2.006 Equation Sheet (Spring 2018)

Constitutive relationships (for specific variables, i.e. per unit mass)

Model	Specific energy	Specific enthalpy (h=u+Pv)	Specific entropy
Incompr. liquid	$u_2 - u_1 = c(T_2 - T_1)$	$h_2 - h_1 = c(T_2 - T_1) + \frac{1}{\rho}(P_2 - P_1)$	$s_2 - s_1 = c \ln \frac{T_2}{T_1}$
Ideal Gas	$u_2 - u_1 = c_v(T_2 - T_1)$	$h_2 - h_1 = c_p(T_2 - T_1)$	$s_2 - s_1 = c_v \ln \frac{T_2}{T_1} + R \ln \frac{v_2}{v_1}$ $s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{P_2}{P_1}$ $s_2 - s_1 = c_v \ln \frac{P_2}{P_1} + c_p \ln \frac{v_2}{v_1}$
Solid	$u_2 - u_1 = c(T_2 - T_1)$		$s_2 - s_1 = c \ln \frac{T_2}{T_1}$

Ideal Gas law: $PV = mRT$ or $P = \rho RT$, Ideal gas specific heats: $c_p = c_v + R$

Conservation Relations for Closed Systems

First law, closed systems: $\Delta E = Q - W$

Second law, closed systems: $\Delta S = \int \frac{\delta Q}{T} + S_{gen}$

Conservation Relations for Open Systems (using ϑ for velocity)

Mass conservation

(integral form):
$$\frac{d}{dt}\int_{CV} \rho \, dV + \int_{CS} \rho\left(\vec{\vartheta}_r . \vec{n}\right) dA = 0$$

(summation form):
$$\frac{dM_{CV}}{dt} = \sum_{in} \dot{m}_{in} - \sum_{out} \dot{m}_{out}$$

(differential form):
$$\frac{\partial \rho}{\partial t} + \vartheta . \nabla \rho + \rho(\nabla . \vartheta) = 0 \qquad \text{or} \qquad \frac{D\rho}{Dt} + \rho \nabla \cdot \vartheta = 0$$

Energy conservation (First Law)

(integral form):
$$\frac{d}{dt}\int_{CV} \rho\left(u + \frac{\vartheta^2}{2} + gz\right) dV = \dot{Q} - \dot{W}_{shaft} - \int_{CS} \rho\left(h + \frac{\vartheta^2}{2} + gz\right)\left(\vec{\vartheta}_r . \vec{n}\right) dA$$

(summation form):
$$\frac{dE_{CV}}{dt} = \dot{Q} - \dot{W}_{shaft} + \sum_{in} \dot{m}_{in}\left(h + \frac{\vartheta^2}{2} + gz\right)_{in} - \sum_{out} \dot{m}_{out}\left(h + \frac{\vartheta^2}{2} + gz\right)_{out}$$

Second Law

(integral form):
$$\frac{d}{dt}\int_{CV} \rho s \, dV = \sum_i \left(\frac{\dot{Q}}{T}\right)_i - \int_{CS} \rho s\left(\vec{\vartheta}_r . \vec{n}\right) dA + \dot{S}_{gen}$$

(summation form): $$\frac{dS_{CV}}{dt} = \sum_i \left(\frac{\dot{Q}}{T} \right)_i + \sum_{in} \left(\dot{m}\, s \right)_{in} - \sum_{out} \left(\dot{m}\, s \right)_{out} + \dot{S}_{gen}$$

Linear momentum conservation

In an inertial frame:

(integral form): $$\sum F_{ext} = \int_{CS} -PndA + \int_{CS} \tau\, dA + \int_{CV} \rho g dV = \frac{d}{dt} \int_{CV} \rho \vartheta dV + \int_{CS} \rho \vartheta (\vartheta_r . n) dA$$

(summation form): $$\frac{dP_{CV}}{dt} = \sum \dot{m}_{in}\, \vartheta_{in} - \sum \dot{m}_{out}\, \vartheta_{out} + \sum F_{ext}$$

In a non-inertial frame: $$\sum F - \int_{CV} \rho a_{rf} dV = \frac{d}{dt} \int_{CV} \rho \vartheta dV + \int_{CS} \rho \vartheta (\vartheta_r . n) dA$$

Angular momentum conservation

(integral form): $$\sum T = \frac{d}{dt} \int_{CV} \rho (r \times \vartheta) dV + \int_{CS} \rho (r \times \vartheta)(\vartheta_r . n) dA$$

(summation form): $$\frac{dL_{CV}}{dt} = \sum \dot{m}_{in} (r \times \vartheta)_{in} - \sum \dot{m}_{out} (r \times \vartheta)_{out} + \sum T_{ext}$$

Bernoulli Equation (using ϑ for velocity)

$$\int_1^2 \frac{\partial \vartheta}{\partial \mathrm{t}} ds + \frac{P_2 - P_1}{\rho} + \frac{\vartheta_2^2 - \vartheta_1^2}{2} + g(z_2 - z_1) = 0\,.$$

In terms of potential: $$\frac{\partial \phi}{\partial \mathrm{t}} + \frac{P}{\rho} + \frac{|\nabla \phi|^2}{2} + gz = \mathrm{const}$$

Viscous Flow (incompressible, constant-viscosity fluid) (using ϑ for velocity)

Cartesian co-ordinates:

Mass conservation (continuity equation): $$\frac{\partial \vartheta_x}{\partial x} + \frac{\partial \vartheta_y}{\partial y} + \frac{\partial \vartheta_z}{\partial z} = 0$$

Navier-Stokes equations:

$$\rho \left(\frac{\partial \vartheta_x}{\partial t} + \vartheta_x \frac{\partial \vartheta_x}{\partial x} + \vartheta_y \frac{\partial \vartheta_x}{\partial y} + \vartheta_z \frac{\partial \vartheta_x}{\partial z} \right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu \left(\frac{\partial^2 \vartheta_x}{\partial x^2} + \frac{\partial^2 \vartheta_x}{\partial y^2} + \frac{\partial^2 \vartheta_x}{\partial z^2} \right)$$

$$\rho \left(\frac{\partial \vartheta_y}{\partial t} + \vartheta_x \frac{\partial \vartheta_y}{\partial x} + \vartheta_y \frac{\partial \vartheta_y}{\partial y} + \vartheta_z \frac{\partial \vartheta_y}{\partial z} \right) = -\frac{\partial P}{\partial y} + \rho g_y + \mu \left(\frac{\partial^2 \vartheta_y}{\partial x^2} + \frac{\partial^2 \vartheta_y}{\partial y^2} + \frac{\partial^2 \vartheta_y}{\partial z^2} \right)$$

$$\rho \left(\frac{\partial \vartheta_z}{\partial t} + \vartheta_x \frac{\partial \vartheta_z}{\partial x} + \vartheta_y \frac{\partial \vartheta_z}{\partial y} + \vartheta_z \frac{\partial \vartheta_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu \left(\frac{\partial^2 \vartheta_z}{\partial x^2} + \frac{\partial^2 \vartheta_z}{\partial y^2} + \frac{\partial^2 \vartheta_z}{\partial z^2} \right)$$

Newtonian viscous shear stresses:

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial \vartheta_x}{\partial y} + \frac{\partial \vartheta_y}{\partial x}\right) \qquad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial \vartheta_x}{\partial z} + \frac{\partial \vartheta_z}{\partial x}\right) \qquad \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial \vartheta_y}{\partial z} + \frac{\partial \vartheta_z}{\partial y}\right)$$

Cylindrical co-ordinates

Mass conservation:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\vartheta_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\vartheta_\theta) + \frac{\partial}{\partial z}(\vartheta_z) = 0$$

Navier-Stokes equations:

$$\rho\left(\frac{\partial \vartheta_r}{\partial t} + \vartheta_r\frac{\partial \vartheta_r}{\partial r} + \frac{\vartheta_\theta}{r}\frac{\partial \vartheta_r}{\partial \theta} - \frac{\vartheta_\theta^2}{r} + \vartheta_z\frac{\partial \vartheta_r}{\partial z}\right) = -\frac{\partial P}{\partial r} + \rho g_r + \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\vartheta_r)\right) + \frac{1}{r^2}\frac{\partial^2 \vartheta_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial \vartheta_\theta}{\partial \theta} + \frac{\partial^2 \vartheta_r}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial \vartheta_\theta}{\partial t} + \vartheta_r\frac{\partial \vartheta_\theta}{\partial r} + \frac{\vartheta_\theta}{r}\frac{\partial \vartheta_\theta}{\partial \theta} + \frac{\vartheta_\theta\vartheta_r}{r} + \vartheta_z\frac{\partial \vartheta_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \rho g_\theta + \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\vartheta_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 \vartheta_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial \vartheta_r}{\partial \theta} + \frac{\partial^2 \vartheta_\theta}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial \vartheta_z}{\partial t} + \vartheta_r\frac{\partial \vartheta_z}{\partial r} + \frac{\vartheta_\theta}{r}\frac{\partial \vartheta_z}{\partial \theta} + \vartheta_z\frac{\partial \vartheta_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \vartheta_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \vartheta_z}{\partial \theta^2} + \frac{\partial^2 \vartheta_z}{\partial z^2}\right)$$

Newtonian viscous shear stresses:

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left(r\frac{\partial}{\partial r}\left(\frac{\vartheta_\theta}{r}\right) + \frac{1}{r}\frac{\partial \vartheta_r}{\partial \theta}\right) \qquad \tau_{z\theta} = \tau_{\theta z} = \mu\left(\frac{\partial \vartheta_\theta}{\partial z} + \frac{1}{r}\frac{\partial \vartheta_z}{\partial \theta}\right) \qquad \tau_{rz} = \tau_{zr} = \mu\left(\frac{\partial \vartheta_r}{\partial z} + \frac{\partial \vartheta_z}{\partial r}\right)$$

Viscous flow in Pipes(using ϑ for velocity)

Total Head Loss: $\left(\frac{P}{\rho g} + \alpha\frac{\vartheta^2}{2g} + z\right)_{in} - \left(\frac{P}{\rho g} + \alpha\frac{\vartheta^2}{2g} + z\right)_{out} = \sum f\frac{L}{D}\frac{\vartheta^2}{2g} + \sum K\frac{\vartheta^2}{2g} - h_{pump} + h_{turbine}$

where α = 2 for laminar flow, 1.06 for turbulent flow, or 1 for uniform flow

Friction factor for fully developed laminar flow in circular pipes: $f = 64/\mathrm{Re}_d$

Friction factor for fully developed turbulent flow: $f = \left[-1.8\log_{10}\left(\frac{6.9}{\mathrm{Re}_d} + \left(\frac{\varepsilon/d}{3.7}\right)^{1.11}\right)\right]^{-2}$

Reynolds Number for pipe flow: $\mathrm{Re} = \frac{\rho\vartheta_{ave}D}{\mu}$ **or** $\mathrm{Re} = \frac{\vartheta_{ave}D}{\nu}$

Darcy's Law for a single pipe flow:

$$\Delta P = f\frac{L}{D}\frac{\rho\vartheta_{ave}^{\ 2}}{2}$$

Velocity profile for laminar flow: $v/v_{\max} = \left(1-(r/R)^2\right)$

Velocity profile for turbulent flow: $v/v_{\max} = (1-r/R)^{1/n}$ $6 < \mathrm{n} < 10$

Pipe Entrance Lengths:

Laminar: $$\frac{L_e}{D_i} \approx 0.06\,\mathrm{Re}_D$$

Turbulent (smooth walls): $$\frac{L_e}{D_i} \approx 4.4\,\mathrm{Re}_D^{\ 1/6}$$

Boundary Layers

Boundary layers on a smooth flat plate:

	Laminar ($10^3<\mathrm{Re}_x<10^6$)		Turbulent ($10^6<\mathrm{Re}_x$)
	Laminar, von Karman (quadratic profile)	**Laminar, exact** (Blasius similarity solution)	**Turbulent** 1/7th power law
BL thickness	$\frac{\delta}{x} = \frac{5.5}{\sqrt{\mathrm{Re}_x}}$	$\frac{\delta}{x} = \frac{5}{\sqrt{\mathrm{Re}_x}}$	$\frac{\delta}{x} \cong \frac{0.16}{\mathrm{Re}_x^{1/7}}$
Displacement thickness	$\frac{\delta^*}{x} = \frac{1.83}{\sqrt{\mathrm{Re}_x}}$	$\frac{\delta^*}{x} = \frac{1.72}{\sqrt{\mathrm{Re}_x}}$	$\frac{\delta^*}{x} \cong \frac{0.020}{\mathrm{Re}_x^{1/7}}$
Momentum thickness	$\theta = \frac{2}{15}\delta$	$\frac{\theta}{x} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$	$\frac{\theta}{x} \cong \frac{0.016}{\mathrm{Re}_x^{1/7}}$
Local skin friction coefficient	$C_{f,x} = \frac{0.73}{\sqrt{\mathrm{Re}_x}}$	$C_{f,x} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$	$C_{f,x} \cong \frac{0.027}{\mathrm{Re}_x^{1/7}}$
Drag coefficient	$C_{D,x} = \frac{1.46}{\sqrt{\mathrm{Re}_x}}$	$C_{D,x} = \frac{1.328}{\sqrt{\mathrm{Re}_x}}$	$C_{D,x} \cong \frac{0.031}{\mathrm{Re}_x^{1/7}}$

Blasius Velocity Profile:

$y[U/(\nu x)]^{1/2}$	u/U	$y[U/(\nu x)]^{1/2}$	u/U
0.0	0.0	2.8	0.81152
0.2	0.06641	3.0	0.84605
0.4	0.13277	3.2	0.87609
0.6	0.19894	3.4	0.90177
0.8	0.26471	3.6	0.92333
1.0	0.32979	3.8	0.94112
1.2	0.39378	4.0	0.95552
1.4	0.45627	4.2	0.96696
1.6	0.51676	4.4	0.97587
1.8	0.57477	4.6	0.98269
2.0	0.62977	4.8	0.98779
2.2	0.68132	5.0	0.99155
2.4	0.72899	∞	1.00000
2.6	0.77246		

For general flat plate boundary layer flow (laminar or turbulent):

Wall shear stress: $\tau_w = \rho U_0^2 \dfrac{d\theta}{dx}$

Momentum thickness: $\theta = \displaystyle\int_0^\delta (1-\frac{u}{U_0})\frac{u}{U_0}\,dy$ Displacement thickness: $\delta^* = \displaystyle\int_0^\delta (1-\frac{u}{U_0})\,dy$

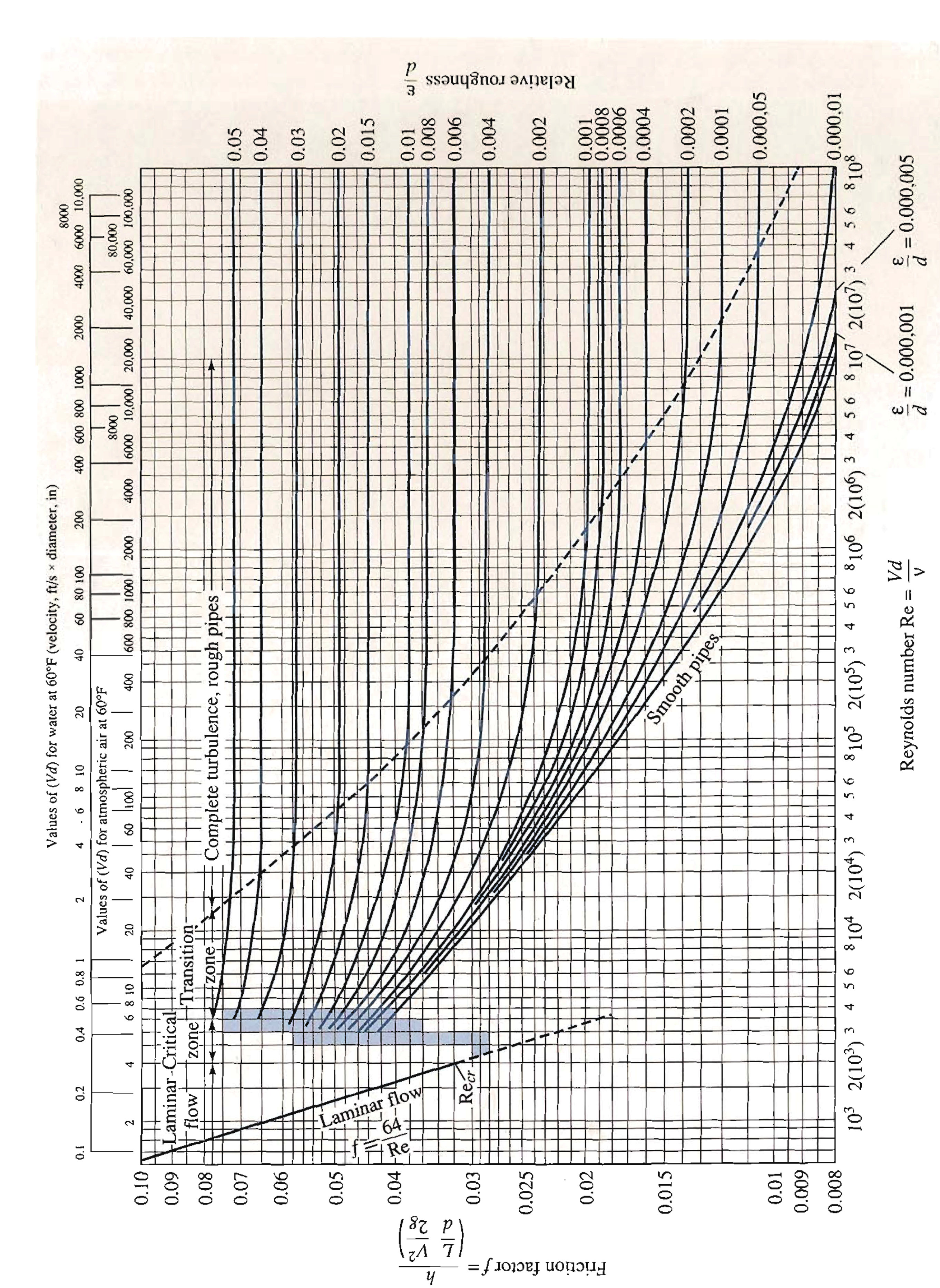
Friction coefficient: $C_f = \dfrac{\tau_w}{\rho U_0^2/2}$

Drag coefficient: $C_D = F_D / (0.5\ \rho\ U^2\ A)$

Law of the Wall: $\dfrac{\bar{u}}{\bar{u}^*} = \dfrac{1}{\kappa}\ln\dfrac{y\bar{u}^*}{\nu} + B$ where $\bar{u}^* = \sqrt{\tau_w/\rho}$, $\kappa = 0.41$ and $B = 5$

and $\nu = \mu / \rho$

Hydraulic Diamater: $D_h = 4A / P$


Values of (Vd) for water at 60°F (velocity, ft/s × diameter, in)
Values of (Vd) for atmospheric air at 60°F
Laminar flow
Critical zone
Transition zone
Complete turbulence, rough pipes
Laminar flow
f = 64/Re
Re_cr
Smooth pipes
Friction factor f = h / ((L/d)(V²/2g))
Reynolds number Re = Vd/ν
Relative roughness ε/d
ε/d = 0.000,001
ε/d = 0.000,005