§1.8 Introduction to Linear Transformations

Def: A transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, $T: \mathbb{R}^n \mapsto \mathbb{R}^m$ is a rule that assigns to each vector in $\mathbb{R}^n$ to a vector $T(\vec{x})$ in $\mathbb{R}^m$.

$\mathbb{R}^n$: domain of T

$\mathbb{R}^m$: codomain of T

For $\vec{x} \in \mathbb{R}^n$, $T(\vec{x})$ is called the image of $\vec{x}$.

The set of all images $T(\vec{x})$ is called the range of T.

A: matrix $m\times n$, $\vec{x} \in \mathbb{R}^n$, $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$T(\vec{x}) \triangleq A\vec{x}$.

Ex: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

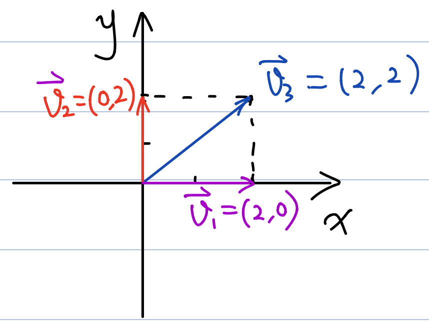
$$T(\vec{x}) = A\vec{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$$

Thus T is the projection from $\mathbb{R}^3$ to x_1x_2-plane

Ex. Let $A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$. $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$,

$T(\vec{x}) = A\vec{x}$ for $\vec{x} \in \mathbb{R}^2$.

Let $\vec{v}_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

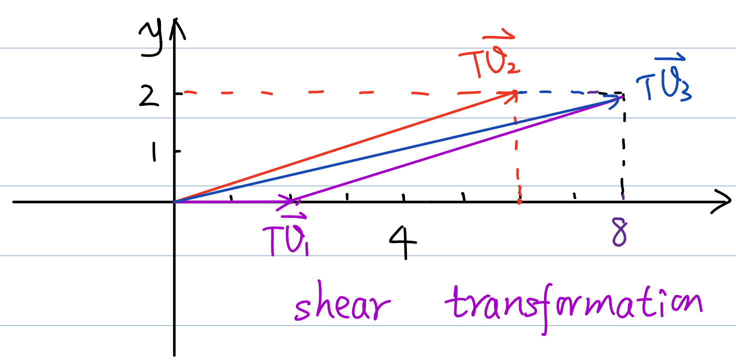


$$T\vec{v}_1 = \begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix}\begin{pmatrix}2\\ 0\end{pmatrix} = \begin{pmatrix}2\\ 0\end{pmatrix}$$

$$T\vec{v}_2 = \begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix}\begin{pmatrix}0\\ 2\end{pmatrix} = \begin{pmatrix}6\\ 2\end{pmatrix}$$

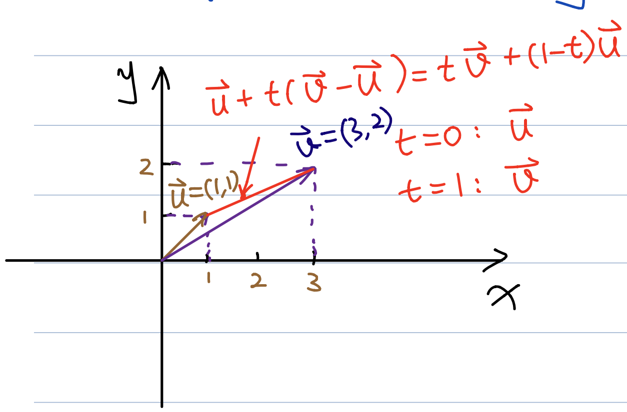
$\downarrow T$

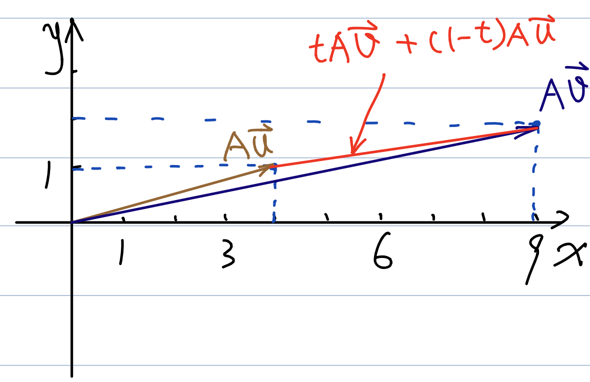
$$T\vec{v}_3 = \begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix}\begin{pmatrix}2\\ 2\end{pmatrix} = \begin{pmatrix}8\\ 2\end{pmatrix}$$



T deforms the square as if the top of the square we pushed to the right while the base is held fixed

T maps a line segment to a line segment





$$A(t\vec{v}+(1-t)\vec{u}) = A(t\vec{v}) + A((1-t)\vec{u}) = tA\vec{v} + (1-t)A\vec{u}$$