MATH2033 Final Notes

May 17, 2025

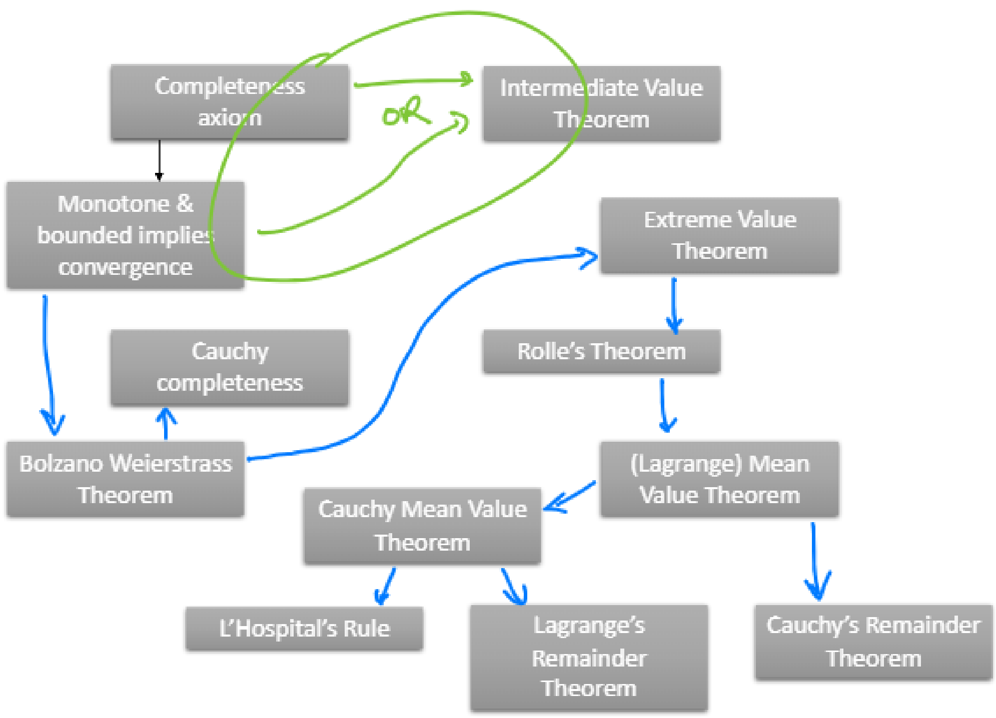
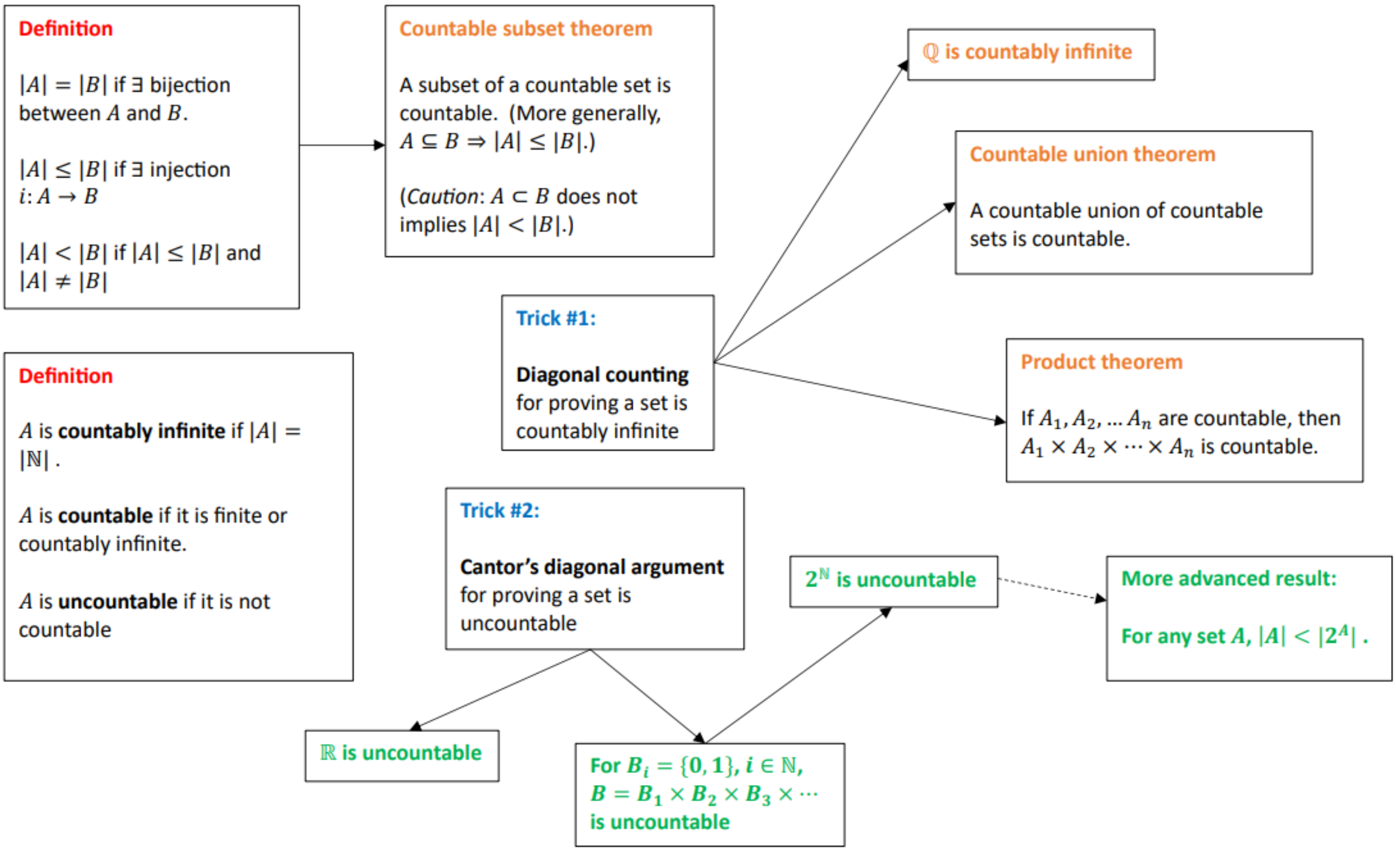


Figure 1: The overall logic of MATH1023

1 Sets, Sequence, Series

1.1 countability



1.2 sup, inf, real numbers

Facts that can be directly used:

- If $x \le C$ or $x < C$ for all $x \in S$, then C is an upper bound of S, and
$$\sup S \le C$$
- $\forall x, y \in A$, we have
$$|x - y| \le \sup A - \inf A$$

1.3 Limits

- **Supreme Limit Theorem**: Given a nonempty set $S \subset \mathbb{R}$ with an upper bound c, then $c = \sup S$ iff there exists a sequence $\{x_n\} \subset S$ such that $\lim_{n\to\infty} x_n = c$.
- **Interwining Sequence Theorem**: If $\{a_{2n}\}$ and $\{a_{2n+1}\}$ converge to the same limit L, then $\{a_n\}$ converges to L.

1.4 series convergence

- **Summation by parts** Suppose $S_j = \sum_{k=1}^{j} a_k$, then
$$\sum_{k=1}^{n} a_k b_k = S_n b_n - \sum_{k=1}^{n-1} S_k (b_{k+1} - b_k).$$
- **Dirichelet's test**: If $\sum_{k=1}^{\infty} a_k$ is bounded and $\{b_k\}$ is monotone decreasing that converges to 0, then $\sum_{k=1}^{\infty} a_k b_k$ converges.
- **Generalized Dirichelet's test**: The above test can be generalized to the case where $\{b_k\}$ is not monotone decreasing, but $\sum_{k=1}^{\infty} |b_k - b_{k+1}|$ converges and $b_k \to 0$.
- **Cauchy Product Theorem**: If $\sum_{k=0}^{\infty} a_k$ converges *absolutely* and $\sum_{k=0}^{\infty} b_k$ converges, then
$$\sum_{k=0}^{\infty} a_k b_k = \sum_{k=0}^{\infty} a_k \cdot \sum_{k=0}^{\infty} b_k.$$

2 ε-type arguments

It's a good practice to separate terms in $f(x)$ and $|x - x_0|$, either by sum or by product.

- Left as sketch paper -

3 Function continuity and differentiability

3.1 Function limits

The following are some theorems that may be negleted but useful:

- **Sequential Limit Theorem**: For $f : S \to \mathbb{R}$, $\lim\limits_{x \to x_0} f(x) = L$ iff

  $$\forall \{x_n\} \subset S \setminus \{x_0\} \text{ s.t. } x_n \to x_0 \implies \lim_{n \to \infty} f(x_n) = L.$$

  In fact, if $\lim\limits_{n \to \infty} f(x_n)$ exists for every such $x_n \to x_0$, then all the limit values are the same. Using this theorem, we can also easily show a function limit do not exist by finding 2 sequences with different limits.

- **Monotone Function Theorem**: If f is increasing on (a, b), then for any $x_0 \in (a, b)$,

  $$f(x_0^-) = \sup\{f(x) | a < x < x_0\},$$

  and if it is bounded below, then

  $$f(a^+) = \inf\{f(x) | x_0 < x < b\}.$$

  Jump Discontinuity: moreover, under the condition above, the following set is countable

  $$\{x \in (a, b) | f(x^+) \neq f(x^-)\}.$$

The following limits can be cited directly:

- Let $f(x) = x \sin(\frac{1}{x})$ whenever $x \neq 0$, and $f(0) = 0$, then

  $$\lim_{x \to 0} f(x) = \lim_{x \to 0} x \sin(\frac{1}{x}) = 0,$$

  which is NOT a result of things like L'Hospital's rule, but a result of the fact that $\sin(\frac{1}{x})$ is bounded plus the Squeeze Theorem.

- For any $n \in \mathbb{N}$, let $p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, then

  $$\lim_{x \to \infty} p_n(x) = \begin{cases} \infty & \text{if } a_n > 0 \\ -\infty & \text{if } a_n < 0 \end{cases}$$

  and

  $$\lim_{x \to 0} \frac{1}{p_n(x)} = 0.$$

3.2 Function continuity

- **Sequential Continuity Theorem**: A function f is continuous at x_0 iff

  $$\forall \{x_n\} \subset S \setminus \{x_0\} \text{ s.t. } x_n \to x_0 \implies \lim_{n \to \infty} f(x_n) = f(x_0).$$

- **Inverse Function Theorem**: covered at the end of `lecture 4` and the beginning of `lecture 5`.

- All of the following require to be continuous on $[a, b]$: EVT, Rolle's, MVT, **Generalized MVT**, which is: f, g continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that

  $$g'(c)(f(b) - f(a)) = f'(c)(g(b) - g(a)).$$

- Relationship between derivative and monotonicity is a little bit complex issue, presented at **Curve Tracing Theorem** and **Local Tracing Theorem** in `lecture 5` Page 6-7.

Remark: When finding things like $f(b) - f(a)$ with $f'(x)$ consider use the MVT or GMVT for proving equations or inequalities.

4 Taylor Approximation, Lagrange Remainder, Taylor Series

4.1 Taylor's Theorem

Let $f : (a, b) \to \mathbb{R}$ be n times differentiable on (a, b). For every $x, c \in (a, b)$, there exists x_0 between x and c such that

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + R_n(x),$$

where $R_n(x)$ is the Lagrange remainder:

$$R_n(x) = \frac{f^{(n+1)}(x_0)}{(n + 1)!}(x - c)^{n+1}.$$

As a corralary, if $f \in C^\infty$, and for $x \in (a, b)$, if $\lim_{n\to\infty} R_n(x) = 0$, then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x - c)^k,$$

which is called the Taylor series of f at c.
Remark: This is the result of MVT and is a property on an interval.

4.2 Taylor Polynomial

Given that f is n-times differentiable at $x = a$, then

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k + o((x - a)^n)$$

as $x \to a$. The polynomial $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k$ which we will denote by $T_a^n(x)$ or simply $T_n(x)$ in this course, is called the n-th degree Taylor polynomial (or approximation) of f near $x = a$. This can be used to approximate $f(x)$ for x near a, and will be useful in n-th derivative tests.
A function f is said to be real analytic at $x = a$ if there exists a neighborhood of a such that

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x - a)^k$$

for all x in that neighborhood.
Remark: This is the result of L'Hospital's rule and is a property at a point.

4.3 Techniques

- **Remark 1:** When dealing with m-times differentiable functions and terms like f, f', f'' with coefficients such as $\frac{1}{2}, \frac{1}{6}$, consider using Taylor's theorem.
- When finding sup or inf, use the facts:

  $$\forall x \in S,\ x \le M \implies \sup S \le M$$
  $$\exists x \in S,\ x \le M \implies \inf S \le M$$

  where M can be any value.
- **Remark 2:** For expressions like $(f(x) + f'(x))$ with information about f, consider constructing a helper function $F(x)$ as a combination of $f(x)$ and e^x.
- **Remark 3:** When applying L'Hospital's rule, remember it cannot be used to prove divergence. For example:

  $$\lim_{x\to 0} \frac{x^2 \sin \frac{1}{x}}{\sin x}, \qquad \lim_{x\to\infty} \frac{x + \sin x}{x}$$

5 Integration

5.1 integral criterion

If f is bounded on $[a,b]$, f is integrable if and only if, given any $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that

$$U(f,P) - L(f,P) < \varepsilon,$$

where

$$U(f,P) = \sum_{i=1}^{n} M_i \Delta x_i, \quad L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i,$$

and M_i and m_i are the supremum and infimum of f on the interval $[x_{i-1}, x_i]$, respectively.

For a bounded function f on $[a,b]$, f is integrable on $[a,b]$ iff f is continuous everywhere except on a set of measure zero, i.e. the set

$$\{x \in [a,b] | S_f \text{ is discontinuous at } x\}$$

is a set of measure zero (make sure it's continuous on other points), i.e. for any $\varepsilon > 0$, there exists countable number of intervals (a_i, b_i) such that

$$\sum_{i=1}^{\infty} (b_i - a_i) < \varepsilon, \text{ and } S_f \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i)$$

Note: A countable union of sets of measure zero is a set of measure zero.

Given this, we may find the following to be integrable:

- By Monotone Function Theorem, if f is monotone on $[a,b]$, then f is integrable on $[a,b]$.
- If f is continuous except a convergent sequence of points $x_n \to c$ in $[a,b]$, then f is integrable on $[a,b]$. This follows directly from choosing proper δ, DO NOT use the tidious function value when encontering some realization of such f.

5.2 integral rules

- **FTC:**
  - If f is integrable on $[a,b]$, continuous at $x_0 \in [a,b]$ and $F(x) = \int_a^x f(t)dt$, then F is continuous on $[a,b]$ and differentiable at x_0, and$F'(x_0) = f(x_0)$.
  - If G is differentiable on $[a,b]$ and $G'(x) = g(x)$ for all $x \in [a,b]$, then$\int_a^b g(x)dx = G(b) - G(a)$.
- **By parts:** If f, g are continuous on $[a,b]$, then$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b g(x)f'(x)dx$.
- **Substitution:** If f is continuous on $[a,b]$, and g is differentiable on $[c,d]$ then $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$

5.3 integral test

- **p-test**

$$\int_1^{\infty} \frac{1}{x^p} dx \text{ converges iff } p > 1$$
$$\int_0^{1} \frac{1}{x^p} dx \text{ converges iff } p < 1$$

- **Comparison Test**
- **Limit Comparison Test**: Suppose f, g are positive functions on $\mathbb{R}$, define $L := \lim_{x\to\infty} \frac{f(x)}{g(x)}$, then
  - If $0 < L < \infty$, then $\int_1^{\infty} f(x)dx$ converges iff $\int_1^{\infty} g(x)dx$ converges.
  - If $L = 0$, $\int_1^{\infty} f(x)dx$ converges $\implies$ $\int_1^{\infty} g(x)dx$ converges.
  - If $L = \infty$, $\int_1^{\infty} g(x)dx$ converges $\implies$ $\int_1^{\infty} f(x)dx$ converges.
- **Absolute Convergence Test**

Remark: When nothing works, consider the function

$$F(x) = \int_1^x f(t)dt$$

and calculate the limit directly. Theoretically, this will always give a result.

A Appendix

A.1 Limits that can be used directly

$$\lim_{x\to 0}\frac{\sin x}{x} = 1,$$
$$\lim_{x\to 0}\frac{x^r}{e^x} = 0 \text{ for any } r\in\mathbb{R},$$
$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e,$$
$$\lim_{x\to 0}(1+x)^{1/x} = e,$$

A.2 Triangular Equations

$$\begin{aligned}
\sin^2\theta + \cos^2\theta &= 1\\
1+\tan^2\theta &= \sec^2\theta\\
1+\cot^2\theta &= \csc^2\theta\\
\tan\theta &= \frac{\sin\theta}{\cos\theta}\\
\sin(\theta\pm\varphi) &= \sin\theta\cos\varphi \pm \cos\theta\sin\varphi\\
\cos(\theta\pm\varphi) &= \cos\theta\cos\varphi \mp \sin\theta\sin\varphi\\
\tan(\theta\pm\varphi) &= \frac{\tan\theta\pm\tan\varphi}{1\mp\tan\theta\tan\varphi}\\
\sin(2\theta) &= 2\sin\theta\cos\theta\\
\cos(2\theta) &= \cos^2\theta - \sin^2\theta\\
&= 1-2\sin^2\theta\\
&= 2\cos^2\theta - 1\\
\tan(2\theta) &= \frac{2\tan\theta}{1-\tan^2\theta}\\
\sin^2\theta &= \frac{1-\cos(2\theta)}{2}\\
\cos^2\theta &= \frac{1+\cos(2\theta)}{2}\\
2\sin\theta\cos\varphi &= \sin(\theta+\varphi)+\sin(\theta-\varphi)\\
2\cos\theta\cos\varphi &= \cos(\theta+\varphi)+\cos(\theta-\varphi)\\
2\sin\theta\sin\varphi &= \cos(\theta-\varphi)-\cos(\theta+\varphi)\\
\sin\theta+\sin\varphi &= 2\sin\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right)\\
\sin\theta-\sin\varphi &= 2\cos\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right)\\
\cos\theta+\cos\varphi &= 2\cos\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right)\\
\cos\theta-\cos\varphi &= -2\sin\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right)\\
\sin m\sin\frac{1}{2} &= \frac{1}{2}(\cos(m-\frac{1}{2})-\cos(m+\frac{1}{2}))\\
\sin(n\theta)\sin\frac{\theta}{2} &= \frac{1}{2}(\cos((n-\frac{1}{2})\theta)-\cos((n+\frac{1}{2})\theta))
\end{aligned}$$