Direct Methods for Optimal Control

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Introduction

Optimal Control Theory

It is a branch of mathematical optimization that deals with finding a control for a dynamical system over a period of time such that an objective function is optimized.

Problem Formulation

$$\min J := \Phi\big(x(t_0), t_0, x(t_f), t_f\big) \; + \; \int_{t_0}^{t_f} \mathcal{L}\big(x(t), u(t), t\big)\; dt$$

subject to

$$\dot{x}(t) = f(t, x(t), u(t)) \qquad \text{system dynamics}$$

$$h(t, x(t), u(t)) \leq 0 \qquad \text{path constraint}$$

$$g(t_0, t_f, x(t_0), x(t_f)) \leq 0 \qquad \text{boundary constraint}$$

Indirect Methods

- We seek a solution to the (closed system of) conditions of optimality.
- The conditions are derived using the well-known calculus of variations and the Euler-Lagrange differential equations, and *Pontryagin's maximum principle* (a necessary condition).
- The boundary value problem is numerically solved to determine candidate optimal trajectories called extremals.
- It thus follows '*optimize, then discretize*' approach.

Direct Methods

- Finitely parameterize the continuous decision variables, notably the controls $u(t)$ and sometimes state $x(t)$ such that the original problem is approximated by a finite dimensional nonlinear program (NLP).
- NLP can then be addressed by structurally exploiting numerical NLP solution methods.
- It thus follows '*discretize, then optimize*' approach.

Types of Direct Methods

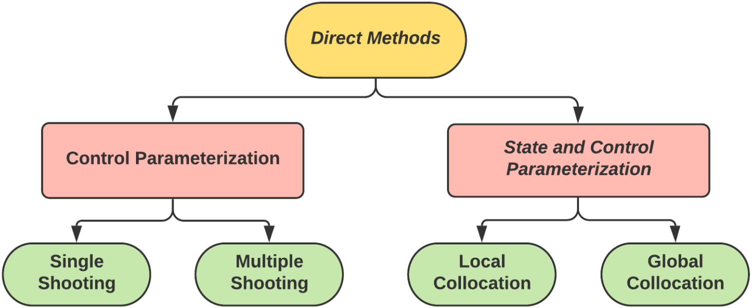

Figure: Types of Direct Methods

Direct Shooting Methods

All shooting methods use an embedded *ordinary differential equation* (ODE) or *differential-algebraic equation* (DAE) solver in order to eliminate the continuous time dynamic system.

Here we first **parameterize the control function** $u(t)$, e.g. by polynomials, by piecewise constant functions, or, more generally, by piecewise polynomials.

Single Shooting

Input: Initial Guess of Parameters in Control Parameterization

Output: Optimal Values of Parameters and Optimal Trajectory

while *Cost is Not at a Minimum and Constraints Not Satisfied* **do**

> Integrate Trajectory from t_0 to t_f;
>
> Compute Error in Terminal Conditions;
>
> Update Unknown Initial Conditions By Driving Cost to Lower Value;

end

Figure: Basic algorithm for single shooting

Integration of System Dynamics

This can be performed using methods like *Euler forward*, *Euler backward*, *Crank-Nicolson*, *Runge-Kutta*, *Hermite-Simpson* and many more depending upon the required accuracy.

Multiple Shooting

- Multiple shooting works by breaking up a trajectory into some number of segments, and using single shooting to solve for each segment.
- As the segments get shorter, the relationship between the decision variables and the objective function and constraints becomes *more linear*.
- The difference between the end of one segment with the start of the next is known as a 'defect', and it is added to the constraint vector.
- Adding all of the segments will increase the number of decision variables (the start of each segment) and the number of constraints (defects).

Multiple Shooting Schematic

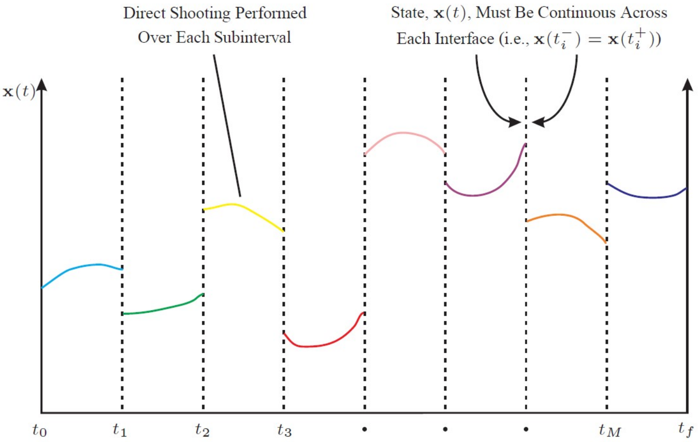

Figure: Schematic of direct multiple shooting

Shooting Methods Discussion

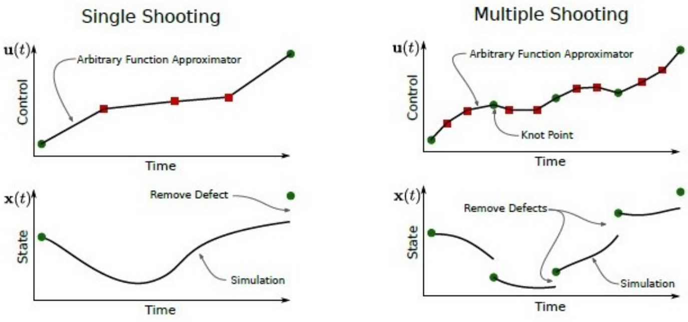

Figure: Single shooting and Multiple shooting

Discussions

Multiple shooting results in a higher dimensional NLP, but it is sparse and more linear than the program produced by single shooting.

Direct Collocation Methods

A direct collocation method is a state and control parameterization method where the state and control are approximated using a specified functional form.

The time derivative of the state is approximated by differentiating the interpolating polynomial and constraining the derivative at a finite set of *collocation points*.

Classification

The two most common forms of collocation are **local collocation** and **global collocation**. The methodology classification for direct collocation is *h-methods* and *p-methods*.

h-method and p-method

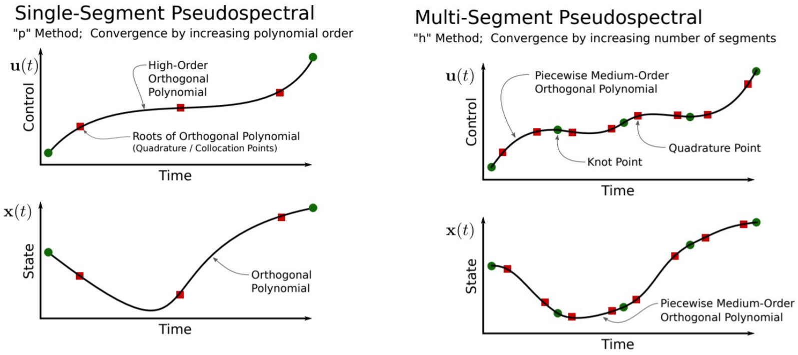

Figure: Comparison between h-method and p-method

Local Collocation

- The time interval is divided into a large number of subintervals called segments and a small number of collocation points are used within each segment.
- The segments are then linked via continuity conditions on the state, the independent variable, and possibly the control.
- The rationale for using local collocation is that a local method provides so-called local support.

Global Collocation

- As a global collocation method, the number of segments is set to unity and the accuracy is assessed as a function of the number of collocation points N.
- Global polynomial methods use a high-degree global polynomial (Chebyshev or Lagrange polynomials) for parameterization.
- The three most commonly used set of collocation points are Legendre-Gauss (LG), Legendre-Gauss-Radau (LGR), and Legendre-Gauss-Lobatto (LGL) points.

Collocation points

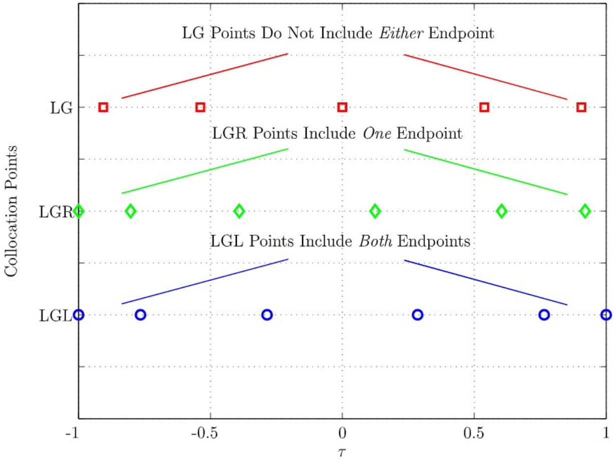

Figure: Differences between LGL, LGR, and LG collocation points

Pseudospectral Method

- In a pseudospectral method the state and control are approximated using orthogonal polynomial and collocation is performed at chosen points.
- The collocation points are the roots of an orthogonal polynomial (or linear combinations of such polynomials and their derivatives).
- For smooth problems, pseudospectral methods typically have faster convergence rates than other methods, exhibiting “spectral accuracy”.

Global and local collocation approaches

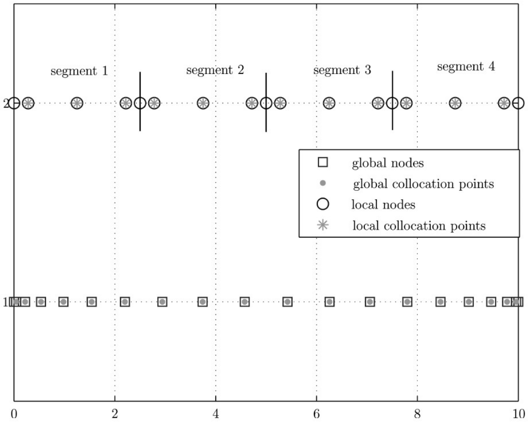

Figure: Distribution of nodes and collocation points for both the global and local approaches ($N = 20$)

Bryson-Denham Problem

The Bryson-Denham optimal control problem is a benchmark test problem for optimal control algorithms. Consider the system:

$$\begin{aligned} \dot{x}(t) &= v(t) \\ \dot{v}(t) &= u(t) \end{aligned} \tag{1}$$

Optimal Control Problem

$$\begin{aligned} &\text{minimize} \quad \frac{1}{2}\int_0^1 u(t)^2 \,\mathrm{d}t \\ &\text{subject to} \quad \begin{cases} \text{underlying system dynamics (1)}, \\ x(0) = x(1) = 0, \\ v(0) = -v(1) = 1, \\ x(t) \leq l, \quad \text{where } l = \frac{1}{9}. \end{cases} \end{aligned} \tag{2}$$

Single Shooting

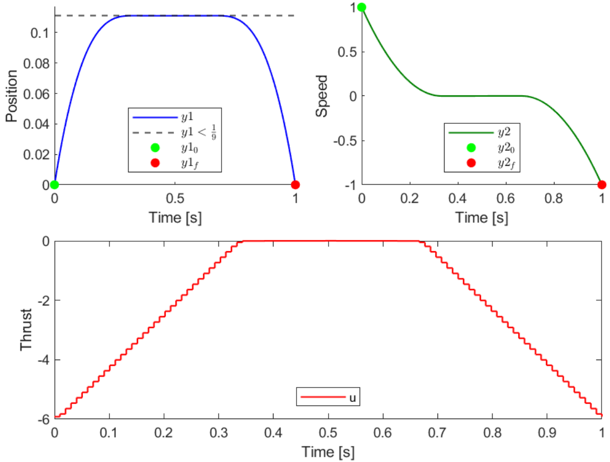

Figure: Bryson-Denham problem by single shooting

Multiple Shooting

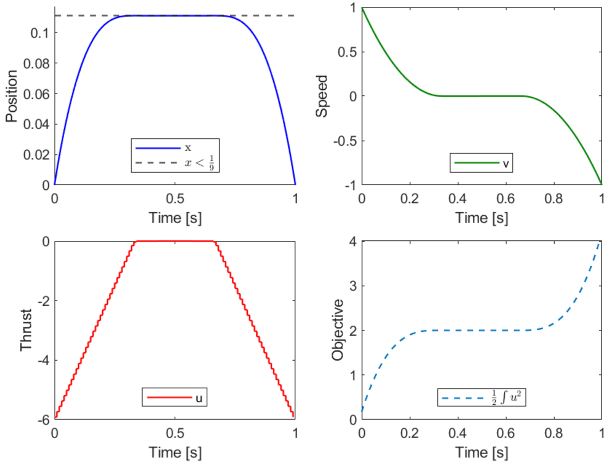

Figure: Bryson-Denham problem by multiple shooting

Direct Collocation

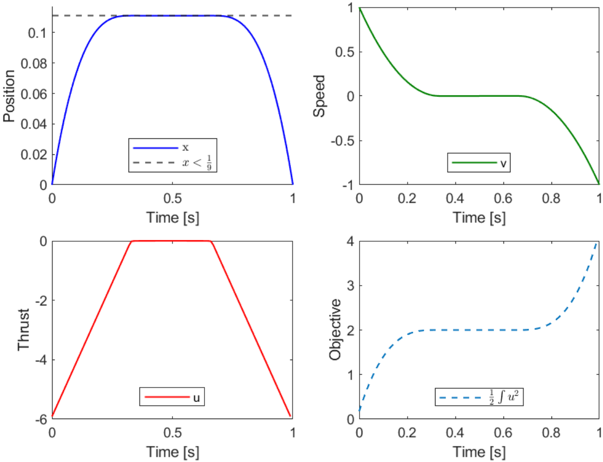

Figure: Bryson-Denham problem by direct collocation

Double Integrator Bang-Bang Problem

Consider the system $\ddot{x} = u, \quad u \in [-1, 1]$ which can represent a car with position $x \in \mathbb{R}$ and with bounded acceleration u acting as the control. Let us study the problem of *parking* the car at the origin, i.e., bringing it to rest at $x = 0$, in **minimal time**.

The dynamics are:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u$$

where $x_1 \in \mathbb{R}$ is the displacement of car and $x_2 \in \mathbb{R}$ is the velocity. We assume that the initial values and final values $x_1(t_0)$, $x_2(t_0)$, $x_1(t_f)$ and $x_2(t_f)$ are given (which are $(10, 0, 0, 0)$ respectively).

Single Shooting

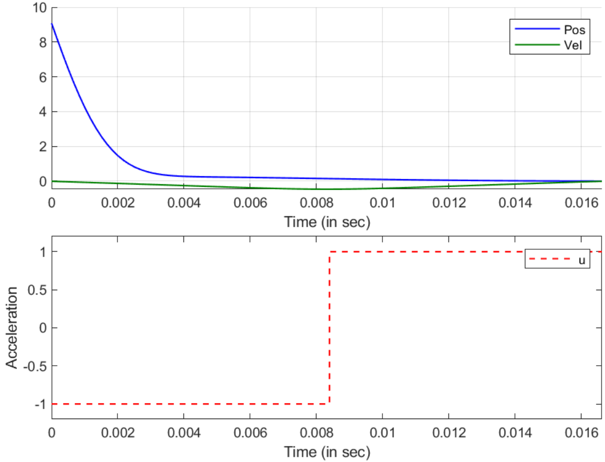

Figure: Double integrator bang-bang problem by single shooting

Multiple Shooting

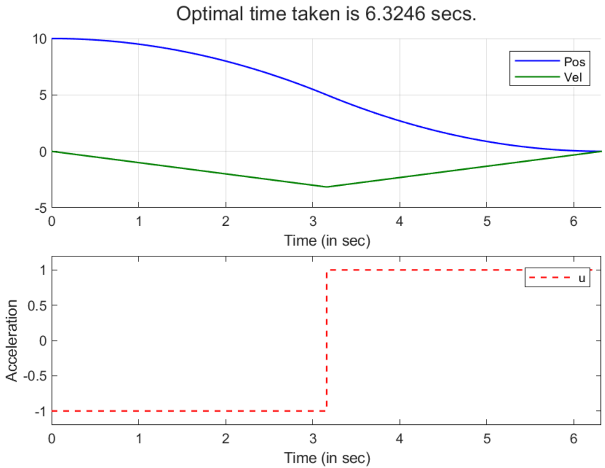

Figure: Double integrator bang-bang problem by multiple shooting

Direct Collocation

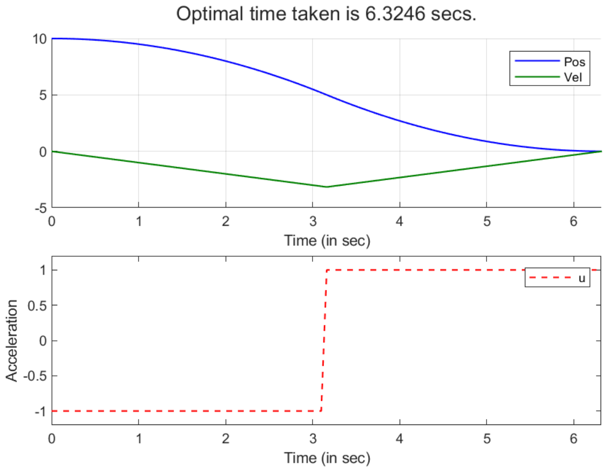

Figure: Double integrator bang-bang problem by direct collocation

Discussions and Conclusions

- The non-smoothness of the optimal control function in bang-bang problems might have caused single-shooting method to fail.
- When we went from direct single shooting to direct multiple shooting we essentially traded non-linearity for problem size.
- Direct collocation is to take one more step in the same direction, adding even more degrees of freedom. The resulting NLP is even larger, but has even more structure that can be exploited.
- Direct methods are easier to implement with effective incorporation of state and control constraints. But, this comes at the cost of accuracy and that is why it is important to choose an appropriate method based on the problem and requirements.

References

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Journal of Guidance, Control, and Dynamics, 31.

[1] Refer the report for more details

[2] All the codes are present in GitHub repository

The End