RBE549 - Homework 12

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Problem 1

In stero imaging, if the rays define by $\vec{X}_L$ and $\vec{X}_R$ do not intersect, we can find $\vec{X}_W$ anyway by minimizing an error measure. One way to do this is to project $\vec{X}_W$ into the left and the right image planes to give $\vec{X}'_L$ and $\vec{X}'_R$. The error E is defined as the squared difference between the observed image points $\vec{X}_L$ $\vec{X}_R$ and the projected points $\vec{X}'_L$ and $\vec{X}'_R$.

A

In this problem, we are asked to show that for the simplified parallel optical axis camera geometry used in class where $\vec{f} \cdot \vec{B} = 0, R = I, \vec{T} = \pm\frac{\vec{B}}{2}$.
We accomplish this by starting with the given definition of error:

$$E = |\vec{X}'_L - \vec{X}_L|^2 + |\vec{X}'_R - \vec{X}_R|^2 \tag{1}$$

...which we can expand the definition to mean literally:

$$E = (x'_L - x_L)^2 + (y'_L - y_L)^2 + (z'_L - z_L)^2 + (x'_R - x_R)^2 + (y'_R - y_R)^2 + (z'_R - z_R)^2 \tag{2}$$

Since we know that X'_L and X'_R are projections we can use the imaging equation with an offset for the camera from x_W by $\frac{\pm b}{2}$.

$$x'_R = \frac{f}{z_W}\left(x^W - \frac{b}{2}\right) \tag{3}$$

$$y'_R = \frac{f}{z_W}\left(y^W\right) \tag{4}$$

$$x'_L = \frac{f}{z_W}\left(x^W + \frac{b}{2}\right) \tag{5}$$

$$y'_L = \frac{f}{z_w}\left(y^W\right) \tag{6}$$

...this allows us to do some introductions:

$$E = (\frac{f}{z^W}\left(x^W + \frac{b}{2}\right) - x_L)^2 + (\frac{f}{z^W}\left(y^W\right) - y_L)^2 + (z'_L - z_L)^2 + (x'_R = \frac{f}{z^W}\left(x^W - \frac{b}{2}\right) - x_R)^2 + (\frac{f}{z^W}\left(y^W\right) - y_R)^2 + (z'_R - z_R)^2 \tag{7}$$

...and we can note that $z'_L = z_L$ and $z'_R = z_R$:

$$E = (\frac{f}{z^W}\left(x^W + \frac{b}{2}\right) - x_L)^2 + (\frac{f}{z^W}\left(y^W\right) - y_L)^2 + (x'_R = \frac{f}{z^W}\left(x^W - \frac{b}{2}\right) - x_R)^2 + (\frac{f}{z^W}\left(y^W\right) - y_R)^2 \tag{8}$$

B

In this problem, we are asked to show that, by differentiating E with respect to x^W and y^W, to show that:

$$x^W = \frac{x_L + x_R}{2}\frac{z^W}{f}, y^W = \frac{y_l + y_R}{2}\frac{z^W}{f} \tag{9}$$

To do this, we take the partial derivative $\frac{\partial}{\partial x^W}E$ and $\frac{\partial}{\partial y^W}E$:

$$\frac{\partial}{\partial x^W}E = \frac{2f}{z^W}\big(\frac{f}{z^W}(x^W + \frac{b}{2}) - x_L\big) + \frac{2f}{z^W}\big(\frac{f}{z^W}(x^W - \frac{b}{2}) - x_R\big) \tag{10}$$

$$\frac{\partial}{\partial x^W}E = \frac{2f}{z^W}\big(\frac{2}{z^W}x^W - x_L - x_R\big) \tag{11}$$

...We can do a similar approach for our $\frac{\partial}{\partial y^W}E$:

$$\frac{\partial}{\partial x^W}E = \frac{2f}{z^W}\big(\frac{f}{z^W}(y^W + \frac{b}{2}) - y_L\big) + \frac{2f}{z^W}\big(\frac{f}{z^W}(y^W - \frac{b}{2}) - y_R\big) \tag{12}$$

$$\frac{\partial}{\partial y^W}E = \frac{2f}{z^W}\big(\frac{2}{z^W}y^W - y_L - y_R\big) \tag{13}$$

Since these are errors, we set them to 0 for minimizing:

$$0 = \frac{2f}{z^W}\big(\frac{2}{z^W}x^W - x_L - x_R\big) \tag{14}$$

$$0 = \frac{2}{z^W}x^W - x_L - x_R \tag{15}$$

$$\frac{2}{z^W}x^W = x_L + x_R \tag{16}$$

$$\frac{x_L + x_R}{2}\frac{z^W}{f} = x^W \tag{17}$$

...and similarly, we'd fine for y^W:

$$\frac{y_L + y_R}{2}\frac{z^W}{f} = y^W \tag{18}$$

C

In this problem, we are asked to show that

$$z^W = \frac{fb}{\Delta x} \tag{19}$$

...concluding that

$$\vec{X}^W = \vec{X}_{AVG}\frac{|\vec{B}|^2}{\vec{B}\vec{\Delta}} \tag{20}$$

We begin by looking at our function E again:

$$E = (\frac{f}{z^W}(x^W + \frac{b}{2}) - x_L)^2 + (\frac{f}{z^W}(y^W) - y_L)^2 + (x_R' = \frac{f}{z^W}(x^W - \frac{b}{2}) - x_R)^2 + (\frac{f}{z^W}(y^W) - y_R)^2 \tag{21}$$

...and then take the partial derivative $\frac{\partial}{\partial z^W}E$:

$$\frac{\partial}{\partial z^W}E = \frac{2f}{z^w}\big(\frac{f}{z^W}(x^W+\frac{b}{2})-x_L\big)\big(x^W+\frac{b}{2}\big)+\frac{2f}{z^w}\big(\frac{f}{z^W}y^W-y_L\big)y^W+\frac{2f}{z^w}\big(\frac{f}{z^W}(x^W-\frac{b}{2}-x_R)\big)\big(x^W-\frac{b}{2}\big)+\frac{2f}{z^w}\big(y^W-y_R\big)y^W \tag{22}$$

$$
\begin{aligned}
\frac{\partial}{\partial z^W}E = \frac{2f}{z^w}\Bigg(& ((\frac{f}{z^w}x^W + \frac{f}{z^w}\frac{b}{2}) - x_L)(x^W + \frac{b}{2}) + ((\frac{f}{z^w}y^W) - y_L)y^W + \\
& ((\frac{f}{z^w}x^W - \frac{f}{z^w}\frac{b}{2}) - x_R)(x^W - \frac{b}{2}) + ((\frac{f}{z^w}y^W - \frac{f}{z^w}\frac{b}{2}) - y_R)(y^W)\Bigg)
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
\frac{2f}{z^w}\Bigg(& (\frac{f}{z^w}x^{W^2} + \frac{f}{z^w}\frac{b}{2}x^W - x_L x^W) + (\frac{b}{2}\frac{f}{z^w}x^W + \frac{f}{z^w}\frac{b^2}{4} - \frac{b}{2}x_L) + (\frac{f}{z^w}y^{W^2} + \frac{f}{z^w}\frac{b}{2}y^W - y_L y^W) + \\
& (\frac{f}{z^w}X^{W^2} - \frac{f}{z^w}\frac{b}{2}x^W - x^W x_R) + (-\frac{f}{z^w}\frac{b}{2}x^W + \frac{f}{z^w}\frac{b^2}{4} + \frac{b}{2}x_R)(\frac{f}{z^w}y^{W^2} - \frac{f}{z^w}\frac{b}{2}y^W - y_R y^W)\Bigg)
\end{aligned} \tag{24}
$$

We can then gather like terms, factor out $\frac{2f}{z^W}$, and set to 0:

$$
0 = \frac{2f}{z^W}\left(x^{W^2} + y^{W^2}\right) - x^W(x_L + x_R) - y^W(y_L + y_R) + \frac{fb^2}{2z^W} + \frac{x_R b}{2} - \frac{x_L b}{2} \tag{25}
$$

...if you recall, we can use our equation replacements from part B here:

$$
\begin{aligned}
0 = \frac{2f}{z^W}\Bigg(& (\frac{x_L + x_R}{2}\frac{z^W}{f}) + (\frac{y_L + y_R}{2}\frac{z^W}{f}) - (\frac{x_L + x_R}{2}\frac{z^W}{f})(x_L + x_R) - \\
& (\frac{y_L + y_R}{2}\frac{z^W}{f})(y_L + y_R) + \frac{fb^2}{2z^W} + \frac{x_R b}{2} - \frac{x_L b}{2}\Bigg)
\end{aligned} \tag{26}
$$

We can simplify, due to massive cancellations, to:

$$
0 = \frac{fb^2}{2z^W} + \frac{x_R b}{2} - \frac{x_L b}{2} \tag{27}
$$

$$
\frac{fb}{z^W} = x_L - x_R \tag{28}
$$

$$
z^W = \frac{fb}{x_L - x_R} \tag{29}
$$

We can define $\Delta x = x_L - x_R$:

$$
z^W = \frac{fb}{\Delta x} \tag{30}
$$

We can then plug into our definitions from above:

$$
\vec{X}^W = \begin{bmatrix} \frac{x_L + x_R}{2}\frac{z^W}{f} \\ \frac{y_L + y_R}{2}\frac{z^W}{f} \\ \frac{fb}{\Delta x} \end{bmatrix} \tag{31}
$$

We can pull out the $\frac{z^W}{f}$:

$$
\vec{X}^W = \frac{z^W}{f}\begin{bmatrix} \frac{x_L + x_R}{2} \\ \frac{y_L + y_R}{2} \\ \frac{fb}{\Delta x}\frac{f}{z^W} \end{bmatrix} \tag{32}
$$

As we stated before, $f = \frac{z_L + z_R}{2}$, so:

$$
\vec{X}^W = \frac{fb}{\frac{\Delta x}{f}}\begin{bmatrix} \frac{x_L + x_R}{2} \\ \frac{y_L + y_R}{2} \\ \frac{z_L + z_R}{2} \end{bmatrix} \tag{33}
$$

...which can be expressed as what we set out to show:

$$
\begin{bmatrix} x^W \\ y^W \\ z^W \end{bmatrix} = \vec{X}_{AVG}\frac{|\vec{B}|^2}{B\Delta x} \tag{34}
$$

1 Problem 2

Another way to determine distance when $\vec{X}_L$ and $\vec{X}_R$ do not intersect is to find the point $\vec{X}^W$ closest to the rays $\vec{X}_L$ and $\vec{X}_R$. Here we define the left image ray as:

$$\vec{X}_L^W = -\frac{\vec{B}}{2} + l\vec{X}_L, 0 \leq l \tag{35}$$

...and the right image ray as:

$$\vec{X}_R^W = \frac{\vec{B}}{2} + r\vec{X}_R, 0 \leq r \tag{36}$$

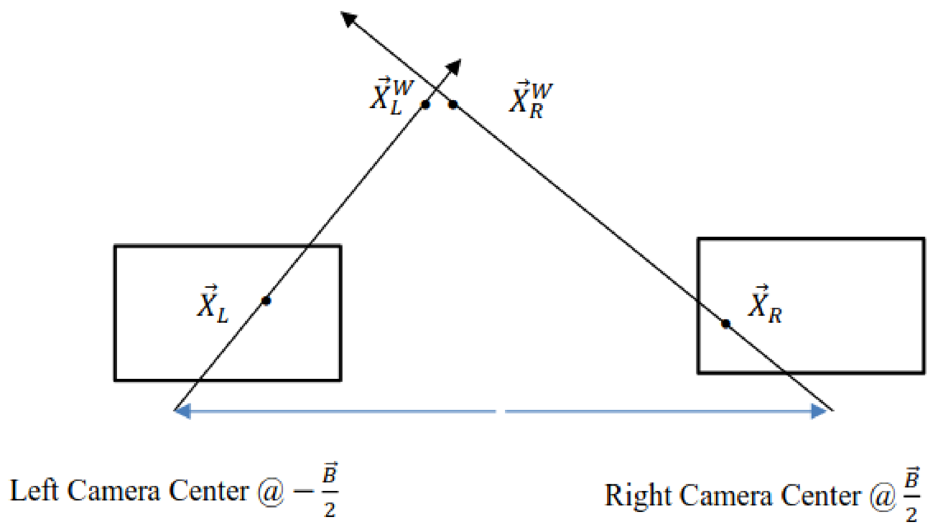

Figure 1: Problem 2

The goal is to find points X_L^W and X_R^W on the left and right image rays that are closest to each other; the point equidistant between them is the desired $\vec{X}^W$. We can find these points by minimizing:

$$min_{l,r}|\vec{X}_L^W - \vec{X}_L^W| \tag{37}$$

...show that the solution is given by:

$$\vec{X}^W = \frac{1}{2}\frac{\big(|\vec{X}_R|^2(\vec{X}_L\vec{B}) - (\vec{X}_L\vec{X}_R)(\vec{X}_R\vec{B})\big)\vec{X}_L + \big((\vec{X}_L\vec{X}_R)(\vec{X}_L\vec{B}) - |\vec{X}_L|^2(\vec{X}_R\vec{B})\big)\vec{X}_R}{|\vec{X}_L|^2|\vec{X}_R|^2 - (\vec{X}_L\vec{X}_R)^w} \tag{38}$$

To approach this, we will start by taking the ∂l and ∂r, then set it equal to 0. Starting with $|X_L^W - X_R^W|^2$, we can subsibtute:

$$|(\frac{-\vec{B}}{2} + l\vec{X}_L) - (\frac{\vec{B}}{2} + r\vec{X}_R)|^2 \tag{39}$$

$$\big(-\frac{\vec{B}}{2} + l\vec{X}_L\big)^2 - 2\big(-\frac{\vec{B}}{2} + l\vec{X}_L\big)\big(\frac{\vec{B}}{2} + r\vec{X}_R\big) + \big(\frac{\vec{B}}{2} + r\vec{X}_R\big)^2 \tag{40}$$

We can fenagle this with algebra to get:

$$\begin{aligned}\frac{|\vec{B}|^2}{2} - \vec{B}l\vec{X}_L + l^2|\vec{X}_L|^2 + \frac{2|\vec{B}|^2}{4} - l\vec{X}_L\vec{B} + \vec{B}r\vec{X}_R \\ -2l\vec{X}_Lr\vec{X}_R + \frac{|\vec{B}|^2}{4} + \vec{B}r\vec{X}_R + r^2|\vec{X}_R|^2\end{aligned} \tag{41}$$

$$|\vec{B}|^2 - 2\vec{B}\vec{X}_L + l^2|\vec{X}_L|^2 - 2l(\vec{X}_L\vec{X}_R) + 2r(\vec{B}\vec{X}_R) + r^2|\vec{X}_R|^2 \tag{42}$$

Now we take ∂l of above:

$$0 = -2\vec{B}\vec{X}_L + 2l|\vec{X}_L|^2 - 2r\vec{X}_L\vec{X}_R \tag{43}$$

Now we solve for r:

$$r = \frac{2\vec{B}\vec{X}_L - 2l|\vec{X}_L|^2}{-2\vec{X}_L\vec{X}_R} \tag{44}$$

We can now subsibtute to solve:

$$\frac{2\vec{B}\vec{X}_L - 2l|\vec{X}_L|^2}{-2\vec{X}_L\vec{X}_R} = \frac{2l\vec{X}_L\vec{X}_R - 2\vec{B}\vec{X}_r}{2|\vec{X}_R|^2} \tag{45}$$

Solving for l in MATLAB:

$$l = \frac{\left(|\vec{X}_R|^2(\vec{B}\cdot\vec{X}_L) - (\vec{X}_L\cdot\vec{X}_R)(\vec{B}\cdot\vec{X}_r)\right)}{|\vec{X}_R|^2|\vec{X}_L|^2 - (\vec{X}_L\cdot\vec{X}_R)^2} \tag{46}$$

$$l = \frac{2(\vec{X}_R\cdot\vec{B}) + 2r|\vec{X}_R|^2}{2(\vec{X}_L\cdot\vec{X}_R)} \tag{47}$$

and having MATLAB solve for r:

$$r = \frac{(\vec{X}_L\cdot\vec{X}_R)(\vec{X}_L\vec{B}) - |\vec{X}_L|^2(\vec{X}_R\cdot\vec{B})}{|\vec{X}_L|^2|\vec{X}_R|^2 - (\vec{X}_L\cdot\vec{X}_R)^2} \tag{48}$$

We can then subsibtute this back into our equations for $\vec{X}^W_{L/R}$:

$$\vec{X}^W_R = \frac{B}{2} + \frac{(\vec{X}_L\cdot\vec{X}_R)(\vec{X}_L\vec{B}) - |\vec{X}_L|^2(\vec{X}_R\cdot\vec{B})}{|\vec{X}_L|^2|\vec{X}_R|^2 - (\vec{X}_L\cdot\vec{X}_R)^2}\vec{X}_R \tag{49}$$

$$\vec{X}^W_L = -\frac{B}{2} + \frac{(\vec{X}_L\cdot\vec{X}_R)(\vec{X}_L\vec{B}) - |\vec{X}_L|^2(\vec{X}_R\cdot\vec{B})}{|\vec{X}_L|^2|\vec{X}_R|^2 - (\vec{X}_L\cdot\vec{X}_R)^2}\vec{X}_L \tag{50}$$

Since we're looking for the midway point, it'd be defined as $\frac{1}{2}$ between these:

$$\frac{1}{2}\left(-\frac{B}{2} + \frac{(\vec{X}_L\cdot\vec{X}_R)(\vec{X}_L\vec{B}) - |\vec{X}_L|^2(\vec{X}_R\cdot\vec{B})}{|\vec{X}_L|^2|\vec{X}_R|^2 - (\vec{X}_L\cdot\vec{X}_R)^2}\vec{X}_L + \frac{B}{2} + \frac{(\vec{X}_L\cdot\vec{X}_R)(\vec{X}_L\vec{B}) - |\vec{X}_L|^2(\vec{X}_R\cdot\vec{B})}{|\vec{X}_L|^2|\vec{X}_R|^2 - (\vec{X}_L\cdot\vec{X}_R)^2}\vec{X}_R\right) \tag{51}$$

...and our $\frac{\pm\vec{B}}{2}$ cancels:

$$\frac{1}{2}\left(\frac{(\vec{X}_L\cdot\vec{X}_R)(\vec{X}_L\vec{B}) - |\vec{X}_L|^2(\vec{X}_R\cdot\vec{B})}{|\vec{X}_L|^2|\vec{X}_R|^2 - (\vec{X}_L\cdot\vec{X}_R)^2}\vec{X}_L + \frac{(\vec{X}_L\cdot\vec{X}_R)(\vec{X}_L\vec{B}) - |\vec{X}_L|^2(\vec{X}_R\cdot\vec{B})}{|\vec{X}_L|^2|\vec{X}_R|^2 - (\vec{X}_L\cdot\vec{X}_R)^2}\vec{X}_R\right) \tag{52}$$

...which is what we sought to find!