Dynamical Black Holes and Gravitational Waves in Quadratic Gravity

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(work in progress)

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Contents

1 Quadratic Gravity in the Strong-Gravity Regime

Einstein's theory of General Relativity (GR) is built upon diffeomorphism symmetry, manifest in its formulation in terms of the Einstein-Hilbert action

$$S_{\mathrm{GR}} = \frac{1}{16\pi G_{\mathrm{N}}} \int_x \sqrt{\det g}\,(2\Lambda - R) \;, \tag{1.1}$$

where G_N and Λ denote the Newton coupling and cosmological constant, respectively. It is a simple theory with only two free parameters and describes all measured gravitational phenomena to date. Apart from simplicity, a là Occam's razor, there is no fundamental principle in classical gravity that would explain the absence of higher-order, diffeomorphism-invariant curvature terms. The next (quadratic) order in curvature invariants, i.e., Quadratic Gravity (QG) is given by

$$S_{\mathrm{gravity}} = S_{\mathrm{GR}} + \int_x \sqrt{\det g}\left(\alpha R_{\mu\nu}R^{\mu\nu} + \beta R^2\right) \;. \tag{1.2}$$

In spacetimes topologically equivalent to flat space the Gauss-Bonnet topological invariant ensures that this is the most general action at quadratic order. Kellog Stelle showed that, opposed to GR, QG is perturbatively renormalizable as a quantum field theory [**?**]. Him, and many subsequent authors, also discuss the appearence of additional massive ghost-like modes, one scalar and one spin-2, which spoil the unitarity of this quantized theory of gravity. Further, the theory can be motivated as a generic infra-red limit of an effective field theory (EFT) treatment of the quantization of gravity, see e.g. [**?**] for a pedagogical review, and [] for how this formulation avoids the issue of unitarity. All subsequent (unresummed local) curvature invariants will be suppressed by powers of the Planck scale.
One might argue for the uniqueness of GR by the fact that it admits a well-posed (numerical) evolution. In fact, as David R. Noakes showed in [**?**], also the dynamics of quadratic gravity can be formulated as a well-posed initial value problem. We regard both these concepts of such crucial importance for all what follows, that we will review them. Readers not interested in a motivation for QG because of its well-posed numerical evolution or for the absence of even higher orders because of effective field theory, can safely skip Sec. 2 and 3, respectively.

Experiments on solar-system scales only test the weak-gravity regime: the existing constraints on higher-order modifications of GR are therefore extremely weak. Submillimeter-tests using pendulums constrain Yukawa-like corrections to the Newtonian potential that arise from higher-derivative terms like $\alpha R_{\mu\nu}R^{\mu\nu}$ and βR^2. But, the constraints are as weak as α, $\beta < 10^{60} \sim 10^{70}$ [**? ? ?**]. Further, they do not allow to distinguish between the two couplings.
The main motivation for this paper is to generate gravitational wave templates from QG, to use experimental data from binary mergers to constraint the QG-couplings α and β, see also [**? ?**] for a similar study in $f(R)$-gravity. A simple comparison of typical curvature scales exemplifies how vastly studies of the strong-gravity regime could improve these bounds. For that, we compare the curvature at the surface of the earth horizon with that of a solar-mass black hole, by use of the Kretschmann scalar $K \sim M^2/r^6$, i.e.,

$$\frac{K_{\oplus\text{-surface}}}{K_{\odot\text{-horizon}}} = \frac{K_{\oplus\text{-surface}}}{K_{\oplus\text{-horizon}}}\,\frac{K_{\oplus\text{-horizon}}}{K_{\odot\text{-horizon}}} \approx 10^{-32} \;. \tag{1.3}$$

1.1 Testing General Relativity

aLIGO/VIRGO has detected gravitational waves (GWs) and these observations extend into the strong-field regime of gravity, where the gravitational field is non-linear and dynamical, precisely where tests of general relativity(GR) are currently lacking. Strong field tests of GR have implications to a large areas in physics and astrophysics. For example, gravitational parity breaking modifies the geometry of spinning black holes (BHs) and the propagation of GWs in these backgrounds. Constrain such a departure from the Kerr geometry of GR in the strong field regime will place constraints on the coupling constants of such theories.[TODO: Need references]

In this work, we are interested in dynamics and stability of BH in QG. In literatures [TODO: Need references], considerable works are done withimposing extra scalar field into the action such that (in the most general form)

$$S \equiv \int d^4s\sqrt{-g}\Bigg\{\frac{1}{16\pi G_N}R + \alpha_1 f_1(\vartheta)R^2 + \alpha_2 f_2(\vartheta)R_{\mu\nu}R^{\mu\nu} + \alpha_3 f_3(\theta)R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} + \alpha_4 f_4(\vartheta)R_{\mu\nu\lambda\sigma}(*R^{\mu\nu\lambda\sigma}) - \frac{\beta}{2}[\nabla_a\vartheta\nabla^a\vartheta + 2V(\vartheta)] + \mathcal{L}_{matter}\Bigg\} \tag{1.4}$$

where g stands for the determinant of the metric $g_{\mu\nu}$. R, $R_{\mu\nu}$, $R_{\mu\nu\lambda\sigma}$, and $*R_{\mu\nu\lambda\sigma}$ are the Ricci scalar, Ricci tensor, Riemann tensor and its dual respectively. ϑ is a dynamical scalar field, $f_i(\vartheta)$ are functionals of this field, (α_i, β) are coupling constants. Coupling to a scalar field ϑ enables to make dynamical theory such as dynamical Chern-Simon (dCS) gravity and Einstein-dilaton-Gauss-Bonnet (EdGB) gravity as examples of QG theories. [What is the major difference (in motivation point of view) having (or not having) dynamical scalar field?]

2 Review: Quadratic Gravity from an Effective-Field-Theory Quantization

[TODO: rewrite] Field-theoretic attempts to quantize gravity have difficulties retaining renormalizability. Without higher-derivative terms, General Relativity is a non-renormalizable theory. The underlying reason can be associated with the negative canonical dimension of the Newton coupling, i.e., $[G_N] = -2$. The quantization of gravity is not just an academic problem. If G_N was of order one at accessible energy scales, then the effects of delocalized quantum matter would directly imply significant quantum-fluctuations in the associated gravitational field. It is only the smallness of the gravitational force, i.e., $G_N \sim 10^{-38}$ GeV, which hides these effects from experimental observation. In fact, this is one particularly insightful way of obtaining the Planck scale. The dimensionless Newton coupling $g_N(\mu) = G_N \times \mu^2$ scales with a quadratic power-law in the characteristic energy scale μ. Since it is dimensionless couplings which determine field-theoretic cross sections, gravitational effects become important whenever $g_N(\mu) \approx 1$, i.e., at a mass scale of $M_{\text{Planck}} = 10^{19}$ GeV – the Planck scale.

The most agnostic approach to the quantization of gravity is to look at quantum gravity as an effective field theory (EFT). In correspondence to, for instance, the Standard Model, this assumes that *all* operators allowed by symmetry are present at M_{Planck}. Neglecting the index structure we can schematically denote those (dimensionful) couplings and the corresponding curvature invariants as $C_N \times R^N$. Since the underlying theory is not known, *all* EFT-couplings are assumed to be of order one at the Planck-scale, i.e., $C_N(\mu = M_{\text{Planck}}) \approx 1$. The EFT draws its predictive power from canonical scaling. Since in four dimensions the dimension of curvature is $[R] = 2$, the associated couplings have dimension $[C_N] = 2N - 4$. The corresponding dimensionless couplings $c_N(\mu) = C_N/\mu^{2N-4}$ scale like $c_N \sim \mu^{4-2N}$. Hence they are suppressed by

$$c_N(M_{\text{exp}}) \approx \left(\frac{M_{\text{exp}}}{M_{\text{Planck}}}\right)^{(2N-4)} . \tag{2.1}$$

In an EFT one would thus conclude that all couplings $c_{N>2}$ are suppressed by increasing powers of the enormously large Planck scale. This does not apply for the quadratic couplings $c_{N=2}$, which constitute so-called marginal couplings of the EFT. These are *not* power-law suppressed and are hence expected to be of similar order as at the Planck-scale.

The EFT for quantum gravity is valid below the Planck scale. It might be possible to embedd it into a non-perturbatively renormalizable quantum field theory of gravity solely defined by diffeomorphism symmetry and a corresponding asymptotically safe fixed point. This theory of Asymptotically Safe Gravity (ASG) was conjectured by Steven Weinberg in 1976 [**?**]. It is supported by the mounting evidence around the corresponding Reuter fixed-point [**?**], cf. [] for reviews. Since the Reuter fixed-point is fully interacting, *all* higher-order couplings will be present at Planckian energies. Below the Planck scale the EFT-description applies and one, again, expects the associated low-energy effective theory to be governed by the four couplings of QG, i.e., G_N, Λ, α and β. It can be added here, that current approximations of the Reuter fixed-point indicate that there are only three so-called relevant couplings. AS restores the predictivity of a quantum field theory of gravity precisely by non-perturbative relations, which express *all* other (irrelevant) couplings in terms of the three relevant ones. Hence, one of the two QG couplings could follow as a prediction from AS, e.g., $\alpha = \alpha(\beta, G_N, \Lambda)$.

3 Review: Well-posed Initial Value Problem for Quadratic Gravity

3.1 Equations of Motion in the Quadratic Gravity

The equations of motion (eom) of QG are given by

$$H_{\mu\nu} = \kappa\, G_{\mu\nu} + E_{\mu\nu} = \frac{1}{2}T_{\mu\nu}\,, \qquad \text{with: } \kappa = \frac{1}{16\pi G_N}\,, \tag{3.1}$$

where $G_{\mu\nu} = R_{\mu\nu} - 1/2Rg_{\mu\nu}$ is the usual Einstein tensor, which is supplemented by its quadratic-order counter-part

$$\begin{aligned} E_{\mu\nu} =& (\alpha - 2\beta)\nabla_\mu\nabla_\nu R - \alpha\Box R_{\mu\nu} - (\frac{1}{2}\alpha - 2\beta)g_{\mu\nu}\Box R + 2\alpha R^{\alpha\beta}R_{\mu\alpha\nu\beta} \\ & - 2\beta R R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(\alpha R_{\alpha\beta}R^{\alpha\beta} - \beta R^2)\,. \end{aligned} \tag{3.2}$$

QG propagates, besides the graviton, an additional massive scalar mode associated with the Ricci-scalar R and a massive spin-2 mode corresponding to the traceless part of the Ricci-tensor $\widetilde{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R$ [?]. Knowing this, it is useful to split the equations of motion into a trace and a traceless part, i.e.,

$$\Box R = \frac{\kappa}{2(3\beta - 2\alpha)} R \, , \textbf{[have to check the factor of 2]} \tag{3.3}$$

$$\begin{aligned}\alpha \, \Box \tilde{R}_{\mu\nu} = & - \kappa \, \tilde{R}_{\mu\nu} - (2\beta - \alpha) \left(\frac{\kappa}{8(3\beta - 2\alpha)} g_{\mu\nu} - \nabla_\mu \nabla_\nu \right) R \\ & - (\alpha - 2\beta) R \widetilde{R}_{\mu\nu} - 2\alpha \left(R_{\mu\rho\nu\sigma} - \frac{1}{4} g_{\mu\nu} \, \widetilde{R}_{\rho\sigma} \right) \widetilde{R}^{\rho\sigma}\end{aligned} \tag{3.4}$$

As anticipated, these equations are fourth order in derivatives. Following [?], there are two more essential steps to cast the equations of motion into a well-posed IVP: (i) we employ harmonic coordinates to treat $g_{\mu\nu}$, $\widetilde{R}_{\mu\nu}$, and R as independet variables, which reduces the system to contain only second-order derivatives; (ii) we use a differentiation procedure to diagonalize the resulting equations and put them in quasilinear form.

3.2 Reduction to a second order system

The Ricci tensor in a general coordinate system is given by

$$R_{\mu\nu} = -\frac{1}{2} g^{\alpha\beta} \partial_\alpha \partial_\beta g_{\mu\nu} + Q_{\mu\nu}(g, \partial g) + \frac{1}{2} \left(g_{\mu\beta} \partial_\nu F^\beta + g_{\nu\beta} \partial_\mu F^\beta \right) , \tag{3.5}$$

$$\text{where} \quad Q^{\mu\nu}(g, \partial g) = g^{\alpha\beta} \left(\Gamma^\mu_{\alpha\gamma} \partial_\beta g^{\nu\gamma} + \Gamma^\nu_{\alpha\gamma} \partial_\beta g^{\mu\gamma} - 2\Gamma^\gamma_{\alpha\beta} \partial_\gamma g^{\nu\mu} \right) \, ,$$

$$\text{and} \quad F^\alpha = g^{\mu\nu} \Gamma^\alpha_{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \left(\sqrt{-g} g^{\alpha\beta} \right) .$$

Harmonic coordinates are defined by $F^\mu = 0$. This choice is advantagous because it allows to reduces the expression of the Ricci tensor in terms of the metric into a quasilinear form, that is the last term in Eq. (3.5) vanishes in harmonic coordinates. Hence,

$$-\frac{1}{2} g^{\alpha\beta} \partial_\alpha \partial_\beta g_{\mu\nu} + Q_{\mu\nu}(g, \partial g) = R_{\mu\nu} = \widetilde{R}_{\mu\nu} + \frac{1}{4} g_{\mu\nu} R \, . \tag{3.6}$$

Adding this to the equations of motion and treating $g_{\mu\nu}$, $\widetilde{R}_{\mu\nu}$, and R as independent variables reduces the system to second order. Notice that Eq. (3.6) and therefore all following equations only hold in harmonic coordinates. The second order set of equations reads

$$\frac{1}{2} g^{\alpha\beta} \partial_\alpha \partial_\beta g_{\mu\nu} = Q_{\mu\nu}(g, \partial g) - \widetilde{R}_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R \, , \tag{3.7}$$

$$\Box R = \frac{\kappa}{2(3\beta - 2\alpha)} R \, , \tag{3.8}$$

$$\begin{aligned}\alpha \, \Box \tilde{R}_{\mu\nu} = & (2\beta - \alpha) \nabla_a \nabla_b R - \frac{1}{16\pi} \tilde{R}_{ab} - \frac{2\beta - \alpha}{128\pi(3\beta - 2\alpha)} g_{ab} R \\ & + (\alpha - 2\beta) \left[g^{cd} g^{mn} g_{mn,cd} - 2Q(g, \partial g) \right] \tilde{R}_{ab} \\ & + \frac{\alpha}{4} g_{ab} \left[g^{cd} g_{mn,cd} - 2Q_{mn}(g, \partial g) \right] g^{om} g^{pn} \tilde{R}_{op}\end{aligned}$$

$$- \alpha\Big[g_{cd,ab} + g_{ab,cd} - g_{cb,ad} - g_{ad,cb} + 4 g_{mn}\Gamma^{m}{}_{c[b}\Gamma^{n}{}_{d]a}\Big]\tilde{R}^{cd}. \tag{3.9}$$

Here we have written $2\beta\, R$ and $R_{\mu\rho\nu\sigma} - \frac{1}{4} g_{\mu\nu}\, \widetilde{R}_{\rho\sigma}$ in Eq. (3.4) in terms of the metric (using again harmonic coordinates). **[motivation not really clear to me]** This set of equations is now only of second order, but the last equation is still not quasilinear.

3.3 Diagonalization to a quasilinear hyperbolic system

We can diagonalize the equations to a quasilinear system by introducing additional variables $V_\mu = \partial_\mu R$ and $h_{\mu\nu\alpha} = g_{\mu\nu,\alpha}$ and adding derivatives of the first two (already quasilinear) equations to the system. We obtain

$$\begin{aligned}
-\frac{1}{2} g^{\eta\delta} g_{\mu\nu,\eta\delta} =& - Q_{\mu\nu}(g,\partial g) + \widetilde{R}_{\mu\nu} + \frac{1}{4} g_{\mu\nu} R, && (3.10)\\
g^{\alpha\beta}\partial_\alpha\partial_\beta R =& \frac{\kappa}{2(3\beta - 2\alpha)} R && (3.11)\\
-\frac{1}{2} g^{\eta\delta} h_{\mu\nu\gamma,\eta\delta} =& \frac{1}{2} g^{\eta\delta}{}_{,\gamma}\, h_{\mu\nu\eta,\delta} - Q_{\mu\nu,\gamma}(g,\partial g, c) + \widetilde{R}_{\mu\nu,\gamma} + \frac{1}{4} g_{\mu\nu,\gamma} R + \frac{1}{4} g_{\mu\nu} V_\gamma, && (3.12)\\
g^{\alpha\beta}\partial_\alpha\partial_\beta V_\gamma =& \frac{\kappa}{2(3\beta - 2\alpha)} V_\gamma, && (3.13)\\
\alpha\Box\widetilde{R}_{\mu\nu} =& (2\beta - \alpha)\nabla_\mu V_\nu - \kappa\, \widetilde{R}_{\mu\nu} - \frac{\kappa(2\beta - \alpha)}{8(3\beta - 2\alpha)} g_{\mu\nu} R && (3.14)\\
&- (\alpha - 2\beta)\left(-g^{\alpha\beta} g^{\sigma\rho} h_{\sigma\rho\alpha,\beta} + 2Q(g,\partial g)\right)\widetilde{R}_{\mu\nu} - \frac{\alpha}{8} g_{\mu\nu}\left(-g^{\alpha\beta} h_{\rho\sigma\alpha,\beta} + 2Q_{\rho\sigma}(g,\partial g)\right)\widetilde{R}^{\rho\sigma}\\
&- \alpha\left(h_{\rho\sigma\mu,\nu} + h_{\mu\nu\rho,\sigma} - h_{\rho\nu\mu,\sigma} - h_{\mu\sigma\rho,\nu} + 2 g_{\alpha\beta}(\Gamma^\alpha_{\rho\nu}\Gamma^\beta_{\mu\sigma} - \Gamma^\alpha_{\rho\sigma}\Gamma^\beta_{\mu\nu})\right)\widetilde{R}^{\rho\sigma}.
\end{aligned}$$

the final well-posed form of the evolution equations.

4 Recast Evolution Equations into 3+1 Form

In this section, we reduce our systems of equations in 3+1 form. There are several reasons to use 3+1 form:

1. We can have a system of equation in terms of first order in time and second order in space i.e. better form as a numerical aspects (Note that this is not always true but at least for me this is true)
2. Easily adapt gauge choice for BH rather than using excision techniques i.e. do not need to solve elliptic problems.
3. Constraints are treated as evolution vars so we can monitor is easily.

We are following usual 3+1 variables such that

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{4.1}$$

where α is lapse, β^i is shift, γ_{ij} is induced metric on spatial hypersurface. Thus, the metric can be expressed as

$$g_{ab} = g_{ij}\gamma^i_a\gamma^j_b - n_a n_b \tag{4.2}$$

where n^a is the covariant normal to the spacelike hypersurface.

Using these defintions, it is obvious to split Ricci tensor

$$R_{ab}n^a n^b = (\partial_\perp K + D_i D^i \alpha)/\alpha - K_{ij}K^{ij} \tag{4.3}$$
$$R_{ab}\gamma_i^a \gamma_j^b = R_{ij} + KK_{ij} - 2K_{ik}K_j^k - (\partial_\perp K_{ij})/\alpha - (D_i D_j \alpha)/\alpha \tag{4.4}$$
$$R_{ab}\gamma_i^a n^b = -D_j K_i^j + D_i K \tag{4.5}$$

Here, we define $\partial_\perp \equiv \partial_t - \pounds_\beta$. For the Ricci scalar

$$^{(4)}R = R + K_{ij}K^{ij} + K^2 - 2(\partial_\perp K)/\alpha - 2(D_i D^i)\alpha/\alpha \tag{4.6}$$

In previous section, we derived the EOM in second order forms (Eqns. 3.7, 3.8, and 3.9).

From Eqn. 3.7. we obtain (Note that this is usual equations from metric)

$$\partial_\perp \gamma_{ij} = -2\alpha K_{ij} \tag{4.7}$$
$$\partial_\perp K_{ij} = \alpha(R_{ij} - 2K_{ik}K_j^k) - D_i D_j \alpha \tag{4.8}$$

There is a gauge freedom for lapse and shift. We will determine these variables later. Now consider Eqns. 3.8 and 3.9. We first consider LHSs of these equations.

$$\begin{aligned}\Box R = \nabla_a \nabla^a R &= g^b{}_a \nabla_b (g^{ac} \nabla_c R) \\ &= (\gamma^b{}_a - n_a n^b)\nabla_b[(\gamma^{ac} - n^a n^c)\nabla_c R]\end{aligned} \tag{4.9}$$

where ∇_a is usual covariant derivative. Note that the derivatives of the physical field are being projected into and normal to the spacelike hypersurfaces. Using this, we will define a new variable $\hat{R} = -n^a \nabla_a R$. Without whole detailed derivations (If you want, I can type it up but basic ideas are pretty standard in 3+1 decomposition...), we can obtain

$$\nabla_a \nabla^a R = n^a \nabla_a \hat{R} + \frac{1}{\alpha} D_a(\alpha D^a R) - K\hat{R} \tag{4.10}$$

where D_a is 3D covariant derivative (or gradient) which lives on spacelike hypersurface. Combining above result with RHS of Eqn. 3.8 provides two first order in time equations for $\hat{R}$ and R which are given by

$$n^a \nabla_a R = -\hat{R} \tag{4.11}$$
$$n^a \nabla_a \hat{R} = -\frac{1}{\alpha} D_a(\alpha D^a R) + K\hat{R} + \frac{1}{36\pi(3\beta_c - 2\alpha_c)} R \tag{4.12}$$

Note that here we invoke α_c and β_c which are slightly different notations than previous sections to avoid conflict between lapse (α) and shift (β). We can re-write above equations also (just for keeping consistency)

$$\partial_\perp R = -\alpha \hat{R} \tag{4.13}$$
$$\partial_\perp \hat{R} = -D_a(\alpha D^a R) + \alpha K \hat{R} + \frac{\alpha\kappa}{2(3\beta_c - 2\alpha_c)} R \tag{4.14}$$

where we use the notation $n^a\nabla_a f = \frac{1}{\alpha}(\partial_t - \beta^i\partial_i)f$ for f a scalar

Similarly, for Eqn. 3.9 (now we deal with tensor not a scalar), we have

$$\begin{aligned}\Box\tilde{R}_{ab} &= \nabla_c\nabla^c\tilde{R}_{ab}\\ &= (\gamma^d{}_c - n_c n^d)\nabla_d[(\gamma^{ce} - n^c n^e)]\nabla_e\tilde{R}_{ab}\\ &= \gamma^d{}_c\nabla_d(\gamma^{ce}\nabla_e\tilde{R}_{ab}) - n_c n^d\nabla_d(\gamma^{ce}\nabla_e\tilde{R}_{ab}) - \gamma^d{}_c\nabla_d(n^c n^e\nabla_e\tilde{R}_{ab}) + n_c n^d\nabla_d(n^c n^e\nabla_e\tilde{R}_{ab})\end{aligned} \tag{4.15}$$

We define $\tilde{V}_{ab} = -n^c\nabla_c\tilde{R}_{ab}$. We argue that this can be considered also as the time derivative (or velocity) as traceless of Ricci tensor (or say tensor flow) like in previous Ricci scalar case. Thus, we will have

$$\nabla_c\nabla^c\tilde{R}_{ab} = n^c\nabla_c\tilde{V}_{ab} + \frac{1}{\alpha}D_c(\alpha D^c\tilde{R}_{ab}) - K\tilde{V}_{ab} \tag{4.16}$$

Now consider RHS of Eqn. 3.9

$$\begin{aligned}Y_{ab} =&(2\beta_c - \alpha_c)\nabla_a\nabla_b R - \frac{1}{16\pi}\tilde{R}_{ab} - \frac{2\beta_c - \alpha_c}{128\pi(3\beta_c - 2\alpha_c)}g_{ab}R\\ &+ (\alpha_c - 2\beta_c)\left[g^{cd}g^{mn}g_{mn,cd} - 2Q(g,\partial g)\right]\tilde{R}_{ab}\\ &+ \frac{\alpha_c}{4}g_{ab}\left[g^{cd}g_{mn,cd} - 2Q_{mn}(g,\partial g)\right]g^{om}g^{pn}\tilde{R}_{op}\\ &- \alpha_c\left[g_{cd,ab} + g_{ab,cd} - g_{cb,ad} - g_{ad,cb} + 4g_{mn}\Gamma^m{}_{c[b}\Gamma^n{}_{d]a}\right]\tilde{R}^{cd}.\end{aligned} \tag{4.17}$$

where we use Y_{ab} just for convenience purpose. In Y_{ab} most of terms are okay (i.e. no involving second derivatives) and most of terms are already decomposed like using Ricci tensors/scalar and metric. So, we split the Y_{ab} into two pieces

$$Y_{ab} = Y^{np}_{ab} + \Delta_c\big[(2\beta_c - \alpha_c)\nabla_a\nabla_b R - \alpha_c(g_{cd,ab} + g_{ab,cd} - g_{cb,ad} - g_{ad,cb})\tilde{R}^{cd}\big] = Y^{np}_{ab} + \Delta_c Y^p_{ab} \tag{4.18}$$

where Δ_c is coupling constant that occurs higher order derivatives and Y^{np}_{ab} contains all remaining terms. Thus, we may choose small value of Δ_c to control how quadratic terms in the theory effects on dynamics of BH. We also use decompositions of Ricci tensors, Ricci scalar, and Christoffel symbols for Y^{np}_{ab} [TODO: check this too]

Using all of these, we obtain

$$n^a\nabla_a\tilde{R}_{ab} = -\tilde{V}_{ab} \tag{4.19}$$

$$n^a\nabla_a\tilde{V}_{ab} = -\frac{1}{\alpha}D_a(\alpha D^a\tilde{R}_{ab}) + K\tilde{V}_{ab} + Y^{np}_{ab} + \Delta_c Y^p_{ab} \tag{4.20}$$

[TODO: Does it make sense?]

4.1 BSSN formulation

So far, we have

$$n^a\nabla_a R = -\hat{R} \tag{4.21}$$

$$
\begin{aligned}
n^a\nabla_a\hat{R} &= -\frac{1}{\alpha}D_a(\alpha D^a R) + K\hat{R} + \frac{1}{36\pi(3\beta_c - 2\alpha_c)}R && (4.22)\\
n^a\nabla_a\tilde{R}_{ab} &= -\tilde{V}_{ab} && (4.23)\\
n^a\nabla_a\tilde{V}_{ab} &= -\frac{1}{\alpha}D_a(\alpha D^a\tilde{R}_{ab}) + K\tilde{V}_{ab} + Y^{np}_{ab} + \Delta_c Y^p_{ab} && (4.24)
\end{aligned}
$$

(Note that I omit the equations from metric because it will be same as usual standard GR). These are the usual 3+1 decomposition (or ADM decomposition) which has been shown to be weakly hyperbolic. Here we recast again these sets of equation in terms of BSSN form. We will follow usual conventions for BSSN variable [TODO: Define vars.. but state here].

After some manipulation, we have

$$
D_a(\alpha D^a R) = \alpha\chi\left[\tilde{\gamma}^{ij}\partial_i\partial_j R + \tilde{\gamma}^{ij}(\partial_i \ln\alpha)\partial_j R - \tilde{\Gamma}^i\partial_i R - \frac{1}{2}\tilde{\gamma}^{ij}\partial_i R\partial_j \ln\chi\right] \tag{4.25}
$$

$$
D_a(\alpha D^a\tilde{R}_{ab}) = \partial_i\alpha\left(\partial_i\tilde{R}^{ab} - \frac{3}{2\chi}\tilde{R}^{ab}\partial_a\chi\right) + \alpha(D^aD_a)_{\text{BSSN}}\tilde{R}^{ab} \tag{4.26}
$$

Thus, we have

$$
\begin{aligned}
n^a\nabla_a R &= -\hat{R} && (4.27)\\
n^a\nabla_a\hat{R} &= -\alpha\chi\left[\tilde{\gamma}^{ij}\partial_i\partial_j R + \tilde{\gamma}^{ij}(\partial_i \ln\alpha)\partial_j R - \tilde{\Gamma}^i\partial_i R - \frac{1}{2}\tilde{\gamma}^{ij}\partial_i R\partial_j \ln\chi\right]\\
&\quad + K\hat{R} + \frac{1}{36\pi(3\beta_c - 2\alpha_c)}R && (4.28)\\
n^a\nabla_a\tilde{R}_{ab} &= -\tilde{V}_{ab} && (4.29)\\
n^a\nabla_a\tilde{V}_{ab} &= -\frac{1}{\alpha}\partial_i\alpha\left(\partial_i\tilde{R}^{ab} - \frac{3}{2\chi}\tilde{R}^{ab}\partial_a\chi\right) - (D^aD_a)_{\text{BSSN}}\tilde{R}^{ab}\\
&\quad + K\tilde{V}_{ab} + Y^{np}_{ab} + \Delta_c Y^p_{ab} && (4.30)
\end{aligned}
$$

4.2 Hyperbolicity of the System

[TODO: Characteristic analysis is required]

4.3 Gauge Choice and Full Evolution System

[TODO: add discussions on $1 + \log$ slicing and Γ-driver]

4.4 Simple Model Problem : Spherical Symmetry

As an example, we impose spherical symmetry in our system. In this section, we adopt $G_N = c = 1$ unit system. We will use our 3+1 decomposed system first.

Choice of Coordinate : Horizon Penetrating coordinate

We present a horizon penetrating coordinate for spherically symmetric metric. Consider usual Schwarzschild line element in Boyer-Lindquist coordinate

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{4.31}$$

where M is a mass. Compare this with usual 3+1 line element form $ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^2 + \beta^i dt)(dx^j + \beta^j dt)$ we can identify the lapse $\alpha^2 = \left(1 - \frac{2M}{r}\right)$. As we know, the line element(Eqn. 4.31) is singular at the horizon ($r = 2M$) and lapse collapse to zero. This can be problematic because equations of motion for the metric can be become exponentially unstable in the presence of a coordinate singularity without some regularization technique.

One way to resolve this problem is to move to a horizon penetrating coordinate system where this singularity is not present. The Kerr-Schild coordinates are one such coordinate system.

For example, Schwarzschild solution in spherical type Kerr-Schild coordinates

$$\alpha = \sqrt{\frac{r}{r + 2M}} \tag{4.32}$$

$$\beta^r = \frac{2M}{r + 2M} \tag{4.33}$$

$$\beta_r = \frac{2M}{r} \tag{4.34}$$

$$\beta^\theta = \beta^\varphi = 0 \tag{4.35}$$

$$K_{ij} = \mathrm{diag}\left[-\frac{2M(r + M)}{\sqrt{r^5(r + 2M)}}, 2M\sqrt{\frac{r}{r + 2M}}, K_{\theta\theta}\sin^2\theta\right] \tag{4.36}$$

Schwarzschild solution in Cartesian type Kerr-Schild coordinate

$$\alpha = \sqrt{\frac{r}{r + 2M}} \tag{4.37}$$

$$\beta^i = \frac{2M}{r}\frac{x^i}{r + 2M} \tag{4.38}$$

$$\beta_i = \frac{2Mx_i}{r^2} \tag{4.39}$$

$$K_{ij} = \frac{2M}{r^4}\sqrt{\frac{r}{r + 2M}}\left[\left(\frac{M}{r} + 2\right) x_i x_j - r^2\delta_{ij}\right] \tag{4.40}$$

where $x^i = (x, y, z)$ which is usual spatial Cartesian coordinate. In both cases, we can see lapse is regular at the horizon.

General spherical symmetric line element in polar-areal form

$$ds^2 = -\alpha(r)^2 dt^2 + a(r)^2 dr^2 + r^2 d\Omega^2 \tag{4.41}$$

where α is referred as lapse function. Compare with above Schwarzschild solution, $\alpha = 1/a$.

Now consider a transformation of the Schwarzschild time t coordinate to a new generic coordinate $\hat{t}$ according to

$$d\hat{t} = dt + a^2\sqrt{1 - \frac{g}{a^2}}dr \tag{4.42}$$

where $g(r)$ is arbitrary function. Substitute this into $ds^2 = -\alpha^2 dt^2 + a^2 dr^2 + r^2 d\Omega^2$ gives

$$\begin{aligned} ds^2 &= -\alpha^2\left(d\hat{t} - a^2\sqrt{1 - \frac{g}{a^2}}dr\right)^2 + a^2 dr^2 + r^2 d\Omega^2 \\ &= -\alpha^2 d\hat{t}^2 + 2\sqrt{1 - \frac{g}{a^2}}d\hat{t}dr + g dr^2 + r^2 d\Omega^2 \end{aligned} \tag{4.43}$$

Compare this with usual 3+1 framework

$$ds^2 = -\alpha^2 d\hat{t}^2 + \gamma_{ij}(dx^i + \beta^i d\hat{t})(dx^j + \beta^j d\hat{t}) \tag{4.44}$$

and so into the lapse $\alpha = 1/\sqrt{g}$, the shift $\beta_i = (\sqrt{1 - g/a^2}, 0, 0)$ or $\beta^i = \gamma^{ij}\beta_j$ and the spatial metric of the constant $\hat{t}$ hypersurface $\gamma_{ij} = diag(g, r^2, r^2\sin^2\theta)$.

If we choose $\alpha = \sqrt{1 - 2M/r} = 1/a$ and $g = 1 + 2M/r$ like in previous (which we will use this), we get

$$ds^2 = -\left(1 - \frac{2M}{r}\right)d\hat{t}^2 + \frac{4M}{r}d\hat{t}dr + \left(1 + \frac{2M}{r}\right)dr^2 + r^2 d\Omega^2 \tag{4.45}$$

which is Schwarzschild in Kerr-Schild coordinate (or Eddington-Finkelstein coordinate). And correspondingly, $\alpha = \sqrt{r/(r + 2M)}$, $\beta_i = (2M/r, 0, 0)$, and $\gamma_{ij} = diag(1 + 2M/r, r^2, r^2\sin^2\theta)$ which are same as above.

As you can see here, the KS (or EF) form of the metric represents an analytic expansion of the Schwarzschild solution from the region $2M < r < \infty$ to $0 < r < \infty$. Thus, we apply this coordinate transformation for our equations.

It is good to rewrite the metric into usual $3+1$ variable form i.e. keep it geometric variables (should be careful of confusion) with considering time dependent case. Here, we use t for time coordinate that we used above.

$$ds^2 = (-\alpha^2 + a^2\beta^2)dt^2 + 2a^2\beta dt dr + a^2 dr^2 + r^2 b^2 d\Omega^2 \tag{4.46}$$

where α, a, b, and β are functions of r and t, and $d\Omega^2$ is the metric of unit sphere. From this, we can calculate non-vanishing components of connection coefficients and Ricci tensors for i, j, and k (spatial indices)

$$\begin{aligned} &\Gamma^r{}_{rr} = \frac{\partial_r a}{a}, \quad \Gamma^r{}_{\theta\theta} = -\frac{rb\partial_r(rb)}{a^2}, \quad \Gamma^\theta{}_{r\theta} = \frac{\partial_r(rb)}{rb} \\ &\Gamma^r{}_{\varphi\varphi} = -\sin^2\theta\frac{rb\partial_r(rb)}{a^2}, \quad \Gamma^\varphi{}_{r\varphi} = \Gamma^\theta{}_{r\theta} \\ &\Gamma^\theta{}_{\varphi\varphi} = -\sin\theta\cos\theta, \quad \Gamma^\varphi{}_{\varphi\theta} = -\cot\theta \end{aligned}$$

$$R^r{}_r = -\frac{2}{arb}\partial_r\left(\frac{\partial_r(rb)}{a}\right) \tag{4.47}$$

$$R^\theta{}_\theta = \frac{1}{ar^2b^2}\left[a - \partial_r\left(\frac{rb\partial_r(rb)}{a}\right)\right] \tag{4.48}$$

$$\tag{4.49}$$

5 Initial Data

Under usual 3+1 decomposition [Still same for QG?], the constraint equations are (in vacua)

$$D_j K^j{}_i - D_i K = 0 \tag{5.1}$$

$$R + K^2 - K_{ij}K^{ij} = 0 \tag{5.2}$$

The spatial metric γ_{ij}, the extrinsic curvature K_{ij}, and any matter field should satisfy the constraints. Thus, we have to specify (γ_{ij}, K_{ij}) on some initial spatial slice Σ that are compatible with the constraint equations. These fields can then be used as initial data for a dynamical evolution obtained by solving the evolution equation.

5.1 Elementary Black Hole Solution

[Might use Brill-Lindquist for testing case.
Put formula for BSSN or GH like Schwarzschild in terms of correct coordinate]

5.2 Black Holes in QG

[We may use some analytic solutions in literatures]

5.3 Binary Black Hole Initial Data

Puncture Method

[Maybe solve elliptic equations or just find puncture like ID]

6 Gravitational Wave Extractions for QG

Here, we calculate the Ψ_4 to extract the gravitational wave information. To do that, we first define tetrad. There are lots of possible ways to do this but we will try to follow the way I know (a way is in `hyperGHSF` code). We define the timelike member of the tetrad to be the normal to our spacelike hypersurfaces. The remaining three are then constructed via a Gram-Schmidt procedure from a set of three independent vectors living on the hypersurfaces. Our demand for them seem to be only that in the asymptotically flat limit, we recover something akin to the usual unit vectors of spherical coordinates.

Indeed, we start with a version of them

$$u^a = (0, x, y, z) \tag{6.1}$$

$$v^a = (0, xz, yz, -x^2 - y^2) \tag{6.2}$$
$$w^a = (0, -y, x, 0) \tag{6.3}$$

and then using the 3-metric, γ_{ij} ,orthonormalize them with respect to it. In particular, we define new orthonormal spacelike vectors

$${}^{(1)}e^i = \frac{u^i}{||u||} \tag{6.4}$$
$${}^{(2)}e^i = \frac{v^i - \langle {}^{(1)}e|v\rangle\ {}^{(1)}e^i}{||v - \langle {}^{(1)}e|v\rangle\ {}^{(1)}e||} \tag{6.5}$$
$${}^{(3)}e^i = \frac{w^i - \langle {}^{(1)}e|w\rangle\ {}^{(1)}e^i - \langle {}^{(2)}e|w\rangle\ {}^{(2)}e^i}{||w - \langle {}^{(1)}e|w\rangle\ {}^{(1)}e - \langle {}^{(2)}e|w\rangle\ {}^{(2)}e||} \tag{6.6}$$

where we are defining the inner product and the norm as

$$\langle u|v\rangle \equiv \gamma_{ij} u^i v^j \tag{6.7}$$
$$||u|| \equiv \sqrt{\langle u|u\rangle} \tag{6.8}$$

with these, we construct a null tetrad according to

$$l^a = \frac{1}{2}(n^a + {}^{(1)}e^a) \tag{6.9}$$
$$\tilde{n}^a = \frac{1}{2}(n^a - {}^{(1)}e^a) \tag{6.10}$$
$$m^a = \frac{1}{2}({}^{(2)}e^a + i\,{}^{(3)}e^a) \tag{6.11}$$
$$\bar{m}^a = \frac{1}{2}({}^{(2)}e^a - i\,{}^{(3)}e^a) \tag{6.12}$$

where, because we are running out of letters, the usual null vector n^a has been written with a tilde to distinguish if from the normal to the foliation. Then we can compute the relevant complex Penrose scalar $\Psi4$ such that

$$\Psi_4 = C_{abcd}\tilde{n}^a \bar{m}^b \tilde{n}^c \bar{m}^d \tag{6.13}$$
$$= C_{abcd}\tilde{n}^a \tilde{n}^c \left[\frac{1}{2}\{{}^{(2)}e^b\,{}^{(2)}e^d -{}^{(3)}e^b\,{}^{(3)}e^d\} + i\,{}^{(2)}e^b\,{}^{(3)}e^d\right] \tag{6.14}$$

Now we must decompose this with respect to an assumed spacelike hypersurface. As usual, we define the normal to the hypersurface as n_a, the metric on the hypersurface as γ_{ij} and the extrinsic curvature as K_{ij}. [Here is subtly. If we do notneed to consider additional higher derivatives of curvature into this manner, we may use same procedure as Einstein GR but not sure]