Example: Let $\vec{a}_1 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$, $\vec{a}_2 = \begin{pmatrix} 5 \\ -13 \\ -3 \end{pmatrix}$, and $\vec{b} = \begin{pmatrix} -3 \\ 8 \\ 1 \end{pmatrix}$. Then

span $\{\vec{a}_1, \vec{a}_2\}$ is a plane through the origin in $\mathbb{R}^3$. Is $\vec{b}$ in that plane?

Solution: We need to find if the equation

$$x_1 \vec{a}_1 + x_2 \vec{a}_2 = \vec{b}$$

have a solution.

We do row reduction on the augmented matrix $(\vec{a}_1, \vec{a}_2, \vec{b})$:

$$\begin{pmatrix} 1 & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & -18 & 10 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & -2 \end{pmatrix}$$

The third equation is $0 = -2$, which shows that the system has no solution. The vector equation $x_1 \vec{a}_1 + x_2 \vec{a}_2 = \vec{b}$ has no solution, and so $\vec{b}$ is not in Span$\{a_1, a_2\}$.

Example: A company manufactures two products. For \$1.00 worth of product B, the company spends \$0.45 on materiables, \$0.25 on labor, and \$0.15 on overhead. For \$1.00 worth of product

C, the company spends \$0.40 on materials, \$0.30 on labor, and \$0.15 on overhead. Let

$$\vec{b} = \begin{pmatrix} 0.45 \\ 0.25 \\ 0.15 \end{pmatrix} \quad \text{and} \quad \vec{c} = \begin{pmatrix} 0.40 \\ 0.30 \\ 0.15 \end{pmatrix}.$$

Then $\vec{b}$ and $\vec{c}$ represent the "costs per dollar of income" for the two products.

a). What economic interpretation can be given to the vector $100\vec{b}$?

b) Suppose the company wishes to manufacture x_1 dollars worth of product B and x_2 dollars worth of product C. Give a vector that describes the various costs the company will have (for materials, labor, and overhead).

Solution: a) $$100\vec{b} = 100\begin{pmatrix} 0.45 \\ 0.25 \\ 0.15 \end{pmatrix} = \begin{pmatrix} 45 \\ 25 \\ 15 \end{pmatrix}$$

The vector $100\vec{b}$ lists the various costs for producing \$100 worth of product B. Namely, \$45 for materials, \$25 for labor, and \$15 for overhead.

b) The costs of manufacturing x_1 dollars worth of B are given by the vector $x_1\vec{b}$, and the costs of manufacturing x_2 dollars worth of C are given by $x_2\vec{c}$. Hence the total costs for both products are given by the vector $x_1\vec{b} + x_2\vec{c}$.

Exercise: For what value(s) of h will $\vec{y}$ be in $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ if

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} \text{ and } \vec{y} = \begin{pmatrix} -4 \\ 3 \\ h \end{pmatrix}.$$

Solution: The vector $\vec{y}$ belongs to $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ if and only if there exist scalars x_1, x_2, x_3 such that

$$x_1 \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} + x_2 \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ h \end{pmatrix}$$

$$\begin{pmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{pmatrix} \sim \begin{pmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{pmatrix}$$

The system is consistent if and only if $h-5=0$
i.e. $h=5$.