RBE549 - Final Exam

Keith Chester

Due date: December 12, 2022

Problem 1

In this problem, we are tasked with solving an iterative optical flow problem. To compute optical flow, we learned an interative method to update $u(x,y)$, $v(x,y)$ at each iteration, according to:

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}^2 = \begin{bmatrix} \lambda I_x^2 + 4 & \lambda I_x I_y \\ \lambda I_x I_y & \lambda I_y^2 + 4 \end{bmatrix}^{-1} \begin{bmatrix} \sum_{n \epsilon neighbors(x,y)} u^{old}(n) - \lambda I_x I_t \\ \sum_{n \epsilon neighbors(x,y)} v^{old}(n) - \lambda I_y I_t \end{bmatrix} \tag{1}$$

We are asked to consider a local coordinate frame (x', y') where x' is aligned with the image gradient and y' is perpendicular to the image gradient. Likewise, $(u', v') = (\frac{dx'}{dt}, \frac{dy'}{dt})$ are the image velocities in this frame. In this coordinate frame,

$$I_{x'} = \sqrt{I_x^2 + I_y^2} \; and \; I_{y'} = 0 \tag{2}$$

We wish to show that the update equations:

$$\begin{bmatrix} u'(x,y) \\ v'(x,y) \end{bmatrix}^2 = \begin{bmatrix} \lambda I_{x'}^2 + 4 & \lambda I_{x'} I_{y'} \\ \lambda I_{x'} I_{y'} & \lambda I_{y'}^2 + 4 \end{bmatrix}^{-1} \begin{bmatrix} \sum_{n \epsilon neighbors(x,y)} u'^{old}(n) - \lambda I_{x'} I_t \\ \sum_{n \epsilon neighbors(x,y)} v'^{old}(n) - \lambda I_{y'} I_t \end{bmatrix} \tag{3}$$

...can reduce to

$$u'^{new}(x,y) = \bar{u}'^{old} - \frac{I_{x'}^2 \bar{u}'^{old} + I_{x'} I_t}{I_{x'}^2 + \frac{4}{\lambda}} \tag{4}$$

$$v'^{new}(x,y) = \bar{v}'^{old} \tag{5}$$

. To do this, we first expand our second term to an equivalnet form to make matters easier for us to work with. Specifically, the inverse of a $2x2$ matrix is:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{6}$$

...and thus...

$$\begin{bmatrix} \lambda I_{x'}^2 + 4 & \lambda I_{x'} I_{y'} \\ \lambda I_{x'} I_{y'} & \lambda I_{y'}^2 + 4 \end{bmatrix}^{-1} = \frac{1}{(\lambda I_{x'}^2 + 4)(\lambda I_{y'}^2 + 4) - \lambda I_{x'} I_{y'} \lambda I_{x'} I_{y'}} \begin{bmatrix} \lambda I_{y'}^2 + 4 & -\lambda I_{x'} I_{y'} \\ -\lambda I_{x'} I_{y'} & \lambda I_{y'}^2 + 4 \end{bmatrix} \tag{7}$$

...which simplifies to:

$$\frac{1}{4\lambda I_{x'}^2 + 4\lambda I_{y'}^2 + 16} \begin{bmatrix} \lambda I_{y'}^2 + 4 & -\lambda I_{x'} I_{y'} \\ -\lambda I_{x'} I_{y'} & \lambda I_{x'}^2 + 4 \end{bmatrix} \tag{8}$$

This in turn leads us to an expanded view from earlier:

$$u'^{new}(x,y) = \frac{1}{4(\lambda I_{x'}^2 + \lambda I_{y'}^2 + 4)} \begin{bmatrix} \lambda I_{y'}^2 + 4 & -\lambda I_{x'} I_{y'} \end{bmatrix} \begin{bmatrix} 4\bar{u}'^{old} - \lambda I_{x'} I_t \\ 4\bar{v}'^{old} - \lambda I_{y'} I_t \end{bmatrix} \tag{9}$$

...We can then expand it into this:

$$u'^{new}(x,y) = \frac{1}{4(\lambda I_{x'}^2 + \lambda I_{y'}^2 + 4)}\big((\lambda I_{y'}^2 + 4)(4\bar{u}'^{old} - \lambda I_{x'} I_t) - (\lambda I_{x'} I_{y'})(4\bar{v}'^{old} + \lambda I_{x'} I_{y'} \lambda I_{y'} I_t)\big) \tag{10}$$

$$u'^{new}(x,y) = \frac{\lambda I_{y'}^2 4\bar{u}^{old} - \lambda I_{y'}^2 \lambda I_{x'} I_t + 16\bar{u}'^{old} - 4\lambda I_{x'} I_t - (\lambda I_{x'} I_{y'})4\bar{v}'^{old} + \lambda^2 I_{y'}^2 I_{x'} I_t}{4(\lambda I_{x'}^2 + \lambda I_{y'}^2 + 4)} \tag{11}$$

$$u'^{new}(x,y) = \frac{4(\lambda I_{y'}^2 \bar{u}'^{old} + 4\bar{u}'^{old} - \lambda I_{x'} I_t - \lambda I_{x'} I_{'} y \bar{v}'^{old})}{4(\lambda I_{x'}^2 + \lambda I_{y'}^2 + 4)} \tag{12}$$

$$u'^{new}(x,y) = \frac{(\lambda I_{y'}^2 \bar{u}'^{old} + 4\bar{u}'^{old} - \lambda I_{x'} I_t - \lambda I_{x'} I_{y'} \bar{v}'^{old})}{(\lambda I_{x'}^2 + \lambda I_{y'}^2 + 4)} \tag{13}$$

Now we can introduce terms to try and help us simplify to our end goal. To this end, we will be adding in the terms $\lambda I_{x'}^2 \bar{u}'^o ld - \lambda I_{x'}^2 \bar{u}'^o ld$. Since the terms cancel out on their own it's the equivalent of adding 0, and thus does not change our definition.

$$u'^{new}(x,y) = \frac{\lambda I_{y'}^2 \bar{u}'^{old} + 4\bar{u}'^{old} - \lambda I_{x'} I_t - \lambda I_{x'} I_{y'} \bar{v}'^{old} + \lambda I_{x'}^2 \bar{u}'^{old} - \lambda I_{x'}^2 \bar{u}'^{old}}{(\lambda I_{x'}^2 + \lambda I_{y'}^2 + 4)} \tag{14}$$

$$u'^{new}(x,y) = \frac{\bar{u}'^{old}\big(\lambda I_{y'}^2 + 4 + \lambda I_{x'}^2\big) - \lambda I_{x'}^2 \bar{u}'^{old} - \lambda I_{x'} I_{y'} \bar{v}'^{old} - \lambda I_{x'} I_t}{(\lambda I_{x'}^2 + \lambda I_{y'}^2 + 4)} \tag{15}$$

$$u'^{new}(x,y) = \bar{u}'^{old} - \frac{\lambda I_{x'}^2 \bar{u}'^{old} + \lambda I_{x'} I_{y'} \bar{v}'^{old} + \lambda I_{x'} I_t}{(\lambda I_{x'}^2 + \lambda I_{y'}^2 + 4)} \tag{16}$$

As specified before, $I_{y'} = 0$, so we can now fill that in:

$$u'^{new}(x,y) = \bar{u}'^{old} - \frac{\lambda I_{x'}^2 \bar{u}'^{old} + \lambda I_{x'} 0 \bar{v}'^{old} + \lambda I_{x'} I_t}{(\lambda I_{x'}^2 + 0 + 4)} = \bar{u}'^{old} - \frac{\lambda I_{x'}^2 \bar{u}'^{old} + \lambda I_{x'} I_t}{(\lambda I_{x'}^2 + 4)} \tag{17}$$

$$u'^{new}(x,y) == \bar{u}'^{old} - \frac{I_{x'}^2 \bar{u}'^{old} + I_{x'} I_t}{(I_{x'}^2 + \frac{4}{\lambda})} \tag{18}$$

...which matches what we were seeking to reduce to for $u'^{new}(x.y)$. Doing this for $v'^{new}(x,y)$ we would follow a similar path - we would find that:

$$v'^{new}(x,y) = \frac{1}{4(\lambda I_{x'}^2 + \lambda I_{y'}^2 + 4)} \begin{bmatrix} -\lambda I_{x'} I_{y'} & \lambda I_{x'}^2 + 4 \end{bmatrix} \begin{bmatrix} 4\bar{u}'^{old} - \lambda I_{x'} I_t \\ 4\bar{v}'^{old} - \lambda I_{y'} I_t \end{bmatrix} \tag{19}$$

$$v'^{new}(x,y) = \frac{1}{4(\lambda I_{x'}^2 + \lambda I_{y'}^2 + 4)}\big((\lambda I_{x'}^2 + 4)(4\bar{v}^{old} - \lambda I_{y'} I_t) - (\lambda I_{x'} I_{y'})(4\bar{u}^{old} - \lambda I_{x'} I_t)\big) \tag{20}$$

...again, we can utilize the specified knowledge that $I_{y'} = 0$:

$$v'^{new}(x,y) = \frac{1}{4(\lambda I_{x'}^2 + \lambda 0^2 + 4)}\big((\lambda I_{x'}^2 + 4)(4\bar{v}^{old} - \lambda 0 I_t) - (\lambda I_{x'} 0)(4\bar{u}^{old} - \lambda I_{x'} I_t)\big) \tag{21}$$

$$v'^{new}(x,y) = \frac{1}{4(\lambda I_{x'}^2 + 4)}((\lambda I_{x'}^2 + 4)(4\bar{v}^{old})) \tag{22}$$

$$v'^{new}(x,y) = \frac{4\bar{v}^{old} \lambda I_{x'}^2 + 16\bar{v}^{old}}{4(\lambda I_{x'}^2 + 4)} \tag{23}$$

$$v'^{new}(x,y) = \bar{v}^{old} \frac{4\lambda I_{x'}^2 + 16}{4(\lambda I_{x'}^2 + 4)} \tag{24}$$

$$v'^{new}(x,y) = \bar{v}^{old} \frac{4(\lambda I_{x'}^2 + 4)}{4(\lambda I_{x'}^2 + 4)} \tag{25}$$

$$v'^{new}(x,y) = \bar{v}^{old} \tag{26}$$

...thus proving that:

$$u'^{new}(x,y) = \bar{u}'^{old} - \frac{I_{x'}^2 \bar{u}'^{old} + I_{x'} I_t}{I_x^2 + \frac{4}{\lambda}} \tag{27}$$

$$v'^{new}(x,y) = \bar{v}'^{old} \tag{28}$$

Problem 2

We can represent an object by its boundary, $(x(s), y(s)), 0 \leq s \leq S$, where S is the length of the object's boundary and s is distance along that boundary from some arbitrary starting point. Combine x and y into a single complex funcion $z(s) = x(s) + jy(s)$. The Discrete Fourier Transform (DFT) of z is:

$$Z(k) = \sum_{s=0}^{S-1} e^{-2\pi j \frac{ks}{S}} z(s), 0 \leq k \leq S-1 \tag{29}$$

We can use the coefficients $Z(k)$ to represent the object boundary. The limit on s is $S-1$ because for a closed contour $z(S) = z(0)$. The Inverse Discrete Fourier Transform is:

$$z(s) = \frac{1}{S} \sum_{k=0}^{S-1} e^{+2\pi j \frac{ks}{S}} Z(k), 0 \leq s \leq S-1 \tag{30}$$

A

Suppose that the object is translated by $(\Delta x, \Delta y)$, that is, $z'(s) = z(s) + \Delta x + j\Delta y$. How is z''s DFT $Z'(k)$ related to $Z(k)$?

$$Z(k) = \sum_{s=0}^{S-1} e^{-2*\pi j \frac{ks}{S}} z(s), 0 \leq k \leq S-1 \tag{31}$$

Which we can define $z'(s)$ as:

$$z'(s) = z(s) + \Delta x + j\Delta y \tag{32}$$

If we plug this into our original equation, we get...

$$Z'(k) = \sum_{s=0}^{S-1} e^{-2\pi j \frac{ks}{S}} z(s) + \sum_{s=0}^{S-1} e^{-2\pi j \frac{ks}{S}} (\Delta x + j\Delta y)(1) \tag{33}$$

Here we see that we have a segment that is equivalent to oru defined $Z(k)$, so we can simplify by expressing:

$$Z(k) + (1)(\Delta x + j\Delta y) \sum_{s=0}^{S-1} e^{-2\pi j \frac{ks}{S}} \tag{34}$$

With Δx and $j\Delta y$ isolated, we can use a table of known Fourier Transforms to identify the resulting conversion. Based on our problem's definition ΔX and Δy are both constants, so we can state that:

$$Z'(k) = Z(k) + (\Delta x + j\Delta y)\sigma(\frac{2\pi k}{S}) \tag{35}$$

...where $\frac{1}{S}$ acts as our scaling factor.

B

In this section, we are asked to suppose that the object is scaled by an integer constant c, that is $z'(s) = cz(s)$. For simplicity, we are to assume that $S' = S$. How is $Z'(k)$ as the DFT of z' related to $Z(k)$?
Starting with our definition of $Z(k)$, which is our Fourier Transform of $z(s)$ as defined earlier:

$$Z(k) = \sum_{s=0}^{S-1} e^{-2\pi j \frac{ks}{S}} z(s), 0 \leq k \leq S-1 \tag{36}$$

If we then define $z'(s)$ as above, $z'(s) = cz(s)$, we can plug it into our Fourier Transform:

$$Z'(k) = \sum_{s=0}^{S-1} e^{-2\pi j \frac{ks}{S}} cz(s) \tag{37}$$

Since c is a constant, we know we can pull it in front of the fourier Transform like so:

$$Z'(k) = c \sum_{s=0}^{S-1} e^{-2\pi j \frac{ks}{S}} z(s) \tag{38}$$

...and further simplified:

$$Z'(k) = cZ(k) \tag{39}$$

...which shows that multiplying our time series function by a given scalar simply multipies our frequncy domain by the same scalar.

C

We wish to know what object has $z(s) = \left[x_0 + R\cos(\frac{2\pi s}{s})\right] + j\left[y_0 + R\sin(\frac{2\pi s}{s})\right]$. Below we created a graph drawing the resulting shape, and include the code utilized to generate it. Arbitrary values were chosen for S, r, x_0, and y_0 for the sake of plotting.

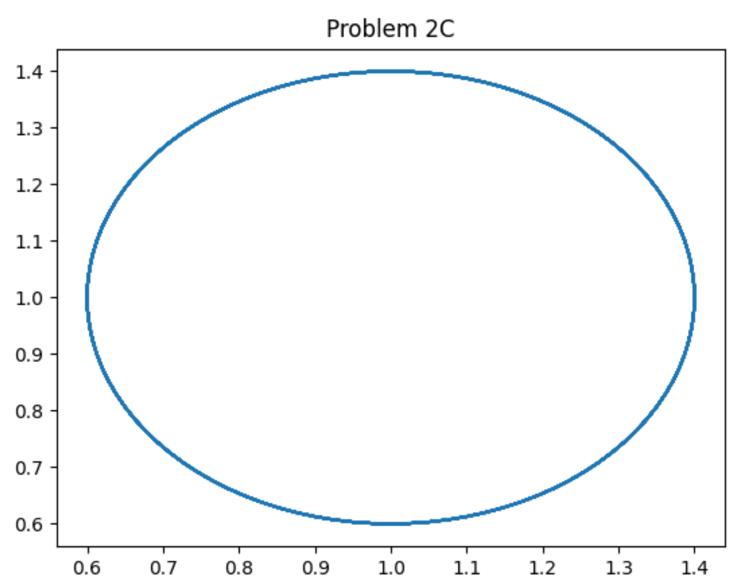

Figure 1: Our resulting shape

```python
import numpy as np
from numpy import cos, sin, pi
import matplotlib.pyplot as plt

figure = plt.figure()
plt.title("Problem 2C")

S = 10
r = 4
x0 = 1
y0 = 1

theta = [theta for theta in np.arange(0, S, 0.01)]
X = [
        x0 + r * cos(2*pi*theta)/S
        for theta in theta
    ]
Y = [
        y0 + r * sin(2*pi*theta)/S
        for theta in theta
    ]

# Plot the results
plt.plot(X, Y)
plt.savefig("./imgs/prob2_c.png")
```

D

What is $Z(k)$ corresponding to $z(s)$ from Part C? To do this, we begin wtih our $z(s)$:

$$z(s) = \left[x_0 + R\cos(\tfrac{2\pi s}{s})\right] + j\left[y_0 + R\sin(\tfrac{2\pi s}{s})\right] \tag{40}$$

We can utilize the inverse of Euler's formula; $\cos(x) = \frac{e^{ix}+e^{-ix}}{2}$ and $\sin(t) = \frac{e^{ix}-e^{-ix}}{2x}$. This allows us to expand our starting equation:

$$z(s) = x_0 + R\frac{e^{j\frac{2\pi s}{S}} + e^{-j\frac{2\pi s}{S}}}{2} + jy_0 + R\frac{e^{j\frac{2\pi s}{S}} - e^{-j\frac{2\pi s}{S}}}{2} \tag{41}$$

$$z(s) = x_0 + jy_0 + \frac{R}{2}(e^{j\frac{2\pi s}{S}} + e^{-j\frac{2\pi s}{S}} + e^{j\frac{2\pi s}{S}} - e^{-j\frac{2\pi s}{S}}) \tag{42}$$

$$z(s) = x_0 + jy_0 + \frac{R}{2}(2e^{j\frac{2\pi s}{S}}) \tag{43}$$

$$z(s) = x_0 + jy_0 + Re^{j\frac{2\pi s}{S}} \tag{44}$$

The Discrete Fourier Transform (DFT) from earlier in our problem we can begin to expand this equation now. Starting with:

$$Z(k) = \sum_{s=0}^{S-1} e^{-2\pi j\frac{ks}{S}} z(s), 0 \leq k \leq S-1 \tag{45}$$

...which can lead us to:

$$Z(k) = \sum_{s=0}^{S-1} e^{-2\pi j\frac{ks}{S}} (x_0 + jy_0 + Re^{j\frac{2\pi s}{S}}) \tag{46}$$

...constants can be pulled out, leaving us with:

$$Z(k) = \sum_{s=0}^{S-1} e^{-2\pi j\frac{ks}{S}} (x_0 + jy_0) + R\sum_{s=0}^{S-1} e^{-2\pi j\frac{ks}{S}} e^{j\frac{2\pi s}{S}} \tag{47}$$

$$Z(k) = (x_0 + jy_0)\sum_{s=0}^{S-1} e^{-2\pi j\frac{ks}{S}} + R\sum_{s=0}^{S-1} e^{-j2\pi(k-1)}S \tag{48}$$

We know from a Fourier Transform lookup table we can then convert this to:

$$Z(k) = (x_0 + jy_0)\sigma(\frac{2\pi k}{S}) + \sigma(2\pi\frac{k-1}{S}) \tag{49}$$