Matrix Algebra and Applications

MATH 2111 Lecture Notes

LI, Hongrui

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3.1 Introduction to Determinants

At the end of Chapter 2, we have discussed the invertibility of a linear transfermation $T : \mathbb{R}^T \mapsto \mathbb{R}^T$, whose standard matrix happens to be a square matrix. In this chapter, we will introduce the concept of determinant, which is a scalar value that can be computed from the elements of a square matrix and encodes certain properties of the linear transformation described by the matrix. The determinant of a matrix is a fundamental concept in linear algebra, and it is used in many other areas of mathematics. Before we introduce the definition and properties of determinants, we claim that the thing called "determinant of a square matrix" has the following property:

Proposition 3.1. Invertibility and Determinants
Let A be an $n \times n$ matrix. Then A is invertible if and only if $\det(A) \neq 0$.
Corollary
$\det A = 0$ if and only if rows of A are linearly dependent.

The next step is to give a defination of determinants. It's easy to observe that linear dependence is obvious when two columns or two rows are the same, or a column or a row is zero. Starting from a 2×2 matrix, we find the following defination of determinant will work perfectly for what we want to be, and then we extend it to $n \times n$ matrix by recursion.

Remark: The handwritten lecture notes presents the proposition above explicitly after defining the determinant of a matrix and showing it's properties. However, I think it's better to present the proposition first, since it gives a clear picture of why we need to introduce the concept of determinant.

Definition 3.1. Determinant of a Matrix
Let A be a 2×2 matrix, then the determinant of A, denoted by $\det(A)$ or $|A|$, is defined as the following

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

In general, the determinant of an $n \times n$ matrix A, and A_{ij} is the $(n-1) \times (n-1)$ matrix obtained by deleting the i-th row and j-th column of A, is defined as $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$.
Cofactor
The (i, j)-cofactor of A is defined as $C_{ij} = (-1)^{i+j} \det(A_{ij})$.

Proof. Given this defination, we can easily verify Prop. 3.1. Suppose A has been reduced to an echelon form U by row replacement and row interchanges. If there are r interchanges, then

$$detA = (-1)^r detU = (-1)^r \prod_{i=1}^{n} u_{ii}.$$

This part to be reconsidered!
The defination of determinant is a recursive defination, which is based on the defination of determinant of 2×2 matrix. The determinant of a matrix can be calculated by the following formula:

Proposition 3.2. Calculation of Determinant
Let A be an $n \times n$ matrix, then the determinant of A can be computed by a cofactor expansion across any row or down any column, that is,

$$\begin{aligned}\det(A) &= \sum_{j=1}^{n} a_{ij}C_{ij} \text{ (expansion across the } i\text{-th row)}\\ &= \sum_{i=1}^{n} a_{ij}C_{ij} \text{ (expansion down the } j\text{-th column).}\end{aligned}$$

Thm for triangular matrix
If A is a triangular matrix, then $\det(A)$ is the product of the main diagonal entries of A.

$$\begin{pmatrix} * & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & * \end{pmatrix} \text{ or } \begin{pmatrix} * & 0 & 0 & \cdots & 0 \\ * & * & 0 & \cdots & 0 \\ * & * & * & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & * \end{pmatrix}$$

Warning:

- Mind the sign of each new determinant - indexes changed after each recursive call !!
- Remember to multiply the coefficient of each a_{ij} in the final results !!

3.2 Properties of Determinants

Row and Column Operations

Proposition 3.3. Row Operations
Let A be an $n \times n$ matrix.

- If a multiple of one row of A is added to another row to produce a matrix B, then $\det(B) = \det(A)$.
- If two rows of A are interchanged to produce a matrix B, then $\det(B) = -\det(A)$.
- If one row of A is multiplied by k to produce a matrix B, then $\det(B) = k\det(A)$.

Example 3.1. Example
Let $A = \begin{pmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{pmatrix}$. Calculate $\det(A)$.
Solution The answer is -36 (please take care of all the calculation details: every step of the row operation my cause mistakes).

Now that we have the theorems for row operations, it's natural to think about whether the same properties works for column operations. The answer is yes, and the justification just follows the following theorem:

If A is an $n \times n$ matrix, then $\det(A^T) = \det(A)$.

Determinants of Products

Proposition 3.4. Determinants of Products
Let A and B be $n \times n$ matrices. Then $\det(AB) = \det(A)\det(B)$.
Remark: $\det(A+B)$ is not $\det(A) + \det(B)$.

Using this proposition, it's easy to show that if A is invertible, then

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

A Linearity Property of Determinant Function

We have the following theorem:

$$\begin{aligned} &\det(\mathbf{a_1}, \cdots, \mathbf{a_{j-1}}, \mathbf{a_j} + \mathbf{b_j}, \mathbf{a_{j+1}}, \cdots, \mathbf{a_n}) \\ = &\det(\mathbf{a_1}, \cdots, \mathbf{a_{j-1}}, \mathbf{a_j}, \mathbf{a_{j+1}}, \cdots, \mathbf{a_n}) + \det(\mathbf{a_1}, \cdots, \mathbf{a_{j-1}}, \mathbf{b_j}, \mathbf{a_{j+1}}, \cdots, \mathbf{a_n}). \end{aligned}$$

3.3 Cramer's Rule, Volume, and Linear Transformations

Cramer's Rule

To present Cramer's Rule, we first introduce a notation: For any $n \times n$ matrix A and any $\mathbf{b}$ in $\mathbb{R}^n$, let $A_i(\mathbf{b})$ be the matrix obtained by replacing the i-th column of A by the column vector b. In other words,

$$A = (a_1, \cdots, a_i, \cdots, a_n) \text{ and } A_i(\mathbf{b}) = (a_1, \cdots, \mathbf{b}, \cdots, a_n).$$

Proposition 3.5. Cramer's Rule
Let A be an $n \times n$ matrix and $\mathbf{b}$ be in $\mathbb{R}^n$. If $\det(A) \neq 0$, then the unique solution of $A\mathbf{x} = \mathbf{b}$ is given by

$$x_i = \frac{\det(A_i(\mathbf{b}))}{\det(A)} \text{ for } i = 1, 2, \cdots, n.$$

Proof. Since the determinant of an arbitary matrix A is quite a complicated problem, we try to get as many "0" as possible in the matrix when calculating the determinant. Hence, we can make use of identity matrix $I_n = (\mathbf{e_1}, \cdots, \mathbf{e_i}, \cdots, \mathbf{e_n})$ and $I_n(\mathbf{b}) = (\mathbf{e_1}, \cdots, \mathbf{b}, \cdots, \mathbf{e_n})$. Then we have

$$\begin{aligned} A \cdot I_n(\mathbf{b}) &= (A\mathbf{e_1}, \cdots, A\mathbf{b}, \cdots, A\mathbf{e_n}) \\ &= (\mathbf{a_1}, \cdots, \mathbf{b}, \cdots, \mathbf{a_n}) \\ &= A_i(\mathbf{b}). \\ det(A \cdot I_n(\mathbf{b})) &= det(A) \cdot det(I_n(\mathbf{b})) \end{aligned}$$

When calculating the of $I_n(\mathbf{b})$, simply look at the i-th row of I_n, which only have one non-zero entry 1 to be multiplied with x_i, then we have

$$det(I_n(\mathbf{b})) = (-1)^{i+i} \cdot x_i \cdot det(I_{n-1}) = x_i.$$

Hence, we have $x_i = \frac{det(A_i(\mathbf{b}))}{det(A)}$.

There are many exercises of solving linear systems in previous chapters, so we don't give more examples here. The readers can solve the previous examples using the Cramer's Rule and check the correctness of the results.

A formula for A^{-1}

Proposition 3.6. Let A be an $n \times n$ matrix. If $\det(A) \neq 0$, then $A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$, where $\text{adj}(A)$ is the adjugate matrix of A:

$$\text{adj}(A) = \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix}$$

, where C_{ij} is the (i,j)-cofactor of A, i.e. $C_{ij} = (-1)^{i+j}\det(A_{ij})$.

Warning DO NOT mistake the meaning of C_{ij}, which is $(-1)^{i+j}$ times the determinant of the $(n-1)\times(n-1)$ matrix obtained by deleting the i-th row and j-th column of A. (NOT simply $(-1)^{i+j}a_{ij}$)
For an example of this rule, please refer to the end of FILE Lecture 10.

Determinants as Area of Volume

Proposition 3.7. Determinants as Area or Volume

- If A is a 2×2 matrix, then $|\det(A)|$ is the area of the parallelogram spanned by the columns of A.
- If A is a 3×3 matrix, then $|\det(A)|$ is the volume of the parallelepiped spanned by the columns of A.

The sign of $\det(A)$ is determined by the orientation of the parallelogram or parallelepiped.

Proof. The theorem is obviously true for any 2×2 diagonal matrix, since $\det A = a_{11} \times a_{22} =$ area of rectangle. Hence, it suffice to show that any 2×2 matrix can be transformed into a diagonal matrix in a way that changes neither the area of the associated parallelogram nor $|\det A|$.

- Let $\mathbf{a_1}$ and $\mathbf{a_2}$ be nonzero vectors, Then for any scalar c, the area of the parallelogram determined by $\mathbf{a_1}$ and $\mathbf{a_2}$ equals the area of the parallelogram determined by $\mathbf{a_1}$ and $\mathbf{a_2} + c\mathbf{a_1}$. Column interchanges do not change the parallelogram at all.
- The absolute value of the determinant is unchanged when two columns are interchanged or a multiple of one column is added to another. Such operation suffice to transform A into a diagonal matrix.

Example 3.2. Calculate the area of he parallelogram determined by the points $(-2,-2)$, $(0,3)$, $(4,-1)$, $(6,4)$
Solution Standard steps below:
First translate the parallelogram to one having the origin as a vertex. We substract the vertex $(-2,-2)$ from each of the other vertices to get the new vertices $(0,0)$, $(2,5)$, $(6,1)$, $(8,6)$. The new parallelogram has the same area, and is determined by the columns of

$$\begin{pmatrix} 2 & 6 \\ 5 & 1 \end{pmatrix}$$

Since $|\det A| = |2 \times 1 - 6 \times 5| = |-28|$, the area of the parallelogram is 28.

Linear Transformations

Proposition 3.8. Linear Transformations and Area or Volume

- Let $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be a linear transformation with standard matrix A. If S is a parrallelogram in $\mathbb{R}^2$, then
$$\text{area of } T(S) = |\det(A)| \cdot \text{area of } S.$$
- Let $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ be a linear transformation with standard matrix A. If S is a parallelepiped in $\mathbb{R}^3$, then
$$\text{volume of } T(S) = |\det(A)| \cdot \text{volume of } S.$$

Proof. Just follows the previous proposition.

Example 3.3. Let a and b be positive numbers. Find the area of the region E bounded by the ellipse whose equation is $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$.