RBE549 - Homework 10

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Problem 1

Here we are tasked with finding what velocity $\vec{V}_{MIN}$ satisfies the Optical Flow Constraint Equation (OFCE) $I_x u + I_y v + I_t = 0$ and has the smallest magnitude $|\vec{V}|$.

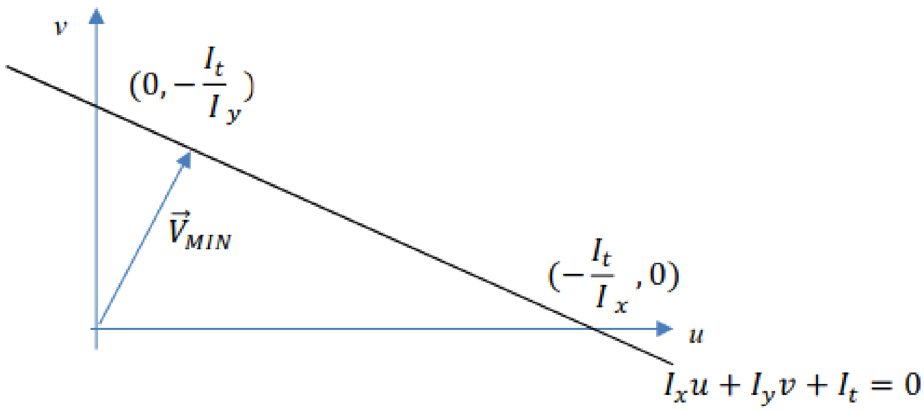

Figure 1: Optical Flow Chart

To do this, we will first solve for the line between the provided points $(0, -\frac{I_t}{I_y})$ and $(-\frac{I_t}{I_x}, 0)$. From this we can determine the $\vec{V}_{MIN}$ vector. To find the line:

$$v = mu + b \tag{1}$$

$$-\frac{I_t}{I_y} = m * 0 + b \tag{2}$$

$$b = -\frac{I_t}{I_y} \tag{3}$$

...and now we can solve for m:

$$0 = m(-\frac{I_t}{I_x}) - \frac{I_t}{I_y} \tag{4}$$

$$-\frac{I_t I_x}{I_y I_t} = m \tag{5}$$

$$-\frac{I_x}{I_y} = m \tag{6}$$

...so our line for the two points is $v = -\frac{I_x}{I_y}u - \frac{I_t}{I_y}$. Since the line perpendicular to a line with slope m is $\frac{-1}{m}$, it follows thats the resulting m for our perpendicular line is thus:

$$\frac{-1}{m} = \frac{-1}{-\frac{I_x}{I_y}} = \frac{I_y}{I_x} \tag{7}$$

Now we need to find the b for our given line that intersects with $(0,0)$, where our $\vec{V}_{MIN}$ comes from. This suggests that the b is thus 0, making a final equation

$$v = \frac{I_y}{I_x}u \tag{8}$$

We then solve for where $\vec{V}_{MIN}$ intersects our line, which would be the minimum velocity required:

$$\frac{I_y}{I_x}u = -\frac{I_x}{I_y}u - \frac{I_t}{I_y} \tag{9}$$

$$(\frac{I_y}{I_x} + \frac{I_x}{I_y})u = \frac{I_t}{I_y} \tag{10}$$

$$\frac{I_x^2 + I_y^2}{I_x I_y}u = \frac{I_t}{I_y} \tag{11}$$

$$u = \frac{I_t I_x}{I_x^2 + I_y^2} \tag{12}$$

Now that we know the u, we need to find the v:

$$v = \frac{I_y}{I_x}\frac{I_t I_x}{I_x^2 + I_y^2} = \frac{I_t I_y}{I_x^2 + I_y^2} \tag{13}$$

Problem 2

Here we are told to suppose that a given image's brightness is given by:

$$I(x, y, t) = I_o + k(tan^{-1}(\frac{x}{y}) - st) \tag{14}$$

A

We are tasked with finding I_x, I_y, I_t, which are the partial derivatives of each:

$$I_x = \frac{k}{y(\frac{x^2}{y^2} + 1)} = \frac{k\,y}{x^2 + y^2} \tag{15}$$

$$I_y = -\frac{kx}{y^2(\frac{x^2}{y^2} + 1)} = -\frac{k\,x}{x^2 + y^2} \tag{16}$$

$$I_t = -ks \tag{17}$$

B

Now, utilizing these partial deriviatives, we wish to find the simplest possible form of $I_x u + I_y v + I_t = 0$. Once we plug in our derivatives, we simplfiy:

$$\frac{kuy}{x^2 + y^2} - \frac{kvx}{x^2 + y^2} - ks = 0 \tag{18}$$

...which we can simplify to:

$$sx^2 + sy^2 + vx = uy \tag{19}$$

C

Now we are tasked with showing that the rotating flow field of $u = sy$ and $v = -sx$ is a solution to the OFCE. To do this we'll plug in these values:

$$\frac{kuy}{x^2+y^2} - \frac{kvx}{x^2+y^2} - ks = 0 \tag{20}$$

$$\frac{ksx^2}{x^2+y^2} + \frac{ksy^2}{x^2+y^2} - ks = 0 \tag{21}$$

$$\frac{sx^2}{x^2+y^2} + \frac{sy^2}{x^2+y^2} - s = 0 \tag{22}$$

$$\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} - 1 = 0 \tag{23}$$

$$\frac{x^2+y^2}{x^2+y^2} - 1 = 0 \tag{24}$$

$$1 - 1 = 0 \tag{25}$$

$$0 = 0 \tag{26}$$

...and thus we have proved that these values do solve this OFCE.

Problem 3

In this problem we are looking at an iterative method for computing optical flow, with each iteration being updated according to:

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}^{new} = \begin{bmatrix} \lambda I_x^2 + 4 & \lambda I_x I_y \\ \lambda I_x I_y & \lambda I_y^2 + 4 \end{bmatrix}^{-1} \begin{bmatrix} \sum_{n \epsilon neighbors(x,y)} u^{old}(n) - \lambda I_x I_t \\ \sum_{n \epsilon neighbors(x,y)} v^{old}(n) - \lambda I_y I_t \end{bmatrix} \tag{27}$$

A

First, we are tasked to show that the equation above is equivalent to

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}^{new} = \frac{1}{4\lambda I_x^2 + 4\lambda I_y^2 + 16} \begin{bmatrix} \lambda I_y^2 + 4 & -\lambda I_x I_y \\ -\lambda I_x I_y & \lambda I_x^2 + 4 \end{bmatrix} \begin{bmatrix} \sum_{n \epsilon neighbors(x,y)} u^{old}(n) - \lambda I_x I_t \\ \sum_{n \epsilon neighbors(x,y)} v^{old}(n) - \lambda I_y I_t \end{bmatrix} \tag{28}$$

The primary difference lies in the second term. First we will define the inverse of a $2x2$ matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{29}$$

Following this definition, we get:

$$\begin{bmatrix} \lambda I_x^2 + 4 & \lambda I_x I_y \\ \lambda I_x I_y & \lambda I_y^2 + 4 \end{bmatrix}^{-1} = \frac{1}{(\lambda I_x^2 + 4)(\lambda I_y^2 + 4) - \lambda I_x I_y \lambda I_x I_y} \begin{bmatrix} \lambda I_y^2 + 4 & -\lambda I_x I_y \\ -\lambda I_x I_y & \lambda I_y^2 + 4 \end{bmatrix} \tag{30}$$

...which simplifies to:

$$\frac{1}{4\lambda I_x^2 + 4\lambda I_y^2 + 16} \begin{bmatrix} \lambda I_y^2 + 4 & -\lambda I_x I_y \\ -\lambda I_x I_y & \lambda I_x^2 + 4 \end{bmatrix} \tag{31}$$

...which thus proves the two equations are equivalent.

B

We are then tasked with showing that this is the equivalent to the update equations:

$$u^{new}(x,y) = \bar{u}^{old} - \frac{I_x}{I_x^2 + I_y^2 + \frac{4}{\lambda}(I_x\bar{u}^old d} + I_t) \tag{32}$$

$$v^{new}(x,y) = \bar{v}^{old} - \frac{I_y}{I_x^2 + I_y^2 + \frac{4}{\lambda}(I_x\bar{u}^old d} + I_t) \tag{33}$$

...where $\bar{u}^{old}$ and $\bar{v}^{old}$ are the averages of the 4 neighbors of $u(x,y)$ and $v(x,y)$.
First, we further define $\bar{u}^{old}$ mathematically:

$$\bar{u}^{old} = \frac{1}{4}\Sigma_{n\epsilon neighbors(x,y)}u^{old}(n) \tag{34}$$

$$4\bar{u}^{old} = \Sigma_{n\epsilon neighbors(x,y)}u^{old}(n) \tag{35}$$

...so if we looked at our base equation now:

$$u^{new}(x,y) = \frac{1}{4(\lambda I_x^2 + \lambda I_y^2 + 4)}\begin{bmatrix}\lambda I_y^2 + 4 & -\lambda I_x I_y\end{bmatrix}\begin{bmatrix}4\bar{u}^{old} - \lambda I_x I_t \\ 4\bar{v}^{old} - \lambda I_y I_t\end{bmatrix} \tag{36}$$

We can then expand this out by performing vector multiplication, getting:

$$u^{new}(x,y) = \frac{1}{4(\lambda I_x^2 + \lambda I_y^2 + 4)}\big((\lambda I_y^2 + 4)(4\bar{u}^{old} - \lambda I_x I_t) - (\lambda I_x I_y)(4\bar{v}^{old} + \lambda I_x I_y \lambda I_y I_t)\big) \tag{37}$$

We can expand this:

$$u^{new}(x,y) = \frac{\lambda I_y^2 4\bar{u}^{old} - \lambda I_y^2 \lambda I_x I_t + 16\bar{u}^{old} - 4\lambda I_x I_t - (\lambda I_x I_y)4\bar{v}^{old} + \lambda^2 I_y^2 I_x I_t}{4(\lambda I_x^2 + \lambda I_y^2 + 4)} \tag{38}$$

$$u^{new}(x,y) = \frac{4(\lambda I_y^2\bar{u}^{old} - \lambda I_x I_t - \lambda I_x I_y \bar{v}^{old})}{4(\lambda I_x^2 + \lambda I_y^2 + 4)} \tag{39}$$

$$u^{new}(x,y) = \frac{(\lambda I_y^2\bar{u}^{old} - \lambda I_x I_t - \lambda I_x I_y \bar{v}^{old})}{(\lambda I_x^2 + \lambda I_y^2 + 4)} \tag{40}$$

$$u^{new}(x,y) = \frac{\bar{u}^{old}(\lambda I_y^2 + 4) - \lambda I_x(I_t + I_y\bar{v}^{old})}{\lambda(I_x^2 + I_y^2 + \frac{4}{\lambda})} \tag{41}$$

Here we can add some terms that help us reduce complexity in the numerator after factoring:

$$u^{new}(x,y) = \frac{\lambda I_y^2\bar{u}^{old} - \lambda I_x I_t - \lambda I_x I_y\bar{v}^{old} + \lambda I_x^2\bar{u}^{old} - \lambda I_x^2\bar{u}^{old}}{\lambda(I_x^2 + I_y^2 + \frac{4}{\lambda})} \tag{42}$$

$$u^{new}(x,y) = \frac{\bar{u}^{old}\lambda(I_y^2 + I_x^2 + \frac{4}{\lambda}) - \lambda I_x^2 - \bar{u}^{old} - \lambda I_x I_t - \lambda I_x I_y\bar{v}^{old}}{\lambda(I_x^2 + I_y^2 + \frac{4}{\lambda})} \tag{43}$$

$$u^{new}(x,y) = \bar{u}^{old} - \frac{\lambda I_x^2\bar{u}^{old} - \lambda I_x I_t - \lambda I_x I_y\bar{v}^{old}}{\lambda(I_x^2 + I_y^2 + \frac{4}{\lambda})} \tag{44}$$

$$u^{new}(x,y) = \bar{u}^{old} - \lambda I_x\left(\frac{I_x\bar{u}^{old} + I_y\bar{v}^{old} + I_t}{\lambda(I_x^2 + I_y^2 + \frac{4}{\lambda})}\right) \tag{45}$$

$$u^{new}(x,y) = \bar{u}^{old} - \frac{I_x}{I_x^2 + I_y^2 + \frac{4}{\lambda}}\left(I_x\bar{u}^{old} + I_y\bar{v}^{old} + I_t\right) \tag{46}$$

And with this, we can presume that the update equation for v^{new} is similarly:

$$v^{new}(x,y) = \bar{v}^{old} - \frac{I_x}{I_x^2 + I_y^2 + \frac{4}{\lambda}}\left(I_x\bar{u}^{old} + I_y\bar{v}^{old} + I_t\right) \tag{47}$$

C

Here we are tasked with finding out what the update equations found above reduce to when $\lambda = 0$.

$$u^{new}(x,y) = \bar{u}^{old} - \frac{I_x}{I_x^2 + I_y^2 + \frac{4}{\lambda}}\left(I_x\bar{u}^{old} + I_y\bar{v}^{old} + I_t\right) \tag{48}$$

We have to adjust the denominator I showed above, because we wish to avoid the undefined nature the $\frac{4}{\lambda}$ introduces. Thus we multiply by $\frac{\lambda}{\lambda}$.

$$u^{new}(x,y) = \bar{u}^{old} - \frac{\lambda I_x}{\lambda I_x^2 + \lambda I_y^2 + 4}\left(I_x\bar{u}^{old} + I_y\bar{v}^{old} + I_t\right) \tag{49}$$

...and replaceing $\lambda = 0$ yields:

$$u^{new}(x,y) = \bar{u}^{old} - \frac{0 * I_x}{0 * I_x^2 + 0 * I_y^2 + 4}\left(I_x\bar{u}^{old} + I_y\bar{v}^{old} + I_t\right) \tag{50}$$

$$u^{new}(x,y) = \bar{u}^{old} - \frac{0}{4}\left(I_x\bar{u}^{old} + I_y\bar{v}^{old} + I_t\right) \tag{51}$$

$$u^{new}(x,y) = \bar{u}^{old} \tag{52}$$

...and it goes to follow that for $v^{new}(x,y) = \bar{v}^{old}$ is for our v update equation.