Matrix Algebra and Applications

MATH 2111 Lecture Notes

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5.1 Eigenvalues and Eigenvectors

Definition 5.1. Let A be an $n \times n$ matrix. A scalar λ is called an **eigenvalue** of A if there exists a *nonzero* vector $\mathbf{x}$ in $\mathbb{R}^n$ such that

$$A\mathbf{x} = \lambda\mathbf{x} \tag{1}$$

The vector $\mathbf{x}$ is called an **eigenvector** of A corresponding to λ.

Example 5.1. Show that 7 is an eigenvalue of the matrix

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \tag{2}$$

and find the corresponding eigenvectors. **Solution.** The scalar 7 is an eigenvalue of A if and only if the equation

$$A\mathbf{x} = 7\mathbf{x} = 7I_2\mathbf{x}$$

has a nontrivial solution, which is equivalent to

$$(A - 7I_2)\mathbf{x} = \mathbf{0}. \tag{3}$$

We perform the row reduction on the augmented matrix (details should be shown in the exam), and get the result that $x_1 - x_2 = 0$. Hence, the general solution is

$$\mathbf{x} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{4}$$

So $(1,1)^T$ is an eigenvector of A corresponding to the eigenvalue 7.

Instead of giving the general solution, we first consider the special case when the matrix is triangular.

Proposition 5.1. Let A be an $n \times n$ upper triangular matrix. Then the eigenvalues of A are the diagonal entries of A.

Proof. Let A be an $n \times n$ upper triangular matrix. Then we have

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}, A - \lambda I_n = \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} - \lambda \end{bmatrix}$$

Hence we have the following statements equivalent:

- λ is an eigenvalue of A.
- The system $(A - \lambda I_n)\mathbf{x} = \mathbf{0}$ has a nontrivial solution.
- The system $(A - \lambda I_n)\mathbf{x} = \mathbf{0}$ has a free variable.
- At least one of the entries on the diagonal of $A - \lambda I_n$ is zero.
- $\lambda = a_{ii}$ for some i.

Remark: For a lower triangular matrix, the proof is similar and we have the same conclusion. 0 can also be an eigenvalue.

Proposition 5.2. If $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k}$ are eigenvectors of a matrix A corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$, then the set $\{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k}\}$ is linearly independent.

Proof. [This proof is given by *Github Copilot* to make the notes more elegent, but the correctness has not yet been verified...] Suppose that there exist scalars $c_1, c_2, \cdots, c_k$ such that

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k} = \mathbf{0}$$

Then we have

$$A(c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k}) = c_1A\mathbf{v_1} + c_2A\mathbf{v_2} + \cdots + c_kA\mathbf{v_k} = \mathbf{0}$$

Since $A\mathbf{v_i} = \lambda_i\mathbf{v_i}$, we have

$$c_1\lambda_1\mathbf{v_1} + c_2\lambda_2\mathbf{v_2} + \cdots + c_k\lambda_k\mathbf{v_k} = \mathbf{0}$$

Multiplying both sides by $\mathbf{v_1}^T$, we get

$$c_1\lambda_1\mathbf{v_1}^T\mathbf{v_1} + c_2\lambda_2\mathbf{v_2}^T\mathbf{v_1} + \cdots + c_k\lambda_k\mathbf{v_k}^T\mathbf{v_1} = 0$$

Since $\mathbf{v_i}^T\mathbf{v_j} = 0$ for $i \neq j$, we have $c_1\lambda_1\mathbf{v_1}^T\mathbf{v_1} = 0$. Since $\mathbf{v_1} \neq \mathbf{0}$, we have $\mathbf{v_1}^T\mathbf{v_1} \neq 0$. Hence $c_1\lambda_1 = 0$. Since $\lambda_1 \neq 0$, we have $c_1 = 0$. Similarly, we can show that $c_2 = c_3 = \cdots = c_k = 0$. Hence the set $\{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k}\}$ is linearly independent.

Example 5.2. Suppose that $\mathbf{b_1}, \mathbf{b_2}$ are eigenvectors corresponding to distinct eigenvalues λ_1, λ_2 of a matrix A. And suppose that $\mathbf{b_3}, \mathbf{b_4}$ are linearly independent vectors corresponding to a third distince eigenvalue λ_3. Show that the set $\{\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}, \mathbf{b_4}\}$ is linearly independent.

Remark: Note that under a same eigenvalue, the eigenvectors are actually the solutions to a homogeneous system of linear equations, which can be independent or dependent.

5.2 Characteristic Equation

In the previous section, we have already tried to get the eigenvalues and eigenvectors of a matrix. Recall the ways we used to find the eigenvalues and eigenvectors, we can generalize the idea behind it as follows:
For any $n \times n$ matrix A, according to the definition of eigenvalues and eigenvectors, we have

$$A\mathbf{x} = \lambda\mathbf{x} \Leftrightarrow (A - \lambda I_n)\mathbf{x} = \mathbf{0}.$$

This equation has a nontrivial solution if and only if the matrix $A - \lambda I_n$ is singular, i.e., $\det(A - \lambda I_n) = 0$. Hence, all possible eigenvalues of A are the roots of the equation $\det(A - \lambda I_n) = 0$, which is called the **characteristic equation** of A, and the eigenvectors are the solutions to the system $(A - \lambda I_n)\mathbf{x} = \mathbf{0}$ with corresponding λ. Furthermore, we summarize this process to get a more systematic method by giving the following definition and proposition.

Definition 5.2. Let A be an $n \times n$ matrix, then the equation

$$\det(A - \lambda I_n) = 0$$

is called the **characteristic equation** of A.
The polynomial

$$p_A(\lambda) = \det(A - \lambda I_n)$$

is called the **characteristic polynomial** of A, a polynomial of degree n in variable λ.

Proposition 5.3. λ is an eigenvalue of A if and only if λ is a root of the characteristic equation $\det(A - \lambda I_n) = 0$.

This proposition has connected the concept of eigenvalues with the determinant of a matrix, and consequently, the invertibility of a matrix. Therefore, we give some new insights into the invertibility of a matrix.

A : a $n \times n$ matrix. Then A is invertible if and only if

- The number 0 is not an eigenvalue of A.
- The determinant $\det(A) \neq 0$, and

$$\det A = (-1)^r \det u = \begin{cases} (-1)^r \cdot (\text{product of pivots of } u) & u \text{ invertible;} \\ 0 & u \text{ singular,} \end{cases}$$

  where u is an echelon form of A by row replacements and row interchanges (without scaling) with r row interchanges.

Example 5.3. The characteristic polynomial of a 6×6 matrix A is given by

$$\lambda^6 - 4\lambda^5 - 12\lambda^4$$

Find the eigenvalues and their multiplicities.
Solution.

- $\lambda = 0$ multiplicity 4.
- $\lambda = 6$ multiplicity 1.
- $\lambda = -2$ multiplicity 1.

Remark: The purpose of putting this example here is to show the concept of *multiplicity* of an eigenvalue, which is the number of times the eigenvalue appears as a root of the characteristic polynomial.

Example 5.4. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix}$$

Solution. -1, 4, -3.

Similarity

Definition 5.3. Two $n \times n$ matrices A and B are called **similar** if there exists an invertible matrix P such that

$$B = P^{-1}AP \text{ or,equivalently } A = PBP^{-1}.$$

Changing A into $P^{-1}AP$ is called a **similarity transformation**.

Proposition 5.4. If A and B are similar matrices, then A and B have the same characteristic polynomial and hence the same eigenvalues(with the same multiplicities).

Proof. Suppose that A and B are similar matrices, then there exists an invertible matrix P such that $B = P^{-1}AP$. Then we have

$$\begin{aligned}
\det(B - \lambda I_n) &= \det(P^{-1}AP - \lambda I_n) \\
&= \det(P^{-1}AP - \lambda P^{-1}IP) \\
&= \det(P^{-1}(A - \lambda I_n)P) \\
&= \det(P^{-1})\det(A - \lambda I_n)\det(P) \\
&= \det(A - \lambda I_n).
\end{aligned}$$

Therefore, A and B have the same characteristic polynomial and hence the same eigenvalues.

Warning:

- Two matrices may have the same eigenvalues but are not similar.
- Similarity is not the same as row equivalence. Row operations on a matrix usually change its eigenvalues.

Application to Dynamical Systems

Example 5.5. Let A be the matrix

$$A = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}$$

Analyze the long-term behavior of the system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ with $x_0 = (0.6, 0.4)^T$. **Solution.** We can first get the eigenvalues of A by solving the characteristic equation $\det(A - \lambda I_2) = 0$, which is

$$\lambda^2 - 1.92\lambda + 0.92 = 0$$

The roots are $\lambda_1 = 1$ and $\lambda_2 = 0.92$, and the corresponding eigenvectors are

$$\mathbf{v_1} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Then $\{v_1, v_2\}$ is obviously a basis for $\mathbb{R}^2$ since they are linearly independent and the dimension of $\mathbb{R}^2$ is 2. Hence, there exists weight c_1, c_2 such that

$$x_0 = c_1\mathbf{v_1} + c_2\mathbf{v_2} = [\mathbf{v_1}, \mathbf{v_2}] \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

By multiplying both sides by $[\mathbf{v_1}, \mathbf{v_2}]^{-1}$, we can get the weights $c_1, c_2 = 0.125, 0.225$. Since $\mathbf{v_1}$ and $\mathbf{v_2}$ are eigenvectors of A corresponding to distinct eigenvalues, we can write

$$\mathbf{x_1} = A\mathbf{x_0} = A(c_1\mathbf{v_1} + c_2\mathbf{v_2}) = c_1A\mathbf{v_1} + c_2A\mathbf{v_2} = c_1\lambda_1\mathbf{v_1} + c_2\lambda_2\mathbf{v_2}$$

$$\mathbf{x_2} = A\mathbf{x_1} = A(c_1\lambda_1\mathbf{v_1} + c_2\lambda_2\mathbf{v_2}) = c_1\lambda_1^2\mathbf{v_1} + c_2\lambda_2^2\mathbf{v_2}$$

$$\cdots$$

In general,

$$\mathbf{x_k} = c_1\lambda_1^k\mathbf{v_1} + c_2\lambda_2^k\mathbf{v_2}$$

As $k \to \infty$, $\lambda_2^k \to 0$ since $0 < \lambda_2 < 1$, and

$$\mathbf{x_k} \to c_1\mathbf{v_1} = 0.125 \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.625 \end{bmatrix}$$

5.3 Diagonalization