RBE549 - Homework 8

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Problem 1

In this problem we are asked if we have two binary images, I_1 and I_2, we want to show that that $|I_1 - I_2|^2 = \sum$ # of all pixels where $I_1 \neq I_2$, with $|I|^2 = \sum i_{jk}^2$ as the sum of all pixels squared in I.

First, we define that binary images are images that have the possible values of $(0, 1)$. Thus we have only a few possibilities for any given pixel amongst our function.

- $I_1 = 0$, $I_2 = 0$: 0
- $I_1 = 1$, $I_2 = 0$: 1
- $I_1 = 1$, $I_2 = 1$: 0
- $I_1 = 1$, $I_2 = 0$: 1, as we square the result of the absolute

Thus we see that, because of the $|result|^2$, wherever both I_1 and I_2 are 1 where the other is 0, we end up with a 1 value. Wherever $I_1 \cup I_2$, or both are 1, we result in an outcome of 0.

Problem 2

In this problem we are exploring Bayesian classifiers. We are assuming that two populations of classes, c_1 and c_2, are equally likely to occur. We state that c_1 has sample pattern vectors $(1, 2, 3), (2, 2, 4), (2, 2, 3), (2, 3, 3)$, and c_2 has sample pattern vectors $(1, 2, 4), (1, 3, 4), (2, 3, 4), (1, 3, 3)$.

A

When we apply $m_j = \frac{1}{n_j} \sum_{x \epsilon c_j}$ to the sample patterns, where n_j is the number of sample pattern vectors from class c_j, what is m_1 and m_2? First, we look again at the equation

$$m_n = \frac{1}{4} \sum_{i=1}^{4} x_{n_i} \tag{1}$$

...given our describe vectors for each class before, we find that:

$$m_1 = \frac{1}{4} \sum_{i=1}^{4} x_{1_i} = \frac{1}{4}((1, 2, 3) + (2, 2, 4) + (2, 2, 3) + (2, 3, 3)) = \frac{1}{4}(7, 9, 13) = (1.75, 2.25, 3.25) \tag{2}$$

$$m_2 = \frac{1}{4} \sum_{i=1}^{4} x_{2_i} = \frac{1}{4}((1, 2, 4) + (1, 3, 4) + (2, 3, 4) + (1, 3, 3)) = \frac{1}{4}(5, 11, 15) = (1.25, 2.75, 3.75) \tag{3}$$

...therefore our $m_1 = (1.75, 2.25, 3.25)$ and our $m_2 = (1.25, 2.75, 3.75)$.

B

If $C_j = \sum_{x \epsilon c_j} xx^T - m_j m_j^T$, what is C_1, C_2, and what is the inverse of this matrix?

$$C_1 = \frac{1}{4}\sum_{i=1}^{4} x_i x_i^T - m_1 m_1^T \tag{4}$$

$$C_1 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \tag{5}$$

$$C_1 = \begin{bmatrix} 0.188 & 0.063 & 0.063 \\ 0.063 & 0.188 & -0.063 \\ 0.063 & -0.063 & 0.188 \end{bmatrix} \tag{6}$$

...with the inverse as:

$$C_1^{-1} = \begin{bmatrix} 8 & -4 & -4 \\ -4 & 8 & 4 \\ -4 & 4 & 8 \end{bmatrix} \tag{7}$$

...and now for C_2:

$$C_2 = \frac{1}{4}\sum_{i=1}^{4} x_i x_i^T - m_2 m_2^T \tag{8}$$

$$C_2 = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} \tag{9}$$

$$C_2 = \begin{bmatrix} 0.188 & 0.063 & 0.063 \\ 0.063 & 0.188 & -0.063 \\ 0.063 & -0.063 & 0.188 \end{bmatrix} \tag{10}$$

...with the inverse as:

$$C_2^{-1} = \begin{bmatrix} 8 & -4 & -4 \\ -4 & 8 & 4 \\ -4 & 4 & 8 \end{bmatrix} \tag{11}$$

Thus we see that $C_1 = C_2$ and $C_1^{-1} = C_2^{-1}$.

C

Here we are tasked with discovering the decision functions, while assuming that classes are equally likely as $d_j(x) = x^T C^{-1} m_j - \frac{1}{2} m_j^T C^{-1} m_j$. First, for d_1:

$$d_1(x) = x^T \begin{bmatrix} 0.188 & 0.063 & 0.063 \\ 0.063 & 0.188 & -0.063 \\ 0.063 & -0.063 & 0.188 \end{bmatrix} \begin{bmatrix} 1.75 \\ 2.25 \\ 3.25 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1.75 & 2.25 & 3.25 \end{bmatrix} \begin{bmatrix} 0.188 & 0.063 & 0.063 \\ 0.063 & 0.188 & -0.063 \\ 0.063 & -0.063 & 0.188 \end{bmatrix} \begin{bmatrix} 1.75 \\ 2.25 \\ 3.25 \end{bmatrix} \tag{12}$$

...leading us to:

$$d_1(x) = x^T \begin{bmatrix} -8 \\ 24 \\ 28 \end{bmatrix} - 65.5 \tag{13}$$

And now for C_2:

$$d_2(x) = x^T \begin{bmatrix} 0.188 & 0.063 & 0.063 \\ 0.063 & 0.188 & -0.063 \\ 0.063 & -0.063 & 0.188 \end{bmatrix} \begin{bmatrix} 1.25 \\ 2.75 \\ 3.75 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1.25 & 2.75 & 3.75 \end{bmatrix} \begin{bmatrix} 0.188 & 0.063 & 0.063 \\ 0.063 & 0.188 & -0.063 \\ 0.063 & -0.063 & 0.188 \end{bmatrix} \begin{bmatrix} 1.25 \\ 2.75 \\ 3.75 \end{bmatrix} \tag{14}$$

...leading us to:

$$d_2(x) = x^T \begin{bmatrix} -16 \\ 32 \\ 36 \end{bmatrix} - 65.5 \tag{15}$$

D

Here we are tasked with showing that the decision boundary of the two classes $d_1(x) - d_2(x)$:

$$d_1(x) - d_2(x) = (x^t \begin{bmatrix} -8 \\ 24 \\ 28 \end{bmatrix} - 65.5) - (x^T \begin{bmatrix} -16 \\ 32 \\ 36 \end{bmatrix} - 65.5) = x^T \begin{bmatrix} 8 \\ -8 \\ -8 \end{bmatrix} + 36 \tag{16}$$

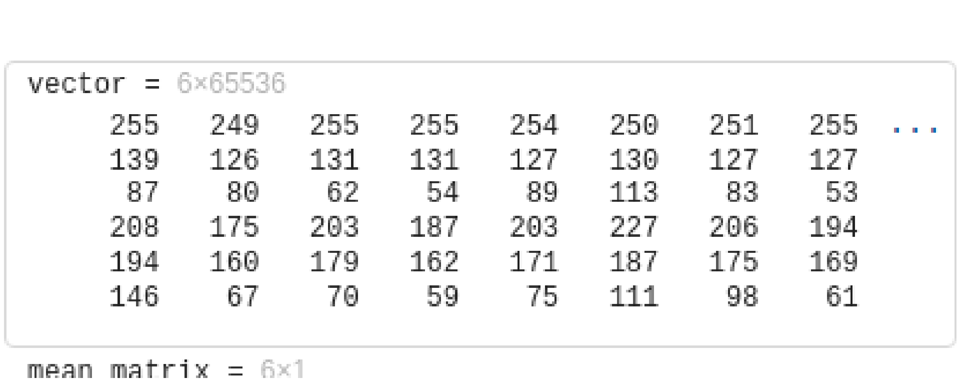
Problem 3

For this problem we explore PCA analysis for a series of satellite images provided, corresponding to six spectral bands. The MATLAB code used for this problem is provided at the end of the section and in the provided file *problem_3.mlx*.

A

First, we are tasked to organize the images into a $256^2 = 65,536$ vectors and find the mean vector $m_x = E\{x\}$, covariance matrix $C_x = E\{(x - m_x)(x - m_x)^T\}$, and eigenvalues and eigenvectors.

This is viewable within *problem_3.mlx*, but we share outputs here. First, a screenshot of the resulting vector of combining images- a screenshot as the actual vector is too large to show:


```
vector = 6×65536
     255   249   255   255   254   250   251   255 ...
     139   126   131   131   127   130   127   127
      87    80    62    54    89   113    83    53
     208   175   203   187   203   227   206   194
     194   160   179   162   171   187   175   169
     146    67    70    59    75   111    98    61

mean matrix = 6×1
```

Figure 1: Combined image vectors

We can see the vector is of the expected size, $6x65536$. We then find that the calculated mean matrix is:

$$\begin{bmatrix} 108.8263 \\ 110.2227 \\ 108.4957 \\ 157.1799 \\ 141.9874 \\ 109.6162 \end{bmatrix} \tag{17}$$

...and our resulting C_x comes to:

$$C_x = 10^3 * \begin{bmatrix} 1.4610 & 1.2925 & 1.1335 & -0.3030 & 0.4125 & 1.0947 \\ 1.2925 & 1.2832 & 1.1826 & -0.2736 & 0.4329 & 1.1341 \\ 1.1335 & 1.1826 & 1.3830 & -0.3537 & 0.3605 & 1.3074 \\ -0.3030 & -0.2736 & -0.3537 & 0.8365 & 0.3419 & -0.3855 \\ 0.4125 & 0.4329 & 0.3605 & 0.3419 & 0.5007 & 0.3596 \\ 1.0947 & 1.1341 & 1.3074 & -0.3855 & 0.3596 & 1.3695 \end{bmatrix} \tag{18}$$

...and our eigenvectors and values, respectively:

$$\begin{bmatrix} 0.3125 & -0.4534 & -0.0681 & -0.6654 & -0.0827 & 0.4925 \\ -0.6413 & 0.5152 & -0.0615 & -0.2757 & -0.0969 & 0.4838 \\ 0.4548 & 0.2810 & -0.5261 & 0.4365 & 0.0457 & 0.4947 \\ -0.2122 & -0.2711 & -0.4210 & 0.1286 & -0.8179 & -0.1367 \\ 0.3710 & 0.3267 & 0.6487 & 0.0682 & -0.5535 & 0.1542 \\ -0.3187 & -0.5195 & 0.3415 & 0.5192 & 0.0795 & 0.4859 \end{bmatrix} \tag{19}$$

$$10^3 * \begin{bmatrix} 0.0373 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0656 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0914 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.4151 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0620 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5.1623 \end{bmatrix} \tag{20}$$

B

Here we are tasked with solving the y vectors where $y = A(x - m_x)$. We must also compute $C_y = AC_xA^T$. First, we solved for A:

$$A = \begin{bmatrix} 0.4925 & 0.4838 & 0.4947 & -0.1367 & 0.1542 & 0.4859 \\ -0.0827 & -0.0969 & 0.0457 & -0.8179 & -0.5535 & 0.0795 \\ -0.6654 & -0.2757 & 0.4365 & 0.1286 & 0.0682 & 0.5192 \\ -0.0681 & -0.0615 & -0.5261 & -0.4210 & 0.6487 & 0.3415 \\ -0.4534 & 0.5152 & 0.2810 & -0.2711 & 0.3267 & -0.5195 \\ 0.3125 & -0.6413 & 0.4548 & -0.2122 & 0.3710 & -0.3187 \end{bmatrix} \tag{21}$$

We can then solve for y via $y = A(x - m_x)$. However, due to its size, we include a screenshot of it from MATLAB - you can get the exact value from *problem_3.mlx*.

```
y = 6×65536
    94.0420   42.2146   39.2458   29.5079  ...
   -83.3214  -42.3525  -77.3355  -56.0788
   -85.6066 -128.6614 -135.4352 -147.8549
    24.3637   -5.8912    4.4250    0.5849
   -73.1711  -40.2386  -48.3822  -46.1326
    14.3625   37.2066   27.8392   24.7946

Cy = 6×6
```

Figure 2: $y = A(x - m_x)$

And finally we can solve for C_y via $C_y = AC_xA^T$:

$$C_y = 10^3 * \begin{bmatrix} 5.1623 & 0.0000 & 0.0000 & 0 & -0.0000 & -0.0000 \\ 0.0000 & 1.0620 & -0.0000 & 0.0000 & 0.0000 & 0.0000 \\ -0.0000 & -0.0000 & 0.4151 & 0.0000 & 0.0000 & 0.0000 \\ -0.0000 & 0.0000 & 0.0000 & 0.0914 & -0.0000 & -0.0000 \\ -0.0000 & 0.0000 & 0.0000 & -0.0000 & 0.0656 & -0.0000 \\ -0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0000 & 0.0373 \end{bmatrix} \tag{22}$$

C

In this section we reform the images and show them to demonstrate the principal component analysis we performed on these images, ordering them by information density.

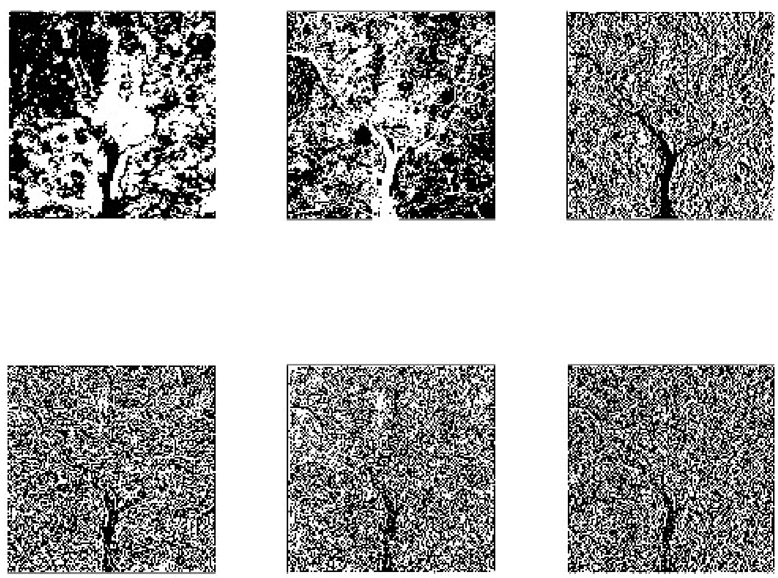


Figure 3: Final images