# RBE549 - Homework 10

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## Problem 1

In this problem ,we are asked to consider two surfaces:

$$z_1 = \frac{1}{2}\ln(x^2 + y^2) \tag{1}$$

...and...

$$z_2 = \tan^- 1(\frac{x}{y}) \tag{2}$$

### A

We are asked to find $p(x, y)$ and $q(x, y)$ for both surfaces. $p(x, y) = \frac{\partial z_n}{\partial x}$ and $q(x, y) = \frac{\partial z_n}{d_y}$. So we solve for each:

$$p_1 = \frac{\partial z_1}{\partial x} = \frac{\partial}{\partial x}\frac{1}{2}\ln(x^2 + y^2) = \frac{x}{x^2 + y^2} \tag{3}$$

$$q_1 = \frac{\partial z_1}{\partial y} = \frac{\partial}{\partial y}\frac{1}{2}\ln(x^2 + y^2) = \frac{y}{x^2 + y^2} \tag{4}$$

$$p_2 = \frac{\partial z_2}{\partial x} = \frac{\partial}{\partial x}\tan^- 1(\frac{x}{y}) = \frac{y}{x^2 + y^2} \tag{5}$$

$$q_2 = \frac{\partial z_2}{\partial y} = \frac{\partial}{\partial y}\tan^- 1(\frac{x}{y}) = -\frac{x}{x^2 + y^2} \tag{6}$$

### B

Here, we are tasked in showing that $z_1$ and $z_2$ gives rise to the same shading when a rotationally symmetric reflectance map applies; $R(p, q) = R(p^2 + q^2)$.

Here, we can show this by demonstrating that $p_1^2 + q_1^2 = p_2^2 + q^2$.

$$p_1^2 + q_1^2 = p_2^2 + q^2 \tag{7}$$

$$\frac{x}{x^2 + y^2}^2 + \frac{y}{x^2 + y^2}^2 = \frac{y}{x^2 + y^2}^2 + -\frac{x}{x^2 + y^2}^2 \tag{8}$$

$$\frac{x^2}{x^2 + y^2} + \frac{y^2}{(x^2 + y^2)^2} = \frac{y^2}{(x^2 + y^2)^2} + \frac{x^2}{x^2 + y^2} \tag{9}$$

$$\frac{1}{x^2 + y^2} = \frac{1}{x^2 + y^2} \tag{10}$$

# 1 Problem 2

In this problem, we're looking at a surface normal $\hat{n} = \begin{bmatrix} n_x & n_y & n_z \end{bmatrix}^T$, where $n_x^2 + n_y^2 + n_z^2 = 1$, and can be represented by surface orientation $p, q. p = \frac{n_x}{\sqrt{1-n_x^2-n_y^2}}$, etc. In addition to the original origin, $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$, we can use another origin from the point $\begin{bmatrix} 0 & 0 & -1 \end{bmatrix}^T$.

### A

Here, we are tasked with solving for $f$ and $g$. We can do this by solving for the line of the vector from our new origin point of $\begin{bmatrix} 0 & 0 & -1 \end{bmatrix}^T$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} + k \begin{bmatrix} l \\ m \\ n \end{bmatrix} \tag{11}$$

Note that the slopes represented by our variables can be further defined as:

$$\begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} \tag{12}$$

Our $x$ and our $y$ are the values we're seeking to solve for, $f$ and $g$, respectively. Now we fill in our known values to solve for our line:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} + k \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} \tag{13}$$

$$\begin{bmatrix} f \\ g \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} + k \begin{bmatrix} n_x - 0 \\ n_y - 0 \\ n_z + 1 \end{bmatrix} \tag{14}$$

We can then break this down into a system of equations. First, we solve for $t$ using our $z$ value as we have an additional known here.

$$1 = -1 + k(n_z + 1) \tag{15}$$

$$k = \frac{2}{n_z + 1} \tag{16}$$

...and now we can solve for $f$ and $g$:

$$f = 0 + k(n_x - 0) = \frac{2n_x}{n_z + 1} \tag{17}$$

$$g = 0 + k(n_y - 0) = \frac{2n}{n_z + 1} \tag{18}$$

Resulting in a final $f$, $g$, $z$ =

$$\begin{bmatrix} f \\ g \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2n_x}{n_z+1} \\ \\ \frac{2n_y}{n_z+1} \\ \\ 1 \end{bmatrix} \tag{19}$$

### B

Now we are taskedwith showing that when $n_z = 0$, $f$ and $g$ are finite and lie on a circle of radius 2.

$$f = \frac{2n_x}{n_z + 1} = \frac{2n_x}{0 + 1} = 2n_x \tag{20}$$

$$g = \frac{2n}{n_z + 1} = \frac{2n_y}{0 + 1} = 2n_y \tag{21}$$

Given these values, we can confirm the statement that $f$ and $g$ are finite as they depend upon $n_x$ and $n_y$ respectively. Now, the radius is the result of the magnitude of the vector. This means we can do the following:

$$|f, g| = \sqrt{f^2 + g^2} = \sqrt{(2n_x)^2 + (2n_y)^2} = \sqrt{4(\sqrt{1 - n_y^2})^2 + (2n_y)^2} = \sqrt{4 - 4n_y^2 + 4n_y^2} = \sqrt{4} = 2 \tag{22}$$

## Problem 3

Here we try our hands at a photometric stereo shading problem. We have an error $E$ defined as:

$$E = \int\int F(p, q, p_x, p_y, q_x, q_y, x, y)dxdy \tag{23}$$

...where the error functional is:

$$F(p, q, p_x, p_y, q_x, q_y) = \lambda(p_x^2 + p_y^2 + q_x^2 + q_y^2) + (I(x, y) - R(p, q))^2 \tag{24}$$

If we use the Euler-Lagrange Equations $F_p - \frac{\partial}{\partial x}F_{p_x} - \frac{\partial}{\partial y}F_{p_y} = 0$ and $F_q - \frac{\partial}{\partial x}F_{q_x} - \frac{\partial}{\partial y}F_{q_y} = 0$, show that $p$ and $q$ according to:

$$p^{new}(x, y) = p_{avg} + \frac{1}{4\lambda}(I - R)\frac{\partial R}{\partial p} \tag{25}$$

$$q^{new}(x, y) = q_{avg} + \frac{1}{4\lambda}(I - R)\frac{\partial R}{\partial q} \tag{26}$$

...where $p_{avg}$ and $q_{avg}$ are teh averages of the neighbors of the $p$, $q$ respectively.

To tackle this, first we will find the partial derivatives of $F$ to $p$, $p_x$, $p_y$, $q$, $q_x$, and $q_y$.

$$F_p = \frac{\partial}{\partial p}F = \frac{\partial}{\partial p}\lambda(p_x^2 + p_y^2 + q_x^2 + q_y^2) + (I(x, y) - R(p, q))^2 = 2(I(x, y) - R(p, q))\frac{\partial R}{\partial p} \tag{27}$$

$$F_q = \frac{\partial}{\partial q}F = \frac{\partial}{\partial q}\lambda(p_x^2 + p_y^2 + q_x^2 + q_y^2) + (I(x, y) - R(p, q))^2 = 2(I(x, y) - R(p, q))\frac{\partial R}{\partial q} \tag{28}$$

$$F_{p_x} = \frac{\partial}{\partial p_x}F = 2\lambda p_x \tag{29}$$

$$F_{p_y} = \frac{\partial}{\partial p_y}F = 2\lambda p_y \tag{30}$$

$$F_{q_x} = \frac{\partial}{\partial q_x}F = 2\lambda q_x \tag{31}$$

$$F_{q_y} = \frac{\partial}{\partial q_y}F = 2\lambda q_y \tag{32}$$

...and we'll need the partial derivative of these for our Euler-Lagrange equations mentioned above as well.

$$\frac{\partial}{\partial x}Fp_x = 2\lambda p_{xx} \tag{33}$$

$$\frac{\partial}{\partial x}Fp_y = 2\lambda p_{yy} \tag{34}$$

$$\frac{\partial}{\partial x}Fq_x = 2\lambda q_{xx} \tag{35}$$

$$\frac{\partial}{\partial x}Fq_y = 2\lambda q_{yy} \tag{36}$$

We can combine these components in the Euler-Lagragne equation mentioned above to get:

$$F_p - \frac{\partial}{\partial x}F_{p_x} - \frac{\partial}{\partial y}F_{p_y} = 2(I(x,y) - R(p,q))\frac{\partial R}{\partial p} - 2\lambda p_{xx} - 2\lambda p_{yy} = 0 \tag{37}$$

$$F_q - \frac{\partial}{\partial x}F_{p_x} - \frac{\partial}{\partial y}F_{p_y} = 2(I(x,y) - R(p,q))\frac{\partial R}{\partial q} - 2\lambda q_{xx} - 2\lambda q_{yy} = 0 \tag{38}$$

If we then say that $\Delta q = q_{xx} + q_{yy}$ we can expand by claiming that $\Delta q = -\frac{1}{\lambda}(I(x,y) - R(p,q))\frac{\partial R}{\partial q}$, our Laplacian.

From here we swap to a discrete space versus the continuous space we've been working on, focusing on pixels. If we label a given $p_{i,j}$ where $(i,j)$ are the coordinates, we can do the following:

$$\Delta p_{ij} = (p_{i-1,j} + p_{i,j+1} + p_{i+1,j} + p_{i,j-1}) - 4p_{ij} = 4(\bar{p}_{i_j} - p_{ij}) \tag{39}$$

...where we define $\bar{p}_{i,j}$ as an average of these points, so:

$$\bar{p}_{i,j} = \frac{1}{4}(p_{i-1,j} + p_{i,j+1} + p_{i+1,j} + p_{i,j-1}) \tag{40}$$

meaning...

$$4(\bar{p}_{i_j} - p_{ij}) = \frac{-1}{\lambda}(I(x,y) - R(p,q))\frac{\partial R}{\partial p} \tag{41}$$

$$p_{ij} = \bar{p}_{i,j} + \frac{-1}{4\lambda}(I(x,y) - R(p,q))\frac{\partial R}{\partial p} \tag{42}$$

And for $q$:

$$4(\bar{q}_{i_j} - q_{ij}) = \frac{-1}{\lambda}(I(x,y) - R(p,q))\frac{\partial R}{\partial q} \tag{43}$$

$$q_{ij} = \bar{q}_{i,j} + \frac{-1}{4\lambda}(I(x,y) - R(p,q))\frac{\partial R}{\partial q} \tag{44}$$

We can apply these to an iterative (because we are working in discrete pixel space as opposed to our earlier continuous space), solving with $p_{i,j}^{k+1}$ and $p_{i,j}^{-k}$ as our iterative steps in place of $p_{i,j}$ and $\bar{p}_{i,j}$, respectively. This approach can also be instead said to be $p^{new}$ and $p_{old}$, respectively:

$$p^{new} = p^{avg} + \frac{1}{\lambda}(I(x,y) - R(p,q))\frac{\partial R}{\partial p} \tag{45}$$

...and similary for $q$:

$$q^{new} = q^{avg} + \frac{1}{\lambda}(I(x,y) - R(p,q))\frac{\partial R}{\partial q} \tag{46}$$