RBE549 - Homework 3

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Problem 1

Problem 2

In this problem, we are tasked with proving that if $f(x)$ is an odd function and purely imaginiary (ie no real component), then the Fourier Transform of $f(x)$, $F(x)$ is both real and odd.

We start with defining $F(x)$:

$$\int_{-\infty}^{\infty} f(x)e^{-j\omega x}dx \tag{1}$$

...and we can split this integral into two parts, specifically utilizing that $\int_{-A}^{A}$ can be split into muliple integrals of along the requested range, such that $\int_{-A}^{0} + \int_{0}^{A}$ is equivalent.

$$\int_{-\infty}^{0} f(x)e^{-j\omega x}dx + \int_{0}^{\infty} f(x)e^{-j\omega x}dx \tag{2}$$

We can then exapnd this with an idenity:

$$\int_{-\infty}^{0} f(x)(\cos(\omega x) - j\sin(\omega x))dx + \int_{0}^{\infty} f(x)(\cos(\omega x) - j\sin(\omega x))dx \tag{3}$$

Similarly to before, we can utilize a property of integrals to further separate this - specifically that $\int(A(x)+B(X)) = \int A(x) + \int B(x)$.

$$\int_{-\infty}^{0} f(x)\cos(\omega x)dx + \int_{-\infty}^{0} -jf(x)\sin(\omega x)dx + \int_{0}^{\infty} f(x)\cos(\omega x)dx + \int_{0}^{\infty} -jf(x)\sin(\omega x)dx \tag{4}$$

We can reorder terms to isolate the cosine terms, and move $-j$ (since it's a constant) out of the integral:

$$\int_{-\infty}^{0} f(x)\cos(\omega x)dx + \int_{0}^{\infty} f(x)\cos(\omega x)dx - j\int_{-\infty}^{0} f(x)\sin(\omega x)dx - j\int_{0}^{\infty} f(x)\sin(\omega x)dx \tag{5}$$

To prove that a function is odd, we must show that $f(-x) = -f(x)$. However, we know that cos is an even function, meaning $f(-x) = f(x)$. thus if we were to plug in $-x$, our negative sign does not escape the cosine, resulting in a cancellation of terms. This ultimately leads us to:

$$F(x) = -j\int_{-\infty}^{\infty} f(x)\sin(xt)dt \tag{6}$$

Since j is our imaginary component, and we have a singular component left, it shows that our Fourier transform is in fact completely imaginary. If we let $f(x) = jg(x)$ as $f(x)$ is purely imaginiary:

$$F(x) = -j\int_{-\infty}^{\infty} jg(x)\sin(xt)dx = -j^2\int_{-\infty}^{\infty} g(x)\sin(xt) = \int_{-\infty}^{\infty} g(x)\sin(xt) \tag{7}$$

This result is an odd function (even or odd functions multiplied by an odd function is odd) with no imaginary components!

Problem 3

Problem 4

For this problem we look at two kernels that are used to detect diagonal edges:

$$NE = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & -1 \end{bmatrix} \tag{8}$$

$$NW = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} \tag{9}$$

A

How are these operates related to Sobel H and Sobel V?

B

Now we aim to suggest two ways to combine these operators into a singular kernel that can identify northeast or northwest diagonals together, and discuss the potential problems with these combinations.

C

This problem asks us to express NW as a convolution of an unknown $2x2$ operator with the kernel of:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{10}$$

...or, if we labeled the above as $h(x)$, find $g(x)$ such that $g(x) * h(x) = NW$. The resulting matrix for $g(x)$ is:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{11}$$

Problem 5

We look at the outcome of convolving a Sobel vertical edge detector (V) with a $3x3$ blurring mask (B)

$$V = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} \tag{12}$$

$$B = \begin{bmatrix} 2 & 3 & 2 \\ 3 & 4 & 3 \\ 2 & 3 & 2 \end{bmatrix} \tag{13}$$

A

We ask - what is the result of the combined convolutional mask for applying V first, and then B? If we perform the convolution, we get a matrix that is a $5x5$:

$$\begin{bmatrix} -2 & -3 & 0 & 3 & 2 \\ -7 & -10 & 0 & 10 & 7 \\ -10 & -14 & 0 & 14 & 10 \\ -7 & -10 & 0 & 10 & 7 \\ -2 & -3 & 0 & 3 & 2 \end{bmatrix} \tag{14}$$

B

We are asked if convolving with B first, then V produces different results than convolving with V first, and then B. The answer is *no* - convolution has a commutative property - $f(x) * g(x) = g(x) * f(x)$. Thus we would have the same result we found in part A of this question.

C

In this part, we're asked if the convolution mask of $V * B$ seperable into a convolution of x-only and y-only masks. The answer here is *no* as well. In order for a convoluton mask to be seperable, the resulting matrix must have a rank of 1. The rank of our resulting mask is 2, thus precluding us from finding a seperable convolution.