Basic Introduction to Mathematical Formulation of Numerical Relativity

Hyun Lim

The purpose of this note is that introduce basic of basic numerical relativity. Introduction includes mathematical formulations mostly. Here, we assume that readers are familiar with basic general relativity and some techniques in differential geometry. There are several books that you may use for further references

- T. Baumgarte & S. Shapiro, Numerical Relativity : Solving Einsteins Equations on the Computer
- M. Alcubierre, Introduction to 3+1 Numerical Relativity
- E. Gourgoulhon, 3+1 Formalism in General Relativity
- C. Bona, C. Palenzuela-Luque, & C. Bona-Caas, Elements of Numerical Relativity and Relativistic Hydrodynamics

Convention
We use some conventions that are widely adopted in this field (As you know, this is not necessarily true always)

- We use geometrized units such that $c = G = 1$
- Usual the Einstein summation convention is used
- Indices $a, b, c, ..., h$ and $o, p, q, ...$ run over spacetime indices, while $i, j, k, ...n$ run over spatial indices only (Sometimes called Fortran convention)
- The flat spacetime (or space) metric is represented by η_{ab} (or η_{ij} for spatial) in any coordinate system. For example, we have $\eta_{ab} = diag(-1, 1, 1, 1)$ in Cartesian (inertial) coordinates
- In general (again not always), we refer to objects associated with
  - the spacetime manifold M as g_{ab}, ${}^{(4)}\Gamma^a_{bc}$, ∇_a, ${}^{(4)}R_{ab}$, etc.
  - a spatial slice Σ as γ_{ij}, Γ^i_{jk}, D_i, R_{ij}, etc.
  - a conformally related space as $\bar{\gamma}_{ij}$, $\bar{\Gamma}^i_{jk}$, $\bar{D}_i$, $\bar{R}_{ij}$, etc.
- The symmetric and antisymmetic parts of a tensor is defined in the way

$$T_{(ab)} \equiv \frac{1}{2}(T_{ab} + T_{ba}) \quad \text{and} \quad T_{[ab]} \equiv \frac{1}{2}(T_{ab} - T_{ba})$$

I. THE 3+1 DECOMPOSITION

There are many discussions about how to determine the dynamical evolution of physical system governed by the Einstein's equations of general relativity. In this note, we do not consider all details about math and physics underneath of these discussions including 3+1 decomposition. We concentrate on the formalisms and techniques to decompose Einstein's equations.

A. Foliation of Spacetime

Consider curved 4D spacetime manifold M and the metric in this spacetime g_{ab}. We assume that the spacetime (M, g_{ab}) can be foliated into a family of nonintersecting spacelike 3-surface Σ (hypersurface), which arise as the level surfaces of a scalar function t that can be interpreted as a global time function. From t, we can define 1-form

$$\Omega_a = \nabla_a t \tag{1}$$

which satisfies by construction

$$\nabla_{[a}\Omega_{b]} = \nabla_{[a}\nabla_{b]}t = 0 \tag{2}$$

The 4-metric g_{ab} defines the norm of Ω_a such that

$$||\Omega||^2 = \Omega_a\Omega^a = \nabla_a t\nabla^a t = g^{ab}\nabla_a t\nabla_b t \equiv -\frac{1}{\alpha^2} \tag{3}$$

α measures how much proper time elapses between neighboring time slices along the normal vector Ω^a to the slice, and is therefore called the *lapse* function. We assume that $\alpha > 0$ so that Ω^a is timelike and the hypersurface Σ is spacelike everywhere.

Further we can define the normalized 1-form

$$\omega_a \equiv \alpha\Omega_a \tag{4}$$

note that ω_a is rotation free i.e. $\omega_{[a}\nabla_b\omega_{c]} = 0$. Using this, we can now define the unit normal to the slices as

$$n^a \equiv -g^{ab}\omega_b \tag{5}$$

so that $n_a = -\omega_a$. Here the negative sign has been chosen then n^a points in the direction of increasing t.

$$n^a\omega_a = -g^{ab}\omega_a\omega_b = -\alpha^2 g^{ab}\Omega_a\Omega_b = 1 \tag{6}$$

n^a is normalized and timelike by construction

$$n^a n_a = g^{ab}\omega_a\omega_b = -1 \tag{7}$$

With the normal vector, we can construct the spatial metric γ_{ab} that is induced by g_{ab} on the 3-dimensional hypersurfaces Σ

$$\gamma_{ab} = g_{ab} + n_a n_b \tag{8}$$

Thus γ_{ab} is considered as a projection tensor that projects out all geometric objects lying along n^a. This metric allows us to compute distances within a slice Σ. We can further see that γ_{ab} is purely spatial i.e. resides entirely in Σ with no piece along n^a

$$n^a\gamma_{ab} = n^a(g_{ab} + n_a n_b) = n^a g_{ab} + n^a n_a n_b = n^a g_{ab} + \underbrace{n^a n_a}_{=-1} n_b = n_b - n_b = 0 \tag{9}$$

The inverse spatial metric is

$$\gamma^{ab} = g^{ac}g^{bd}\gamma_{cd} = g^{ac}g^{bd}(g_{cd} + n_c n_d) = g^{ab} + n^a n^b \tag{10}$$

We want to break up 4-dimensional tensors by decomposing them into a purely spatial part, which lies in the hypersurfaces Σ, and a timelike part, which is normal to the spatial surface. To do that, we need two projection operators. The first one, which projects a 4-dimensional tensor into a spatial slice, can be found by raising only one index of the spatial metric γ_{ab}

$$\gamma^a_b = g^a_b + n^a n_b = \delta^a_b + n^a n_b \tag{11}$$

To project higher rank tensors into the spatial surface, each free index has to be contracted with a projection operator. We sometimes denote this projection with a symbol $\perp$. For example,

$$\perp T_{ab} = \gamma^c_a\gamma^d_b T_{cd} \tag{12}$$

Similarly, we define the normal projection operator

$$N^a_b \equiv -n^a n_b = \delta^a_b - \gamma^a_b \tag{13}$$

Using these two projection operators, we can decompose any tensor into its spatial and timelike parts. For example, arbitrary vector v^a can be decomposed

$$v^a = \delta^a_b v^b = (\gamma^a_b + N^a_b)v^b = \perp v^a - n^a n_b v^b \tag{14}$$

For second rank tensor,

$$\begin{aligned} T_{ab} &= \delta^c_a \delta^d_b T_{cd} = (\gamma^c_a + N^c_a)(\gamma^d_b + N^d_b)T_{cd} \\ &= (\gamma^c_a \gamma^d_a + \gamma^c_a N^d_b + \gamma^d_b N^c_a + N^c_a N^d_b)T_{cd} \\ &= \gamma^c_a \gamma^d_a T_{cd} - \gamma^c_a n^d n_b T_{cd} - \gamma^d_b n^c n_a T_{cd} + n^c n_a n^d n_b T_{cd} \\ &= \perp T_{ab} - n^c n_a \perp T_{cb} - n^d n_b \perp T_{ad} + n_a n_b n^c n^d T_{cd} \end{aligned} \tag{15}$$

Note that, in this example, the $\perp$ symbol has to be used with some case, since it applies only to the free indices of the tensor that it operates on. To avoid confusion, we will usually write out the projection operators explicitly.

We also need a 3-dimensional covariant derivative that maps spatial tensors into spatial tensor. It is uniquely defined by requiring that it be compatible with the 3-dimensional metric γ_{ab}. We can construct this derivative by projecting all indices present in a 4-dimensional covariant derivative into Σ. For example, for a scalar, we define

$$D_a f \equiv \gamma^b_a \nabla_b f \tag{16}$$

For tensor

$$D_a T^b_c \equiv \gamma^d_a \gamma^b_e \gamma^f_c \nabla_d T^e_f \tag{17}$$

Other type tensors follow same rule as above. Further, the 3-dimensional covariant derivative is compatible with the spatial metric i.e. $D_a \gamma_{bc} = 0$ like usual 4-dimensional covariant derivative. Also, for purely spatial vectors, the 3-dimensional covariant derivative satisfies the Leibniz rule.

Like 4-dimensional case, we can define connection coefficients, Riemann tensors, Ricci tensors, and Ricci scalar by usual way with D_a and γ_{ab}

$$\Gamma^a{}_{bc} = \frac{1}{2}\gamma^{ad}(\partial_c \gamma_{db} + \partial_b \gamma_{dc} - \partial_d \gamma_{bc}) \tag{18}$$

$$2D_{[a}D_{b]}w_c = R^d{}_{cba} w_d \qquad R^d{}_{cba} n_d = 0 \tag{19}$$

$$R_{abc}{}^d = \partial_b \Gamma^d{}_{ac} - \partial_a \Gamma^d{}_{bc} + \Gamma^e{}_{ac}\Gamma^d{}_{eb} - \Gamma^e{}_{bc}\Gamma^d{}_{ea} \tag{20}$$

$$R_{ab} = R^c{}_{acb} \tag{21}$$

$$R = R^a{}_a \tag{22}$$

B. The Extrinsic Curvature

Einstein's equations relate contractions for the 4-dimensional Riemann tensor ${}^{(4)}R^a{}_{bcd}$ to the stress-energy tensor T_{ab}. The goal of 3+1 decomposition is breaking ${}^{(4)}R^a{}_{bcd}$ into spatial tensors. This decomposition involves its 3-dimensional Riemann tensor $R^a{}_{bcd}$, but this cannot contain all the information needed. $R^a{}_{bcd}$ is a purely spatial object and can be computed from spatial derivatives of the spatial metric alone, while ${}^{(4)}R^a{}_{bcd}$ is a spacetime creature which also contains time derivatives of the 4-dimensional metric. So, the 3-dimensional curvature $R^a{}_{bcd}$ only contains information about the curvature intrinsic to a slice Σ, but it gives no information about what shape this slices take in the spacetime M in which it is embedded. This information is contained in a tensor called *extrinsic* curvature.

The extrinsic curvature K_{ab} can be found by projecting gradients of the normal vector into the slice Σ. The projection of the gradient of the normal vector $\gamma^c_a \gamma^d_b \nabla_c n_d$ can be split into a symmetric part, also known as the expansion tensor

$$\theta_{ab} = \gamma^c_a \gamma^d_b \nabla_{(c} n_{d)} \tag{23}$$

and an antisymmetric part, also known as the rotation 2-form or twist

$$\omega_{ab} = \gamma^c_a \gamma^d_b \nabla_{[c} n_{d]} \tag{24}$$

We now define the extrinsic curvature, K_{ab}, as the negative expansion

$$K_{ab} = -\gamma_a^c \gamma_b^d \nabla_{(c} n_{d)} = -\gamma_a^c \gamma_b^d \nabla_c n_d \tag{25}$$

By definition, the extrinsic curvature is symmetric and purely spatial. It measures the gradient of the normal vector n^a. Since the latter are normalized, they can only differ in the direction in which they are pointing, and the extrinsic curvature therefore provides information on how much this direction changes from point to point across a spatial hypersurface. As a consequence, the extrinsic curvature measures the rate at which the hypersurface deforms as it is carried forward along a normal.

Furthermore, we can express the extrinsic curvature in terms of the acceleration of the unit normal vector field

$$a_a \equiv n^b \nabla_b n_a \tag{26}$$

By definition, the acceleration is purely spatial i.e. $n^a a_a = 0$. Note that we can find a relationship between lapse and acceleration. From definition, $a_a = n^b \nabla_b n_a$ and $n_a = -\omega_a = -\alpha \Omega_a = -\alpha \nabla_a t$

$$\begin{aligned}
a_a &= n^b \nabla_b n_a = -n^b \nabla_b (\alpha \nabla_a t) \\
&= -n^b (\nabla_b \alpha)(\nabla_a t) - n^b \alpha \nabla_b \nabla_a t \\
&= \frac{1}{\alpha} n^b n_a \nabla_b \alpha - n^b \alpha \nabla_a \nabla_b t = \frac{1}{\alpha} n^b n_a \nabla_b \alpha - n^b \alpha \nabla_a \left(-\frac{1}{\alpha} n_b \right) \\
&= \frac{1}{\alpha} n^b n_a \nabla_b \alpha + n^b \alpha \frac{1}{\alpha} \nabla_a n_b + n^b n_b \alpha \nabla_a \frac{1}{\alpha}
\end{aligned} \tag{27}$$

here we use the fact that $\nabla_{[a} \nabla_{b]} t = 0$ to swap the order of derivative. And using the relations $n^b \nabla_a n_b = 0$ and $n_b n^b = -1$ to simply the expression

$$\begin{aligned}
a_a &= \frac{1}{\alpha} n^b n_a \nabla_b \alpha + n^b \alpha \frac{1}{\alpha} \nabla_a n_b + n^b n_b \alpha \nabla_a \frac{1}{\alpha} = \frac{1}{\alpha} n^b n_a \nabla_b \alpha - (-\alpha \frac{1}{\alpha^2} \nabla_a \alpha) \\
&= \frac{1}{\alpha} (\nabla_a \alpha + n^b n_a \nabla_b \alpha) = \frac{1}{\alpha} (\delta_a^b \nabla_b \alpha + n^b n_a \nabla_b \alpha) = \frac{1}{\alpha} (\delta_a^b + n^b n_a) \nabla_b \alpha \\
&= \frac{1}{\alpha} \gamma_a^b \nabla_b \alpha = \frac{1}{\alpha} D_a \alpha \\
&= D_a \ln \alpha
\end{aligned} \tag{28}$$

Expanding RHS of Eqn. 25

$$\begin{aligned}
K_{ab} &= -\gamma_a^c \gamma_b^d \nabla_c n_d = -(\delta_a^c + n_a n^c)(\delta_b^d + n_b n^d) \nabla_c n_d \\
&= -(\delta_a^c \delta_b^d \nabla_c n_d + \delta_a^c n_b n^d \nabla_c n_d + \delta_b^d n_a n^c \nabla_c n_d + n_a n^c n_b n^d \nabla_c n_d) \\
&= -(\nabla_a n_b + \delta_a^c n_b \underbrace{n^d \nabla_c n_d}_{=0} + n_a n^c \nabla_c n_b + n_a n^c n_b \underbrace{n^d \nabla_c n_d}_{=0}) \\
&= -(\nabla_a n_b + n_a \underbrace{n^c \nabla_c n_b}_{=a_b}) \\
&= -\nabla_a n_b - n_a a_b
\end{aligned} \tag{29}$$

We can also express the extrinsic curvature via Lie derivative of spatial metric along n^a

$$\begin{aligned}
\pounds_n \gamma_{ab} &= n^c \nabla_c \gamma_{ab} + \gamma_{ab} \nabla_b n^c + \gamma_{cb} \nabla_a n^c \\
&= n^c \nabla_c (g_{ab} + n_a n_b) + (g_{ac} + n_a n_c) \nabla_b n^c + (g_{cb} + n_b n_c) \nabla_a n^c \\
&= n^c [\underbrace{\nabla_c g_{ab}}_{=0} \nabla_c (n_a n_b)] + g_{ac} \nabla_b n^c + n_a \underbrace{n_c \nabla_b n^c}_{=0} + g_{cb} \nabla_a n^c + n_b \underbrace{n_c \nabla_a n^c}_{=0} \\
&= n^c n_b \nabla_c n_a + n^c n_a \nabla_c n_b + \nabla_b (g_a c n^c) + \nabla_a (g_c b n^c) \\
&= n_b \underbrace{n^c \nabla_c n_a}_{=a_a} + n_a \underbrace{n^c \nabla_c n_b}_{=a_b} + \nabla_b n_a + \nabla_a n_b \\
&= n_b a_a + n_a a_b + \nabla_b n_a + \nabla_a n_b = 2(n_{(a} a_{b)} + \nabla_{(a} n_{b)})
\end{aligned} \tag{30}$$

Using Eqn. 29 and the fact that the extrinsic curvature is symmetric then we can conclude

$$\pounds_n \gamma_{ab} = -2K_{ab} \tag{31}$$

equivalently

$$K_{ab} = -\frac{1}{2}\pounds_n \gamma_{ab} \tag{32}$$

To sum up, we have three expressions that define the extrinsic curvature

$$\begin{aligned} K_{ab} &= -\gamma_a^c \gamma_b^d \nabla_c n_d \\ &= -\nabla_a n_b - n_a a_b \\ &= -\frac{1}{2}\pounds_n \gamma_{ab} \end{aligned}$$

The mean curvature is defined by the trace of the extrinsic curvature

$$K = g^{ab} K_{ab} = \gamma^{ab} K_{ab} \tag{33}$$

Take the trace of Eqn. 32 then

$$K = \gamma^{ab} K_{ab} = -\frac{1}{2}\gamma^{ab}\pounds_n \gamma_{ab} = -\frac{1}{2\gamma}\pounds_n \gamma = -\frac{1}{2\sqrt{\gamma}}\pounds_n \sqrt{\gamma} = -\pounds_n \ln\sqrt{\gamma} \tag{34}$$

This expression has a geometrical interpretation. Since $\sqrt{\gamma} d^3x$ is the proper volume element in the spatial slice Σ, the negative of the mean curvature measures the fractional change in the proper 3-volume along n^a.

C. The Equations of Gauss, Codazzi, and Ricci

From previous sections, we define the spatial metric γ_{ab} and the extrinsic curvature K_{ab}. Unfortunately, γ_{ab} and K_{ab} cannot be chosen arbitrarily. Instead, they should satisfy certain constrains so the the spatial slices fit into the spacetime M. To find these relations, we need to relate the 3-dimensional Riemann tensor $R^a{}_{bcd}$ to the 4-dimensional Riemann tensor ${}^{(4)}R^a{}_{bcd}$.

First, we need to rewrite ${}^{(4)}R^a{}_{bcd}$ in terms of combinations of projections

$$\begin{aligned} {}^{(4)}R_{abcd} &= \delta_a^p \delta_b^q \delta_c^r \delta_d^s \, {}^{(4)}R_{pqrs} \\ &= (\gamma_a^p - n^p n_a)(\gamma_b^q - n^q n_b)(\gamma_c^r - n^r n_c)(\gamma_d^s - n^s n_d)^{(4)}R_{pqrs} \\ &= (\gamma_a^p \gamma_b^q - \gamma_b^q n^p n_a - \gamma_a^p n^q n_b + n_a n_b n^p n^q)(\gamma_c^r - n^r n_c)(\gamma_d^s - n^s n_d)^{(4)}R_{pqrs} \\ &= (\gamma_a^p \gamma_b^q \gamma_c^r - \gamma_a^p \gamma_b^q n^r n_c - \gamma_a^p \gamma_c^r n^q n_b + \gamma_a^p n^q n_b n^r n_c - \gamma_b^q \gamma_c^r n^p n_a + \gamma_b^q n^p n_a n^r n_c \\ &\quad + \gamma_c^r n_a n_b n^p n^q - n^r n_c n_a n_b n^p n^q)(\gamma_d^s - n^s n_d)^{(4)}R_{pqrs} \\ &= (\gamma_a^p \gamma_b^q \gamma_c^r \gamma_d^s - \gamma_a^p \gamma_b^q \gamma_c^r n^s n_d - \gamma_a^p \gamma_b^q \gamma_d^s n^r n_c + \gamma_a^p \gamma_b^q n^r n_c n^s n_d - \gamma_a^p \gamma_c^r \gamma_d^s n^q n_b \\ &\quad + \gamma_a^p \gamma_c^r n^q n_b n^s n_d + \gamma_a^p \gamma_d^s n^q n_b n^r n_c - \gamma_a^p n^q n_b n^r n_c n^s n_d - \gamma_b^q \gamma_c^r \gamma_d^s n^p n_a - \gamma_b^q \gamma_c^r n^p n_a n^s n_d \\ &\quad + \gamma_b^q \gamma_d^s n^p n_a n^r n_c - \gamma_b^q n^p n_a n^r n_c n^s n_d + \gamma_c^r \gamma_d^s n_a n_b n^p n^q - \gamma_c^r n_a n_b n^p n^q n^s n_d - \gamma_d^s n^r n_c n_a n_b n^p n^q \\ &\quad + n_a n_b n_c n_d n^p n^q n^r n^s)^{(4)}R_{pqrs} \end{aligned} \tag{35}$$

There are a lot of terms in above expression but we can simplify it. Using the symmetry properties of Riemann tensor, we can vanish the terms like $n^p n^q \, {}^{(4)}R_{pqrs} = 0$ because $n^p n^q$ is symmetric under swapping p, q and ${}^{(4)}R_{pqrs}$ is anti-symmetric under this swapping. Thus

$$
\begin{aligned}
{}^{(4)}R_{abcd} = {} & \gamma^p_a\gamma^q_b\gamma^r_c\gamma^s_d\,{}^{(4)}R_{pqrs} - \gamma^p_a\gamma^q_b\gamma^r_c n^s n_d\,{}^{(4)}R_{pqrs} - \gamma^p_a\gamma^q_b\gamma^s_d n^r n_c\,{}^{(4)}R_{pqrs} + \underbrace{\gamma^p_a\gamma^q_b n^r n_c n^s n_d\,{}^{(4)}R_{pqrs}}_{=0} \\
& - \gamma^p_a\gamma^r_c\gamma^s_d n^q n_b\,{}^{(4)}R_{pqrs} + \gamma^p_a\gamma^r_c n^q n_b n^s n_d\,{}^{(4)}R_{pqrs} + \gamma^p_a\gamma^s_d n^q n_b n^r n_c\,{}^{(4)}R_{pqrs} \\
& \underbrace{-\gamma^p_a n^q n_b n^r n_c n^s n_d\,{}^{(4)}R_{pqrs}}_{=0} - \gamma^q_b\gamma^r_c\gamma^s_d n^p n_a\,{}^{(4)}R_{pqrs} - \gamma^q_b\gamma^r_c n^p n_a n^s n_d\,{}^{(4)}R_{pqrs} \\
& + \gamma^q_b\gamma^s_d n^p n_a n^r n_c\,{}^{(4)}R_{pqrs} - \underbrace{\gamma^q_b n^p n_a n^r n_c n^s n_d\,{}^{(4)}R_{pqrs}}_{=0} + \underbrace{\gamma^r_c\gamma^s_d n_a n_b n^p n^q\,{}^{(4)}R_{pqrs}}_{=0} \\
& - \underbrace{\gamma^r_c n_a n_b n^p n^q n^s n_d}_{{}^{(4)}R_{pqrs}} - \underbrace{\gamma^s_d n^r n_c n_a n_b n^p n^q\,{}^{(4)}R_{pqrs}}_{=0} + \underbrace{n_a n_b n_c n_d n^p n^q n^r n^s\,{}^{(4)}R_{pqrs}}_{=0}
\end{aligned} \tag{36}
$$

Above shows that we have three types of projection : completely spatial projection, projection with one index projected in the normal direction, and projection with two indices projected in the normal direction. All other projections vanish. Collect the term by types of projection and rearrange dummy indices

$$
\begin{aligned}
{}^{(4)}R_{abcd} = {} & \gamma^p_a\gamma^q_b\gamma^r_c\gamma^s_d\,{}^{(4)}R_{pqrs} \\
& - \gamma^p_a\gamma^q_b\gamma^r_c n^s n_d\,{}^{(4)}R_{pqrs} - \underbrace{\gamma^p_a\gamma^q_b\gamma^s_d n^r n_c\,{}^{(4)}R_{pqrs}}_{=\gamma^p_a\gamma^q_b\gamma^r_d n^s n_c\,{}^{(4)}R_{pqsr}} - \underbrace{\gamma^p_a\gamma^r_c\gamma^s_d n^q n_b\,{}^{(4)}R_{pqrs}}_{\gamma^r_a\gamma^p_c\gamma^q_d n^s n_b\,{}^{(4)}R_{rspq}} - \underbrace{\gamma^q_b\gamma^r_c\gamma^s_d n^p n_a\,{}^{(4)}R_{pqrs}}_{\gamma^r_b\gamma^p_c\gamma^q_d n^s n_a\,{}^{(4)}R_{srpq}} \\
& + \gamma^p_a\gamma^r_c n^q n_b n^s n_d\,{}^{(4)}R_{pqrs} + \underbrace{\gamma^p_a\gamma^s_d n^q n_b n^r n_c\,{}^{(4)}R_{pqrs}}_{\gamma^p_a\gamma^r_d n^q n_b n^s n_c\,{}^{(4)}R_{pqsr}} - \underbrace{\gamma^q_b\gamma^r_c n^p n_a n^s n_d\,{}^{(4)}R_{pqrs}}_{\gamma^p_b\gamma^r_c n^q n_a n^s n_d\,{}^{(4)}R_{qprs}} + \underbrace{\gamma^q_b\gamma^s_d n^p n_a n^r n_c\,{}^{(4)}R_{pqrs}}_{\gamma^p_b\gamma^r_d n^q n_a n^s n_c\,{}^{(4)}R_{qpsr}} \\
= {} & \gamma^p_a\gamma^q_b\gamma^r_c\gamma^s_d\,{}^{(4)}R_{pqrs} \\
& - \gamma^p_a\gamma^q_b\gamma^r_c n_d n^s\,{}^{(4)}R_{pqrs} - \gamma^p_a\gamma^q_b\gamma^r_d n_c n^s \underbrace{{}^{(4)}R_{pqsr}}_{=-{}^{(4)}R_{pqrs}} - \gamma^p_c\gamma^q_d\gamma^r_a n_b n^s \underbrace{{}^{(4)}R_{rspq}}_{={}^{(4)}R_{pqrs}} - \gamma^p_c\gamma^q_d\gamma^r_b n_a n^s \underbrace{{}^{(4)}R_{srpq}}_{={}^{(4)}R_{pqsr}=-{}^{(4)}R_{pqrs}} \\
& + \gamma^p_a\gamma^r_c n_b n_d n^q n^s\,{}^{(4)}R_{pqrs} + \gamma^p_a\gamma^r_d n_b n_c n^q n^s \underbrace{{}^{(4)}R_{pqsr}}_{=-{}^{(4)}R_{pqrs}} - \gamma^p_b\gamma^r_c n_a n_d n^q n^s \underbrace{{}^{(4)}R_{qprs}}_{=-{}^{(4)}R_{pqrs}} + \gamma^p_b\gamma^r_d n_a n_c s n^q n^s \underbrace{{}^{(4)}R_{qpsr}}_{={}^{(4)}R_{pqrs}} \\
= {} & \gamma^p_a\gamma^q_b\gamma^r_c\gamma^s_d\,{}^{(4)}R_{pqrs} \\
& - \gamma^p_a\gamma^q_b \underbrace{(\gamma^r_c n_d - \gamma^r_d n_c)}_{=2\gamma^r_{[c}n_{d]}} n^s\,{}^{(4)}R_{pqrs} - \gamma^p_c\gamma^q_d \underbrace{(\gamma^r_a n_b - \gamma^r_b n_a)}_{=2\gamma^r_{[a}n_{b]}} n^s\,{}^{(4)}R_{pqrs} \\
& + \gamma^p_a \underbrace{(\gamma^r_c n_d - \gamma^r_d n_c)}_{=2\gamma^r_{[c}n_{d]}} n_b n^q n^s\,{}^{(4)}R_{pqrs} - \gamma^p_b \underbrace{(\gamma^r_c n_d - \gamma^r_d n_c)}_{=2\gamma^r_{[c}n_{d]}} n_a n^q n^s\,{}^{(4)}R_{pqrs}
\end{aligned} \tag{37}
$$

So, 4-dimensional Riemann tensor ${}^{(4)}R_{abcd}$ can be written as

$$
\begin{aligned}
{}^{(4)}R_{abcd} = {} & \gamma^p_a\gamma^q_b\gamma^r_c\gamma^s_d\,{}^{(4)}R_{pqrs} - 2\gamma^p_a\gamma^q_b 2\gamma^r_{[c}n_{d]} n^s\,{}^{(4)}R_{pqrs} - 2\gamma^p_c\gamma^q_d\gamma^r_{[a}n_{b]} n^s\,{}^{(4)}R_{pqrs} \\
& + 2\gamma^p_a\gamma^r_{[c}n_{d]} n_b n^q n^s\,{}^{(4)}R_{pqrs} - 2\gamma^p_b\gamma^r_{[c}n_{d]} n_a n^q n^s\,{}^{(4)}R_{pqrs}
\end{aligned} \tag{38}
$$

Using above projection, we make the relation between ${}^{(4)}R^a{}_{bcd}$ and $R^a{}_{bcd}$ which gives Gauss, Codazzi, and Ricci equations. The Riemann tensor is defined in terms of second covariant derivatives of a vector. So, we need to relate between 4-dimensional covariant derivatives and 3-dimensional covariant derivatives.

Consider first derivative of a spatial vector V^b. By definition we made before

$$
\begin{aligned}
D_a V^b &= \gamma^p_a\gamma^b_q\nabla_p V^q = \gamma^p_a(\delta^b_q + n^b n_q)\nabla_p V^q \\
&= \gamma^p_a\delta^b_q\nabla_p V^q + \gamma^p_a n^b n_q\nabla_p V^q = \gamma^p_a\nabla_p V^b + \gamma^p_a n^b(\nabla_p(V^q n_q) - V^q\nabla_p n_q)
\end{aligned} \tag{39}
$$

V^q is spatial so $n_qV^q = 0$ then

$$D_aV^b = \gamma_a^p\nabla_pV^b - \gamma_a^pn^bV^q\nabla_pn_q = \gamma_a^p\nabla_pV^b - n^bV^e\underbrace{\gamma_e^q\gamma_e^p\nabla_pn_q}_{=-K_{ae}}$$
$$= \gamma_a^p\nabla_pV^b - n_bV^eK_{ae} \tag{40}$$

For second derivative

$$\begin{aligned} D_aD_bV^c &= \gamma_a^p\gamma_b^q\gamma_r^c\nabla_p\nabla_qV^r = \gamma_a^p\gamma_b^q\gamma_r^c\nabla_p(\gamma_q^d\gamma_e^r\nabla_dV^e) \\ &= \gamma_a^p\gamma_b^q\gamma_r^c\gamma_q^d\gamma_e^r\nabla_p\nabla_dV^e + \gamma_a^p\gamma_b^q\gamma_r^c\gamma_e^r\nabla_p\gamma_q^d\nabla_dV^e + \gamma_a^p\gamma_b^q\gamma_r^c\gamma_q^d\nabla_p\gamma_e^r\nabla_dV^e \\ &= \gamma_a^p\gamma_b^d\gamma_e^c\nabla_p\nabla_dV^e + \gamma_a^p\gamma_b^q\gamma_e^c\nabla_p(\delta_q^d + n^dn_q)\nabla_dV^e + \gamma_a^p\gamma_b^d\gamma_r^c\nabla_p(\delta_e^r + n^rn_e)\nabla_dV^e \end{aligned} \tag{41}$$

δ_b^a is just number so derivative of δ_b^a is zero. Evaluate $\nabla_a(n_bn^c)$

$$\begin{aligned} \nabla_a(n_bn^c) &= n_b\nabla_an^c + n^c\nabla_an_b \\ &= -n_b(K_a^c + n_aa^c) - n^c(K_{ab} + n_aa_b) \end{aligned} \tag{42}$$

Here we use Eqn. 29, $K_{ab} = -\nabla_an_b - n_aa_b$ so

$$\begin{aligned} D_aD_bV^c = \gamma_a^p\gamma_b^d\gamma_e^c\nabla_p\nabla_dV^e &- \gamma_a^p\gamma_b^q\gamma_e^c[n^d(K_{pq} + n_pa_q) + n_q(K_p^d + n_pa^d)]\nabla_dV^e \\ &- \gamma_a^p\gamma_b^d\gamma_r^c[n^r(K_{pe} + n_pa_e) + n_e(K_p^r + n_pa^r)]\nabla_dV^e \end{aligned} \tag{43}$$

Use the fact the $n_a\gamma^{ab} = 0$ we can simplify further

$$\begin{aligned} D_aD_bV^c &= \gamma_a^p\gamma_b^d\gamma_e^c\nabla_p\nabla_dV^e - [\gamma_a^p\gamma_b^q\gamma_e^cn^dK_{pq} + n^d\gamma_b^q\gamma_e^c\underbrace{\gamma_a^pn_p}_{=0}a_q + \gamma_a^p\gamma_e^c\underbrace{\gamma_b^qn_q}_{=0}(K_p^d + n_pa^d)]\nabla_dV^e \\ &\quad - [\gamma_a^p\gamma_b^d\underbrace{\gamma_r^cn^r}_{=0}(K_{pe} + n_pa_e) + \gamma_a^p\gamma_b^d\gamma_r^cn_eK_p^r + \gamma_b^d\gamma_r^c\underbrace{\gamma_a^pn_p}_{=0}a^r]\nabla_dV^e \\ &= \gamma_a^p\gamma_b^d\gamma_e^c\nabla_p\nabla_dV^e - \gamma_e^cn^dK_{ab}\nabla_dV^e - \gamma_b^dK_a^c\underbrace{n_e\nabla_dV^e}_{=-V^e\nabla_dn_e} \\ &= \gamma_a^p\gamma_b^d\gamma_e^c\nabla_p\nabla_dV^e - K_{ab}\gamma_e^cn^d\nabla_dV^e + \gamma_b^dK_a^cV^e\nabla_d(\gamma_e^sn_s) \\ &= \gamma_a^p\gamma_b^d\gamma_e^c\nabla_p\nabla_dV^e - K_{ab}\gamma_e^cn^d\nabla_dV^e + V^eK_a^c\underbrace{\gamma_b^d\gamma_e^s\nabla_dn_s}_{=-K_{be}} \end{aligned} \tag{44}$$

Just rearrange dummy indices (for convenience) then second derivative is written as

$$D_aD_bV^c = \gamma_a^p\gamma_b^q\gamma_r^c\nabla_p\nabla_qV^r - K_{ab}\gamma_r^cn^p\nabla_pV^r - K_a^cK_{bp}V^p \tag{45}$$

The definition of the 3-dimensional Riemann tensor is

$$R^{dc}{}_{ba}V_d = 2D_{[a}D_{b]}V^c \tag{46}$$

Insert Eqn. 45 into RHS of with anti-symmetrization then

$$R^{dc}{}_{ba}V_d = 2\gamma_a^p\gamma_b^q\gamma_r^c\nabla_{[p}\nabla_{q]}V^r - 2K_{[ab]}\gamma_r^cn^p\nabla_pV^r - 2K_{[a}^cK_{b]p}V^p \tag{47}$$

Since K_{ab} is symmetric so $K_{[ab]} = 0$. And, $\nabla_{[p}\nabla_{q]}V^r$ give 4-dimensional Riemann tensor such that

$$2\nabla_{[p}\nabla_{q]}V^r = {}^{(4)}R^{dr}{}_{qp}V_d \tag{48}$$

Thus

$$R^{dc}{}_{ba}V_d = \gamma_a^p\gamma_b^q\gamma_r^c\,{}^{(4)}R^{dr}{}_{qp}V_d - 2K_{[a}^cK_{b]p}V^p \tag{49}$$

$$\begin{aligned} R_{dcba}V^d &= \gamma_a^p\gamma_b^q\gamma_c^r\,{}^{(4)}R_{drqp}V^d - 2K_{c[a}K_{b]d}V^d \\ &= \gamma_a^p\gamma_b^q\gamma_c^r\gamma_d^s\,{}^{(4)}R_{srqp}V^d - 2K_{c[a}K_{b]d}V^d \end{aligned} \tag{50}$$

This relation has to hold for any arbitrary spatial vector so we can conclude

$$R_{abcd} + K_{ab}K_{bd} - K_{ad}K_{cb} = \gamma_a^p \gamma_b^q \gamma_c^r \gamma_d^s \, {}^{(4)}R_{pqrs} \tag{51}$$

The Eqn. 51 is called the *Gauss' equations*. It relates the full spatial projection of ${}^{(4)}R^a{}_{bcd}$ to the $R^a{}_{bcd}$ and terms quadratic in the extrinsic curvature.

Next, consider a spatial derivative of the extrinsic curvature

$$\begin{aligned} D_a K_{bc} &= \gamma_a^p \gamma_b^q \gamma_c^r \nabla_p K_{qr} = -\gamma_a^p \gamma_b^q \gamma_c^r \nabla_p (\nabla_q n_r + n_q a_r) \\ &= -(\gamma_a^p \gamma_b^q \gamma_c^r \nabla_p \nabla_q n_r + \gamma_a^p \gamma_c^r \underbrace{\gamma_b^q n_q}_{=0} \nabla_p a_r + \gamma_c^r a_r \underbrace{\gamma_a^p \gamma_b^a \nabla_p n_q}_{=-K_{ab}}) \\ &= -\gamma_a^p \gamma_b^q \gamma_c^r \nabla_p \nabla_q n_r + a_c K_{ab} \end{aligned} \tag{52}$$

Make it anti-symmetrize

$$\begin{aligned} D_{[a} K_{b]c} &= -\gamma_a^p \gamma_b^q \gamma_c^r \underbrace{\nabla_{[p} \nabla_{q]} n_r}_{=\frac{1}{2}\,{}^{(4)}R_{srqp} n^s} + a_c \underbrace{K_{[ab]}}_{=0} \\ &= -\frac{1}{2} \gamma_a^p \gamma_b^q \gamma_c^r n^s \, {}^{(4)}R_{pqrs} \end{aligned} \tag{53}$$

This can be rewritten as

$$D_b K_{ac} - D_a K_{bc} = \gamma_a^p \gamma_b^q \gamma_c^r n^s \, {}^{(4)}R_{pqrs} \tag{54}$$

The Eqn. 54 is called the *Codazzi's equation*. This shows projection with one index projected in the normal direction. Now, we consider the last remaining projection namely with two indices projected in the normal direction. This will involve a time derivative of K_{ab}. Compute

$$\begin{aligned} \pounds_n K_{ab} &= n^c \nabla_c K_a b + K_{ac} \nabla_b n^c + K_c b \nabla_a n^c \\ &= -n^c \nabla_c (\nabla_a n_b + n_a a_b) - K_{ac}(K_b^c + n_b a^c) - K_{cb}(K_a^c + n_a a^c) \\ &= -n^c \nabla_c \nabla_a n_b - n^c a_b \nabla_c n_a - n^c n_a \nabla_c a_b - (K_{ac} K_b^c + K_{cb} K_a^c) - (K_{ab} n_b + K_{cb} n_a) a^c \end{aligned} \tag{55}$$

Using the definition of Riemann tensors

$${}^{(4)}R_{dbac} n^d = 2\nabla_{[c} \nabla_{a]} n_b = \nabla_c \nabla_a n_b - \nabla_a \nabla_c n_b \tag{56}$$

$$\rightarrow \nabla_c \nabla_a n_b = {}^{(4)}R_{dbac} n^d + \nabla_a \nabla_c n_b \tag{57}$$

Substitue Eqn. 57 into Eqn. 55 then

$$\pounds_n K_{ab} = -{}^{(4)}R_{dbac} n^d n^c - n^c \nabla_a \nabla_c n_b - n^c a_b \nabla_c n_a - n^c n_a \nabla_c a_b - (K_{ac} K_b^c + K_{cb} K_a^c) - (K_{ab} n_b + K_{cb} n_a) a^c \tag{58}$$

Using the definition of $a_b = n^c \nabla_c n_b$ then we can find the relation

$$\nabla_a a_b = \nabla_a (n^c \nabla_c n_b) = n^c \nabla_a \nabla_c n_b + \nabla_a n^c \nabla_c n_b \tag{59}$$

$$\begin{aligned} \rightarrow n^c \nabla_a \nabla_c n_b &= \nabla_a a_b - \nabla_a n^c \nabla_c n_b \\ &= \nabla_a a_b - (K_a^c + n_a a^c)(K_{cb} + n_c a_b) \\ &= \nabla_a a_b - K_a^c K_{cb} - \underbrace{K_a^c n_c}_{=0} a_b - n_a a^c K_{cb} - n_a \underbrace{n_c a^c}_{=n_c n^d \nabla_d n^c = 0} a_b \\ &= \nabla_a a_b - K_a^c K_{cb} - n_a a^c K_{cb} \end{aligned} \tag{60}$$

Insert Eqn. 60 into Eqn. 58 and simplify it

$$\begin{aligned} \pounds_n K_{ab} &= -{}^{(4)}R_{dbac} n^d n^c - (\nabla_a a_b - K_a^c K_{cb} - n_a a^c K_{cb}) - n^c a_b \nabla_c n_a - n^c n_a \nabla_c a_b \\ &\quad - (K_{ac} K_b^c + K_{cb} K_a^c) - (K_{ab} n_b + K_{cb} n_a) a^c \\ &= -{}^{(4)}R_{dbac} n^d n^c - \nabla_a a_b - a_b \underbrace{n^c \nabla_c n_a}_{=a_a} - n^c n_a \nabla_c a_b - K_b^c K_{ac} - K_{ca} n_b a^c \\ &= -n^c n^d \, {}^{(4)}R_{dbac} - \nabla_a a_b - n^c n_a \nabla_c a_b - a_a a_b - K_b^c K_{ac} - K_{ca} n_b a^c \end{aligned} \tag{61}$$

Note that Eqn. 61 is purely spatial. To see that

$$\begin{aligned} n^a \pounds_n K_{ab} &= -\underbrace{n^a n^c n^d\, {}^{(4)}R_{dbac}}_{=0} - n^a \nabla_a a_b - n^c \underbrace{n^a n_a}_{=-1} \nabla_c a_b - \underbrace{n^a a_a}_{=0} a_b - K^c_b \underbrace{n^a K_{ac}}_{=0} - \underbrace{n^a K_{ca}}_{=0} n_b a^c \\ &= -n^a \nabla_a a_b + n^c \nabla_c a_b = 0 \end{aligned} \tag{62}$$

The last line came from swapping dummy index (c to a). Since $\pounds_n K_{ab}$ is purely spatial, projecting the two free indicies in Eqn. 61 leaves the LHS unchanged

$$\gamma^q_a \gamma^r_b \pounds_n K_{qr} = \pounds_n (\gamma^q_a \gamma^r_b K_{qr}) = \pounds_n K_{ab} \tag{63}$$

And, RHS is

$$\begin{aligned} \pounds_n K_{ab} &= -\gamma^q_a \gamma^r_b n^c n^d\, {}^{(4)}R_{drqc} - \gamma^q_a \gamma^r_b \nabla_q a_r - \gamma^r_b n^c \underbrace{\gamma^q_a n_q}_{=0} \nabla_c a_r - \gamma^q_a \gamma^r_b a_q a_r - \gamma^q_a \gamma^r_b K^c_r K_{qc} - K_{cq} \gamma^q_a \underbrace{\gamma^r_b n_r}_{=0} a^c \\ &= -n^d n^c \gamma^q_a \gamma^r_b\, {}^{(4)}R_{drqc} - D_a a_b - a_a a_b - K^c_b K_{ac} \end{aligned} \tag{64}$$

We know that $a_a = D_a \ln \alpha$ so

$$\begin{aligned} D_a a_b &= D_a D_b \ln \alpha = D_a \left(\frac{1}{\alpha} D_b \alpha \right) \\ &= \frac{1}{\alpha} D_a D_b \alpha - \frac{1}{\alpha^2} D_a \alpha D_b \alpha = \frac{1}{\alpha} D_a D_b \alpha - \underbrace{\left(\frac{1}{\alpha} D_a \alpha \right)}_{=a_a} \underbrace{\left(\frac{1}{\alpha} D_b \alpha \right)}_{=a_b} \\ &= -a_a a_b + \frac{1}{\alpha} D_a D_b \alpha \end{aligned} \tag{65}$$

Therefore, we can conclude

$$\pounds_n K_{ab} = n^d n^c \gamma^q_a \gamma^r_b\, {}^{(4)}R_{drcq} - \frac{1}{\alpha} D_a D_b \alpha - K^c_b K_{ac} \tag{66}$$

The Eqn. 66 is called the *Ricci's equation* which shows projection namely two indices projected in the normal direction as we stated previously.

D. The Constraints and Evolution Equations

Now, we can rewrite Einstein's equations in a 3+1 form. Basically, we just need to take the equations of Gauss, Codazzi, and Ricci and eliminate the 4-dimensional Riemann tensor using Einstein's equations

$$G_{ab} = {}^{(4)}R_{ab} - \frac{1}{2}\, {}^{(4)}R g_{ab} = 8\pi T_{ab} \tag{67}$$

In this section, we will derive the constraint equations from the Gauss's equation (Eqn. 51) and the Codazzi's equations (Eqn. 54), and will then derive the evolution equations from Eqn. 32 and the Ricci's equation (Eqn. 66).

From Gauss's equation (Eqn. 51)

$$R^a{}_{bcd} + K^a_c K_{bd} - K^a_d K_{cb} = \gamma^{pa} \gamma^q_b \gamma^r_c \gamma^s_d\, {}^{(4)}R_{pqrs} \tag{68}$$

Contract this with γ^c_a

$$R^a{}_{bad} + K^a_a K_{bd} - K^c_d K_{cb} = \gamma^{pc} \gamma^q_b \gamma^r_c \gamma^s_d\, {}^{(4)}R_{pqrs} \tag{69}$$

Denote $K^a_a = K$ which trace of extrinsic curvature and use the 3-dimensional Ricci tensor then

$$\gamma^{pr} \gamma^q_b \gamma^s_d\, {}^{(4)}R_{pqrs} = R_{bd} + K K_{bd} - K^c_d K_{cb} \tag{70}$$

Rearrange index

$$\gamma^{pr}\gamma^q_b\gamma^{sd\,(4)}R_{pqrs} = R^b{}_d + KK^b_d - K^{cd}K_{cb} \tag{71}$$

Contract again with γ^b_d

$$\begin{aligned}\gamma^{pr}\gamma^b_d\gamma^q_b\gamma^{sd\,(4)}R_{pqrs} &= \gamma^b_d R^b{}_d + \gamma^b_d KK^b_d - \gamma^b_d K^{cd}K_{cb} \\ \gamma^{pr}\gamma^{qs\,(4)}R_{pqrs} &= R + K^2 - K^{cb}K_{cb}\end{aligned} \tag{72}$$

LHS of above equation can be written as

$$\begin{aligned}(g^{pr}+n^pn^r)(g^{qs}+n^qn^s)\,^{(4)}R_{pqrs} &= g^{pr}g^{qs\,(4)}R_{pqrs} + g^{pr}n^qn^{s\,(4)}R_{pqrs} + g^{qs}n^pn^{r\,(4)}R_{pqrs} + \underbrace{n^pn^qn^rn^{s\,(4)}R_{pqrs}}_{=0} \\ &= \,^{(4)}R + n^qn^{s\,(4)}R_{pq} + n^pn^{r\,(4)}R_{pr} \\ &= \,^{(4)}R + 2n^pn^{r\,(4)}R_{pr}\end{aligned} \tag{73}$$

Last line came from swapping dummy indices (q, s to p, r). So, we have

$$^{(4)}R + 2n^pn^{r\,(4)}R_{pr} = R + K^2 - K^{ab}K_{ab} \tag{74}$$

Again here, we swap the dummy index (c to a). Contract the Einstein tensor with n^pn^r gives

$$\begin{aligned}2n^pn^rG_{pr} &= 2n^pn^r\left(^{(4)}R_{pr} - \frac{1}{2}g_{pr}\,^{(4)}R\right) \\ &= 2n^pn^{r\,(4)}R_{pr} - n^pn^r(\gamma_{pr}+n_pn_r)\,^{(4)}R \\ &= 2n^pn^{r\,(4)}R_{pr} - \underbrace{n^pn^r\gamma_{pr}}_{=0}\,^{(4)}R - \underbrace{n_pn^p}_{=-1}\underbrace{n^rn_r}_{=-1}\,^{(4)}R \\ &=^{(4)} R + 2n^pn^{r\,(4)}R_{pr}\end{aligned} \tag{75}$$

Thus

$$2n^pn^rG_{pr} = R + K^2 - K^{ab}K_{ab} \tag{76}$$

And from Einstein's equation

$$2n^pn^rG_{pr} = 16\pi n^pn^rT_{pr} \tag{77}$$

We now define the energy density ρ to be the total energy density as measured by a normal observer n^a

$$\rho \equiv n_an_bT^{ab} = n^an^bT_{ab} \tag{78}$$

Finally, we get

$$R + K^2 - K^{ab}K_{ab} = 16\pi\rho \tag{79}$$

The Eqn. 79 is called the *Hamiltonian constraint*

Now, consider the Codazzi's equation (Eqn. 54)

$$D_bK^c_a - D_aK^c_b = \gamma^p_a\gamma^q_b\gamma^{rc}n^{s\,(4)}R_{pqrs} \tag{80}$$

Contract this with γ^b_c

$$\begin{aligned}D_bK^b_a - D_aK^b_b &= \gamma^p_a\gamma^q_b\underbrace{\gamma^b_c\gamma^{rc}}_{=\gamma^{rb}}n^{s\,(4)}R_{pqrs} \\ D_bK^b_a - D_aK &= \gamma^p_a\underbrace{\gamma^q_b\gamma^{rb}}_{\gamma^{qr}}n^{s\,(4)}R_{pqrs} \\ &= \gamma^p_a(g^{qr}+n^qn^r)n^{s\,(4)}R_{pqrs} \\ &= -\gamma^p_an^sg^{qr\,(4)}R_{qprs} + \gamma^p_a\underbrace{n^qn^rn^{s\,(4)}R_{pqrs}}_{=0} \\ &= -\gamma^p_an^{s\,(4)}R_{ps}\end{aligned} \tag{81}$$

Contract the Einstein's tensor with n^s and γ_a^p

$$\begin{aligned}\gamma_a^p n^s G_{ps} &= \gamma_a^p n^s \left({}^{(4)}R_{ps} - \frac{1}{2} g_{ps} \, {}^{(4)}R \right) \\ &= \gamma_a^p n^s \, {}^{(4)}R_{ps} - \frac{1}{2} \underbrace{\gamma_a^p g_{ps} n^s}_{=\gamma_{as} n^s = 0} \, {}^{(4)}R \\ &= \gamma_a^p n^s \, {}^{(4)}R_{ps}\end{aligned} \tag{82}$$

So

$$D_b K_a^b - D_a K = -\gamma_a^p n^s G_{ps} \tag{83}$$

From the Einstein's equations again

$$\gamma_a^p n^s G_{ps} = 8\pi \gamma_a^p n^s T_{ps} \tag{84}$$

We now define S_a to be the momentum density as measured by a normal observer n^a

$$S_a \equiv -\gamma_a^b n^c T_{bc} \tag{85}$$

and finally find

$$D_b K_a^b - D_a K = 8\pi S_a \tag{86}$$

The Eqn. 86 is called the *momentum constraint.*

Now we need to evaluate the evolution equations that evolve the data (γ_{ab}, K_{ab}) forward in time can be found from Eqn. 32, which can be considered as the definition of the extrinsic curvature, and the Ricci's equation (Eqn. 66). These equations involve the Lie derivative along n^a. However, the $\pounds_n$ is not a natural time derivative since n^a is not dual to the surface 1-form Ω_a i.e. their dot product is not unity but rahter

$$n^a \Omega_a = -\alpha g^{ab} \nabla_a t \nabla_b t = \frac{1}{\alpha} \tag{87}$$

Instead, consider a new vector

$$t^a = \alpha n^a + \beta^a \tag{88}$$

which is dual to Ω_a for any spatial vector β^a

$$t^a \Omega_a = \alpha \underbrace{n^a \Omega_a}_{=1/\alpha} + \underbrace{\beta^a \Omega_a}_{=0} = 1 \tag{89}$$

We called the vector β^a is *shift vector.* The vector t^a will connect points with the same spatial coordinate on neighboring time slices. Then the shift vector β^a will measure the amount by which the spatial coordinates are shifted within a slice with respect to the normal vector. As we discussed before, the lapse function α measures how much proper time elapses between neighboring time slices along the normal vector. Thus, the lapse and shift determine how the coordinates evolve in time. Note that the choice of α and β^a is quite arbitrary and there are many discussions of choices for lapse and shift in literatures. Recall that β^a is spatial and so $n^a \beta_a = 0$, hence only three of its components may be freely specified. Therefore, we have total four freedom to chose gauge functions α and β^a completely arbitrarily embodies the four-fold coordinate degrees of freedom.

Consider now the Lie derivative of K_{ab} along t^a

$$\pounds_t K_{ab} = \pounds_{\alpha n + \beta} K_{ab} = \alpha \pounds_n K_{ab} + \pounds_\beta K_{ab} \tag{90}$$

which follows from the definition of the Lie derivative. Insert this into the Ricci's equation to eliminate $\pounds_n K_{ab}$

$$\pounds_n K_{ab} = \frac{1}{\alpha} (\pounds_t K_{ab} - \pounds_\beta K_{ab}) \tag{91}$$

From Ricci's equation

$$\pounds_n K_{ab} = n^d n^c \gamma^q_a \gamma^r_b \, {}^{(4)}R_{drcq} - \frac{1}{\alpha} D_a D_b \alpha - K^c_b K_{ac} \tag{92}$$

Rewrite the first term of RHS

$$\begin{aligned} n^d n^c \gamma^q_a \gamma^r_b \, {}^{(4)}R_{drcq} &= (\gamma^{dc} - g^{dc}) \gamma^q_a \gamma^r_b \, {}^{(4)}R_{drcq} \\ &= \gamma^{dc} \gamma^q_a \gamma^r_b \, {}^{(4)}R_{drcq} - \gamma^q_a \gamma^r_b \, {}^{(4)}R_{rq} \end{aligned} \tag{93}$$

During the calculation to get Hamiltonian constraint (Eqn. 70), we have the relation

$$\gamma^{cd} \gamma^q_a \gamma^r_b \, {}^{(4)}R_{drcq} = R_{ab} + K K_{ab} - K^c_b K_{ac} \tag{94}$$

Contract the Einstein's equations with g^{rq}

$$\begin{aligned} g^{rq} \left({}^{(4)}R_{rq} - \frac{1}{2} g_{rq} \, {}^{(4)}R \right) &= g^{rq} \, {}^{(4)}R_{rq} - \frac{1}{2} \underbrace{g^{rq} g_{rq}}_{=4} \, {}^{(4)}R = - \, {}^{(4)}R \\ &= 8\pi g^{rq} T_{rq} = 8\pi T \\ &\rightarrow \, {}^{(4)}R = -8\pi T \end{aligned} \tag{95}$$

Again using Einstein's equations

$$\begin{aligned} &{}^{(4)}R_{rq} - \frac{1}{2} g_{rq} \, {}^{(4)}R = 8\pi T_{rq} \\ &\rightarrow \, {}^{(4)}R_{rq} = \frac{1}{2} g_{rq} \, {}^{(4)}R + 8\pi T_{rq} = 8\pi \left(T_{rq} - \frac{1}{2} g_{rq} T \right) \end{aligned} \tag{96}$$

Using these results, Eqn. 93 can be expressed

$$\begin{aligned} n^d n^c \gamma^q_a \gamma^r_b \, {}^{(4)}R_{drcq} &= R_{ab} + K K_{ab} - K^c_b K_{ac} - 8\pi \gamma^q_a \gamma^r_b \left(T_{rq} - \frac{1}{2} g_{rq} T \right) \\ &= R_{ab} + K K_{ab} - K^c_b K_{ac} - 8\pi \left(\gamma^q_a \gamma^r_b T_{rq} - \frac{1}{2} \gamma^q_a \gamma^r_b g_{rq} g^{ef} T_{ef} \right) \\ &= R_{ab} + K K_{ab} - K^c_b K_{ac} - 8\pi \left(\gamma^q_a \gamma^r_b T_{rq} - \frac{1}{2} \gamma_{ab} (\gamma^{ef} - n^e n^f) T_{ef} \right) \end{aligned} \tag{97}$$

We now define the spatial stress and its trace by

$$S_{ab} \equiv \gamma^c_a \gamma^d_b T_{cd} \qquad S \equiv S^a_a = \gamma^{ab} T_{ab} \tag{98}$$

Thus

$$n^d n^c \gamma^q_a \gamma^r_b \, {}^{(4)}R_{drcq} = R_{ab} + K K_{ab} - K^c_b K_{ac} - 8\pi \left(S_{ab} - \frac{1}{2} \gamma_{ab} (S - \rho) \right) \tag{99}$$

Then Ricci's equation is rewritten as

$$\begin{aligned} \pounds_n K_{ab} &= R_{ab} + K K_{ab} - K^c_b K_{ac} - 8\pi \left(S_{ab} - \frac{1}{2} \gamma_{ab} (S - \rho) \right) - \frac{1}{\alpha} D_a D_b \alpha - K^c_b K_{ac} \\ &= R_{ab} + K K_{ab} - 2K^c_b K_{ac} - 8\pi \left(S_{ab} - \frac{1}{2} \gamma_{ab} (S - \rho) \right) - \frac{1}{\alpha} D_a D_b \alpha \end{aligned} \tag{100}$$

In terms of the Lie derivative along t^a

$$\frac{1}{\alpha} (\pounds_t K_{ab} - \pounds_\beta K_{ab}) = R_{ab} + K K_{ab} - 2K^c_b K_{ac} - 8\pi \left(S_{ab} - \frac{1}{2} \gamma_{ab} (S - \rho) \right) - \frac{1}{\alpha} D_a D_b \alpha \tag{101}$$

Finally, we have the first evolution equation

$$\pounds_t K_{ab} = \pounds_\beta K_{ab} - D_a D_b \alpha + \alpha(R_{ab} + KK_{ab} - 2K^c_b K_{ac}) - 8\pi\alpha \left(S_{ab} - \frac{1}{2}\gamma_{ab}(S - \rho) \right) \tag{102}$$

The Eqn. 102 is the evolution equation for the extrinsic curvature. Note that here all differential operators and the Ricci tensor R_{ab} are associated with the spatial metric γ_{ab} not full 4-dimensional metric g_{ab}.

From Eqn. 32

$$\begin{aligned} K_{ab} &= -\frac{1}{2}\pounds_n \gamma_{ab} = -\frac{1}{2}\frac{1}{\alpha}(\pounds_t \gamma_{ab} - \pounds_\beta \gamma_{ab}) \\ \rightarrow \pounds_t \gamma_{ab} &= \pounds_\beta \gamma_{ab} - 2\alpha K_{ab} \end{aligned} \tag{103}$$

The Eqn. 103 is the evolution equations for the spatial metric. In summary, we have four constrain equations (Eqns. 79 and 86) and twelve evolution equations (Eqns. 102 and 103). These sets of equation are completely equivalent to Einstein's equations.

E. The ADM Equations

So far, we derived our equations in a covariant, coordinate-independent manner i.e. the basis vector have been completely arbitrary and have no particular relationship to the 1-form Ω_a or to the congruence defined by t^a. Here we consider specific choice of basis vector (There is a motivation about this choice but here we are concentrate on the formulation alone. We refer the references that are showed at the beginning of this note for reader)

$$t^a = (1, 0, 0, 0) \tag{104}$$

This implies that the Lie derivative along t^a reduces to a partial derivative with respect to t i.e. $\pounds_t = \partial_t$. Since spatial tensors vanish when contracted with the normal vector, this also means that all components of spatial tensors with a contravariant index equal to zero must vanish. For the shift vector, this implies $n_a \beta^a = n_0 \beta^0 = 0$. And, spatial basis vector $e^a_{(i)}$ where i distinguishes the vectors, not the component should satisfy $\Omega_a e^a_{(i)} = 0$. From this choice and condition, we can conclude $\Omega_a e^a_{(i)} = -\frac{1}{\alpha} n_a e^a_{(i)} = 0$ which implies that the covariant spatial components of the normal vector have to vanish

$$n_i = 0 \tag{105}$$

Consequently, shift vector should have the form

$$\beta^a = (0, \beta^i) \tag{106}$$

Then form $t^a = \alpha n^a + \beta^a$ gives the contravariant components

$$n^a = \left(\frac{1}{\alpha}, -\frac{\beta^i}{\alpha} \right) \tag{107}$$

and from normalization condition $n_a n^a = -1$ we find

$$n_a = (-\alpha, 0, 0, 0) \tag{108}$$

Then from the definition of the spatial metric $\gamma_{ab} = g_{ab} + n_a n_b$ we have

$$\gamma_{ij} = g_{ij} \tag{109}$$

this implies that γ_{ij}, the metric on Σ, is just the spatial part of the 4-metric g_{ab}. Since zeroth component of spatial contravariant tensor have to vanish, we also have $\gamma^{a0} = 0$. The inverse metric can therefore be expressed as

$$g^{ab} = \gamma^{ab} - n^a n^b = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^i}{\alpha^2} \\ \frac{\beta^j}{\alpha^2} & \gamma^{ij} - \frac{\beta^i \beta^j}{\alpha^2} \end{pmatrix} \tag{110}$$

From this, we can check

$$\gamma^{ik}\gamma_{kj} = (g^{ik} + n^i n^k)g_{kj} = g^{ik}g_{kj} + n^i \underbrace{n^k g_{kj}}_{=0} = \delta^i_j \tag{111}$$

This implies that γ^{ij} and γ_{ij} are 3-dimensional inverses, and can hence be used to raise and lower spatial indices of spatial tensors i.e. $\beta_i = \gamma_{ij}\beta^j$. Inverting g^{ab} gives g_{ab}

$$g_{ab} = \begin{pmatrix} -\alpha^2 + \beta_l\beta^l & \beta_i \\ \beta_j & \gamma_{ij} \end{pmatrix} \tag{112}$$

So the line element may be decomposed as

$$ds^2 = g_{ab}dx^a dx^b = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{113}$$

which is often referred to as the metric in 3+1 form. The entire content of any spatial tensor is available from their spatial components. This is true for contravariant components, since their zeroth component vanishes, but it is also true for covariant components. Therefore, the entire content of the decomposed Einstein's equations is contained in their spatial components alone. Then, we can rewrite the constraint and evolution equations in the same form as previous but only contain spatial indices.

For the Hamiltonian constraint (Eqn. 79)

$$\begin{aligned} R + K^2 - K^{ab}K_{ab} &= 16\pi\rho \\ \Rightarrow R + K^2 - K^{ij}K_{ij} &= 16\pi\rho \end{aligned} \tag{114}$$

For the momentum constraint (Eqn. 86)

$$\begin{aligned} D_b K^b_a - D_a K &= 8\pi S_a \\ \Rightarrow D_j K^j_i - D_i K &= 8\pi S_i \end{aligned} \tag{115}$$

or equivalently

$$\begin{aligned} &D_j K^{ji} - D^i K = 8\pi S^i \\ &\Rightarrow D_j K^{ij} - \gamma^{ij} D_j K = D_j(K^{ij} - \gamma^{ij}K) = 8\pi S^i \end{aligned} \tag{116}$$

For the evolution equation of the extrinsic curvature (Eqn. 102)

$$\begin{aligned} \pounds_t K_{ab} &= \pounds_\beta K_{ab} - D_a D_b \alpha + \alpha(R_{ab} + KK_{ab} - 2K^c_b K_{ac}) - 8\pi\alpha\left(S_{ab} - \frac{1}{2}\gamma_{ab}(S-\rho)\right) \\ \Rightarrow \partial_t K_{ij} &= \pounds_\beta K_{ij} - D_i D_j \alpha + \alpha(R_{ij} + KK_{ij} - 2K^k_j K_{ik}) - 8\pi\alpha\left(S_{ij} - \frac{1}{2}\gamma_{ij}(S-\rho)\right) \\ &= \beta^k D_k K_{ij} + K_{ik}D_j\beta^k + K_{kj}D_i\beta^k - D_i D_j\alpha + \alpha(R_{ij} + KK_{ij} - 2K^k_j K_{ik}) - 8\pi\alpha\left(S_{ij} - \frac{1}{2}\gamma_{ij}(S-\rho)\right) \end{aligned} \tag{117}$$

For the evolution equation of the spatial metric (Eqn. 103)

$$\begin{aligned} \pounds_t \gamma_{ab} &= \pounds_\beta \gamma_{ab} - 2\alpha K_{ab} \\ \Rightarrow \partial_t \gamma_{ij} &= \pounds_\beta \gamma_{ij} - 2\alpha K_{ij} \\ &= -2\alpha K_{ij} + \beta^k \underbrace{D_k\gamma_{ij}}_{=0} + \gamma_{ik}D_j\beta^k + \gamma_{kj}D_i\beta^k \\ &= -2\alpha K_{ij} + D_j(\gamma_{ik}\beta^k) + D_i(\gamma_{kj}\beta^k) \\ &= -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \end{aligned} \tag{118}$$

with the matter sources terms appearing in the above equations are defined by

$$\rho = n_a n_b T^{ab} \qquad S^i = -\gamma^{ij} n^a T_{aj} \qquad S_{ij} = \gamma_{ia}\gamma_{jb} T^{ab} \qquad S = \gamma^{ij} S_{ij} \tag{119}$$

Eqns 114, 116, 117, and 118 are equivalent to Einstein's equations and comprise the standard 3+1 equations. They are also referred as the ADM equations after Arnowiit, Deser, and Misner.

Using these ADM equations, we can have other equations for the determinant of the spatial metric $\gamma = det(\gamma_{ij})$ and the trace of the extrinsic curvature $K = K^i_i$.

Consider a version of Jacobi's formula for γ

$$\partial_t \ln\sqrt{\gamma} = \frac{1}{2\gamma}\partial_t\gamma = \frac{1}{2}\gamma^{ij}\partial_t\gamma_{ij} \tag{120}$$

Then use Eqn. 118

$$\begin{aligned}
\partial_t \ln\sqrt{\gamma} &= \frac{1}{2}\gamma^{ij}(-2\alpha K_{ij} + D_i\beta_j + D_j\beta_i) \\
&= -\alpha K + \frac{1}{2}(\underbrace{D^j\beta_j}_{=D^i\beta_i} + D^i\beta_i) \\
&= -\alpha K + D^i\beta_i = -\alpha K + D_i\beta^i
\end{aligned} \tag{121}$$

Contract Eqn. 117 with γ^{ij}

$$\begin{aligned}
\gamma^{ij}\partial_t K_{ij} &= \gamma^{ij}\left[\beta^k D_k K_{ij} + K_{ik}D_j\beta^k + K_{kj}D_i\beta^k - D_iD_j\alpha + \alpha(R_{ij} + KK_{ij} - 2K^k_j K_{ik}) - 8\pi\alpha\left(S_{ij} - \frac{1}{2}\gamma_{ij}(S-\rho)\right)\right] \\
&= \beta^k D_k(\underbrace{\gamma^{ij}K_{ij}}_{=K}) + \underbrace{\gamma^{ij}K_{ik}D_j\beta^k}_{=K^j_k D_j\beta^k = K^i_k D_i\beta^k} + \underbrace{\gamma^{ij}K_{kj}D_i\beta^k}_{=K^i_k D_i\beta^k} - \underbrace{\gamma^{ij}D_iD_j}_{\equiv D^2}\alpha + \alpha(\underbrace{\gamma^{ij}R_{ij}}_{=R} + \underbrace{K\gamma^{ij}K_{ij}}_{K^2} - 2\underbrace{\gamma^{ij}K^k_j}_{K^{ik}}K_{ik}) \\
&\quad - 8\pi\alpha\left(\underbrace{\gamma^{ij}S_{ij}}_{=S} - \frac{1}{2}\underbrace{\gamma^{ij}\gamma_{ij}}_{=3}(S-\rho)\right) \\
&= \beta^k D_k K + 2K^i_k D_i\beta^k - D^2\alpha + \alpha(\underbrace{R + K^2 - K^{ik}K_{ik}}_{=16\pi\rho} - K^{ik}K_{ik}) - 4\pi\alpha(3\rho - S) \\
&= \beta^k D_k K + 2K^i_k D_i\beta^k - D^2\alpha - \alpha K^{ik}K_{ik} + 4\pi\alpha(S+\rho)
\end{aligned} \tag{122}$$

Then evaluate LHS $\gamma^{ij}\partial_t K_{ij}$

$$\gamma^{ij}\partial_t K_{ij} = \partial_t(\gamma^{ij}K_{ij}) - K_{ij}\partial_t\gamma^{ij} = \partial_t K - K_{ij}\partial_t\gamma^{ij} \tag{123}$$

We need to find the evolution equations for inverse of the spatial metric. To do that, using the fact $\gamma^{ik}\gamma_{kj} = \delta^i_j$

$$\begin{aligned}
&\partial_t(\gamma^{ik}\gamma_{kj}) = \partial_t\delta^i_j = 0 = \gamma^{ik}\partial_t\gamma_{kj} + \gamma_{kj}\partial_t\gamma^{ik} \\
&\Rightarrow \gamma_{kj}\partial_t\gamma^{ik} = -\gamma^{ik}\partial_t\gamma_{kj}
\end{aligned} \tag{124}$$

Contract both sides with γ^{jl} and use Eqn. 118

$$\begin{aligned}
&\gamma^{jl}\gamma_{kj}\partial_t\gamma^{ik} = -\gamma^{jl}\gamma^{ik}\partial_t\gamma_{kj} \\
&\Rightarrow \delta^l_k\partial_t\gamma^{ik} = \partial_t\gamma^{il} = -\gamma^{jl}\gamma^{ik}(-2\alpha K_{kj} + D_k\beta_j + D_j\beta_k) = 2\alpha K^{il} - \gamma^{ik}D_k\beta^j - \gamma^{kj}D_k\beta^i
\end{aligned} \tag{125}$$

Thus

$$\begin{aligned}
\partial_t K - K_{ij}\partial_t\gamma^{ij} &= \partial_t K - K_{ij}(2\alpha K^{ij} - \gamma^i k D_k\beta^j - \gamma^{kj}D_k\beta^i) \\
&= \partial_t K - 2\alpha K^{ij}K_{ij} + K^k_j D_k\beta^j + K^k_i D_k\beta^i \\
&= \partial_t K - 2\alpha K^{ij}K_{ij} + 2K^k_i D_k\beta^i
\end{aligned} \tag{126}$$

Rearrange dummy indices in Eqns. 122 and 126 then we get

$$\partial_t K - 2\alpha K_{ij}K^{ij} + 2K^i_k D_i\beta^k = \beta^i D_i K + 2K^i_k D_i\beta^k - D^2\alpha - \alpha K^{ij}K_{ij} + 4\pi\alpha(S+\rho)$$
$$\Rightarrow \partial_t K = -D^2\alpha + \alpha\left[K_{ij}K^{ij} + 4\pi(S+\rho)\right] + \beta^i D_i K \tag{127}$$

Here we obtain additional evolution equations for the determinant of the spatial metric γ and trace of the extrinsic curvature K. Eqns 121 and 127 will be used for some calculations in later sections.

II. CONFORMAL TRANSFORMATION

From previous section, we decompose the Einstein's equations in the 3+1 form (or the ADM form) that provides evolution and constraint equations. Constraint equations will provide initial data that we can use for black hole evolution and evolution equations provide dynamics of metrics with certain initial data from constraint equations. Often time, solving those equations could be hard so we treat them differently with some transformations.

In this note, we consider a conformal transformation (Again here we do not discuss about details of these treatment). Further, we only consider conformal transformations of the spatial metric. In a different context it may also be useful to study conformal transformation of the spacetime metric.

A. Conformal Transformation of the Spatial Metric

Let's writhe the spatial metric γ_{ij} as a product of some power of a positive scaling factor ψ and a background metric $\bar{\gamma}_{ij}$

$$\gamma_{ij} = \psi^p \bar{\gamma}_{ij} \tag{128}$$

This identification is a conformal transformation of the spatial metric. We call ψ as the conformal factor and $\bar{\gamma}_{ij}$ as the conformally related metric. In many practice, we choose $p = 4$ for a conformal power. From the property of the metric $\gamma_{ik}\gamma^{kj} = \bar{\gamma}_{ik}\bar{\gamma}^{kj} = \delta^i_j$, conformal transformation for inverse metric should be

$$\gamma^{ij} = \psi^{-4}\bar{\gamma}^{ij} \tag{129}$$

Superficially, the conformal transformation just shows rewriting one unknown as a product of two unknowns in order to make solving some equations easier but the conformal transformation serves to define an equivalence class of manifolds and metric (Check Wald's GR book for more deep discussions).

Using these transformation, we can find a transformation rule for connection coefficients and etc. In three dimensions, the connection coefficients must transform according to

$$\begin{aligned}
\Gamma^i{}_{jk} &= \frac{1}{2}\gamma^{il}(\partial_k\gamma_{lj} + \partial_j\gamma_{lk} - \partial_l\gamma_{jk}) \\
&= \frac{1}{2}\psi^{-4}\bar{\gamma}^{il}(\partial_k(\psi^4\bar{\gamma}_{lj}) + \partial_j(\psi^4\bar{\gamma}_{lk}) - \partial_l(\psi^4\bar{\gamma}_{jk})) \\
&= \frac{1}{2}\psi^{-4}\bar{\gamma}^{il}(\psi^4\partial_k\bar{\gamma}_{lj} + \bar{\gamma}_{lj}\partial_k\psi^4 + \psi^4\partial_j\bar{\gamma}_{lk} + \bar{\gamma}_{lk}\partial_j\psi^4 - \psi^4\partial_l\bar{\gamma}_{jk} - \bar{\gamma}_{jk}\partial_l\psi^4) \\
&= \frac{1}{2}\psi^{-4}\bar{\gamma}^{il}\psi^4(\partial_k\bar{\gamma}_{lj} + \partial_j\bar{\gamma}_{lk} - \partial_l\bar{\gamma}_{jk}) + \frac{1}{2}\psi^{-4}(\bar{\gamma}^{il}\bar{\gamma}_{lj}\partial_k\psi^4 + \bar{\gamma}^{il}\bar{\gamma}_{lk}\partial_j\psi^4 - \bar{\gamma}^{il}\bar{\gamma}_{jk}\partial_l\psi^4) \\
&= \bar{\Gamma}^i{}_{jk} + \frac{1}{2}\psi^{-4}(\delta^i_j\partial_k\psi^4 + \delta^j_k\partial_j\psi^4 - \bar{\gamma}^{il}\bar{\gamma}_{jk}\partial_l\psi^4)
\end{aligned} \tag{130}$$

where we define

$$\bar{\Gamma}^i{}_{jk} \equiv \frac{1}{2}\gamma^{il}(\partial_k\bar{\gamma}_{lj} + \partial_j\bar{\gamma}_{lk} - \partial_l\bar{\gamma}_{jk}) \tag{131}$$

and we can further simplify

$$\psi^{-4}\partial_i\psi^4 = \psi^{-4}(4\psi^3\partial_i\psi) = \frac{4}{\psi}\partial_i\psi = 4\partial_i \ln\psi \tag{132}$$

Therefore

$$\begin{aligned}
\Gamma^i{}_{jk} &= \bar{\Gamma}^i{}_{jk} + 2(\delta^i_j \partial_k \ln\psi + \delta^j_k \partial_j \ln\psi - \bar{\gamma}_{jk}\bar{\gamma}^{il}\partial_l \ln\psi) \\
&= \bar{\Gamma}^i{}_{jk} + 2(\delta^i_j \bar{D}_k \ln\psi + \delta^j_k \bar{D}_j \ln\psi - \bar{\gamma}_{jk}\bar{\gamma}^{il}\bar{D}_l \ln\psi)
\end{aligned} \tag{133}$$

In the last line, since ψ is the scalar function, we can replace partial derivative to spatial covariant derivative for conformally related metric. And, the $\bar{D}_i$ is compatible with the conformally related metric i.e. $\bar{D}_i\bar{\gamma}_{jk} = 0$ as usual sense.

Without detail proof, we can easily make relationship between D_i and $\bar{D}_i$. For arbitrary vector v^i, $D_j v^i$ and $\bar{D}_j v^i$ is related by the formula

$$D_j v^i = \bar{D}_j v^i + C^i{}_{jk} v^k \tag{134}$$

where $C^i{}_{jk}$ is

$$C^i{}_{jk} \equiv \Gamma^i{}_{jk} - \bar{\Gamma}^i{}_{jk} = 2(\delta^i_j \bar{D}_k \ln\psi + \delta^j_k \bar{D}_j \ln\psi - \bar{\gamma}_{jk}\bar{\gamma}^{il}\bar{D}_l \ln\psi) \tag{135}$$

Thus $C^i{}_{jk}$ has same property as connection coefficients. For example, $C^i{}_{jk} = C^i{}_{kj}$ for torsion free. Without loss of generality, this relation still works for higher rank tensor. For example,

$$D_i F^j_k = \bar{D}_i F^j_k + C^j{}_{il} F^l_k - C^l{}_{ki} F^j_l \tag{136}$$

This relation useful for many calculation. Next consider the Ricci tensor. From the Riemann tensor relation, $R_{dbac}n^i = 2D_{[l}D_{k]}n_j$, Ricci tensor can be expressed

$$R_{ij}v^j = D_j D_i v^j - D_i D_j v^j \tag{137}$$

Express D_i in terms of $\bar{D}_i$

$$\begin{aligned}
R_{ij}v^j &= D_j D_i v^j - D_i D_j v^j \\
&= \bar{D}_j(D_i v^j) + C^j{}_{jk} D_i v^k - C^k{}_{ij} D_k v^j - (\bar{D}_i(D_j v^j) + \underbrace{C^j{}_{ik} D_j v_k}_{=C^k{}_{ji} D_k v^j} - C^k{}_{ji} D_k v^j) \\
&= \bar{D}_j(\bar{D}_i v^j + C^j{}_{ik} v^k) + C^j{}_{jk}(\bar{D}_i v^k + C^k{}_{il} v^l) - \bar{D}_i(\bar{D}_j v^j + C^j{}_{jk} v^k) - C^k{}_{ji}(\bar{D}_k v^j + C^j{}_{kl} v^l) \\
&= \bar{D}_j\bar{D}_i v^j + \underbrace{(\bar{D}_j C^j{}_{ik})v^k}_{=(\bar{D}_k C^k{}_{ij})v^j} + C^j{}_{ik}\bar{D}_j v^k + C^j{}_{jk}\bar{D}_i v^k + \underbrace{C^j{}_{jk} C^k{}_{il} v^l}_{=C^l{}_{lk} C^k{}_{ij} v^j} \\
&\quad - \bar{D}_i\bar{D}_j v^j - \underbrace{(\bar{D}_i C^j{}_{jk})v^k}_{=(\bar{D}_i C^k{}_{kj})v^j} - C^j{}_{jk}\bar{D}_i v^k - \underbrace{C^k{}_{jk}\bar{D}_k v^j}_{=C^j{}_{ki}\bar{D}_j v^k} - \underbrace{C^k{}_{ji} C^j{}_{kl} v^l}_{=C^k{}_{li} C^l{}_{kj} v^j} \\
&= \bar{D}_j\bar{D}_i v^j - \bar{D}_i\bar{D}_j v^j + (\bar{D}_k C^k{}_{ij})v^j - (\bar{D}_i C^k{}_{kj})v^j + C^l{}_{lk} C^k{}_{ij} v^j - C^k{}_{li} C^l{}_{kj} v^j
\end{aligned} \tag{138}$$

during evaluation we relabel some dummy indices ($j \leftrightarrow k$ or $j \leftrightarrow l$). Define a conformally related Ricci tensor

$$\bar{R}_{ij}v^j = \bar{D}_j\bar{D}_i v^j - \bar{D}_i\bar{D}_j v^j \tag{139}$$

and this is true for all arbitrary v^j

$$R_{ij} = \bar{R}_{ij} + \bar{D}_k C^k{}_{ij} - \bar{D}_i C^k{}_{kj} + C^l{}_{lk} C^k{}_{ij} - C^k{}_{li} C^l{}_{kj} \tag{140}$$

We can express this relation in terms of conformally related metric and conformal scalar function. Let's evaluate term by term in Eqn. 140

$$\begin{aligned}
\bar{D}_k C^k{}_{ij} &= 2\bar{D}_k(\delta^k_i \bar{D}_j \ln\psi + \delta^k_i \bar{D}_i \ln\psi - \bar{\gamma}_{ij}\bar{\gamma}^{kl}\bar{D}_l \ln\psi) \\
&= 2(\underbrace{\delta^k_i \bar{D}_k}_{=\bar{D}_i} \bar{D}_j \ln\psi + \underbrace{\delta^k_i \bar{D}_k}_{=\bar{D}_j} \bar{D}_i \ln\psi - \bar{\gamma}_{ij}\bar{\gamma}^{kl}\bar{D}_k\bar{D}_l \ln\psi) \\
&= 4\bar{D}_i\bar{D}_j \ln\psi - 2\bar{\gamma}_{ij}\bar{\gamma}^{kl}\bar{D}_k\bar{D}_l \ln\psi
\end{aligned} \tag{141}$$

Here we use the fact that $\bar{D}_i\bar{D}_j f = \bar{D}_j\bar{D}_i f$ i.e. covariant derivative commutes with scalar function

$$\begin{aligned} {C^k}_{kj} &= 2(\underbrace{\delta^k_k}_{=3}\bar{D}_j\ln\psi + \underbrace{\delta^k_j\bar{D}_k}_{=\bar{D}_j}\ln\psi - \underbrace{\bar{\gamma}_{kj}\gamma^{kl}\bar{D}_l}_{=\delta^l_j\bar{D}_l=\bar{D}_j}\ln\psi) \\ &= 6\bar{D}_j\ln\psi \qquad (142) \\ &\Rightarrow \bar{D}_i{C^k}_{kj} = 6\bar{D}_i\bar{D}_j\ln\psi \qquad (143) \end{aligned}$$

$$\begin{aligned} {C^l}_{lk}{C^k}_{ij} &= 12\bar{D}_k\ln\psi(\delta^k_i\bar{D}_j\ln\psi + \delta^k_j\bar{D}_i\ln\psi - \bar{\gamma}_{ij}\bar{\gamma}^{kl}\bar{D}_l\ln\psi) \\ &= 12(\underbrace{\delta^k_i\bar{D}_k}_{\bar{D}_i}\ln\psi\bar{D}_j\ln\psi + \underbrace{\delta^k_j\bar{D}_k}_{\bar{D}_j}\ln\psi\bar{D}_i\ln\psi - \bar{\gamma}_{ij}\bar{\gamma}^{kl}\bar{D}_k\ln\psi\bar{D}_l\ln\psi) \\ &= 24\bar{D}_i\ln\psi\bar{D}_j\ln\psi - 12\bar{\gamma}_{ij}\bar{\gamma}^{kl}\bar{D}_k\ln\psi\bar{D}_l\ln\psi \qquad (144) \end{aligned}$$

$$\begin{aligned} {C^k}_{li}{C^l}_{kj} &= 4(\delta^k_l\bar{D}_i\ln\psi + \delta^k_i\bar{D}_l\ln\psi - \bar{\gamma}_{li}\bar{\gamma}^{km}\bar{D}_m\ln\psi)(\delta^l_k\bar{D}_j\ln\psi + \delta^l_j\bar{D}_k\ln\psi - \bar{\gamma}_{kj}\bar{\gamma}^{lm}\bar{D}_m\ln\psi) \\ &= 4(\underbrace{\delta^k_l\delta^l_k}_{=3}\bar{D}_i\ln\psi\bar{D}_j\ln\psi + \underbrace{\delta^k_l\delta^l_j}_{=\delta^k_j}\bar{D}_i\ln\psi\bar{D}_k\ln\psi - \underbrace{\delta^k_l\bar{\gamma}_{kj}\bar{\gamma}^{lm}}_{\bar{\gamma}_{lj}\bar{\gamma}^{lm}=\delta^m_j}\bar{D}_i\ln\psi\bar{D}_m\ln\psi + \underbrace{\delta^k_i\delta^l_k}_{=\delta^l_i}\bar{D}_l\ln\psi\bar{D}_j\ln\psi \\ &\quad + \delta^k_i\delta^l_j\bar{D}_l\ln\psi\bar{D}_k\ln\psi - \underbrace{\delta^k_i\bar{\gamma}_{kj}}_{=\bar{\gamma}_{ij}}\bar{\gamma}^{lm}\bar{D}_l\ln\psi\bar{D}_m\ln\psi - \underbrace{\delta^l_k\bar{\gamma}_{ki}\bar{\gamma}^{km}}_{=\bar{\gamma}_{li}\bar{\gamma}^{lm}=\delta^m_i}\bar{D}_m\ln\psi\bar{D}_j\ln\psi \\ &\quad - \underbrace{\delta^l_j\bar{\gamma}_{lj}}_{\bar{\gamma}_{ij}}\bar{\gamma}^{km}\bar{D}_k\ln\psi\bar{D}_m\ln\psi + \underbrace{\bar{\gamma}_{li}\bar{\gamma}^{lm}\bar{\gamma}_{kj}\bar{\gamma}^{km}}_{=\delta^m_i\delta^m_j}\bar{D}_m\ln\psi\bar{D}_m\ln\psi) \\ &= 4(3\bar{D}_i\ln\psi\bar{D}_j\ln\psi + \bar{D}_i\ln\psi\bar{D}_j\ln\psi - \bar{D}_i\ln\psi\bar{D}_j\ln\psi + \bar{D}_i\ln\psi\bar{D}_j\ln\psi \\ &\quad + \bar{D}_j\ln\psi\bar{D}_i\ln\psi - \bar{\gamma}_{ij}\bar{\gamma}^{lm}\bar{D}_l\ln\psi\bar{D}_m\ln\psi - \bar{D}_i\ln\psi\bar{D}_j\ln\psi \\ &\quad - \underbrace{\bar{\gamma}_{ij}\bar{\gamma}^{km}\bar{D}_k\ln\psi\bar{D}_m}_{=\bar{\gamma}_{ij}\bar{\gamma}^{lm}\bar{D}_l\ln\psi\bar{D}_m}\ln\psi + \bar{D}_i\ln\psi\bar{D}_j\ln\psi) \\ &= 20\bar{D}_i\ln\psi\bar{D}_j\ln\psi - 8\bar{\gamma}_{ij}\bar{\gamma}^{lm}\bar{D}_l\ln\psi\bar{D}_m\ln\psi \qquad (145) \end{aligned}$$

Combine these results then finally we get

$$\begin{aligned} R_{ij} &= \bar{R}_{ij} + \bar{D}_k{C^k}_{ij} - \bar{D}_i{C^k}_{kj} + {C^l}_{lk}{C^k}_{ij} - {C^k}_{li}{C^l}_{kj} \\ &= 4\bar{D}_i\bar{D}_j\ln\psi - 2\bar{\gamma}_{ij}\underbrace{\bar{\gamma}^{kl}\bar{D}_k}_{=\bar{\gamma}^{ml}\bar{D}_m}\bar{D}_l\ln\psi - 6\bar{D}_i\bar{D}_j\ln\psi + 24\bar{D}_i\ln\psi\bar{D}_j\ln\psi - 12\bar{\gamma}_{ij}\underbrace{\bar{\gamma}^{kl}\bar{D}_k}_{=\bar{\gamma}^{ml}\bar{D}_m}\ln\psi\bar{D}_l\ln\psi \\ &\quad - (20\bar{D}_i\ln\psi\bar{D}_j\ln\psi - 8\bar{\gamma}_{ij}\bar{\gamma}^{lm}\bar{D}_l\ln\psi\bar{D}_m\ln\psi) \\ &= \bar{R}_{ij} - 2(\bar{D}_i\bar{D}_j\ln\psi + \bar{\gamma}_{ij}\bar{\gamma}^{lm}\bar{D}_l\bar{D}_m\ln\psi) + 4(\bar{D}_i\ln\psi\bar{D}_j\ln\psi - \bar{\gamma}_{ij}\bar{\gamma}^{lm}\bar{D}_l\ln\psi\bar{D}_m\ln\psi) \qquad (146) \end{aligned}$$

For Ricci scalar

$$\begin{aligned} R &= \gamma^{ij}R_{ij} \\ &= \psi^{-4}\bar{\gamma}^{ij}[\bar{R}_{ij} - 2(\bar{D}_i\bar{D}_j\ln\psi + \bar{\gamma}_{ij}\bar{\gamma}^{lm}\bar{D}_l\bar{D}_m\ln\psi) + 4(\bar{D}_i\ln\psi\bar{D}_j\ln\psi - \bar{\gamma}_{ij}\bar{\gamma}^{lm}\bar{D}_l\ln\psi\bar{D}_m\ln\psi)] \\ &= \psi^{-4}[\bar{\gamma}^{ij}\bar{R}_{ij} - 2(\bar{\gamma}^{ij}\bar{D}_i\bar{D}_j\ln\psi + \underbrace{\bar{\gamma}^{ij}\bar{\gamma}_{ij}}_{=3}\bar{\gamma}^{lm}\bar{D}_l\bar{D}_m\ln\psi) + 4(\bar{\gamma}^{ij}\bar{D}_i\ln\psi\bar{D}_j\ln\psi - \underbrace{\bar{\gamma}^{ij}\bar{\gamma}_{ij}}_{=3}\bar{\gamma}^{lm}\bar{D}_l\ln\psi\bar{D}_m\ln\psi)] \qquad (147) \end{aligned}$$

Define conformally related Ricci scalar as $\bar{R} = \bar{\gamma}^{ij}\bar{R}_{ij}$ and define the covariant Laplacian associated $\bar{\gamma}_{ij}$ as $\bar{D}^2 = \bar{D}^i\bar{D}_i = \bar{\gamma}^{ij}\bar{D}_i\bar{D}_j$ then

$$R = \psi^{-4}[\bar{R} - 8(\bar{D}^2\ln\psi - \bar{\gamma}^{ij}\bar{D}_i\ln\psi\bar{D}_j\ln\psi)] \qquad (148)$$

Here again we do some relabeling for dummy indices. We can rewrite this using

$$\begin{aligned}
\bar{D}^2 \ln\psi = \bar{\gamma}^{ij}\bar{D}_i\bar{D}_j \ln\psi &= \bar{\gamma}^{ij}\bar{D}_i\left(\frac{1}{\psi}\bar{D}_j\psi\right) \\
&= \bar{\gamma}^{ij}\left(\frac{1}{\psi}\bar{D}_i\bar{D}_j\psi - \frac{1}{\psi^2}\bar{D}_i\psi\bar{D}_j\psi\right) \\
&= \frac{1}{\psi}\bar{\gamma}^{ij}\bar{D}_i\bar{D}_j\psi - \bar{\gamma}^{ij}\underbrace{\left(\frac{1}{\psi}\bar{D}_i\psi\right)}_{=\bar{D}_i \ln\psi}\underbrace{\left(\frac{1}{\psi}\bar{D}_j\psi\right)}_{\bar{D}_j \ln\psi} \\
&= \frac{1}{\psi}\bar{D}^2\psi - \bar{\gamma}^{ij}\bar{D}_i \ln\psi\bar{D}_j \ln\psi
\end{aligned} \tag{149}$$

Therefore, we can get

$$R = \psi^{-4}\bar{R} - 8\psi^{-5}\bar{D}^2\psi \tag{150}$$

B. Conformal Transformation of the Extrinsic Curvature

It is useful to rescale the extrinsic curvature K_{ij} conformally. Practically, it is convenient to split K_{ij} into it trace part K and a traceless part A_{ij} such that

$$K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij}K \tag{151}$$

It is easy to check above definition makes A_{ij} is traceless

$$\gamma^{ij}A_{ij} = \underbrace{\gamma^{ij}K_{ij}}_{=K} - \frac{1}{3}\underbrace{\gamma^{ij}\gamma_{ij}}_{=3}K = 0 \tag{152}$$

Consider the transformations

$$a^{ij} = \psi^{\alpha}\bar{A}^{ij} \tag{153}$$

$$K = \psi^{\beta}\bar{K} \tag{154}$$

where α and β are arbitrary exponents. The choices of α and β are completely free. The purpose of these transformations is that the transformation should bring the systems of equation into a simple and solvable form. We can choose different exponents subject to which system we want to solve. For example, $\alpha = -10$ is widely used to solve constraint equations. $\alpha = -4$ is chosen to build standard BSSN formalism (It will be discussed later).

III. REVISITING : THE EVOLUTION EQUATIONS

At this point, we derive the 3+1 decomposition of the Einstein's equations (Eqns. 114, 115, 117, and 118 i.e. the ADM equations) that provides initial value problem (or Cauchy problem). Unfortunately, despite all of efforts (including best numerical implementation and methodical preparation, again these will not be discussed in this note), most likely simulations would crash a rather short time.

The ADM equations are not yet in a form that is suitable for stable numerical integration. The failure of these equations can be understood in terms of their mathematical properties, notions of hyperbolicity and well-posedness. Detail discussions of these problem can be found in many literatures. Here, we can summarize that the ADM equations are indeed only weakly hyperbolic. As a consequence, the evolution problem is not well-posed, and we have no guarantees to expect the solutions of numerical implementations to be well-behaved.

In the following, we will discuss a different approach that leads to different reformulations of evolution equations that are both strongly hyperbolic and that have been used successfully in numerical simulations. There are many different approaches about that but we first concentrate on the BSSN (Baumgarte, Shapiro, Shibata, Nakamura) formulation.

A. The BSSN Equations

To derive this formulation, let's choose the conformal factor ψ as $\psi = e^{-\phi}$ so that we have a conformal transformation for the spatial metric.

$$\gamma_{ij} = e^{4\phi}\bar{\gamma}_{ij} \tag{155}$$

Then, we require that the determinant of conformally related metric $\bar{\gamma}_{ij}$ be equal to that of the flat metric η_{ij} in whatever coordinate system we are using such that

$$\phi = \frac{1}{12}\ln\frac{\gamma}{\eta} \tag{156}$$

In the following, we adopt a Cartesian coordinate system so that $\bar{\gamma} = \eta = 1$.

Also, we choose a conformal rescaling for $\tilde{A}_{ij}$

$$A_{ij} = e^{4\phi}\tilde{A}_{ij} \tag{157}$$

For inverse A^{ij}

$$\begin{aligned} A^{ij} &= \gamma^{ik}\gamma^{jl}A_{kl} = e^{-4\phi}\bar{\gamma}^{ik}e^{-4\phi}\bar{\gamma}_{jl}e^{4\phi}\tilde{A}_{kl} \\ &= e^{-4\phi}\tilde{A}^{ij} \end{aligned} \tag{158}$$

Here we use $\tilde{A}$ instead of $\bar{A}$ to avoid confusion between different conformal transformations for evolution scheme and constraint equations (This will be discussed in later section). So, the extrinsic curvature can be expressed

$$K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij}K = e^{4\phi}\left(\tilde{A}_{ij} + \frac{1}{3}\bar{\gamma}_{ij}K\right) \tag{159}$$

Now we need to find evolution equations for new variables $\bar{\gamma}_{ij}$, $\tilde{A}_{ij}$, and ϕ. From evolution equation for γ_{ij} (Eqn. 118)

$$\partial_t\gamma_{ij} = \pounds_\beta\gamma_{ij} - 2\alpha K_{ij} \tag{160}$$

Applying the transformation gives

$$\begin{aligned} \partial_t(e^{4\phi}\bar{\gamma}_{ij}) &= \pounds_\beta(e^{4\phi}\bar{\gamma}_{ij}) - 2\alpha e^{4\phi}\left(\tilde{A}_{ij} + \frac{1}{3}\bar{\gamma}_{ij}K\right) \\ 4e^{4\phi}\bar{\gamma}_{ij}\partial_t\phi + e^{4\phi}\partial_t\bar{\gamma}_{ij} &= 4e^{4\phi}\bar{\gamma}_{ij}\pounds_\beta\phi + e^{4\phi}\pounds_\beta\bar{\gamma}_{ij} - 2\alpha e^{4\phi}\left(\tilde{A}_{ij} + \frac{1}{3}\bar{\gamma}_{ij}K\right) \end{aligned} \tag{161}$$

Multiply both sides with $e^{-4\phi}$ then

$$4\bar{\gamma}_{ij}\partial_t\phi + \partial_t\bar{\gamma}_{ij} = 4\bar{\gamma}_{ij}\pounds_\beta\phi + \pounds_\beta\bar{\gamma}_{ij} - 2\alpha\left(\tilde{A}_{ij} + \frac{1}{3}\bar{\gamma}_{ij}K\right) \tag{162}$$

Now we need to find the evolution equation for ϕ. To do that contract both sides with $\bar{\gamma}^{ij}$

$$4\underbrace{\bar{\gamma}^{ij}\bar{\gamma}_{ij}}_{=3}\partial_t\phi + \bar{\gamma}^{ij}\partial_t\bar{\gamma}_{ij} = 4\underbrace{\bar{\gamma}^{ij}\bar{\gamma}_{ij}}_{=3}\pounds_\beta\phi + \bar{\gamma}^{ij}\pounds_\beta\bar{\gamma}_{ij} - 2\alpha\left(\underbrace{\bar{\gamma}^{ij}\tilde{A}_{ij}}_{=0} + \frac{1}{3}\underbrace{\bar{\gamma}^{ij}\bar{\gamma}_{ij}}_{=3}K\right) \tag{163}$$

Also we use a version of Jacobi's formula

$$\bar{\gamma}^{ij}\partial_t\bar{\gamma}_{ij} = \frac{1}{2\bar{\gamma}}\partial_t\bar{\gamma} = 0 \tag{164}$$

This is because $\bar{\gamma} = 1$. So, we can get

$$\begin{aligned} 12\partial_t\phi &= 12\pounds_\beta\phi + \bar{\gamma}^{ij}(\beta^k \underbrace{\bar{D}_k\bar{\gamma}_{ij}}_{=0} + \bar{\gamma}_{ik}\bar{D}_j\beta^k + \bar{\gamma}_{kj}\bar{D}_i\beta^k) - 2\alpha K \\ &= 12\beta^i\bar{D}_i\phi + \underbrace{\bar{\gamma}^{ij}\bar{\gamma}_{ik}}_{\delta^j_k}\bar{D}_j\beta^k + \underbrace{\bar{\gamma}^{ij}\bar{\gamma}_{kj}}_{\delta^i_k}\bar{D}_i\beta^k - 2\alpha K \end{aligned} \tag{165}$$

Here we require that the determinant of conformally related metric be equal to the flat metric so we can replace conformally related covariant derivative $\bar{D}_i$ to usual partial derivative ∂_i. Rearrange the indices and simplify it

$$\partial_t\phi = -\frac{1}{6}\alpha K + \beta^k\partial_k\phi + \frac{1}{6}\partial_k\beta^k \tag{166}$$

Substitute Eqn. 166 into Eqn. 162

$$\begin{aligned} 4\bar{\gamma}_{ij}\left(-\frac{1}{6}\alpha K + \beta^k\partial_k\phi + \frac{1}{6}\partial_k\beta^k\right) + \partial_t\bar{\gamma}_{ij} &= 4\bar{\gamma}_{ij}\pounds_\beta\phi + \pounds_\beta\bar{\gamma}_{ij} - 2\alpha\left(\tilde{A}_{ij} + \frac{1}{3}\bar{\gamma}_{ij}K\right) \\ \Rightarrow \partial_t\bar{\gamma}_{ij} &= \frac{2}{3}\bar{\gamma}_{ij}\alpha K - 4\gamma_{ij}\beta^k\partial_k\phi - \frac{2}{3}\bar{\gamma}_{ij}\partial_k\beta^k + 4\bar{\gamma}_{ij}\underbrace{\pounds_\beta\phi}_{=\beta^k\partial_k\phi} + \pounds_\beta\bar{\gamma}_{ij} - 2\alpha\tilde{A}_{ij} - \frac{2}{3}\bar{\gamma}_{ij}\alpha K \\ &= -2\alpha\tilde{A}_{ij} + \beta^k\partial_k\bar{\gamma}_{ij} + \bar{\gamma}_{ik}\partial_j\beta^k + \bar{\gamma}_{kj}\partial_i\beta^k - \frac{2}{3}\bar{\gamma}_{ij}\partial_k\beta^k \end{aligned} \tag{167}$$

Again partial derivatives come from the Lie derivative of $\bar{\gamma}_{ij}$. Next consider splitting the evolution equation for the extrinsic curvature. From Eqn. 127

$$\partial_t K = -D^2\alpha + \alpha[K_{ij}K^{ij} + 4\pi(S+\rho)] + \beta^i\partial_i K \tag{168}$$

Note that we change D_iK to ∂_iK because K is scalar. Only term that affects from the transformation is $K^{ij}K_{ij}$ so

$$K^{ij}K_{ij} = \left(e^{4\phi}\tilde{A}_{ij} + \frac{1}{3}e^{4\phi}\bar{\gamma}_{ij}K\right)\left(e^{-4\phi}\tilde{A}^{ij} + \frac{1}{3}e^{-4\phi}\bar{\gamma}^{ij}K\right) = \tilde{A}^{ij}\tilde{A}_{ij} + \frac{1}{3}K^2 \tag{169}$$

Thus

$$\begin{aligned} \partial_t K &= -D^2\alpha + \alpha\left[\tilde{A}^{ij}\tilde{A}_{ij} + \frac{1}{3}K^2 + 4\pi(S+\rho)\right] + \beta^i\partial_i K \\ &= -\gamma^{ij}D_jD_i\alpha + \alpha\left(\tilde{A}^{ij}\tilde{A}_{ij} + \frac{1}{3}K^2\right) + 4\pi\alpha(\rho+S) + \beta^i\partial_i K \end{aligned} \tag{170}$$

From Eqn. 117

$$\begin{aligned} \partial_t K_{ij} &= \beta^k D_k K_{ij} + K_{ik}D_j\beta^k + K_{kj}D_i\beta^k - D_iD_j\alpha + \alpha(R_{ij} + KK_{ij} - 2K^k_jK_{ik}) - 8\pi\alpha\left(S_{ij} - \frac{1}{2}\gamma_{ij}(S-\rho)\right) \\ &= \pounds_\beta K_{ij} - D_iD_j\alpha + \alpha(R_{ij} + KK_{ij} - 2K^k_jK_{ik}) - 8\pi\alpha\left(S_{ij} - \frac{1}{2}\gamma_{ij}(S-\rho)\right) \end{aligned}$$

We define an operator for convenience $d_t \equiv \partial_t - \pounds_\beta$ then we can rewrite

$$d_tK_{ij} = -D_iD_j\alpha + \alpha(R_{ij} + KK_{ij} - 2K^k_jK_{ik}) - 8\pi\alpha\left(S_{ij} - \frac{1}{2}\gamma_{ij}(S-\rho)\right) \tag{171}$$

Using d_t we can also rewrite

$$\begin{aligned} &\partial_t\gamma_{ij} = \pounds_\beta\gamma_{ij} - 2\alpha K_{ij} \\ &\Rightarrow d_t\gamma_{ij} = -2\alpha K \end{aligned} \tag{172}$$

$$\begin{aligned} &\partial_t K = -D^2\alpha + \alpha[K_{ij}K^{ij} + 4\pi(S+\rho)] + \underbrace{\beta^i\partial_i K}_{=\pounds_\beta K} \\ &\Rightarrow d_t K = -D^2\alpha + \alpha[K_{ij}K^{ij} + 4\pi(S+\rho)] \end{aligned} \tag{173}$$

LHS of Eqn. 171 can be splited

$$\begin{aligned}
d_t K_{ij} &= d_t\left(A_{ij} + \frac{1}{3}\gamma_{ij}K\right) = d_t A_{ij} + \frac{1}{3}\gamma_{ij} d_t K + \frac{1}{3}K d_t \gamma_{ij} \\
&= d_t A_{ij} + \frac{1}{3}\gamma_{ij}(-D^2\alpha + \alpha[\underbrace{K_{ij}K^{ij}}_{R+K^2-16\pi\rho} + 4\pi(S+\rho)]) + \frac{1}{3}K(-2\alpha K_{ij}) \\
&= d_t A_{ij} - \frac{1}{3}\gamma_{ij}(D^2\alpha - \alpha[R + K^2 + 4\pi(S - 3\rho)]) - \frac{2}{3}\alpha K K_{ij}
\end{aligned} \tag{174}$$

So Eqn. 171 can be rewritten as

$$\begin{aligned}
d_t A_{ij} &= \frac{1}{3}\gamma_{ij}D^2\alpha - \frac{1}{3}\gamma_{ij}\alpha R - \frac{1}{3}\gamma_{ij}K^2 + \frac{4}{3}\pi\gamma_{ij}\alpha(S - 3\rho) - \frac{2}{3}\alpha K K_{ij} - D_i D_j \alpha + \alpha(R_{ij} + K K_{ij} - 2K_j^k K_{ik}) \\
&\quad - 8\pi\alpha S_{ij} - 4\pi\alpha\gamma_{ij}(S - \rho) \\
&= -\left(D_i D_j \alpha - \frac{1}{3}\gamma_{ij}D^2\alpha\right) + \alpha\left(R_{ij} - \frac{1}{3}\gamma_{ij}R\right) - 8\pi\alpha\left(S_{ij} - \frac{1}{3}\gamma_{ij}S\right) + \alpha\left(\frac{5}{3}K K_{ij} - 2K_j^k K_{ik} - \frac{1}{3}\gamma_{ij}K^2\right)
\end{aligned} \tag{175}$$

where we can further splitting

$$\begin{aligned}
\frac{5}{3}K K_{ij} - 2K_j^k K_{ik} - \frac{1}{3}\gamma_{ij}K^2 &= \frac{5}{3}K\left(A_{ij} + \frac{1}{3}K\gamma_{ij}\right) - 2\left(A_{ik} + \frac{1}{3}\gamma_{ik}K\right)\left(A_j^k + \frac{1}{3}K\delta_j^k\right) - \frac{1}{3}\gamma_{ij}K^2 \\
&= \frac{5}{3}K A_{ij} + \frac{5}{9}K^2\gamma_{ij} - 2\left(A_{ik}A_j^k + \frac{1}{3}K\underbrace{A_{ik}\delta_j^k}_{A_{ij}} + \frac{1}{3}K\underbrace{\gamma_{ik}A_j^k}_{A_{ij}} + \frac{1}{9}\underbrace{\gamma_{ik}\delta_j^k}_{\gamma_{ij}}K^2\right) - \frac{1}{3}\gamma_{ij}K^2 \\
&= \frac{5}{3}K A_{ij} + \frac{5}{9}\gamma_{ij}K^2 - 2A_{ik}A_j^k - \frac{4}{3}K A_{ij} - \frac{2}{9}\gamma_{ij}K^2 - \frac{1}{3}\gamma_{ij}K^2 \\
&= \frac{1}{3}K A_{ij} - 2A_{ik}A_j^k
\end{aligned} \tag{176}$$

And define trace-free part of a tensor such that

$$D_i D_j \alpha - \frac{1}{3}\gamma_{ij}D^2\alpha = (D_i D_j \alpha)^{TF} \tag{177}$$

$$R_{ij} - \frac{1}{3}\gamma_{ij}R = R_{ij}^{TF} \tag{178}$$

$$S_{ij} - \frac{1}{3}\gamma_{ij}S = S_{ij}^{TF} \tag{179}$$

Therefore

$$d_t A_{ij} = -(D_i D_j \alpha)^{TF} + \alpha(R_{ij}^{TF} - 8\pi S_{ij}^{TF}) + \alpha\left(\frac{1}{3}K A_{ij} - 2A_{ik}A_j^k\right) \tag{180}$$

Now perform conformal transformation for A_{ij}. LHS of above equation gives

$$d_t(e^{4\phi}\tilde{A}_{ij}) = 4e^{4\phi}\tilde{A}_{ij}d_t\phi + e^{4\phi}d_t\tilde{A}_{ij} \tag{181}$$

From Eqn. 166, we can get $d_t\phi$

$$\begin{aligned}
d_t(e^{4\phi}\tilde{A}_{ij}) &= 4e^{4\phi}\tilde{A}_{ij}\left(\frac{1}{6}\partial_k\beta^k - \frac{1}{6}\alpha K\right) + e^{4\phi}d_t\tilde{A}_{ij} \\
&= e^{4\phi}d_t\tilde{A}_{ij} + \frac{2}{3}e^{4\phi}\tilde{A}_{ij}\partial_k\beta^k - \frac{2}{3}e^{4\phi}\alpha K\tilde{A}_{ij}
\end{aligned} \tag{182}$$

For $A_{ik}A^k_j$

$$
\begin{aligned}
A_{ik}A^k_j &= \gamma^{kl}A_{ik}A_{jl} = e^{-4\phi}\bar{\gamma}^{kl}e^{4\phi}\tilde{A}_{ik}e^{4\phi}\tilde{A}_{jl} \\
&= e^{4\phi}\bar{\gamma}^{kl}\tilde{A}_{ik}\tilde{A}_{jl} = e^{4\phi}\tilde{A}_{ik}\tilde{A}^k_j
\end{aligned} \tag{183}
$$

Combine these results then we get

$$
\begin{aligned}
e^{4\phi}d_t\tilde{A}_{ij} + \frac{2}{3}e^{4\phi}\tilde{A}_{ij}\partial_k\beta^k - \frac{2}{3}e^{4\phi}\alpha K\tilde{A}_{ij} &= -(D_iD_j\alpha)^{TF} + \alpha(R^{TF}_{ij} - 8\pi S^{TF}_{ij}) + \alpha\left(\frac{1}{3}Ke^{4\phi}\tilde{A}_{ij} - 2e^{4\phi}\tilde{A}_{ik}\tilde{A}^k_j\right) \\
\Rightarrow e^{4\phi}d_t\tilde{A}_{ij} &= -\frac{2}{3}e^{4\phi}\tilde{A}_{ij}\partial_k\beta^k - (D_iD_j\alpha)^{TF} + \alpha(R^{TF}_{ij} - 8\pi S^{TF}_{ij}) + \alpha\left(Ke^{4\phi}\tilde{A}_{ij} - 2e^{4\phi}\tilde{A}_{ik}\tilde{A}^k_j\right)
\end{aligned} \tag{184}
$$

Multiply both sides by $e^{-4\phi}$ and return $d_t = \partial_t - \pounds_\beta$

$$
\begin{aligned}
\partial_t\tilde{A}_{ij} &= e^{-4\phi}\left[-(D_iD_j\alpha)^{TF} + \alpha(R^{TF}_{ij} - 8\pi S^{TF}_{ij})\right] + \alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j\right) + \pounds_\beta\tilde{A}_{ij} - \frac{2}{3}\tilde{A}_{ij}\partial_k\beta^k \\
&= e^{-4\phi}\left[-(D_iD_j\alpha)^{TF} + \alpha(R^{TF}_{ij} - 8\pi S^{TF}_{ij})\right] + \alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j\right) \\
&\quad + \beta^k\partial_k\tilde{A}_{ij} + \tilde{A}_{ik}\partial_j\beta^k + \tilde{A}_{kj}\partial_i\beta^k - \frac{2}{3}\tilde{A}_{ij}\partial_k\beta^k
\end{aligned} \tag{185}
$$

Note that the divergence of shift $\partial_i\beta^i$ appears because the choice $\bar{\gamma} = 1$ makes ϕ as a tensor density of weight 1/6, and $\bar{\gamma}_{ij}$ and $\tilde{A}_{ij}$ as tensor densities of weight $-2/3$.

From Eqn. 146, we can split the Ricci tensor into two terms

$$
R_{ij} = \bar{R}_{ij} + R^\phi_{ij} \tag{186}
$$

where R^ϕ_{ij} only depends on the conformal function such that

$$
R^\phi_{ij} = -2(\bar{D}_i\bar{D}_j\phi + \bar{\gamma}_{ij}\bar{\gamma}^{lm}\bar{D}_l\bar{D}_m\phi) + 4(\bar{D}_i\phi\bar{D}_j\phi - \bar{\gamma}_{ij}\bar{\gamma}^{lm}\bar{D}_l\phi\bar{D}_m\phi) \tag{187}
$$

Define conformal connection function

$$
\begin{aligned}
\bar{\Gamma}^i \equiv \bar{\gamma}^{jk}\bar{\Gamma}^i{}_{jk} &= \bar{\gamma}^{jk}\frac{1}{2}\bar{\gamma}^{il}(\partial_j\bar{\gamma}_{lk} + \partial_k\bar{\gamma}_{jl} - \partial_l\bar{\gamma}_{jk}) \\
&= \frac{1}{2}(\bar{\gamma}^{il}\bar{\gamma}^{jk}\partial_j\bar{\gamma}_{jk} + \bar{\gamma}^{il}\bar{\gamma}^{jk}\partial_k\bar{\gamma}_{jl} - \bar{\gamma}^{jl}\underbrace{\bar{\gamma}^{jk}\partial_l\bar{\gamma}_{jk}}_{=0}) \\
&= \frac{1}{2}(\bar{\gamma}^{il}\partial^k\bar{\gamma}_{lk} + \bar{\gamma}^{il}\underbrace{\partial^j\bar{\gamma}_{jl}}_{\partial^k\bar{\gamma}_{lk}}) = \bar{\gamma}^{il}\partial^k\bar{\gamma}_{lk} \\
&= \underbrace{\partial^k(\bar{\gamma}^{il}\bar{\gamma}_{lk})}_{=\partial^k\delta^i_k=0} - \bar{\gamma}_{lk}\partial^k\bar{\gamma}^{il} = -\partial_l\bar{\gamma}^{il}
\end{aligned} \tag{188}
$$

Using this, we can express Ricci tensor in terms of conformal connection function (**Need to figure out**)

$$
\bar{R}_{ij} = -\frac{1}{2}\bar{\gamma}^{lm}\partial_m\partial_l\bar{\gamma}_{ij} + \bar{\gamma}_{k(i}\partial_{j)}\bar{\Gamma}^k + \bar{\Gamma}^k\bar{\Gamma}_{(ij)k} + \bar{\gamma}^{lm}(2\bar{\Gamma}^k{}_{l(i}\bar{\Gamma}_{j)km} + \bar{\Gamma}^k{}_{im}\bar{\Gamma}_{klj}) \tag{189}
$$

We treat $\bar{\Gamma}^i$ as a new independent function. To obtain the dynamical information of $\bar{\Gamma}^i$, we need a time evolution scheme of it. Take the time derivative gives

$$
\partial\bar{\Gamma}^i = \partial_t(-\partial_j\bar{\gamma}^{ij}) = -\partial_j(\partial_t\bar{\gamma}^{ij}) \tag{190}
$$

Here we interchange a partial time and spatial derivatives because they commute. Now we need to evaluate $\partial_t\bar{\gamma}^{ij}$. From Eqn. 167 and using the similar ways in previous calculation

$$
\begin{aligned}
\partial_t\bar{\gamma}^{ij} &= -\bar{\gamma}^{ik}\bar{\gamma}^{jl}\partial_t\bar{\gamma}_{kl} \\
&= -\bar{\gamma}^{ik}\bar{\gamma}^{jl}\left(-2\alpha\tilde{A}_{kl} + \beta^m\partial_m\bar{\gamma}_{kl} + \bar{\gamma}_{ml}\partial_k\beta^m + \bar{\gamma}_{mk}\partial_l\beta^m - \frac{2}{3}\bar{\gamma}_{kl}\partial_m\beta^m\right) \\
&= 2\alpha\underbrace{\bar{\gamma}^{ik}\bar{\gamma}^{jl}\tilde{A}_{kl}}_{=\tilde{A}^{ij}} - \beta^m\underbrace{\bar{\gamma}^{ik}\bar{\gamma}^{jl}\partial_m\bar{\gamma}_{kl}}_{=-\partial_m\bar{\gamma}^{ij}} - \bar{\gamma}^{ik}\bar{\gamma}^{jl}\bar{\gamma}_{ml}\partial_k\beta^m - \bar{\gamma}^{ik}\bar{\gamma}^{jl}\bar{\gamma}_{mk}\partial_l\beta^m + \frac{2}{3}\underbrace{\bar{\gamma}^{ik}\bar{\gamma}^{jl}\bar{\gamma}_{kl}}_{=\bar{\gamma}^{ij}}\partial_m\beta^m
\end{aligned} \tag{191}
$$

We perform some re-arrangement of metrics and indices to simplify further

$$\begin{aligned}\bar{\gamma}^{ik}\bar{\gamma}^{jl}\bar{\gamma}_{ml}\partial_k\beta^m &= \delta^j_m\partial^i\beta^m = \delta^j_m\partial^i(\delta^m_i\beta^i) = \delta^j_m\delta^m_i\partial^i\beta^i = \delta^j_m\delta^m_i\bar{\gamma}^{im}\partial_m\beta^i \\ &= \bar{\gamma}^{jm}\partial_m\beta^i\end{aligned} \tag{192}$$

Similarly

$$\begin{aligned}\bar{\gamma}^{ik}\bar{\gamma}^{jl}\bar{\gamma}_{mk}\partial_l\beta^m &= \delta^i_m\partial^j\beta^m = \delta^i_m\partial^j(\delta^m_j\beta^j) = \delta^i_m\delta^m_j\partial^j\beta^j = \delta^i_m\delta^m_j\bar{\gamma}^{jm}\partial_m\beta^j \\ &= \bar{\gamma}^{im}\partial_m\beta^j\end{aligned} \tag{193}$$

Thus $\partial_t\bar{\gamma}^{ij}$ is

$$\partial_t\bar{\gamma}^{ij} = 2\alpha\tilde{A}^{ij} - \bar{\gamma}^{mj}\partial_m\beta^i - \bar{\gamma}^{mi}\partial_m\beta^j + \frac{2}{3}\bar{\gamma}^{ij}\partial_l\beta^l + \beta^l\partial_l\bar{\gamma}^{ij} \tag{194}$$

where we relabel some dummy indices. So $\partial_t\bar{\Gamma}^i$ is

$$\begin{aligned}\partial_t\bar{\Gamma}^i &= -\partial_j\left(2\alpha\tilde{A}^{ij} - \bar{\gamma}^{mj}\partial_m\beta^i - \bar{\gamma}^{mi}\partial_m\beta^j + \frac{2}{3}\bar{\gamma}^{ij}\partial_l\beta^l + \beta^l\partial_l\bar{\gamma}^{ij}\right) \\ &= -2\tilde{A}^{ij}\partial_j\alpha - 2\alpha\partial_j\tilde{A}^{ij} + \underbrace{\partial_j\bar{\gamma}^{mj}}_{=-\bar{\Gamma}^m}\partial_m\beta^i + \bar{\gamma}^{mj}\partial_j\partial_m\beta^i + \partial_j\bar{\gamma}^{mi}\partial_m\beta^j + \underbrace{\bar{\gamma}^{mi}\partial_j\partial_m\beta^j}_{=\bar{\gamma}^{ji}\partial_j\partial_l\beta^l} \\ &\quad - \frac{2}{3}\underbrace{\partial_j\bar{\gamma}^{ij}}_{=-\bar{\Gamma}^i}\partial_l\beta^l - \frac{2}{3}\bar{\gamma}^{ij}\partial_j\partial_l\beta^l - \partial_j\beta^l\partial_l\bar{\gamma}^{ij} - \underbrace{\beta^l\partial_j\partial_l\bar{\gamma}^{ij}}_{=\partial_m\beta^j\partial_j\bar{\gamma}^{im}}\end{aligned} \tag{195}$$

where we apply the definition of $\bar{\Gamma}^i$. Relabel indices and simply it

$$\partial_t\bar{\Gamma}^i = -2(\alpha\partial_j\tilde{A}^{ij} + \tilde{A}^{ij}\partial_j\alpha) - \bar{\Gamma}^j\partial_j\beta^i + \bar{\gamma}^{lj}\partial_l\partial_j\beta^i + \frac{1}{3}\bar{\gamma}^{li}\partial_l\partial_j\beta^j + \frac{2}{3}\bar{\Gamma}^i\partial_j\beta^j + \beta^j\partial_j\bar{\Gamma}^i \tag{196}$$

The divergence of $\tilde{A}^{ij}$ can be simplified by using the momentum constraint. From Eqn. 115, we have

$$D_j(K^{ij} - \gamma^{ij}K) = 8\pi S^i \tag{197}$$

Split the extrinsic curvature via $K^{ij} + \gamma^{ij}K/3$ then

$$D_j\left(A^{ij} - \frac{2}{3}\gamma^{ij}K\right) = D_jA^{ij} - \frac{2}{3}\gamma^{ij}D_jK = 8\pi S^i \tag{198}$$

We need to find a conformal relation between D_jA^{ij} and $\bar{D}_j\tilde{A}^{ij}$. From the conformal transformation section, we know the relation between D_jA^{ij} and $\bar{D}_jA^{ij}$

$$D_jA^{ij} = \bar{D}_jA^{ij} + C^i{}_{jk}A^{kj} + C^j{}_{jk}A^{ik} \tag{199}$$

where again $C^i{}_{jk} \equiv \Gamma^i{}_{jk} - \bar{\Gamma}^i{}_{jk}$ and with our choice$\psi = e^\phi$

$$C^i{}_{jk} = 2(\delta^i_j\bar{D}_k\phi + \delta^i_k\bar{D}_j\phi - \bar{\gamma}_{jk}\bar{\gamma}^{il}\bar{D}_l\phi) \tag{200}$$

so $C^j{}_{jk} = 6\bar{D}_k\phi$. Substitute these into Eqn. 199

$$\begin{aligned}D_jA^{ij} &= \bar{D}_jA^{ij} + 2(\delta^i_j\bar{D}_k\phi + \delta^i_k\bar{D}_j\phi - \bar{\gamma}_{jk}\bar{\gamma}^{il}\bar{D}_l\phi)A^{kj} + 6A^{ik}\bar{D}_k\phi \\ &= \bar{D}_jA^{ij} + 2(\delta^i_jA^{kj}\bar{D}_k\phi + \delta^i_kA^{kj}\bar{D}_j\phi - \underbrace{A^{kj}\bar{\gamma}_{jk}}_{=0}\bar{\gamma}^{il}\bar{D}_l\phi) + 6A^{ik}\bar{D}_k\phi \\ &= \bar{D}_jA^{ij} + 2(A^{ik}\bar{D}_k\phi + A^{ij}\bar{D}_j\phi) + 6A^{ik}D_k\phi = \bar{D}_jA^{ij} + 10A^{ij}\bar{D}_j\phi\end{aligned} \tag{201}$$

Note that we can further simplify $\bar{D}_j A^{ij} + 10A^{ij}\bar{D}_j\phi = e^{-10\phi}\bar{D}_j(e^{10\phi}A^{ij})$. From this relation, we can rescale A^{ij} as $A^{ij} = e^{-10\phi}\bar{A}^{ij}$ and this is widely used in solving momentum constraints. This will be discussed later for initial data. Here we keep our choice $A^{ij} = e^{-4\phi}\tilde{A}^{ij}$

$$
\begin{aligned}
D_j A^{ij} &= \bar{D}_j(e^{-4\phi}\tilde{A}^{ij}) + 10e^{-4\phi}\tilde{A}^{ij}\bar{D}_j\phi \\
&= e^{-4\phi}\bar{D}_j\tilde{A}^{ij} - 4e^{-4\phi}\tilde{A}^{ij}\bar{D}_j\phi + 10e^{-4\phi}\tilde{A}^{ij}\bar{D}_j\phi \\
&= e^{-4\phi}(\partial_j\tilde{A}^{ij} + \bar{\Gamma}^i{}_{jk}\tilde{A}^{kj} + \bar{\Gamma}^j{}_{jk}\tilde{A}^{ij} + 6e^{-4\phi}\tilde{A}^{ij}\bar{D}_j\phi)
\end{aligned} \tag{202}
$$

Using a formula

$$
\bar{\Gamma}^j{}_{jk} = \frac{1}{2}\bar{\gamma}^{lm}\partial_k\bar{\gamma}_{lm} = \frac{1}{2\bar{\gamma}}\partial_k\bar{\gamma} = 0 \tag{203}
$$

Therefore

$$
D_j A^{ij} = e^{-4\phi}(\partial_j\tilde{A}^{ij} + \bar{\Gamma}^i{}_{jk}\tilde{A}^{kj} + 6\tilde{A}^{ij}\partial_j\phi) \tag{204}
$$

Substitute this into the momentum constraint then

$$
\begin{aligned}
&D_j A^{ij} - \frac{2}{3}\gamma^{ij}D_j K = e^{-4\phi}(\partial_j\tilde{A}^{ij} + \bar{\Gamma}^i{}_{jk}\tilde{A}^{kj} + 6\tilde{A}^{ij}\partial_j\phi) - \frac{2}{3}e^{-4\phi}\bar{\gamma}^{ij}D_j K = 8\pi S^i \\
&\Rightarrow e^{-4\phi}(\partial_j\tilde{A}^{ij} + \bar{\Gamma}^i{}_{jk}\tilde{A}^{kj} + 6\tilde{A}^{ij}\bar{D}_j\phi) - \frac{2}{3}e^{-4\phi}\bar{\gamma}^{ij}D_j K = 8\pi S^i = 8\pi\gamma^{ij}S_j = 8\pi e^{-4\phi}\bar{\gamma}^{ij}S_j
\end{aligned} \tag{205}
$$

Multiply both sides by $e^{4\phi}$ and replace covariant derivatives for ϕ and K to partial derivatives (because they are scalars) then $\partial_j\tilde{A}^{ij}$ can be expressed

$$
\partial_j\tilde{A}^{ij} = -\bar{\Gamma}^i{}_{jk}\tilde{A}^{kj} - 6\tilde{A}^{ij}\partial_j\phi + \frac{2}{3}\bar{\gamma}^{ij}\partial_j K + 8\pi\bar{\gamma}^{ij}S_j \tag{206}
$$

Substitute this into Eqn. 196

$$
\begin{aligned}
\partial_t\bar{\Gamma}^i = &-2\tilde{A}^{ij}\partial_j\alpha + 2\alpha\left(\bar{\Gamma}^i{}_{jk}\tilde{A}^{kj} + 6\tilde{A}^{ij}\partial_j\phi - \frac{2}{3}\bar{\gamma}^{ij}\partial_j K - 8\pi\bar{\gamma}^{ij}S_j\right) \\
&+ \beta^j\partial_j\bar{\Gamma}^i - \bar{\Gamma}^j\partial_j\beta^i + \frac{2}{3}\bar{\Gamma}^i\partial_j\beta^j + \bar{\gamma}^{lj}\partial_l\partial_j\beta^i + \frac{1}{3}\bar{\gamma}^{li}\partial_l\partial_j\beta^j
\end{aligned} \tag{207}
$$

Eqns. 166, 167, 170, 185, and 207 are called the *BSSN equations*. Below is summary of that.

$$
\partial_t\phi = -\frac{1}{6}\alpha K + \beta^k\partial_k\phi + \frac{1}{6}\partial_k\beta^k \tag{208}
$$

$$
\partial_t\bar{\gamma}_{ij} = -2\alpha\tilde{A}_{ij} + \beta^k\partial_k\bar{\gamma}_{ij} + \bar{\gamma}_{ik}\partial_j\beta^k + \bar{\gamma}_{kj}\partial_i\beta^k - \frac{2}{3}\bar{\gamma}_{ij}\partial_k\beta^k \tag{209}
$$

$$
\partial_t K = -\gamma^{ij}D_j D_i\alpha + \alpha\left(\tilde{A}^{ij}\tilde{A}_{ij} + \frac{1}{3}K^2\right) + 4\pi\alpha(\rho + S) + \beta^i\partial_i K \tag{210}
$$

$$
\begin{aligned}
\partial_t\tilde{A}_{ij} = &\, e^{-4\phi}\left[-(D_i D_j\alpha)^{TF} + \alpha(R_{ij}^{TF} - 8\pi S_{ij}^{TF})\right] + \alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j\right) \\
&+ \beta^k\partial_k\tilde{A}_{ij} + \tilde{A}_{ik}\partial_j\beta^k + \tilde{A}_{kj}\partial_i\beta^k - \frac{2}{3}\tilde{A}_{ij}\partial_k\beta^k
\end{aligned} \tag{211}
$$

$$
\begin{aligned}
\partial_t\bar{\Gamma}^i = &-2\tilde{A}^{ij}\partial_j\alpha + 2\alpha\left(\bar{\Gamma}^i{}_{jk}\tilde{A}^{kj} + 6\tilde{A}^{ij}\partial_j\phi - \frac{2}{3}\bar{\gamma}^{ij}\partial_j K - 8\pi\bar{\gamma}^{ij}S_j\right) \\
&+ \beta^j\partial_j\bar{\Gamma}^i - \bar{\Gamma}^j\partial_j\beta^i + \frac{2}{3}\bar{\Gamma}^i\partial_j\beta^j + \bar{\gamma}^{lj}\partial_l\partial_j\beta^i + \frac{1}{3}\bar{\gamma}^{li}\partial_l\partial_j\beta^j
\end{aligned} \tag{212}
$$

The ADM equations have (γ_{ij}, K_{ij}) as evolved variables and the BSSN equations split these variables into $(\phi, \tilde{\gamma}_{ij}, K, \tilde{A}_{ij}, \bar{\Gamma}^i)$. Note that there are variants of the BSSN formulation. For example, people let $\chi = e^{-4\phi}$ then (this is used in order to regularize the system near puncture)

$$
\partial_a\chi = -4e^{-4\phi}\partial_a\phi = -4\chi\partial_a\phi \Rightarrow \partial_a\phi = -\frac{1}{4\chi}\partial_a\chi \tag{213}
$$

So, Eqn. 166 is changed

$$-\frac{1}{4\chi}\partial_t\chi = -\frac{1}{6}\alpha K + \beta^k\left(-\frac{1}{4\chi}\partial_k\chi\right) + \frac{1}{6}\partial_k\beta^k \tag{214}$$

$$\Rightarrow \partial_t\chi = \frac{2}{3}\chi(\alpha K - \partial_k\beta^k) + \beta^k\partial_k\chi \tag{215}$$

And there are minor changes for equations of $\tilde{A}_{ij}$ and $\bar{\Gamma}^i$. The χ method is used for binary black hole simulation a lot (Our code is also based on this). The BSSN equations for black hole case, i.e.vacuum Einstein's equations, with this method are

$$\partial_t\chi = \frac{2}{3}\chi(\alpha K - \partial_k\beta^k) + \beta^k\partial_k\chi \tag{216}$$

$$\partial_t\bar{\gamma}_{ij} = -2\alpha\tilde{A}_{ij} + \beta^k\partial_k\bar{\gamma}_{ij} + \bar{\gamma}_{ik}\partial_j\beta^k + \bar{\gamma}_{kj}\partial_i\beta^k - \frac{2}{3}\bar{\gamma}_{ij}\partial_k\beta^k \tag{217}$$

$$\partial_t K = -\gamma^{ij}D_jD_i\alpha + \alpha\left(\tilde{A}^{ij}\tilde{A}_{ij} + \frac{1}{3}K^2\right) + \beta^i\partial_i K \tag{218}$$

$$\begin{aligned}\partial_t\tilde{A}_{ij} = {} & \chi\left[-(D_iD_j\alpha)^{TF} + \alpha R_{ij}^{TF}\right] + \alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}_j^k\right) \\ & + \beta^k\partial_k\tilde{A}_{ij} + \tilde{A}_{ik}\partial_j\beta^k + \tilde{A}_{kj}\partial_i\beta^k - \frac{2}{3}\tilde{A}_{ij}\partial_k\beta^k\end{aligned} \tag{219}$$

$$\begin{aligned}\partial_t\bar{\Gamma}^i = {} & -2\tilde{A}^{ij}\partial_j\alpha + 2\alpha\left(\bar{\Gamma}^i{}_{jk}\tilde{A}^{kj} - \frac{3}{2\chi}\tilde{A}^{ij}\partial_j\chi - \frac{2}{3}\bar{\gamma}^{ij}\partial_j K\right) \\ & + \beta^j\partial_j\bar{\Gamma}^i - \bar{\Gamma}^j\partial_j\beta^i + \frac{2}{3}\bar{\Gamma}^i\partial_j\beta^j + \bar{\gamma}^{lj}\partial_l\partial_j\beta^j + \frac{1}{3}\bar{\gamma}^{li}\partial_l\partial_j\beta^j\end{aligned} \tag{220}$$

These are standard form of the BSSN equations for the binary black hole problems.

IV. THE GENERALIZED HARMONICS FORMULATION

There is an alternative hyperbolic formalism for Einstein's equation called the generalized harmonic formalism. The idea is adapting generalization of the harmonic coordinate condition of a form

$$\Box x^a = H^a \tag{221}$$

where $\Box$ is usual wave operator, x^a is coordinate, and H^a is arbitrary source functions. Starting from Eqn. 221, we can have

$$H^a = \Box x^a = \frac{1}{\sqrt{-g}}\partial_c(\sqrt{-g}\; g^{cb}\underbrace{\partial_b x^a}_{=\delta^a_b}) = \frac{1}{\sqrt{-g}}\partial_c(\sqrt{-g}\; g^{ca}) \tag{222}$$

Therefore

$$\begin{aligned}H_a &= g_{ab}H^b = g_{ab}\frac{1}{\sqrt{-g}}\partial_c(\sqrt{-g}\; g^{cb}) \\ &= g_{ab}g^{cb}\underbrace{\frac{1}{\sqrt{-g}}\partial_c\sqrt{-g}}_{=\partial_c \ln\sqrt{-g}} + g_{ab}\frac{1}{\sqrt{-g}}\sqrt{-g}\partial_c g^{cb} \\ &= \partial_a \ln\sqrt{-g} + g_{ab}\partial_c g^{cb}\end{aligned} \tag{223}$$

Use the fact that $g_{ab}\partial_c g^{cb} = -g^{cb}\partial_c g_{ab}$ we can also write

$$H_a = \partial_a \ln\sqrt{-g} - g^{cb}\partial_c g_{ab} \tag{224}$$

Using this, take the gradient

$$\partial_b H_a = \partial_b \partial_a \ln\sqrt{-g} - \partial_b g^{cd}\ \partial_c g_{ad} - g^{cd}\partial_b\partial_c g_{ad} \tag{225}$$

So we can easily symmetrize it

$$H_{(a,b)} = (\ln\sqrt{-g})_{,ab} - g^{cd} g_{c(a,b)d} - g^{cd}{}_{(,a} g_{b)c,d} \tag{226}$$

where $f_{,a} = \partial_a f$ which is usual partial derivative

Now consider the Einstein's equation

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi T_{ab} \tag{227}$$

Rewrite this into trace reversed form (I will not derive this here because this is obvious GR exercise...)

$$R_{ab} = 4\pi(2T_{ab} - g_{ab}T) \tag{228}$$

So, we only need to consider Ricci tensor decomposition because RHS is just showing matter terms. Ricci tensor is expressed

$$R_{ab} = \Gamma^c{}_{ab,c} - \Gamma^c{}_{cb,a} + \Gamma^d{}_{ab}\Gamma^c{}_{dc} - \Gamma^d{}_{cb}\Gamma^c{}_{da} \tag{229}$$

Using definition of the connection coefficient, we can express Ricci tensor in terms of metric. Let's evaluate term by term

$$\begin{aligned} \Gamma^c{}_{ab,c} &= \partial_c \left[\frac{1}{2} g^{cd}(\partial_a g_{db} + \partial_b g_{da} - \partial_d g_{ab})\right] \\ &= \frac{1}{2}\partial_c g^{cd}(\partial_a g_{db} + \partial_b g_{da} - \partial_d g_{ab}) + \frac{1}{2} g^{cd}(\partial_c\partial_a g_{db} + \partial_c\partial_b g_{da} - \partial_c\partial_d g_{ab}) \end{aligned} \tag{230}$$

Using the identity $\Gamma^a{}_{ab} = \partial_b \ln\sqrt{-g}$, the second and third terms of Eqn. 229 can be expressed

$$\Gamma^c{}_{cb,a} = \partial_a\partial_b \ln\sqrt{-g} \tag{231}$$

$$\Gamma^d{}_{ab}\Gamma^c{}_{dc} = \partial_d \ln\sqrt{-g}\ \Gamma^d{}_{ab} \tag{232}$$

Substitute all these into Eqn. 229 gives

$$\begin{aligned} R_{ab} &= \frac{1}{2}\partial_c g^{cd}(\partial_a g_{db} + \partial_b g_{da} - \partial_d g_{ab}) + \frac{1}{2} g^{cd}(\partial_c\partial_a g_{db} + \partial_c\partial_b g_{da} - \partial_c\partial_d g_{ab}) \\ &- \partial_a\partial_b \ln\sqrt{-g} + \partial_d \ln\sqrt{-g}\ \Gamma^d{}_{ab} + \Gamma^d{}_{cb}\Gamma^c{}_{da} \\ &= -\frac{1}{2} g^{cd}\partial_c\partial_d g_{ab} + \underbrace{\frac{1}{2} g^{cd}(\partial_c\partial_a g_{db} + \partial_c\partial_b g_{da})}_{=g^{cd}\partial_c\partial_{(a}g_{b)d}=g^{cd}g_{c(a,b)d}} + \partial_c g^{cd}\underbrace{\frac{1}{2}(\partial_a g_{db} + \partial_b g_{da} - \partial_d g_{ab})}_{=\Gamma_{dab}} \\ &- \partial_a\partial_b \ln\sqrt{-g} + \partial_d \ln\sqrt{-g}\ \Gamma^d{}_{ab} + \Gamma^d{}_{cb}\Gamma^c{}_{da} \\ &= -\frac{1}{2} g^{cd} g_{ab,cd} + g^{cd}{}_{,c}\Gamma_{dab} - (\ln\sqrt{-g})_{,ab} + g^{cd} g_{c(a,b)d} + (\ln\sqrt{-g})_{,d}\Gamma^d{}_{ab} + \Gamma^d{}_{cb}\Gamma^c{}_{da} \end{aligned} \tag{233}$$

Using Eqn. 226, we can have

$$-(\ln\sqrt{-g})_{,ab} + g^{cd} g_{c(a,b)d} = -H_{(a,b)} - g^{cd}{}_{(,a} g_{b)c,d} \tag{234}$$

Also using Eqn. 223, we can rewrite

$$(\ln\sqrt{-g})_{,d}\Gamma^d{}_{ab} = (H_d - g_{de} g^{ce}{}_{,c})\Gamma^d{}_{ab} = H_d\Gamma^d{}_{ab} - g^{ce}{}_{,c}\Gamma_{eab} \tag{235}$$

Substitute these into Eqn 233 then

$$R_{ab} = -\frac{1}{2}g^{cd}g_{ab,cd} + g^{cd}{}_{,c}\Gamma_{dab} - H_{(a,b)} - g^{cd}{}_{(,a}g_{b)c,d} + H_d\Gamma^d{}_{ab} - \underbrace{g^{ce}{}_{,c}\Gamma_{eab}}_{=g^{cd}{}_{,c}\Gamma_{dab}} + \Gamma^d{}_{cb}\Gamma^c{}_{da}$$
$$= -\frac{1}{2}g^{cd}g_{ab,cd} - H_{(a,b)} - g^{cd}{}_{(,a}g_{b)c,d} + H_d\Gamma^d{}_{ab} + \Gamma^d{}_{cb}\Gamma^c{}_{da} \tag{236}$$

Thus, the Einstain's equations can be written as

$$-\frac{1}{2}g^{cd}g_{ab,cd} - H_{(a,b)} - g^{cd}{}_{(,a}g_{b)c,d} + H_d\Gamma^d{}_{ab} + \Gamma^d{}_{cb}\Gamma^c{}_{da} = 4\pi(2T_{ab} - g_{ab}T) \tag{237}$$

More explicitly

$$g^{cd}\partial_c\partial_d g_{ab} + \partial_a g^{cd}\partial_d g_{bc} + \partial_b g^{cd}\partial_d g_{ac} + 2\partial_{(b}H_{a)} - 2H_d\Gamma^d{}_{ab} + \Gamma^d{}_{cb}\Gamma^c{}_{da} = -8\pi(2T_{ab} - g_{ab}T) \tag{238}$$

V. CONSTRUCT INITIAL DATA

Previously, we found that the spatial metric, the extrinsic curvature, and any matter field should satisfy the Hamiltonian constraint

$$R + K^2 - K_{ij}K^{ij} = 16\pi\rho \tag{239}$$

and the momentum constraint

$$D_j(K^{ij} - \gamma^{ij}K) = 8\pi S^i \tag{240}$$

on every hypersurface Σ. Thus, we have to specify (γ_{ij}, K_{ij}) on some initial spatial slice Σ that are compatible with the constraint equations. These fileds can then be used as initial data for a dynamical evolution obtained by solving the evolution equations. q Recall that the conformal transformation of the spatial metric and extrinsic curvature

$$\gamma_{ij} = \psi^4\bar{\gamma}_{ij} \tag{241}$$
$$K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij}K \tag{242}$$
$$A^{ij} = \psi^\alpha\bar{A}^{ij} \tag{243}$$
$$K = \psi^\beta\bar{K} \tag{244}$$

From Eqn. 201, we saw that the divergence of any symmetric, traceless tensor A^{ij} satisfies

$$D_jA^{ij} = \bar{D}_jA^{ij} + 10A^{ij}\bar{D}_j\ln\psi \tag{245}$$

here we use $\psi = e^{phi}$ as a conformal factor for this section. This relation also can be written as

$$D_jA^{ij} = \psi^{-10}\bar{D}_j(\psi^{10}A^{ij}) = \psi^{-10}\bar{D}_j(\psi^{10+\alpha}\bar{A}^{ij}) \tag{246}$$

Thus, we choose $\alpha = -10$ i.e. $A^{ij} = \psi^{-10}\bar{A}^{ij}$. Inverse can be found

$$A_{ij} = \gamma_{ik}\gamma_{jl}A^{kl} = \psi^4\bar{\gamma}_{ik}\psi^4\bar{\gamma}_{jl}\psi^{-10}\bar{A}^{kl}$$
$$= \psi^{-2}\bar{\gamma}_{ik}\bar{\gamma}_{jl}\bar{A}^{kl} = \psi^{-2}\bar{A}_{ij} \tag{247}$$

Substitute this expression into the momentum constraint

$$D_j(K^{ij} - \gamma^{ij}K) = D_j\left(A^{ij} - \frac{2}{3}\gamma^{ij}K\right) = D_jA^{ij} - \frac{2}{3}\gamma^{ij}D_jK = \psi^{-10}\bar{D}_j\bar{A}^{ij} - \frac{2}{3}\psi^{-4}\bar{\gamma}^{ij}\bar{D}_j(\psi^\beta\bar{K}) = 8\pi S^i \tag{248}$$

Note that K is a scalar function so $D_i K = \bar{D}_i K$. Thus we can expand further

$$\psi^{-10}\bar{D}_j\bar{A}^{ij} - \frac{2}{3}\psi^{\beta-4}\bar{\gamma}^{ij}\bar{D}_j\bar{K} - \frac{2}{3}\beta\psi^{\beta-5}\bar{\gamma}^{ij}\bar{K}\bar{D}_j\psi = 8\pi S^i \tag{249}$$

We would like to simplify the equations so the choice for β is $\beta = 0$. This implies that we treat K as a conformal invariant function $K = \bar{K}$. With these choices for A^{ij} and K, the momentum constraints become

$$\bar{D}_j\bar{A}^{ij} - \frac{2}{3}\psi^6\bar{\gamma}^{ij}\bar{D}_j\bar{K} = 8\pi\psi^{10}S^i \tag{250}$$

For Hamiltonian constraint, recall Eqn. 150, $R = \psi^{-4}\bar{R} - 8\psi^{-5}\bar{D}^2\psi$

$$\begin{aligned}
&R + K^2 - K_{ij}K^{ij} = 16\pi\rho \\
&\Rightarrow \psi^{-4}\bar{R} - 8\psi^{-5}\bar{D}^2\psi + K^2 - \left(\psi^{-2}\bar{A}_{ij} + \frac{1}{3}\psi^4\bar{\gamma}_{ij}K\right)\left(\psi^{-10}\bar{A}^{ij} + \frac{1}{3}\psi^{-4}\bar{\gamma}^{ij}K\right) = 16\pi\rho \\
&\psi^{-4}\bar{R} - 8\psi^{-5}\bar{D}^2\psi + K^2 - \psi^{-12}\bar{A}_{ij}\bar{A}^{ij} - \frac{1}{3}K^2 = 16\pi\rho \\
&8\bar{D}^2\psi - \psi\bar{R} - \frac{2}{3}\psi^2K^2 + \psi^{-7}\bar{A}^{ij}\bar{A}_{ij} = 16\pi\psi^5\rho
\end{aligned} \tag{251}$$

A. Simple Black Hole Solution

Consider vacuum case for which the matter source terms vanish i.e. $\rho = S^i = 0$ and focus on a moment of time symmetry. At a moment of time symmetry, all time derivative of γ_{ij} are zero and 4-dimensional line interval has to be invariant under time reversal, $t \to -t$. The latter condition implies that the shift must be vanish $\beta^i = 0$ and so by Eqn. 118, the extrinsic curvature also has to vanish everywhere on the slice $K_{ij} = 0$. As a consequence, $K = 0$. Note that a 4-geometry is said to be time symmetric if there exists such a time slice. Thus, the momentum constraints $D_j(K^{ij} - \gamma^{ij}K) = 0$ are satisfied automatically. The Hamiltonian constraint reduces in these choices

$$\bar{D}^2\psi = \frac{1}{8}\psi\bar{R} \tag{252}$$

Further, choose the conformally related metric to be flat

$$\bar{\gamma}_{ij} = \eta_{ij} \tag{253}$$

Then this assumption makes $\bar{D}_i$ reduces to the flat covariant derivative i.e. partial derivative in Cartesian coordinates and Ricci tensor and scalar associated with $\bar{\gamma}_{ij}$ must vanish. So, the Hamiltonian constraint becomes

$$\bar{D}^2\psi = 0 \tag{254}$$

which is simple Laplace equation with flat Laplace operator $\bar{D}^2$. Here, we are interested in asymptotically flat solutions which satisfy

$$\psi \to 1 + \mathcal{O}(r^{-1}) \qquad \text{as} \qquad r \to \infty \tag{255}$$

where r is the coordinate radius. For spherically symmetric, one can be

$$\psi = 1 + \frac{\mathcal{M}}{2r} \tag{256}$$

B. Conformal Transverse-traceless Decomposition

Any symmetric, traceless tensor can be split into a transverse-traceless part that is divergence-less and a longitudinal part that can be written as a symmetric, traceless gradient of a vector. so, we can decompose $\bar{A}^{ij}$ as

$$\bar{A}^{ij} = \bar{A}^{ij}_{TT} + \bar{A}^{ij}_{L} \tag{257}$$

where $\bar{A}^{ij}_{TT}$ is transverse-traceless part and it is divergence-less such that

$$\bar{D}_j \bar{A}^{ij}_{TT} = 0 \tag{258}$$

and $\bar{A}^{ij}_L$ is longitudinal para that satisfies

$$\bar{A}^{ij}_L \equiv (\bar{L}W)^{ij} = \bar{D}^i W^j + \bar{D}^j W^i - \frac{2}{3}\bar{\gamma}^{ij}\bar{D}_k W^k \tag{259}$$

where W^i is a vector potential and $\bar{L}$ is called the longitudinal operator or vector gradient. It is easy to see $\bar{L}$ produces a symmetric tensor. Further

$$\begin{aligned}\bar{\gamma}_{ij}(\bar{L}W)^{ij} &= \bar{\gamma}_{ij}\left(\bar{D}^i W^j + \bar{D}^j W^i - \frac{2}{3}\bar{\gamma}^{ij}\bar{D}_k W^k\right) = \bar{D}_j W^j + \bar{D}_i W^i - \frac{2}{3}\underbrace{\bar{\gamma}_{ij}\bar{\gamma}^{ij}}_{=3}\bar{D}_k W^k \\ &= \bar{D}_j W^j + \bar{D}_i W^i - 2\bar{D}_k W^k = 0\end{aligned} \tag{260}$$

Thus $\bar{L}$ produces also traceless tensor. Note that vectors ξ^i satisfying $(\bar{L}\xi)^{ij}$ are called conformal Killing vector which suggests why $\bar{L}$ is also called the conformal Killing operator. Now we can write the divergence of $\bar{A}^{ij}$ as

$$\begin{aligned}\bar{D}_j\bar{A}^{ij} &= \underbrace{\bar{D}_j\bar{A}^{ij}_{TT}}_{=0} + \bar{D}_j\bar{A}^{ij}_L = \bar{D}_j(\bar{L}W)^{ij} = \bar{D}_j\bar{D}^i W^j + \bar{D}_j\bar{D}^j W^i - \frac{2}{3}\bar{D}_j(\bar{\gamma}^{ij}\bar{D}_k W^k) \\ &= \bar{D}^i\bar{D}_j W^j + \bar{R}^i{}_j W^j + \bar{D}_j\bar{D}^j W^i - \frac{2}{3}\underbrace{\bar{\gamma}^{ij}\bar{D}_j}_{=\bar{D}^i}\bar{D}_k W^k \\ &= \bar{D}^2 W^i + \frac{1}{3}\bar{D}^i\bar{D}_j W^j + \bar{R}^i{}_j W^j\end{aligned} \tag{261}$$

where $\bar{D}^2 = \bar{D}_i\bar{D}^i$ which is 3-dim Laplacian. Here we use

$$\begin{aligned}&\bar{D}_i\bar{D}_j W^k - \bar{D}_j\bar{D}_i W^k = \bar{R}^{lk}{}_{ji} W_l \\ &\Rightarrow \bar{D}_i\bar{D}^j W^i - \bar{D}^j\bar{D}_i W^i = \bar{R}^{lij}{}_i V_l = R^{lj} W_l = R^j{}_l W^l \\ &\Rightarrow \bar{D}_i\bar{D}^j W^i = \bar{D}^j\bar{D}_i W^i + R^j{}_l W^l\end{aligned} \tag{262}$$

with re-labeling dummy indicies. We also can define the vector Laplacian such that

$$(\bar{\Delta}_L W)^i \equiv \bar{D}^2 W^i + \frac{1}{3}\bar{D}^i\bar{D}_j W^j + \bar{R}^i{}_j W^j \tag{263}$$

Substitute Eqn. 263 into Eqn. 250 gives

$$(\bar{\Delta}_L W)^i - \frac{2}{3}\psi^6\bar{\gamma}^{ij}\bar{D}_j\bar{K} = 8\pi\psi^{10}S^i \tag{264}$$

Hamiltonian constraint has same form as Eqn. 251.

C. Conformal Thin-sandwich Decomposition

VI. THE LAPSE AND SHIFT CONDITIONS

In previous sections, we performed a 3+1 decomposition of Einstein'equations. However, we have not specified the lapse function α and the shift vector β^i that appear in these equations. The lapse and shift can be freely specifiable gauge variables that need to be chosen in order to advance the field data from one time slice to the next. Also, choosing the lapse and shift specify coordinate conditions that we impose to the equations. Therefore, finding kinematical conditions for the coordinates that allow for a well-behaved, long time evolution is important and nontrivial in general. In this section, we introduce several different gauge choices that are commonly used in numerical relativity.

A. Geodesic Slicing

Consider one of the simplest possible choice of α and β^i

$$\alpha = 1 \qquad \beta^i = 0 \tag{265}$$

This gauge choice is called geodesic slicing and the resulting coordinates are also known as the Gaussian normal coordinates.

B. Maximal Slicing

C. Harmonic Coordinates and $1 + \log$ Slicing

D. Minimal Distortion