Resistive MHD: going beyond ideal MHD

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I. THE EVOLUTION EQUATIONS

In this section it is presented the relativistic fluid equations coupled to the Maxwell ones. Later some attention will be devoted to the Ohm's law, which is the equation needed to close the system. This equation basically relates the dynamics of the matter with the electromagnetic field and viceversa. Finally, the ideal MHD limit is presented brevely. During these notes we have chosen gaussian units such that $c = 1$ and we adopt the convention where roman indices a,b,c,... denote spacetime components (ie, from 0 to 3), while i,j,k,... denote spatial ones. Bold letters will represent vectors. We will also consider the following sign convention for the normal to the hypersurfaces, namely

$$n_\mu = (-\alpha, 0) \quad , \quad n^\mu = \frac{1}{\alpha}(1, -\beta^i) \tag{1}$$

A. The Maxwell equations

Let us consider from the very beginning the extended Maxwell equations

$$\nabla_\mu (F^{\mu\nu} + g^{\mu\nu}\psi) = -I^\nu + \kappa n^\nu \psi \tag{2}$$
$$\nabla_\mu ({}^*F^{\mu\nu} + g^{\mu\nu}\phi) = \kappa n^\nu \phi \tag{3}$$

where $F^{\mu\nu}$ is the Maxwell tensor, ${}^*F^{\mu\nu}$ is the Faraday one, I^ν is the electric current and (ϕ, ψ) are scalars to control the constraints. When both the electric and magnetic susceptibility of the medium vanish, like in vacuum or in a highly ionized plasma, the Faraday tensor is simply the dual of the Maxwell one, that is

$${}^*F^{\mu\nu} = \frac{1}{2}\,\epsilon^{\mu\nu\alpha\beta}\,F_{\alpha\beta}\ , \qquad F^{\mu\nu} = -\frac{1}{2}\,\epsilon^{\mu\nu\alpha\beta}\,{}^*F_{\alpha\beta} \tag{4}$$

where $\epsilon^{\mu\nu\alpha\beta}$ is the Levi-Civita pseudotensor of the spacetime, which can be written in terms of the 4-indices Levi-Civita symbol $\eta^{\mu\nu\alpha\beta}$ as

$$\epsilon^{\mu\nu\alpha\beta} = \frac{1}{\sqrt{g}}\,\eta^{\mu\nu\alpha\beta} \qquad \epsilon_{\mu\nu\alpha\beta} = -\sqrt{g}\,\eta_{\mu\nu\alpha\beta}\ . \tag{5}$$

In this case, these tensors can be decomposed in terms of the electric and magnetic fields,

$$F^{\mu\nu} = n^\mu E^\nu - n^\nu E^\mu + \epsilon^{\mu\nu\alpha\beta}\,B_\alpha\,n_\beta \tag{6}$$
$${}^*F^{\mu\nu} = n^\mu B^\nu - n^\nu B^\mu - \epsilon^{\mu\nu\alpha\beta}\,E_\alpha\,n_\beta \tag{7}$$

so that E^μ and B^μ are the electric and magnetic fields measured by a normal observer n^μ, where both fields are purely spatial (ie, $E^\mu n_\mu = B^\mu n_\mu = 0$). Notice that the electric and magnetic fields depend strongly on the observer. For instance, a moving charged particle produce a measurable electric and magnetic field for most of the observers. However, a observer comoving with the particle will measure only a electric field, since the particle is in rest with respect to it. There are some scalars which can be constructed from the Maxwell and Faraday tensors which do not depend on the coordinates. The simpler ones are just quadratic combinations, namely

$${}^*F_{\mu\nu}F^{\mu\nu} = 4E^\mu B_\mu \quad , \quad F_{\mu\nu}F^{\mu\nu} = 2(B^2 - E^2) \quad . \tag{8}$$

Since $F^{\mu\nu}$ is antisymmetric, the four-divergence of equation (2) leads to the current conservation

$$\nabla_\mu I^\mu = 0\ . \tag{9}$$

The electric current I^ν can also be decomposed into components along and perpendicular to the vector n^ν,

$$I^\nu = n^\nu q + J^\nu \quad , \tag{10}$$

where q and J^ν are the charge density and the current as observed by the normal observer n^ν. Again, J^ν is purely spatial so $J^\nu n_\nu = 0$. Current conservation (9)can be expressed, with the decomposition (10), like

$$(\partial_t - \mathcal{L}_\beta)q + \nabla_i(\alpha J^i) = \alpha trKq \tag{11}$$

So, only a prescription for the spatial components J^i is necessary in order to closed the equations. This relation, which is commonly known as the Ohm's law, will be discussed in detail in the next section.

By expanding with the definition (6) in the firs equation (2) we obtain

$$\nabla_\mu F^{\mu\nu} = E^\nu \nabla_\mu n^\mu + n^\mu \nabla_\mu E^\nu - E^\mu \nabla_\mu n^\nu - n^\nu \nabla_\mu E^\mu + \nabla_\mu(\epsilon^{\mu\nu\alpha} B_\alpha) \tag{12}$$

where we have used the contraction $\epsilon^{\mu\nu\alpha} \equiv \epsilon^{\mu\nu\alpha\beta} n_\beta$. Many of these terms are included in the Lie derivative of the electric field,

$$\mathcal{L}_{\alpha\mathbf{n}} E^\nu = \alpha(n^\mu \nabla_\mu E^\nu - E^\mu \nabla_\mu n^\nu - E^\mu n^\nu \nabla_\mu(ln\alpha)) \tag{13}$$

Developing the other terms and performing the 3+1 decomposition of the equations, as it is done in [1], we can arrive to the following final expressions for the electric part (and for the magnetic part, repeating the process for the dual equation (3)

$$(\partial_t - \mathcal{L}_\beta)E^i \;-\; \epsilon^{ijk}\nabla_j(\alpha B_k) + \alpha\gamma^{ij}\nabla_j\psi = \alpha trKE^i - \alpha J^i \tag{14}$$
$$(\partial_t - \mathcal{L}_\beta)\psi \;+\; \alpha\nabla_i E^i = \alpha q - \alpha\kappa\psi \tag{15}$$
$$(\partial_t - \mathcal{L}_\beta)B^i \;+\; \epsilon^{ijk}\nabla_j(\alpha E_k) + \alpha\gamma^{ij}\nabla_j\phi = \alpha trKB^i \tag{16}$$
$$(\partial_t - \mathcal{L}_\beta)\phi \;+\; \alpha\nabla_i B^i = -\alpha\kappa\phi \tag{17}$$

B. The characteristic structure

The characteristic structure of the system (14-17) can be found easily by solving the Runkine-Hugoniot jump conditions

$$\tilde{v}[E^i] \;=\; -\epsilon^{ijk} n_j[\alpha B_k] + f\alpha[J^i] + (1-f)\alpha n^i[\psi] \tag{18}$$
$$\tilde{v}[B^i] \;=\; \epsilon^{ijk} n_j[\alpha E_k] + \alpha\, n^i[\phi] \tag{19}$$
$$\tilde{v}[\psi] \;=\; \alpha\, n_i[E^i] - \alpha[q] \tag{20}$$
$$\tilde{v}[\phi] \;=\; \alpha\, n_i[B^i] \tag{21}$$

where we have defined $\tilde{v} \equiv v + \beta^n$ and f is a parameter with values either 0 or 1.

1. The electrovacuum case

Let us consider first the simplest case in which there are no charges and no currents (i.e., the so-called electrovacuum). In this limit $J^i = q = 0$ and so we will take $f = 0$. It is now trivial to compute the following list of eigenvectors

- the constraint modes, which propagates with light speed $v = -\beta^n \pm \alpha$,

$$[E^n] \pm [\psi] \quad , \quad [B^n] \pm [\phi] \tag{22}$$

- the transversal modes, corresponding to the EM waves, which also propagates with light speed $v = -\beta^n \pm \alpha$, can be written in two equivalent forms

$$[E^i - E^n n^i] \mp [\epsilon^{ijk} n_j B_k] \quad = \quad [B^i - B^n n^i] \pm [\epsilon^{ijk} n_j E_k] \tag{23}$$

C. The conservation of the stress-energy tensor

Let us assume that the perfect fluid and the electromagnetic fields are minimally coupled, that is, that there are no mixed terms and so the total stress-energy tensor can be written just by addition

$$T_{\mu\nu} = T^{fluid}_{\mu\nu} + T^{em}_{\mu\nu} \tag{24}$$

so explicitely it can be written as

$$T_{\mu\nu} = [\rho(1+\epsilon) + p]\, u_\mu u_\nu + p g_{\mu\nu} + F_\mu{}^\lambda F_{\nu\lambda} - \frac{1}{4} g_{\mu\nu}\ F^{\lambda\alpha} F_{\lambda\alpha} \tag{25}$$

The rest mass density ρ is the density of the fluid measured by a comoving observer. The total energy density of the fluid also contains contributions from the internal degrees of freedom of the particles,

$$e = \rho(1+\epsilon) \tag{26}$$

where the internal energy ϵ accounts, for instance, for the thermal energy, the binding energy,... The pressure p is described by an Equation of State as a function of the rest mass density and the internal energy, and it is a propertie of the type of fluid which is being considered. Within the pressure we can construct the enthalpy h,

$$h = \rho(1+\epsilon) + p \quad . \tag{27}$$

It is important to stress that the set of thermodynamic quantities $\{\rho, \epsilon, P\}$ are all measured in the rest frame of the fluid element, although in general we will use an Eulerian perspective where the coordinates are not tied to the flow of the fluid. Therefore, we will need the four-velocity u^μ to describe how the fluid moves with respect to the Eulerian observers. The 4-velocity follows the usual normalization relation

$$u^\mu u_\mu = -1 \quad . \tag{28}$$

It is more useful to deal with the standard velocity vectors, so the velocity u^μ will be decomposed into spatial and temporal components, namely

$$u^\mu = W n^\mu + W v^\mu \tag{29}$$

where v^μ corresponds to the familiar three-dimensional quantities as measured by Eulerian observers (ie, $v^\mu n_\mu = 0$). Notice that the time component is not independent due to the normalization relation (28), so

$$W = -n_\mu u^\mu = (1 - v_i v^i)^{-1/2} \quad , \quad u^i = W(v^i - \frac{\beta^i}{\alpha}) \tag{30}$$

where we can recognize now that W is the standard Lorentz factor. The set of fluid variables $U = (\rho, \epsilon, p, v^i)$ are the primitive quantities which describe the state of a perfect fluid.

In addition to the conservation of energy and momentum, when there are neither creation nor destruction of particles, the fluids has to conserve also the total number of baryons. This law is expressed in terms of the baryon number density ρu^μ, and is written as

$$\nabla_\mu (\rho u^\mu) = 0 \tag{31}$$

which is just the relativistic generalization of the conservation of mass. Then, the primitive quantites U has to be computed from the conservation of energy, momentum and baryonic number. The EOS for the pressure close the system of equations.

In order to capture properly the weak solutions of the non-linear equations in the presence of shocks it is important to write them in local conservation law form. Let us start with the simpler one. The baryon conservation can be written as a conservation law depending on v^i in a straigth manner,

$$\partial_t(\sqrt{\gamma} D) + \partial_j[\sqrt{\gamma} D(\alpha v^j - \beta^j)] = 0 \tag{32}$$

where we have defined $D \equiv \rho W$.

The evolution of matter must comply with the conservation of the stress-energy tensor

$$\nabla_\nu T^{\mu\nu} = 0, \tag{33}$$

which can be expressed as a system of conservation laws for the energy and momentum densities,namely,

$$\partial_t(\sqrt{\gamma}U) + \partial_j[\sqrt{\gamma}(\alpha S^j - \beta^j U)] = \sqrt{\gamma}[\alpha S^{ij} K_{ij} - S^j \partial_j \alpha] \tag{34}$$

$$\partial_t(\sqrt{\gamma}S_i) + \partial_j[\sqrt{\gamma}(\alpha S^j{}_i - \beta^j S_i)] = \sqrt{\gamma}[\alpha \Gamma^j{}_{ik} S^k{}_j + S_j \partial_i \beta^j - U \partial_i \alpha] \tag{35}$$

The physical system is now described with the physical field $(\rho, p, \epsilon, v_i, E_i, B_i, q)$. The evolution equations for the electromagnetic fields are given by the Maxwell equations and the conservation of charge, while the fluid fields are still governed by the conservation of the total energy and momentum and the baryonic number. By using (132) we can write the projections of the stress-energy tensor,

$$U = hW^2 - p + \frac{1}{2}(E^2 + B^2) \quad , \tag{36}$$

$$S_i = hW^2 v_i + \epsilon_{ijk} E^j B^k \quad , \tag{37}$$

$$S_{ij} = hW^2 v_i v_j + \gamma_{ij} p - E_i E_j - B_i B_j + \frac{1}{2}\gamma_{ij}(E^2 + B^2) \tag{38}$$

In general it will be useful to describe the energy conservation in terms of the quantity

$$\tau \equiv U - D = hW^2 - p + \frac{1}{2}(E^2 + B^2) - \rho W \tag{39}$$

since the newtonian limit of the energy is recovered in this way. Basically, we are substracting the mass rest energy density from the total energy. The evolution equation for τ is obtained from the substraction of (34) and (32).

D. The case without EM shock: exploiting the Lorentz force

In some cases we do not expect the formation of shocks in the EM sector, so it is not necessary to write the EM contribution in a conservation law form. In such a case the full system of fluid plus EM fields can be described as a conservation equations for the fluid with some additional source terms corresponding to the Lorentz force

$$\nabla_\mu T^{\mu\nu}_{fluid} = -\nabla_\mu T^{\mu\nu}_{em} = F^{\nu\mu} I_\mu \tag{40}$$

Let us develop in detail these equations. For the momentum contribution, it is more convenient to use the mixed form

$$\nabla_\mu T^\mu{}_\nu = F_{\nu\mu} I^\mu \tag{41}$$

The first term can be written explicitely in conserved form by expanding the covariant derivative conveniently

$$\nabla_\mu T^\mu{}_\nu = \partial_\mu T^\mu{}_\nu + \hat{\Gamma}^\mu_{\mu\alpha} T^\alpha{}_\nu - \hat{\Gamma}^\mu_{\nu\alpha} T^\alpha{}_\mu = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} T^\mu{}_\nu\right) - \hat{\Gamma}^\mu_{\nu\alpha} T^\alpha{}_\mu \tag{42}$$

where $\hat{\Gamma}^\mu_{\mu\alpha}$ are the 4-dimensional Christophel symbols.

By multiplying by $\sqrt{-g}$ the eq.(41), for $\nu = i$, we obtain

$$\partial_t \left(\sqrt{-g} T^0{}_i\right) + \partial_k \left(\sqrt{-g} T^k{}_i\right) = \sqrt{-g} \hat{\Gamma}^\mu_{\nu\alpha} T^\alpha{}_\mu + \sqrt{-g} F_{i\mu} I^\mu \tag{43}$$

By considering the relation between the 3+1 projections of the stress-energy tensor and its components

$$S_i = -n_\mu T^\mu{}_i = \alpha T^0{}_i \tag{44}$$

$$S^k{}_i = h^k{}_\mu T^\mu{}_i = T^k{}_i + \beta^k S_i / \alpha \tag{45}$$

with $h^k{}_\mu = g^k{}_\mu + n^k n_\mu$ the projection operator, we recover exactly the equation (35) with an additional term $+\sqrt{-g} F_{i\mu} I^\mu$ in the rhs. We will later discuss this term.

For the energy equation is more convenient to start from the contravariant form

$$\nabla_\mu T^{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} T^{\mu\nu}\right) + \hat{\Gamma}^\nu_{\mu\alpha} T^{\mu\alpha} \tag{46}$$

By multiplying (40) by the lapse, introducing it inside the derivatives, and with $\nu = 0$, we obtain

$$\partial_t \left(\alpha^2 \sqrt{\gamma} T^{00}\right) + \partial_k \left(\alpha^2 \sqrt{\gamma} T^{0i}\right) = \alpha \sqrt{\gamma} T^{00} \partial_t \alpha + \alpha \sqrt{\gamma} T^{0i} \partial_i \alpha - \alpha \sqrt{-g} \hat{\Gamma}^0_{\mu\alpha} T^{\mu\alpha} + \alpha \sqrt{-g} F^{0\mu} I_\mu \tag{47}$$

By considering again the relation between the 3+1 projections of the stress-energy tensor and its components

$$U = n_\mu n_\nu T^{\mu\nu} = \alpha^2 T^{00} \tag{48}$$
$$S^i = -n_\mu h^i{}_\nu T^{\mu\nu} = \alpha \left(T^{i0} + \beta^i T^{00}\right) \tag{49}$$

and cancelling a lot of terms coming from $\hat{\Gamma}^0_{\mu\alpha} T^{\mu\alpha}$, we recover exactly the equation (34) with an additional term $+\alpha\sqrt{-g} F^{0\mu} I_\mu$ in the rhs.

The Lorentz force can be written, by using the relations for the Maxwell tensor (6) and for the current (10), as

$$F_{\nu\mu} I^\mu = qE_\nu + n_\nu J^\mu E_\mu + \epsilon_{\nu\mu\alpha} J^\mu B_\alpha \tag{50}$$

where most of the other terms are zero either because $E^\mu n_\mu = B^\mu n_\mu = J^\mu n_\mu = 0$ or because $\epsilon_{\mu\nu\alpha\beta}\, n^\alpha\, n^\beta = 0$. This implies that contribution of the Lorentz force to the momentum equation will be

$$\sqrt{-g} F_{i\mu} I^\mu = \alpha\sqrt{\gamma} \left(qE_i + \epsilon_{ijk} J^j B^k\right) \tag{51}$$

while that to the energy equation will be

$$\sqrt{-g}\alpha F^{0\mu} I_\mu = -\alpha\sqrt{\gamma} n_\nu F^{\nu\mu} I_\mu = \alpha\sqrt{\gamma} J^i E_i \quad . \tag{52}$$

E. The Ohm law

As mentioned above, Maxwell equations are coupled to the fluid ones by means of the current 4-vector I^a, whose explicit form will depend in general on the electromagnetic fields and on the local fluid properties, measured on the comoving frame. For this reason, it is convenient to introduce the electric and magnetic fields ($e^\mu \equiv F^{\mu\nu} u_\nu, b^\mu \equiv^* F^{\mu\nu} u_\nu$) measured by an observer comoving with the fluid, which have only three independent components since $e^\mu u_\mu = b^\mu u_\mu = 0$. The Maxwell and Faraday tensor can be written, in terms of the comoving fields, as:

$$F^{\mu\nu} = u^\mu e^\nu - u^\nu e^\mu + \epsilon^{\mu\nu\alpha\beta}\, b_\alpha\, u_\beta \tag{53}$$
$$^*F^{\mu\nu} = u^\mu b^\nu - u^\nu b^\mu - \epsilon^{\mu\nu\alpha\beta}\, e_\alpha\, u_\beta \tag{54}$$

We can also decompose the electric current into a component along u^ν and components normal to it:

$$I^\mu = u^\mu \tilde{q} + j^\mu \quad , \tag{55}$$

where $j^\mu u_\mu = 0$ and $\tilde{q}$ is the charge density measured by the comoving observer. The relation with the Eulerian quantities (10) can be obtained easily by contracting

$$-q = n_\mu I^\mu = n_\mu u^\mu \tilde{q} + n_\mu j^\mu \quad , \tag{56}$$

which implies that $\tilde{q} = (q + n_\mu j^\mu)/W$. Substituting this relation in (55) and using the decomposition of the 4-velocity, it is obtained the final relation

$$I_i = J_i = (q + j^\mu n_\mu) v_i + j_i \quad . \tag{57}$$

A standard prescription, known as the Ohm law, is to consider the current to be proportional to the Lorentz force acting on a charged particle, that is, a linear relation between $j^\mu = \sigma^{\mu\nu} e_\nu$, with $\sigma^{\mu\nu}$ being the electrical conductivity of the medium. This general linear relation can account for the induction and the Hall terms,

$$\sigma^{\mu\nu} = \chi(g^{\mu\nu} + \xi^2 b^\mu b^\nu + \xi \epsilon^{\mu\nu\alpha\beta} u_\alpha b_\beta) \tag{58}$$

The first term leads to the well known isotropic scalar Ohm law, while the other two represents the anisotropies due to the presence of a magnetic field. The conductivity χ can be calculated in the collision-time approximation [2]

$$\xi = e\tau/m \quad , \chi = \frac{n_e e \xi}{1 + \xi^2 b^2} \quad , \tag{59}$$

where τ is the collision time, n_e is the electron density, e and m are the electron's charge and mass and $b^2 = b^\mu b_\mu$. Another option comes from a more theoretical basis, which can be understood as a redefinition of the previous parameters [3]

$$\chi = \frac{\sigma}{1 + \xi^2 b^2} \quad , \sigma = 1/(n_e e \xi), \quad \xi = 1/R \tag{60}$$

being R the proportionality constant in the dissipative force between the two components of the fluid.

Let us now write the general relativistic Ohm law in terms of fields measured by an Eulerian observer. First, we will write the full current

$$j_\mu = \frac{\sigma}{1+\xi^2 b^2}[e_\mu + \xi^2 (e_\nu b^\nu) b_\mu + \xi \epsilon_{\mu\nu\alpha\beta} e^\nu u^\alpha b^\beta] \tag{61}$$

We will neglect in general the third term with respect to the second one, since the ratio e/m is of order $\sim O(10^{21})$ in geometrized. With this assumption, and using that the electric and magnetic fields in the fluid frame can be written as

$$e^\mu \;=\; F^{\mu\nu} u_\nu = W n^\mu (E^\nu v_\nu) + W E^\nu + W \epsilon^{\mu\nu\alpha} v_\nu B_\alpha \tag{62}$$

$$b^\mu \;=\; {}^*F^{\mu\nu} u_\nu = W n^\mu (B^\nu v_\nu) + W B^\nu - W \epsilon^{\mu\nu\alpha} v_\nu E_\alpha \tag{63}$$

we can obtain easily the relations

$$j_\mu n^\mu = \frac{\sigma}{1+\xi^2 b^2}[e_\mu n^\mu + \xi^2 (e_\nu b^\nu) b_\mu n^\mu] = \frac{\sigma}{1+\xi^2 b^2}[-W(E^k v_k) - W\xi^2 (E^j B_k)(B^k v_k)] \tag{64}$$

We can now substitute eqs.(62-64) in (57) to obtain the prescription for the (spatial) current

$$J_i = q v_i + \frac{\sigma}{1+\xi^2 b^2}[\mathcal{E}_i + \xi^2 (E^k B_k) \mathcal{B}_i] \tag{65}$$

where we have introduced the shortcuts

$$\mathcal{E}_i = W\left[E_i + \epsilon_{ijk} v^j B^k - (v_k E^k) v_i\right] \quad , \quad \mathcal{B}_i = W\left[B_i - \epsilon_{ijk} v^j E^k - (v_k B^k) v_i\right] \tag{66}$$

It is important to recall that in deriving these relations for Ohm's law we are implicitly assuming that the collision frequency of the constituent particles of our fluid is much larger that the typical oscillation frequency of the plasma. Stated differently, the timescale for the electrons and ions to come into equilibrium is much shorter than any other timescale in the problem, so that no charge separation is possible and the fluid is globally neutral. This assumption is a key aspect of the MHD approximation.

The general system of relativistic resistive MHD equations brings about a delicate issue when the conductivity in the plasma undergoes very large spatial variations. In the regions with high conductivity, in fact, the system will evolve on timescales which are very different from those in the low-conductivity region. Mathematically, therefore, the problem can be regarded as a hyperbolic one with stiff relaxation terms which requires special care to capture the dynamics in a stable and accurate manner.

F. The final evolution equations

The final set of evolution equations to solve, in conservation law form, is

$$\partial_t(\sqrt{\gamma} B^i) \;+\; \partial_k[\sqrt{\gamma}\left(-\beta^k B^i + \alpha(\epsilon^{ikj} E_j + \gamma^{ik}\phi)\right)] = -\sqrt{\gamma} B^k (\partial_k \beta^i) + \sqrt{\gamma}\phi\left(\gamma^{ij}\partial_j \alpha - \alpha \gamma^{jk}\Gamma^i_{jk}\right) \tag{67}$$

$$\partial_t(\sqrt{\gamma} E^i) \;+\; \partial_k[\sqrt{\gamma}\left(-\beta^k E^i + \alpha(-\epsilon^{ikj} B_j + \gamma^{ik}\psi)\right)] = -\sqrt{\gamma} E^k (\partial_k \beta^i) + \sqrt{\gamma}\psi\left(\gamma^{ij}\partial_j \alpha - \alpha \gamma^{jk}\Gamma^i_{jk}\right) - \alpha\sqrt{\gamma} J^i \tag{68}$$

$$\partial_t(\sqrt{\gamma}\phi) \;+\; \partial_k[\sqrt{\gamma}(-\beta^k \phi + \alpha B^k)] = \sqrt{\gamma}[-\alpha\,\phi\, trK + B^k(\partial_k \alpha) - \alpha\kappa\phi] \tag{69}$$

$$\partial_t(\sqrt{\gamma}\psi) \;+\; \partial_k[\sqrt{\gamma}(-\beta^k \psi + \alpha E^k)] = \sqrt{\gamma}[-\alpha\,\psi\, trK + E^k(\partial_k \alpha) + \alpha q - \alpha\kappa\psi] \tag{70}$$

$$\partial_t(\sqrt{\gamma} q) \;+\; \partial_k[\sqrt{\gamma}(-\beta^k q + \alpha J^k)] = 0 \tag{71}$$

$$\partial_t(\sqrt{\gamma} D) \;+\; \partial_k[\sqrt{\gamma}(-\beta^k + \alpha v^k) D] = 0 \tag{72}$$

$$\partial_t(\sqrt{\gamma}\tau) \;+\; \partial_k[\sqrt{\gamma}\left(-\beta^k \tau + \alpha(S^k - v^k D)\right)] = \sqrt{\gamma}[\alpha S^{ij} K_{ij} - S^j \partial_j \alpha] \tag{73}$$

$$\partial_t(\sqrt{\gamma} S_i) \;+\; \partial_k[\sqrt{\gamma}(-\beta^k S_i + \alpha S^k{}_i)] = \sqrt{\gamma}[\frac{\alpha}{2} S^{jk}\partial_i \gamma_{jk} + S_j \partial_i \beta^j - (\tau + D)\partial_i \alpha] \tag{74}$$

where

$$
\begin{aligned}
J^i &= qv^i + \frac{\sigma W}{1+\xi^2 b^2}\left\{\left[E^i + \epsilon^{ijk} v_j B_k - (v_k E^k) v^i\right] + \xi^2 (E^k B_k)\left[B^i - \epsilon^{ijk} v_j E_k - (v_k B^k) v^i\right]\right\} \quad , &(75)\\
D &= \rho W \quad , &(76)\\
\tau &= hW^2 - p + \frac{1}{2}(E^2 + B^2) - \rho W \quad , &(77)\\
S_i &= hW^2 v_i + \epsilon_{ijk} E^j B^k \quad , &(78)\\
S_{ij} &= hW^2 v_i v_j + \gamma_{ij} p - E_i E_j - B_i B_j + \frac{1}{2}\gamma_{ij}(E^2 + B^2) &(79)
\end{aligned}
$$

II. SOLVING THE MAXWELL EQUATIONS FOR FINITE CONDUCTIVITIES

In this section we will explain the problems related to the evolution of the full Maxwell equations in high conductivity mediums, which is a type of hyperbolic system with relaxation terms. Then we will introduce the implicit-explicit (IMEX) Runge Kutta methods to deal with these kind of systems, by treating the non-stiff part by a Strong-Stability Preserving (SSP) scheme, while the stiff part is evolved with an L-stable diagonally implicit Runge Kutta (DIRK). After presenting the scheme, its properties and some examples, it will discussed in detail its application to the EMHD equations. Finally, the conversion between conserved and primitive fields is explained.

A. The hyperbolic systems with relaxation terms

Althought the system of equations describing the evolution of fluid in presence of electromagnetic fields are known (ie,the resistive MHD system (67-74)), they are not solved numerically in the general case. The main reason is that the full maxwell equations in high conductivity mediums are in general a stiff system of partial differential equations. The prototype of these systems can be written as

$$
\partial_t \mathbf{U} = F(\mathbf{U}) + \frac{1}{\epsilon} R(\mathbf{U}) \tag{80}
$$

where $\epsilon > 0$ is the relaxation time. We will consider that in the limit $\epsilon \to \infty$ the system is hyperbolic with a spectral radius c_h(ie, the absolute value of the maximum eigenvalue). At the other limit $\epsilon \to 0$ the system is clearly stiff since the time scale ϵ of the relaxation (or stiff term) $R(\mathbf{U})$ is very different from the speeds c_h of the hyperbolic (or non-stiff) part $F(\mathbf{U})$.

In the stiff limit ($\epsilon \to 0$) the stability of an explicit time evolution scheme [13] is only achieved with a time step size $\Delta t \leq \epsilon$. This restriction is much stronger than the one given by the CFL condition $\Delta t \leq \Delta x / c_h$ and in practice it can become a too oncrous a constraint. The development of efficient numerical schemes for such systems is challenging, since in many applications the relaxation time can vary many orders of magnitude, from order one to a very small values compared to the time scale determined by the characteristic speeds of the system. There are several ways to deal with the hyperbolic systems with relaxation. The simplest one is considered only the stiff limit $\epsilon \to 0$, where the system is well approximated by a suitable reduced set of conservation laws called equilibrium system such that

$$
\begin{aligned}
R(\bar{\mathbf{U}}) &= 0 &(81)\\
\partial_t \bar{\mathbf{U}} &= G(\bar{\mathbf{U}}) \quad . &(82)
\end{aligned}
$$

This is exactly the situation in the so-called ideal MHD limit, where the full Maxwell equations reduce to a equilibrium system which is valid only for vanishing resistivity $\eta = 0$.

Usually it is very difficult to split the problem in separate regimes and to use different solvers in the stiff and non stiff regions. Thus one has to use the original relaxation system in the whole computational domain. The construction of schemes that work for all ranges of the relaxation time, using coarse grids that do not resolve the small relaxation time, has been studied by using different techniques.

One of the most popular because of their simplicity and robustness are the splitting methods. Strang splitting provides second order accuracy if each step is at least second order accurate [4], and this property is maintained under some assumptions even for stiff problems [5]. However, higher order accuracy is difficult to obtain even in non-stiff regimes with this kind of splitting. Moreover, when applied to hyperbolic systems with relaxation, Strang splitting scheme reduces to first order accuracy since the kernel of the relaxation operator is non-trivial and corresponds to a singular matrix in the linear case, therefore the assumptions in [5] are not satisfied.

Another approach is based on the IMEX Runge-Kutta methods, which deals with some the problems previously mentioned and can be implemented easily. These kind of methods are still under development and have few drawbacks. The most serious one is a degradation, for some range of values of the relaxation time ϵ, in the accuracy order for most of the Strong Stability Preserving schemes available in the literature. The next subsections will explained in detail these kind of methods.

B. The IMEX Runge-Kutta methods

An IMEX Runge-Kutta scheme consists of applying an implicit discretization to the stiff terms and an explicit one to the non stiff ones. When applied to system (80) it takes the form

$$\begin{aligned} \mathbf{U}^{(i)} = \mathbf{U}^n \; &+ \; \Delta t \sum_{j=1}^{i-1} \tilde{a}_{ij} F(\mathbf{U}^{(j)}) \\ &+ \; \Delta t \sum_{j=1}^{\nu} a_{ij} \frac{1}{\epsilon} R(\mathbf{U}^{(j)}) \\ \mathbf{U}^{n+1} = \mathbf{U}^n \; &+ \; \Delta t \sum_{i=1}^{\nu} \tilde{\omega}_i F(\mathbf{U}^{(i)}) + \Delta t \sum_{i=1}^{\nu} \omega_i \frac{1}{\epsilon} R(\mathbf{U}^{(i)}) \end{aligned} \tag{83}$$

where $\mathbf{U}^{(i)}$ are the auxiliary intermediate values of the Runge-Kutta. The matrices $\tilde{A} = (\tilde{a}_{ij})$, $\tilde{a}_{ij} = 0$ for $j \geq i$ and $A = (a_{ij})$ are $\nu \times \nu$ matrices such that the resulting scheme is explicit in F and implicit in R. An IMEX Runge-Kutta is characterized by these two matrices and the coefficient vectors $\tilde{\omega}_i$ and ω_i. Since the simplicity and efficency of solving the implicit part at each step is of great importance, it is natural to consider diagonally implicit Runge-Kutta (DIRK) schemes ($a_{ij} = 0$ for $j > i$) for the stiff terms.

IMEX Runge-Kutta schemes can be represented by a double tableau in the usual Butcher notation ([6, 7])

$$\begin{array}{c|c} \tilde{c} & \tilde{A} \\ \hline & \tilde{\omega}^T \end{array} \qquad\qquad \begin{array}{c|c} c & A \\ \hline & \omega^T \end{array}$$

where the coefficients $\tilde{c}$ and c used for the treatment of non-autonomous systems are given by the following relation

$$\tilde{c}_i = \sum_{j=1}^{i-1} \tilde{a}_{ij} \quad , \quad c_i = \sum_{j=1}^{i} a_{ij} \quad . \tag{84}$$

Let us consider a simple example in order to fix ideas. Let us consider the following IMEX scheme, written in the tableau form

TABLE I: Tableau for the explicit (left) implicit (right) IMEX-SSP2(2,2,2) L-stable scheme

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array} \qquad \begin{array}{c|cc} \gamma & \gamma & 0 \\ 1-\gamma & 1-2\gamma & \gamma \\ \hline & 1/2 & 1/2 \end{array} \quad \gamma = 1 - \frac{1}{\sqrt{2}}$$

so the IMEX Runge-Kutta scheme would be written explicitly as

$$\begin{aligned} \mathbf{U}^{(1)} = \mathbf{U}^n \; &+ \; \frac{\Delta t}{\epsilon} \gamma R(\mathbf{U}^{(1)}) \\ \mathbf{U}^{(2)} = \mathbf{U}^n \; &+ \; \Delta t F(\mathbf{U}^{(1)}) \\ &+ \; \frac{\Delta t}{\epsilon} [(1-2\gamma) R(\mathbf{U}^{(1)}) + \gamma R(\mathbf{U}^{(2)})] \\ \mathbf{U}^{n+1} = \mathbf{U}^n \; &+ \; \frac{\Delta t}{2} [F(\mathbf{U}^{(1)}) + F(\mathbf{U}^{(2)})] \\ &+ \; \frac{\Delta t}{2\epsilon} [R(\mathbf{U}^{(1)}) + R(\mathbf{U}^{(2)})] \end{aligned}$$

Notice that at each substep it has to be solved an implicit equation for the auxiliary intermediate values $\mathbf{U}^{(i)}$. The complexity on inverting this equation will depend on the particular form of the operator $R(\mathbf{U})$

C. Asymptotic properties of IMEX schemes

We will introduce now some definitions and theorems which will describe some of the useful properties of the IMEX schemes.

Definition 1 *An IMEX scheme in the form (83) for the system (80) is said to be asymptotic preserving (AP) if in the limit $\epsilon \to 0$ the scheme becomes a consistent discretization of the equilibrium system (81).*

This definition does not imply that the scheme is asymptotically accurate, that is, that preserves the order of accuracy in t in the stiff limit $\epsilon \to 0$. In order to give sufficient conditions for the AP and asympotically accurate property, we make use of the following lemma [8]:

Lemma 1 *If all the diagonal elements of the triangular matrix A that characterize the DIRK scheme are non zero, then*

$$\lim_{\epsilon \to 0} R(U^{(i)}) = 0 \quad . \tag{85}$$

This means that in the limit $\epsilon \to 0$ the equilibrium system solution is recovered. In order to apply the previous Lemma, the vectors of c and $\tilde{c}$ cannot be equal. In fact $\tilde{c}_1 = 0$ whereas $c_1 \neq 0$. Note that if $c_1 = 0$ but $a_{ii} \neq 0$ for $i > 1$, then we still have $\lim_{\epsilon \to 0} R(U^{(i)}) = 0$ for $i > 1$ but $\lim_{\epsilon \to 0} R(U^{(1)}) \neq 0$ in general. The corresponding scheme may be inaccurate if the initial condition is not "well prepared", that is, $R(U^{(0)}) = 0$. In this case the scheme is not able to treat the so called "initial layer" problem and degradation of accuracy in the stiff limit is expected. Now we can state the following [8]:

Theorem 1 *If $\det A \neq 0$, then in the limit $\epsilon \to 0$, the IMEX scheme (83) applied to the system (80) becomes the explicit RK scheme characterized by $(\tilde{A}, \tilde{\omega}, \tilde{c})$ applied to the equilibrium system (81).*

The theorem guarantees that in the stiff limit the numerical scheme becomes the explicit RK scheme applied to the equilibrium system, and therefore the order of accuracy of the limiting scheme is greater (if the order of the implicit RK is higher than the order of the explicit one) or equal to the order of accuracy of the original IMEX scheme.

However, this result does not guarantee the accuracy of the solution for the non conserved quantities. In fact, since the very last step in the scheme it is not a projection towards the local equilibrium, a final layer effect ocurrs. The use of stiffly accurate schemes (ie, schemes for which $a_{\nu j} = \omega_j$ for $j = 1, ..., \nu$) in the implicit step may serve as a remedy to this problem.

D. Strong Stability Preserving IMEX schemes

Solutions of conservation equations have some norm that decreases in time. It would be desirable, in order to avoid spurious numerical oscillations arising near discontinuities of the solution, to maintain such property at a discrete level by the numerical method. If U^n represents a vector of solution values at the time $t = n\ \Delta t$, then

Definition 2 *A sequence $\{U^n\}$ is said to be strongly stable in a given norm $\|\cdot\|$ provided that $\|U^{n+1}\| \leq \|U^n\|$ for all $n \geq 0$.*

The most commonly used norms are the TV-norm and the infinity norm. A numerical scheme that maintains strong stability at discrete level is called Strong Stability Preserving (SSP) (see [9] for a detailed description of optimal SSP schemes and their properties). From the previous subsection remarks it follows that if the explicit part of the IMEX scheme is SSP, then in the stiff limit we will obtain an SSP method for the equilibrium system. This property is essential to avoid spurious oscillations during the evolution of non-smooth data.

There are different IMEX RK schemes available in the literature. We have considered only some of the second and third order IMEX schemes which satisfies the Theorem 1, given in its Butcher tableau form [8]. In all these schemes the implicit tableau corresponds to an L-stable scheme (that is, $\omega^T A^{-1} e = 1$, being e a vector whose components are all equal to 1), whereas the explicit tableau is SSPk, where k denotes the order of the SSP scheme. We shall use the notation SSP$k(s, \sigma, p)$, where the triplet (s, σ, p) characterizes the number of s stages of the implicit scheme, the number σ of stages of the explicit scheme and the order p of the IMEX scheme.

TABLE II: Tableau for the explicit (left) implicit (right) IMEX-SSP2(3,2,2) stiffly accurate scheme

$$
\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 \\
1 & 1/2 & 1/2 & 0 \\
\hline
 & 1/3 & 1/3 & 1/3
\end{array}
\qquad\qquad
\begin{array}{c|ccc}
1/4 & 1/4 & 0 & 0 \\
1/4 & 0 & 1/4 & 0 \\
1 & 1/3 & 1/3 & 1/3 \\
\hline
 & 1/3 & 1/3 & 1/3
\end{array}
$$

TABLE III: Tableau for the explicit (left) implicit (right) IMEX-SSP3(3,3,2) L-stable scheme

$$
\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1/2 & 1/4 & 1/4 & 0 \\
\hline
 & 1/6 & 1/6 & 2/3
\end{array}
\qquad
\begin{array}{c|ccc}
\gamma & \gamma & 0 & 0 \\
1-\gamma & 1-2\gamma & \gamma & 0 \\
1/2 & 1/2-\gamma & 0 & \gamma \\
\hline
 & 1/6 & 1/6 & 2/3
\end{array}
$$

TABLE IV: Tableau for the explicit (left) implicit (right) IMEX-SSP3(4,3,3) L-stable scheme

$$
\begin{array}{c|cccc}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
1/2 & 0 & 1/4 & 1/4 & 0 \\
\hline
 & 0 & 1/6 & 1/6 & 2/3
\end{array}
\qquad
\begin{array}{c|cccc}
\alpha & \alpha & 0 & 0 & 0 \\
0 & -\alpha & \alpha & 0 & 0 \\
1 & 0 & 1-\alpha & \alpha & 0 \\
1/2 & \beta & \eta & 1/2-\beta-\eta-\alpha & \alpha \\
\hline
 & 0 & 1/6 & 1/6 & 2/3
\end{array}
$$

$$
\begin{aligned}
\alpha &= 0.24169426078821\ ,\ \beta = 0.06042356519705\ , \\
\eta &= 0.12915286960590
\end{aligned}
$$

There is an efficient way to implement the IMEX RK when the explicit part is given by the TVD third order RK of Shu and Osher. In that case, the memory can be reused for the explicit part so that only two levels of fields and one of rhs is needed, namely

1. first step (the relaxation time ϵ is included in R_V)

$$
\begin{aligned}
\mathbf{W}^{(1)} &= \mathbf{W}^n \\
\mathbf{V}^{(1)} &= \mathbf{V}^n + \Delta t\, a_{11} R_V^{(1)}
\end{aligned}
\tag{86}
$$

2. second step

$$
\begin{aligned}
\mathbf{W}^{(2)} &= \mathbf{W}^n \\
\mathbf{V}^{(2)} &= \mathbf{V}^n + \Delta t\,(a_{21} R_V^{(1)} + a_{22} R_V^{(2)})
\end{aligned}
$$

3. third step

$$
\begin{aligned}
\mathbf{W}^{(3)} &= \mathbf{W}^n + \Delta t\, F_W^{(2)} \\
\mathbf{V}^{(3)} &= \mathbf{V}^n + \Delta t\, F_V^{(2)} + \Delta t\,(a_{31} R_V^{(1)} + a_{32} R_V^{(2)} + a_{33} R_V^{(3)})
\end{aligned}
$$

4. fourth step

$$
\begin{aligned}
\mathbf{W}^{(4)} &= \frac{3}{4}\mathbf{W}^n + \frac{1}{4}\mathbf{W}^{(3)} + \frac{1}{4}\Delta t\, F_W^{(3)} \\
\mathbf{V}^{(4)} &= \frac{3}{4}\mathbf{V}^n + \frac{1}{4}\mathbf{V}^{(3)} + \frac{1}{4}\Delta t\, F_V^{(3)} \\
&+ \Delta t\,[(a_{41} - a_{31}/4) R_V^{(1)} + (a_{42} - a_{32}/4) R_V^{(2)} + (a_{43} - a_{33}/4) R_V^{(3)} + a_{44} R_V^{(4)}]
\end{aligned}
$$

5. fifth step

$$
\begin{aligned}
\mathbf{W}^{(5)} &= \frac{1}{3}\mathbf{W}^n + \frac{2}{3}\mathbf{W}^{(4)} + \frac{2}{3}\Delta t\, F_W^{(4)} \\
\mathbf{V}^{(5)} &= \frac{1}{3}\mathbf{V}^n + \frac{2}{3}\mathbf{V}^{(4)} + \frac{2}{3}\Delta t\, F_V^{(4)} \\
&+ \Delta t\,[(a_{51} - 2\,a_{41}/3) R_V^{(1)} + (a_{52} - 2\,a_{42}/3) R_V^{(2)} + (a_{53} - 2\,a_{43}/3) R_V^{(3)} + (a_{54} - 2\,a_{44}/3) R_V^{(4)} + a_{55} R_V^{(5)}]
\end{aligned}
$$

TABLE V: Tableau for the explicit (left) implicit (right) IMEX-SSP3(4,3,3) L-stable scheme

$$
\begin{array}{c|ccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1/2 & 0 & 1/4 & 1/4 & 0 & 0 \\
1 & 0 & 1/6 & 1/6 & 2/3 & 0 \\
\hline
 & 0 & 1/6 & 1/6 & 2/3 & 0
\end{array}
\qquad
\begin{array}{c|ccccc}
\alpha & \alpha & 0 & 0 & 0 & 0 \\
0 & -\alpha & \alpha & 0 & 0 & 0 \\
1 & 0 & 1-\alpha & \alpha & 0 & 0 \\
1/2 & \beta & \eta & 1/2-\beta-\eta-\alpha & \alpha & 0 \\
1 & 0 & 1/6 & 1/6 & 2/3 & 0 \\
\hline
 & 0 & 1/6 & 1/6 & 2/3 & 0
\end{array}
$$

$$\alpha = 0.24169426078821 \quad , \quad \beta = 0.06042356519705 \quad , \quad \eta = 0.12915286960590$$

TABLE VI: Tableau for the explicit (left) implicit (right) IMEX-SSP2(4,3,3) L-stable scheme

$$
\begin{array}{c|ccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1/2 & 0 & 1/4 & 1/4 & 0 & 0 \\
1 & 0 & 1/6 & 1/6 & 2/3 & 0 \\
\hline
 & 0 & 1/6 & 1/6 & 2/3 & 0
\end{array}
\qquad
\begin{array}{c|ccccc}
\alpha & \alpha & 0 & 0 & 0 & 0 \\
0 & -\alpha & \alpha & 0 & 0 & 0 \\
1 & 0 & 1-\alpha & \alpha & 0 & 0 \\
1/2 & a_{41} & a_{42} & a_{43} & \alpha & 0 \\
1 & 0 & 1/6 & 0 & 2/3 & 1/6 \\
\hline
 & 0 & 1/6 & 0 & 2/3 & 1/6
\end{array}
$$

$$a_{41} = \frac{1}{8\alpha}(2\alpha^2 + 2\alpha - 1) \quad , \quad a_{42} = \frac{1}{8\alpha}(-4\alpha^2 + 1) \quad , \quad a_{43} = \frac{1}{4}(-3\alpha + 1) \quad , \quad \alpha = 1/3 \quad .$$

E. The application to equations with linear stiff terms

Notice that we can split the vector of fields $\mathbf{U}$ in two parts, one containing some stiff terms and other which all the rhs can be treated explicitely, respectively $\mathbf{U} = (\mathbf{V}, \mathbf{W})$. So, in general the evolution equations can be written as

$$\partial_t \mathbf{W} = F_W(\mathbf{V}, \mathbf{W}) \tag{87}$$

$$\partial_t \mathbf{V} = F_V(\mathbf{V}, \mathbf{W}) + \frac{1}{\epsilon(\mathbf{W})} R_V(\mathbf{V}, \mathbf{W}) \quad . \tag{88}$$

where we have considered that the relaxation parameter ϵ can depend also on the $\mathbf{W}$ fields. The evolution procedure to compute each step $U^{(i)}$ can be split in two substeps

1. compute the explicit intermediate values $\{\mathbf{V}^*, \mathbf{W}^*\}$, that is,

$$
\begin{aligned}
\mathbf{W}^* = \mathbf{W}^n \; &+ \; \Delta t \sum_{j=1}^{i-1} \tilde{a}_{ij} F_W(\mathbf{U}^{(j)}) \\
\mathbf{V}^* = \mathbf{V}^n \; &+ \; \Delta t \sum_{j=1}^{i-1} \tilde{a}_{ij} F_V(\mathbf{U}^{(j)}) \\
&+ \; \Delta t \sum_{j=1}^{i-1} a_{ij} \frac{1}{\epsilon^{(j)}} R_V(\mathbf{U}^{(j)})
\end{aligned} \tag{89}
$$

   where we have defined $\epsilon^{(j)} = \epsilon(\mathbf{W}^{(j)})$.

2. compute the implicit part, involving only $\mathbf{V}$, by solving the implicit equation

$$
\begin{aligned}
\mathbf{V}^{(i)} &= \mathbf{V}^* + a_{ii} \frac{\Delta t}{\epsilon^{(i)}} R_V(\mathbf{V}^{(i)}, \mathbf{W}^{(i)}) \\
\mathbf{W}^{(i)} &= \mathbf{W}^*
\end{aligned} \tag{90}
$$

There are different ways to solve the equation in the second step (90). We will consider here only the simplest ones, which does not require solving any implicit equations.

TABLE VII: Tableau for the explicit (left) implicit (right) IMEX-SSP3(4,3,3) L-stable scheme

0	0	0	0	0	0	1/2	1/2	0	0	0	0
0	0	0	0	0	0	0	-1/2	1/2	0	0	0
1	0	1	0	0	0	1	0	1/2	1/2	0	0
1/2	0	1/4	1/4	0	0	1/2	1/8	0	-1/8	1/2	0
1	0	1/6	1/6	2/3	0	1	0	1/6	1/6	2/3	0
	0	1/6	1/6	2/3	0		0	1/6	1/6	2/3	0

1. *Assuming R_V linear in* $\mathbf{V}$

It will also be assumed that the stiff part can be written in the following way,

$$R_V(\mathbf{V},\mathbf{W}) = A(\mathbf{W})\mathbf{V} + S(\mathbf{W}) \tag{91}$$

In this way, the implicit equation can be solved just by inverting the matrix, namely

$$\begin{aligned}
\mathbf{V}^{(i)} &= [I - a_{ii}\,\frac{\Delta t}{\epsilon^{(i)}} A(\mathbf{W}^*)]^{-1}\ (\mathbf{V}^* + a_{ii}\,\frac{\Delta t}{\epsilon^{(i)}}\ S(\mathbf{W}^*)) \\
\mathbf{W}^{(i)} &= \mathbf{W}^*
\end{aligned} \tag{92}$$

2. *Linearizing R_V around* $\{\mathbf{V}',\mathbf{W}'\}$

Another option would be to linearize the stiff term around $\{\mathbf{V}',\mathbf{W}'\}$, namely

$$R_V(\mathbf{V}^{(i)},\mathbf{W}^{(i)}) \approx R_V(\mathbf{V}',\mathbf{W}') + \left(\frac{\partial R_V}{\partial \mathbf{V}}\right)_{\mathbf{V}',\mathbf{W}'} (\mathbf{V}^{(i)} - \mathbf{V}') + \left(\frac{\partial R_V}{\partial \mathbf{W}}\right)_{\mathbf{V}',\mathbf{W}'} (\mathbf{W}^{(i)} - \mathbf{W}') \tag{93}$$

Since the solution for $\mathbf{W}^{(i)}$ is known, we can consider only the case $\mathbf{W}' = \mathbf{W}^{(i)}$, which implies that

$$R_V(\mathbf{V}^{(i)},\mathbf{W}^{(i)}) \approx R_V(\mathbf{V}',\mathbf{W}^{(i)}) + \left(\frac{\partial R_V}{\partial \mathbf{V}}\right)_{\mathbf{V}',\mathbf{W}^{(i)}} (\mathbf{V}^{(i)} - \mathbf{V}') \tag{94}$$

By defining $A(\mathbf{V}',\mathbf{W}^{(i)}) \equiv \left(\frac{\partial R_V}{\partial \mathbf{V}}\right)_{\mathbf{V}',\mathbf{W}^{(i)}}$ and substituting the previous expansion (94) in (90), we obtain

$$\mathbf{V}^{(i)} = \mathbf{V}^* + a_{ii}\,\frac{\Delta t}{\epsilon^{(i)}}[R_V(\mathbf{V}',\mathbf{W}^{(i)}) + A(\mathbf{V}',\mathbf{W}^{(i)})(\mathbf{V}^{(i)} - \mathbf{V}')] \tag{95}$$

By adding and substracting $\mathbf{V}'$ on the right-hand-side, and rearranging the terms, it is obtained

$$[I - a_{ii}\,\frac{\Delta t}{\epsilon^{(i)}} A(\mathbf{V}',\mathbf{W}^{(i)})]\mathbf{V}^{(i)} = \mathbf{V}^* - \mathbf{V}' + a_{ii}\,\frac{\Delta t}{\epsilon^{(i)}} R_V(\mathbf{V}',\mathbf{W}^{(i)}) + [I - a_{ii}\,\frac{\Delta t}{\epsilon^{(i)}} A(\mathbf{V}',\mathbf{W}^{(i)})]\mathbf{V}' \tag{96}$$

In this way, the implicit equation can be solved explicitely by inverting the matrix $M \equiv [I - a_{ii}\,\frac{\Delta t}{\epsilon^{(i)}} A(\mathbf{V}',\mathbf{W}^{(i)})]^{-1}$, namely

$$\mathbf{V}^{(i)} = \mathbf{V}' + M[\mathbf{V}^* - \mathbf{V}' + a_{ii}\,\frac{\Delta t}{\epsilon^{(i)}}\ R_V(\mathbf{V}',\mathbf{W}^{(i)})] \tag{97}$$

III. DIFFERENT LIMITS OF THE RESISTIVE MHD

In this section we will explain two different limits of the previous resistive MHD equations, which allow for some simplifications in the evolution equations. The limits are formally obtained by setting the conductivities to infinite.

A. The ideal MHD

Let us assume that there lots of free electrons so $n_e e\xi$ is large, while at the same time $\xi|b| << 1$ (ie, the collision time τ is small compared with the electron Larmor period). In this limit the conductivity (128) reduces to

$$\sigma^{\mu\nu} \approx (n_e e^2 \tau/m) g^{\mu\nu} = \sigma_I g^{\mu\nu} \tag{98}$$

which describes accurately an isotropic highly conducting hot fluid when the magnetic field is not too large, since the electron density is high, the collision time is short but the product $n_e \tau$ is not small.

The well-known ideal-MHD limit of Ohm's law can be obtained by requiring the current to be finite even in the limit of infinite conductivity ($\sigma_I \to \infty$). This implies that the electric field measured by the comoving observer has to vanish,

$$e^\mu = 0 \longrightarrow E^i + \epsilon^{ijk} v_j B_k - (E^k v_k) v^i = 0 \quad . \tag{99}$$

Projecting this equation along $\mathbf{v}$ one finds that the electric field does not have a component along that direction and then from the rest of the equation one recovers the well-known ideal-MHD condition

$$E^i = -\epsilon^{ijk} v_j B_k \, , \tag{100}$$

stating that in this limit the electric field is orthogonal to both $\mathbf{B}$ and $\mathbf{v}$. Such a condition also expresses the fact that in ideal MHD the electric field is not an independent variable since it can be be computed via a simple algebraic relation from the velocity and magnetic vector fields.

The current I_ν becomes undetermined in the relation (**??**) and the redundant equations (14,15) may be used to compute it properly. The only non-redundant evolution equation for the magnetic field can be obtained by substituting (100) in (16), namely

$$\begin{aligned} \partial_t(\sqrt{\gamma} B^i) \;&+\; \partial_k[\sqrt{\gamma}\{(\alpha v^k - \beta^k) B^i - \alpha v^i B^k + \alpha \gamma^{ki} \phi\}] \\ &= \sqrt{\gamma}[-B^k \partial_k \beta^i + \phi \gamma^{ik} (\partial_k \alpha + \Gamma^j_{jk})] \end{aligned} \tag{101}$$

together with the conservation law evolution of the constraint violation ϕ,

$$\partial_t \phi \;+\; \partial_i(-\beta^i \phi + \alpha B^i) = B^i \partial_i \alpha - \alpha B^i \Gamma^k_{ki} - \phi \partial_i \beta^i - \alpha \kappa \phi \tag{102}$$

The electric field can be removed also from the conserved fields in order to convert to primitive quantities in a more robust way,

$$\tau = hW^2 + B^2 - p - D - \frac{1}{2}[(B^k v_k)^2 + \frac{B^2}{W^2}] \quad , \tag{103}$$

$$S_i = [hW^2 + B^2] v_i - (B^k v_k) B_i \quad . \tag{104}$$

The transformation from conserved to primitive is simplified by the elimination of the electric field as an independent variable. The first step is to notice that the scalar product $\mathbf{B} \cdot \mathbf{v}$ can be solved from the contraction of (104) with $\mathbf{v}$,

$$v_i B^i = \frac{S_i B^i}{hW^2} \tag{105}$$

Using this relation the scalar product $S^i S_i$ can be solved for the Lorentz factor again, obtaining

$$c \equiv \frac{1}{W^2} = 1 - \frac{x^2 S^2 + (2x + B^2)(S_i B^i)^2}{x^2 (x + B^2)^2} \tag{106}$$

where we have defined $x \equiv hW^2$.

The second step involves to use some particular Equation Of State. We will use here an hybrid EoS with a cold and thermal components

$$\begin{aligned} \epsilon &= \epsilon_{cold} + \epsilon_{th} \quad , \quad \epsilon_{cold} = \epsilon_{cold}(\rho) \\ p &= p_{cold} + p_{th} \quad , \quad p_{cold} = p_{cold}(\rho) \quad , \quad p_{th} = (\Gamma - 1)\rho \epsilon_{th} \end{aligned} \tag{107}$$

where Γ stands for the thermal component Γ_{th}. The enthalpy h, by using the ideal gas EOS, is just

$$h = \rho + \rho \epsilon_{cold} + p_{cold} + \frac{\Gamma}{\Gamma - 1} p_{th} \quad , \tag{108}$$

so, by inverting the relation we can write that

$$p = \frac{\Gamma - 1}{\Gamma} \left[\frac{x}{W^2} - \frac{D}{W} - p_{cold} - \rho\epsilon_{cold} \right] \quad . \tag{109}$$

A direct substitution of (105) and (109) in the definition of τ (103) leads to the final expression

$$\begin{aligned} f(x) &= \left[1 - \frac{(\Gamma-1)}{\Gamma W^2}\right] x + \left[\frac{\Gamma - 1}{\Gamma W} - 1\right] D + \left[1 - \frac{1}{2W^2}\right] B^2 - \frac{1}{2x^2}(S_i B^i)^2 - \tau - \frac{1}{\Gamma}\left[p_{cold} - (\Gamma-1)\rho\epsilon_{cold}\right] \quad (110) \\ &= \left[1 - \frac{(\Gamma-1)c}{\Gamma}\right] x + \left[\frac{(\Gamma-1)\sqrt{c}}{\Gamma} - 1\right] D + \left[1 - \frac{c}{2}\right] B^2 - \frac{1}{2x^2}(S_i B^i)^2 - \tau - \frac{1}{\Gamma}\left[p_{cold} - (\Gamma-1)D\sqrt{c}\epsilon_{cold}\right] \end{aligned}$$

The procedure for a tabulated EoS shares several steps. Let us write them all for completeness. Remember that we solved for $x \equiv hW^2$, with $h = \rho(1+\epsilon)+p$. First we calculate the Lorentz factor, as a function of conserved quantities and x by computing S^2

$$W^{-2}(x) = 1 - \frac{x^2 S^2 + (2x + B^2)(S_i B^i)^2}{x^2(x+B^2)^2} \tag{111}$$

Afterwards, the density is obtained easily from

$$\rho(x) = \frac{D}{W(x)} \tag{112}$$

On the other hand, we can compute the internal energy by using the energy equation (to solve for the pressure p) and the definition of the entalpy, resulting into the equation

$$\epsilon(x) = (W(x) - 1) + \frac{x}{DW(x)}\left(1 - W^2(x)\right) + \frac{W(x)}{D}\left(\tau - B^2\right) + \frac{W(x)}{2D}\left[\left(\frac{S_k B^k}{x}\right)^2 + \frac{B^2}{W^2(x)}\right] \tag{113}$$

Notice that it is easy to check that $\rho(x)$ is within the table, and set it to either its lower/upper bound if not. For $\epsilon(x)$ it is a bit more complicated, but still straightforward. The lower bound can be obtained by setting $\epsilon_{min} = (\rho(x), T_{min}, Y_e)$, being T_{min} the minimum temperature in the table. Same argument holds for the upper bound. Once we have make sure that both $(\rho(x), \epsilon(x))$ are within the table limits, we can actually compute the pressure

$$p(x) = p(\rho(x), \epsilon(x), Y_e) \tag{114}$$

The final equation to solve is the definition of $x = hW^2$, that is,

$$f(x) = x - \left[\rho(x)\left(1 + \epsilon(x)\right) + p(x)\right] W^2(x) \tag{115}$$

Unfortunately, this unknown x can be really small outside the star, and the mathematical operations involved be under round-off error. Therefore, it is convenient to rescale both the unknown and the conserved fields, so that all of them are order $O(1)$, in the following way:

$$x \equiv hW^2/D \quad , \quad q \equiv \tau/D \quad , \quad r \equiv S^2/D^2 \quad , \quad s \equiv B^2/D \quad , \quad t \equiv (S_i B^i)/D^{3/2} \tag{116}$$

Within these fields , the equation 111 reduces to

$$W^{-2}(x) = 1 - \frac{x^2 r + (2x+s)t^2}{x^2(x+s)^2} \tag{117}$$

while that the internal energy 113 is just

$$\epsilon(x) = (W(x) - 1) + \frac{x}{W(x)}\left(1 - W^2(x)\right) + W(x)\left(q - s + \frac{1}{2}\left[\left(\frac{t}{x}\right)^2 + \frac{s}{W^2(x)}\right]\right) \tag{118}$$

The final equation to solve is the definition of $x = hW^2/D$, that is,

$$f(x) = x - \left(1 + \epsilon(x) + \frac{p(x)}{\rho(x)}\right) W(x) \tag{119}$$

1. Causal structure

The hyperbolic structure of the ideal MHD equations is much more involved that in the non-magnetic field case. As in the classical MHD case, there are seven physical waves: two Alfvn waves, two fast and two slow magnetosonic waves and one entropy wave. The characteristic structure of these equations in the fully relativistic case hyperbolic system was studied by Anile [**?**]. It was found that only the entropic waves and the Alfvn waves can be explicitly written in closed form, while the other four velocities are found by solving a quartic polynomial:

- **one entropic wave**

$$\lambda_{entropic} = -\beta^n + \alpha v^n \quad , \tag{120}$$

  which reduces to one of the material waves for vanishing magnetic field.

- **two Alfven waves**

$$\lambda^{\pm}_{alfven} = -\beta^n + \alpha v^n - \frac{B^n}{hW^2 + B^2}\left[B^k v_k \pm \sqrt{h(B^k v_k)^2 + \frac{B^2}{W^2}}\right] \tag{121}$$

  In the case without magnetic field these modes dissapear.

- **four magnetosonic waves**: the two solutions with maximum and minimum speeds are called fast magnetosonic waves, while the two solutions in between the slow magnetosonic waves. Their speeds are given by the solution of the following quartic characteristic equation which can be solved numerically.

$$\begin{aligned} 0 &= hW^4(1-c_s^2)(\alpha v^n - \beta^n - \lambda)^4 \\ &+ [(\beta^n+\lambda)^2 - \alpha^2]\,[(\alpha v^n - \beta^n - \lambda)^2(hW^2c_s^2 + B^2 + W^2(B^k v_k)^2) \\ &\qquad - c_s^2(W(B^k v_k)(\alpha v^n - \beta^n - \lambda) + \alpha\frac{B^n}{W})^2] \end{aligned} \tag{122}$$

  In the case of $B = 0$ the fast magnetosonic modes reduce to the acoustic waves while the slow ones reduce to two of the material waves.

The seven waves can be ordered as follows

$$\lambda^{+}_{fast} \geq \lambda^{+}_{alfven} \geq \lambda^{+}_{slow} \geq \lambda_{entropic} \geq \lambda^{-}_{slow} \geq \lambda^{-}_{alfven} \geq \lambda^{-}_{fast} \tag{123}$$

A very useful upper bound for fast waves (which have the maximum speed) can be found by considering the degenerate case of normal propagation [**?**]. In that case there is an analytical expression for the two fast magnetosonic waves, namely

$$\lambda^{n\pm}_{fast} = -\beta^n + \frac{\alpha}{1-v^2a^2}\left[(1-a^2)v^n \pm \sqrt{a^2(1-v^2)[(1-v^2a^2)-(1-a^2)v_n^2]}\right] \tag{124}$$

$$\lambda^{x\pm}_{fast} = -\beta^x + \frac{\alpha}{1-v^2a^2}\left[(1-a^2)v^x \pm \sqrt{a^2(1-v^2)[(1-v^2a^2)\gamma^{xx}-(1-a^2)(v^x)^2]}\right] \tag{125}$$

where

$$a^2 = c_s^2 + c_a^2 - c_s^2 c_a^2 \tag{126}$$

The sound speed c_s can be written easily for the ideal EOS, while the Alfven speed c_a can be written easily in terms of the comoving magnetic four-vector b^μ,namely

$$c_s^2 = \frac{\Gamma p}{h} \quad , \quad c_a^2 = \frac{b^2}{h+b^2} \quad , \quad b^2 = B^2/W^2 + (v_k B^k)^2 \tag{127}$$

where we have used that $e^\mu = 0$ in the ideal MHD, so that b^2 is a lorentz invariant which is valid also in the frame of the eulerian observers, namely $b^2 = B^2 - E^2$.

B. The force-free limit

Let us assume now that the magnetic field is large and the fluid is not so hot (ie, the collision time is large), so $\xi^2 b^2 >> 1$. In this case the second term dominates over the first one in eq. (128) and it can be reduced to

$$\sigma^{\mu\nu} \approx (n_e e \xi / b^2) b^\mu b_\nu = \sigma_B g^{\mu\nu} \quad . \tag{128}$$

which approximates reasonably well highly-conducting cold fluids with low densities. In this case, the Lorentz force computed in terms of the comoving fields is just

$$F^{\mu\nu} I_\nu = \tilde{q} e^\mu + \sigma_B u^\mu (b_\nu e^\nu)^2 \ . \tag{129}$$

The analogous of the ideal MHD limit can be obtained by requiring the current to be finite even in the limit of infinite conductivity ($\sigma_B \to \infty$). Allowing for (60), this implies that the electric and magnetic fields must be perpendicular, that is

$$b^\mu e_\mu = B^\mu E_\mu = \gamma^{ij} \, B_i E_j = 0 \, . \tag{130}$$

This condition amounts to the vanishing of the second term in the Lorentz force (129), namely

$$F^{\mu\nu} I_\nu = \tilde{q} e^\mu . \tag{131}$$

Since the charge density and the electric field in the comoving frame are usually small, we can assume that in general the Lorentz force will be neglectible in this limit. This is the reason why it is called force-free limit, since the Lorentz force is very small.

The force-free limit is usually introduced in a different way. In the magnetospheres of the neutron stars or black holes the density is so low so that even moderate magnetic fields stresses will dominate over the pressure gradients. Mathematically, this means that the stress-energy tensor is mainly dominated by the electromagnetic part,

$$T_{\mu\nu} \approx T^{em}_{\mu\nu} \tag{132}$$

which conservation law implies that the Lorentz force in neglectible

$$\nabla_\nu T^{\mu\nu}_{em} = -F^{\mu\nu} I_\nu \approx 0 \tag{133}$$

Let us study in more detail the last relation. The spatial projection, written in terms of Eulerian observers, is just

$$q E^i + \epsilon^{ijk} J_j B_k = 0 \tag{134}$$

which leads to the relation (130) after contracting with B_i. This is a vague evidence to show how the force-free equations are recovered from a infinite conductivity limit, in a similar way to the ideal MHD.

By performing the vector and the scalar product of the previous equation (134) with $\mathbf{B}$ one can obtain that

$$J^i = q \frac{\epsilon^{ijk} E_j B_k}{B^2} + (J^k B_k) \frac{B^i}{B^2} \quad , \quad E^i B_i = 0 \quad . \tag{135}$$

On the other hand, by using the Maxwell equations on can compute the time evolution of the previous equation. The relation $\partial_t (E^i B_i) = 0$ imposes a equation for $J^i B_i$ which can be substituted into (135).

In this case, the magnetic field evolution is given by the standard Maxwell equation. On the other side, the electric field can not be computed properly since the indetermined current. Instead, we can use the conservation of the momentum, which in the force-free limit has reduced to the Poynting vector

$$S^i = \epsilon^{ijk} E_j B^k \tag{136}$$

After evolving the Poynting flux, the electric field can be reconstructed by means of the force-free condition (130)

$$E^i = -\frac{1}{B^2} \epsilon^{ijk} S_j B_k \tag{137}$$

As in the case of the ideal MHD, neither the charge density q nor the scalar field ψ appears on the equations so they do not have to be evolved. Notice that in this limit we have neglected the effects of the fluid, so it does not have to be evolved at all.

Finally, just mention that the characteristic speeds are given by two Alfven waves and two magnetosonic waves, moving at the speed of light. Thus, we can still use the expression (124) with $a = 1$.

IV. THE RESISTIVE MAGNETOHYDRODYNAMICS EQUATIONS

In order to solve the full Maxwell and Hydrodynamic equations, we have to use the IMEX Runge-Kuttas introduced before in order to deal with the very short timescales, which are present when the conductivity is high. For this particular set of equations, the fields to be evolved can be splitt into stiff $\mathbf{V} = \{\mathbf{E}\}$ and non-stiff ones $\mathbf{W} = \{\mathbf{B}, \psi, \phi, q, \tau, \mathbf{S}, D\}$. The evolution for the potentially stiff field is

$$\partial_t(\sqrt{\gamma}E^i) \;+\; \partial_k[-\beta^k\sqrt{\gamma}E^i - \alpha\epsilon^{ikj}\sqrt{\gamma}B_j] = -\sqrt{\gamma}E^k(\partial_k\beta^i) - \alpha\sqrt{\gamma}\gamma^{ij}\partial_j\psi - \alpha\sqrt{\gamma}J^i \tag{138}$$
$$\tag{139}$$

where we can use a generalized tensorial Ohm law

$$J^i = J^i_{exp} + J^i_{stiff} = qv^i + \frac{\sigma}{1+\xi^2 b^2}[\mathcal{E}^i + \xi^2(E^k B_k)\mathcal{B}^i] \quad , \tag{140}$$

The rhs of this evolution equation can be splitted in potentially stiff terms and regular ones,

$$\partial_t(\sqrt{\gamma}\mathbf{E}) \;=\; F_E + (\sqrt{\gamma}R_E) \;\; . \tag{141}$$

where we have introduced the factor $1/\epsilon$ on the definition of R_E, so finally we obtain

$$F_E \;=\; -\partial_k[-\beta^k\sqrt{\gamma}E^i - \alpha\epsilon^{ikj}\sqrt{\gamma}B_j] - \sqrt{\gamma}E^k(\partial_k\beta^i) - \alpha\sqrt{\gamma}\gamma^{ij}\partial_j\psi - \alpha\sqrt{\gamma}qv^i \tag{142}$$
$$R_E \;=\; -\alpha J^i_{stiff} = -\alpha\frac{\sigma}{1+\xi^2 b^2}[\mathcal{E}^i + \xi^2(E^k B_k)\mathcal{B}^i] \quad . \tag{143}$$

Notice that q also has stiff terms in the sources. In principle they are not really stiff, since there are no terms like σq. However, they can be pretty big and small errors in the reconstruction of the variables will lead to large errors in the fluxes and in the charge. A dirty way of curing this is by removing the charge from the list of evolved variables by using the constraint $q = \nabla_i E^i$. The only place where it appears is in a term of the current qv^i, since the divergence cleaning equation becomes trivial. If we substitute the constraint there we get that the system of equations is still hyperbolic, although there are two equations less (i.e., the one for the ψ and for q). Another approach, much more sofisticated, is to compute the currents J^i at the same time that the stiff sources are computed, and reconstruct them directly. It is ensured that the fluxes of q at the interfaces will be finite and well defined. This is the numerical approach that we will follow.

A. The inversion from conserved to primitive

In order to evolve this system numerically, the fluxes $\{\mathbf{F}_\tau, \mathbf{F_S}, \mathbf{F_D}\}$ have to be computed in each timestep. This implies that the primitive quantities $u_{primitive} = \{\rho,\ p,\ \mathbf{v},\ \mathbf{E},\ \mathbf{B}\}$ has to be recovered from the conserved fields $u_{conserved} = \{D,\ \tau,\ \mathbf{S},\ \mathbf{E},\ \mathbf{B}\}$. Due to the enthalpy and the Lorentz factor these quantities are related by complicated equations which become trascendental except for specially simple equation of state (EOS).

Notice that the solution of the conserved quantities $\{D,\ \tau,\ \mathbf{S},\ \mathbf{B}\}$ at $t = (n+1)\Delta t$ is obtained by evolving their evolution equations. However, we only have an approximate solution for the electric field $\{\mathbf{E}^*\}$ since only the non-stiff terms have been considered on its evolution equation. As it was shown in the previous section, the solution $\mathbf{E}$ of the complete evolution equation involves inverting a implicit equation, and as a result the complete solution is a function of the velocity $\mathbf{v}$ and the fields $\{\mathbf{B}, \mathbf{E}^*\}$. In the case where $\xi = 0$ the stiff part is linear in $\mathbf{E}$, so it can be developed and solve explicitely like

$$\mathbf{E} = M^{-1}(\mathbf{v})\ [\mathbf{E}^* + a_{ii}\ \frac{\Delta t}{\epsilon^{(i)}}\ S_E(\mathbf{v}, \mathbf{B})] \tag{144}$$

In general, for $\xi \neq 0$, we can still linearize around the solution. There are two different ways of doing this. The straightest one, but not the best, is by means of the Newton-Raphson method. Assuming that the other fiels (ie, not only the conserved but also the primitive) are given, we can expand the function

$$f_E = \mathbf{E} - \mathbf{E}^* - a_{ii}\ \Delta t\ R_E \tag{145}$$

which should vanish for the physical solutions. The expansion around an initial guess $\mathbf{E}'$, setting the final point to be the solution, leads to the following equation

$$\mathbf{E} = \mathbf{E}' - M^{-1}\ f_{E'} \quad , \quad M \equiv \frac{\partial f_E}{\partial E} \quad . \tag{146}$$

Another way of performing such a linearization is to linearize the stiff part and then invert the system analytically, like in the $\xi = 0$ case.

$$\mathbf{E} = \mathbf{E}' + M[\mathbf{E}^* - \mathbf{E}' + a_{ii}\ \frac{\Delta t}{\epsilon^{(i)}}\ R_E(\mathbf{v},\mathbf{B},\mathbf{E}')]\ ,\ M \equiv \left[I - a_{ii}\ \frac{\Delta t}{\epsilon^{(i)}} \frac{\partial R_E}{\partial E}\right]^{-1}_{\mathbf{E}'} \quad . \tag{147}$$

The problem is that the velocity is a primitive quantity, so it is yet not know at the time $t = (n+1)\Delta t$. The inversion of the conserved quantities into the primitive ones involves all the fields, included the electric field E. This way, the only way to keep consistency is by performing the evolution of the stiff part and the inversion from conserved to primitive quantities at the same time. We will describe in the following the iterative procedure to evolve the stiff part and recover the primitive fields for an hybrid EOS.

1. let us consider that the guess for electric field $\bar{\mathbf{E}}$ is given by one of the following ways:
    - the values of the previous time level $\bar{\mathbf{E}} = \mathbf{E^n}$
    - the values from the ideal MHD case $\bar{\mathbf{E}} = \mathbf{E_{ideal}}$. This implies solve for the ideal MHD and then set $\mathbf{E_{ideal}} = -\mathbf{v} \times \mathbf{B}$.
    - the values of the explicit and previous implicit parts $\bar{\mathbf{E}} = \mathbf{E^n_*}$.
    - the safest values $\bar{\mathbf{E}} = 0$.
2. we will solve for the combination $x \equiv hW^2$ (except in the case of the isentropic solver where we will solve for $x \equiv \rho$, see the appendix A). Let us assume that its value is given also by the previous step of the iterative process. Again, at the first step the initial guess will be given by $\bar{x} = x^n$. It will be checked that $W^2 > 0$ for this guess $\bar{x}$, otherwise the guess will be increased by a factor until $W^2 > 0$ is fulfilled. This is just making sure that the function to solve is real at $\bar{x}$.
3. compute the energy and momentum densities without the electromagnetic contribution:
$$\begin{aligned} \tilde{\tau} &= \tau - \frac{1}{8\pi}(E^k E_k + B^k B_k) \quad , & (148)\\ \tilde{S}_i &= S_i - \frac{1}{4\pi}\epsilon_{ijk} E^j B^k & (149)\end{aligned}$$
The norm of $\tilde{S}_i$ leads to the following relation
$$W^2 = \frac{x^2}{x^2 - \tilde{S}^i \tilde{S}_i} \quad , \quad c \equiv \frac{1}{W^2} = 1 - \frac{\tilde{S}^2}{x^2} \tag{150}$$
4. The hybrid EOS splits the pressure in two contributions. The cold part corresponds to a polytrope, while the thermal part is associated with an ideal gas EOS. It can be written then as
$$\begin{aligned} p &= K\rho^\Gamma + (\Gamma_{th} - 1)\rho\epsilon_{th} \quad , & (151)\\ \epsilon &= \epsilon_{th} + \epsilon_{cold} = \epsilon_{th} + \frac{K}{\Gamma - 1}\rho^{\Gamma-1} \quad . & (152)\end{aligned}$$
The pressure can be expressed in terms of only the primitive fields by manipulating these expressions,
$$p = \frac{\Gamma - \Gamma_{th}}{\Gamma - 1} K\rho^\Gamma + (\Gamma_{th} - 1)\rho\epsilon \tag{153}$$
By combining this equation with the definition of the enthalpy h we can write the pressure as a function of the conserved quantities and the unknown x,
$$p = \frac{\Gamma_{th} - 1}{\Gamma_{th}} \left(\frac{x}{W^2} - \frac{D}{W}\right) + \frac{\Gamma - \Gamma_{th}}{\Gamma_{th}(\Gamma - 1)} K \left(\frac{D}{W}\right)^\Gamma \tag{154}$$
5. substituting the previous expression (154) in the definition of $\tilde{\tau}$ (150) we obtain
$$f(x) = [1 - \frac{(\Gamma_{th} - 1)}{W^2\Gamma_{th}}]x + [\frac{\Gamma_{th} - 1}{\Gamma_{th} W} - 1]D + \frac{\Gamma_{th} - \Gamma}{\Gamma_{th}(\Gamma - 1)} K \left(\frac{D}{W}\right)^\Gamma - \tilde{\tau} \quad . \tag{155}$$

The derivative of the function $f(x)$ needed for the Newton-Raphson solver can be computed analytically. For the ideal EOS case it is just

$$f'(x) = 1 - \frac{2(\Gamma-1)\tilde{S}^2}{\Gamma x^2} - \frac{(\Gamma-1)c}{\Gamma} + \frac{(\Gamma-1)D\tilde{S}^2}{\sqrt{c}\Gamma x^3} \tag{156}$$

6. solve numerically the eq. (110) by means of an iterative Newton-Raphson solver, so the solution in the iteration $m+1$ can be computed as

$$x_{(m+1)} \ = \ x_{(m)} - \frac{f(x_{(m)})}{f'(x_{(m)})} \quad . \tag{157}$$

7. For the next iterations we need to compute the updated velocities, which can be obtained from

$$v_i = \frac{\tilde{S}_i}{x} \quad , \quad W^2 = \frac{x^2}{x^2 - \tilde{S}^2} \quad , \quad \rho = \frac{D}{W} \quad . \tag{158}$$

8. the electric field will be updated with the new velocities and densities following eq. (144). Writting this equation in components, we have to solve

$$E^i \ = \ E^i_* + a_{ii} \ \Delta t \ R^i_E \quad , \tag{159}$$

$$R^i_E \ = \ -\alpha \frac{\sigma}{1+\xi^2 b^2}[\mathcal{E}^i + \xi^2 (E^k B_k)\mathcal{B}^i] \tag{160}$$

or, by defining the shortcut $\tilde{\sigma} \equiv a_{ii}\, \Delta t \, \alpha \, \sigma/(1+\xi^2 b^2)$, like

$$E^i = E^i_* - \tilde{\sigma}[\mathcal{E}^i + \xi^2(E^k B_k)\mathcal{B}^i] \quad . \tag{161}$$

- The case $\xi = 0$ is relatively easy. After some manipulations, it can be written like

$$[\delta^i_k + \tilde{\sigma}(\delta^i_k - v_k v^i)]E^k = E^i_* + \tilde{\sigma} S^i \quad , \quad S^i = -W \epsilon^{ijk} v_j B_k \quad . \tag{162}$$

  This equation can be solved directly by inverting the matrix. The result is

$$E^i = M^i_k [E^k_* + \tilde{\sigma} S^k] \quad , \quad M^i_k = \frac{\tilde{\sigma} v_k v^i W^2 + \delta^i_k(\tilde{\sigma} + W^2)}{\tilde{\sigma}(1 + W^2 + \tilde{\sigma}) + W^2} \quad . \tag{163}$$

- The case $\xi \neq 0$ is more complicated, and will involve an expansion around some initial guess. One way to solve it is by defining a function

$$f^i_E = E^i - E^i_* + \tilde{\sigma}[\mathcal{E}^i + \xi^2(E^k B_k)\mathcal{B}^i] = 0 \quad . \tag{164}$$

  and then solving with a Newton-Raphson method the function. The Jacobian M is constructed by varying only the electric field, so we assume implicitely that in this step all the other variables are fixed. Then we can solve numerically the eq. (164) by means of an iterative Newton-Raphson solver, so the solution in the iteration $m+1$ can be computed as

$$E^i_{(m+1)} = E^i_{(m)} - \lambda \, \left(M^{-1}_{(m)}\right)^i_j \, f^j_{E_{(m)}} \quad . \tag{165}$$

  where λ is a parameter which can be tuned in order to obtain global convergence of the method. It start from a value $\lambda = 1$, and it is decreased to half of its value if the error is larger than in the previous step (ie, if the solution does not converge).

- Another option for the case $\xi \neq 0$ is to linearize the stiff terms around some initial guess and then solve the system by inverting again the matrix. Following eq. (147) using bars instead of tildes,

$$E^i = \bar{E}^i + M[E^{*i} - \bar{E}^i + a_{ii} \ \Delta t \ R^i_E] = \bar{E}^i + M\left[E^{*i} - \bar{E}^i - \tilde{\sigma}\{\mathcal{E}^i + \xi^2(E^k B_k)\mathcal{B}^i\}\right] \tag{166}$$

In order to construct the matrix M we need to compute the derivatives of the stiff part with respect to the electric field, namely

$$A \equiv \frac{\partial R_E^i}{\partial E^j} = -\frac{\alpha\sigma}{1+\xi^2 b^2}\left[\frac{\partial \mathcal{E}^i}{\partial E^j} + \xi^2\{(\delta_j^k B_k)\mathcal{B}^i + (E^k B_k)\frac{\partial \mathcal{B}^i}{\partial E^j}\}\right] - [\mathcal{E}^i + \xi^2 (E^k B_k)\mathcal{B}^i]\alpha\sigma \frac{\partial}{\partial E^j}\left(\frac{1}{1+\xi^2 b^2}\right) \quad (167)$$

By using the definitions (66) it is easy to compute

$$\frac{\partial \mathcal{E}^i}{\partial E^j} = W(\delta_j^i - v^i v_j) \quad , \quad \frac{\partial \mathcal{B}^i}{\partial E^j} = -W\epsilon^{irs} v_r \gamma_{js} \quad , \quad \frac{\partial}{\partial E^j}\left(\frac{1}{1+\xi^2 b^2}\right) = \frac{2W\xi^2 \epsilon_{jnr} b^n v^r}{(1+\xi^2 b^2)^2} \quad (168)$$

where we have used that

$$\frac{\partial b^2}{\partial E^j} = 2b^n \frac{\partial b_n}{\partial E^j} = 2b^n(-W\epsilon_{nrs} v^r \delta_j^s) = -2W\epsilon_{jnr} b^n v^r \quad . \quad (169)$$

It is now possible to construct the matrix to be inverted

$$M^{-1} = \delta_j^i + \tilde{\sigma} W \left\{\delta_j^i - v^i v_j + \xi^2 [\mathcal{B}^i B_j / W - (E^k B_k)\epsilon^{irs} v_r \gamma_{js}] + \frac{2\xi^2}{1+\xi^2 b^2}\epsilon_{jrs} b^r v^s [\mathcal{E}^i + \xi^2 (E^k B_k)\mathcal{B}^i]\right\} \quad (170)$$

This Ohm's law, although it is quite general, it still suffers from a coupling to the velocity in regions with high magnetizations. Additionally, the parameter ξ is not the most convenient to describe these regions, since it would have to increase in order to balance the natural decrease of the magnetic field with the distance to the source. For these reasons we will postulate the following Ohm's law

$$J^i = q[(1-F)v^i + F v_d^i] + \frac{\sigma}{1+\zeta^2}\left[\mathcal{E}^i + \zeta^2\{(E^k B_k)B^i + \chi(E^2 - B^2)E^i\}/B^2\right] \quad , \quad (171)$$

where F is a function which vanishes inside the star and it is unity outside, with a smooth transition from inner to the outer region. We have chosen

$$F(\rho\,,\rho_o) = \frac{2}{1+e^{2\,K\,(\rho-\rho_o)}} \quad (172)$$

where typically we adopt $K \approx 0.001/\rho_{atm}$ and $\rho_o \approx 200 - 2000\,\rho_{atm}$, being ρ_{atm} the value for the density of the magnetosphere. Notice that one can reinterpret the parameter ζ in terms of the old ξ (i.e., $\xi^2 b^2 \equiv \zeta^2$). This definition is more convenient as it is constant in the magnetosphere. I am taking $\zeta = F\sigma$, with $\sigma \approx 10^5$. In this way we ensure that the interior of the star (i.e., $F=0$) is dominated by a large isotropic conductivity, reducing the system of equations to the ideal MHD limit. The exterior of the star (i.e., $F=1$) is dominated by the anisotropic term which enforces the force-free limit. The condition $B^2 > E^2$ is imposed through the new terms proportional to the conductivity χ, which only non-zero when $E^2 - B^2 > 0$. In order to not overdamp the electric field, this conductivity is estimated at every step as the characteristic time for $E^2 - B^2$ to decay to zero, namely, $\chi \approx (\alpha\sqrt{\gamma}\sigma a_{ii}\Delta t)^{-1}$. Finally, we have also defined the drift velocity as $v_d^i \equiv \epsilon^{ijk} E_j B_k / B^2$. The stiff part will be now

$$R_E = -\alpha J_{stiff}^i = -\alpha \frac{\sigma}{1+\zeta^2}\left[\mathcal{E}^i + \zeta^2\{(E^k B_k)B^i + \chi(E^2 - B^2)E^i\}/B^2\right] \quad , \quad (173)$$

We define the shortcut $\tilde{\sigma} \equiv a_{ii}\,\Delta t\,\alpha\,\sigma/(1+\zeta^2)$, and write the matrix to be inverted as

$$M^{-1} = \delta_j^i + \tilde{\sigma}\left[W(\delta_j^i - v^i v_j) + \zeta^2\left\{B^i B_j + \chi[2E^i E_j + \delta_j^i (E^2 - B^2)]\right\}/B^2\right] \quad (174)$$

9. After we have obtained the final x by requiring $f(x) \le f_{thr}$ and $c - 1 + \tilde{S}^2/x^2 \le c_{thr}$, we can obtain all the primitive fields with the following relations

$$W^2 = \frac{x^2}{x^2 - \tilde{S}^2} \quad , \quad h = \frac{x}{W^2} \quad , \quad (175)$$

$$\rho = \frac{D}{W} \quad , \quad p = \frac{\Gamma - 1}{\Gamma}(h - \rho) \quad , \quad v_i = \frac{\tilde{S}_i}{x} \quad . \quad (176)$$

Sometime the solver for the hybrid EoS (or only ideal gas EoS) will fail because the energy density is too small, leading to a negative $x = hW^2$ which implies a negative pressure. In these cases one could consider an isentropic process (i.e., the entropy does not change). Remember that the polytropic EoS is written without any loss of generality as $p = K(S)\rho^\Gamma$, so that in an isentropic process $K(S) = cte.$ and $p = p(\rho)$. Since the internal energy is then also a function of the density (i.e., $\rho\epsilon = p/(\Gamma - 1)$), there is an overdeterminacy in the conserved quantities. We can take advantage of this by neglecting the equation for the energy density when recovering the primitive fields from the conserved ones.

1. let us consider that the guess for electric field $\bar{\mathbf{E}}$ is given as in the ideal MHD case.
2. we will solve for the combination $x \equiv \rho$, with a initial guess from the previous step.
3. compute the momentum densities without the electromagnetic contribution:
$$\tilde{S}_i = S_i - \frac{1}{4\pi}\epsilon_{ijk}E^j B^k \tag{177}$$
The norm of $\tilde{S}_i$ leads to the following equation for ρ
$$f(x) = h^2 W^2 (W^2 - 1) - \tilde{S}^2 \tag{178}$$
since $W = D/x$ and $h = x + \frac{\Gamma K}{\Gamma - 1} x^\Gamma$. The derivative of the function $f(x)$ needed for the Newton-Raphson solver can be computed analytically.
$$f'(x) = 2hh'W^2(W^2 - 1) + 2h^2 W W'(2W - 1) \tag{179}$$
where $h' = 1 + \frac{\Gamma^2 K}{\Gamma - 1}\rho^{\Gamma - 1}$ and $W' = -D/\rho^2$.
4. For the next iterations we need to compute the updated velocities, which can be obtained from $v_i = \frac{\tilde{S}_i}{hW^2}$
5. the electric field will be updated with the new velocities and densities following eq. (144). The rest of the process is the same as in the ideal EoS case. Note that one can also use the isentropic solver in the MHD limit, although the equations are more involved in that case because they are written in terms of S_i instead of $\tilde{S}_i$. The norm of S_i leads to the following equation for ρ
$$f(x) = (hW^2 + B^2)^2 (1 - \frac{1}{W^2}) - \frac{(2hW^2 + B^2)}{(hW^2)^2}(S_i B^i)^2 - S^2 \tag{180}$$
and after few lines of calculations, the derivative is just
$$\begin{aligned} f'(x) &= 2(hW^2 + B^2)(h'W^2 + 2hWW')(1 - \frac{1}{W^2}) + 2(hW^2 + B^2)^2 \frac{W'}{W^3} \\ &+ 2\frac{(S_i B^i)^2}{hW^2}\left[(hW^2 + B^2)(W^2 h' + 2hWW')\right] \end{aligned} \tag{181}$$

Appendix A. Exact solutions involving magnetic fields

Let us consider some exact solutions of the Maxwell equations either considering the test field case (ie, the EM fields does not change the geometry) or the full Einstein-Maxwell equations. The simplest solution is the Reisner-Nordstrom charged black hole, which in the Kerr-Schild coordinates can be written as:

$$\begin{aligned} \alpha &= \sqrt{\frac{r^2}{r^2 + 2Mr - Q^2}} \;\;,\;\; \beta^r = \frac{2Mr - Q^2}{r^2 + 2Mr - Q^2} \;\;,\;\; \gamma_{rr} = 1 + \frac{2Mr - Q^2}{r^2} \;\;,\;\; \gamma_{\theta\theta} = r^2 \;\;,\;\; \gamma_{\phi\phi} = r^2 sin^2\theta \\ K_{rr} &= -\frac{(Mr - Q^2)(\Delta + r^2)}{r^4 \Delta^{1/2}} \;\;,\;\; K_{\theta\theta} = \frac{2Mr - Q^2}{\Delta^{1/2}} \;\;,\;\; K_{\phi\phi} = (2Mr - Q^2) r \;\;,\;\; E^r = -\alpha\frac{Q}{r^2} \end{aligned} \tag{A-1}$$

with $\Delta = r^2 + 2Mr - Q^2$.

A interesting solution was given by Wald, which describes a solution of the Maxwell equations in the test field case (ie, the EM fields does not change the geometry). We assume a kerr black hole in Boyer-Lindquist coordinates, inmersed in a uniform magnetic field (Wald 1974):

$$ds^2 = -(1 - \frac{2Mr}{\Sigma})dt^2 - \frac{4Marsin^2\theta}{\Sigma}dtd\phi + [\frac{(r^2+a^2)^2 - \Delta a^2 sin^2\theta}{\Sigma}]sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \tag{A-2}$$

where $\Sigma = r^2 + a^2cos^2\theta$ and $\Delta = r^2 + a^2 - 2Mr$. Within this background, the following Maxwell tensor is a solution:

$$\begin{aligned}
F &= F_{10}\omega^1 \wedge \omega^0 + F_{13}\omega^1 \wedge \omega^3 + F_{20}\omega^2 \wedge \omega^0 + F_{23}\omega^2 \wedge \omega^3 && \text{(A-3)}\\
\omega^0 &= (\frac{\Delta}{\Sigma})^2(dt - asin^2\theta d\phi) \quad , \quad \omega^1 = (\frac{\Sigma}{\Delta})^2 dr \quad , \quad \omega^2 = \Sigma^{1/2}d\theta \quad , \quad \omega^3 = \frac{sin\theta}{\Sigma^{1/2}}[(r^2+a^2)d\phi - adt] && \text{(A-4)}\\
F_{10} &= B_0[\frac{arsin^2\theta}{\Sigma} - \frac{Ma(r^2 - a^2cos^2\theta)(1+cos^2\theta)}{\Sigma^2}] \quad , \quad F_{13} = B_0\frac{\Delta^{1/2}rsin\theta}{\Sigma} && \text{(A-5)}\\
F_{20} &= B_0\frac{\Delta^{1/2}asin\theta cos\theta}{\Sigma} \quad , \quad F_{23} = B_0\frac{cos\theta}{\Sigma}[r^2 + a^2 - \frac{2Mra^2(1+cos^2\theta)}{\Sigma}] && \text{(A-6)}
\end{aligned}$$

where we use that $\omega^a \wedge \omega^b \equiv \frac{1}{2}(\omega^a \otimes \omega^b - \omega^b \otimes \omega^a)$. We can write the solution in the standard base $\{t, r, \theta, \phi\}$, that is,

$$\begin{aligned}
F &= F_{rt}\omega^r \wedge \omega^t + F_{r\phi}\omega^r \wedge \omega^\phi + F_{\theta t}\omega^\theta \wedge \omega^t + F_{\theta\phi}\omega^\theta \wedge \omega^\phi && \text{(A-7)}\\
F_{rt} &= F_{10} - \frac{asin\theta}{\Delta^{1/2}}F_{13} \quad , \quad F_{r\phi} = -asin^2\theta F_{10} + \frac{(r^2+a^2)}{\Delta^{1/2}}sin\theta F_{13} && \text{(A-8)}\\
F_{\theta t} &= \Delta^{1/2}F_{20} - asin\theta F_{23} \quad , \quad F_{\theta\phi} = -a\Delta^{1/2}sin^2\theta F_{20} + (r^2+a^2)sin\theta F_{23} \quad . && \text{(A-9)}
\end{aligned}$$

The Schwarszchild black hole (ie, the case with no spin $a = 0$) is particularly simple. The line element in the BL coordinates reduces to

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(\frac{r^2}{r^2 - 2Mr}\right)dr^2 + r^2d\theta^2 + r^2sin^2\theta d\phi^2 \quad . \tag{A-10}$$

The electromagnetic tensor is just:

$$F = B_0rsin^2\theta\omega^r \wedge \omega^\phi + r^2sin\theta cos\theta\omega^\theta \wedge \omega^\phi \tag{A-11}$$

and using that $\sqrt{\gamma} = r^2sin\theta/\alpha$ we can explicitely write the magnetic field components, still in Boyer-Lindquist coordinates

$$\begin{aligned}
B^r &= B_0\alpha cos\theta \quad , \quad B^\theta = -B_0\frac{\alpha sin\theta}{r} \quad , \quad B^\phi = 0 \quad , && \text{(A-12)}\\
E^r &= E^\theta = E^\phi = 0 \quad . && \text{(A-13)}
\end{aligned}$$

The same solution can be written in Kerr-Schild coordinates. The line element in the KS coordinates reduces to

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{4M}{r}drdt + \left(1 + \frac{2M}{r}\right)dr^2 + r^2d\theta^2 + r^2sin^2\theta d\phi^2 \quad . \tag{A-14}$$

The electric and magnetic fields in KS coordinates are:

$$\begin{aligned}
B^r &= B_0\alpha cos\theta \quad , \quad B^\theta = -B_0\frac{\alpha sin\theta}{r} \quad , \quad B^\phi = 0 \quad , && \text{(A-15)}\\
E^r &= E^\theta = 0 \quad , \quad E^\phi = \frac{2B_0\alpha M}{r^2} \quad . && \text{(A-16)}
\end{aligned}$$

The lapse and the shift are just $\alpha = \sqrt{\frac{r}{r+2M}}$ and $\beta^r = \frac{2M}{r+2M}$, $\beta^\theta = \beta^\phi = 0$. The metric can be written as

$$\gamma_{ij} = \begin{pmatrix} 1 + \frac{2M}{r} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2sin^2\theta \end{pmatrix}$$

while the extrinsic curvature is just

$$K_{ij} = \begin{pmatrix} -\frac{2M}{r^3}\alpha(r+M) & 0 & 0 \\ 0 & 2M\alpha & 0 \\ 0 & 0 & 2M\alpha sin^2\theta \end{pmatrix}$$

.

The cartesian coordinates are related to the spherical ones by the following transformation:

$$x = r\,sin\theta\,cos\phi \quad \text{(A-17)}$$
$$y = r\,sin\theta\,sin\phi \quad \text{(A-18)}$$
$$z = r\,cos\theta \quad . \quad \text{(A-19)}$$

or, in terms of the basis,

$$e_r = sin\theta\,cos\phi\,e_x + sin\theta\,sin\phi\,e_y + cos\theta\,e_z \quad \text{(A-20)}$$
$$e_\theta = r\,cos\theta\,cos\phi\,e_x + r\,cos\theta\,sin\phi\,e_y - r\,sin\theta\,e_z \quad \text{(A-21)}$$
$$e_\phi = -r\,sin\theta\,sin\phi\,e_x + r\,sin\theta\,cos\phi\,e_y \quad \text{(A-22)}$$

so that we can write the previous solution in this coordinates. By using $r = \sqrt{x^2+y^2+z^2}$:

$$\gamma_{ij} = \begin{pmatrix} 1+\frac{2Mx^2}{r^3} & \frac{2Mxy}{r^3} & \frac{2Mxz}{r^3} \\ \frac{2Mxy}{r^3} & 1+\frac{2My^2}{r^3} & \frac{2Myz}{r^3} \\ \frac{2Mzx}{r^3} & \frac{2Mzy}{r^3} & 1+\frac{2Mz^2}{r^3} \end{pmatrix}$$

the extrinsic curvature is

$$K_{ij} = \begin{pmatrix} [(\frac{M}{r}+2)x^2 - r^2]Q & (\frac{M}{r}+2)xyQ & (\frac{M}{r}+2)xzQ \\ (\frac{M}{r}+2)yxQ & [(\frac{M}{r}+2)y^2 - r^2]Q & (\frac{M}{r}+2)yzQ \\ (\frac{M}{r}+2)zxQ & (\frac{M}{r}+2)zyQ & [(\frac{M}{r}+2)z^2 - r^2]Q \end{pmatrix}$$

where we have used $Q \equiv -2Mr^{-4}\frac{1}{\sqrt{1+2M/r}}$. The lapse and the shift are $\alpha = \sqrt{\frac{r}{r+2M}}$ and $\beta^i = \frac{2Mx^i}{r(r+2M)}$. The electric and magnetic fields are:

$$B^x = B^y = 0 \;,\; B^z = \alpha B_0 \;, \quad \text{(A-23)}$$
$$E^x = -2\alpha M B_0 \frac{y}{r^2} \;,\; E^y = 2\alpha M B_0 \frac{x}{r^2} \;,\; E^z = 0 \;. \quad \text{(A-24)}$$

Appendix B. The Kerr-Schild solution and the Blandfor-Znajeck mechanism

By following Hawley's thesis, the Kerr line element written in Boyer-Lindquist coordinates is just

$$\begin{aligned} ds^2 &= -(\frac{\Delta - a^2 sin^2\theta}{\rho^2})dt^2 - \frac{4Mar sin^2\theta}{\rho^2}dtd\phi \\ &+ \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\Sigma}{\rho^2}sin^2\theta d\phi^2 \end{aligned} \quad \text{(B-1)}$$

with

$$\begin{aligned} \Delta &\equiv r^2 + a^2 - 2Mr \\ \rho^2 &\equiv r^2 + a^2 cos^2\theta \\ \Sigma &\equiv (r^2+a^2)^2 - a^2\Delta sin^2\theta \end{aligned} \quad \text{(B-2)}$$

Kerr-Schild coordinates, the rotational analogue of the Ingoing Eddington-Finkelstein coordinates, are obtained by transforming the Boyer-Lindquist coordinates t and ϕ into the Kerr-Schild coordinates $\tilde{t}$ and $\tilde{\phi}$ according to:

$$d\tilde{t} = dt + \frac{2Mr}{\Delta}dr \;\;,\;\; d\tilde{\phi} = d\phi + \frac{a}{\Delta}dr \;\;. \quad \text{(B-3)}$$

Thus we arrive at the Kerr line element in Kerr-Schild form:

$$\begin{aligned} ds^2 &= -(1-\frac{2Mr}{\rho^2})d\tilde{t}^2 - \frac{4Mar}{\rho^2}sin^2\theta d\tilde{t}d\tilde{\phi} + \frac{4Mr}{\rho^2}d\tilde{t}dr + (1+\frac{2Mr}{\rho^2})dr^2 \\ &- 2a(1+\frac{2Mr}{\rho^2})sin^2\theta dr d\tilde{\phi} + \rho^2 d\theta^2 + sin^2\theta[\rho^2 + a^2(1+\frac{2Mr}{\rho^2})sin^2\theta]d\tilde{\phi}^2 \end{aligned} \tag{B-4}$$

The Kerr solution possesses two event horizons $r_\pm$ and two surfaces of infinite redshift $S_\pm$. The event horizons occur where the surfaces of constant r become null, which corresponds to where g^{rr} is zero. From this we find the event horizons are given by roots of

$$\Delta = r^2 + a^2 - 2Mr = 0 \rightarrow r_\pm = M \pm \sqrt{M^2 - a^2} \quad . \tag{B-5}$$

So the event horizons are always surfaces of constant r, but have smaller radii for larger values of a. Since we will be concerned with solutions exterior to the black hole, we will only need to retain the outer event horizon r_+. The location of surfaces of infinite redshift are found where the $t-t$ component of the metric is zero. Inspecting the metric (B-4), we see that this occurs for

$$\rho^2 - 2Mr = 0 \rightarrow r_{S\pm} = M \pm \sqrt{M^2 - a^2 cos^2\theta} \quad . \tag{B-6}$$

Thus, as the black hole increases, the surface of infinite redshift is pinched along the axis of rotation. The region between r_+ and S_+ is called the ergosphere, in which the asymptotic time translation Killing field $\xi^\mu = (\partial/\partial t)^\mu$ becomes spacelike. In this region, an observer cannot remain stationary with respect to observers at spatial infinity, but must orbit in the direction of the black hole's rotation. For this reason, S_+ is also known as the stationary limit surface. The ergosphere allows for some interesting physics, because is this region the energy of a test particle is not necessarily positive. Penrose was the first to point that, in principle, this implies that energy can be mechanically extracted from the spin of the black hole.

Appendix C. Transformation from cartesian to (Kerr-Schild) spherical coordinates

In this appendix we are going to describe in detail how to transform tensors expressed in cartesian coordinates to spherical ones. We are interested in the Kerr-Schild coordinates, which are defined as:

$$x + i\,y = (r - i\,a)\,e^{i\phi}\,sin\theta \quad , \quad z = r\,cos\theta \quad . \tag{C-1}$$

or

$$x = \sqrt{r^2 + a^2} sin\theta\, cos\tilde{\phi} \tag{C-2}$$
$$y = \sqrt{r^2 + a^2} sin\theta\, sin\tilde{\phi} \tag{C-3}$$
$$z = r\,cos\theta \quad . \tag{C-4}$$

where $\tilde{\phi} \equiv \phi - arctg(a/r)$. Notice that these expressions reduce to the standard case in the non-spinning case $a = 0$. The transformation is in general complicated, so we will restrict to the case $a = 0$, where $r \equiv \sqrt{x^2 + y^2 + z^2}$.

$$dx = sin\theta\, cos\phi\, dr + r\, cos\theta\, cos\phi\, d\theta - r\, sin\theta\, sin\phi\, d\phi \tag{C-5}$$
$$dy = sin\theta\, sin\phi\, dr + r\, cos\theta\, sin\phi\, d\theta + r\, sin\theta\, cos\phi\, d\phi \tag{C-6}$$
$$dz = cos\theta\, dr - r\, sin\theta\, d\theta \tag{C-7}$$

It is convenient to define the cylindrical radius $\bar{\omega} \equiv \sqrt{x^2 + y^2} = rsin\theta$. The different components of the Jacobian will are:

$$\frac{\partial x}{\partial r} = \frac{x}{r} \;,\; \frac{\partial y}{\partial r} = \frac{y}{r} \;,\; \frac{\partial z}{\partial r} = \frac{z}{r} \;, \tag{C-8}$$
$$\frac{\partial x}{\partial \theta} = \frac{x\,z}{\bar{\omega}} \;,\; \frac{\partial y}{\partial \theta} = \frac{y\,z}{\bar{\omega}} \;,\; \frac{\partial z}{\partial \theta} = -\bar{\omega} \;, \tag{C-9}$$
$$\frac{\partial x}{\partial \phi} = -y \;,\; \frac{\partial y}{\partial \phi} = x \;,\; \frac{\partial z}{\partial \phi} = 0 \;. \tag{C-10}$$

Let us use it for instace to compute the rotation frequency $\Omega_F \equiv \frac{F_{tr}}{F_{r\phi}} = \frac{F_{t\theta}}{F_{\theta\phi}}$. Our code evolves the cartesian components $x^a = \{t, x, y, z\}$ but we need the generalized spherical ones $x^{\bar{a}} = \{t, r, \theta, \phi\}$. We will use of the common relation to transform the components, namely

$$F_{\bar{a}\bar{b}} = \frac{\partial x^a}{\partial x^{\bar{a}}} \frac{\partial x^b}{\partial x^{\bar{b}}} F_{ab} \tag{C-11}$$

Using the previous relations (C-8), we can compute

$$\begin{aligned}
F_{tr} &= \frac{\partial x}{\partial r} F_{tx} + \frac{\partial y}{\partial r} F_{ty} + \frac{\partial z}{\partial r} F_{tz} && \text{(C-12)}\\
F_{r\phi} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \phi} F_{xx} + \frac{\partial x}{\partial r}\frac{\partial y}{\partial \phi} F_{xy} + \frac{\partial y}{\partial r}\frac{\partial x}{\partial \phi} F_{yx} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \phi} F_{yy} + \frac{\partial z}{\partial r}\frac{\partial x}{\partial \phi} F_{zx} + \frac{\partial z}{\partial r}\frac{\partial y}{\partial \phi} F_{zy} && \text{(C-13)}\\
F_{t\theta} &= \frac{\partial x}{\partial \theta} F_{tx} + \frac{\partial y}{\partial \theta} F_{ty} + \frac{\partial z}{\partial \theta} F_{tz} && \text{(C-14)}\\
F_{\theta\phi} &= \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \phi} F_{xx} + \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial \phi} F_{xy} + \frac{\partial y}{\partial \theta}\frac{\partial x}{\partial \phi} F_{yx} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \phi} F_{yy} + \frac{\partial z}{\partial \theta}\frac{\partial x}{\partial \phi} F_{zx} + \frac{\partial z}{\partial \theta}\frac{\partial y}{\partial \phi} F_{zy} && \text{(C-15)}
\end{aligned}$$

We may need also to do the transformation of vectors, in order to get the components

$$B^{\bar{a}} = \frac{\partial x^{\bar{a}}}{\partial x^a} B^a \tag{C-16}$$

We need now the inverse transformation. We need to write the radius and the azimutal angle as a function of $\{x, y, z\}$, namely

$$r(x,y,z) = \sqrt{x^2 + y^2 + z^2} \quad , \quad \phi(x,y,z) = tg^{-1}(y/x) \tag{C-17}$$

$$\begin{aligned}
\frac{\partial r}{\partial x} &= \frac{x}{r} \quad , \quad \frac{\partial r}{\partial y} = \frac{y}{r} \quad , \quad \frac{\partial r}{\partial z} = \frac{z}{r} \quad , \\
\frac{\partial \phi}{\partial x} &= \frac{-y}{\bar{\omega}} \quad , \quad \frac{\partial \phi}{\partial y} = \frac{x}{\bar{\omega}} \quad , \quad \frac{\partial \phi}{\partial z} = 0 \quad .
\end{aligned} \tag{C-18}$$

Using the previous relations (C-18), we can compute

$$\begin{aligned}
B^r &= \frac{\partial r}{\partial x} B^x + \frac{\partial r}{\partial y} B^y + \frac{\partial r}{\partial z} B^z && \text{(C-19)}\\
B^\phi &= \frac{\partial \phi}{\partial x} B^x + \frac{\partial \phi}{\partial y} B^y + \frac{\partial \phi}{\partial z} B^z . && \text{(C-20)}
\end{aligned}$$

Appendix D. The matter terms of Einstein equations

Einstein equations contain the stress-energy tensor in the source terms. Depending on the particular formalism being used, one have to compute the corresponding projections of this tensor. In particular, in the 3+1 formalisms...

Appendix E.The speed of sound in the relativistic case

Let us consider a perfect fluid stress-energy tensor

$$T^{ab}_{fluid} = (e + p) u^a u^b + p\ g^{ab} \tag{E-1}$$

where u^a is the fluid four-velocity, $e = \rho\ (1 + \epsilon)$ is the total energy and p is the pressure. This stress-energy tensor follows a conservation law $\nabla_a T^{ab} = 0$ which determines the motion equations for the fluid. Let us assume also the special relativistic limit with $g_{ab} = diag(-1, 1, 1, 1)$ in cartesian coordinates. It is straightforward (althought tedious) to check, from the fluid motion equations, that an adiabatic perturbation will propagate with a sound speed given by

$$c_s^2 = (\frac{dp}{de})_{s=cte} \tag{E-2}$$

where the differentials are total because the process is done at constant entropy.

On the other side, from the thermodinamic law, we know that

$$d\epsilon = -p \; d(\frac{1}{\rho}) + T \; ds \tag{E-3}$$

so in an adiabatic process it can be written

$$d\epsilon|_s = \frac{p}{\rho^2} \; d\rho|_s \tag{E-4}$$

We are interested on the total energy e, so

$$de|_s = d\rho|_s + d(\rho\epsilon)|_s = (1+\epsilon) \; d\rho|_s + \rho \; d\epsilon|_s \tag{E-5}$$

Now, substituting (E-4) into (E-5), we obtain the useful relation

$$de|_s = (1 + \epsilon + \frac{p}{\rho}) \; d\rho|_s \tag{E-6}$$

so the sound speed can be written in general for any EOS as

$$c_s^2 = \frac{\rho}{\rho + \rho\epsilon + p} \quad . \tag{E-7}$$

Substituting the ideal gas EOS $p = (\Gamma - 1)\rho\epsilon$ in (E-7) we obtain the equation (**??**).

Appendix F. Conversion to physical units

It is customary in general relativity to adopt geometrized units $G = c = 1$, such that all quantities, including mass (M) and time (T), have units of length (L). Vacuum solutions are invariant under changes in this fundamental length scale L. A quantity X that scales as $L^l M^m T^t$ can be converted into geometrized units by multiplying with the factor $c^t \; (G/c^2)^m$. After the conversion to geometrized units, X scales as L^{l+m+t}.

Most equations of state break this intrinsic scale-invariance, and the fundamental length-scale must be fixed by additional choices. Once the new scale is chosen, transformations between geometrized and physical units can be easily made. In the following, we summarize the basic procedure detailed in [10] to account for the proper scaling of quantities.

There are two common approaches in the literature to set this additional length scale. The first one is obtained by fixing a constant physical quantity, e.g., the solar mass $M_\odot = 1$, and from it deduce the appropriate conversion factors. That is, if a quantity $\hat{X}$ has dimensions of $L^l M^m T^t$, its dimensionless counterpart, $\bar{X}$, is obtained from the following equation:

$$\hat{X} = \left(\frac{G \; M_\odot}{c^2}\right)^{l+t} \frac{M_\odot^m}{c^t} X \, . \tag{F-1}$$

where the constant in the MKS (meter-kg-second) and the cgs (centimeter-g-second) systems are

$$\begin{aligned} G &= 6.67 \times 10^{-11} m^3 kg^{-1} s^{-2} = 6.67 \times 10^{-8} cm^3 g^{-1} s^{-2} && \text{(F-2)} \\ c &= 3.0 \times 10^8 ms^{-1} = 3.0 \times 10^{10} cms^{-1} && \text{(F-3)} \\ M_\odot &= 1.989 \times 10^{30} kg = 1.989 \times 10^{33} g && \text{(F-4)} \end{aligned}$$

Remember that in the MKS system the magnetic field is in Teslas and the energy in Jules, while that in the cgs the magnetic field is in Gauss and the energy in ergs.

There is still the freedom to choose κ, and all dimensions are scaled with this parameter. Usually the choice $\kappa = 100$ is preferred because it leads to physical units which are close to the current observations. For instance, TOV stars constructed with these parameters have a maximum stable mass of $\hat{M}_{\max} = 1.64 M_\odot$ with a radius of $\hat{R}_{\max} = 14.11$ km.

For a polytropic EoS $p = K\rho^\Gamma$ with $\Gamma = 1 + 1/n$, the adimensional quantities $\bar{X}$ are related to the physical ones through the polytropic constant K, namely

$$\bar{M} = K^{-n/2} M \quad , \bar{R} = K^{-n/2} R \quad , \bar{T} = K^{-n/2} T \quad , \bar{\Omega} = K^{n/2} \Omega \quad , \bar{B} = K^{n/2} B \quad . \tag{F-5}$$

where the last relation can be obtained easily from the energy $B^2R^3 = M$, the luminosity $L = M/T = B^2R^6\Omega^4$ or from the magnetic pressure. This way, we can convert easily the values from any K to a different one. For instance, if the mass of a star with $K = 372$ and $\Gamma = 2$ is $M = 3.15$, then $\bar{M} = 372^{-1/2}M = 0.1633$. We can again go back to physical units (with $M_\odot$=1) with an arbitrary K. A common choice which leads to a masses and sizes similar to the observed for neutron stars is $K = 100$, so that $M = 100^{1/2}\bar{M} = 1.633M_\odot$.

The second method for choosing the length scale is explained in detail in [10], and is more involved. It is based on fixing the maximum stable mass for a family of solutions (with given $\{\kappa = 1, \Gamma\}$) to a physically motivated value. Thus, a quantity $\hat{X}$ with dimensions $L^lM^mT^t$ is obtained by using the relation:

$$\hat{X} = \hat{\kappa}^x c^y G^z X, \tag{F-6}$$

where

$$x = \frac{l+m+t}{2(\Gamma-1)}, \quad y = \frac{(\Gamma-2)l + (3\Gamma-4)m - t}{\Gamma-1}, \quad z = -\frac{l+3m+t}{2}. \tag{F-7}$$

In this method $\hat{\kappa}$ has dimensions. We now identify the maximum stable mass for the given polytrope to some physical maximum mass. Although this second method for fixing the fundamental length scale generally leads to different results from the first, it can be checked that for $\Gamma = 2$ both methods (the first one with $\kappa = 100$, while the second one always has $\kappa = 1$) provide the same scaling factors when the physical maximum stable mass is set to $\hat{M} = 1.64M_\odot$. Since the dimensionless maximum stable mass is $M = 0.164$, Eq. (F-6) can be solved for $\hat{\kappa}$ with $\{l = 0, m = 1, t = 0\}$, giving $\hat{\kappa} = 1.456 \times 10^5 \mathrm{cm}^5/\left(\mathrm{g\,s}^2\right)$. With this value, (F-6) can again be used to recover the dimensions of any quantity.

For the Lorene initial data for a single star (Magstar), the easiest thing to do is to calculate the corresponding $\hat{\kappa}$, considering that the maximum mass of the non-rotating star computed with numerically has a mass $M_{max} = 3.16$ and making it correspond in physical units to $M_{max} = 1.64M_\odot$. This way we obtain $\hat{\kappa} = 390.36\mathrm{cm}^5/\left(\mathrm{g\,s}^2\right)$. Then, we can convert all the unitless quantities to unit by using (F-6), namely

$$\hat{T}(l=0, m=0, t=1) \Rightarrow (x=1/2, y=-1, z=-1/2) \Rightarrow \hat{T} = (2.55\times10^{-6}T)s = (2.55\times10^{-3}T)ms \tag{F-8}$$

$$\hat{M}(l=0, m=1, t=0) \Rightarrow (x=1/2, y=2, z=-3/2) \Rightarrow \hat{M} = (1.031\times10^{30}M)kg = (0.519M)M_\odot \tag{F-9}$$

$$\hat{L}(l=1, m=0, t=0) \Rightarrow (x=1/2, y=0, z=-1/2) \Rightarrow \hat{L} = (764L)m = (0.764L)km \tag{F-10}$$

$$\hat{\rho}(l=-3, m=1, t=0) \Rightarrow (x=-1, y=2, z=0) \Rightarrow \hat{\rho} = (2.3\times10^{21}\rho)kgm^{-3} = (2.3\times10^{18}\rho)g/cm^3 \tag{F-11}$$

$$\hat{B}(l=-1/2, m=1/2, t=-1) \Rightarrow (x=-1/2, y=2, z=0) \Rightarrow \hat{B} = \sqrt{8\pi}(4.5\times10^{19}B)G = (2.3\times10^{20}B)G \tag{F-12}$$

where we have used that the magnetic field pressure is $B^2/(8\pi)$ and has units in the cgs system of $dyn/cm^2 = g/(cms^2)$.

We could repeat the same calculation for the Lorene initial data for a binary system with $K = p^\Gamma/\rho = 123$, reescale the adimensional maximum mass of the non-rotating star of mass 0.164 to be $M_{max} = 1.81$, and making it correspond in physical units to $M_{max} = 1.64M_\odot$. This way we obtain $\hat{\kappa} = 1191.4\mathrm{cm}^5/\left(\mathrm{g\,s}^2\right)$.

Another way of doing it is to consider $M_\odot = 1$, so the mass is given naturally in solar mass units. We will follow this convention, in order to compare our simulations with the AEI results [11, 12]. The maximum mass for a non-rotating star with $K = 123$ is $M = 1.82$ (for the rotating is $M = 2.09$), which implies that $\hat{\kappa} = 1465.4\mathrm{cm}^5/\left(\mathrm{g\,s}^2\right)$. Then, we can convert all the unitless quantities to unit by using (F-6), namely

$$\hat{T}(l=0, m=0, t=1) \Rightarrow (x=1/2, y=-1, z=-1/2) \Rightarrow \hat{T} = (4.94\times10^{-3}T)ms \tag{F-13}$$

$$\hat{M}(l=0, m=1, t=0) \Rightarrow (x=1/2, y=2, z=-3/2) \Rightarrow \hat{M} = M\,M_\odot \tag{F-14}$$

$$\hat{L}(l=1, m=0, t=0) \Rightarrow (x=1/2, y=0, z=-1/2) \Rightarrow \hat{L} = (1.48L)km \tag{F-15}$$

$$\hat{\rho}(l=-3, m=1, t=0) \Rightarrow (x=-1, y=2, z=0) \Rightarrow \hat{\rho} = (6.14\times10^{17}\rho)g/cm^3 \tag{F-16}$$

$$\hat{B}(l=-1/2, m=1/2, t=-1) \Rightarrow (x=-1/2, y=2, z=0) \Rightarrow \hat{B} = (2.3\sqrt{8\,\pi}\times10^{19}B)G \tag{F-17}$$

$$\hat{\mathcal{E}}(l=2, m=1, t=-2) \Rightarrow (x=1/2, y=4, z=-3/2) \Rightarrow \hat{\mathcal{E}} = (1.8\times10^{54}\mathcal{E})erg \tag{F-18}$$

where we have used that the energetic units comes from $\mathcal{E} = Mc^2$.

A simple way to estimate the emission of a optically thick source is by means of the effective temperature T_{eff} of the corresponding black body, which is defined as

$$\mathcal{L} = 4\pi R^2\sigma T_{eff}^4 \quad , \quad \sigma = 5.67\times10^{-5} ergs\,cm^{-2}s^{-1}K^{-4} \tag{F-19}$$

where σ is the Stefan-Boltzmann constant and $\mathcal{L}$ the total luminosity of a source of size R. This effective temperature allows to compute the peak frequency of the black body radiation $\nu_{peak}(Hz) = 5.88\times10^{10}T(K)$. In order to know if the magnetosphere is really optically thick we need to know its density and temperature. The charge density can be estimated roughly from the Julian-Goldreich one $\nabla \cdot (\Omega r B)$. Dividing by the electric charge of the electron $e = 4.8 \times 10^{-10} statc$ and multiplying by its mass $m_e = 9.1 \times 10^{-28} g$ we can calculate the density,

$$\rho_{JG} = \frac{\Omega B\, m_e}{2\pi c\, e} [g/cm^3] \quad . \tag{F-20}$$

For a $\Omega = 1.5 rad/ms$ and $B = 10^{12} G$ the density is just $1.5 \times 10^{-14} g/cm^3$.

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[13] except for some schemes which iteratively approach to an implicit scheme, like for instance a particular version of the ICN (Choptuik, private communication)