Solving the Stellar Structure Equations for Generating Spherically Symmetric Initial Data

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1 The Stellar Structure Equations

The Tolman-Oppenheimer-Volkoff (TOV) equations describe the structure of a spherically symmetric body of isotropic material. Here, we constraint ourselves as Newtonian case which gives the simple stellar structure equations that are a simple special form of Poisson equation

$$\frac{\partial P}{\partial r} = -\frac{Gm(r)\rho(r)}{r^2} \tag{1}$$
$$\frac{\partial m}{\partial r} = 4\pi r^2 \rho(r) \tag{2}$$

where $m(r)$ is the mass contained in a spherical shell of radius r, ρ is the density of the object, and P is the pressure. To find pressure, we match these with an equation of state (EOS)

$$f(\rho) = P \tag{3}$$

so we can find the densith ρ via inversing of this function such that $\rho = f^{-1}(P)$. Here we assume piecewise polytope EOS as an analytic model which is a piecewise function where each of the pieces is of the form

$$P = K\rho^{\Gamma} \tag{4}$$

where $\Gamma = C_p/C_v = n + 1/n$ that depends some polytropic index n and the specific heats C_p and C_v.

Instead of using this, we can use more complicated EOS to describe different model also can use more sophisticated tabulated EOS for realistic simulation. For polytrope case, the value K can be derived by calculating the degeneracy pressure from degenerated Fermi gas with the appropriate compositions.

We use conventional finite differencing with 4th order Runge-Kutta method for time integration manner.

2 Internal Energy

In our case, we should derive the internal energy. (If we use tabulated EOS, internal energy is included in that form) From the first law of thermodynamics:

$$dU = TdS - PdV \tag{5}$$

We want to express this in terms of specific internal energy u which is energy per unit of mass. We assume a (baryon) mass conservation so we can express the first law of thermodynamics in terms of u and ρ such that

$$\begin{aligned}
dU &= d(um) = mdu = TdS - PdV \\
mdu &= Td(ms) - Pd\left(\frac{m}{\rho}\right) = mTds + \frac{mP}{\rho^2}d\rho \\
du &= Tds + \frac{P}{\rho^2}d\rho
\end{aligned} \tag{6}$$

where s is specific entropy (entropy per unit of mass). A star without heating source in therodynamical equilibrium is isothermal case i.e. $T = constant$ so that $dT = 0$. Using that, we can integrate a thermodynamical potentail which depends on T. Consider specific Helmholtz free energy, denote f, $f = u - Ts$ and its differential is

$$df = du - sdT - Tds = Tds + \frac{P}{\rho^2}d\rho - sdT - Tds = \frac{P}{\rho^2}d\rho \tag{7}$$

From thermodynamical relation, we obtain

$$\left(\frac{\partial f}{\partial \rho}\right)_T = \frac{P}{\rho^2} \tag{8}$$

We consider polytrope EOS i.e. $P = K\rho^\Gamma$ so

$$\left(\frac{\partial f}{\partial \rho}\right)_T = K\rho^{\Gamma-2} \tag{9}$$

Integrate this with respect to ρ gives a form of specific Helmholtz free energy

$$f(\rho, T) = \frac{K}{\Gamma - 1}\rho^{\Gamma-1} + c(T) \tag{10}$$

where $c(T)$ is a arbitrary constant of integration. This depends on T only because integrations is belongs to ρ only. Using the relation, $f = u - Ts$, we can obtain the specific internal energy

$$\begin{aligned}
u(T, \rho) = f + Ts &= \frac{K}{\Gamma - 1}\rho^{\Gamma-1} + c(T) - T\left(\frac{\partial f}{\partial T}\right)_\rho \\
&= \frac{K}{\Gamma - 1}\rho^{\Gamma-1} + c(T) - T\frac{dc(T)}{dT}
\end{aligned} \tag{11}$$

Define $u_{th}(T) = c(T) - T\frac{dc(T)}{dT}$ as a thermal part of the specific energy that depends on temperature only and using $P = K\rho^\Gamma$ then we have

$$u(T, \rho) = \frac{1}{1 - \Gamma}\frac{P}{\rho} + u_{th}(T) \tag{12}$$

If there is no thermal specific energy at initial i.e. $u_{th}(T) = 0$, we obtain usual eqaution of state $P = (\Gamma - 1)\rho u$

3 Generating Particle Data

After solve the problem, we need to create the file as a particle-based file. To do that, we need to calculate the radial positions of the particles at first. We assume that each particle has equal mass so the mass of an individual particle is

$$m_{part} = \frac{m(R)}{n_{part}} \tag{13}$$

where R is the radius of the star and n_{part} is the number of particle. Sum of the masses of all particles i within radius r_i less than r

$$m_{sum}(r) = \sum_i m_{part} \qquad \text{for } r_i \leq r \tag{14}$$

This should be approximately the same as $m(r)$

$$m_{sum}(r) \approx m(r) = \int r^2 \rho dr \tag{15}$$

We use a rejection sampling algorithm to satisfy this. That provides the information about the probability of finding a particle at radius r be given mass equation in stellar structure equation with appropriate normalization.

Then, we sample the θ and ϕ coordinates of the paricles form the uniform distribution. After all this, we convert this from spherical coordinates to Cartesian coordinates via usual coordinate transformation

3.1 Smoothing Length

For smoothing length, root finding algorithm is used to demand that each particle has 10 neighbors within a distance $2h$ (of course, this is chosen by convention that is the radius of support for most smoothing kernels.)

Also, we set particle velocity to zero and set particle internal energy to be $u_i = u(r_i)$ for a particle i at position r_i