Notes about dilaton theory

(Dated: August 22, 2016)

I. EVOLUTION EQUATIONS

Let us start with an action given by a Lagrangian in the physical or Jordan frame defined on a string metric g^{st}_{ab} (**is this the full Lagrangian? see for instance [1]** **[ewh: It depends on what you are asking. No, it is not the "full" Lagrangian of low energy string theory. But even that is a truncation of the string worldsheet action to consider only tree level (as opposed to higher order loop effects). Even considering different string theories, the truncated actions (or Lagrangians) are different between type I, IIA, IIB, heterotic and so on. But that is probably not what you are asking nor is that relevant to what we want to do. What you have is (mostly) the correct Lagrangian for a subsector of low energy heterotic string theory in which some of the fields have been set to zero. The reason it is mostly correct is that the α_0 does not belong. It's insertion usually comes at the stage that the Einstein frame is used and an association is made between that Lagrangian (in the Einstein frame) and between other closely related theories. But what you have is correct provided what you are interested in is a parametrized version of Einstein-Maxwell-dilaton theory.]**)

$$\mathcal{L}_{st} = e^{-2\alpha_0\varphi}\left[R_{st} + 4\left(\nabla\varphi\right)^2 - F^2\right] \tag{1}$$

that can be converted, by a conformal transformation $g_{ab} = e^{-2\alpha_0\varphi} g^{st}_{ab}$, into a Lagrangian in the Einstein frame

$$\mathcal{L} = R - 2\left(\nabla\varphi\right)^2 - 2V(\varphi^2) - e^{-2\alpha_0\varphi}F^2 \tag{2}$$

where we have included a massive potential $V(\varphi^2) = m^2\varphi^2$.

We will assume that this is the correct Lagrangian in the Einstein frame from which we will derive the equations of motion (see for instance [2, 3]). By varying the action with respect to g^{ab} we obtain the following equations

$$\begin{aligned} R_{ab} - \frac{1}{2}g_{ab}R &= 2T^{\varphi}_{ab} + 2e^{-2\alpha_0\varphi}T^{em}_{ab} && (3)\\ T^{\varphi}_{ab} &= \nabla_a\varphi\nabla_b\varphi - \frac{1}{2}g_{ab}\left[\nabla_c\phi\nabla^c\varphi + V(\varphi^2)\right] && (4)\\ T^{em}_{ab} &= F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd} && (5)\end{aligned}$$

Notice that we can write this in terms of the Einstein equations ($G_{ab} = 8\pi T_{ab}$) with a standard stress-energy tensor given by

$$T_{ab} = \frac{1}{4\pi}\left(T^{\varphi}_{ab} + e^{-2\alpha_0\varphi}T^{em}_{ab}\right) \tag{6}$$

Just for completeness, these equations can be written in the Ricci formulation in a way consistent with Eric's equations, namely

$$R_{ab} = 2\nabla_a\varphi\nabla_b\varphi + g_{ab}V(\varphi^2) + 2e^{-2\alpha_0\varphi}\left(F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}\right) \tag{7}$$

For the other equations we can use the Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial\lambda} - \partial_a\left[\frac{\mathcal{L}}{\partial(\partial_a\lambda)}\right] = 0 \tag{8}$$

The part of the Lagrangian containing scalar field terms is

$$\mathcal{L}_{\phi} = -2(\nabla\varphi)^2 - 2V(\varphi^2) - e^{-2\alpha_0\varphi}F_{cd}F^{cd} \tag{9}$$

so, the final evolution equations for the scalar field are just

$$g^{ab}\nabla_a\nabla_b\varphi = \frac{\partial V}{\partial\varphi^2}\varphi - \frac{\alpha_0}{2}e^{-2\alpha_0\varphi}F_{cd}F^{cd} \tag{10}$$

which can be written in terms of the old equations by substituting $\frac{\partial V}{\partial\varphi^2}\varphi \to \frac{\partial V}{\partial\varphi^2}\varphi - \frac{\alpha_0}{2}e^{-2\alpha_0\varphi}F^2$. Notice that we can expand this term by using $F^2 = 2(B^2 - E^2)$.

We can write down explicitly the KG equations in general

$$\begin{aligned} \partial_t\varphi &= \beta^k\partial_k\varphi - \alpha\Pi & (11)\\ \partial_t\Pi &= \beta^k\partial_k\Pi + \alpha\left[-\gamma^{ij}\nabla_i\nabla_j\varphi + \Pi trK + V'\varphi - \frac{\alpha_0}{2}e^{-2\alpha_0\varphi}F^2\right] - \gamma^{ij}\nabla_i\varphi\nabla_j\alpha & (12)\end{aligned}$$

where we have defined $V' \equiv dV/d\varphi^2$. We could rewrite easily this system for a conformal decomposition like the BSSN one as

$$\begin{aligned} \partial_t\phi &= \beta^k\partial_k\varphi - \alpha\Pi & (13)\\ \partial_t\Pi &= \beta^k\partial_k\Pi + \alpha\left[-\chi\tilde{\gamma}^{ij}\partial_i\partial_j\varphi + \chi\tilde{\Gamma}^k\partial_k\varphi + \frac{1}{2}\tilde{\gamma}^{ij}\partial_i\varphi\partial_j\chi + \Pi trK + V'\varphi - \frac{\alpha_0}{2}e^{-2\alpha_0\varphi}F^2\right] & \\ &- \chi\tilde{\gamma}^{ij}\partial_i\varphi\partial_j\alpha & (14)\end{aligned}$$

Finally, the lagrangian containing the EM field is

$$\mathcal{L}_{em} = -e^{-2\alpha_0\varphi}F_{cd}F^{cd} \tag{15}$$

where $F_{ab} = \partial_a A_b - \partial_b A_a$. The evolution equations by varying A_a are

$$\nabla_a\left(e^{-2\alpha_0\varphi}F^{ab}\right) = 0 \tag{16}$$

which can be written in terms of the standard EM fields with a current

$$\nabla_a F^{ab} = 2\alpha_0 F^{ab}\nabla_a\varphi = -I^b \tag{17}$$

We can introduce the constraint damping fields, so that the Maxwell equations will be written as

$$\begin{aligned} \nabla_a(F^{ab} + g^{ab}\psi) &= -I^b + \kappa n^b\psi & (18)\\ \nabla_a({}^*F^{ab} + g^{ab}\phi) &= \kappa n^b\phi & (19)\end{aligned}$$

Therefore, we only need to compute the projections of the 4-current, namely the 3-current and the electric charge density, namely

$$\begin{aligned} J_i &\equiv I_i = -2\alpha_0 F^a{}_i\partial_a\varphi & (20)\\ q &\equiv -n_b I^b = 2\alpha_0 E^i\partial_i\varphi = -2\alpha\alpha_0 F^{a0}\partial_a\varphi & (21)\end{aligned}$$

[1] T. Damour and A. M. Polyakov, "String theory and gravity," *General Relativity and Gravitation* **26** (Dec., 1994) 1171–1176, gr-qc/9411069.

[2] G. T. Horowitz, "The Dark Side of String Theory: Black Holes and Black Strings," in *String Theory and Quantum Gravity '92*, J. Harvey, R. Iengo, K. S. Narain, S. Randjbar-Daemi, and H. Verlinde, eds., p. 55. 1993. hep-th/9210119.

[3] G. W. Gibbons and M. Rogatko, "Decay of Dirac hair around a dilaton black hole," *Phys. Rev. D.* **77** no. 4, (Feb., 2008) 044034, arXiv:0801.3130 [hep-th].