RBE549 - Homework 7

Keith Chester

Due date: November 1, 2022

Problem 1

In this problem we are shown a number of shapes and then wish to find the $r(\theta)$ for the given shapes. The image below shows this:

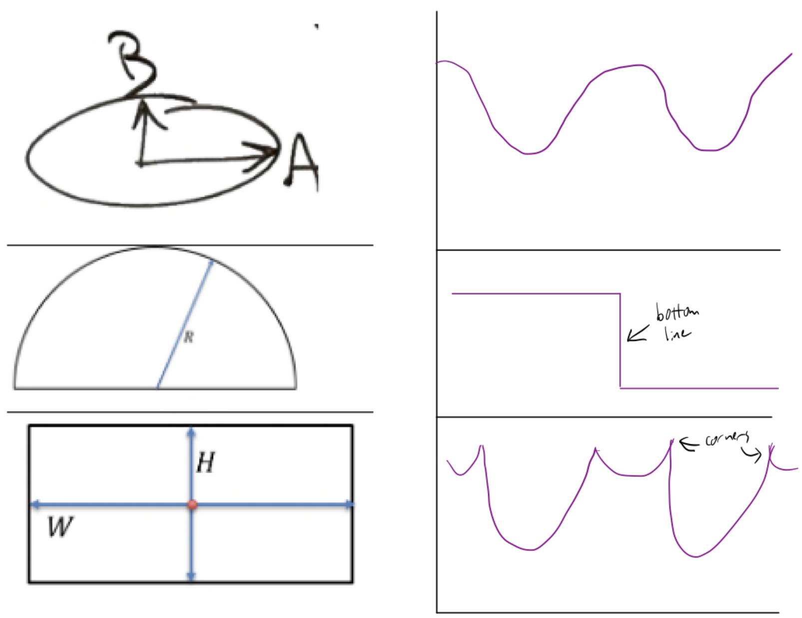

Figure 1: $r(\theta)$

Problem 2

In this problem we are aske a hypothetical problem. Supposing that we have a 2-class classification problem with centroid points μA and μB, and that each classes are equally likely, we need to show that the Nearest Mean classifier decision boundary is midway along the line segment connecting μA to μB.

First, we can look at the problem in 1, 2, and 3 dimensions. For a $1D$ classifier, the resulting points would fall on a line and the resulting bounary would be between the centroids. If we looked at a $2D$ classifier, we would see the resulting decision boundary would be a line. Finally, for a $3D$ classifier, the decision boundary would be a plane. In each we depict the boundaries at a set distance d between centroids.

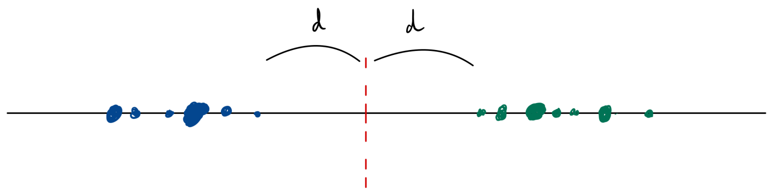

Figure 2: $1D$ Classifier

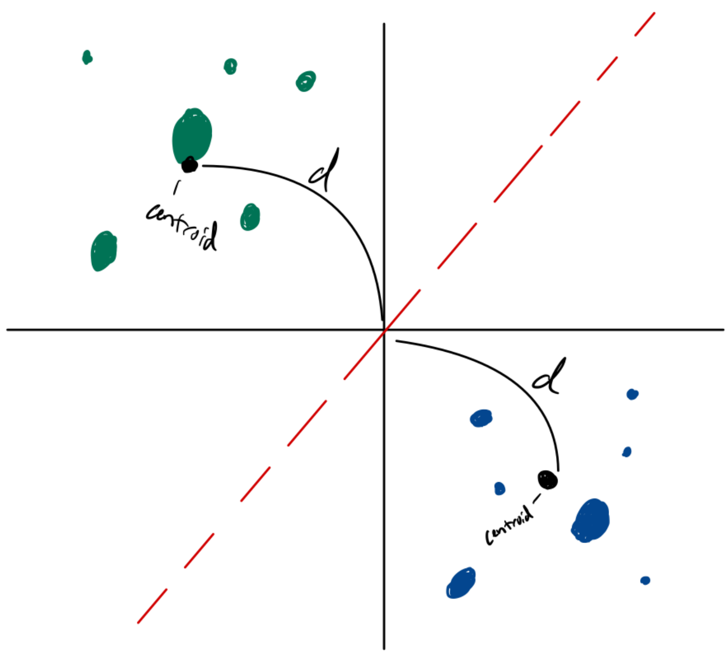

Figure 3: $1D$ Classifier

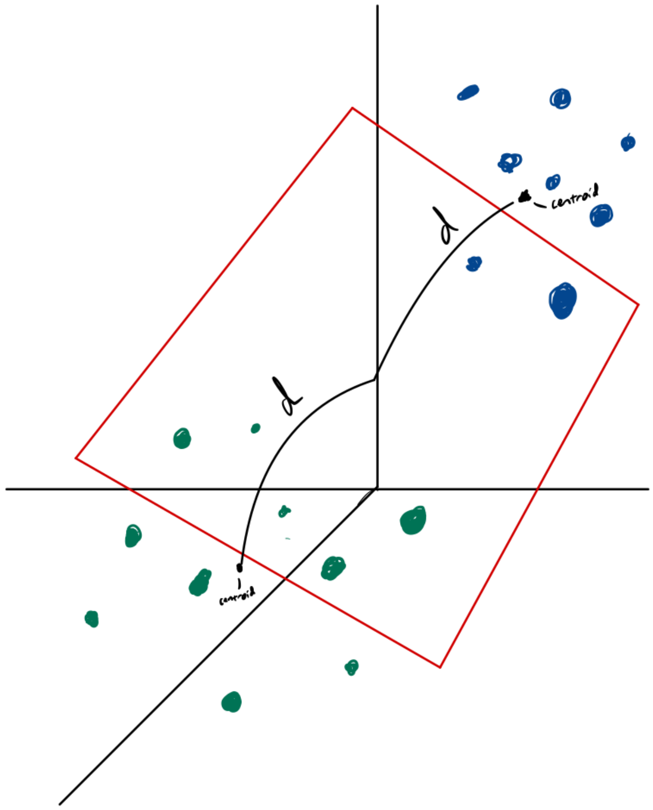

Figure 4: $3D$ Classifier

Since the problem states that the two classes A and B are equally likely, it means the the resulting centroids would be equidistant from a properly calculated Nearest Mean decision boundary. If either class had a higher probability than the other, the decision boundary as calculated via Nearest Mean would be closer to the centroid of that class.

Problem 3

In this problem we are asked to represent an object by its boundary $(x(s), y(s)), 0 \leq s \leq S$ where S is the length of the object's boundary and s is distance along that bonudary from some arbitrary starting point. Combine x and y into a single complex function $z(a) = x(s) + jy(s)$. The Discrete Fourier Transform (DFT) of z is:

$$Z(k) = \sum_{s=0}^{S-1} e^{-2*\pi j \frac{ks}{S}} z(s), 0 \leq k \leq S-1 \tag{1}$$

We can use the coefficients $Z(k)$ to represent the object boundary. The limit onf s is $S-1$ because for a closed contour $Z(S) = z(0)$. The Inverse Discrete Fourier Transform is:

$$z(s) = \frac{1}{S} \sum_{k=0}^{S-1} e^{+2\pi j \frac{ks}{S}} Z(k), 0 \leq s \leq S-1 \tag{2}$$

A

Here we suppose that the object is translated by $(\Delta x, \Delta y)$, such that $z'(s) = z(s) + \Delta x + j\Delta y$. How is z''s DFT $Z'(k)$ related to $Z(k)$? Stating again, that we start with:

$$Z(k) = \sum_{s=0}^{S-1} e^{-2*\pi j \frac{ks}{S}} z(s), 0 \leq k \leq S-1 \tag{3}$$

Which we can define $z'(s)$ as:

$$z'(s) = z(s) + \Delta x + j\Delta y \tag{4}$$

If we plug this into our original equation, we get...

$$Z'(k) = \sum_{s=0}^{S-1} e^{-2\pi j \frac{ks}{S}} z(s) + \sum_{s=0}^{S-1} e^{-2\pi j \frac{ks}{S}} (\Delta x + j\Delta y)(1) \tag{5}$$

Here we see that we have a segment that is equivalent to oru defined $Z(k)$, so we can simplify by expressing:

$$Z(k) + (1)(\Delta x + j\Delta y) \sum_{s=0}^{S-1} e^{-2\pi j \frac{ks}{S}} \tag{6}$$

With Δx and $j\Delta y$ isolated, we can use a table of known Fourier Transforms to identify the resulting conversion. Based on our problem's definition ΔX and Δy are both constants, so we can state that:

$$Z'(k) = Z(k) + (\Delta x + j\Delta y)\sigma(\frac{2\pi k}{S}) \tag{7}$$

...where $\frac{1}{S}$ acts as our scaling factor.

B

What object has $z(s) = R\cos(\frac{2\pi s}{S}) + jR\sin(\frac{4\pi s}{S})$? Below we see a graph drawing the shape, and included code to generate it. Random values were chosen for the S and r for the sake of plotting it.

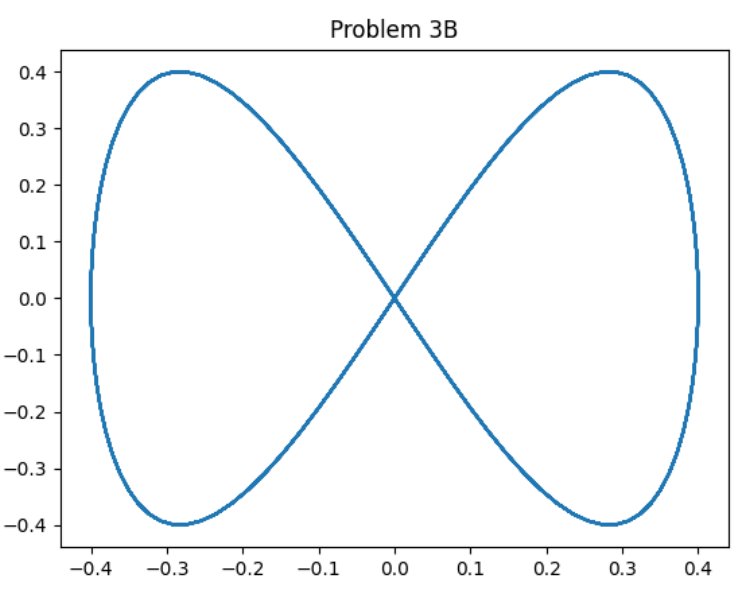

Figure 5: Our resulting shape

```python
import numpy as np
from numpy import cos, sin, pi
import matplotlib.pyplot as plt

figure = plt.figure()
plt.title("Problem 3B")

S = 10
r = 4

theta = [theta for theta in np.arange(0, S, 0.01)]
X = [
        r * cos(2*pi*theta)/S
        for theta in theta
    ]
Y = [
        r * sin(4*pi*theta)/S
        for theta in theta
    ]

# Plot the results
plt.plot(X, Y)
plt.savefig("./imgs/prob3_b.png")
```

C

In this problem we are tasked with showing what $Z(k)$ is corresponding to $z(s)$. We begin with our equation from B:

$$z(s) = R\cos(\frac{2\pi s}{S}) + jR\sin(\frac{4\pi s}{S}) \tag{8}$$

We can utilize the inverse of Euler's formula; $\cos(x) = \frac{e^{ix}+e^{-ix}}{2}$ and $\sin(t) = \frac{e^{ix}-e^{-ix}}{2x}$. This allows us to expand our starting equation:

$$z(s) = \frac{R(e^{j\frac{2\pi s}{S}} + e^{-j\frac{2\pi s}{S}})}{2} + \frac{R(e^{j\frac{4\pi s}{S}} - e^{-j\frac{4\pi s}{S}})}{2} \tag{9}$$

We can then begin simplifying our equation by moving out constants ($\frac{R}{2}$) and repositioning terms:

$$\frac{R}{2}(e^{j\frac{2\pi s}{S}} + e^{-j\frac{2\pi s}{S}} + e^{j\frac{4\pi s}{S}} - e^{-j\frac{4\pi s}{S}}) \tag{10}$$

Looking at the discrete fourier transform (DFT) from earlier in our problem, specifically, we can then begin to expand on this equation. Our starting equation is:

$$Z(k) = \sum_{s=0}^{S-1} e^{-2*\pi j\frac{ks}{S}} z(s), 0 \leq k \leq S-1 \tag{11}$$

...which can lead us to:

$$Z(k) = \sum_{s=0}^{S-1} e^{-2*\pi j\frac{ks}{S}} (\frac{R}{2}(e^{j\frac{2\pi s}{S}} + e^{-j\frac{2\pi s}{S}} + e^{j\frac{4\pi s}{S}} - e^{-j\frac{4\pi s}{S}})) \tag{12}$$

$$Z(k) = \frac{R}{2}\sum_{s=0}^{S-1} e^{-j2\pi(k-1)}S + e^{-j2\pi(k+1)}S + e^{-j2\pi(k-2)}S + e^{-j2\pi(k+2)}S \tag{13}$$

We know from a Fourier Transform lookup table that $e^{j2\pi\omega x_0}$ transforms to $2\pi\sigma(x - x_0)$, so applying this here:

$$Z(k) = \sigma(2\pi\frac{k-1}{S}) + \sigma(2\pi\frac{k+1}{S}) + \sigma(2\pi\frac{k-2}{S}) + \sigma(2\pi\frac{k+2}{S}) \tag{14}$$