

# RBE549 - Midterm Exam

Keith Chester

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## Problem 1

In this problem we are presented with a Hough transform problem, wherein  $x$ ,  $y$ ,  $b$ , and  $m$  are positive or negative real numbers. 2 points are given in  $(m, b)$  space by  $P_1 : (m, b) = (0.5, 4)$  and  $P_2 : (m, b) = (0.9, 0)$ .

### A

What line  $L_1$  in  $(m, b)$  space through  $P_1$  and  $P_2$ ?

We can set the line equation ( $y = mx + b$ ) for  $P_1$  and  $P_2$  equal to each other to solve this.

$$P_1 = P_2 \quad (1)$$

$$0.5x + 4 = 0.9x + 0 \quad (2)$$

$$4 = 0.4x \quad (3)$$

$$10 = x \quad (4)$$

...and then we plug in  $x$  to solve for  $y$ :

$$0.5(10) + 4 = y \quad (5)$$

$$y = 9 \quad (6)$$

...thus the equation for  $L_1$ , a line that passes through points  $P_1$  and  $P_2$ , is  $b = -10m + 9$ .

### B

The point  $P_3$  in  $(x, y)$  space that corresponds to line  $L_1$  is the point  $(10, 9)$ , as calculated above in part A. We can also just deduce it from our solved equation mentioned in A - specifically that since  $y = mx + b$  for  $b = -mx + y$ , and we stated that  $b = -10m + 9$ , we can also deduce that  $(x, y) = (10, 9)$ .

### C

Since it is a horizontal line, which means the slope  $m$  must be  $m = 0$ , we now know that the resulting point,  $P_4$ , must fall along the line we calculated earlier,  $L_1$ , specifically where  $m = 0$ , or the vertical axis of  $(m, b)$  space. Thus we can solve:

$$b = -10m + 9 \quad (7)$$

$$b = -10 * 0 + 9 \quad (8)$$

$$b = 9 \quad (9)$$

...thus we know that  $P_4$  will fall on  $(m, b) = (0, 9)$ . This means that the equation for  $L_2$  is  $y = 9$ .

## Problem 2

In this problem we are aiming to discover a structuring element  $SE$  such that  $II \oplus SE = OI$

### A

A  $3 \times 3$  structuring element  $SE$  that satisfies the given  $II$  and  $OI$  would be:

$$\begin{array}{|c|c|c|} \hline 1 & 0 & 1 \\ \hline 0 & \mathbf{0} & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array} \quad (10)$$

...where the central 0 is the point of origin for the  $SE$ .

### B

Another structuring element  $SE_2$  can satisfy  $II \oplus SE_2 = II$  but  $SE \neq SE_2$ , where again the central value (now a 1) is the point of origin::

$$\begin{array}{|c|c|c|} \hline 1 & 0 & 1 \\ \hline 0 & \mathbf{1} & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array} \quad (11)$$

## Problem 3

In this problem we are solving equations having to do with image focus and lenses.

### A

In this problem, we attempt to prove with similar triangles that:

$$\frac{1}{-z_0} + \frac{1}{z_c} = \frac{1}{f} \quad (12)$$

First, we look at the original diagram provided. I have added markers for the lengths ( $z_0$  and  $z_c$ ) assigned to their lengths, and marked the two lengths of  $f$  to its distance from the center of the lens.

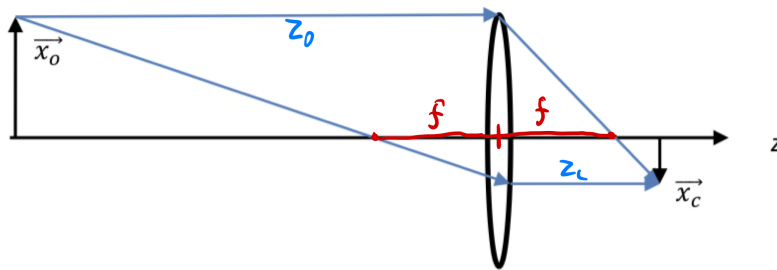


Figure 1: Thin Lens Diagram

We will note that the triangle provided by  $z_0$  and be represented on the other side of the lens by flipping it along the lens as an axis. From there we can begin to mark angles.

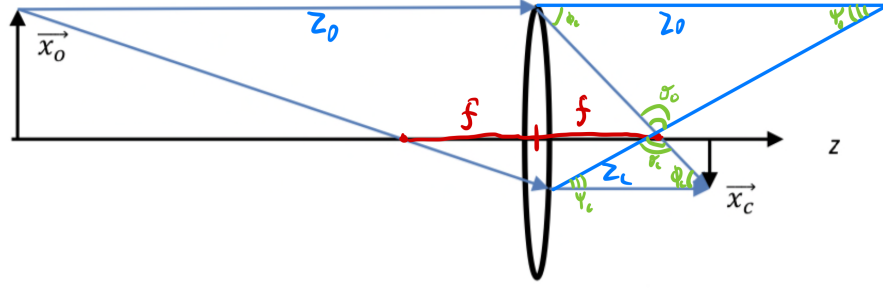


Figure 2: Thin Lens Diagram Expanded

Because the  $z$  axis,  $z_c$ , and  $z_0$  are parallel lines, we can thus know that opposite angles of lines crossing them are equivalent. Thus we see that  $\phi_c$  and  $\phi_0$  are equivalent and marked as such. The same goes for  $\psi_c$  and  $\psi_0$ , as marked. Since the lines passing through the focal point  $f$  are from parallel lines and continuous, we can also determine that  $\theta_0$  and  $\theta_c$  are also equivalent.

With this, we know that these are similar triangles, and can begin to form ratios to represent the relationships. Thus we have:

$$\frac{-z_0}{z_c} = \frac{f}{z_c - f} \quad (13)$$

From here we can perform algebra to isolate the variables into a manner that is closer to what we sought to originally prove:

$$-z_0 z_c - z_0 f = z_c f \quad (14)$$

$$-z_0 z_c = z_c f - z_0 f \quad (15)$$

$$-z_0 z_c = f(z_c - z_0) \quad (16)$$

$$\frac{-z_0 z_c}{f} = z_c - z_0 \quad (17)$$

$$\frac{1}{f} = \frac{z_c - z_0}{-z_0 z_c} \quad (18)$$

$$\frac{1}{f} = \frac{z_c}{-z_0 z_c} + \frac{z_0}{z_0 z_c} \quad (19)$$

$$\frac{1}{f} = \frac{1}{-z_0} + \frac{1}{z_c} \quad (20)$$

## B

In this problem we are tasked with finding the number of receptors on which the image of Mars falls while looking up into the night sky. We say that a human eyeball has a radius of  $12\text{mm}$  and contains roughly  $1.5 \times 10^8$  receptors. We assume that the receptors are uniform across a  $160^\circ$  partial sphere in the back of our eye. The planet of Mars has a  $4 \times 10^3\text{km}$  radius with an average distance of  $2.25 \times 10^8\text{km}$ . We use an  $f$  focal value equal to the eye's diameter ( $24\text{mm}$ ).

To solve this, first we use similar triangles using a lens diagram (similar to the one provided for the question in 3A); this allows us to determine the size of the image of Mars on our eye.

$$\frac{z_0}{z_c} = \frac{h_0}{h_1} \quad (21)$$

In this we have  $z_0$  as the distance from Mars to our eye, and  $h_0$  the width of Mars, and  $h_i$  is the size of the image of Mars on our eye. We need to solve for  $z_c$  before we can figure out  $h_1$ , so to do this we look at the equation utilized earlier:

$$\frac{1}{-z_0} + \frac{1}{z_c} = \frac{1}{f} \quad (22)$$

$$\frac{1}{-2.25 \times 10^8 km} + \frac{1}{z_c} = \frac{1}{24mm} \quad (23)$$

$$\frac{1}{z_c} \approx 83.33 + 4.44 \times 10^{-9} \quad (24)$$

$$z_c \approx 0.024 \approx 24mm \quad (25)$$

Now we can determine the size of the image of Mars on our eye:

$$\frac{z_0}{z_c} = \frac{h_0}{h_1} \quad (26)$$

$$\frac{2.25 \times 10^8 km}{12mm} = \frac{8,000km}{h_1} \quad (27)$$

$$h_1 = 8.53 \times 10^{-7} m = 8.53 \times 10^{-4} \quad (28)$$

...which is an incredibly tiny size. So let's figure out how many receptors we have per square  $mm$  of our eye. First we need to find the area of a spherical cap, such that we can determine the average number of receptors per  $mm$  in the eye. We know that the area of a spherical cap ( $S$ ) is:

$$S = 2rh\pi \quad (29)$$

...where  $h$  is equal to:

$$h = r - \sqrt{r^2 - a^2} \quad (30)$$

...but because of the partial sphere,  $a$  is not straight but rather angled to reach the  $160^\circ$ ; thus we have an actual interior angle of  $80^\circ$ , so  $a = r * \cos(10^\circ)$

$$h = 12mm - \sqrt{(12mm)^2 - (12mm * \cos(10^\circ))^2} \quad (31)$$

$$h \approx 0.00992 \approx 9.92mm \quad (32)$$

Now that we have  $h$ , we can solve for  $S$ :

$$S = 2rh\pi = 2 * 12mm * 9.92mm * \pi \approx 747.95mm^2 \quad (33)$$

Since we know that there are approximately  $1.5 \times 10^8$  receptors, we can use a ratio to get the approximate receptors per  $mm$ , or  $r_{mm}$ :

$$\frac{1mm^2}{r_{mm}} = \frac{747.95mm^2}{1.5 \times 10^8} \quad (34)$$

$$r_{mm} \approx 200,548 \quad (35)$$

Now that we know the number of receptors per  $mm$ , we need to find the area that the Mars image sits on. With an image height of  $8.53 \times 10^{-7}$ , which means a radius of half that, or  $4.27 \times 10^{-7}$ , which is our  $a$ . Using the area formulas we used earlier:

$$h = r - \sqrt{r^2 - a^2} = 12mm - \sqrt{(12mm)^2 - (4.27 \times 10^{-4})^2} = 7.60 \times 10^{-9} \quad (36)$$

$$S = 2\pi rh = 2\pi(12mm)7.11 \times 10^{-7}mm = 5.73 \times 10^{-7}mm^2 \quad (37)$$

So this area times our known receptors per *mm*:

$$x_{receptors} = 200,548 \times 5.73 \times 10^{-7} = 0.115 \quad (38)$$

...and now, we have  $\approx 0.115$  receptors seeing the image of Mars - which is why you can't see Mars with your naked eye on a clear night when it's at the average distance.