}) + -(\frac{1}{2}\log_2\frac{1}{2}) = 1.0 \tag{65}$$

$$\frac{1}{3} * 0.0 + \frac{2}{3} * 1.0 = 0.67 \tag{66}$$

Input	Entropy	Split Entropy	Gain
A_1	0.92	0.0	0.92
A_3	0.92	0.67	0.25

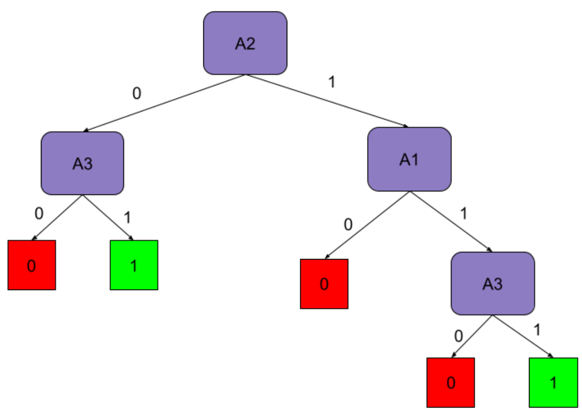
Thus we would further split on A_1. This can be mapped directly to $A_3 = y$ at this point as we've hit minimum entropy, and end this node here.

Then we can begin to explore what happens for when $A_2 = 0$, based on our first split.

$A_2 = 0$**:**

Example	A_1	A_2	A_3	Output y
x_1	1	0	0	0
x_2	1	0	1	0

Here we don't even have to calculate much - if $A_3 = 0$, then $y = 0$.



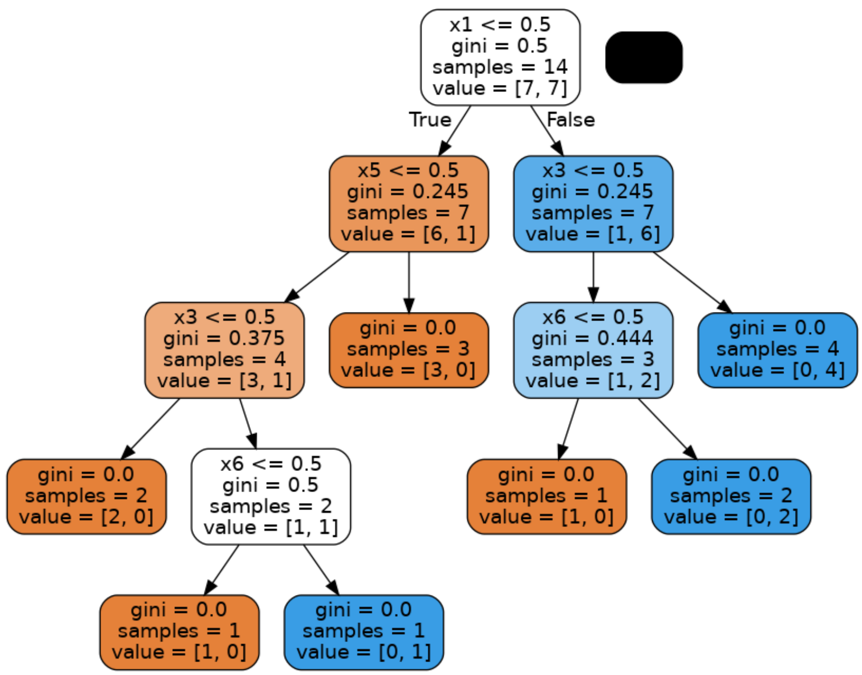
Problem 4

In this section, we consider the following data input with six inputs and a singular target output:

Example	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_10	A_11	A_12	A_13	A_14
x_1	1	1	1	1	1	1	1	0	0	0	0	0	0	0
x_2	0	0	0	1	1	0	0	1	1	0	1	0	1	1
x_3	1	1	1	0	1	0	0	1	1	0	0	0	1	1
x_4	0	1	0	0	1	0	0	1	0	1	1	1	0	1
x_5	0	0	1	1	0	1	1	0	1	1	0	0	1	0
x_6	0	0	0	1	0	1	0	1	1	0	1	1	1	0
T	1	1	1	1	1	1	0	1	0	0	0	0	0	0

When we run our code found in *problem4.py*, we find that we can train both a perceptron and decision tree on this data. If given unseen data of $A_{15} :< 1, 1, 0, 0, 1, 1 >$ with result $T_{15} = 1$, we find that both accurately predict this result.

When we generate the decision tree, we can output its resulting branches, seen as follows:



Our code that utilizes the perceptron makes a perceptron makes a perceptron enacting the following linear equation:

$$T = 8x_1 + 0x_2 + 4x_3 - x_4 - 54x_5 + 3x_6 - 4 \tag{67}$$