Weekly Status Report

Development of a 1D Adaptive Wavelet Collocation Code
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1 Summary of method

The adaptive wavelet collocation method for differential equations has a number of merits which may make it a viable alternative to traditional finite element, finite volume, or high order finite difference methods for certain problems. Some important advantages include the following:

- the multiresolution properties of wavelets allow for natural grid adaptation
- the existence of a fast $\mathcal{O}(\mathcal{N})$ scheme for computation of wavelet coefficients and spatial derivatives
- the ability of second-generation wavelets to adapt to complex geometries

1.1 Dyadic Grid

The method makes use of a dyadic grid. Each grid level is defined by

$$\mathcal{G}^j = \{x_k^j \in \Omega : k \in \mathcal{K}^j\}, j \in \mathcal{Z}, \tag{1}$$

where $\mathcal{K}^j$ is the integer set representing the spatial locations in the grid at level j. The grids are nested, implying that $\mathcal{G}^j \subset \mathcal{G}^{j+1}$. In other words, the points at level x^j are a perfect subset of the points at level x^{j+1}. This can also be demonsrated by the relation that $x_k^j = x_{2k}^{j+1}$.

1.2 Interpolating Subdivision

The interpolating subdivision scheme is central to the second-generation wavelet collocation approach. The scheme is used to approximate values at odd points x_{2k+1}^{j+1} by constructing interpolating polynomials of order $2N-1$ with the points at grid level j. Lagrange polynomials are used, and the method can be used with a uniform grid or with nonuniform points such as Chebychev points. The interpolating scheme is

$$f^j(x_{2k+1}^{j+1}) = \sum_{l=-N+1}^{N} w_{k,l}^j f(x_{k+l}^j), \tag{2}$$

where the coefficients $w^j_{k,l}$ are the lagrange polynomials $L_{2N,k+l}(x)$ evaluated at $x = x^{j+1}_{2k+1}$. The lagrange polynomial in this notation is given by

$$L_{2N,k+l}(x) = \prod_{i=k+l}^{2N+k+l} \tag{3}$$

For a uniform grid, the coefficients should be constant irrespective of the grid level j, reducing computational cost.

1.3 Wavelet Construction

The construction of second-generation wavelets makes use of the dyadic grid. A fast interpolating subdivision scheme is used to interpolate functional values defined at points on level j, to odd points (i.e. x^{j+1}_{2k+1}) at the next higher level of resolution. This scheme is used to construct the scaling and detail wavelet functions.

2 This week's goals

1. To use the scaling and detail wavelet functions, in conjunction with the forward wavelet transform, to approximate some initial function $u(x)$. A function $u(x)$ may be approximated by
$$u^J(x) = \sum_{k \in \mathcal{K}'} c^0_k \phi^0_k(x) + \sum_{j=0}^{J-1} \sum_{l \in \mathcal{L}^j} d^j_l \psi^j_l(x). \tag{4}$$
2. Once a function can be approximated, an algorithm to throw away small detail coefficients can be developed.
3. The grid points can then be altered using the binary tree structure.
4. Develop a code to calculate spatial derivatives as in Vasilyev & Bowman (2000).