Orlando Alvarez

Physics 12a -- Fall 1976

General Information

Physics 12a is designed for the student who is planning to take more advanced courses in physics or physical chemistry. It is not intended to stand on its own as a one-year survey of general physics.

Physics Prerequisite: There is no physics prerequisite. If you have not had high school physics you will obviously be less familiar with the central concepts, and therefore should make special efforts to gain familiarity and practice as we go along. The Tipler text may prove especially helpful to you by emphasizing the more basic aspects of the material.

Math Prerequisite: You must have a good working knowledge of differential and integral calculus of one variable. Any one-year course ought to suffice. You should also be taking Applied Math 21 or Math 21 concurrently if you have not already had a second-year calculus course. Physics 12b will become involved with functions of more than one variable and Physics 112 with partial differential equations and Fourier integrals. If your math background doesn't keep generally apace, you may get into trouble.

Although either Applied Math 21 or Math 21 is appropriate, we call your attention to the existence of a special section of Math 21 which is designed to relate more closely to Physics 12 and 55, both in the examples used and the order in which the topics are covered.

Staff:				
	Professor Sidney Coleman	333	Lyman	Tue, Thurs. 2-3
	Professor Bill Skocpol	362a	Jefferson	Mon, Wed. 9-10
	Dr. Jay Blake	258	Jefferson	Mon, Fri. 2-3
	Teaching Fellows -- office hours to be announced			

Textbook: Kleppner and Kolenkow, Introduction to Mechanics is the primary textbook used in the course and should be purchased by everyone. Tipler's *Physics* is an interesting new book which covers many of the topics in this course at a more basic level. It is rich in explanation and exercises and is strongly recommended as supplementary reading, particularly for those whose previous exposure to mechanics is weak.

Requirements: The course work will be organized into twelve units, each of which will include lectures, a problem set, a section meeting, and a set of three equivalent unit tests (you may take more than one for practice or to improve your score). In a given week, you will be reading and attending lectures for the next unit, completing your problem set and bringing it to your section meeting on the present unit, and taking unit tests on previous units. This allows lots of practice and several cuts at the material. In reality, this course is highly paced, not "self-paced" as it is sometimes referred to. You must commit yourself to a heavy steady workload or you will fall hopelessly behind.

In addition to the above routine, there will be opportunities for laboratory work, and a final examination.

Problem sets for each unit will consist of approximately eight homework problems. They will be graded in the section meeting for that unit on the following somewhat informal scale:

Attempted	Correct	Score
All	6	4
6	4	3
4	2	2
Anything at all right		1

Sections (two hours once a week) will discuss the problem sets and other material related to the unit. Available laboratory experiments will also be introduced. You should fill out the enclosed section request form at the first meeting of the course. Preliminary assignments will be handed out at the second meeting and posted outside the Physics Focus office. To request a subsequent change, fill out a form available at lecture or Physics Focus. Sections will begin the second week of the course (Oct. 5).

Unit tests: Several tests for each unit will be available, beginning Thursday of the week the unit is being discussed in section. The Test Center will be located in Science Center 224 and will be open on Mondays and Thursdays from 2pm - 10pm. You will check out a version of the test and work on it for no more than one hour. When it is completed bring it to a grader who will check it over and assign a score or give you advice and a second chance. The score will be as follows: Perfect first time -- 6 points; minor error -- 5 points; correct after some help -- 4 points; correct after lots of help -- 3 points. You make take additional versions of the test on other days to improve your score. (The highest score will be counted.)

Laboratory: You will have the opportunity to earn up to 20 additional points by submitting "scientific essays" based on observational or experimental data combined with an appropriate analysis of the situation. You will be introduced to some of the available experiments in section and can work with the apparatus on a first-come first-served basis during the hours that the lab rooms are open (to be announced). A standard treatment of a standard experiment, such as measuring g by two different methods and comparing the results will be worth 5 points. More extensive projects can earn a larger fraction of the twenty points. Your "scientific essays" need not be based on predetermined laboratory experiments. For example, an expedition to an amusement park with

measuring tapes and a stopwatch could provide the basis for a very interesting paper. You may work together in small groups gathering the data, but each person should write his or her own report. Turn the reports in to your section instructor.

Grading: During the term you will have the opportunity to earn up to 140 points (10 for each unit, and 20 for labs), although a score of 110 might be more typical. The course will be graded on the basis of 200 points, with the final exam counting the difference between what you already have and 200. Thus a person who gets only 50% on the final will end up with 170 points if he or she has earned all 140 prefinal points, but will get only 100 points if he or she has done nothing else all term. Based on standards used in the recent past, an overall total of 160 or so ought to be worth a B minus, and a total of 185 or so an A minus.

Mini-syllabus (unit structure as presently projected)

Unit 1 -- Vectors and Kinematics (K&K Ch. 1)
Unit 2 -- Conservation laws and isolated systems (Ch. 3)
Unit 3 -- Fundamental applications of $\vec{F} = m\vec{a}$ (Ch. 2)
Unit 4 -- Applications of $\vec{F} = m\vec{a}$ which lead to simple differential equations (Ch. 2)
Unit 5 -- Energy concepts and conservative systems (Ch. 4; 5 optional)
Unit 6 -- The harmonic oscillator system (Ch. 10)
Unit 7 -- The central force system (Ch. 9)
Units 8-10 -- Special Relativity (Chs. 11-14)
Units 11-12 -- Rotational motion (Chs. 6-7)

EXTRA COPIES OF ALL MATERIALS HANDED OUT AT LECTURES
USUALLY WILL BE AVAILABLE AT THE PHYSICS FOCUS OFFICE (101b S.C.)

Messages for the staff may also be left there.

PHYSICS 12a MATHEMATICAL HANDBOOK

(Cut this up and fasten it into your 3"x5" notebook)

QUADRATIC EQUATIONS: $ax^2 + bx + c = 0 \qquad x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

BINOMIAL EXPANSION: $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \cdots$

The series terminates if α is a positive integer. Otherwise it converges for $|x| < 1$.

TRIGONOMETRIC IDENTITIES: $\sin^2 x + \cos^2 x = 1$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \qquad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\sin 2x = 2 \sin x \cos x \qquad \cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

$\sin^2 \tfrac{1}{2}x = \frac{1-\cos x}{2} \qquad \cos^2 \tfrac{1}{2}x = \frac{1+\cos x}{2} \qquad \tan^2 \tfrac{1}{2}x = \frac{1-\cos x}{1+\cos x}$

$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} \qquad \sec^2 x = \tan^2 x + 1 \qquad \csc^2 x = \cot^2 x + 1$

For small x: $\sin x \approx x \qquad \cos x \approx 1 - \tfrac{1}{2}x^2 \qquad \tan x \approx x$

EXPONENTIALS AND LOGARITHMS:

If $e^y = x$ then $y = \ln x$

$e^{x+y} = e^x e^y \qquad e^{x-y} = e^x/e^y \qquad e^{\alpha x} = [e^x]^\alpha$

$\ln(xy) = \ln x + \ln y \qquad \ln(x/y) = \ln x - \ln y \qquad \ln(x^\alpha) = \alpha \ln x$

$e = 2.718 \qquad e^{-1} = 0.368 \qquad \ln 2 = 0.7 \qquad \ln 3 = 1.1 \qquad \ln 10 = 2.3$

HYPERBOLIC FUNCTIONS:

$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad e^{\pm x} = \cosh x \pm \sinh x$

$\cosh^2 x - \sinh^2 x = 1$

DIFFERENTIATION and INTEGRATION:

$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx} \qquad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ Chain rule: If $y = y(u)$ and $u = u(x)$ then $dy/dx = dy/du \; du/dx$

If $u(x) = dy/dx$ then $\int u(x')\,dx' = y(x) + \text{constant}$

DERIVATIVES:

$y(x)$	dy/dx
x^n	nx^{n-1}
$\ln x$	$1/x$
$\sin kx$	$k \cos kx$
$\cos kx$	$-k \sin kx$
e^{kx}	ke^{kx}
$\sin^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{a^2-x^2}}$
$\cos^{-1} x/a$	$-\frac{1}{\sqrt{a^2-x^2}}$
$\tan^{-1} x/a$	$\frac{1}{a}\frac{1}{1+x^2/a^2}$
$\sinh^{-1} x/a$	$\frac{1}{\sqrt{a^2+x^2}}$

INTEGRALS:

$\int x^n dx = \frac{1}{n+1}x^{n+1} \qquad \int \frac{dx}{x} = \ln x$

$\int \cos kx\,dx = \frac{1}{k}\sin kx \qquad \int \sin kx\,dx = -\frac{1}{k}\cos kx$

$\int e^{kx}dx = \frac{1}{k}e^{kx} \qquad \int \tan x\,dx = -\ln|\cos x|$

$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} = -\cos^{-1}\frac{x}{a}$

$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

$\int \frac{dx}{a^2-x^2} = \frac{1}{a}\tanh^{-1}\frac{x}{a}$

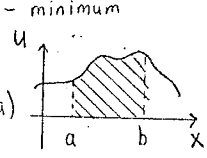
$\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a}$

$\int_0^\infty e^{-ax^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \qquad \int_0^\infty e^{-ax}dx = 1/a$

Maxima and minima of $y(x)$ occur where $dy/dx = 0$

If $d^2y/dx^2 < 0$ – maximum, if $d^2y/dx^2 > 0$ – minimum

Area under curve $u(x)$ between a and b is $\int_a^b u(x)\,dx$. To convert to average divide by $(b-a)$



Averages for a full cycle: $\overline{\sin \omega t} = \overline{\cos \omega t} = 0 \qquad \overline{\sin^2 \omega t} = \overline{\cos^2 \omega t} = \tfrac{1}{2}$

POWER SERIES EXPANSIONS:

Taylor's series: $f(a+x) = f(a) + f'(a)x + \frac{1}{2}f''(a)x^2 + \cdots$

$\sin x = x - x^3/6 + x^5/120 + \cdots \qquad \cos x = 1 - \frac{x^2}{2} + x^4/24 + \cdots$

$\sinh x = x + x^3/6 + x^5/120 + \cdots \qquad \cosh x = 1 + x^2/2 + x^4/24 + \cdots$

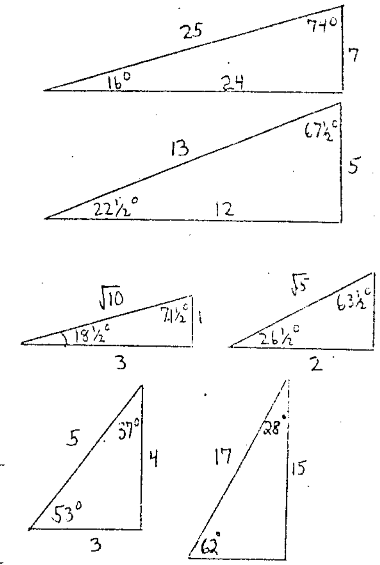
$e^x = 1 + x + x^2/2 + x^3/6 + \cdots \qquad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$(1 \pm x)^{1/2} = 1 \pm \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \cdots \qquad (1 \pm x)^{-1/2} = 1 \mp \tfrac{1}{2}x + \tfrac{3}{8}x^2 + \cdots$

$(1 \pm x)^{-1} = 1 \mp x + x^2 \mp x^3 + \cdots \qquad \frac{1-x^{n+1}}{1-x} = 1 + x + x^2 + \cdots + x^n$

TRIGONOMETRY and TRIANGLES:

θ	θ(rad)	$\sin\theta$	$\cos\theta$	$\tan\theta$
16°	0.28	7/25	24/25	7/24
18½°		$1/\sqrt{10}$	$3/\sqrt{10}$	1/3
22½°	$\pi/8$	5/13	12/13	5/12
26½°		$1/\sqrt{5}$	$2/\sqrt{5}$	1/2
28°	0.49	8/17	15/17	8/15
30°	$\pi/6$	1/2	$\sqrt{3}/2$	$\sqrt{3}/3$
37°	0.65	3/5	4/5	3/4
45°	$\pi/4$	$1/\sqrt{2}$	$1/\sqrt{2}$	1
53°	0.93	4/5	3/5	4/3
60°	$\pi/3$	$\sqrt{3}/2$	1/2	$\sqrt{3}$
62°	1.08	15/17	8/17	15/8
63½°		$2/\sqrt{5}$	$1/\sqrt{5}$	2
67½°	$3\pi/8$	12/13	5/13	12/5
71½°		$3/\sqrt{10}$	$1/\sqrt{10}$	3
74°	1.29	24/25	7/25	24/7



$\sqrt{2} = 1.414 \qquad \frac{1}{\sqrt{2}} = .707 \qquad \sqrt{3} = 1.732 \qquad \sqrt{3}/2 = .866 \qquad \sqrt{3}/3 = .577 \qquad \sqrt{5} = 2.24 \qquad \sqrt{10} = 3.18$

VECTORS: ($\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ are a right-handed triad of unit vectors)

$\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = 1 \qquad \hat{\imath}\cdot\hat{\jmath} = \hat{\imath}\cdot\hat{k} = \hat{\jmath}\cdot\hat{k} = 0$

$\hat{\imath}\times\hat{\jmath} = \hat{k} \qquad \hat{\jmath}\times\hat{k} = \hat{\imath} \qquad \hat{k}\times\hat{\imath} = \hat{\jmath}$

$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z = \vec{B}\cdot\vec{A} \qquad \vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$

$\vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\hat{\imath} + (A_zB_x - A_xB_z)\hat{\jmath} + (A_xB_y - A_yB_x)\hat{k}$

$\vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C} = \vec{B}\cdot(\vec{C}\times\vec{A})$

$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{C}\times\vec{B})\times\vec{A} = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

$\frac{d}{dt}(\vec{A}\cdot\vec{B}) = \frac{d\vec{A}}{dt}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{dt}$

$\frac{d}{dt}(\vec{A}\times\vec{B}) = \frac{d\vec{A}}{dt}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{dt}$

DIFFERENTIAL EQUATIONS:

To solve $\dot{x} = f(t)$, just integrate both sides

To solve $\dot{x} = f(x)$, separate variables:

$dx/dt = f(x)$; $dx/f(x) = dt$ – now integrate both sides

To solve $\ddot{x} = f(x)$ or $\ddot{x} = f(\dot{x})$, introduce $v \equiv \dot{x}$

$\ddot{x} = \dot{v} = dv/dt = dv/dx \; dx/dt = v\,dv/dx$

Now solve $v\,dv/dx = f(x)$ or $v\,dv/dx = f(v)$ by separating variables.

Linear equations: If $\dot{x} = -\gamma x$, $x = x_0 e^{-\gamma t}$

If $\ddot{x} = -\omega_0^2 x$, $x = A\cos(\omega_0 t + \varphi)$

For other linear equations use trial solution $e^{\alpha t}$

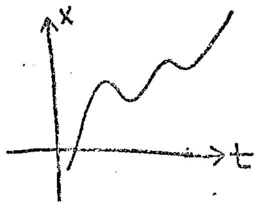
①

Newton's First Law

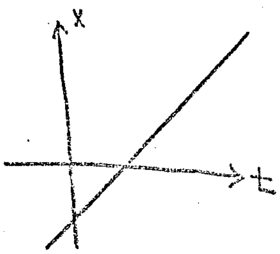
"An isolated particle moves in a straight line with constant velocity."

Isolated?

Particle? $x(t)$



NOT ISOLATED



ISOLATED

②

One-Dimensional World

① $x = vt + b$

v, b: constants

② $\frac{dx}{dt} \equiv v = \text{constant}$ ← "velocity"

③ $\frac{d^2x}{dt^2} \equiv a = 0$ ← "acceleration"

Three Equivalent Statements

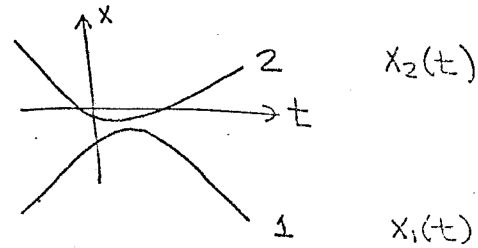
③ is an "equation of motion"

- differential equation that ennables us to predict $x(t)$ in terms of $x(t=0)$ [b] and $\frac{dx}{dt}\Big|_{t=0}$ $\Rightarrow$ [v].

③

Mechanics "From the Outside"

(A quick look at asymptotic conservation laws)



$$x_1(t) = v_1^{in}\,t + b_1^{in} \qquad \text{(far past)}$$

$$= v_1^{out}\,t + b_1^{out} \qquad \text{(far future)}$$

likewise for x_2

④

First guess:

$$v_1^{in} + v_2^{in} \stackrel{?}{=} v_1^{out} + v_2^{out}$$

"Law of (Asymptotic) Conservation of Total Velocity"

No good!

Try again:

$$m_1 v_1^{in} + m_2 v_2^{in} \stackrel{?}{=} m_1 v_1^{out} + m_2 v_2^{out}$$

"Law of (Asymptotic) Conservation of Total Momentum"

m is called "mass"

This is <u>the</u> definition of mass.

mv is called "momentum" sometimes denoted by p

$$p = mv$$

(5)

Three Comments

(1) Can be generalized to many incoming + outgoing particles

$$m_1^{in} V_1^{in} + m_2^{in} V_2^{in} + \cdots m_{N^{in}}^{in} V_{N^{in}}^{in}$$
$$\equiv \sum_{a=1}^{N^{in}} m_a^{in} V_a^{in}$$
$$= \sum_{a=1}^{N^{out}} m_a^{out} V_a^{out}$$

N. B. Incoming + Outgoing particles need not be equal in number or character.

(6)

(2) Can there be two equally valid definitions of mass?

$$m_1 V_1^{in} + m_2 V_2^{in} = m_1 V_1^{out} + m_2 V_2^{out}$$
and $$m_1' V_1^{in} + m_2' V_2^{in} = m_1' V_1^{out} + m_2' V_2^{out}$$

multiply second equation by $\frac{m_1}{m_1'}$

$$m_1 V_1^{in} + \frac{m_2' m_1}{m_1'} V_2^{in} = m_1 V_1^{out} + \frac{m_2' m_1}{m_1'} V_2^{out}$$

subtract

$$\left(\frac{m_2' m_1}{m_1'} - m_2\right)(V_2^{in} - V_2^{out}) = 0$$

Either $V_2^{in} = V_2^{out}$ (A lie!)

or $$\frac{m_2'}{m_1'} = \frac{m_2}{m_1}$$

(7)

But we can always choose $m_1 = m_1'$
(Choice of unit of mass)

$\therefore$ $m_2 = m_2'$.

(3) Galilean Invariance

$$X_a(t) \rightarrow X_a(t) + \underbrace{U}_{\text{constant}} t$$

Turns physically possible motions into " " "

$$\sum_{a=1}^{N^{in}} m_a^{in} V_a^{in} = \sum_{a=1}^{N^{out}} m_a^{out} V_a^{out}$$
$$\Rightarrow \sum_{a=1}^{N^{in}} m_a^{in} (V_a^{in} + U) = \sum_{a=1}^{N^{out}} m_a^{out} (V_a^{out} + U)$$
subtract $$\sum_{a=1}^{N^{in}} m_a^{in} U = \sum_{a=1}^{N^{out}} m_a^{out} U$$
$$\Rightarrow \sum_{a=1}^{N^{in}} m_a^{in} = \sum_{a=1}^{N^{out}} m_a^{out}$$

(Asymptotic) Conservation of Mass!

Another Conservation Low

$$\tfrac{1}{2} M_1 (V_1^{in})^2 + \tfrac{1}{2} M_2 (V_2^{in})^2 = \tfrac{1}{2} M_1 (V_1^{out})^2 + \tfrac{1}{2} M_2 (V_2^{out})^2$$

(Asymptotic) Conservation of (Kinetic) Energy

Not universally true; valid only for "elastic collisions" (orphic remark)

Apply Galilean Invariance

$$\tfrac{1}{2} M_1 (V_1^{in} + U)^2 + \tfrac{1}{2} M_2 (V_2^{in} + U)^2 = \tfrac{1}{2} M_1 (V_1^{out} + U)^2 + \tfrac{1}{2} M_2 (V_2^{out} + U)^2$$

Extract Coeficient of U

$$M_1 V_1^{in} + M_2 V_2^{in} = M_1 V_1^{out} + M_2 V_2^{out}$$

$\therefore$ $M_1 = m_1$, $M_2 = m_2$ (up to choice of units)

9

Mechanics from the Inside
(the rest of the course)

Newtons Second Law

$$\underset{\substack{\uparrow \\ \text{"force"}}}{F} = ma$$

$$= m\frac{d^2x}{dt^2}$$

$$= m\frac{dv}{dt}$$

$$= \frac{dp}{dt}$$

Strictly speaking just <u>definition</u> of F. <u>However</u>

(1) Focusses attention on $\frac{dp}{dt}$.

(2) In most (if not all) cases we will study

$$m\frac{d^2x_a}{dt^2} = \underset{\substack{\uparrow \\ \text{force on} \\ \text{ath particle}}}{F_a} = \sum_{b\neq a} \underset{\substack{\uparrow \\ \text{sum of forces exerted by} \\ \text{all other particles}}}{F_{ab}}$$

10

Furthermore,

F_{ab} is a function only of $x_a - x_b$ (relative position) and $v_a - v_b$ (relative velocity)

N.B. Under these conditions both sides of the second law are unchanged by Galilean transformations.

Newton's Third Law

$$F_{ab} = -F_{ba}$$

"Action Equals Reaction"

11

Consequence

$$\underset{\substack{\uparrow \\ \text{total momentum}}}{P} = \sum_{a=1}^{N} p_a \equiv \sum_{a=1}^{N} m_a v_a$$

$$\frac{dP}{dt} = \sum_{a=1}^{N} \frac{dp_a}{dt}$$

$$= \sum_{a=1}^{N} \sum_{\substack{b\neq a \\ b=1}}^{N} F_{ab}$$

The terms in this sum cancel pairwise

$F_{12} + F_{21} = 0$

$F_{13} + F_{31} = 0$ etc.

$$\boxed{\frac{dP}{dt} = 0}$$ Conservation of Momentum

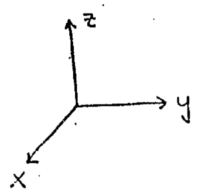
12

Warning:

There is no guarantee (expess or implied) that all lectures will be presented in this format with notes handed out.

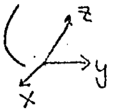
Be prepared to take your own notes at any time.

A few words on (Cartesian, right-handed) coordinates



Three Criteria:

(1) Axes are perpendicular

(

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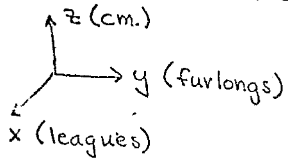
(2) Length measured in same units in all three directions

(

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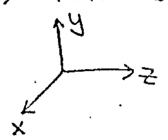
(3) Must be right-handed

(

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Given a coordinate system, we associate with every point a triplet of real numbers (x, y, z), its coordinates.

Some freedom in choosing coordinates:

(1) Origin can be anywhere

(We can <u>spatially translate</u> the coordinate system)

(2) Axes can be oriented any which way (consistent with $\perp$ condition, rt. hnd. condition)

(We can <u>rotate</u> the coord. system)

Vectors

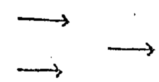
Prototype: Displacement in space

$\vec{A}$

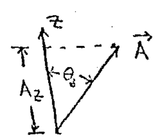
has magnitude and direction

$|\vec{A}| \equiv$ length of $\vec{A}$

Vectors have <u>only</u> mag. + dir.
If two vectors have same mag. + dir. they are <u>equal</u>


} Three equal Vectors

Components of Vectors



$A_z = |\vec{A}| \cos\theta_z$

Likewise for A_x, A_y

$$\vec{A} \underset{\text{system}}{\overset{\text{coordinate}}{\Longleftrightarrow}} (A_x, A_y, A_z)$$

Note: Origin of coordinates irrelevant;
Orientation of axes relevant.

Contrast with <u>scalars</u> quantities that have magnitude but not direction — neither origin of coord. nor orientation of axes relevant to associating a number with a scalar.

5

Some Scalars

- Length
- Temperature
- Mass
- Pressure

Some Vectors

- Displacement
- Velocity
- Acceleration *
- Force *

* to be shown

Position is usually treated as an "honorary vector" (displacement from the origin of coord.)

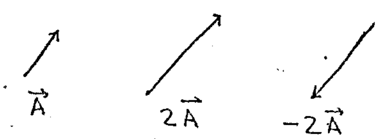
$$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2} \quad \text{(Pythagoras)}$$

$|\vec{A}| = 0$ if and only if $\vec{A} = 0$.

6

Operations on Vectors

(1) Multiplication by a scalar



$$\vec{A} \iff (A_x, A_y, A_z)$$

$$\underset{\uparrow \text{ Scalar}}{\alpha} \vec{A} \iff (\alpha A_x, \alpha A_y, \alpha A_z)$$

Associative Law:

$$\alpha(\beta \vec{A}) = (\alpha\beta)\vec{A}$$

α, β ← scalars

$$|\alpha \vec{A}| = |a|\,|\vec{A}|$$

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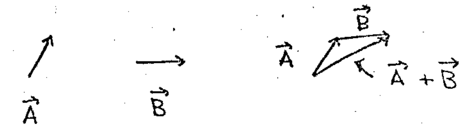
(2) Addition of Vectors



$$\vec{A} \iff (A_x, A_y, A_z)$$

$$\vec{B} \iff (B_x, B_y, B_z)$$

$$\vec{A}+\vec{B} \iff (A_x+B_x, A_y+B_y, A_z+B_z)$$

$$\vec{A}-\vec{B} \equiv \vec{A} + (-\vec{B}) \equiv \vec{A} + [(-1)\vec{B}]$$

Trivia:

$$\vec{A}+\vec{B} = \vec{B}+\vec{A}$$

$$\vec{A}+(\vec{B}+\vec{C}) = (\vec{A}+\vec{B})+\vec{C}$$

$$\alpha(\vec{A}+\vec{B}) = (\alpha\vec{A}) + (\alpha\vec{B})$$

$$\vec{A}+(-\vec{A}) = \vec{0}$$

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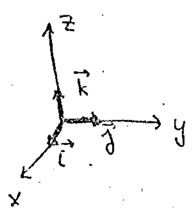
Basis Vectors

$\vec{i}$, $\vec{j}$, and $\vec{k}$ are three vectors of unit length oriented along the x, y, and z axes



$$\vec{i} \iff (1,0,0)$$

$$\vec{j} \iff (0,1,0)$$

$$\vec{k} \iff (0,0,1)$$

Warning: What three vectors we call $\vec{i}$, $\vec{j}$, $\vec{k}$ depends on how the coordinates are oriented.

$$\boxed{\vec{A} = A_x \vec{i} + A_y \vec{j} + A_z \vec{k}}$$

Don't have to write no $\iff$'s no more

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Operations (Con't)

Products

(3) Scalar (Dot) Product $\Rightarrow$ Scalar

(4) Vector (Cross) Product $\Rightarrow$ Vector

Nice Criteria for a Product.

(a) $P(\vec{A},\vec{B})$ function of pairs of vectors

(b) $P(\alpha\vec{A},\vec{B}) = \alpha P(\vec{A},\vec{B})$

$P(\vec{A},\beta\vec{B}) = \beta P(\vec{A},\vec{B})$

(c) $P(\vec{A}+\vec{B},\vec{C}) = P(\vec{A},\vec{C}) + P(\vec{B},\vec{C})$

$P(\vec{A},\vec{B}+\vec{C}) = P(\vec{A},\vec{B}) + P(\vec{A},\vec{C})$

(d) $P(\vec{A},\vec{B})$ should not depend on the choice of coordinates

N.B. If $P(\vec{A},\vec{B})$ satisfies (a)-(d), so does $\lambda P(\vec{A},\vec{B})$

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From (a), (b) + (c)

$$P(\vec{A},\vec{B}) = P(A_x\vec{i} + A_y\vec{j} + A_z\vec{k},\ B_x\vec{i} + B_y\vec{j} + B_z\vec{k})$$

$$= A_xB_x P(\vec{i},\vec{i}) + A_xB_y P(\vec{i},\vec{j}) + A_xB_z P(\vec{i},\vec{k})$$

$$+ A_yB_x P(\vec{j},\vec{i}) + A_yB_y P(\vec{j},\vec{j}) + A_yB_z P(\vec{j},\vec{k})$$

$$+ A_zB_x P(\vec{k},\vec{i}) + A_zB_y P(\vec{k},\vec{j}) + A_zB_z P(\vec{k},\vec{k})$$

Thus we need only know

$$\begin{array}{ccc} P(\vec{i},\vec{i}) & P(\vec{i},\vec{j}) & P(\vec{i},\vec{k}) \\ P(\vec{j},\vec{i}) & P(\vec{j},\vec{j}) & P(\vec{j},\vec{k}) \\ P(\vec{k},\vec{i}) & P(\vec{k},\vec{j}) & P(\vec{k},\vec{k}) \end{array}$$

to compute $P(\vec{A},\vec{B})$ for <u>any</u> $\vec{A}$ and $\vec{B}$

11

To exploit (d)

If in one system of coord. three vectors are called

$\vec{i}\quad \vec{j}\quad \vec{k}$

there is another in which they are

$\vec{j}\quad \vec{k}\quad \vec{i}$

one in which they are

$\vec{k}\quad \vec{i}\quad \vec{j}$

one in which they are

$-\vec{i}\quad -\vec{j}\quad \vec{k}$

one in which they are

$\vec{i},\ -\vec{j},\ -\vec{k}$

and one in which they are

$\vec{j},\ -\vec{i},\ \vec{k}$

12

Scalar (Dot Product)

$\vec{A}\cdot\vec{B}$ = a scalar

$\vec{i}\cdot\vec{i} = \vec{j}\cdot\vec{j} = \vec{k}\cdot\vec{k} = 1$ (choice of scale)*

$\vec{i}\cdot\vec{k} = (-\vec{i})\cdot\vec{k} = -(\vec{i}\cdot\vec{k}) = 0$

likewise $\vec{k}\cdot\vec{i} = \vec{i}\cdot\vec{j} = \vec{j}\cdot\vec{i} = \vec{j}\cdot\vec{k} = \vec{k}\cdot\vec{j} = 0$

$$\boxed{\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z}$$

*λ on p.9

Some Properties of the Dot Product

(13)

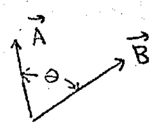
$$\vec A\cdot\vec A = |\vec A|^2$$

$$\vec A\cdot\vec B = \vec B\cdot\vec A$$

$$\vec A\cdot\vec i = A_x, \quad \vec A\cdot\vec j = A_y, \quad \vec A\cdot\vec k = A_z$$

$$|\vec A+\vec B|^2 = (\vec A+\vec B)\cdot(\vec A+\vec B)$$
$$= \vec A\cdot\vec A + \vec B\cdot\vec A + \vec A\cdot\vec B + \vec B\cdot\vec B$$

$$|\vec A+\vec B|^2 = |\vec A|^2+|\vec B|^2+2\,\vec A\cdot\vec B$$



choose coord's so $\vec A = \vec i\,|\vec A|$.

$$\vec A\cdot\vec B = |\vec A|\ \vec i\cdot\vec B = |\vec A|\,B_x$$

$$\boxed{\vec A\cdot\vec B = |\vec A|\,|\vec B|\cos\theta}$$

If $\vec A\cdot\vec B = 0$ either

(a) $\vec A = 0$

(b) $\vec B = 0$

or (c) $\vec A$ and $\vec B$ are $\perp$.

Cross Product

(14)

$\vec A\times\vec B$ is a vector

$$\vec i\times\vec i = \alpha\vec i + \beta\vec j + \gamma\vec k$$
$$\vec i\times\vec i = \alpha\vec i - \beta\vec j - \gamma\vec k \quad\Rightarrow\quad \beta=\gamma=0$$
$$(-\vec i)\times(-\vec i) = \alpha(-\vec i)$$
$$\vec i\times\vec i = -\alpha\vec i \quad\Rightarrow\quad \alpha = 0$$

$$\boxed{\vec i\times\vec i = \vec j\times\vec j = \vec k\times\vec k = \vec 0}$$

$$i\times\vec j = \delta\vec i + \epsilon\vec j + \vec k$$ ← chosen $=1$ by choice of λ (as before)

$$(-i)\times(-\vec j) = -\delta\vec i - \epsilon\vec j + \vec k \quad\Rightarrow\quad \delta=\epsilon=0$$

$$\boxed{\vec i\times\vec j = \vec k \qquad \vec j\times\vec k = \vec i \qquad \vec k\times\vec i = \vec j}$$

$$\vec j\times(-\vec i) = \vec k = -\vec j\times\vec i$$

$$\boxed{\vec j\times\vec i = -\vec k \qquad \vec k\times\vec j = -\vec i \qquad \vec i\times\vec k = -\vec j}$$

$k \overset{\nearrow i \searrow}{\longleftarrow} j$ ⟸ MNEMONIC

(15)

$$\vec A\times\vec B = (A_x\vec i + A_y\vec j + A_z\vec k)\times(B_x\vec i + B_y\vec j + B_z\vec k)$$

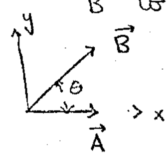
$$\boxed{\begin{aligned}\vec A\times\vec B = &(A_yB_z - B_yA_z)\vec i\\ &+(A_zB_x - B_zA_x)\vec j\\ &+(A_xB_y - A_yB_x)\vec k\end{aligned}}$$

Properties of Cross Product

$$\vec A\times\vec B = -\vec B\times\vec A$$

choose $\vec A = |\vec A|\,\vec i$

$\vec B$ to lie in x,y plane.



$$\vec A\times\vec B = B_yA_x\,\vec k$$
$$= |\vec A|\,|\vec B|\sin\theta\,\vec k$$

(16)

Thus

(1) $\vec A\times\vec B$ is $\perp$ to both $\vec A$ and $\vec B$

(2) $|\vec A\times\vec B| = |\vec A|\,|\vec B|\,|\sin\theta|$

(3) $\vec A, \vec B$ and $\vec A\times\vec B$ form a right-handed triplet

Note: If $\vec A\times\vec B = \vec 0$, then

Either (a) $\vec A = 0$

(b) $\vec B = 0$

or (c) $\vec A$ and $\vec B$ are $\parallel$.

(17)

Operations (Concluded)

(5) Differentiation of Vectors

Def. of Derivative of a scalar function of a single variable, $x(t)$

$$\frac{dx}{dt} \equiv \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

Same thing for a vector function, $\vec{A}(t)$

$$\frac{d\vec{A}}{dt} \equiv \lim_{\Delta t \to 0} \frac{\vec{A}(t+\Delta t) - \vec{A}(t)}{\Delta t}$$

Note $\frac{d\vec{A}}{dt}$ is a vector

Trivia:

$$\frac{d}{dt}(\vec{A}+\vec{B}) = \frac{d\vec{A}}{dt} + \frac{d\vec{B}}{dt}$$

$$\frac{d}{dt}(\alpha\vec{A}) = \frac{d\alpha}{dt}\vec{A} + \alpha\frac{d\vec{A}}{dt}$$

$$\frac{d}{dt}(\vec{A}\cdot\vec{B}) = \frac{d\vec{A}}{dt}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{dt}$$

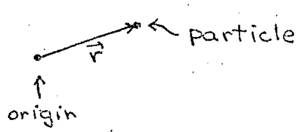
$$\frac{d}{dt}(\vec{A}\times\vec{B}) = \frac{d\vec{A}}{dt}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{dt}$$

(18)

$$\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{k}$$

$$\frac{d\vec{A}}{dt} = \frac{dA_x}{dt}\vec{i} + \frac{dA_y}{dt}\vec{j} + \frac{dA_z}{dt}\vec{k}$$

Velocity + Acceleration in Three Dimensions



$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{V} \equiv \frac{d\vec{r}}{dt}$$

"speed" $= |\vec{V}|$

$$\vec{a} \equiv \frac{d\vec{V}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

(19)

Kinematic Exercise I

Newton's First Law

$$\vec{a} \equiv \frac{d^2\vec{r}}{dt^2} = \vec{0}$$

$$a_x = a_y = a_z = 0$$

$x = V_x t + b_x$ etc. for y and z

$$\vec{r} = \vec{V}t + \vec{b}$$

(20)

Kinematic Exercise II

Falling Particle

$$\vec{a} \equiv \frac{d^2\vec{r}}{dt^2} = -g\vec{k}$$

$g \approx 980$ cm/sec^2 ≈ 1 m/sec^2

$$\frac{d^2}{dt^2}\left(\tfrac{1}{2}gt^2\vec{k}\right)$$

$$= \frac{d}{dt}g\vec{k}t = g\vec{k}$$

$$\frac{d^2}{dt^2}\left(\vec{r} + \tfrac{1}{2}gt^2\vec{k}\right) = -g\vec{k} + g\vec{k} = \vec{0}$$

$$\vec{r} + \tfrac{1}{2}gt^2\vec{k} = \vec{V}_0 t + \vec{r}_0$$

$$\vec{r} = -\tfrac{1}{2}gt^2\vec{k} + \vec{V}_0 t + \vec{r}_0$$

If we choose coordinates such that $\vec{r}_0 = 0$ (choice of center)

$\vec{V}_0 = V_{0z}\vec{k} + V_{0y}\vec{j}$ (y,x rotation)

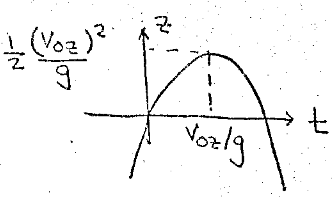
(21)

$r_x \equiv x = 0$

$r_y \equiv y = V_{0y} t$

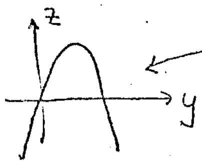
$r_z \equiv z = -\frac{1}{2} g t^2 + V_{0z} t$

$= -\frac{1}{2} g \left(t - \frac{V_{0z}}{g}\right)^2 + \frac{1}{2} \frac{(V_{0z})^2}{g}$



parabola
(badly drawn)

since y is just a multiple of t
also a parabola in z-y plane

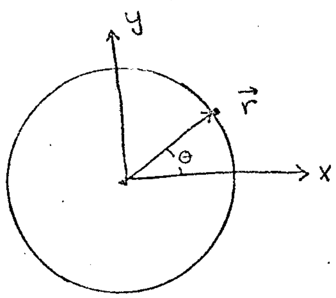


trajectory
(worsely drawn)

MOVIE TIME!

(22)

Kinematic Exercise III
Motion in a Circle



$|\vec{r}| = R$

$r_x \equiv x = R\cos\theta$

$r_y \equiv y = R\sin\theta$

$\vec{r} = (\vec{i}\cos\theta + \vec{j}\sin\theta) R$

$\vec{V} \equiv \frac{d\vec{r}}{dt} = [-\vec{i}\sin\theta + \vec{j}\cos\theta] \frac{d\theta}{dt} R$

$\vec{r}\cdot\vec{V} = \frac{d\theta}{dt} R[-\sin\theta\cos\theta + \cos\theta\sin\theta]$

$= 0$

(23)

Another Way of Seeing the
Same Thing

$\frac{d}{dt} R^2 = 0 = \frac{d}{dt} \vec{r}\cdot\vec{r}$

$= \frac{d\vec{r}}{dt}\cdot\vec{r} + \vec{r}\cdot\frac{d\vec{r}}{dt}$

$= \vec{V}\cdot\vec{r} + \vec{r}\cdot\vec{V}$

$= 2\,\vec{r}\cdot\vec{V}$

(Also works for motion on
a sphere)

(Also works backwards:

$\vec{r}\cdot\vec{V} = 0 \Rightarrow \frac{d}{dt}\vec{r}\cdot\vec{r} = 0$

$\Rightarrow |\vec{r}| = \text{constant}$)

(24)

Special Case $\theta = \omega t$

$\vec{V} = R\omega[-\vec{i}\sin\theta + \vec{j}\cos\theta]$

$\vec{a} = -R\omega^2[\vec{i}\cos\theta + \vec{j}\sin\theta]$

$= -\omega^2\vec{r}$

(1)

A lot of lecture I can be made 3d just by putting arrows on things.

E.g. Mechanics "from the outside"

$$\vec{r}(t) = \vec{v}^{\,in}\, t + \vec{b}^{\,in} \quad \text{(far past)}$$
$$= \vec{v}^{\,out}\, t + \vec{b}^{\,out} \quad \text{(far future)}$$

$$m_1 \vec{v}_1^{\,in} + m_2 \vec{v}_2^{\,in} = m_1 \vec{v}_1^{\,out} + m_2 \vec{v}_2^{\,out}$$

(Asymptotic Law of Conservation of Momentum)

Note: Mom. is a <u>vector</u>

(We will shortly derive this from Newton's Laws.)

(Demonstration)

(2)

E.g., Asym. Cons. of Kinetic Energy

$$\tfrac{1}{2} M_1 |\vec{V}_1^{in}|^2 + \tfrac{1}{2} M_2 |\vec{V}_2^{in}|^2 = \tfrac{1}{2} M_1 |\vec{V}_1^{out}|^2 + \tfrac{1}{2} M_2 |\vec{V}_2^{out}|^2$$

Note: For "elastic" collisions only

(Think of as pure empirical statement)

Galilean Transformation:

$$\vec{r}_a(t) \to \vec{r}_a(t) + \vec{U}\, t$$
$$\vec{V}_a(t) \to \vec{V}_a + \vec{U}$$

$$|\vec{V}_a|^2 = \vec{V}_a \cdot \vec{V}_a \longrightarrow (\vec{V}_a + \vec{U}) \cdot (\vec{V}_a + \vec{U})$$
$$= |\vec{V}_a|^2 + 2\vec{U} \cdot \vec{V}_a + |\vec{U}|^2$$

(3)

$$\tfrac{1}{2} M_1 |\vec{V}_1^{in} + \vec{U}|^2 + \tfrac{1}{2} M_2 |\vec{V}_2^{in} + \vec{U}|^2$$
$$= \tfrac{1}{2} M_1 |\vec{V}_1^{out} + \vec{U}|^2 + \tfrac{1}{2} M_2 |\vec{V}_2^{out} + \vec{U}|^2$$

const. term yields old result

coeff of $|\vec{U}|^2$ yields $M_1 + M_2 = M_1 + M_2$

$$\vec{U} \cdot (M_1 \vec{V}_1^{in} + M_2 \vec{V}_2^{in} - M_1 \vec{V}_1^{out} - M_2 \vec{V}_2^{out}) = 0$$

Since $\vec{U}$ is arbitrary

$$M_1 \vec{V}_1^{in} + M_2 \vec{V}_2^{in} = M_1 \vec{V}_1^{out} + M_2 \vec{V}_2^{out}$$

therefore, as before

$m_1 = M_1 \qquad m_2 = M_2$ (up to scale of mass)

(4)

Mechanics from the Inside

Newtons Second Law

$$\vec{F} = m \vec{a}$$
$$= m \frac{d^2 \vec{r}}{dt^2} \equiv m \ddot{\vec{r}}$$
$$= m \frac{d\vec{V}}{dt} \equiv m \dot{\vec{V}}$$
$$= \frac{d\vec{p}}{dt} \equiv \dot{\vec{p}}$$

$$\underbrace{\vec{F}_a}_{\text{force on particle } a} = \sum_{b \neq a} \vec{F}_{ab}$$

$\vec{F}_{ab}$: force exerted on a by ~~all other the~~ particle b

$\sum_{b \neq a}$: summed over all other particles

⑤

Newton's Third Law

$$\vec F_{ab} = -\vec F_{ba}$$

Conservation of Momentum

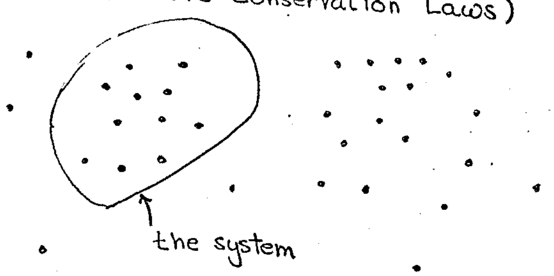
$$\left.\begin{aligned} P &\equiv \sum_a \vec p_a \\ \frac{d\vec p_a}{dt} &= \sum_{b\neq a}\vec F_{ba} \\ \vec F_{ba} &= -\vec F_{ab}\end{aligned}\right\} \text{ ingredients}$$

$$\frac{d\vec P}{dt} = \sum_a \sum_{b\neq a} \vec F_{ba} = \vec 0 \qquad \text{result}$$

END OF RERUN OF
FIRS- LECTURE

⑥

Systems of Particles (and Some More Conservation Laws)



everthing else is «the external world». Division is arbitrary (But some arbitrary divisions are more useful than others).

Notation: If $a=1,2\dots N$ particle a is in the system

If $a>N$ in the ext. world.

⑦

$$m_a\frac{d^2\vec r_a}{dt^2} = \sum_{\substack{b\neq a\\ b\le N}} \vec F_{ab} + \sum_{b>N}\vec F_{ab}$$

$(a\le N)$

$$\dot{\vec p}_a = \sum_{\substack{b\neq a\\ b\le N}} \vec F_{ab} + \vec F_a^{\,ext}$$

↑ Definition of External Force

Notes: (1) For shorthand, in future I will write

$$\sum_{\substack{b\neq a\\ b\le N}} \quad \text{just as} \quad \sum_b$$

remember it's just over internal ~~forces~~ particles.

(2) If

$$\vec F_a^{\,ext}=0$$

system is isolated.

⑧

$$\vec P \equiv \sum_a \vec p_a \qquad \text{(Total Momentum of the System)}$$

$$\dot{\vec P} = \sum_a \dot{\vec p}_a = \sum_a\sum_b \vec F_{ab} + \sum_a \vec F_a^{\,ext}$$

$$\sum_a\sum_{a\neq b}\vec F_{ab} = \vec 0 \qquad \text{(as before)}$$

$$\sum_a \vec F_a^{\,ext} \equiv \vec F^{ext} \; \text{(Total External Force on the System)}$$

$$\boxed{\frac{d\vec P}{dt} = \vec F^{ext}}$$

If system is isolated $\vec F^{ext}=\vec 0$ and we get mom. cons. again.

(9)

How to compute $\vec{P}$?

$$\vec{P} = \sum_a \vec{p}_a$$
$$= \sum_a m_a \vec{v}_a = \sum_a m_a \dot{\vec{r}}_a$$
$$= \frac{d}{dt} \sum_a m_a \vec{r}_a$$

Define $\quad M \equiv \sum_a m_a \quad$ (total mass)

$$\vec{R} \equiv \frac{\sum_a m_a \vec{r}_a}{M} = \frac{\sum_a m_a \vec{r}_a}{\sum_a m_a}$$

(Position of center of mass)

$$\boxed{\vec{P} = M \frac{d}{dt} \vec{R} = M \dot{\vec{R}}}$$

(10)

More on center of mass.

(1) Center of mass position is a sort of "average position"

Shift all particles by same amount, $\vec{c}$,

$$\vec{r}_a \longrightarrow \vec{r}_a + \vec{c}$$

(can be either physical shift or shift of coordinate origin)

$$\vec{R} = \frac{\sum_a m_a (\vec{r}_a)}{\sum_a m_a}$$
$$\longrightarrow \frac{\sum_a m_a (\vec{r}_a + \vec{c})}{\sum_a m_a}$$
$$= \frac{\sum_a m_a \vec{r}_a}{\sum_a m_a} + \frac{\sum_a m_a}{\sum_a m_a} \vec{c}$$

$$\vec{R} \longrightarrow \vec{R} + \vec{c}$$

(This is why we divide by M)

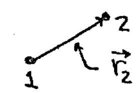
(11)

(2) Likewise, under Galilean transf.,

$$\vec{r}_a \rightarrow \vec{r}_a + \vec{U}t$$
$$\vec{R} \rightarrow \vec{R} + \vec{U}t$$

(3) Simple Example: Two-Particle System

Choose origin ~~center~~ of coordinates so $\vec{r}_1 = 0$.



$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} = \frac{m_2}{m_1 + m_2} \vec{r}_2$$

Halfway between if $m_1 = m_2$

Closer to 1 if $m_1 > m_2$

Clorer to 2 if $m_2 > m_1$

(As always: $m_1 > 0$, $m_2 > 0$)

(12)

(4) $\quad \vec{P} = M \dot{\vec{R}}$

$$\dot{\vec{P}} = \vec{F}^{ext}$$

$$\boxed{M \ddot{\vec{R}} = \vec{F}^{ext}} \quad \text{(Newton's 2nd Law for Center-of-mass)}$$

Super-Important

(a) If $\vec{F}^{ext} = 0$, $\quad M \ddot{\vec{R}} = 0$

The Center-of-mass of an isolated systems moves like an isolated particle.

(Demonstrate)

(13)

(b) (Still $\vec{F}^{ext} = \vec{0}$)

$$\dot{\vec{R}} = \frac{\vec{P}}{M}$$

$$\vec{R} = \frac{\vec{P}}{M} t + \vec{B}$$

Choose center of coords. so $\vec{B} = 0$

Make a Galilean transformation

$$\vec{r}_a \to \vec{r}_a + \vec{U} t$$

with $\quad \vec{U} = -\frac{\vec{P}}{M}$

$$\vec{R} = \frac{\vec{P}}{M} t \to \vec{R} - \frac{\vec{P}}{M} t = \vec{0}$$

"Going to the center-of-mass frame"

Makes great simplifications for two-particle isolated systems

(We can always transform back after we have solved the problem in c-o-m frame.)

(14)

(c) $\quad M \ddot{\vec{R}} = \vec{F}^{ext}$

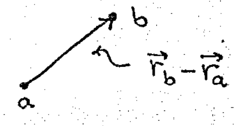
justifies treating a composite system as a particle (if you're so nearsighted that all you can measure is $\vec{R}$)

Also, $\quad M = \sum_a m_a$

explains why mass "defined" as "quantity of matter" seems OK.

(15)

A new assumption about $\vec{F}_{ab}$ and a new conservation law



Assume: $\quad \vec{F}_{ab}$ is $\parallel$ to $\vec{r}_b - \vec{r}_a \quad$ *

$$(\vec{r}_b - \vec{r}_a) \times \vec{F}_{ab} = 0$$

Define $\quad \boxed{\vec{L} = \sum_a \vec{r}_a \times \vec{p}_a} \quad$ Angular Momentum

$$\dot{\vec{L}} = \sum_a \dot{\vec{r}}_a \times \vec{p}_a + \sum_a \vec{r}_a \times \dot{\vec{p}}_a$$

$$= \vec{0} + \sum_a \vec{r}_a \times \vec{F}_a$$

$$= \sum_a \vec{r}_a \times \left(\sum_b \vec{F}_{ab} + \vec{F}_a^{ext}\right)$$

Note: Independent of Coord. System

Invariant under Galilean transf.

(16)

$$\dot{\vec{L}} = \sum_a \vec{r}_a \times \left(\sum_b \vec{F}_{ab} + \vec{F}_a^{ext}\right)$$

We can pair together terms in the first sum, e.g.

$$\vec{r}_1 \times \vec{F}_{12} + \vec{r}_2 \times \vec{F}_{21}$$

$$= \vec{r}_1 \times \vec{F}_{12} - \vec{r}_2 \times \vec{F}_{12}$$

$$= (\vec{r}_1 - \vec{r}_2) \times \vec{F}_{12} = \vec{0}$$

thus,

$$\dot{\vec{L}} = \sum_a \vec{r}_a \times \vec{F}_a^{ext}$$

For an isolated system

$$\boxed{\dot{\vec{L}} = 0}$$

We can sometimes get part of this even if the system is not isolated. Eg., if $\vec{F}_a^{ext}$ are all $\parallel$ to $\vec{k}$

$$\vec{k} \cdot (\vec{r}_a \times \vec{F}_a^{ext}) = 0$$

and $\quad \vec{k} \cdot \dot{\vec{L}} = \dot{L}_z = 0$

(17)

Important: $\vec{L}$ depends on origin of coordinates

$$\vec{r}_a \to \vec{r}_a + \vec{c}$$

$$\vec{L} = \sum_a \vec{r}_a \times \vec{p}_a$$

$$\to \sum_a (\vec{r}_a + \vec{c}) \times \vec{p}_a$$

$$= \sum_a \vec{r}_a \times \vec{p}_a + \vec{c} \times \sum_a \vec{p}_a$$

$$\vec{L} \to \vec{L} + \vec{c} \times \vec{P}$$

However, simultaneous $\vec{P} + \vec{L}$ conservation give the same consequences whatever the origin of coordinates

$$\{\dot{\vec{L}} = 0, \quad \dot{\vec{P}} = 0\}$$

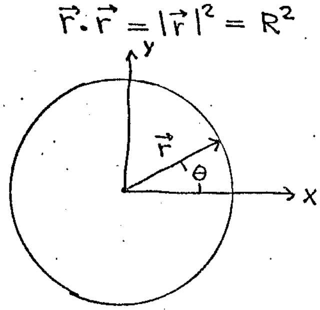
$$\Leftrightarrow \{\dot{\vec{L}} + \vec{c} \times \dot{\vec{P}} = 0, \quad \dot{\vec{P}} = 0\}$$

(18)

Example: Particle moving in circle.

Choose origin of coord. to be center of circle, circle to lie in x-y plane

$$\vec{r} \cdot \vec{r} = |\vec{r}|^2 = R^2$$



$\theta = \omega t$
$\dot{\theta} = \omega$
period of motion $= \frac{2\pi}{|\omega|}$

$x = R\cos\theta \qquad y = R\sin\theta$

$$\vec{r} = R(\cos\theta\,\vec{i} + \sin\theta\,\vec{j})$$

$$\vec{v} \equiv \dot{\vec{r}} = R\dot{\theta}(-\sin\theta\,\vec{i} + \cos\theta\,\vec{j})$$

$$\vec{P} = m\vec{v} = mR\omega(-\sin\theta\,\vec{i} + \cos\theta\,\vec{j})$$

(19)

[Digression:

Note $\vec{r} \cdot \vec{v} = 0$

This also follows from

$$\vec{r} \cdot \vec{r} = \text{const.}$$

$$\frac{d}{dt}\vec{r} \cdot \vec{r} = 0 = \dot{\vec{r}} \cdot \vec{r} + \vec{r} \cdot \dot{\vec{r}} = 2\vec{r} \cdot \dot{\vec{r}}$$

(Thus also true for motion on a sphere)

$$\ddot{\vec{r}} = \dot{\vec{v}} = -R\omega^2(\vec{i}\cos\theta + \vec{j}\sin\theta)$$

$$= -R\omega^2\,\vec{r}$$

(Force points towards center of circle.)

$$\vec{L} = \vec{r} \times \vec{p} = m\omega R[\cos^2\theta(\vec{i} \times \vec{j}) - \sin^2\theta(\vec{j} \times \vec{i})]$$

$$= m\omega R\,\vec{k}\,\underbrace{[\cos^2\theta + \sin^2\theta]}_{1}$$

(Demonstration)

(1)

Erratum

Lecture III, last line should be

$$\vec{L} = m\omega R^2 \vec{k}$$

Outline of Lecture IV

Two small topics

1) Ang. Mom + COM
2) Asymptotic Conservation Laws
   2a) Independence.
   2b) Relations

One big topic.

Collisions + Sequences of Collisions

Concepts not discussed here but which you should be able to understand after this lecture: (1) Impulse (K+K 3.4)
(2) The examples of momentum transport (K+K Examples 3.16 to 3.19).

(2)

Ang. Mom. + C-O-M

$$\vec{L} = \sum_a \vec{r}_a \times \vec{p}_a$$ depends on origin of coord.

$$\vec{R} = \frac{\sum_a m_a \vec{r}_a}{\sum_a m_a}$$ gives a "natural" origin of coord.

$$\vec{L} = \sum_a (\vec{r}_a - \vec{R}_{\text{⊗}}) \times \vec{p}_a + \sum_a \vec{R}_{\text{⊗}} \times \vec{p}_a$$

$$= \vec{L}_{CM} + \vec{R} \times \vec{P}$$

ang mom with com as origin — "ang mom of com"

$\vec{L}_{CM}$ is Galilean-Invariant. Proof:

$$\vec{r}_a \to \vec{r}_a + \vec{U}t \qquad \vec{R}_{\text{⊗}} \to \vec{R}_{\text{⊗}} + \vec{U}t$$
$$\vec{v}_a \to \vec{v}_a + \vec{U} \qquad \vec{p}_a \to \vec{p}_a + m_a\vec{U}$$

$$\therefore \quad \vec{L}_{CM} \to \sum_a (\vec{r}_a - \vec{R}) \times (m_a\vec{U} + \vec{p}_a)$$
$$= \vec{L}_{CM} + \sum_a m_a \vec{r}_a \times \vec{U} - \sum_a m_a \vec{R} \times \vec{U}$$
$$= \vec{L}_{CM} + M\vec{R} \times \vec{U} - M\vec{R} \times \vec{U}$$
$$= \vec{L}_{CM}$$

(3)

Asymptotic Conservation Laws can be deduced from local ones

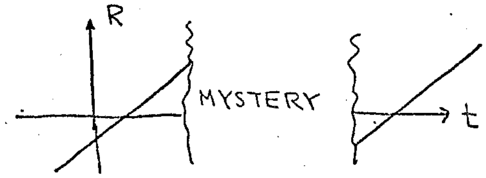
"in" ≡ computed in far past

"out" ≡ computed in far future

$$\dot{\vec{P}} = \vec{0} \Rightarrow \vec{P}^{in} = \vec{P}^{out} \qquad (1)$$
$$\dot{\vec{L}} = \vec{0} \Rightarrow \vec{L}^{in} = \vec{L}^{out} \qquad (2)$$

$$\dot{\vec{R}} = \vec{P}/M \Rightarrow \frac{d}{dt}\left(\vec{R} - \frac{\vec{P}}{M}t\right) = 0$$
$$\Rightarrow \left(\vec{R} - \frac{\vec{P}}{M}t\right)^{in} = \left(\vec{R} - \frac{\vec{P}}{M}t\right)^{out} \qquad (3)$$

(1) and (3) are <u>independent</u> statements



ALLOWED BY (1)
BUT FORBIDDEN BY (3)

(4)

For Elastic Collisions we have, in addition

$$K \equiv \sum_a \tfrac{1}{2} m_a |\vec{v}_a|^2$$
$$K^{in} = K^{out} \qquad (4)$$

[Derived from nowhere, so far]

We already showed that

$$(4) \xRightarrow[\text{Invariance}]{\text{Galilean}} (1) \xRightarrow[\text{Invariance}]{\text{Galilean}} \left(\text{Cons. of Mass}\right)$$

I will now show that

$$(2) \xRightarrow[\text{Invariance}]{\text{Galilean}} (3)$$

$$\vec{L} = \vec{L}_{CM} + \vec{R} \times \vec{P}$$

$$\vec{r}_a \to \vec{r}_a + \vec{U}t \qquad \vec{p}_a \to \vec{p}_a + m_a\vec{U}$$
$$\vec{R} \to \vec{R} + \vec{U}t \qquad \vec{P} \to \vec{P} + M\vec{U}$$
$$\vec{L}_{CM} \to \vec{L}_{CM}$$

(5)

$$\therefore \quad \vec{L} \to \vec{L}_{CM} + (\vec{R} + \vec{U}t) \times (\vec{P} + M\vec{U})$$

$$= \vec{L}_{CM} + \vec{R} \times \vec{P} + t\,\vec{U} \times \vec{P} + M\vec{R} \times \vec{U} + \underbrace{Mt\,\vec{U} \times \vec{U}}_{\vec{0}}$$

$$= \vec{L} + (M\vec{R} - \vec{P}t) \times \vec{U}$$

$$\vec{L}^{in} = \vec{L}^{out}$$

$$(\vec{L}^{in} + (M\vec{R} - \vec{P}t)^{in} \times \vec{U}) = (\vec{L}^{out} + (M\vec{R} - \vec{P}t)^{out} \times \vec{U})$$

subtract

$$(M\vec{R} - \vec{P}t)^{in} = (M\vec{R} - \vec{P}t)^{out}$$

QED

(6)

Collisions in one Dimension

(a) Two Equal Mass Particles; Elastic Collision, COM Frame

$$\vec{P} = m(V_1^{in} + V_2^{in}) = 0$$

$$V_1^{in} = -V_2^{in}$$

① ⟶   ⟵ ②   incoming situation

$\vec{P} = 0$ after collision so

$$\boxed{V_2^{out} = -V_1^{out}}$$

$$\tfrac{1}{2}m(V_1^{in})^2 + \tfrac{1}{2}m(V_2^{in})^2 = m(V_1^{in})^2 = m(V_1^{out})^2$$

$$\therefore \quad \boxed{V_1^{out} = \pm V_1^{in}}$$

(7)

Thus there are two possibilities for the outgoing situation

⟵ ②   ① ⟶

(a) $V_1^{out} = V_1^{in}$ $V_2^{out} = +V_2^{in}$

(b) $V_1^{out} = -V_1^{in}$ $V_2^{out} = -V_2^{in}$

⟵ ①   ② ⟶

For sliders on air track (b) prevails (but (a) is also possible in some situations)

(8)

Assume (b), for simplicity, and go on. The "lab frame" is that in which one of the particles (say, 2) is at rest

	1	2
in	V	−V
out	−V	V

} What we know

Make a Galilean transformation and add V to all velocities

	1	2
in	2V	0
out	0	2V

} Answer in Lab Frame

① ⟶   ②   in

①   ② ⟶   out

Consistency Check: $\vec{P} = 2mv$, kinetic energy $= \tfrac{1}{2}m(2v)^2$.   (Demonstr

9

One Dimension, Two Particles, Unequal Masses

Useful Identity $\frac{1}{2}mv^2 = \frac{1}{2}\frac{(mv)^2}{m} = \frac{p^2}{2m}$

$$p_1^{in} = -p_2^{in} \qquad \text{(COM frame)}$$

$$p_1^{out} = -p_2^{out} \qquad (\vec{P} \text{ cons.})$$

$$\frac{(p_1^{in})^2}{2m_1} + \frac{(p_2^{in})^2}{2m_2} = \frac{1}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)(p_1^{in})^2$$

$$= \frac{1}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)(p_1^{out})^2$$

(a) $p_1^{in} = p_1^{out} \qquad p_2^{in} = p_2^{out}$

$\boxed{\text{(b) } p_1^{in} = -p_1^{out} \qquad p_2^{in} = -p_2^{out}}$

10

$$p_1^{in} \equiv p$$

	1	2
in	p/m_1	$-p/m_2$
out	$-p/m_1$	p/m_2

Velocities in COM frame

Add p/m_2 to all velocities

	1	2
in	$p\left(\frac{1}{m_1}+\frac{1}{m_2}\right)$	0
out	$p\left(\frac{1}{m_2}-\frac{1}{m_1}\right)$	$2p/m_2$

(Velocities in Lab frame)

11

If we define

$$V \equiv p\left(\frac{1}{m_1}+\frac{1}{m_2}\right) = p\,\frac{m_1+m_2}{m_1 m_2}$$

This becomes

	1	2
in	V	0
out	$V\left(\frac{m_1-m_2}{m_1+m_2}\right)$	$\frac{2m_1 V}{m_1+m_2}$

Special Case (a) $m_2 = 3m_1$

	1	2
in	V	0
out	$-\frac{1}{2}V$	$\frac{1}{2}V$

(Demonstrate)

Special Case (b) m_2 much greater than m_1 (we write $m_2 \gg m_1$) More precisely, $\lim_{m_1/m_2 \to 0}$

	1	2
in	V	0
out	$-V$	0

(Demonstrate)

12

Three Dimensions, Two Particles, Elastic Collision

COM Frame

$$\vec{p}_1^{\,in} = -\vec{p}_2^{\,in}$$

$$\vec{p}_1^{\,out} = -\vec{p}_2^{\,out}$$

Kinetic Energy Conservation

$$\frac{|\vec{p}_1^{\,in}|^2}{2m_1} + \frac{|\vec{p}_2^{\,in}|^2}{2m_2} = \frac{|\vec{p}_1^{\,in}|^2}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)$$

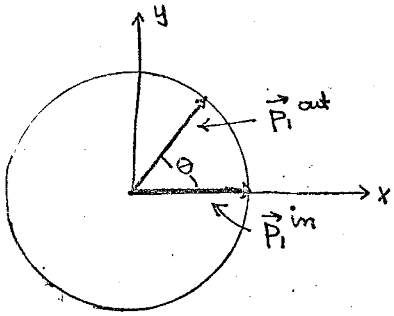
$$= \frac{|\vec{p}_1^{\,out}|^2}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)$$

$$\boxed{|\vec{p}_1^{\,in}| = |\vec{p}_1^{\,out}|} \Rightarrow$$

$$|\vec{p}_2^{\,in}| = |\vec{p}_2^{\,out}|$$

Choose coordinates so $\vec{p}_1^{\,in}$ lies along the x-axis

$\vec{p}_1^{\,out}$ in the x-y plane



θ is called the (center-of-mass) scattering angle. We can always choose coordinates so $p_{1y}^{out} \geq 0$ $(\vec{i} \to \vec{i}, \vec{j} \to -\vec{j}, \vec{k} \to -\vec{k})$, i.e., so

$$0 \leq \theta \leq \pi$$

$\theta = 0$ $\quad \vec{p}_1^{\,out} = \vec{p}_1^{\,in}$ $\quad$ "forward scattering"

$\theta = \pi$ $\quad \vec{p}_1^{\,out} = -\vec{p}_1^{\,in}$ $\quad$ "backward scattering"

Possibilities (a) and (b) in one dimension

Non-Asymptotic Digression

If forces are such that $\dot{\vec{L}} = 0$ (remember, needs additional assumption) then choose coord. so

$$\vec{L} = |\vec{L}|\,\vec{k}$$

In COM frame

$$m_1\vec{r}_1 + m_2\vec{r}_2 = \vec{0}\,, \qquad \vec{r}_2 = -\frac{m_1}{m_2}\vec{r}_1$$

$$\vec{p}_1 + \vec{p}_2 = \vec{0}\,, \qquad \vec{p}_2 = -\vec{p}_1$$

$$\vec{L} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2$$

$$= \vec{r}_1 \times \vec{p}_1 + \frac{m_1}{m_2}\vec{r}_1 \times \vec{p}_1$$

$$= \left(1 + \frac{m_1}{m_2}\right)\vec{r}_1 \times \vec{p}_1$$

$\therefore$ $\vec{r}_1 \cdot \vec{L} = 0 \Rightarrow \vec{r}_1 \cdot \vec{k} = 0 \Rightarrow$

$\vec{r}_1$ lies in the x,y plane for all time. $\Rightarrow$ same for $\vec{r}_2$

[This derivation assumes $|\vec{L}| \neq 0$. What happens if $\vec{L} = \vec{0}$?]

Back to Asymptopia

$$\vec{p}_1^{\,in} = -\vec{p}_2^{\,in} \equiv \vec{p}$$

$$\vec{p}_1^{\,out} = -\vec{p}_2^{\,out} \equiv \vec{p}\,'$$

$$|\vec{p}| = |\vec{p}\,'| \qquad \vec{p} \cdot \vec{p}\,' = |\vec{p}|^2 \cos\theta$$

Velocities in COM frame

	1	2
in	$\vec{p}/m_1$	$-\vec{p}/m_2$
out	$\vec{p}\,'/m_1$	$-\vec{p}\,'/m_2$

Velocities in Lab frame

	1	2
in	$\vec{p}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)$	$\vec{0}$
out	$\frac{\vec{p}\,'}{m_1} + \frac{\vec{p}}{m_2}$	$-\frac{\vec{p}\,'}{m_2} + \frac{\vec{p}}{m_2}$

From this we can compute all lab frame quantities (but the answers are <u>ugly</u>).

Sometimes a complex process can be broken up into something we understand, followed by a brief collision, followed by something we understand, etc.

Example: Bead sliding on a vertical pole. Assume:

(1) Except when it hits the floor, bead falls freely
(2) Bead collides elastically with floor $(z=0)$
(3) Earth much more massive than bead
(4) Bead at rest at $t=0$ at position z_0

Until $z=0$

$$z = z_0 - \tfrac{1}{2} g t^2$$

Bead hits floor at $t_1 = \sqrt{2z_0/g}$

$$z(t_1) = 0$$
$$v(t_1) = -g t_1$$

(17)

Bead bounces elastically

$$z(t_1+\epsilon)=0$$
$$v(t_1+\epsilon) = +g t_1$$

ϵ very small time (time of collision) Consider $\lim_{\epsilon\to 0}$

for $t>t_1$

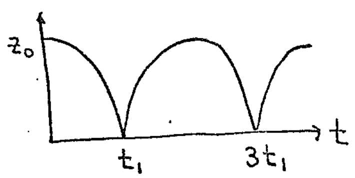
$$z(t) = -\tfrac{1}{2} g (t-t_1)^2 + \underset{g t_1}{\underset{\shortparallel}{v(t_1)}}(t-t_1) + \underset{0}{\underset{\shortparallel}{z(t_1)}}$$

$$= -\tfrac{1}{2} g (t-t_1)^2 + g t_1 (t-t_1)$$

$$= -\tfrac{1}{2} g t^2 + 2 g t t_1 - \tfrac{3}{2} g t_1^2$$

$$= -\tfrac{1}{2} g (t-2t_1)^2 + \tfrac{1}{2} g t_1^2$$

$$= -\tfrac{1}{2} g (t-2t_1)^2 + z_0$$



etc.

(18)

Sometimes a complex process can be thought of as a sequence of collisions.

Example 1: Newton's balls (Demonstrate)

Initial Configuration (Spacing Exagerrated):

① → ② ③

First Collision (just after):

① ② → ③

Just after second collision:

① ② ③ →

etc. All we can see (because spacing is small) is

① → ②③④⑤

going into

①②③④ ⑤ →

(19)

We can compound this again

① → ② → ③ ④ ⑤

after first compound collision:

① → ② ③ ④ ⑤ →

after second:

① ② ③ ④ → ⑤ →

What we see is this:

①② → ③④⑤

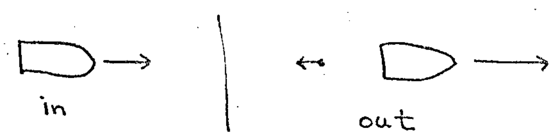
going into this:

①②③ ④⑤ →

(20)

Example 2: The Rocket

Rocket (mass m, velocity v) ejects exhaust particle (mass $-\Delta m$, velocity $v-u$). What happens to the rocket? (Note: This is <u>inelastic</u>. [Consider $v=0$])



$$p^{in} = mv$$

$$p^{out} = (m+\Delta m)\underbrace{(v+\Delta v)}_{v^{out}} + (v-u)(-\Delta m)$$

$$= mv + m\Delta v + u\,\Delta m + (\Delta m)(\Delta v)$$

4th term << 2nd term if $\Delta m \ll m$ neglect it

$$p^{in} = p^{out} \Rightarrow$$

$$mv = mv + m\Delta v + u\,\Delta m$$

$$\Delta v = -u \frac{\Delta m}{m}$$

Let this happen in time Δt

$$\frac{\Delta V}{\Delta t} = -u \;\frac{\Delta m}{\Delta t}\;\frac{1}{m}$$

Let $\Delta t \to 0$

$$\boxed{\frac{dV}{dt} = -\frac{u}{m}\,\frac{dm}{dt}} \leftarrow$$ Rocket Equation

$$\frac{dV}{dt} = -u\,\frac{d}{dt}\ln m$$

$$\frac{d}{dt}(V + u\ln m) = 0$$

$$V + u\ln m = \text{const} \equiv u\ln m_0$$

[m_0 = mass of rocket when $V=0$]

$$\boxed{V = u\ln\left(\frac{m_0}{m}\right)} \leftarrow$$ Integrated Rocket Eq.

(demonstrate)

Prologue: The Forces of Nature ①

(As seen from the microworld — between two protons 10^{-13} cm. apart, a typical separation inside a nucleus)

Force	$\lvert\vec F\rvert$ (rounded off to order-of-magnitude)	$\lvert\vec F\rvert$ falls off at large $\lvert\vec r\rvert \equiv r$ like
Strong Interaction (Nuclear Force)	1 (choice of scale)	$\frac{e^{-r/r_s}}{r^2}$ $r_s \approx 10^{-13}$ cm
Electromagnetism (Physics 12b)	10^{-3}	$1/r^2$
Weak Interaction	10^{-6}	$\frac{e^{-r/r_w}}{r^2}$ $r_w \approx 10^{-15}$ cm
Gravity	10^{-39}	$1/r^2$

(Explain + Demonstrate)

The Forces of Nature ②

(As seen from this course)

Gravity

Elastic Forces (Associated with the resistance of solids to deformation)*

Friction (Solid against solid)*

Viscous Forces (Solid through fluid)*

Etc.

* All of these in fact consequence of electromagnetism, but this is for physics 143 (and beyond)

Note on Units: $F = ma = m\frac{d^2x}{dt^2}$

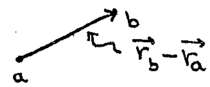
If we measure mass in $\left\{\begin{matrix}\text{kg.}\\ \text{grams}\end{matrix}\right\}$, length in $\left\{\begin{matrix}\text{meters}\\ \text{cm.}\end{matrix}\right\}$, time in seconds, force is measured in $\left\{\begin{matrix}\text{kg.-m}/(\text{sec})^2\\ \text{g-cm.}/(\text{sec})^2\end{matrix}\right\}$. For short these units are called $\left\{\begin{matrix}\text{Newtons}\\ \text{dynes}\end{matrix}\right\}$.

Lecture V ③

Gravity

$$\vec F_{ab} = \frac{G m_a m_b (\vec r_b - \vec r_a)}{|\vec r_b - \vec r_a|^3} \qquad \text{(Newton 1666)}$$

↳ force b exerts on a



$G = 6.67 \times 10^{-11}$ N$\cdot$m^2/(kg)^2

(Only measured in 1771)

Notes:

(1) Attractive force

(2) $|\vec F_{ab}| = \frac{G m_a m_b}{|\vec r_a - \vec r_b|^2}$

(3) Consistent with 3d law $\vec F_{ab} = -\vec F_{ba}$

(4) " " $\dot{\vec L} = \vec 0$

(5) $\vec F_{ab}/m_a$ independent of m_a !!!

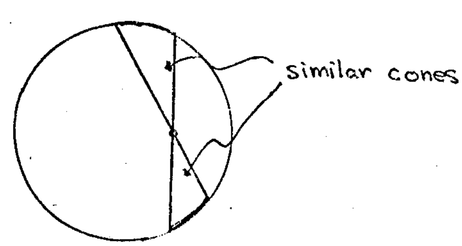
④ A theorem about gravity.

Given a uniform spherical thin shell of matter of mass M and radius R, with center at the origin of coord., the gravitational force it exerts on a particle of mass m and location $\vec r$ is

$$\vec F = \vec 0 \qquad |\vec r| < R \qquad \text{(a)}$$

$$\vec F = -\frac{GMm\,\vec r}{|\vec r|^3} \qquad |\vec r| > R \qquad \text{(b)}$$

Proof of (a)



For (b) see K+K Appendix 2.1

(Criticism of <u>Ringworld</u>)

5

Corollary: For any spherically symmetric distribution of matter let $M(r)$ be the total mass inside a sphere of radius r. Then

$$\vec{F} = -\frac{G M(|\vec r|)\, m \vec r}{|\vec r|^3}$$

Application: World of radius R, uniform mass density μ. $(M = \frac{4}{3}\pi R^3 \mu)$

$$M(r) = \frac{4\pi}{3} r^3 \mu \qquad r \le R$$
$$= M \qquad r \ge R$$

$$\vec F = -\frac{4\pi G \mu m \vec r}{3} \qquad |\vec r| \le R \qquad (a)$$
$$= -\frac{GMm}{|\vec r|^3}\vec r \qquad |\vec r| \ge R \qquad (b)$$

6

Reconciling Galileo + Newton

Outside the earth

$$\vec F = -\frac{G m \overset{\text{mass of earth}}{M_e} \vec r}{|\vec r|^3}$$

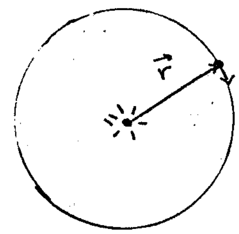
$$\frac{\vec r}{|\vec r|} \equiv \vec k \qquad |\vec r| = R_e \text{ (radius of earth)}$$

$$\frac{\vec F}{m} = \vec a = -\frac{G M_e}{R_e^2}\vec k \qquad \text{(Newton)}$$
$$= -g\vec k \qquad \text{(Galileo)}$$

$$\boxed{g = \frac{G M_e}{R_e^2}}$$

7

Weighing the sun (in units of earth mass)



approximate orbit by circle of radius $R_0 = |\vec r|$

$$\vec F = -M_e \vec r \omega^2 \quad \text{(Lecture III)}$$
$$= -\frac{G \overset{\text{mass of sun}}{M_s} M_e \vec r}{|\vec r|^3} \quad \text{(Newton)}$$

$\therefore \quad G M_s = \omega^2 R_0^3$

$G M_e = g R_e^2$

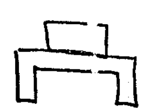
$$\therefore \quad \boxed{\frac{M_s}{M_e} = \frac{\omega^2 R_0^3}{g R_e^2} = 3.3 \times 10^5}$$

$$\left[\begin{array}{ll} g = 10\ \text{m/sec}^2 & R_e = 6.4\times 10^6\ \text{m} \\ R_0 = 1.5\times 10^{11}\ \text{m} & \omega = \frac{2\pi}{1\ \text{yr}} = \frac{2\cancel{\pi}}{\cancel{\pi}\times 10^7\ \text{sec}} \\ \frac{M_s}{M_e} = \frac{2^2 \times (1.5)^3}{(6.4)^2} \times \frac{10^{33}}{10^{13}\times 10^{14}} = 3.3\times 10^5 & \end{array}\right]$$

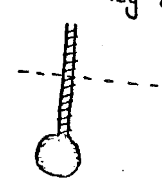
8

Contact Forces:

If we neglect gravity and electromagnetism, ordinary solid bodies exert forces on each other only when they are in contact



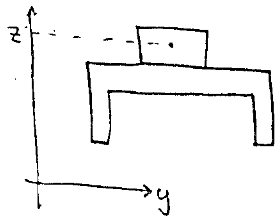
What is force table exerts on block?



What is force part of rope above dashed line exerts on part below the line?

The theory of these forces can be very complex, but there is one simple (+ common) situation: when the force simply enforces a constraint on the motion of the system. Then the constraint can be used to solve for the force.

Example: Block moving on frictionless table



Block does not sink into table.
Constraint: $z = \text{const.}$

$\therefore$ Total $F_z = \ddot{z}\, m$ ← mass of block
$= 0$

$\therefore$ Table <u>must</u> exert force on block ~~in~~ in z-direction just sufficient to cancel all other F_z's.*
This is called <u>normal</u> <u>force</u>.

(Linguistic note: "normal" here means $\perp$, not "usual".)

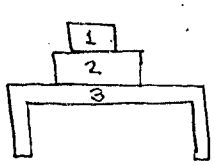
* provided their sum is ≤ 0. (It is possible to lift the block).

If table is frictionless, this is the only force table exerts on block.

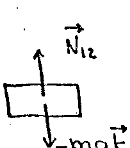
Likewise for ball rolling on hill, etc.

If there is no friction, only force hill exerts on ball is one $\perp$ to surface of hill.

Example 1: Two blocks on table



Forces acting on block 1:



$\vec{N}_{12}$ normal force of 2 on 1

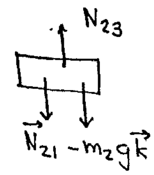
$-mg\vec{k}$ gravity

(This is a "force diagram")

must sum to zero (block is at rest)

$$\vec{N}_{12} + (-m_1 g\vec{k}) = \vec{0}$$
$$\vec{N}_{12} = m_1 g\vec{k}$$

Force Diagram for block 2:

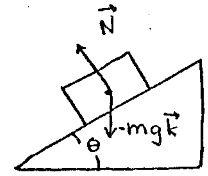


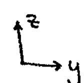
$$\vec{N}_{21} - m_2 g\vec{k} + \vec{N}_{23} = \vec{0}$$
$$\vec{N}_{21} = -\vec{N}_{12} = -m_1 g\vec{k} \quad \text{(3d law)}$$
$$\therefore \vec{N}_{23} = (m_1 + m_2) g\vec{k}$$

Etc.

Note we could compute $\vec{N}_{23}$ directly by considering 1+2 as a single system.

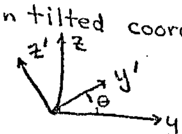
Example 2: Block on tilted table





$$m\ddot{\vec{r}} = \vec{N} - mg\vec{k}$$

Convenient to work in tilted coordinates



$\vec{k}_{z'} = \cos\theta$
$\vec{k}_{y'} = \sin\theta$
$N_{z'} = ?$ $\quad N_{y'} = 0$

$m\ddot{z}' = 0$ (constraint)

$N_{z'} - mg\cos\theta = 0$ (fixes $N_{z'}$)

$m\ddot{y}' = -mg\sin\theta$

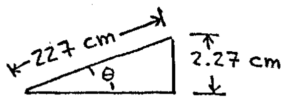
$\ddot{y}' = -g\sin\theta$

(13)

Experimental Check:

Assume $y'(0)=0$ $\quad \dot{y}'(0)=0$

$$y' = -\tfrac{1}{2} g t^2 \sin\theta$$



$\sin\theta = .01$

$g = 9.8 \text{ m/sec}^2$

We measure time it takes slider to travel 2 meters

$$-2\text{ m} = -\tfrac{1}{2} \times (9.8)\frac{\text{m}}{\text{sec}^2} \times t^2 \times .01$$

$$t = \sqrt{\frac{400}{9.8}} \text{ sec}$$

$$= 6.4 \text{ sec.}$$

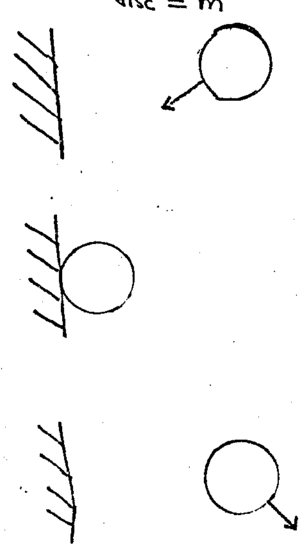
Does it work?

(14)

[I may skip this if time is short]

Example 3: Frictionless disc ~~rotating~~ moving on table collides elastically with wall.

$m_{wall} \gg m_{disc} \equiv m$



(15)

Using methods of Lecture IV, we find

$$|\vec{p}^{\,in}| = |\vec{p}^{\,out}|$$

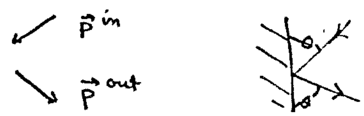
but scattering ∠ is arbitrary.

We can now go further:

Only force is normal force, directed along x-axis

$$\therefore \quad p_y^{in} = p_y^{out}$$

Hence: either $p_x^{in} = p_x^{out}$ (impossible — ~~ball~~ disc penetrates wall) or $p_x^{in} = -p_x^{out}$



"∠ of incidence = ∠ of reflection"

(16)

Contact Forces, Con't.: Tension

Mathematical description of string:

Label points along string by ℓ, length from end of string [ideal strings don't stretch (constraint)]

$$0 \le \ell \le L$$

Configuration of string is given by giving position of each point

$$\vec{r}(\ell)$$

$$\left|\frac{d\vec{r}}{d\ell}\right| = \sqrt{\left(\frac{dx}{d\ell}\right)^2 + \left(\frac{dy}{d\ell}\right)^2 + \left(\frac{dz}{d\ell}\right)^2} = 1$$

(Pythagoras)

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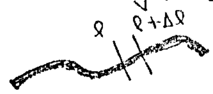
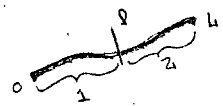
String dynamics

(1) Mass density $\lambda(\ell)$:



Mass of small segment between ℓ and $\ell+\Delta\ell$ is $\lambda(\ell)\Delta\ell$ (in $\lim_{\Delta\ell\to 0}$). Mass between ℓ_1 and ℓ_2 is $\int_{\ell_1}^{\ell_2}\lambda(\ell)d\ell$.

(2) Tension



$$\vec{F}_{12} = T(\ell)\frac{d\vec{r}}{d\ell} = -\vec{F}_{21}$$

$$|\vec{F}_{12}| = |\vec{F}_{21}| = T(\ell)$$

18

(3) External Forces acting on string

$\vec{F}^{ext}$ on small segment between ℓ and $\ell+\Delta\ell$ is $\vec{f}^{ext}(\ell)\,\Delta\ell$
"force density"

(in $\lim_{\Delta\ell\to 0}$). $\vec{F}^{ext}$ ~~betwe~~ on segment between ℓ_1 and ℓ_2 is $\int_{\ell_1}^{\ell_2}\vec{f}^{ext}(\ell)d\ell$.

E.g. Gravity (at Earth's surface)

$$\vec{f}^{ext}(\ell) = -\lambda(\ell)g\vec{k}$$

(4) Equation of motion (Newton's 3d Law)

$$\lambda(\ell)\Delta\ell\,\frac{d^2\vec{r}(\ell)}{dt^2} = \vec{f}^{ext}(\ell)\Delta\ell + T(\ell+\Delta\ell)\frac{d\vec{r}}{d\ell}(\ell+\Delta\ell) - T(\ell)\frac{d\vec{r}}{d\ell}(\ell)$$

Divide by $\Delta\ell$ and let $\Delta\ell\to 0$

$$\lambda(\ell)\frac{d^2\vec{r}(\ell)}{dt^2} = \vec{f}^{ext}(\ell) + \frac{d}{d\ell}\left[T\frac{d\vec{r}}{d\ell}\right]$$

19

Important Note:

If string is stationary $\frac{d^2\vec{r}(\ell)}{dt^2} = \vec{0}$

or massless $\lambda(\ell) = 0$

then $\vec{f}^{ext}(\ell) + \frac{d}{d\ell}\left[T\frac{d\vec{r}}{d\ell}\right] = \vec{0}$

(5) Forces acting on the ends ($\ell = 0$ or L) must cancel tension force

$$\vec{F}^{ext} + T(0)\frac{d\vec{r}}{d\ell}(0) = \vec{0}$$

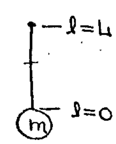
acting on end "0"

$$\vec{F}^{ext} - T(L)\frac{d\vec{r}}{d\ell}(L) = \vec{0}$$

acting on end "L"

20

Example 1: Weight hanging from uniform ($\lambda(\ell) = $ const.) string



$$\frac{d\vec{r}}{d\ell} = \vec{k}$$

Lower end (+ weight) $T(0)\vec{k} + (-mg\vec{k}) = 0$

$T(0) = mg$

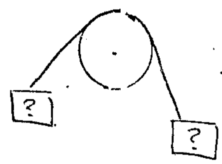
$$\frac{d}{d\ell}\left[T(\ell)\vec{k}\right] - \lambda(\ell)g\vec{k} = \vec{0}$$

$$\frac{dT}{d\ell} = \lambda \qquad T(\ell) = mg + \lambda\ell g$$

$$T(L) = mg + \lambda L$$

$\vec{F}^{ext}$ at top $= (mg + g\lambda L)\vec{k}$

Example 2: Massless string moving over frictionless pulley



$$\vec{n}(\ell) + \frac{d}{d\ell}\left[T \frac{d\vec{r}}{d\ell}\right] = 0$$

normal force of pulley on string

$$\vec{n}(\ell) + \frac{dT}{d\ell}\frac{d\vec{r}}{d\ell} + \frac{d^2\vec{r}}{d\ell^2} = 0$$

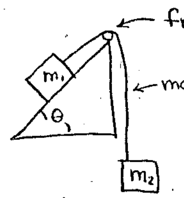
$$\vec{n}(\ell) \cdot \frac{d\vec{r}}{d\ell} = 0 \qquad (\text{"normal"})$$

$$\frac{d\vec{r}}{d\ell} \cdot \frac{d\vec{r}}{d\ell} = 1 \qquad (\text{Pythagoras})$$

$$\frac{d}{d\ell}\left(\frac{d\vec{r}}{d\ell} \cdot \frac{d\vec{r}}{d\ell}\right) = 0 = 2\,\frac{d^2\vec{r}}{d\ell^2} \cdot \frac{d\vec{r}}{d\ell}$$

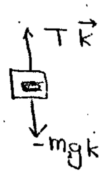
$$0 + \frac{dT}{d\ell} + 0 = 0$$

or $T = \text{const.}$



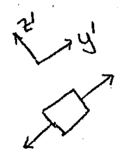
What must m_2 be so nothing moves?

Force diagram for m_2 plus end of string



$$T = m_2 g$$

Force diagram for ~~m~~ m_1 plus end of string



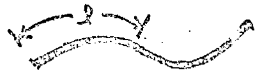
$T\vec{j}'$ (same T as before— example 2)

$-m_1 g \sin\theta\, \vec{j}'$ (from slide 12 – combined effect of gravity + normal force)

$$m_1 g \sin\theta = T = m_2 g$$

$$m_1 \sin\theta = m_2$$

Contact Forces, Con't. : Tension



Mathematical description of string:

Label points along string by ℓ, length from end of string [ideal strings don't stretch (constraint)]

$$0 \le \ell \le L$$

Configuration of string is given by giving position of each point

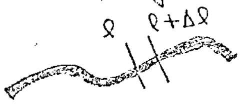
$$\vec{r}(\ell)$$

$$\left|\frac{d\vec{r}}{d\ell}\right| = \sqrt{\left(\frac{dx}{d\ell}\right)^2 + \left(\frac{dy}{d\ell}\right)^2 + \left(\frac{dz}{d\ell}\right)^2} = 1$$

(Pythagoras)

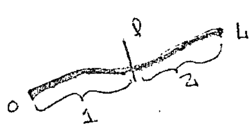
String dynamics

(1) Mass density $\lambda(\ell)$:



Mass of small segment between ℓ and $\ell+\Delta\ell$ is $\lambda(\ell)\Delta\ell$ (in $\lim\limits_{\Delta\ell\to 0}$). Mass between ℓ_1 and ℓ_2 is $\int_{\ell_1}^{\ell_2} \lambda(\ell)\,d\ell$.

(2) Tension



$$\vec{F}_{12} = T(\ell)\frac{d\vec{x}}{d\ell} = -\vec{F}_{21}$$

$$|\vec{F}_{12}| = |\vec{F}_{21}| = |T(\ell)|$$

If $T > 0$ (usual case) force is attractive (a "pull")

(3) External Forces acting on string

$\vec{F}^{ext}$ on small segment between ℓ and $\ell+\Delta\ell$ is $\vec{f}^{ext}(\ell)\,\Delta\ell$
↑ "force density"

(in $\lim\limits_{\Delta\ell\to 0}$). $\vec{F}^{ext}$ ~~between~~ on segment between ℓ_1 and ℓ_2 is $\int_{\ell_1}^{\ell_2} \vec{f}^{ext}(\ell)\,d\ell$.

E.g. Gravity (at Earth's surface)

$$\vec{f}^{ext}(\ell) = -\lambda(\ell)\, g\, \vec{k}$$

(4) Ecuation of motion (Newton's 3d Law)

$$\lambda(\ell)\,\Delta\ell\,\frac{d^2\vec{r}(\ell)}{dt^2} = \vec{f}^{ext}(\ell)\Delta\ell + T(\ell+\Delta\ell)\frac{d\vec{r}}{d\ell}(\ell+\Delta\ell) - T(\ell)\frac{d\vec{r}}{d\ell}(\ell)$$

Divide by $\Delta\ell$ and let $\Delta\ell \to 0$

$$\lambda(\ell)\frac{d^2\vec{r}(\ell)}{dt^2} = \vec{f}^{ext}(\ell) + \frac{d}{d\ell}\left[T\frac{d\vec{r}}{d\ell}\right]$$

Important Note:

If string is at rest $\quad \frac{d^2\vec{r}(\ell)}{dt^2} = \vec{0}$

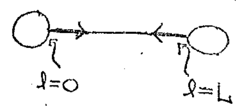
or massless $\quad \lambda(\ell) = 0$

then $\vec{f}^{ext}(\ell) + \frac{d}{d\ell}\left[T\frac{d\vec{r}}{d\ell}\right] = \vec{0}$

(5) If a string is fixed at an end to another body, the force it exerts on the other body is

$T(0)\frac{d\vec{r}}{d\ell}(0)$ at end "0"

$-T(L)\frac{d\vec{r}}{d\ell}(L)$ at end "L"



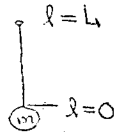
Always a "pull" if $T>0$

If the end is "free" [connected to nothing] $T=0$ at the end

[$\vec{F}=m\vec{a}$ for other body in $\lim\limits_{m\to 0}$]

⑤

Example 1: Weight hanging from uniform ($\lambda(\ell) = \text{const} \equiv \lambda$) string



$$\frac{d\vec{r}}{d\ell} = \vec{k}$$

Force balance for weight

$$T(0)\,\vec{k} + (-mg\vec{k}) = 0 \qquad T(0) = mg$$

Body of string:

$$\frac{d}{d\ell}\left[T(\ell)\,\vec{k}\right] - \lambda g \vec{k} = \vec{0}$$

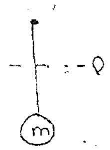
$$\frac{dT}{d\ell} = \lambda g \qquad T = \lambda g \ell + mg$$

$$\vec{F}^{ext} \text{ at top} = (\lambda g L + mg)\,\vec{k}$$

(force ceiling exerts on string)

⑥

A even more trivial way to get the same answer



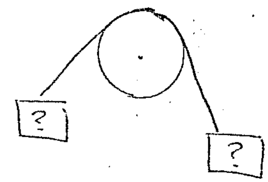
Consider all of system below the dotted line as a subsystem. All external forces must sum to zero.

$$\vec{O} = \underset{\text{gravity on weight}}{-mg\vec{k}} \; \underset{\text{gravity on string}}{-\lambda \ell g \vec{k}} \; \underset{\text{upper part of string on lower part}}{+T(\ell)\,\vec{k}}$$

$$T(\ell) = mg + \lambda \ell$$

⑦

Example 2: Massless string moving over frictionless pulley



$$\vec{n}(\ell) + \frac{d}{d\ell}\left[T\frac{d\vec{r}}{d\ell}\right] = 0$$

$\vec{n}(\ell)$: normal force density of pulley on string

$$\vec{n}(\ell) + \frac{dT}{d\ell}\frac{d\vec{r}}{d\ell} + T\frac{d^2\vec{r}}{d\ell^2} = 0$$

$$\vec{n}(\ell)\cdot\frac{d\vec{r}}{d\ell} = 0 \qquad (\text{"normal"})$$

$$\frac{d\vec{r}}{d\ell}\cdot\frac{d\vec{r}}{d\ell} = 1 \qquad (\text{Pythagoras})$$

$$\frac{d}{d\ell}\left(\frac{d\vec{r}}{d\ell}\cdot\frac{d\vec{r}}{d\ell}\right) = 0 = 2\,\frac{d^2\vec{r}}{d\ell^2}\cdot\frac{d\vec{r}}{d\ell}$$

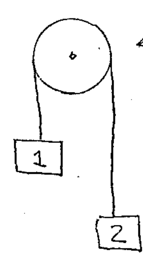
⑧

$$\frac{d\vec{r}}{d\ell}\cdot\left[\vec{n}(\ell) + \frac{dT}{d\ell}\frac{d\vec{r}}{d\ell} + T\frac{d^2\vec{r}}{d\ell^2}\right] = 0$$

$$0 \quad + \quad \frac{dT}{d\ell} \quad + \; 0 \quad = 0$$

$$\frac{dT}{d\ell} = 0 \quad \text{or} \quad \boxed{T = \text{constant}}$$

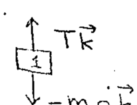
Example 3:



← massless string frictionless pully

Given m_1, what must m_2 be so nothing moves?

Force diagram for 1

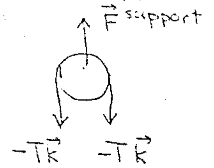


$$T = m_1 g$$

Likewise, for 2 $\quad T = m_2 g$

same T, by Example 2. Thus $m_1 = m_2$

Of course, the pulley hangs from a support. What is $\vec{F}^{\,support}$, the force the support exerts on the pulley?

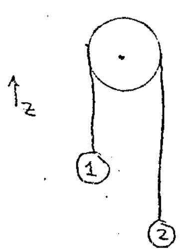


← Force diagram for pulley + part of string (assuming massless pulley)

$\vec{F}^{\,supp} - 2T\vec{k} = 0 \qquad \vec{F}^{\,supp} = 2T\vec{k} = 2m_1 g\,\vec{k}$

(Could also be obtained by considering pulley + string + weights as a single system.)

Example 4: Same as before, except $m_1 \neq m_2$



$$m_1 \ddot{z}_1 = T - m_1 g$$
$$m_2 \ddot{z}_2 = T - m_2 g$$
} Same T, by Example 2

$$m_1 \ddot{z}_1 - m_2 \ddot{z}_2 = (m_2 - m_1) g$$

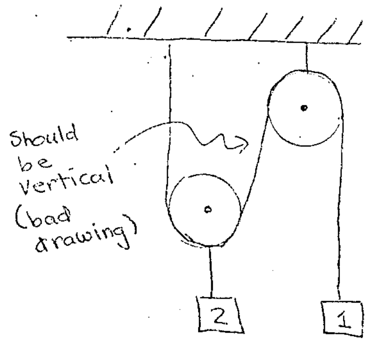
but $z_1 + z_2 = \text{const}$ [ideal strings don't stretch]

$$\ddot{z}_1 + \ddot{z}_2 = 0 \qquad \text{or } -\ddot{z}_1 = +\ddot{z}_2$$

$$(m_1 + m_2)\ddot{z}_1 = (m_2 - m_1) g$$

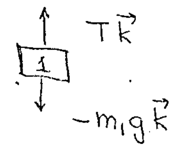
$$\ddot{z}_1 = \frac{m_2 - m_1}{m_1 + m_2} g$$

Example 5: Double pulley (both massless)



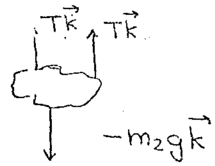
Given m_1, what is m_2 so nothing moves?

Force diagram for ~~m~~ 1



$T = m_1 g$

Force diagram for 2 + pulley

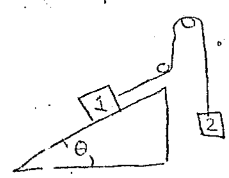


$2T = m_2 g$

$\boxed{m_2 = 2m_1}$

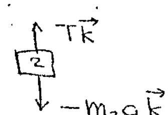
(Demonstrate)

Example 6:



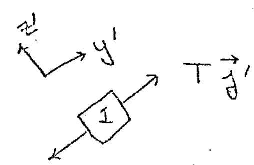
Given m_1, What is m_2 such that nothing moves?

Force diagram for 2



$T = m_2 g$

Force diagram for 1



$-m_1 g\,\vec{j}'\sin\theta$
(from lecture V)

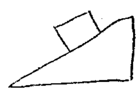
$T = m_1 g \sin\theta$

$m_2 = m_1 \sin\theta$
$= (.01)\, m_1$
in case at hand

(Demonstrate)

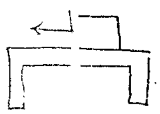
Contact Forces, Con't.: Friction

Friction is a force that acts between bodies in contact. It is $\perp$ to the normal force. It may keep a body from moving (relative to the other body)



static friction

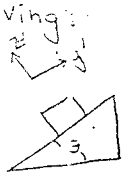
or slow one down that is in (relative) motion



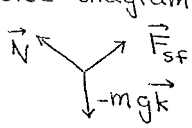
sliding friction

An accurate description of friction is very complicated. We will settle here for a crude (but serviceable) approximate description.

Static Friction is a "constraint force" like normal force. It is just big enough to keep the body from moving: Example: Block on tilted table



Force diagram for block:



nothing moves: $\vec{N} + \vec{F}_{sf} - mg\vec{k} = \vec{0}$

$\vec{N}_{z'} = mg\cos\theta$ (as before)

$\vec{F}_{sf\,y'} = mg\sin\theta$

The only interesting feature is that this does not go on forever as we increase θ. Eventually the block slips. What determines the $\angle$ of slippage?

(Rough, Empirical) Answer:

$$|\vec{F}_{sf}| \le \mu\, |\vec{N}|$$

μ ↳ "coeficient of friction" (depends on composition of bodies)

Note: Area of contact is irrelevant (Surprising but (roughly) true)

Determination of angle of slippage

$$mg\sin\theta_0 \le \mu\, mg\cos\theta_0 \quad [|\vec{F}_{sf}| \le \mu|\vec{N}|]$$

$\tan\theta_0 \le \mu$ no slippage

$\tan\theta > \mu$ it slips

(Independent of mass of block!)

Do the demonstration yourself!

Ingredients:
- One quarter
- One dime
- One copy K+K
- Detergent to remove grease spots from above

(Rough, Empirical) Description of Sliding Friction

$$\vec{F}_{sdf} = -\mu \frac{\vec{V}}{|\vec{V}|} |\vec{N}|$$

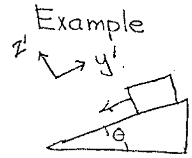
$\vec{V}$ = (relative) velocity of two surfaces

[Not quite right for very small $|\vec{V}|$]

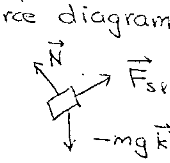
Note: (1) Tends to slow things down

(2) $|\vec{F}_{sdf}| = \mu\, |\vec{N}|$

Example



Force diagram



$|\vec{N}| = mg\cos\theta$ (as before)

$m\ddot{y}' = -mg\sin\theta + \mu|\vec{N}|$

$\cancel{m}\ddot{y}' = -\cancel{m}g(\sin\theta - \mu\cos\theta)$

(Demonstrate)

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$$\dot{y}' = \dot{y}'(0) - g(\sin\theta - \mu\cos\theta)t$$

↳ better be ≤ 0

as long as $\dot{y}' \le 0$. [If $\dot{y}'$ changes sign, $\vec{F}_{s\&f}$ reverses direction]

Case (a) $\sin\theta - \mu\cos\theta \ge 0$

AOK block accelerates downward forever

Case (b) $\sin\theta - \mu\cos\theta \le 0$

block <u>decelerates</u> until it comes to rest at

$$t = \frac{\dot{y}'(0)}{g(\sin\theta - \mu\cos\theta)}$$

What happens then?

$\sin\theta - \mu\cos\theta \le 0$ $\quad$ $\tan\theta - \mu \le 0$

$\tan\theta \le \mu$

It stays at rest! (See 15)

18

Prologue to Next Week

Differential Equations and Initial Value Data.

Prototype Equation

$$\frac{dX}{dt} = f(X,t)$$

How to solve (approximately). Pick a small number, ϵ, and replace $\frac{dX}{dt}$ by $\frac{X(t+\epsilon) - X(t)}{\epsilon}$.

$$X(t+\epsilon) = X(t) + \epsilon f(X(t), t)$$

Given $X(0)$ compute $X(\epsilon)$

From $X(\epsilon)$ " $X(2\epsilon)$

etc.

For <u>every</u> $X(0)$ we get a <u>unique</u> solution.

$X(0)$ is "initial value data".

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Example $\frac{dX}{dt} = X$ $\quad$ $X(0) = 1$

$\epsilon = 0.1$

t	$X(t)$
0	1
0.1	1.10
0.2	1.21
0.3	1.33
0.4	1.46
0.5	1.61
0.6	1.77
0.7	1.95
0.8	2.14
0.9	2.36
1.0	2.59

⟸ Computed from $X(t+\epsilon) = X(t) + \epsilon X(t) = (1.1)X(t)$

Exact Solution $\frac{1}{X}\frac{dX}{dt} = 1$

$$\frac{d}{dt}\ln X = 1$$

$$\ln X(t) - \underbrace{\ln X(0)}_{0} = t$$

$$X(t) = e^t$$

$$X(1) = e = 2.72\ldots$$

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Generalization: Many X's

X_a $\quad$ $a = 1, 2 \ldots N$

$$\frac{dX_1}{dt} = f_1(X_1, X_2 \cdots X_n, t)$$

$$\frac{dX_2}{dt} = f_2(\quad " \quad) \quad \text{etc.}$$

Same story, just more columns in table.

Initial value data: All X's at $t=0$.

For <u>every</u> set of values of this set of N real variables there is a <u>unique</u> solution.

Special Case: Diff. Eqs. for the motion of n particles

$$m_1 \frac{d^2\vec{x}_1}{dt^2} = \vec{F}_1(\vec{r}_1, \vec{v}_1, \vec{r}_2, \vec{v}_2 \cdots \vec{r}_n, \vec{v}_n, t) \quad \text{etc}$$

Equivalently

$$\frac{d\vec{r}_1}{dt} = \vec{v}_1$$

$$\frac{d\vec{v}_1}{dt} = \frac{1}{m_1}\vec{F}_1(\vec{r}_1, \vec{v}_1, \vec{r}_2, \vec{v}_2 \cdots \vec{r}_n, \vec{v}_n, t) \text{ etc.}$$

$6n$ X's

$$X_1 = x_1 \quad X_2 = y_1 \quad X_3 = z_1 \quad X_4 = v_{x1} \quad X_5 = v_{y1} \quad X_6 = v_{z1}$$

etc.

Moral: For every value of $\vec{x}_a(0), \vec{v}_a(0)$ (all a's) there is a unique solution. $[a = 1 \cdots n]$

Corollary: If (by guesswork or whatever) we can find a ~~set~~ family of solutions rich enough so that there is one of them for any value of $\vec{x}_1(0), \vec{x}_2(0) \cdots \vec{v}_1(0), \vec{v}_2(0) \cdots \vec{v}_n(0)$, we have all the solutions.

You might read Feynman Lectures, Vol. 1, Chapter 9
Feynman solves for planetary orbit numerically.

1

Solving Eqs. of Motion

Method 1: Direct Integration.

Suitable for: $\frac{d^2x}{dt^2} = f(t)$ ← known function of t

$$V(t) = V(0) + \int_0^t f(t')dt'$$

$$x(t) = x(0) + \int_0^t V(t')dt'$$

Of course, whether we can write these $\int$'s in terms of known functions depends on how exotic f is (and how big our $\int$ tables are). In general, we will be happy if we can "reduce the solution to quadratures" ["quadratures" is an old word for $\int$'s] — happier still, of course, if we can do the $\int$'s.

Example: Free fall (Lecture 2 et seq.)

2

Method 2: Guesswork

We know we have guessed the most general solution if we can fit arbitrary initial value data. (See end of Lecture 6)

Example: One dimensional Harmonic Oscillator

$F = -kx$ $k > 0$ (Attractive Force)

$m\ddot{x} = -kx$ ⇐ One of the most important Eqs. in physics!

Why? Consider any force dependent only on x

$$m\ddot{x} = F(x)$$

Suppose $x = x_0$ is a pt. of equilibrium

[i.e. $x(0) = x_0$, $\dot{x}(0) = 0 \Rightarrow x(t) = x_0$]

then $F(x_0) = 0$

Consider small deviations from equilibrium

$$|x - x_0| < \epsilon$$

Note: Even if this is true at the start we always have to check that it remains true as time goes on

3

from $\frac{dF}{dx}(x_0) \equiv \left.\frac{dF}{dx}\right|_{x_0} = \lim_{x\to x_0} \frac{F(x) - F(x_0)}{x - x_0}$

we approximate

$$F(x) = \underbrace{F(x_0)}_{0} + (x - x_0)\left.\frac{dF}{dx}\right|_{x_0}$$

("Linear approximation") Should be good for sufficiently small ε. How good? + How small is sufficient? depends on details of F(x). Example: Lin. approx. to sin x about $x_0 = 0$ is x

x	sin x
.1	.10
.2	.20
.3	.30
.4	.39
.5	.48
.6	.56
.7	.64

10% accuracy for $|x| < .7$ rad $\approx 40°$

For $F(x) = \sin ax$ 10% accuracy for $|x| < \frac{.7}{a}$. May be large or small range — depends on a.

4

In linear approx

$$m\frac{d^2x}{dt^2} = F(x) = \left.\frac{dF}{dx}\right|_{x_0}(x - x_0)$$

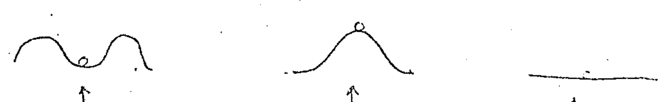
$$m\frac{d^2}{dt^2}(x - x_0) = -k(x - x_0) \qquad \left[k \equiv -\left.\frac{dF}{dx}\right|_{x_0}\right]$$

$x - x_0$ obeys harmonic oscillator Eq!

Why is $k > 0$?

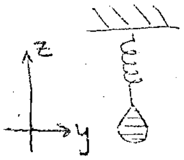
If $k < 0$, force is repulsive away from equilibrium ≡ equilibrium is unstable ≡ small departures from equilibrium grow with time.

Example: Ball on track in earth's gravity



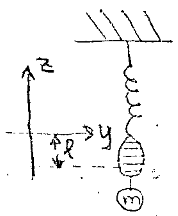
stable equilibrium $k > 0$ | unstable equilibrium $k < 0$ | neutral equilibrium $k = 0$

Linear approx tends to be very good for weights on springs



Equilibrium $z=0$

Add new weight of mass m, find new equilibrium position



$$-mg\vec{k} + k\ell\,\vec{k} = \vec{0}$$

added force of gravity

force of spring in linear approx

$$\ell = \frac{mg}{k}$$

(Demonstrate and measure k)

Now for solution of

$$m\ddot{x} = -kx$$

Define $\omega = \sqrt{k/m}$

$$\ddot{x} = -\omega^2 x$$

Not hard to guess two solutions:

$x = \sin\omega t$ $\qquad$ $x = \cos\omega t$

We can obtain ~~an~~ more general solution by forming a general linear combination:

$$x = A\sin\omega t + B\cos\omega t$$

[Note: This works only because Eq. is linear -- couldn't do this trick if there were, e.g., x^2 or $x\dot{x}$ terms.]

$x(0) = B$ $\qquad$ $\dot{x}(0) = \omega A$

$$\therefore\quad x = x(0)\cos\omega t + \frac{\dot{x}(0)}{\omega}\sin\omega t$$

General solution

Many ways of writing this:

$$C\cos(\omega t + \theta) \qquad (C>0)$$

$$= C\cos\omega t\cos\theta - C\sin\omega t\sin\theta$$

$$= A\sin\omega t + B\cos\omega t$$

if $A = -C\sin\theta$ $\quad$ $B = C\cos\theta$

or $C = \sqrt{A^2+B^2}$ $\quad$ $\tan\theta = -\frac{A}{B}$

Another way:

$$C\cos(\omega t+\theta) = \mathrm{Re}\left[Ce^{i(\omega t+\theta)}\right] \quad *$$

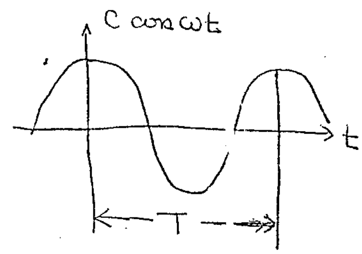
$$= \mathrm{Re}\left[\alpha e^{i\omega t}\right]$$

$\alpha = Ce^{i\theta}$ arbitrary complex number

Note: Linear approximation meets a minimal consistency check — if A + B are sufficiently small $|x| < \epsilon$ for all t.

*[In case you didn't know:

$$e^{i\theta} = \cos\theta + i\sin\theta\,]$$



$$T = \text{period} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

(Demonstrate)

Variations on the Harmonic Oscillator

1. Isotropic Oscillator in 3d

$$m\ddot{\vec{r}} = -k\vec{r}$$

$m\ddot{x} = -kx$ $\qquad$ likewise for y + z

$$\therefore\quad x = A_x\sin\omega t + B_x\cos\omega t$$

$$y = A_y\sin\omega t + B_y\cos\omega t$$

$$z = A_z\sin\omega t + B_z\cos\omega t$$

$$\vec{r} = \vec{A}\sin\omega t + \vec{B}\cos\omega t$$

$$\vec{r}(0) = \vec{B}$$

$$\dot{\vec{r}}(0) = \omega\vec{A}$$

(9)

What does the orbit look like?

(1) $|\vec{r}|^2 = \vec{r}\cdot\vec{r}$ is bounded so there is certainly ~~once~~ at least one time at which

$$\frac{d}{dt}\,\vec{r}\cdot\vec{r} = 2\,\vec{r}\cdot\dot{\vec{r}} = 0$$

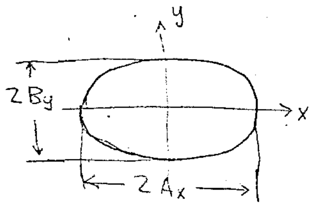
Choose this to be $t=0$.

$$\Rightarrow \quad \vec{A}\cdot\vec{B} = 0$$

(2) Choose coord. so $\vec{A}$ lies along the x axis and $\vec{B}$ along the y axis

$$x = A_x \sin\omega t$$
$$y = B_y \cos\omega t$$
$$z = 0$$

$$\frac{x^2}{A_x^2} + \frac{y^2}{B_y^2} = 1$$

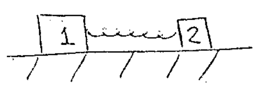


An ellipse! (Center at origin)

[We will demonstrate this next lecture]

(10)

2. Push Me – Pull You



$$m_1 = m_2 \equiv m$$

$$F_{12} = -F_{21}$$
$$= k\,(x_2 - x_1 - \ell), \quad \ell, k > 0$$

ℓ is equilibrium value of $x_2 - x_1$

First solve in com frame

$[\,m x_1 + m x_2 = 0 \iff x_1 + x_2 = 0\,]$

$$m\ddot{x}_1 = k\,(x_2 - x_1 - \ell) = -2k\,(x_1 + \tfrac{\ell}{2})$$

$$x_1 + \frac{\ell}{2} = A\sin\omega t + B\cos\omega t \qquad [\omega = \sqrt{2k/m}]$$

$$x_1 = -\frac{\ell}{2} + A\sin\omega t + B\cos\omega t = -x_2$$

In general

$$x_1 = -\frac{\ell}{2} + A\sin\omega t + B\cos\omega t + Ct + D$$

$$x_2 = \frac{\ell}{2} - A\sin\omega t - B\cos\omega t + \underbrace{Ct + D}_{\text{com motion}}$$

(11)

Consider Initial Value Data at $t=0$

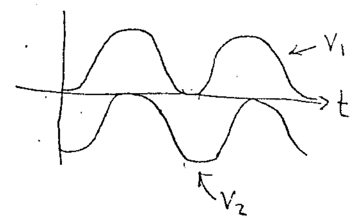
$$\underbrace{x_1 = -\ell/2 \qquad x_2 = \ell/2}_{\Downarrow\; B = D = 0} \qquad \dot{x}_1 = 0 \quad \dot{x}_2 \neq 0$$

$$\Downarrow$$

$$C + \omega A = 0$$

$$x_1 = -\frac{\ell}{2} + A\,(\sin\omega t - \omega t)$$
$$x_2 = \frac{\ell}{2} - A\,(\sin\omega t + \omega t)$$

$$V_1 = A\omega\,(\cos\omega t - 1)$$
$$V_2 = -A\omega\,(\cos\omega t + 1)$$



(12)

3. Unstable Equilibrium (in linear approx)

$$m\ddot{x} = Kx \qquad K > 0$$

$$\ddot{x} = \lambda^2 x \qquad \lambda = \sqrt{K/m}$$

Can we guess solutions?

Trig Functions	Hyperbolic Functions
$\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$	$\sinh\phi = \dfrac{e^{\phi} - e^{-\phi}}{2}$
$\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$	$\cosh\phi = \dfrac{e^{\phi} + e^{-\phi}}{2}$
$\dfrac{d}{d\theta}\sin\theta = \cos\theta$	$\dfrac{d}{d\phi}\sinh\phi = \cosh\phi$
$\dfrac{d}{d\theta}\cos\theta = -\sin\theta$	$\dfrac{d}{d\phi}\cosh\phi = \sinh\phi$

$$x = A\sinh\lambda t + B\cosh\lambda t$$

$$x(0) = B \qquad \dot{x}(0) = \lambda A$$

as $t \to \infty$ $\quad \sinh\lambda t \to \frac{1}{2}e^{\lambda t} \quad \cosh\lambda t \to \frac{1}{2}e^{\lambda t}$

$$x \to \tfrac{1}{2}(A+B)\,e^{\lambda t}$$

Note: Even for very small $A+B$ this is not to be trusted forever. Eventually x gets so large the linear approximation breaks down.

Solving Eqs. (Con't.) (13)

Method 3: Separation Method

Good for Eqs. of the form

$$\frac{dx}{dt} = \frac{f(t)}{g(x)}$$

$$g(x)\,dx = f(t)dt \qquad \text{(Separation)}$$

$$\int_{x(0)}^{x(t)} g(x')dx' = \int_0^t f(t')dt'$$

"Reduced to Quadratures"

(14)

Viscous Drag

A body moving through a fluid (air, water, oil, etc.) is subject to a force

$$\vec{F} = -\underset{\substack{\uparrow\\ \text{constant}}}{c}\,\vec{V}$$

$\vec{V}$: Velocity (rel. to fluid).

This is "viscous drag".

Like sliding friction, this "opposes motion"

Unlike sliding friction

(1) $|\vec{F}| = c|\vec{v}|$

(2) C is independent of composition of body (but does depend on comp. of fluid)

(3) C depends on shape of body + on orientation (rel to $\vec{v}$). Thus this Eq. is good for falling spheres + automobiles moving in still air but not for tumbling leaves.

Also: This is linear approx. — breaks down for suff. high $|\vec{v}|$. [when turbulence appears.]

(15)

Example 1: Free motion in fluid (1 dim)

$$m\,a = -cv$$

$$\frac{dv}{dt} = -\frac{c}{m}v$$

$$\frac{dv}{v} = -\frac{c}{m}dt \qquad \text{(Separate)}$$

$$\int_{v(0)}^{v(t)} \frac{dv'}{v'} = -\frac{c}{m}\int_0^t dt'$$

$$\ln\frac{v(t)}{v(0)} = -\frac{c}{m}t$$

$$v(t) = v(0)e^{-\frac{c}{m}t}$$

$$x(t) = \int_0^t v(t')dt' + x(0)$$

$$= x(0) + \frac{mv(0)}{c}\left[1 - e^{-(c/m)t}\right]$$

as $t\to\infty$ $\quad v\to 0 \quad x\to x(0) + \frac{mv(0)}{c}$

(16)

Example 2: Same Eq., Alternative Method

$$\frac{dv}{dt} = -\frac{c}{m}v$$

Consider v as a function of x (why not?)

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v$$

$$\frac{dv}{dx} = -\frac{c}{m}$$

Integrate starting from $x(0)$ $\quad v(x(0)) \equiv v(0)$

$$v(x) = v(0) - \frac{c(x-x(0))}{m}$$

From this we see directly that as $v\to 0$

$$x - x(0) \to \frac{mv(0)}{c}$$

If we want more information we can go on:

$$\frac{dx}{dt} = v(0) - \frac{c}{m}[x - x(0)]$$

Can solve by separation, get same result as before.

16.5

Example 3: Same thing in 3d

$$\frac{d\vec{v}}{dt} = -\frac{c}{m}\vec{v}$$

$$\frac{dv_x}{dt} = -\frac{c}{m}v_x \qquad \text{likewise for } y+z$$

$$x(t) = x(0) + \frac{m v_x(0)}{c}\left[1 - e^{-ct/m}\right]$$

likewise for $y+z$

$$\vec{r}(t) = \vec{r}(0) + \frac{m\vec{v}(0)}{c}\left[1 - e^{-ct/m}\right]$$

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Example 4: Falling Ball (with air resistance)

$$m\frac{d^2\vec{r}}{dt^2} = m\frac{d\vec{v}}{dt} = -c\vec{v} - mg\hat{k}$$

$x+y$ components as in Example 3.
Concentrate on z component $v \equiv v_z$

$$\frac{dv_z}{dt} = -\frac{c}{m}v - g = -\frac{c}{m}\left(v + \frac{m}{c}g\right)$$

$$\frac{dv}{v + \frac{m}{c}g} = -\frac{c}{m}dt$$

$$\int_{v(0)}^{v(t)} \frac{dv'}{v' + \frac{m}{c}g} = -\frac{c}{m}\int_0^t dt'$$

$$\ln\left[\frac{v(t) + \frac{m}{c}g}{v(0) + \frac{m}{c}g}\right] = -\frac{c}{m}t$$

$$v(t) = \left[v(0) + \frac{m}{c}g\right]e^{-ct/m} - \frac{m}{c}g$$

as $t \to \infty$ $\qquad v \to -\frac{m}{c}g$ ("terminal velocity")

~~Very different from sliding f...~~

Very different from sliding friction!

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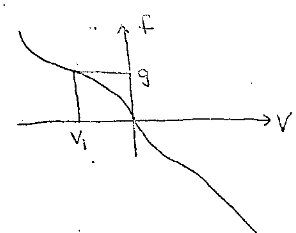
General Discussion of Terminal Velocity

$$\frac{dv}{dt} = f(v) - g$$

Assume: (1) $f(0) = 0$
(2) $\frac{df}{dv} < 0$ for all v
(3) $f(v) \to \mp\infty$ as $v \to \pm\infty$

All obeyed by $f = -\frac{cv}{m}$

I will show that under these conditions there is a unique terminal velocity v_1, such that $v \to v_1$ as $t \to \infty$.
I will not need to do any $\int$'s

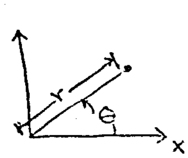


Under stated conditions there is a unique solution to $f(v_1) = g$

If $v < v_1$ $\qquad \frac{dv}{dt} = f(v) - g > 0$ $\qquad$ v increases
If $v > v_1$ $\qquad \frac{dv}{dt} = f(v) - g < 0$ $\qquad$ v decreases

QED

Polar Coordinates ①



Instead of labelling pts. by (x,y,z) we use (r,θ,z). [Note: θ ill-defined at $r=0$ — no serious problem, but something to keep in mind.]

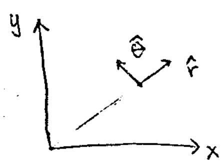
$$r=\sqrt{x^2+y^2} \qquad \theta=\tan^{-1}(x/y) \qquad z=z$$

$$x=r\cos\theta \qquad y=r\sin\theta \qquad z=z$$

[Note: r is not $|\vec{r}|=\sqrt{x^2+y^2+z^2}$
$=\sqrt{r^2+z^2}$

This is confusing notation but it is that used in K+K and we will stick to it.]

Vectors in Polar Coordinates ②



$\hat{r}$ is a unit vector plane pointing along the direction of $\vec{r}=x\vec{i}+y\vec{j}$

$$\hat{r}=\frac{\vec{r}}{|\vec{r}|}=\frac{x\vec{i}+y\vec{j}}{r}=\cos\theta\,\vec{i}+\sin\theta\,\vec{j}$$

$\hat{\theta}$ is an orthogonal unit vector

$$\hat{\theta}=-\sin\theta\,\vec{i}+\cos\theta\,\vec{j}$$

It points in the direction of increasing θ. $\hat{r},\hat{\theta},\vec{k}$ form a right-handed triad of orthogonal unit vectors, just like $\vec{i},\vec{j},\vec{k}$

$$|\hat{r}|=|\hat{\theta}|=|\vec{k}|=1$$

$$\hat{r}\cdot\hat{\theta}=\hat{r}\cdot\vec{k}=\hat{\theta}\cdot\vec{k}=0$$

$$\hat{r}\times\hat{\theta}=\vec{k} \qquad \hat{\theta}\times\vec{k}=\hat{r} \qquad \vec{k}\times\hat{r}=\hat{\theta}$$

③

They are different in that $\vec{i},\vec{j}$ and $\vec{k}$ are constants while $\hat{r}$ and $\hat{\theta}$ are functions of position

(1) $\hat{r}$ and $\hat{\theta}$ are ill-defined at $r=0$

(2) At all other positions, they are functions of θ

$$\frac{d\hat{r}}{d\theta}=\hat{\theta} \qquad \frac{d\hat{\theta}}{d\theta}=-\hat{r}$$

In studying the motion of a particle, we will always express the vectors associated with the particle $(\vec{r},\vec{p},\vec{a},\vec{F})$ in terms of $\hat{r}$ and $\hat{\theta}$ for the position of the particle. Thus we will write Eqs. like

$$\vec{F}=F_z\vec{k}+F_r\hat{r}+F_\theta\hat{\theta}$$

or

$$F_r=\vec{F}\cdot\hat{r}$$

without bothering to explicitly state which $\hat{r}$ and $\hat{\theta}$ we are using.

Velocity and Acceleration in Polar Coordinates $(z=0)$ ④

$$\vec{r}=r\hat{r}$$

$$\vec{v}=\dot{\vec{r}}=\dot{r}\hat{r}+r\frac{d}{dt}\hat{r}$$

$$\frac{d}{dt}\hat{r}=\frac{d\hat{r}}{d\theta}\frac{d\theta}{dt}=\hat{\theta}\dot{\theta}$$

$$\vec{v}=\dot{r}\hat{r}+r\dot{\theta}\hat{\theta}$$

$$\boxed{v_r=\dot{r} \qquad v_\theta=\dot{\theta}r}$$ (no surprise)

$$\vec{a}=\frac{d\vec{v}}{dt}=\ddot{r}\hat{r}+r\ddot{\theta}\hat{\theta}+\dot{r}\dot{\theta}\hat{\theta}+\dot{r}\frac{d\hat{r}}{dt}+r\dot{\theta}\frac{d\hat{\theta}}{dt}$$

$$\frac{d\hat{\theta}}{dt}=\frac{d\hat{\theta}}{d\theta}\frac{d\theta}{dt}=-\hat{r}\dot{\theta}$$

$$\vec{a}=(\ddot{r}-r\dot{\theta}^2)\hat{r}+(r\ddot{\theta}+2\dot{r}\dot{\theta})\hat{\theta}$$

$$\boxed{\begin{aligned}a_r&=\ddot{r}-r\dot{\theta}^2\\ a_\theta&=r\ddot{\theta}+2\dot{r}\dot{\theta}\end{aligned}}$$ (ugh!) (but useful)

Check: $r=\text{const}$ $\quad\theta=\omega t$

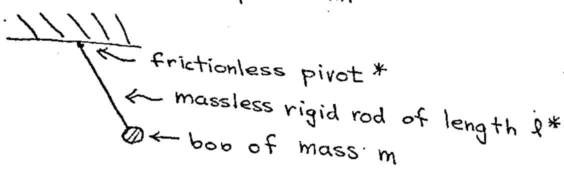
$$a_r=-r\omega^2 \qquad a_\theta=0$$

What we found in lecture 3.

(5)

Sometimes Eqs. of Motion that are complicated in Cartesian coordinates are simple in polar coordinates.

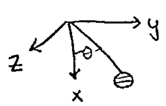
Example 1: The pendulum



*or any other arrangement that constrains the bob to move on a circle of radius ℓ (e.g. a massless string or a circular track)

We will also restrict ourselves to motions in the plane of the paper — we will relax this restriction shortly

Coordinates:

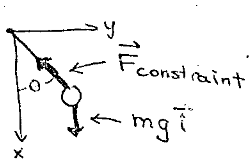


Unconventional, but right-handed

$\vec{F}_{grav} = mg\vec{i}$

(6)

Force Diagram



[No force in z-direction — thus it is consistent to assume motion is in x-y plane]

Note constraint force is ∥ to $\hat{r}$. Thus it makes no contribution to the θ-component of

$$\vec{F} = m\vec{a}$$

$$F_\theta = m a_\theta$$

$$F_\theta = mg\,\vec{i}\cdot\vec{e}_\theta = mg\,\vec{i}\cdot[-\sin\theta\,\vec{i} + \cos\theta\,\vec{j}\,]$$

$$= -mg\sin\theta$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \ell\ddot{\theta}$$

$$\therefore \quad \cancel{m}\ell\ddot{\theta} = -\cancel{m}g\sin\theta$$

Make linear approximation $\sin\theta \approx \theta$

[10% accuracy for $|\theta| < 40°$] and obtain

(7)

the linear approx to the Eq. of Motion

$$\ddot{\theta} = -\omega^2\theta \qquad \omega = \sqrt{g/\ell}$$

The Harmonic Oscillator Again!

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\ell/g} \qquad \text{(Do experiment)}$$

Historical Note:

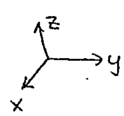
As Galileo pointed out, the linearity of the pendulum Eq. can be checked directly. (He used somewhat different language). If Eq. is linear, two pendulums of same ℓ, released from rest at different positions, will describe same motion (up to a scale). You don't need a good clock to check this prediction — just see if the pendulums keep in step. Same method can be used to check independence of mass of bob

(Demonstrate)

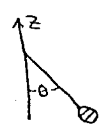
(8)

Example 2:

Spherical Pendulum (in Linear Approx)



return to conventional coordinates



pendulum free to move anywhere on a sphere of radius ℓ with center at pivot

$|\vec{r}|^2 = \ell^2$

We will describe position by giving x+y; z then determined: $z = \sqrt{\ell^2 - x^2 - y^2}$

We must find $F_x + F_y$.

(1) Problem obviously invariant under rotations in x-y plane — thus it suffices to find $F_x + F_y$ for

$x = 0$ y arbitrary

This we can figure out from Example 1 [and $\vec{F} = m\vec{a}$]

(9)

Since there is a motion that stays in the y-z plane

$$F_x = m\ddot{x} = 0$$

$$F_y = m\ddot{y} = m\ell \frac{d^2}{dt^2}\sin\theta$$

$$= m\ell \frac{d^2}{dt^2}\theta \qquad \text{(linear approx)}$$

$$= -m\ell\omega^2\theta \qquad (\;\text{"}\;)$$

$$= -m\omega^2 y \qquad (\;\text{"}\;)$$

If we define $\vec{r}\,' = x\vec{i} + y\vec{j}$ this is equivalent to

$$\ddot{\vec{r}}\,' = -\omega^2 \vec{r}\,'$$

A two-dimensional isotropic harmonic oscillator. Bob traces out ellipse in the horizontal plane.

(Demonstrate)

(10)

Example 3: An introduction to the problem of 2 particles interacting through a "central force"

$$\vec{F}_{12} = -\vec{F}_{21} = \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|} f(|\vec{r}_1 - \vec{r}_2|)$$

arbitrary function

e.g., gravity $\quad f = \dfrac{1}{|\vec{r}_1 - \vec{r}_2|^2}$

① $\ddot{\vec{r}}_1 = \dfrac{1}{m_1}\vec{F}_{12}$

$$\ddot{\vec{r}}_2 = \frac{1}{m_2}\vec{F}_{21} = -\frac{1}{m_2}\vec{F}_{12}$$

Define $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$

$$\ddot{\vec{r}} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\vec{F}_{12} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\frac{\vec{r}}{|\vec{r}|} f(|\vec{r}|)$$

Define μ by

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \quad \text{or} \quad \mu = \frac{m_1 m_2}{m_1 + m_2} \quad (\text{«reduced mass»})$$

$$\boxed{\mu\ddot{\vec{r}} = \frac{\vec{r}}{|\vec{r}|} f(\vec{r})} \Leftarrow \text{Single particle problem}$$

(11)

Since in center-of-mass frame

$$m_1\vec{r}_1 + m_2\vec{r}_2 = \vec{0}$$

once we know $\vec{r} = \vec{r}_1 - \vec{r}_2$

We can easily compute

$$\vec{r}_1 = \frac{m_2\vec{r} + m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} = \frac{m_2}{m_1 + m_2}\vec{r} = \frac{\mu}{m_1}\vec{r}$$

and $\vec{r}_2 = \dfrac{\mu}{m_2}\vec{r}$

We need study only this 1-particle problem

② Force is such that $\dot{\vec{L}} = \vec{0}$

$\therefore$ motion is in a plane

③ Choose this to be the x-y plane and introduce polar coordinates

$$\vec{F}_{12} = \hat{r} f(r)$$

$$\mu a_r = \mu(\ddot{r} - r\dot{\theta}^2) = f(r)$$

$$\mu a_\theta = \mu(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0$$

(12)

④ Consider

$$\frac{d}{dt}(r^2\dot{\theta}) = 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = r(2\dot{r}\dot{\theta} + r\ddot{\theta})$$

Thus $\mu a_\theta = 0 \Longleftrightarrow \mu r^2\dot{\theta} = \text{const.}$

[Is this a new result?

$$\vec{L} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2$$

in com frame $\quad \vec{p}_2 = -\vec{p}_1$

$$\vec{L} = (\vec{r}_1 - \vec{r}_2) \times \vec{p}_1 = \vec{r} \times \vec{p}_1$$

$$\vec{p}_1 = m_1\dot{\vec{r}}_1 = m_1\frac{\mu}{m_1}\dot{\vec{r}} = \mu\dot{\vec{r}}$$

$$\vec{L} = \mu\,\vec{r} \times \dot{\vec{r}}$$

$$\vec{r} = r\hat{r} \qquad [\hat{r} \times \hat{\theta} = \vec{k}]$$

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\mu\vec{r} \times \dot{\vec{r}} = \mu r^2\dot{\theta}\,\vec{k}$$

No, it is not a new result — just $|\vec{L}| = \text{constant}$]

(13)

(A) (con't). However, though not new, it is useful

$$\mu r^2 \dot\theta = |\vec L|$$

$$\boxed{\dot\theta = \frac{|\vec L|}{\mu r^2}}$$

$$\mu(\ddot r - r\dot\theta^2) = f(r)$$

$$\boxed{\mu \ddot r = f(r) + \frac{|\vec L|^2}{\mu r^3}}$$

Thus we have reduced the problem to solving a one-dimensional Eq. of motion. Once we have solved this, we can then find θ by integration.

We will beat the two boxed Eqs. into the ground in week 7.

(14)

Solving Diff. Eqs. (Resumed)

Method 4: Energy Method

Works for any Eq. of the form

$$m\ddot x = F(x)$$

↑ function of x only, not of v or t

Step 1: Construct $V(x)$, the solution to

$$\frac{dV}{dx} = -F(x)$$

$$V(x) = -\int_0^x F(x')\,dx' + V(0)$$

↑ arbitrary constant — how we choose it will not affect the method

(15)

Step 2: Consider

$$\frac{d}{dt}\left[\tfrac{1}{2}m\dot x^2 + V(x)\right]$$

$$= m\dot x\ddot x + \frac{dV}{dx}\dot x$$

$$= \dot x F(x) - F(x)\dot x = 0$$

$\therefore$ $\frac{1}{2}m\dot x^2 + V(x) =$ constant, conventionally called E

Step 3:

$$\tfrac{1}{2}m\dot x^2 + V(x) = E$$

$$\dot x^2 = \frac{2}{m}(E-V)$$

$$\dot x = \pm\sqrt{\frac{2}{m}(E-V(x))} = \frac{dx}{dt}$$

Since $\dot x$ is continuous (+) can change to (−) or (−) to (+) only at times when $\dot x = 0$. We will analyze in detail what happens when $\dot x = 0$ next lecture.

(16)

If $\dot x(0) \neq 0$, the constant E and the $\pm$ sign is determined by $\dot x(0)$. The Eq. can now be "solved" (i.e. reduced to quadratures) by the separation method

$$\frac{dx}{\sqrt{\frac{2}{m}(E-V(x))}} = \pm dt$$

$$\int_{x(0)}^{x(t)} \frac{dx'}{\sqrt{\frac{2}{m}(E-V(x'))}} = \pm\int_0^t dt' = \pm t$$

Note: Solution given in terms of $x(0)$, $\dot x(0)$ — the usual initial value data.

Is this just a mathematical trick? Or is it a clue to a deep physical insight?

And what about more than one particle in more than one dimension?

1

IX Work and Energy

$$m\frac{d\vec{V}}{dt} = \vec{F}$$

$$K = \tfrac{1}{2} m |\vec{V}|^2 = \tfrac{1}{2} m \vec{V}\cdot\vec{V}$$ ("Kinetic Energy")

$$\frac{dK}{dt} = m\vec{V}\cdot\dot{\vec{V}} = \vec{V}\cdot\vec{F}$$

$$K(t_1) - K(t_2) = \int_{t_1}^{t_2} \frac{dK}{dt}\,dt$$

$$= \int_{t_1}^{t_2} \vec{V}\cdot\vec{F}\,dt$$

"Work done on particle"

(sometimes called $W(t_2,t_1)$)

"Change in kinetic energy equals work done"

2

This is the work-energy theorem. It is trivial, but it can be useful.

Example: Falling body ($\vec{F} = -mg\vec{k}$)

At $z=0$, the body is falling in the z-direction with velocity $V_{0z}\vec{k}$. What is V_z at arbitrary z?

$$\underset{\substack{\uparrow\\ \text{time when}\\ \text{position is } z}}{K(t_2)} = \underset{\substack{\uparrow\\ \text{time when}\\ \text{position is } 0}}{K(t_1)} + \int_{t_1}^{t_2} \frac{dz}{dt}(-mg)\,dt$$

$$\tfrac{1}{2} m (V_z)^2 = \tfrac{1}{2} m (V_{0z})^2 - mg\left[z(t_2) - z(t_1)\right]$$

$$(V_z)^2 = (V_{0z})^2 - mgz$$

[Note: Could have been obtained by algebra and $z = -\tfrac{1}{2}gt^2 + V_{0z}t$, $V_z = -gt + V_{0z}$ but this way is <u>much</u> faster (no need to solve quadratic Eq. for t)]

3

Units: If we measure mass in $\left\{\begin{matrix}\text{kg}\\ \text{g}\end{matrix}\right\}$, distance in $\left\{\begin{matrix}\text{m}\\ \text{cm}\end{matrix}\right\}$ and time in sec., we measure work + energy in $\left\{\begin{matrix}\text{kg}\cdot\text{m}^2/\text{sec}^2\\ \text{g}\,(\text{cm})^2/\text{sec}^2\end{matrix}\right\}$

These are called $\left\{\begin{matrix}\text{Joules}\\ \text{ergs}\end{matrix}\right\}$.

Example: A 75 kg physicist falls 2 meters. How much work has been done on him by the earth's gravity?

Ans.: $75\text{ kg} \times 2\text{ m} \times 10\text{ m/sec}^2$

$= 1500$ Joules

$= 1.5 \times 10^{10}$ ergs.

4

Two small topics:

1. Power is work / unit time

$$P \equiv \vec{F}\cdot\vec{V}$$

Power is measured in $\left\{\begin{matrix}\text{Joules/sec}\\ \text{ergs/sec}\end{matrix}\right\}$ called $\left\{\begin{matrix}\text{watts}\\ \text{ergs/sec}\end{matrix}\right\}$.

2. Extension to system of particles:

$$m_a\dot{\vec{V}}_a = \vec{F}_a \qquad a = 1, 2, \dots n$$

$$K = \tfrac{1}{2}\sum_a m_a |\vec{V}_a|^2$$

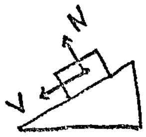
$$K(t_2) - K(t_1) = \sum_a \int dt\, \vec{F}_a\cdot\vec{V}_a$$

Note: Internal forces <u>do</u> <u>not</u> cancel, in general. More on this in next lecture — for now we stick to a single particle (or, equivalently, to motion of the c.o.m. of a composite system).

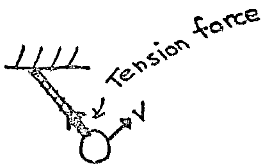
(5)

The work-energy theorem is trivial, but it can be made slightly less trivial by the following important principle:

$\boxed{\text{Forces of Constraint Do No Work}}$



Sliding Block

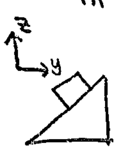


Pendulum

A constraint force (normal force, tension in an ideal string) is always $\perp$ to allowed motions — $\therefore$ $\vec{F}\cdot\vec{v}=0$.

(6)

Example: Block sliding without friction in earth's gravity



Block starts from rest at $z=0$. What is $|\vec{v}|$ at arbitrary z?

$$\tfrac{1}{2}m\,|\vec{v}|^2 - 0 = \int_{t_1}^{t_2}(-mg\vec{k})\cdot\frac{d\vec{r}}{dt}\,dt \quad *$$

$$= -mg\int_{t_1}^{t_2}\frac{dz}{dt}\,dt$$

$$\tfrac{1}{2}\cancel{m}|\vec{v}|^2 = -\cancel{m}g\,z$$

Note: No need to consider normal force
" " " " tilted coord.

* $z(t_1)=0$, $z(t_2)=z$

(7)

The work-energy theorem is trivial but it can be made highly non-trivial by the following important principle:

$\boxed{\text{Many of the forces of nature are conservative}}$

Def. A force is conservative if and only if

$$\int_{t_1}^{t_2}\vec{F}\cdot\vec{v}\,dt$$

depends <u>only</u> on $\vec{r}(t_1)$ and $\vec{r}(t_2)$, <u>whatever</u> the motion between these two points.

Note: <u>Whatever</u> means <u>whatever</u>: we consider <u>general</u> motions from $\vec{r}(t_1)$ to $\vec{r}(t_2)$, <u>not</u> just soln's of the Eqs. of motion.

(8)

Example: (And #1 in a catalog of cons. forces) Constant force

$$\vec{F} = -mg\vec{k}$$

$$\int_{t_1}^{t_2}\vec{F}\cdot\vec{v}\,dt = -mg\int_{t_1}^{t_2}\frac{dz}{dt}\,dt$$

$$= -mg\left[z(t_2)-z(t_1)\right]$$

$$\equiv -mg\left[z_2-z_1\right]$$

Note: We never had to use the information that $\vec{r}(t)$ is a solution of some Eq.

What's so hot about conservative F's?

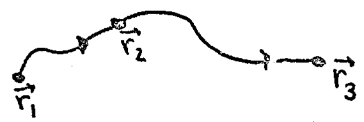
Define $\vec{r}(t_1) \equiv \vec{r}_1$, $\vec{r}(t_2) \equiv \vec{r}_2$

$$W(\vec{r}_2, \vec{r}_1) \equiv + \int_{t_1}^{t_2} \vec{F} \cdot \vec{v}\, dt$$

Theorem:

$$W(\vec{r}_3, \vec{r}_1) = W(\vec{r}_3, \vec{r}_2) + W(\vec{r}_2, \vec{r}_1)$$

Proof:



Equivalently,

$$W(\vec{r}_2, \vec{r}_1) = W(\vec{r}_3, \vec{r}_1) - W(\vec{r}_3, \vec{r}_2)$$

Pick some arbitrary standard point $\vec{r}_0$

Define $V(\vec{r}) \equiv W(\vec{r}_0, \vec{r})$

then $\int_{t_1}^{t_2} \vec{F} \cdot \vec{v}\, dt = W(\vec{r}_2, \vec{r}_1)$

$$= V(\vec{r}_1) - V(\vec{r}_2)$$

Fold into work-energy theorem *

$$\left.\begin{aligned} K(t_2) - K(t_1) &= \int_{t_1}^{t_2} \vec{F} \cdot \vec{v}\, dt \\ &= V(\vec{r}_1) - V(\vec{r}_2) \end{aligned}\right\} K = \tfrac{1}{2} m \vec{v}^2$$

$$\boxed{K(t_2) + V(\vec{r}_2) = K(t_1) + V(\vec{r}_1)}$$

"Law of conservation of energy"

$K + V =$ "total energy" $= E$

$K =$ "kinetic energy"

$V =$ "potential energy"

* Assuming all forces conservative

Remarks:

① Law valid whatever $\vec{r}_0$ – changing $\vec{r}_0$ to $\vec{r}_0'$ just adds a constant $-W(\vec{r}_0', \vec{r}_0)$ – to V and hence to E.

Example: Constant Force (page ⑩)

$$W(\vec{r}_0, \vec{r}) = mg(z - z_0)$$

We usually choose $z_0 = 0$ and write

$$V = mgz$$

but this is pure convention

② Law valid also if only forces are cons. F's and constraint F's (Constraint F's do no work.) (Demonstrate)

③ If $\vec{F} = \vec{F}^{\text{cons}} + \vec{F}^{\text{non-cons}}$

we can define V from $\vec{F}^{\text{cons}}$ only + get

$$E(t_2) - E(t_1) = \int_{t_1}^{t_1} \vec{F}^{\text{non-cons}} \cdot \vec{v}\, dt$$

(Not very useful, but sometimes of some help).

Friction + Viscous Damping are non-conservative

$$\vec{F} = -\mu |\vec{N}| \frac{\vec{v}}{|\vec{v}|} \text{ or } -c\vec{v} \qquad \mu, c > 0$$

$$-\vec{F} \cdot \vec{v} \geq 0 \qquad [= 0 \text{ only if } \vec{v} = 0]$$

$\int \vec{F} \cdot \vec{v} \neq 0$

but $\vec{r}_1 = \vec{r}_2$

Catalog of Conservative Forces #2

Any one-dimensional force that only depends on x (no dependence on v, t)

$$m\frac{dv}{dt} = F(x)$$

$$\int_{t_1}^{t_2} F\cdot v\, dt = \int_{t_1}^{t_2} F(x)\frac{dx}{dt}\,dt$$

$$= \int_{x_1}^{x_2} F(x)dx$$

$$V(x) = \int_{x}^{x_0} F(x')\,dx'$$

$$= -\int_{x_0}^{x} F(x')\,dx'$$

$$\boxed{F(x) = -dV/dx}$$

Example: Harmonic Oscillator

$$F = -kx \qquad \text{Choose } x_0 = 0$$

$$V = \int_0^{x} kx'dx'$$

$$= \frac{1}{2}kx^2$$

Check

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

$$x = C\cos(\omega t+\theta) \qquad \omega^2 = k/m$$

$$\dot{x} = -\omega C\sin(\omega t+\theta)$$

$$E = \frac{1}{2}m\omega^2C^2\sin^2(\omega t+\theta) + \frac{1}{2}kC^2\cos^2(\omega t+\theta)$$

$$= \frac{1}{2}kC^2$$

Indeed, indep. of time.

In one dim, E-cons is almost equivalent to Eq. of Motion

$$\frac{d}{dt}\left[\frac{1}{2}m\dot{x}^2 + V(x)\right] = 0$$

$$m\dot{x}\ddot{x} + \frac{dV}{dx}\dot{x} = 0$$

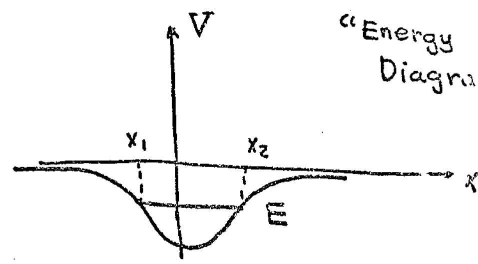
if $\dot{x} \neq 0$,

$$m\ddot{x} = -\frac{dV}{dx} = F(x)$$

[Not so in more dimensions (no way to divide $\vec{v}$ out of $\dot{\vec{v}}\cdot\vec{v}$)]

E-cons. can be used to reduce Eqs. of motion to quadratures (see last lecture). Almost as important, it is a quick way to get a qualitative idea of the motion.

"Energy Diagram"



$$E - V = \frac{1}{2}m\dot{x}^2 \geq 0$$

Thus if E is as shown, motion is restricted to $x_1 \leq x \leq x_2$ ("bounded motion") Let us assume $\dot{x} > 0$. Then $\dot{x} > 0$ until $x = x_2$. What happens then? $\dot{x} = 0$

$$m\ddot{x} = -\frac{dV}{dx} < 0$$

Particle turns around! $\dot{x} < 0$ until $x = x_1$, when it turns around again, etc. x_1, x_2 called "turning points". On the other hand, if $E > 0$, $\dot{x}$ never changes sign ("unbounded motion")

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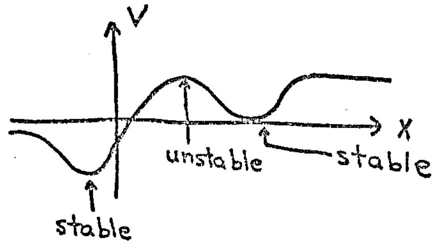
We can also use V to study points of equilibrium

$$F=0 \Leftrightarrow \frac{dV}{dx}=0$$



Near equilibrium point, x_0, we can approximate

$$V(x) = V(x_0) + \underbrace{\frac{dV}{dx}\Big|_{x_0}}_{0}(x-x_0) + \frac{1}{2}\frac{d^2V}{dx^2}\Big|_{x_0}(x-x_0)^2$$

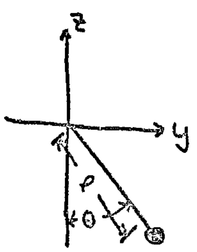
$$F(x) = -\frac{d^2V}{dx^2}\Big|_{x_0}(x-x_0)$$

$$k = +\frac{d^2V}{dx^2}\Big|_{x_0} \quad \text{(linear approx)}$$

$\omega^2 = k/m$ (frequency of small oscillations about stable equilibrium) (see last week)

18

All of this goes through if the motion is one-dimensional because of constraints. Example: Pendulum

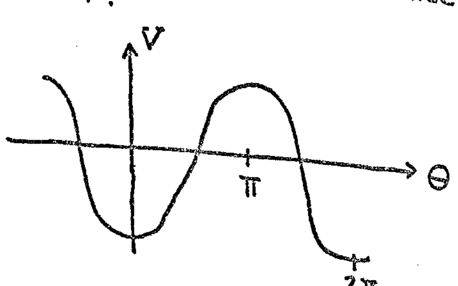


$$K = \tfrac{1}{2}m|\vec{v}|^2 = \tfrac{1}{2}m\ell^2\dot{\theta}^2$$

$$V = mgz = -mg\ell\cos\theta$$

$$E = \tfrac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell\cos\theta$$

Just like previous case, except "mass" is $m\ell^2$.



E always $> -mg\ell$

bounded if $E < mg\ell$

unbounded if $E > mg\ell$

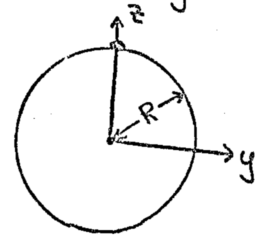
19

$\theta = 0$ stable equilibrium

$$\omega^2 = \frac{d^2V/d\theta^2|_0}{\underbrace{m\ell^2}_{\text{"mass"}}} = \frac{mg\ell}{m\ell^2} = \frac{g}{\ell}$$

$\theta = \pi$ unstable equilibrium

It also all goes through if the motion just happens to be one-dimensional because of choice of initial-value data. Example: Escape velocity



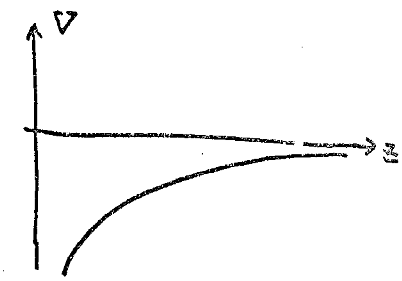
Particle fired from surface of isolated airless non-rotating planet, straight up. Motion stays on z-axis.

20

$$F = -\frac{\overset{\text{mass of planet}}{M}mG}{z^2} \qquad \text{choose } z_0 = \infty$$

$$V = \int_z^\infty -\frac{MmG}{z^2}\,dz = -\frac{MmG}{z}$$

Particle escapes to ∞ if $E \geq 0$. If $E<0$, it falls back.



What does this mean in terms of initial v?

$$E = \tfrac{1}{2}mv^2 - \frac{MmG}{R} \geq 0 \quad \text{(condition for escape)}$$

$$v^2 \geq \frac{2MG}{R}$$

If we define $g = \dfrac{MG}{R^2}$

$$v^2 \geq 2gR \quad \text{("escape velocity")}$$

21

Catalog of Conservative F's (con't.)

#3: Central Forces

$$\vec{F} = \frac{F(r)\,\vec{r}}{r} \qquad r \equiv |\vec{r}|$$

$$\int_{t_1}^{t_2} \vec{F}\cdot\vec{v}\,dt = \int_{t_1}^{t_2} F(r)\,\frac{\vec{r}}{r}\cdot\dot{\vec{r}}\,dt$$

$$\vec{r}\cdot\dot{\vec{r}} = \frac{1}{2}\frac{d}{dt}\,\vec{r}\cdot\vec{r} = \frac{1}{2}\frac{d}{dt}r^2 = r\dot{r}$$

$$\int_{t_1}^{t_2} \vec{F}\cdot\vec{v}\,dt = \int_{t_1}^{t_2} F(r)\,\frac{dr}{dt}\,dt$$

$$= \int_{r_1}^{r_2} F(r)\,dr$$

Not only independent of path, but V only depends on r!

HOMEWORK COMMENTARY

4-H1 a) Use separation of variables as in lecture.

b) $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$, then sep. of variables.

4-H2 $\vec{a} = -\omega^2 r\hat{r} + \dot{\omega} r\hat{\theta}$ (p 36); $\vec{F} = m\vec{a}$ N ~ radial, F_f is tang.

4-H3 (problem 2.27 is even more remarkable)

4-H4 Motion is known $\overset{(p.36)}{\Longrightarrow}$ $\vec{a}$; $\vec{F}_f = m\vec{a}$;

Starts to skid when $\mu N \leq |F_f| = m|\vec{a}|$

4-H5 a) Because $\vec{F}$ is purely radial, $a_\theta = 0$.

b) Radial component of $\vec{F} = m\vec{a}$ gives T

4-H6 a) The three conditions on tension (2 ends, pulley) determine the two arbitrary constants and the acceleration.

b) Equation should lead to hyperbolic functions

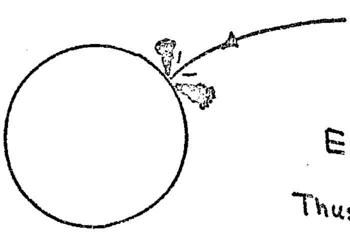
4-H7 $a_r = \ddot{r} - \omega^2 r$; $\ddot{r}$ is constant along the rope because it doesn't stretch, a_r is not constant along rope.

22

Thus, for the planet of the previous example

$$V = -\frac{MmG}{r}$$

Example: Particle falls in "from infinity" and hits the planet. What is its minimum velocity at moment of hit?



If particle was at infinity

$$E = \frac{1}{2}m|\vec{v}|^2 - \frac{GmM}{r} \geq 0$$

Thus, answer is escape velocity.

Homework Commentary

5-H1 See Ex. 4.7; Work-energy theorem

5-H2 Based on $P = \vec{F}\cdot\vec{v}$ (sec. 4.13)

5-H3 See Ex. 4.3, 4.5; in our notation $GM = V_0^2 R$

5-H4 PE + KE = const gives speed of beads at θ

F = ma for bead then gives reaction force on ring

Vertical forces on ring balance with T = 0 at critical condition

5-H5 Force is conservative if $\vec{\nabla}\times\vec{F} = 0$ (p. 219)

To construct $U(x,y,z)$ see Ex. 5.10 or integrate along specific path such as $(0,0,0)\to(x,0,0)\to(x,y,0)\to(x,y,z)$

5-H6 a) Check $\vec{\nabla}\times\vec{F}$

b) Construct $U(\vec{r})$ for conservative force, calculate work by non-cons. force on actual path

Work = Change of KE + PE

5-H7 See Sec. 4.9, 4.10

5-H8 Specific case of Ex. 4.15 note $\mu = \frac{m}{2}$.

21

Catalog of Conservative F's (con't)

#3: Central Forces

$$\vec{F} = \frac{F(r)\,\vec{r}}{r} \qquad r \equiv |\vec{r}|$$

$$\int_{t_1}^{t_2} \vec{F}\cdot\vec{v}\,dt = \int_{t_1}^{t_2} F(r)\,\frac{\vec{r}}{r}\cdot\dot{\vec{r}}\,dt$$

$$\vec{r}\cdot\dot{\vec{r}} = \frac{1}{2}\frac{d}{dt}\,\vec{r}\cdot\vec{r} = \frac{1}{2}\frac{d}{dt}r^2 = r\dot{r}$$

$$\int_{t_1}^{t_2} \vec{F}\cdot\vec{v}\,dt = \int_{t_1}^{t_2} F(r)\,\frac{dr}{dt}\,dt = \int_{r_1}^{r_2} F(r)\,dr$$

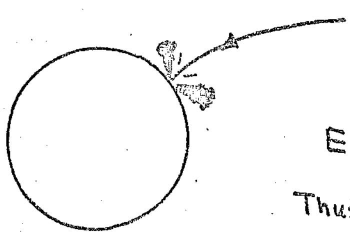
Not only independent of path, but V only depends on r!

22

Thus, for the planet of the previous example

$$V = -\frac{MmG}{r}$$

Example: Particle falls in "from infinity" and hits the planet. What is its minimum velocity at moment of hit?



If particle was at infinity

$$E = \frac{1}{2}m|\vec{v}|^2 - \frac{GmM}{r} \geq 0$$

Thus, answer is escape velocity.

1

For a force depending only on x, $F(x)$, in one dim

$$V(x) = \int_{x}^{x_0} F(x')\,dx' = -\int_{x_0}^{x} F(x')\,dx'$$

and we have the inverse formula,

$$F = -\frac{dV}{dx}$$

In 3d for a conservative force depending only on $\vec{r}$, $\vec{F}(\vec{r})$, we have

$$V(\vec{r}) = \int_{t_1}^{t_2} \vec{F}(\vec{r}(t))\cdot\vec{v}(t)\,dt$$

for any motion such that $\vec{r}(t_1) = \vec{r}$, $\vec{r}(t_2) = \vec{r}_0$ (arb. fixed pt.)

What is the inverse formula?

To answer, we need a new mathematical concept:

2

Partial Derivatives + The Gradient Operator

Given a function of x, $f(x)$, we define the derivative of f with respect to x by

$$\frac{df}{dx} \equiv \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Given a function of x, y, z, $f(x,y,z)$, we define the partial derivative of f with respect to x by

$$\frac{\partial f}{\partial x} \equiv \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z) - f(x,y,z)}{\Delta x}$$

Likewise,

$$\frac{\partial f}{\partial y} \equiv \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y, z) - f(x,y,z)}{\Delta y}$$

Etc. for z. (Math. Note: Obviously can be generalized to function of n variables, $f(x_1, x_2 \cdots x_n)$)

3

Higher Partial Derivatives

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \lim_{\Delta y\to 0} \frac{\frac{\partial f}{\partial x}(x,y+\Delta y,z)-\frac{\partial f}{\partial x}(x,y,z)}{\Delta y}$$

$$= \lim_{\Delta y\to 0}\lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x,y+\Delta y,z)-f(x,y+\Delta y,z)}{\Delta x\,\Delta y} - \frac{f(x+\Delta x,y,z)-f(x,y,z)}{\Delta x\,\Delta y}\right]$$

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \lim_{\Delta x\to 0}\lim_{\Delta y\to 0}\left[\text{same thing}\right]$$

For sufficiently smooth f's, the lims can be interchanged, + these two are equal. Thus we write

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} \equiv \frac{\partial^2 f}{\partial x\partial y} \equiv \frac{\partial^2 f}{\partial y\partial x}$$

Likewise for (x,z) + (y,z).

4

Linear Approx. + Partial Derivatives

For one variable

$$F(x+\Delta x) = F(x) + \left.\frac{dF}{dx}\right|_x \Delta x \qquad \left(\lim_{\Delta x\to 0}\right)$$

What about two variables?

$$F(x+\Delta x, y+\Delta y) = F(x,y+\Delta y) + \left.\frac{\partial F}{\partial x}\right|_{(x,y+\Delta y)} \Delta x$$

$$= F(x,y) + \left.\frac{\partial F}{\partial y}\right|_{(x,y)}\Delta y + \left.\frac{\partial F}{\partial x}\right|_{(x,y)}\Delta x + \underbrace{\left.\frac{\partial^2 F}{\partial y\partial x}\right|_{(x,y)}\Delta x\,\Delta y}_{\text{neglect (quadratic in small quantities)}}$$

Likewise, for three variables

$$F(x+\Delta x, y+\Delta y, z+\Delta z) = \frac{\partial F}{\partial x}\Delta x + \frac{\partial F}{\partial y}\Delta y + \frac{\partial F}{\partial z}\Delta z$$

(All deriv's evaluated at (x,y,z)).

5

A "Chain Rule" for Partial Derivatives

Given a motion of a particle in (x,y,z) space

$$x = x(t),\ y = y(t),\ z = z(t),$$

we define, from $f(x,y,z)$,

$$f(t) \equiv f(x(t), y(t), z(t))$$

↑ value of f at position of particle at time t

$$\frac{df}{dt} = \lim_{\Delta t\to 0}\frac{f(t+\Delta t)-f(t)}{\Delta t}$$

$$f(t+\Delta t) = f\left(x(t)+\frac{dx}{dt}\Delta t,\ y(t)+\frac{dy}{dt}\Delta t, \text{etc.}\right)$$

$$= f(t) + \Delta t\left[\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}+\frac{\partial f}{\partial z}\frac{dz}{dt}\right]$$

$$\therefore \boxed{\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}+\frac{\partial f}{\partial z}\frac{dz}{dt}}$$

6

We can put this in vector language

$$f(x,y,z) = f(\vec r)$$

$$\vec i\,\frac{dx}{dt} + \vec j\,\frac{dy}{dt} + \vec k\,\frac{dz}{dt} = \frac{d\vec r}{dt} = \vec v$$

$$\vec i\,\frac{\partial f}{\partial x} + \vec j\,\frac{\partial f}{\partial y} + \vec k\,\frac{\partial f}{\partial z} \equiv \vec\nabla f$$

↑ "gradient of f" or "grad f"

$$\boxed{\frac{df}{dt} = \frac{d\vec r}{dt}\cdot\vec\nabla f}$$

$$\vec\nabla = \vec i\,\frac{\partial}{\partial x} + \vec j\,\frac{\partial}{\partial y} + \vec k\,\frac{\partial}{\partial z} \qquad (\text{"gradient operator"})$$

is a "vector differential operator"

It turns scalar functions of $\vec r$ into vector functions of $\vec r$.

Some Trivial Properties

$$\vec\nabla(fg) = (\vec\nabla f)g + f(\vec\nabla g)$$

$$\vec\nabla f(g(\vec r)) = \frac{df}{dg}\vec\nabla g$$

7

Now we can answer the question on 1:

$$V(\vec{r}(t)) - V(\vec{r}(0)) = -\int_0^t dt' \, \vec{v}\cdot\vec{F}$$

(for arbitrary motions)

$$\frac{d}{dt} V(\vec{r}(t)) = -\vec{v}(t)\cdot\vec{F}(\vec{r}(t))$$

$$\frac{d}{dt} V(\vec{r}(t)) = \frac{d\vec{r}}{dt}\cdot\vec{\nabla} V = \vec{v}\cdot\vec{\nabla} V$$

(everything evaluated at t)

$$\vec{v}\cdot\vec{\nabla} V = -\vec{v}\cdot\vec{F}$$

but $\vec{v}$ is arbitrary

$$\therefore \quad \boxed{\vec{F} = -\vec{\nabla} V}$$

(3d analog of 1d $F = -dV/dx$)

8

Example 1: Constant Force

$$V = mgz$$

$$-\vec{\nabla} V = -\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(mgz)$$

$$= -\vec{k}\, mg \qquad \text{OK}$$

Example 2: Gravitational Field of Planet

$$V = -\frac{GMm}{r} \qquad r = \sqrt{x^2+y^2+z^2}$$

Preliminary Computation

$$\vec{\nabla} r = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\sqrt{x^2+y^2+z^2}$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}}\left(\vec{i}x + \vec{j}y + \vec{k}z\right) = \frac{\vec{r}}{r}$$

Now we can compute

$$-\vec{\nabla} V = GMm \frac{d}{dr}\left(\frac{1}{r}\right)\vec{\nabla} r$$

$$= -\frac{GMm}{r^2}\vec{r} \qquad \text{OK}$$

9

More about $\vec{\nabla} V$: (1) Geometrical Interpretation

Consider arbitrary curve in 3-space

$\vec{r} = \vec{r}(\ell)$ ← length along curve $\qquad \left|\frac{d\vec{r}}{d\ell}\right| = 1$

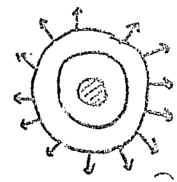
$$\frac{d}{d\ell} V(\ell) \equiv \frac{d}{d\ell} V(\vec{r}(\ell)) = \frac{d\vec{r}}{d\ell}\cdot\vec{\nabla} V$$

If $\vec{r}(\ell)$ lies in a surface of constant V

$$\frac{d\vec{r}}{d\ell}\cdot\vec{\nabla} V = 0$$

$\therefore$ $\vec{\nabla} V$ is $\perp$ to surface of constant V

Example: Gravity of Planet



Surface of constant V are spheres — $\vec{\nabla} V$ is $\parallel$ to $\vec{r}$

10

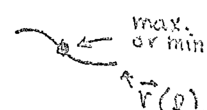
$$\frac{dV}{d\ell} = \frac{d\vec{r}}{d\ell}\cdot\vec{\nabla} V = \underbrace{\left|\frac{d\vec{r}}{d\ell}\right|}_{1}\left|\vec{\nabla} V\right| \cos\theta$$

← $\angle$ between $\frac{d\vec{r}}{d\ell}$ + $\vec{\nabla} V$

& maximum when $\cos\theta = 1$, i.e., $\theta = 0$

$\therefore$ $\vec{\nabla} V$ points in direction of fastest increase of V with ℓ. $|\vec{\nabla} V| = \frac{dV}{d\ell}$ in that direction.

(2) $\vec{\nabla} V$ and equilibria



At a maximum or minimum of V

$\frac{dV}{d\ell} = 0$ for any curve passing through the max or min. $\Rightarrow \vec{\nabla} V = 0$

However, not all pts where $\vec{\nabla} V = 0$ are max or min. Example:

$$V = x^2 - y^2 + z^2 \qquad \vec{\nabla} V = 0 \text{ at } x=y=z=0$$

but this is a min along curves $\parallel$ to x or z axis, max along curves $\parallel$ to y axis.

(11)

An equilibrium pt. is one where

$$\vec{F} = -\vec{\nabla} V = \vec{0}$$

But only minima are pts. of <u>stable</u> equilibrium.

Proof: Assume pt. is a min. Then for any curve passing through pt, V increases as we move away from the pt. Thus, in some neighborhood of the pt.

$$\frac{dV}{d\ell} \geq 0 \qquad \text{(Assume } \ell \text{ increases as we move away)}$$

but $\frac{dV}{d\ell} = \frac{d\vec{r}}{d\ell} \cdot \vec{\nabla} V = -\frac{d\vec{r}}{d\ell} \cdot \vec{F}$

$\vec{F}$ tends to push particle back to min. On the other hand, if pt. is not min, there is at least one curve for which sign is reversed — for displacements in this direction, force tends to push particle farther away.

(12)

Catalog of Conservative Forces: Final Entry

Assume $\vec{F}$ is conservative

$$\vec{F} = -\vec{\nabla} V$$

$$F_x = -\frac{\partial V}{\partial x}, \quad F_y = -\frac{\partial V}{\partial y}, \quad F_z = -\frac{\partial V}{\partial z}$$

then

$$\frac{\partial F_x}{\partial y} = -\frac{\partial^2 V}{\partial y \partial x} = -\frac{\partial^2 V}{\partial x \partial y} = \frac{\partial F_y}{\partial x}$$

Likewise for x+z, y+z. Thus, <u>necessary</u> conditions that $\vec{F}$ be cons. is

$$\boxed{\begin{aligned} \frac{\partial F_x}{\partial y} &= \frac{\partial F_y}{\partial x} \\ \frac{\partial F_x}{\partial z} &= \frac{\partial F_z}{\partial x} \\ \frac{\partial F_y}{\partial z} &= \frac{\partial F_z}{\partial y} \end{aligned}}$$

These are also <u>sufficient</u> conditions! (For proof see K+K, 5.6)

(13)

These equations can be written in vector form

$$\vec{\nabla} \times \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \left(\vec{i} F_x + \vec{j} F_y + \vec{k} F_z\right)$$

$$= \vec{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \vec{j}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \vec{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

$$\boxed{\vec{\nabla} \times \vec{F} = \vec{0}}$$

$\vec{\nabla} \times \vec{F}$ is called "curl of $\vec{F}$"

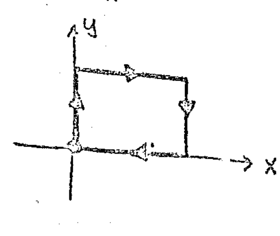
(14)

Example of "Curl Test"

$\vec{F} = y\vec{i}$, i.e. $F_x = y \quad F_y = F_z = 0$

$$\frac{\partial F_x}{\partial y} = 1 \neq \frac{\partial F_y}{\partial x} = 0 \qquad \text{fails test}$$

Check



Motion from origin to origin. If $\vec{F}$ is cons., work done should be same as for standing still at origin — i.e., zero.

However, $\vec{F} \cdot \vec{v} = 0$ except on the top ~~leg~~ side of the rectangle, where $\vec{F} \cdot \vec{v} > 0$ $\therefore \int dt\, \vec{F} \cdot \vec{v} \neq 0$. Checks.

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Energy Conservation for an isolated system of particles with conservative internal forces

Assumptions:

(1) $m\dot{\vec{V}}_a = \vec{F}_a \qquad a = 1 \cdots n$

(2) $\vec{F}_a = \sum_b \vec{F}_{ab} \qquad \vec{F}_{aa} = 0$

(3) $\vec{F}_{ab} = -\vec{\nabla}_a V_{ab}(\vec{r}_a - \vec{r}_b)$

Meaning: (a) $\vec{\nabla}_a = \vec{i}\frac{\partial}{\partial x_a} + \vec{j}\frac{\partial}{\partial y_a} + \vec{k}\frac{\partial}{\partial z_a}$

(b) $\vec{F}_{ab}$ function of $\vec{r}_a - \vec{r}_b$ only (only relative positions matter)

(c) $\vec{F}_{ab}$ would be conservative force acting on a in previous sense if b were nailed down.

True, for example, for gravity

$$V_{ab} = -\frac{G m_a m_b}{|\vec{r}_a - \vec{r}_b|}$$

Also assume

(4) $V_{ab} = V_{ba}$ (implies $\vec{F}_{ab} = -\vec{F}_{ba}$)

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Define

$$V = \frac{1}{2}\sum_a \sum_b V_{ab} \qquad (\text{"total potential energy"})$$

takes care of double counting ($V_{ab} = V_{ba}$)

then $\vec{F}_a = -\vec{\nabla}_a V$

Define

$$K = \sum_a \frac{1}{2} m_a |\vec{V}_a|^2 \qquad (\text{"total kinetic energy"})$$

$$\frac{dK}{dt} = \vec{F}_a \cdot \vec{V}_a \qquad \text{(previous lecture)}$$

$$= -\vec{\nabla}_a V \cdot \frac{d\vec{r}_a}{dt}$$

$$= -\frac{dV}{dt} \qquad \text{(chain rule)}$$

$$\frac{d}{dt}(K+V) = 0$$

Conservation of total energy

[Actually much more general than assumptions (1)-(4)]

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Consequence: Assume motion is such that as $t \to \pm\infty$

$$|\vec{r}_a - \vec{r}_b| \to \infty \qquad V_{ab} \to 0$$

then $\lim_{t\to+\infty} K(t) = \lim_{t\to-\infty} K(t)$

Asymptotic Conservation of Kinetic Energy!

Then why are there inelastic collisions? The assumption isn't always true — particles <u>may</u> stick together as $t \to \pm\infty$.

18

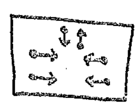
Contrast $\vec{P}$ and E:

A near-sighted observer can always determine $\vec{P}$ of an isolated system

$$\vec{P} = M\dot{\vec{R}}$$

Thus, even if he mistakes a composite system for a particle, he will always correctly compute $\vec{P}$. You can't hide $\vec{P}$.

But you can hide E!

 ← Box full of K but no way to tell from outside.

Thus a near-sighted observer can lose sight of "internal energy" of composite system.

HOMOGENEOUS LINEAR EQUATIONS, MATRICES, AND DETERMINANTS

(This hand-out is to fill you in on some mathematical background that we will need at the end of the second lecture of next week. You may already know this stuff from Math 21 or similar courses, but I still suggest you glance over this for Thursday's lecture just to make sure you understand my notation. If you are interested in learning more about this subject, look in any book on linear algebra).

Consider a set of n simultaneous linear homogeneous equations for n real variables, $x_1, x_2, \ldots x_n$:

$$\sum_j A_{ij} x_j = 0 , \qquad i, j = 1, 2 \cdots n ,$$

where the A's are real coefficients. Such a system of equations always possesses a trivial solution: all the x's are zero. For example, the following system of equations has only the trivial solution:

$$\begin{aligned} 2x_1 + 4x_2 &= 0 \\ x_1 \qquad &= 0 . \end{aligned}$$

In contrast, this system admits non-trivial solutions:

$$\begin{aligned} 2x_1 + 4x_2 &= 0 \\ x_1 + 2x_2 &= 0 . \end{aligned}$$

(Find them!) So does this system:

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 0 \\ x_1 + x_2 \qquad &= 0 \\ 2x_2 + x_3 &= 0 \end{aligned}$$

(Find them!)

I shall now describe a method for determining when a system of linear equations has a non-trivial solution. I will give no proofs. (Again, go to a text on linear algebra if you want a proof.)

The first step is to construct the <u>matrix</u> associated with the system of equations. This is simply the A's arranged in an n by n square array, with A_{ij} placed in the ith row and jth column. (Mnemonic: <u>R</u>oman <u>C</u>atholic.) For example, the matrices associated with the three systems above are

$$\begin{pmatrix} 2 & 4 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 3 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix},$$

respectively. As you can see, the matrices are simply the left-hand sides of the systems with all the x's erased. I will denote the matrix associated with a given system simply by A.

The next step is to construct a function of the matrix called the <u>determinant</u>, denoted by det A. I will give the definition of the determinant only for two by two and three by three matrices. These are the only cases we will need for the purposes of this course. (However, take my word that the definition, and all that follows, can be extended to general matrices.)

In the two by two case:

$$\det A = A_{11} A_{22} - A_{12} A_{21} .$$

In the three by three case:

$$\begin{aligned} \det A = A_{11} &(A_{22} A_{33} - A_{23} A_{32}) \\ - A_{21} &(A_{12} A_{33} - A_{13} A_{32}) \\ + A_{31} &(A_{12} A_{23} - A_{13} A_{22}) . \end{aligned}$$

I can now state the general rule: A system of simultaneous homogeneous linear equations has non-trivial solutions if and only if the determinant of the associated matrix vanishes.

It is easy to check the rule for the three examples above. For the first example,

$$\det A = 2\times0 - 4\times1 = -2,$$

non-zero, as it should be. For the second example,

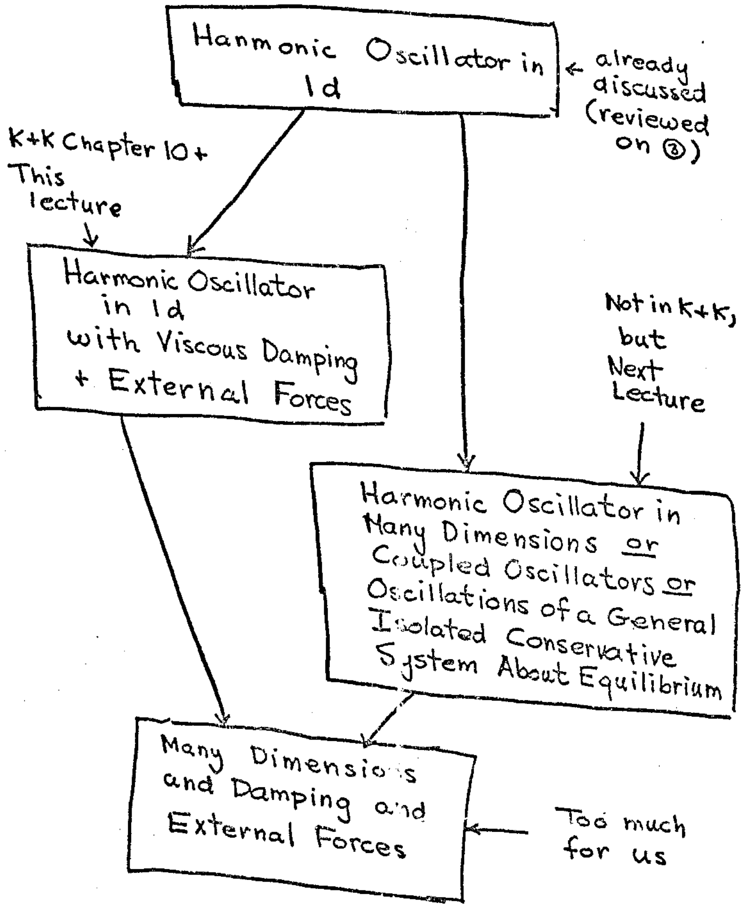
$$\det A = 2\times2 - 4\times1 = 0,$$

zero, as it should be. Check the third example yourself.

Also, you will find it instructive to do the following excercise: Consider the following system of equations:

$$\begin{aligned} x_1 + \alpha x_2 &= 0 \\ \alpha x_1 + x_2 &= 0 \end{aligned}$$

where α is an adjustable number. For what values of α does the determinant vanish? Check that in these cases the system has non-trivial solutions.



(Arrows indicate increasing generality)

②

Damped Oscillator

$$m\ddot{x} = -kx - b\dot{x}$$

$\uparrow$ viscous damping

This is linear approx. to general force $F(x,v)$ if $x=0$ is pt. of equilibrium

$$F(x,v) = \underbrace{F(0,0)}_{=0} + \underbrace{\frac{\partial F}{\partial x}\Big|_{x=v=0}}_{=-k} x + \underbrace{\frac{\partial F}{\partial v}\Big|_{x=v=0}}_{=-b} v$$

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0 \qquad \left[\gamma \equiv \frac{b}{m},\ \omega_0 \equiv \sqrt{k/m}\right]$$

If we measure time in sec, both γ and ω_0 are measured in $(\text{sec})^{-1}$. γ/ω_0 is "dimensionless number" – i.e., independent of our choice of units.

$\gamma/\omega_0 \gg 1$ "strongly damped"

$\gamma/\omega_0 \ll 1$ "weakly damped"

We have already solved the problem for two extreme cases

(1) $\gamma/\omega_0 = 0$ (undamped oscillator)

(2) $\omega_0/\gamma = 0$ (free particle with viscous damping)

(1) $\gamma = 0$

$$x = \text{Re}\, \alpha e^{i\omega_0 t}$$

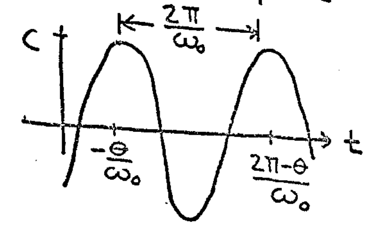
$\uparrow$ arb. complex no.

If we write $\alpha = Ce^{i\theta}$, $C \geq 0$

then (using $e^{i\phi} = \cos\phi + i\sin\phi$)

$$x = C\cos(\omega_0 t + \theta)$$

"amplitude" (C) "phase" (θ)



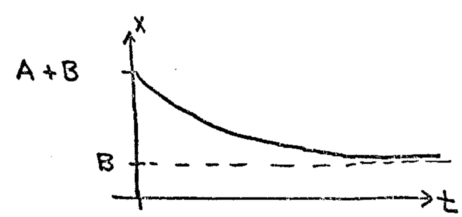
$$\frac{E}{m} = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\frac{k}{m}x^2 = C^2\omega_0^2/2 \quad \text{(from last Tuesday)}$$

Guess: The motion of a weakly damped oscillator should look, qualitatively, like this, at least if observed between times t_1 and t_2 such that $\gamma(t_1-t_2) \ll 1$

④

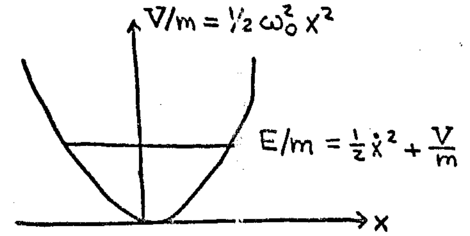
(2) $\omega_0^2 = 0$ $\quad \ddot{x} + \gamma\dot{x} = 0$

$$x = A + Be^{-\gamma t}$$



Guess: The motion of a strongly damped oscillator should look, qualitatively, like this, at least for time intervals such that $\omega_0(t_1-t_2) \ll 1$.

Energetics



$$\frac{d}{dt}(E/m) = F\dot{x}/m$$

non-conservative force only,

$$= -\gamma \dot{x}^2 \leq 0$$

E diminishes with t,

$\therefore$, $\lim_{t\to\infty} x(t) = 0.$

Damping force indeed damps oscillator.

For $\gamma/\omega_0 \ll 1$, energetics + guesswork yields new result:

We have guessed that over one period, we can approximate

$$x \approx C\cos(\omega_0 t + \theta).$$

Over many periods, damping will become noticeable — the "best fit" for C will change with time. How to estimate this?

$$\frac{d}{dt}\left(\frac{E}{m}\right) = -\gamma\dot{x}^2 \approx -\gamma C^2\omega_0^2\cos^2(\omega_0 t+\theta).$$

Since we are only interested in cumulative effects over long time, we might as well replace the r.h.s. by its time average

$$av(\cos^2\omega_0 t) = av(\sin^2\omega_0 t)$$
$$= \frac{1}{2}\,av(\cos^2\omega_0 t + \sin^2\omega_0 t) = \frac{1}{2}.$$

Thus

$$\frac{d}{dt}\left(\frac{E}{m}\right) \approx -\frac{\gamma}{2}C^2\omega_0^2 = -\gamma E/m \text{ (from 3)}$$

$$E(t) = E(0)\,e^{-\gamma t}$$

and $C(t) = C(0)\,e^{-\gamma t/2}$

We will shortly check this sloppy reasoning against the exact result.

Prologue to Exact Solution — Review of Complex Algebra

(1) $z = x + iy$ — z: complex no.; x, y: real no.'s

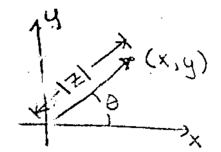
$x \equiv \mathrm{Re}\, z$

$y \equiv \mathrm{Im}\, z$

(2) $z = |z|\,e^{i\theta}$ ← phase of z (defined only modulo 2π)

$|z|$: "magnitude of z" ≥ 0

$$z = |z|\cos\theta + i|z|\sin\theta$$



(3) $z^* \equiv x - iy = |z|\,e^{-i\theta}$ ("Complex conjugate of z")

$(z_1 z_2)^* = z_1^* z_2^*$, $|z|^2 = z z^*$

(4) $\frac{1}{z} = \frac{1}{|z|}e^{-i\theta} = \frac{|z|}{|z|^2}e^{-i\theta} = \frac{z^*}{|z|^2}$

Trick for solving $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$:

Look for solutions of the form $x = e^{\beta t}$

If we can find two independent solutions by this trick, $x_1 = e^{\beta_1 t}$, $x_2 = e^{\beta_2 t}$, then we can match arbitrary initial value data with

$$x(t) = \alpha_1 e^{\beta_1 t} + \alpha_2 e^{\beta_2 t}$$

for appropriate α_1, α_2. We then have the most general solution.

(Appropriately generalized, this trick works for

$$\sum_n c_n \frac{d^n x}{dt^n} = 0 \quad , \; c_n\text{'s constants.})$$

$$\frac{d}{dt}e^{\beta t} = \beta e^{\beta t}$$

$$\therefore, \; (\beta^2 + \gamma\beta + \omega_0^2)\,e^{\beta t} = 0$$

Thus trick works if we can find two roots of

$$\boxed{\beta^2 + \gamma\beta + \omega_0^2 = 0}$$

(9)

Solution of quadratic equation:

$$\beta = \tfrac{1}{2}\left[-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}\right]$$

Three possibilities:

(a) $\gamma^2 > 4\omega_0^2$ Two real roots; program on (8) goes through.

(b) $\gamma^2 < 4\omega_0^2$ Complex (conjugate) roots — what to do?

(c) $\gamma^2 = 4\omega_0^2$ Only one root — what to do?

(a) called "overdamped" oscillator (includes strongly damped case)

(b) called "underdamped" oscillator (includes weakly damped case)

(c) called "critically damped" oscillator (borderline case)

(10)

(a) Overdamped Oscillator $\gamma^2 > 4\omega_0^2$

$$\beta_\pm = \tfrac{1}{2}\left[-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}\right]$$

$$\beta_- < \beta_+ < 0$$

General solution is sum of two ~~damp~~ decreasing (must be so because of energetics) exponentials:

$$x(t) = \alpha_+ e^{\beta_+ t} + \alpha_- e^{+\beta_- t}$$

Strongly damped limit: $\gamma \gg \omega_0$

$$\sqrt{\gamma^2 - 4\omega_0^2} = \gamma\sqrt{1 - (4\omega_0^2/\gamma^2)} \approx \gamma\left(1 - \frac{2\omega_0^2}{\gamma^2}\right) = \gamma - \frac{2\omega_0^2}{\gamma}$$

$$\beta_+ = -\frac{\omega_0^2}{\gamma} \qquad \beta_- = -\gamma + \frac{\omega_0^2}{\gamma}$$

$$x(t) = \underset{\substack{\uparrow \\ \text{slow decrease}}}{\alpha_+ e^{-\omega_0^2 t/\gamma}} + \underset{\substack{\uparrow \\ \text{fast decrease}}}{\alpha_- e^{-\gamma t}} e^{\omega_0^2 t/\gamma}$$

Guess on (4) is verified $\omega_0 t \ll 1$ + $\omega_0/\gamma \ll 1$ implies $\omega_0^2 t/\gamma \ll 1$, i.e. looks much like free particle with viscous damping.

(11)

(b) Underdamped Case: $\gamma^2 < 4\omega_0^2$

$$\beta_\pm = \tfrac{1}{2}\left[-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}\right] = \tfrac{1}{2}\left[-\gamma \pm i\sqrt{4\omega_0^2 - \gamma^2}\right]. \quad (\beta_+ = \beta_-^*)$$

If $x(t)$ were a complex number, the general solution would be

$$x(t) = \alpha_+ e^{\beta_+ t} + \alpha_- e^{+\beta_- t}$$

with $\alpha_\pm$ arbitrary complex nos. But $x(t)$ is real! So what? Real is a special case of complex. Just choose $\alpha_\pm$ so x is real!

$$\alpha_+ = \underset{\substack{\uparrow \\ \text{for later convenience}}}{\tfrac{1}{2}} \overset{\text{arb. complex no.}}{\alpha} = \tfrac{1}{2}Ce^{i\theta} \qquad \alpha_- = \tfrac{1}{2}\alpha^* = \tfrac{1}{2}Ce^{-i\theta}$$

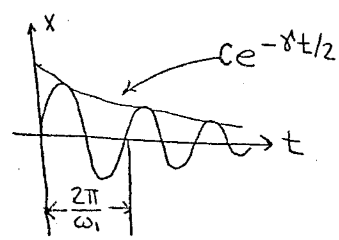
$$x(t) = \tfrac{1}{2}Ce^{-\gamma t/2}\left[e^{i\omega_1 t}e^{i\theta} + e^{-i\omega_1 t}e^{-i\theta}\right]$$

where $\omega_1 \equiv \sqrt{\omega_0^2 - \tfrac{1}{4}\gamma^2}$

$$x(t) = Ce^{-\gamma t/2}\cos(\omega_1 t + \theta)$$

Damped Harmonic Motion!

(12)



(Demonstrate)

Verifies slopply reasoning on (6), in limit $\gamma/\omega_0 \ll 1$, $\omega_1 = \omega_0\sqrt{1 - \frac{\gamma^2}{4\omega_0^2}} \approx \omega_0$

What does E do?

$$x\left(t + \frac{2\pi}{\omega_1}\right) = e^{-\pi\gamma/\omega_1}x(t)$$

$$\dot{x}\left(t + \frac{2\pi}{\omega_1}\right) = e^{-\pi\gamma/\omega_1}\dot{x}(t)$$

$$E\left(t + \frac{2\pi}{\omega_1}\right) = e^{-2\pi/Q}E(t)$$

Where $Q \equiv \frac{\omega_1}{\gamma}$ ("Quality factor")

This is what we would get by integrating

$$\frac{1}{\omega_1}\frac{\dot{E}}{E} = -\frac{\gamma}{\omega_1} = -\frac{1}{Q}$$

This Eq. is false, but gives right answer "on the average". Thus $1/Q$ is "average rate of fractional energy loss per radian".

(c) Critically Damped Oscillator $\quad \gamma^2 = 4\omega_0^2$ 13

$$\beta = -\frac{\gamma}{2} \pm \sqrt{0}$$

Only one β! What is to be done?

Consider this as limit of overdamped case $\quad \beta_\pm = -\frac{\gamma}{2} \pm \epsilon \quad , \; \epsilon = \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\,,$ as $\epsilon \to 0$.

If we say the two independent soln's. are

$$x_\pm(t) = e^{\beta_\pm t}$$

these coalesce as $\epsilon \to 0$.

However, we could just as well say two independent sol'ns are

$$e^{\beta_+ t} \quad \text{and} \quad (e^{\beta_+ t} - e^{\beta_- t})/2\epsilon$$

these go to $\quad e^{-(\gamma/2)t} \quad$ and

$$\lim_{\epsilon \to 0} \; e^{-\gamma t/2} \; \frac{1}{2\epsilon}\left[e^{\epsilon t} - e^{-\epsilon t}\right] = t\, e^{-\gamma t/2}$$

It is easy to check explicitly that general sol'n. is indeed

$$e^{-\gamma t/2}\left[A + Bt\right].$$

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Forced Damped Harmonic Oscillator

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{F(t)}{m}$$

given function of t

Two general remarks about this inhomogeneous differential equation:

(1) Let $x_0(t)$ be some particular sol'n. of the inhom. eq. Then

$$\frac{d^2}{dt^2}\big(x(t) - x_0(t)\big) + \gamma \frac{d}{dt}\big(x(t) - x_0(t)\big) + \omega_0^2 \big(x(t) - x_0(t)\big) = 0$$

I.e. general sol'n. of inhom. eq. = (general sol'n of hom. eq.) + (particular sol'n of inhom. eq)

Since we already know the 1st term, we need only find, by hook or crook, some <u>one</u> particular sol'n. of the inhom. eq.

(2) Let $x_{10}(t)$ be particular solution for $F(t) = F_1(t)$
" $x_{20}(t)$ " " " for $F(t) = F_2(t)$

then $\alpha_1 x_{10}(t) + \alpha_2 x_{20}(t)$ is solution for

$$F(t) = \alpha_1 F_1(t) + \alpha_2 F_2(t)$$

where α_1 and α_2 are arbitrary constants

15

From now on we will concentrate on

$$F(t) = F_0 \cos\omega t$$

with F_0, ω arbitrary >0. We will solve this by writing

$$F_0 \cos\omega t = \tfrac{1}{2} F_0 \left(e^{i\omega t} + e^{-i\omega t}\right)$$

and solving Eq. for each of the two terms on the right.

[<u>Digression on Fourier Series</u>

This is more general than might appear. Not only the cos, but <u>any</u> continuous function of period T

$$f(t+T) = f(t)$$

can be written

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{i2\pi n t/T} \qquad (n \text{ integer})$$

"Fourier Series"

where

$$a_n = \frac{1}{T}\int_0^T e^{-i2\pi n t/T} f(t)\,dt\,.$$

If f is real, $a_n = a_{-n}^*$.]

16

Back to the main problem:

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = e^{\pm i\omega t}$$

$$\frac{d}{dt} e^{\pm i\omega t} = \pm i\omega e^{\pm i\omega t}\,, \quad \frac{d^2}{dt^2} e^{\pm i\omega t} = -\omega^2 e^{\pm i\omega t}$$

Thus, it is reasonable to guess a particular sol'n.

$$x_0(t) = A_\pm e^{\pm i\omega t}$$

$$\left[-\omega^2 \pm i\omega\gamma + \omega_0^2\right] A_\pm e^{\pm i\omega t} = e^{\pm i\omega t}$$

The guess works if

$$A_\pm = \frac{1}{-\omega^2 + \omega_0^2 \pm i\gamma\omega}$$

For the case of interest

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

a particular solution is

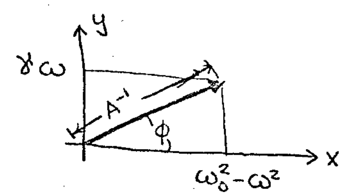
$$x_0(t) = \frac{F_0}{2m}\left[A_+ e^{i\omega t} + A_- e^{-i\omega t}\right]$$

(17)

To gain more insight into this particular solution, write A_+ as

$$A_+ = A\, e^{-i\phi} \qquad A \equiv |A_+| > 0$$

$$\frac{1}{A_+} = \frac{1}{A} e^{+i\phi} = -\omega^2+\omega_0^2 + i\gamma\omega \equiv x+iy$$



Note:

(1) $$A^2 = \frac{1}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$$

(2) If $\omega^2 < \omega_0^2$ $\qquad \frac{\pi}{2} > \phi > 0$

If $\omega^2 = \omega_0^2$ $\qquad \phi = \pi/2$

If $\omega^2 > \omega_0^2$ $\qquad \pi > \phi > \pi/2$

(3) $A_- = A\, e^{i\phi}$

(1) + (2) will be important to us shortly

(3) leads to

(18)

$$x_0(t) = \frac{F_0}{2m}\left[A_+ e^{i\omega t} + A_- e^{-i\omega t}\right]$$

$$= \frac{F_0 A}{m} \cos(\omega t - \phi)$$

"phase lag"

This is particular solution – what about general solution? It is the same, if we wait for a while, because general sol'n. of homogeneous equation is damped (i.e $\to 0$ as $t\to\infty$) if $\gamma \neq 0$. Effects of initial conditions are transient.

Closer look at response

As $\omega^2 \to \pm\infty$ $\qquad A^2 \approx \frac{1}{\omega^4}$

Where is maximum?

$$\frac{d}{d\omega^2}\left[(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2\right] = 0$$

$$2(\omega^2-\omega_0^2) + \gamma^2 = 0$$

$$\omega^2 = \omega_0^2 - \frac{\gamma^2}{2}$$

(19)

Thus A is monotone decreasing for $\omega_0^2 \leqslant \frac{\gamma^2}{2}$. For $\omega_0^2 \geqslant \frac{\gamma^2}{2}$ there is a maximum at $\omega = \sqrt{\omega_0^2 - \frac{1}{2}\gamma^2}$. This is the resonance peak.

The resonance peak is most striking for weak damping $\gamma/\omega_0 \ll 1$.

It occurs at $\sqrt{\omega_0^2-\frac{1}{2}\gamma^2} \approx \omega_0$

At this max,

$$A^2 = \frac{1}{\frac{\gamma^4}{4} + \gamma^2(\omega_0^2 - \frac{\gamma^2}{2})} \approx \frac{1}{\omega_0^2\gamma^2}$$

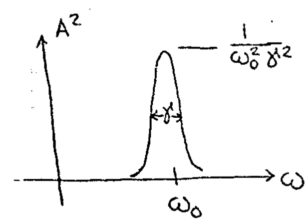
At $\omega = \omega_0 \pm \gamma/2$

$$A^2 = \frac{1}{(\omega_0^2 - [\omega_0 \pm \frac{\gamma}{2}]^2)^2 + \gamma^2[\omega_0 \pm \frac{\gamma}{2}]^2}$$

$$\approx \frac{1}{\gamma^2\omega_0^2 + \gamma^2\omega_0^2}$$

i.e., half the maximum value

(20)



γ is full width at half maximum

As $\gamma \to 0$ peak gets higher and narrower.

Also, phase lag goes rapidly from 0 at right of peak to π at left of peak, passing through $\pi/2$ at center of peak.* (Demonstrate).

Read K+K, Example 10.5, to see how this theory can be applied to "practical" situations.

*See Note (2) on (17).

Small Oscillations of a System of N Particles, Interacting Through Conservative Forces, About A Pt. of Equilibrium

$$m_a \frac{d^2\vec{r}_a}{dt^2} = -\vec{\nabla}_a V(\vec{r}_1, \vec{r}_2 \cdots \vec{r}_N)$$

Assume that at $\vec{r}_a = \vec{r}_{a0}$ $\quad \vec{\nabla}_a V = \vec{0}$ (equilibrium). Also assume this is minimum of V (stable equilibrium)

What can we say about small oscillations?

$$\vec{r}_a - \vec{r}_{a0} \qquad a = 1 \cdots N$$

forms a set of n numbers where

$n = \left\{ \begin{matrix} N \\ 2N \\ 3N \end{matrix} \right\}$ if motion is in $\left\{ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \right\}$ dimensions.

it will be convenient to relabel these as

$$x_i \qquad i = 1 \cdots n$$

and write the equations of motion as

$$m_i \ddot{x}_i = -\frac{\partial}{\partial x_i} V(x_1 \cdots x_n)$$

$x_i = 0$ (all i) is a minimum of V.

We now expand V up to quadratic order only

$$\underset{\substack{\uparrow \\ \text{constant}}}{V = V_0} + \frac{1}{2}\sum_{i,j} V_{ij} x_i x_j$$

(ignoring terms of cubic order + higher)

$$V_{ij} \equiv \frac{\partial^2 V}{\partial x_i \partial x_j} = V_{ji}$$

(derivatives evaluated at all x's = 0)

Important property (Minimum condition)

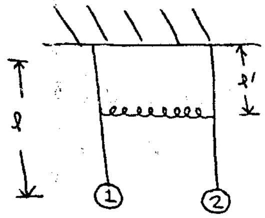
$\sum_{ij} V_{ij} x_i x_j \geq 0$ for any choice of x's

$= 0$ only if all x's vanish

$$\boxed{m_i \ddot{x}_i = -\sum_j V_{ij} x_j} \leftarrow \text{"coupled oscillators"}$$

We will study these eq's for general V_{ij} obeying $V_{ij} = V_{ji}$ and the minimum condition.

Example: Two Pendulums Connected by Spring

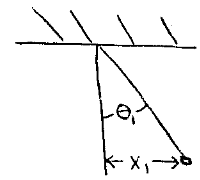


Assume equilibrium is as shown

$m_1 = m_2 = m$

k is spring constant

Let $x_{1,2}$ be horizontal displacements of bobs from equilibrium



$x_1 = l \sin\theta_1 \approx l\,\theta_1$

(error of order θ_1^3 or x_1^3)

Likewise for x_2

$$V = -mgl(\cos\theta_1 + \cos\theta_2) \leftarrow \text{gravity}$$
$$+\frac{1}{2}k\left(\frac{l'}{l}\right)^2 (x_1 - x_2)^2 \leftarrow \text{spring}$$

ignoring terms of order x^3 or higher throughout

$$\cos\theta_1 = 1 - \frac{\theta_1^2}{2} = 1 - \frac{x_1^2}{2l^2}$$

$$V = -2mgl + \frac{mg x_1^2}{2l} + \frac{mg x_2^2}{2l} + \frac{1}{2}k\left(\frac{l'}{l}\right)^2 (x_1^2 - 2x_1x_2 + x_2^2)$$

$$V_{11} = \frac{mg}{l} + k\left(\frac{l'}{l}\right)^2 = V_{22}$$

$$V_{12} = -k\left(\frac{l'}{l}\right)^2 = V_{21}$$

$$m_i \ddot{x}_i = -\sum_j V_{ij} x_j$$

Define $y_i \equiv \sqrt{m_i}\, x_i \qquad x_i = y_i/\sqrt{m_i}$

$$\sqrt{m_i}\,\ddot{y}_i = -\sum_j V_{ij}\, y_j/\sqrt{m_j}$$

$$\boxed{\ddot{y}_i = -\sum_j W_{ij}\, y_j} \leftarrow \text{simplified form of coupled oscillators}$$

where $W_{ij} = \frac{1}{\sqrt{m_i}} V_{ij} \frac{1}{\sqrt{m_j}}$

Note: $W_{ij} = W_{ji}$

$\sum_{ij} W_{ij} y_i y_j \geq 0$ for arbitrary y's

$= 0$ only if all y's vanish

For the example

$$W_{11} = \frac{g}{l} + \frac{k}{m}\left(\frac{l'}{l}\right)^2 = W_{22}$$

$$W_{12} = -\frac{k}{m}\left(\frac{l'}{l}\right)^2 = W_{21}$$

(Trivial check: Gives right result if $k=0$: $\omega = \sqrt{g/l}$)

To go further we need some new mathematics.

Vector Algebra in n Dimensions

All the 3d concepts — vectors, dot product, basis vectors, components, etc. — trivially extend to n dimensions. Only two things are non-trivial — right-handed bases + cross product — and we won't need these in what follows.

We will label basis vectors

$$\vec{e}^{(1)}, \vec{e}^{(2)} \ldots \vec{e}^{(n)}$$

[In 3d $\vec{e}^{(1)}=\vec{i}$, $\vec{e}^{(2)}=\vec{j}$, $\vec{e}^{(3)}=\vec{k}$]

$$\vec{e}^{(i)} \cdot \vec{e}^{(j)} = 0 \quad (i \neq j), \quad = 1 \quad (i=j)$$

Any vector $\vec{a}$ can be written

$$\vec{a} = \sum_i a_i \vec{e}^{(i)}$$

where $a_i = \vec{e}^{(i)} \cdot \vec{a}$

[In 3d, $a_1 = a_x$, $a_2 = a_y$, $a_3 = a_z$]

We will call the a_i's the components of $\vec{a}$ — they depend upon our coordinates (i.e., upon our choice of basis vectors)

Linear Operators

A linear operator is a linear function from vectors to vectors. Insted of writing this as $f(\vec{a})$ or $g(\vec{a})$, we write it as $\underset{\sim}{A}\vec{a}$ or $\underset{\sim}{B}\vec{a}$

$$\underset{\sim}{A}(\alpha\vec{a}+\beta\vec{b}) = \alpha\underset{\sim}{A}\vec{a} + \beta\underset{\sim}{A}\vec{b}$$

(α and β arb. numbers)

We know how a linear op. acts on arb. vectors if we know how it acts on the basis vectors, $\vec{e}^{(1)}, \vec{e}^{(2)} \ldots \vec{e}^{(n)}$

$$\underset{\sim}{A}\vec{e}^{(j)} = \sum_i A_{ij}\vec{e}^{(i)} \qquad \text{(def of } A_{ij}\text{)}$$

$$\vec{e}^{(i)} \cdot \underset{\sim}{A}\vec{e}^{(j)} = A_{ij}$$

Given a basis, the numbers a_i determine $\vec{a}$.
" " " , " A_{ij} " $\underset{\sim}{A}$.

Let $\vec{b} = \underset{\sim}{A}\vec{a}$. How are the components of $\vec{b}$ connected to those of $\vec{a}$?

$$b_i = \vec{e}^{(i)} \cdot \vec{b} = \vec{e}^{(i)} \cdot \underset{\sim}{A}\vec{a}$$

$$= \vec{e}^{(i)} \cdot \underset{\sim}{A} \sum_j a_j \vec{e}^{(j)}$$

$$= \sum_j \vec{e}^{(i)} \cdot \underset{\sim}{A}\vec{e}^{(j)} \, a_j$$

$$\boxed{b_i = \sum_j A_{ij} a_j}$$

Note that our oscillator eq.

$$\ddot{y}_i = -\sum_j W_{ij} y_j$$

can be thought of as an Eq. for vectors involving linear operators

$$\boxed{\ddot{\vec{y}} = -\underset{\sim}{W}\vec{y}}$$

with basis vectors aligned along the coordinate axes.

There is a "graphical" way of writing $b_i = \sum_j A_{ij} a_j$

We arrange the numbers a_j in a column, e.g.

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \qquad \text{for } n=2$$

We arrange A_{ij} in a square array ("a matrix") with i labelling the row + j the column, e.g., for n=2

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

then we write

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \leftarrow \text{def of this expression}$$

$$= \begin{pmatrix} A_{11}a_1 + A_{12}a_2 \\ A_{21}a_1 + A_{22}a_2 \end{pmatrix}$$

We "multiply rows times columns"

Mnemonic: Roman Catholic

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Examples of operators + their associated matrices:

(1) The identity operator $\underset{\sim}{I}$:

$$\underset{\sim}{I}\,\vec{a} = \vec{a}$$

$$I_{ij} = \vec{e}^{(i)} \cdot I\,\vec{e}^{(j)} = \vec{e}^{(i)} \cdot \vec{e}^{(j)} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Thus, in 2d, the matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(2) $\underset{\sim}{W}$ for the pendulum problem:

$$\begin{pmatrix} \frac{g}{\ell} + \frac{k}{m}\left(\frac{\ell'}{\ell}\right)^2 & -\frac{k}{m}\left(\frac{\ell'}{\ell}\right)^2 \\ -\frac{k}{m}\left(\frac{\ell'}{\ell}\right)^2 & \frac{g}{\ell} + \frac{k}{m}\left(\frac{\ell'}{\ell}\right)^2 \end{pmatrix}$$

[from 4]

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A linear op. is "symmetric" if

$\vec{a} \cdot \underset{\sim}{A}\,\vec{b} = \vec{b} \cdot \underset{\sim}{A}\,\vec{a}$ for any $\vec{a}$ and $\vec{b}$

$\Leftrightarrow$ $A_{ij} = A_{ji}$ for all i,j

(Both preceding examples are symmetric)

** Great Diagonalization Theorem **

For any symmetric operator, $\underset{\sim}{A}$, we can find a set of n vectors, $\vec{e}^{(1)\prime}$, $\vec{e}^{(2)\prime} \dots \vec{e}^{(n)\prime}$, such that

(1) $$\vec{e}^{(i)\prime} \cdot \vec{e}^{(j)\prime} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

(2) $$\underset{\sim}{A}\,\vec{e}^{(i)\prime} = \lambda^{(i)}\,\vec{e}^{(i)\prime}$$

for some set of numbers $\lambda^{(1)} \dots \lambda^{(n)}$.

[The $\vec{e}\,'$'s are called eigenvectors of $\underset{\sim}{A}$, the λ's eigenvalues]

Proof on 17, 18, + 19

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[This is called diagonalization because if we use the $\vec{e}\,'$'s as our basis vectors

$$A_{ij} = \vec{e}^{\prime(i)\prime} \cdot \underset{\sim}{A}\,\vec{e}^{(j)\prime} = \begin{matrix} \lambda^{(i)} & i = j \\ 0 & i \neq j \end{matrix}.$$

I.e., the matrix looks like this (for $n=2$):

$$\begin{pmatrix} \lambda^{(1)} & 0 \\ 0 & \lambda^{(2)} \end{pmatrix} \Leftarrow \text{non-zero only on main diagonal}]$$

We apply this theorem to $\underset{\sim}{W}$

$$\underset{\sim}{W}\,\vec{e}^{(i)\prime} = \lambda^{(i)}\,\vec{e}^{(i)\prime}$$

$\sum_{i,j} y_i W_{ij} y_j > 0$ unless all y_i's $= 0$

$\Leftrightarrow$ $\vec{y} \cdot \underset{\sim}{W}\,\vec{y} > 0$ unless $\vec{y} = 0$

Take $\vec{y} = \vec{e}^{(i)\prime}$ $(\neq \vec{0})$

$$\lambda^{(i)} > 0.$$

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Define

$$y_i' \equiv \vec{e}^{(i)\prime} \cdot \vec{y} \qquad \vec{y} = \sum_i y_i'\,\vec{e}^{(i)\prime}$$

$$\ddot{\vec{y}} = -\underset{\sim}{W}\,\vec{y}$$

$$\sum_i \ddot{y}_i'\,\vec{e}^{(i)\prime} = -\sum_i \lambda^{(i)}\,y_i'\,\vec{e}^{(i)\prime}$$

$$\ddot{y}_i' = -\lambda^{(i)}\,y_i'$$

n <u>independent</u> harmonic oscillators

[The big payoff!]

$$y_i' = C_i \cos(\omega_i t + \theta_i) \qquad [\omega_i = \sqrt{\lambda^{(i)}}]$$

$$\boxed{\vec{y} = \sum_i C_i \cos(\omega_i t + \theta_i)\,\vec{e}^{(i)\prime}}$$

Motions where only one C_i is nonzero are called "normal modes". The ω_i's are called "normal frequencies".

Example: The pendulums

One obvious normal mode is $x_1 = x_2$ ($\Leftrightarrow y_1 = y_2$). $\vec{e}^{(1)\prime}$ is represented by

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$\uparrow$ for $\vec{e}^{(1)\prime}\cdot\vec{e}^{(1)\prime} = 1$

Check:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}\frac{g}{\ell}+\frac{k}{m}\left(\frac{\ell'}{\ell}\right)^2 & -\frac{k}{m}\left(\frac{\ell'}{\ell}\right)^2\\ -\frac{k}{m}\left(\frac{\ell'}{\ell}\right)^2 & \frac{g}{\ell}+\left(\frac{k}{m}\right)\left(\frac{\ell'}{\ell}\right)^2\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}g/\ell\\ g/\ell\end{pmatrix} = \frac{g}{\ell}\,\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$\omega_1 = \sqrt{g/\ell}$$

(Demonstrate)

What about $\vec{e}^{(2)\prime}$? Let $\vec{e}^{(2)\prime}$ be represented by $\begin{pmatrix}a\\b\end{pmatrix}$. $\vec{e}^{(2)\prime}\cdot\vec{e}^{(1)\prime} = 0$

$$\Rightarrow \quad a + b = 0 \qquad a^2 + b^2 = 1 \quad \therefore$$

$$\begin{pmatrix}a\\b\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

(minus this will do as well)

Thus, the other normal mode is $y_1 = -y_2$ ($\Leftrightarrow x_1 = -x_2$).

Check:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}\frac{g}{\ell}+\frac{k}{m}\left(\frac{\ell'}{\ell}\right)^2 & -\frac{k}{m}\left(\frac{\ell'}{\ell}\right)^2\\ -\frac{k}{m}\left(\frac{\ell'}{\ell}\right)^2 & \frac{g}{\ell}+\frac{k}{m}\left(\frac{\ell'}{\ell}\right)^2\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}\frac{g}{\ell}+2\frac{k}{m}\left(\frac{\ell'}{\ell}\right)^2\\ -\left[\frac{g}{\ell}+2\frac{k}{m}\left(\frac{\ell'}{\ell}\right)^2\right]\end{pmatrix}$$

$$\omega_2 = \sqrt{\frac{g}{\ell}+2\frac{k}{m}\left(\frac{\ell'}{\ell}\right)^2} = \sqrt{\omega_1^2+2\frac{k}{m}\left(\frac{\ell'}{\ell}\right)^2}$$

for weak springs, $\frac{k}{m}\left(\frac{\ell'}{\ell}\right)^2 \ll \omega_1^2$, $\omega_1 \approx \omega_2$

(Demonstrate)

$$\vec{y} = \sum_i C_i \cos(\omega_i t + \theta_i)\,\vec{e}^{(i)\prime}$$

If more than one C is non-zero motion can look complicated.

Example: $\theta_1 = \theta_2 = 0 \qquad C_1 = C_2 = \sqrt{2}\,C$

$$\begin{pmatrix}y_1\\y_2\end{pmatrix} = C\begin{pmatrix}\cos\omega_1 t + \cos\omega_2 t\\ \cos\omega_1 t - \cos\omega_2 t\end{pmatrix}$$

Define $\omega \equiv \frac{1}{2}(\omega_1+\omega_2) \qquad \Delta \equiv \frac{1}{2}(\omega_2-\omega_1)$

$\omega_1 = \omega - \Delta \qquad \omega_2 = \omega + \Delta$

Because $\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$

$$\begin{pmatrix}y_1\\y_2\end{pmatrix} = 2C\begin{pmatrix}\cos\omega t\,\cos\Delta t\\ \sin\omega t\,\sin\Delta t\end{pmatrix}$$

for weak springs, $\Delta \ll \omega$. Thus, for a long time, 1 swings and 2 stays still. However, after $t = \frac{\pi}{2\Delta}$, the situation is reversed.

(Demonstrate)

If you know determinants, you can automate the search for eigenvectors + eigenvalues.

$$\underset{\sim}{A}\,\vec{e} = \lambda\,\vec{e}$$

λ some eigenvalue
$\vec{e}$ associated eigenvector

$$(\underset{\sim}{A} - \lambda\,\underset{\sim}{I})\,\vec{e} = \vec{0}$$

or $$\sum_j (A_{ij} - \lambda I_{ij})\,e_j = 0$$

system of homogeneous linear eqs. with non-trivial solution

$$\therefore \quad \det(\underset{\sim}{A} - \lambda\,\underset{\sim}{I}) = 0$$

nth order algebraic Eq. for λ!

Example: Pendulum. Matrix for $\underset{\sim}{W} - \lambda\,\underset{\sim}{I}$ is

$$\begin{pmatrix}\frac{g}{\ell}+\frac{k}{m}\left(\frac{\ell'}{\ell}\right)^2-\lambda & -\frac{k}{m}\left(\frac{\ell'}{\ell}\right)^2\\ -\frac{k}{m}\left(\frac{\ell'}{\ell}\right)^2 & \frac{g}{\ell}+\frac{k}{m}\left(\frac{\ell'}{\ell}\right)^2-\lambda\end{pmatrix}$$

determinant $= 0$ is

$$\left[\frac{g}{\ell}+\frac{k}{m}\left(\frac{\ell'}{\ell}\right)^2-\lambda\right]^2 - \left[\frac{k}{m}\left(\frac{\ell'}{\ell}\right)^2\right]^2 = 0$$

$$\lambda - \left[\frac{g}{\ell}+\frac{k}{m}\left(\frac{\ell'}{\ell}\right)^2\right] = \pm\,\frac{k}{m}\left(\frac{\ell'}{\ell}\right)^2$$

$$\lambda = \frac{g}{\ell} \qquad \lambda = \frac{g}{\ell} + \frac{2k}{m}\left(\frac{\ell'}{\ell}\right)^2 \qquad \text{OK}$$

Proof of Diagonalization Theorem

Let $\underset{\sim}{A}$ be symmetric linear op.

Define $Q(\vec{x})$, a quadratic form, by

$$Q(\vec{x}) \equiv \tfrac{1}{2}\vec{x}\cdot \underset{\sim}{A}\,\vec{x}$$

If $\vec{x} = \sum_i x_i \vec{e}_i$

$$Q = \frac{1}{2}\sum_{ij} x_i A_{ij} x_j$$

We can reconstruct $\underset{\sim}{A}$ from Q

$$A_{ij} = \frac{\partial^2 Q}{\partial x_i \partial x_j}$$

If we use a coordinate system where the coord. axes are aligned along the eigenvectors of $\underset{\sim}{A}$

$$\underset{\sim}{A}\,\vec{e}^{(i)\prime} = \lambda^{(i)}\vec{e}^{(i)\prime}$$

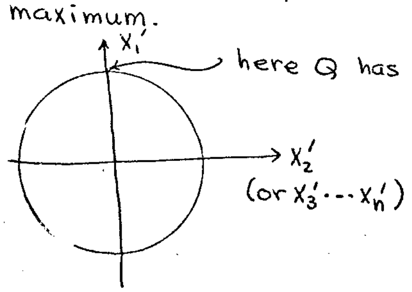
$$\vec{x} = \sum_i x_i' \vec{e}^{(i)\prime}$$

then $Q = \sum_i \lambda^{(i)} (x_i')^2$

Thus, the problem is to find orthogonal coordinates, $x_1', x_2' \cdots x_n'$ such that

$$\frac{\partial^2 Q}{\partial x_i' \partial x_j'} = 0 \qquad i \neq j$$

Consider Q restricted to the unit sphere $\vec{x}\cdot\vec{x} = 1$. Somewhere on the sphere Q must have a maximum. Choose the x_1' axis to pass through this maximum.



Note we are still free to rotate $x_2', x_3' \cdots x_n'$ as we will.

The maximum is at $x_1' = 1$, $x_2' = x_3' = \cdots x_n' = 0$

Since this is a maximum on the sphere the partial derivatives of Q in directions tangent to the sphere must vanish

$$\frac{\partial Q}{\partial x_2'} = \frac{\partial Q}{\partial x_3'} = \cdots \frac{\partial Q}{\partial x_n'} = 0$$

$$Q \equiv \frac{1}{2}\sum_{ij} x_i' A_{ij}' x_j' \qquad (\text{def of } A_{ij}')$$

$$\frac{\partial Q}{\partial x_j'} = \sum_i A_{ij}' x_i'$$

$$A_{12}' = A_{13}' = \cdots A_{1n}' = 0$$

$$A_{21}' = A_{31}' = \cdots A_{n1}' = 0$$

Thus $\vec{e}^{\prime(1)}$ is an eigenvector

$$A_{11}' = \lambda^{(1)}$$

Now define $\overline{Q} = \frac{1}{2}\sum_{\substack{i\neq 1\\ j\neq 1}} A_{ij}' x_i' x_j'$

restricted to the unit sphere of one dimension less $\sum_{i\neq 1} (x_i')^2 = 1$

and play the same game. This fixes x_2' such that

$$A_{23}' = A_{24}' = \cdots A_{2n}' = 0$$

$$A_{32}' = A_{43}' = \cdots A_{n2}' = 0,$$

Insures $\vec{e}^{\prime(2)}$ is an eigenvector

$$A_{22}' = \lambda^{(2)}$$

Etc. until you run out of dimensions.

QED

HOMEWORK COMMENTARY

6-H1 Horizontal momentum is conserved during collision. Only one of the collisions changes the energy of the oscillator.

6-H2. Show $\gamma_{syrup} = g/v_T$, $\omega_0 = \sqrt{g/\Delta\ell}$ Check: $\omega_{syr}/\omega_{air} = 3/5$

6-H3. Match initial conditions to General solution for the critically damped and undamped cases. $v=0$ at max displacement.

6-H4 New equilibrium position gives m/k, and frequency gives $(m+M)/k$. Solve for m. b) Match initial conditions to General solution, then differentiate.

c) $F_{employee} = -F_{damping}$ to maintain steady motion at natural frequency ω_0. $P_e = \vec{F}_e\cdot\vec{v}$, $\overline{\sin^2\omega t} = 1/2$.

6-H5. Show $\omega = \sqrt{2g/\ell}$, $\gamma = B/\rho A^2$, then amplitude decays like $e^{-\gamma/2 t}$.

6-H6 The total distance traveled in one cycle from x_i back to $x_f < x_i$ is approximately $2(x_i + x_f)$. Work $= \Delta KE$. Factor the difference of two squares

6-H7 Slight generalization of lecture example. $m_i \neq m_j$ but $\ell' = \ell$.

6-H8 Set up 3X3 W matrix.

Central Force Motion ①

The Problem
$$\begin{cases} m_1\ddot{\vec{r}}_1 = \vec{F}_{12} \\ m_2\ddot{\vec{r}}_2 = \vec{F}_{21} = -\vec{F}_{12} \\ \vec{F}_{12} = f(|\vec{r}|)\,\vec{r}/|\vec{r}| \qquad \vec{r} = \vec{r}_1 - \vec{r}_2 \end{cases}$$

We will worry a lot about the special case of attractive inverse-square forces

$$f = -\frac{C}{|\vec{r}|^2} \qquad C > 0$$

because this is gravity, if $C = +Gm_1m_2$.

We know a lot about this problem already:

②

(1) $\mu\ddot{\vec{r}} = f(|\vec{r}|)\,\vec{r}/|\vec{r}|$

where $\mu = m_1m_2/(m_1+m_2)$ ("reduced mass")

Also, if we work in the center-of-mass frame

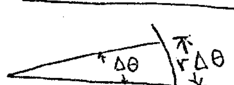
$$m_1\vec{r}_1 + m_2\vec{r}_2 = \vec{0}$$

$$\vec{r}_1 = \mu\vec{r}/m_1 \,, \quad \vec{r}_2 = -\mu\vec{r}/m_2 .$$

(Note that if $m_2 \gg m_1$, $\mu \approx m_1$, $\vec{r}_1 \approx \vec{r}$, $\vec{r}_2 \approx \vec{0}$. This is the typical case for planet + star or satellite + planet)

(2) By $\vec{L}$ conservation, we can always choose $\vec{r}$ to lie in the x-y plane. If we introduce polar coordinates in the plane

$$\boxed{\mu r^2\dot{\theta} = L_z \equiv L} \leftarrow \text{angular momentum equation}$$



$$\Delta A = \tfrac{1}{2} r\,(r\,\Delta\theta)$$

$$\boxed{\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{L}{2\mu}} \leftarrow \text{"Law of Equal Areas" (Kepler)}$$

③

(3) Central Forces are Conservative

$$\tfrac{1}{2}\mu|\dot{\vec{r}}|^2 + V(r) = E \quad \text{is const.}$$

where V is defined by
$$f(r) = -\frac{d}{dr}V(r) \qquad \left\{ \begin{array}{l} \text{If } f = -\frac{C}{r^2} \\ V = -\frac{C}{r} \end{array} \right.$$

$$\tfrac{1}{2}\mu|\dot{\vec{r}}|^2 = \tfrac{1}{2}\mu\dot{r}^2 + \tfrac{1}{2}\mu\dot{\theta}^2r^2$$
$$= \tfrac{1}{2}\mu\dot{r}^2 + \tfrac{1}{2}\frac{L^2}{\mu r^2} \leftarrow \text{by ang. mom eq.}$$

$$\therefore \boxed{E = \tfrac{1}{2}\mu\dot{r}^2 + \tfrac{1}{2}\frac{L^2}{\mu r^2} + V(r)} \leftarrow \text{energy equation}$$
$$\equiv \tfrac{1}{2}\mu\dot{r}^2 + V_{eff}(r)$$

↳ "effective potential" energy

The E-Eq. + the L-Eq. are <u>all</u> we need:

Given $r(0), \theta(0), \dot{r}(0), \dot{\theta}(0)$, we can

(1) Find E + L

(2) Find $r(t)$ by solving E-Eq. by separation

(3) Find $\theta(t)$ by integrating L-Eq.

④

The E-Eq. enables us to use the methods of 1d dynamics to discuss r-motion.

Example 1: Bounded + Unbounded Motion for $V = -C/r$.

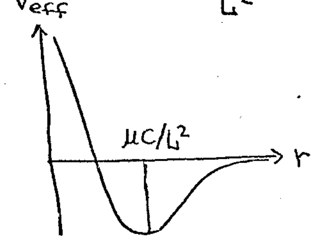
$$V_{eff} = \frac{1}{2}\frac{L^2}{\mu r^2} - \frac{C}{r} \qquad (\text{assume } L \neq 0)$$

as $r \to \infty$ $V_{eff} \to 0$ through negative values

as $r \to 0$ $V_{eff} \to \infty$

Only one minimum: $\dfrac{dV_{eff}}{dr} = -\dfrac{L^2}{\mu r^3} + \dfrac{C}{r^2} = 0$

$$\Longrightarrow r = \frac{\mu C}{L^2}$$



If $E \geq 0$, motion is <u>unbounded</u> $r(t) \to \infty$ as $t \to \pm\infty$

If $E < 0$, motion is <u>bounded</u>

(5)

Example 2: Circular motion for general $f(r) < 0$.

$r = \text{const.} \equiv R$ is equilibrium pt. for r–motion

$$\left.\frac{dV_{eff}}{dr}\right|_R = 0 = +\left.\frac{dV}{dr}\right|_R - \frac{L^2}{\mu R^3}$$

$$-\left.\frac{dV}{dr}\right|_R \equiv f(R) = -\frac{L^2}{\mu R^3} \quad \left(\begin{array}{l}\text{always has} \\ \text{unique sol'n} \\ \text{for } |L| \text{ if } f<0\end{array}\right)$$

$$\dot\theta = \frac{L}{\mu R^2}$$

$$\dot\theta^2 = \frac{L^2}{\mu^2 R^4} = -\frac{f(R)}{\mu R} \quad \text{(constant)}$$

Period of orbit $= T = \dfrac{2\pi}{|\dot\theta|} = 2\pi\sqrt{-\dfrac{\mu R}{f(R)}}$

However, circular motion is stable only if R is minimum of V_{eff}

$$\left.\frac{d^2V_{eff}}{dr^2}\right|_R = -\left.\frac{df}{dr}\right|_R + \frac{3L^2}{\mu R^4} = -\left.\frac{df}{dr}\right|_R - \frac{3f(R)}{R} > 0$$

For $f = -Cr^n$, $C>0$, this is $C R^{n-1}(n+3)$.

Circular motion is unstable if $n < -3$.

(6)

As another example, consider $f = -Ce^{-r/a}$, $C, a > 0$.

$$\frac{d^2V_{eff}}{dr^2} = Ce^{-r/a}\left[-\frac{1}{a} + \frac{3}{r}\right]$$

Circular motion stable for $r < 3a$, unstable for $r > 3a$.

Example 3: Near-circular motion (assuming

$$\left.\frac{d^2V_{eff}}{dr^2}\right|_R = -\left.\frac{df}{dr}\right|_R - \frac{3f(R)}{R} \equiv k > 0)$$

$$r = R + A\cos\left(\sqrt{\tfrac{k}{\mu}}\,t + B\right)$$

A, B ← arb. consts.

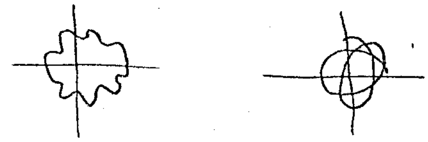
Period of θ-motion is still given by

$$T = 2\pi\sqrt{-\frac{\mu R}{f(R)}}$$

+ tiny correction (if A is tiny).

(7)

The orbit closes if $r(T) = r(0)$



Condition for closing is

$$T\sqrt{\frac{k}{\mu}} = 2\pi N \qquad N = 1, 2, 3 \ldots$$

$$T^2 \frac{k}{\mu} = (2\pi N)^2$$

$$-\frac{\mu R}{f(R)}\left[-\left.\frac{df}{dr}\right|_R - \frac{3f(R)}{R}\right]\frac{1}{\mu} = N^2$$

What is condition that orbit closes for all R?

$$r\frac{d\ln f}{dr} + 3 = N^2$$

$$\frac{d\ln f}{dr} = \frac{N^2 - 3}{r}$$

$$\ln f = (N^2 - 3)\ln r + \text{const.}$$

$$f = -C\,r^{N^2-3} \qquad C > 0$$

$$f = -\frac{C}{r^2},\; -Cr,\; -Cr^6,\; -Cr^{15},\; \ldots$$

isotropic oscillator —consistent with what we know

(8)

More About Computing the Orbit (i.e. $r(\theta)$)

$$\frac{1}{2}\mu\dot r^2 + V(r) + \frac{L^2}{2\mu r^2} = E$$

$$\dot r = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\frac{L}{\mu r^2}$$

$$\therefore \boxed{\frac{L^2}{2\mu r^4}\left(\frac{dr}{d\theta}\right)^2 + V(r) + \frac{L^2}{2\mu r^2} = E} \leftarrow \text{“orbit Eq.”}$$

Define $u \equiv 1/r$.

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$$

$$\boxed{\frac{L^2}{2\mu}\left(\frac{du}{d\theta}\right)^2 + V\left(\frac{1}{u}\right) + \frac{L^2}{2\mu}u^2 = E} \leftarrow \text{“transformed orbit equation”}$$

This is just like the energy Eq. for one-dimensional motion except

“mass” is $\dfrac{L^2}{\mu}$

“time” is θ

9

Solving the Orbit Eq. for $V = -\frac{C}{r}$

$$\frac{L^2}{2\mu}\left(\frac{du}{d\theta}\right)^2 - Cu + \frac{L^2}{2\mu}u^2 = E$$

$$\frac{L^2}{2\mu}\left(\frac{du}{d\theta}\right)^2 + \frac{L^2}{2\mu}\left(u - \frac{C\mu}{L^2}\right)^2 = E + \frac{C^2\mu}{2L^2}$$

Harmonic Oscillator (Again!) $\omega = 1$ (All orbits close)

"Equilibrium" at $u = -\frac{C\mu}{L^2} \equiv \frac{1}{r_0}$

$$u - \frac{1}{r_0} = \frac{1}{r} - \frac{1}{r_0} = \frac{\epsilon}{r_0}\cos\theta$$

def. of ϵ, $\epsilon \geq 0$

[I have chosen $\theta = 0$ to be the max. of u (min of r); K+K choose $\theta = \pi$]

$$\text{"Energy"} = \frac{1}{2} m\omega^2 \frac{\epsilon^2}{r_0^2}$$

$$E + \frac{C^2\mu^2}{2L^2} = \frac{1}{2}\frac{L^2}{\mu}\frac{\epsilon^2}{r_0^2} = \frac{1}{2}\frac{C^2\mu}{L^2}\epsilon^2$$

$$\epsilon^2 = 1 + \frac{2EL^2}{C^2\mu}$$

10

$$\boxed{\epsilon^2 = 1 + \frac{2EL^2}{C^2\mu} \qquad r_0 = \frac{L^2}{C\mu}}$$

These Eqs. give the "geometrical" parameters ϵ + r_0 in terms of the "dynamical parameters" E + L.

$$u - \frac{1}{r_0} = \frac{1}{r} - \frac{1}{r_0} = \frac{\epsilon}{r_0}\cos\theta$$

$$\frac{1}{r} = \frac{1}{r_0}(1 + \epsilon\cos\theta)$$

$$\boxed{r = \frac{r_0}{1 + \epsilon\cos\theta}}$$

This Eq. gives the shape of the orbit.

These boxes contain all we need about the orbit. If we also want to know how things depend on t, we need

$$\boxed{\mu r^2 \dot{\theta} = L}$$

11

More about the shape of the orbit.

$$r(1 + \epsilon\cos\theta) = r_0$$

$$r^2 = (r_0 - r\epsilon\cos\theta)^2$$

$$x^2 + y^2 = (r_0 - \epsilon x)^2 = r_0^2 - 2\epsilon x r_0 + \epsilon^2 x^2$$

$$(1 - \epsilon^2)x^2 + 2\epsilon x r_0 + y^2 = r_0^2$$

$$(1 - \epsilon^2)\left(x + \frac{\epsilon r_0}{1 - \epsilon^2}\right)^2 + y^2 = r_0^2\left(1 + \frac{\epsilon^2}{1 - \epsilon^2}\right) = \frac{r_0^2}{1 - \epsilon^2}$$

$\epsilon = 0$	$\left(E = -\frac{C^2\mu}{2L^2}\right)$	circle	bounded
$1 > \epsilon > 0$	$\left(0 > E > -\frac{C^2\mu}{2L^2}\right)$	ellipse	bounded
$\epsilon = 1$	$(E = 0)$	parabola	unbounded
$\epsilon > 1$	$(E > 0)$	hyperbola	unbounded

[Geometrical Note: In the theory of conic sections, ϵ is called "the eccentricity"]

(Demonstrate)

12

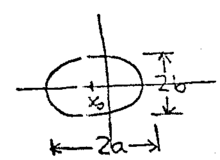
Three Properties of Bounded Motion

(1) Major and Minor Axes

$$\frac{(1 - \epsilon^2)^2}{r_0^2}\left(x + \frac{\epsilon r_0}{1 - \epsilon^2}\right)^2 + \frac{1 - \epsilon^2}{r_0^2}y^2 = 1.$$

"Standard form" of ellipse (with center on x-axis):

$$\frac{1}{a^2}(x - x_0)^2 + \frac{1}{b^2}y^2 = 1$$



$$a = \frac{r_0}{1 - \epsilon^2}, \quad b = \frac{r_0}{\sqrt{1 - \epsilon^2}}, \quad x_0 = \frac{-\epsilon r_0}{1 - \epsilon^2}$$

$a \geq b$ a is "semi-major axis"
b is "semi-minor axis"

Amusing fact: $a = \frac{r_0}{1 - \epsilon^2} = \frac{L^2/C\mu}{-2EL^2/C^2\mu} = \boxed{\frac{C}{-2E} = a}$

All orbits with same major axis have same E.

Unproved fact:

$r = 0$ is a focus of the ellipse (for proof see K+K)

Also: Width at focus $= 2r_0 = 2L^2/C\mu$ depends only on L.

(2) Period of Bounded Motion

$$\frac{dA}{dt} = \frac{L}{2\mu} \qquad (\text{from } ②)$$

$$\therefore \quad \frac{L}{2\mu} \underset{\substack{\uparrow \\ \text{period}}}{T} = \text{Area of ellipse} = \pi ab$$

$$T = \frac{2\pi\mu ab}{L} = \frac{2\pi\mu r_0^2}{L(1-\epsilon^2)^{3/2}}$$

$$= \frac{2\pi\mu \; L^4/C^2\mu^2}{L\,(-2EL^2/C^2\mu)^{3/2}}$$

$$= \frac{2\pi\,\mu^{1/2}\, C}{(-2E)^{3/2}}$$

$$= \frac{2\pi\,\mu^{1/2}}{C^{1/2}}\, a^{3/2}$$

Consistency Check: On ⑤, we found for circular motion of radius R

$$T = 2\pi\sqrt{\frac{-\mu R}{f(R)}}$$

$R = a \qquad f(R) = -C/R^2 \qquad$ OK

(3) Energy Balance:

On the average (over time), how much energy is K $(+\frac{1}{2}m\vec{v}^2)$ and how much is V?

$$av(K) = \frac{1}{T}\int_0^T dt\, K(t)$$

$$av(V) = \frac{1}{T}\int_0^T dt\, V(t)$$

$$av(E) = av(K) + av(V) = E$$

Consider

$$\frac{d}{dt}\vec{p}\cdot\vec{r} = \vec{F}\cdot\vec{r} + \vec{p}\cdot\vec{v}$$

$$= -\frac{C}{|\vec{r}|} + 2K$$

$$= V + 2K$$

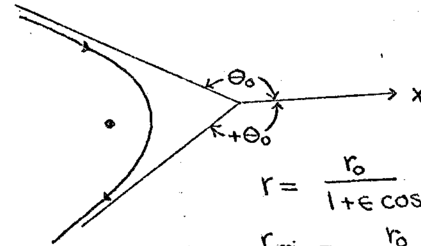
$$av\left(\frac{d}{dt}\vec{p}\cdot\vec{r}\right) = 0$$

$$\therefore \quad av(V) = -2\,av(K)$$

$$\boxed{\begin{aligned} av(K) &= -E \\ av(V) &= 2E \end{aligned}}$$

Clever trick due to Clausius; can be extended to system of particles with inverse square interactions

Unbounded Orbits and scattering



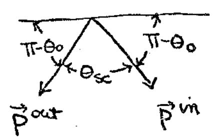
$$r = \frac{r_0}{1+\epsilon\cos\theta} \qquad \epsilon \geqslant 1$$

$$r_{min} = \frac{r_0}{1+\epsilon}$$

$r \to \infty$ at $\pm\theta_0$ defined by

$$\cos\theta_0 = -\frac{1}{\epsilon} \qquad \pi \geqslant \theta_0 \geqslant \frac{\pi}{2}$$

gives asymptotes of hyperbola. C.o.m. scattering ∠, θ_{sc}, is ∠ between $\vec{p}^{\,in}$ and $\vec{p}^{\,out}$.



$$\theta_{sc} + 2(\pi - \theta_0) = \pi$$

$$\theta_{sc} = 2\theta_0 - \pi$$

$$\cos\theta_{sc} = -\cos(2\theta_0)$$

$$= 1 - 2\cos^2\theta_0$$

$$= 1 - \frac{2}{\epsilon^2}$$

As $\epsilon \to \infty \qquad r_{min} \to 0 \qquad \theta_{sc} \to 0$

As $\epsilon \to 1 \qquad r_{min} \to r_0/2 \qquad \theta_{sc} \to \pi$

(demonstrate)

Two Problems in Astronautics

(1) Impulsive Orbit Transfer. A spaceship in orbit fires its engines for a short time. What is the new orbit after the engines are turned off?

Assume: (1) Engine fired in plane of orbit

$$\Delta\vec{p} = \Delta p_r\,\hat{r} + \Delta p_\theta\,\hat{\theta}$$

↳ change in $\vec{p}$ of spaceship (not assumed infinitesimal)

(2) Time for which engines are on $\ll T$

(3) Change in mass of spaceship, μ, negligible.

[(2)+(3) realistic for Apollo]

$$E = \frac{\vec{p}^{\,2}}{2\mu} + V(\vec{r})$$

$$\Delta E = \frac{1}{2\mu}\left[(\vec{p}+\Delta\vec{p})^2 - \vec{p}^{\,2}\right]$$

$$= \frac{1}{2\mu}\left[2p_r\Delta p_r + 2p_\theta\Delta p_\theta + (\Delta p_r)^2 + (\Delta p_\theta)^2\right]$$

$$L = \mu r^2\dot{\theta} = r\,p_\theta$$

$$\Delta L = r\,\Delta p_\theta$$

(Demonstrate)

11-9 Circular Orbit in gravitational field

Assume $M >> m$, so fixed

the gravitational attraction must just provide centripetal acceleration

$$\frac{GMm}{r^2} = m\frac{v^2}{r} \qquad \boxed{\frac{GM}{r} = v^2}$$

orbital period $\equiv T = \dfrac{2\pi r}{v} = \dfrac{2\pi r}{\sqrt{\frac{GM}{r}}}$

$$T = 2\pi\sqrt{\frac{r^3}{GM}}$$

$$\boxed{T^2 = \frac{4\pi^2}{GM}r^3}$$ Kepler's 3rd Law (must be generalized for ellipse)

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = \frac{GMm}{2r} - \frac{GMm}{r}$$

$$\boxed{E = -\frac{GMm}{2r}}$$

For circular orbit,

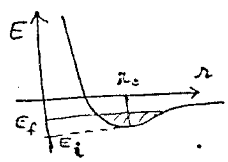
$$V_{orbit} = \sqrt{\frac{GM}{r}}$$

For escape from orbit, $E \geq 0$

$$V_{escape} = \sqrt{\frac{2GM}{r}}$$

11-10. Perturbing the circular orbit

$$E = \frac{1}{2}m\dot{r}^2 + \underbrace{\frac{L^2}{2mr^2} - \frac{GmM}{r}}_{U_{eff}(r)}$$



Circular orbit has $\dot{r} = 0$. Now we will give small $\dot{r}$, leaving L unchanged.

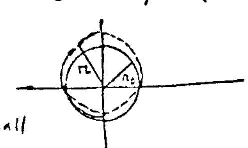
$$L_i = mvr = m\sqrt{\frac{GM}{r_0}}\,r_0 = m\sqrt{GMr_0} = L_f$$

$$k = \left.\frac{d^2U_{eff}(r)}{dr^2}\right|_{r=r_0} = \frac{3L^2}{mr_0^4} - \frac{2GmM}{r_0^3}$$
$$= \frac{3m^2GMr_0}{mr_0^4} - \frac{2GmM}{r_0^3} = \frac{GmM}{r_0^3}$$

$$\omega_{\substack{small\\osc}} = \sqrt{\frac{k}{m}} = \sqrt{\frac{GmM}{mr_0^3}} = \sqrt{\frac{GM}{r_0^3}}$$

$$T_{\substack{small\\osc}} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{r_0^3}{GM}} = \text{orbital period}$$

So $r = r_0 + A\sin\omega t$
$= r_0\,(1 + A/r_0\,\sin\omega t)$

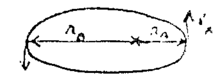


is new orbit, which for small A is almost a circle, but with displaced center.

note $r = \dfrac{r_0}{1 - \frac{A}{r_0}\sin\theta}$ is equation of ellipse

same as perturbed orbit to first order in $\frac{A}{r_0}$

11-11. Some properties of elliptic orbits



At point of closest and furthest approach, $\vec{r}$ and $\vec{v}$ are perpendicular.

So $|\vec{L}| = mv_Ar_A = mv_Br_B$

Energy conservation:

$$E = \tfrac{1}{2}mv_A^2 - \frac{GMm}{r_A} \qquad Er_A^2 = \tfrac{1}{2}mv_A^2r_A^2 - GMmr_A$$

$$E = \tfrac{1}{2}mv_B^2 - \frac{GMm}{r_B} \qquad Er_B^2 = \tfrac{1}{2}mv_B^2r_B^2 - GMmr_B$$

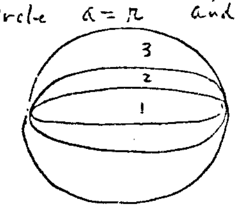
subtract $\quad E(r_A^2 - r_B^2) = -GMm(r_A - r_B)$

$$E = -\frac{GMm(r_A - r_B)}{r_A^2 - r_B^2} = \frac{-GMm}{(r_A + r_B)}$$

define $r_A + r_B \equiv 2a$ $\quad$ a is semi-major axis

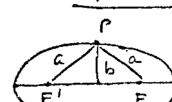
$$\boxed{E = -\frac{GMm}{2a}}$$ for elliptic orbit

For circle $a = r$ and $E = -\frac{GMm}{2r}$ as before



1, 2, 3 all have the same total energy.

11-12. Kepler's Third Law For Ellipses



Planet moving in an ellipse with sun at one focus. The area of the ellipse is $\boxed{A = \pi ab}$ where b is the semiminor axis.

Kepler's Second law: $\dfrac{dA}{dt} = \dfrac{L}{2m}$ = constant

In one period T, planet sweeps out A, so

$$A = \frac{L}{2m}T \quad \text{or} \quad T = \frac{A}{L/2m}$$

at point P, planet is equal distance from F, F'

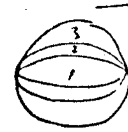
$$\tfrac{1}{2}mv_P^2 - \frac{GMm}{a} = -\frac{GMm}{2a}$$

or $\tfrac{1}{2}mv_p^2 = \dfrac{GMm}{2a}$ $\qquad \boxed{V_p^2 = \frac{GM}{a}}$

$$\frac{L}{2m} = \frac{mv_pb}{2m} = \frac{1}{2}v_pb$$

$$T^2 = \frac{A^2}{(L/2m)^2} = \frac{\pi^2a^2b^2}{\frac{1}{4}v_p^2b^2} = \frac{4\pi^2a^2}{v_p^2}$$

$$T^2 = \frac{4\pi^2a^2}{\frac{GM}{a}} = \boxed{\frac{4\pi^2}{GM}a^3}$$ Kepler's Third Law



1, 2, 3 have same period

11-13 Solving for the general orbits

$M >> m \qquad E = \frac{1}{2} m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GmM}{r}$

so $\dot{r} = \sqrt{\frac{2}{m}\left(E - \frac{L^2}{2mr^2} + \frac{GmM}{r}\right)}$

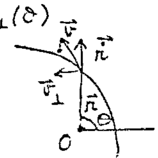
$$\frac{d\theta}{dr} = \frac{d\theta/dt}{dr/dt} = \frac{L/mr^2}{\dot{r}}$$

$$\frac{d\theta}{dr} = \frac{L}{mr^2} \frac{1}{\sqrt{\frac{2}{m}\left(E - \frac{L^2}{2mr^2} + \frac{GmM}{r}\right)}}$$

$$\int d\theta = L \int \frac{dr}{r\sqrt{2mEr^2 + 2Gm^2Mr - L^2}}$$

Doable, but not very illuminating

Notice that since $L = mv_\perp r$, it is just as good to solve for $v_\perp(\theta)$



$$\frac{dL}{d\theta} = mv_\perp \frac{dr}{d\theta} + mr\frac{dv_\perp}{d\theta} = 0$$

so $\frac{dv_\perp}{d\theta} = -\frac{v_\perp}{r}\frac{dr}{d\theta} = -\frac{L}{mr^2}\frac{dr}{d\theta}$

$$= -\frac{d\theta}{dt}\frac{dr}{d\theta} = -\frac{dr}{dt}$$

$$\boxed{\dot{r} = -\frac{dv_\perp}{d\theta}}$$

11-14. $\frac{GmM}{r} = \frac{GmM}{L/mv_\perp} = \frac{Gm^2M}{L} v_\perp$

So $E = \frac{1}{2}m(\dot{r}^2 + v_\perp^2) - \frac{GmM}{r}$

$$E = \frac{1}{2}m\left[\left(\frac{dv_\perp}{d\theta}\right)^2 + v_\perp^2\right] - \frac{GMm^2}{L}v_\perp$$

$$\frac{dE}{d\theta} = 0 = \frac{1}{2}m\left[2\frac{dv_\perp}{d\theta}\frac{d^2v_\perp}{d\theta^2} + 2v_\perp\frac{dv_\perp}{d\theta}\right] - \frac{GMm^2}{L}\frac{dv_\perp}{d\theta}$$

$$\boxed{\frac{d^2v_\perp}{d\theta^2} + v_\perp = \frac{GMm}{L}}$$

We solve this for $v_\perp(\theta)$, $r(\theta) = \frac{L}{mv_\perp(\theta)}$

Solution: $v_\perp = \underbrace{\frac{GMm}{L}}_{\text{constant solution}} + \underbrace{v_1 \cos\theta}_{\text{solution to homogeneous Diff. equation}}$

$$\boxed{\frac{GMm}{L} \equiv v_L = \text{constant } v_\perp \text{ for circular orbit of angular momentum } L}$$

So $v_\perp = v_L + v_1\cos\theta$

($v_\perp$ maximum and r minimum at $\theta = 0$)

$$\dot{r} = -\frac{dv_\perp}{d\theta} = v_1 \sin\theta$$

$$r = \frac{L}{mv_\perp} = \frac{L}{m(v_L + v_1\cos\theta)} = \frac{L/mv_L}{1 + v_1/v_L \cos\theta}$$

$$\boxed{r = \frac{r_L}{1 + v_1/v_L \cos\theta}}$$

r_L = Circular orbit radius $= L/mv_L$

11-15. Interpreting the solution

$\boxed{r = \frac{r_L}{1 + \frac{v_1}{v_L}\cos\theta}}$ $\boxed{v_\perp = v_L + v_1\cos\theta}$ $\boxed{\dot{r} = v_1 \sin\theta}$

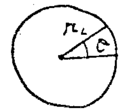
r_L = radius of circular orbit of ang mom. L

v_L = speed in same circular orbit

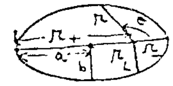
v_1 = maximum $\dot{r}$ in orbit

$\epsilon \equiv$ eccentricity of orbit $= \frac{v_1}{v_L}$

orbits are conic sections



$\epsilon = 0$
circle

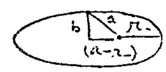


$\epsilon < 1$
ellipse

$$r_- = \frac{r_L}{1+\epsilon}$$

$$r_+ = \frac{r_L}{1-\epsilon}$$

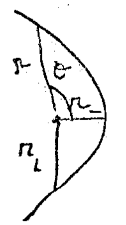
$$a = \frac{r_- + r_+}{2} = \frac{r_L}{1-\epsilon^2}$$



$$b^2 = a^2 - (a - r_-)^2 = a^2 - a^2 + 2ar_- - r_-^2$$

$$= \frac{2r_L^2}{(1+\epsilon)(1-\epsilon^2)} - \frac{r_L^2}{(1+\epsilon)^2} = \frac{r_L^2}{1-\epsilon^2}$$

So $b = \frac{r_L}{\sqrt{1-\epsilon^2}}$



Parabola $r \to \infty$ as $\theta \to \pi$ $(\epsilon = 1)$

Hyperbola $r \to \infty$ as $\theta \to \arccos(-\frac{1}{\epsilon})$ $(\epsilon > 1)$

11-16 Energy of orbits

$$K = \frac{1}{2}m(v_\perp^2 + \dot{r}^2) = \frac{1}{2}m(v_L^2 + 2v_Lv_1\cos\theta + v_1^2\cos^2\theta + v_1^2\sin^2\theta)$$

$$= \frac{1}{2}m(v_L^2 + v_1^2) + mv_Lv_1\cos\theta$$

$$U = -\frac{GMm}{r} = -\frac{Gm^2M}{L}v_\perp = -mv_Lv_\perp$$

$$= -mv_L(v_L + v_1\cos\theta)$$

$E = K + U$

$$\boxed{E = -\frac{1}{2}mv_L^2 + \frac{1}{2}mv_1^2}$$

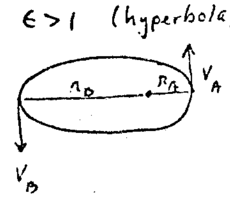
$$E = -\frac{1}{2}mv_L^2\left(1 - \frac{v_1^2}{v_L^2}\right) = -\frac{1}{2}mv_L^2(1-\epsilon^2)$$

$\epsilon = 0$ (circle) $\quad E = -\frac{1}{2}mv_L^2 = -\frac{GMm}{2r}$

$0 < \epsilon < 1$ (ellipse) $\quad -\frac{1}{2}mv_L^2 < E < 0$

$\epsilon = 1$ (parabola) $\quad E = 0$

$\epsilon > 1$ (hyperbola) $\quad E > 0$



$V_A = V_L + V_1 \qquad V_B = V_L - V_1$

$V_AV_B = V_L^2 - V_1^2$

So $E = -\frac{1}{2}V_AV_B$

11-17

Summary of Useful Relations

$$\left.\begin{aligned} |\vec{L}| = L_z &= mr^2\dot{\theta} = 2m\frac{dA}{dt} \\ E &= \tfrac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GmM}{r} \end{aligned}\right\} \text{ All orbits}$$

$$V_{circular} = \sqrt{\frac{GM}{r}} = \frac{GMm}{L} \equiv v_L$$

For ellipse, $T^2 = \dfrac{4\pi^2}{GM}a^3$, $E = \dfrac{-GMm}{2a}$

where a is semi-major axis

$$r = \frac{r_L}{1 + {v_1}/{v_L}\cos\theta} \quad \text{for all orbits}$$

where $r_L \equiv$ radius of circular orbit $= \dfrac{L}{mv_L}$

$v_1 =$ Maximum value of $\dot{r}$

$\epsilon \equiv v_1/v_L =$ eccentricity of orbit

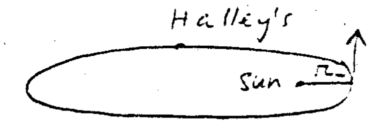
Ellipse: $r_- = \dfrac{r_L}{1+\epsilon}$ $\quad r_+ = \dfrac{r_L}{1-\epsilon}$

$$a = \frac{r_L}{1-\epsilon^2} \qquad b = \frac{r_L}{\sqrt{1-\epsilon^2}}$$

$$v_\perp = v_L + v_1\cos\theta$$
$$\dot{r} = v_1\sin\theta$$
$$E = -\tfrac{1}{2}mv_L^2 + \tfrac{1}{2}mv_1^2$$

11-18

Prob 9.9 from text



$\epsilon = 0.967$
$T = 76$ years
$M_\odot = 2\times10^{30}$

(a) Find r_- and r_+

Since we know period, we can get a

$$a = \left(\frac{GMT^2}{4\pi^2}\right)^{1/3}$$
$$= \left[\frac{\tfrac{2}{3}\times10^{-10}\times2\times10^{30}\times(76\times\pi\times10^7)^2}{4\pi^2}\right]^{1/3}$$
$$= 5.4\times10^{12}\ \text{m}$$

$$r_L = a(1-\epsilon^2) = 5.4\times10^{12}[1-(.967)^2]$$
$$= 3.6\times10^{11}\ \text{m}$$

$r_- = \dfrac{r_L}{1+\epsilon} \cong \dfrac{r_L}{2} = 1.8\times10^{11}$ perihelion

$r_+ = \dfrac{r_L}{1-\epsilon} = \dfrac{3.6\times10^{11}}{.033} = 1.1\times10^{13}$ m aphelion

(b) Find speed at perihelion

$$v_\perp = v_L(1+\epsilon\cos\theta)$$

So $v_- = \sqrt{\dfrac{GM}{r_-}}(1+\epsilon)$

$$= 4.2\times10^4\ \text{m/sec}$$

So we know a lot from some rather simple observations.

(17)

(2) The Fall of Sputnik I

A spherical satellite orbits the earth. What happens if we take account of viscous drag due to the upper atmosphere?

$$\vec{F}_{\text{viscous drag}} = -\gamma\mu\vec{v}$$ (Notation to conform to damped oscillator)

Assume: (1) γ independent of altitude (unrealistic). (2) $\gamma T \ll 1$, so drag has tiny effect over one period (realistic — at least for early stage of orbit decay)

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r}\times\vec{p}) = \vec{r}\times\vec{F} = -\gamma\mu\,\vec{r}\times\vec{v}$$
$$= -\gamma\vec{L}$$

$$\frac{dL}{dt} = -\gamma L$$

$$\frac{dE}{dt} = \vec{v}\cdot\vec{F}_{\text{non cons}} = -\gamma\mu\vec{v}^{\,2} = -2\gamma K$$

(18)

Since we are only interested in effects over many orbits, we may replace K by its average (cf. last Tuesday)

$$\text{av}(K) = -E \qquad \text{(from (14))}$$

$$\frac{dE}{dt} = +2\gamma E$$

$$E(t) = E(0)\,e^{2\gamma t} \qquad L(t) = L(0)\,e^{-\gamma t}$$

[Note: $E<0$, so first of these is OK. Note also: av(K) <u>increases</u> with time — viscous drag <u>speeds up</u> the satellite!]

$$\epsilon^2 = 1 + \frac{2EL^2}{C^2\mu}$$ is <u>independent of time</u>. Shape of orbit does not change!

$$r_0 = \frac{L^2}{C\mu}$$

$$r_0(t) = r_0(0)\,e^{-2\gamma t}$$

Satellite spirals in.

HOMEWORK COMMENTARY

7-H1. E changes, L doesn't. Find new ϵ and r_0 from eq. 9.19 and 9.20. In these problems the "sun" is very massive, so that $\mu \equiv \frac{mM}{M+m} = m$.

7-H2. Here E + L both change because v_θ changes. There is a relation between α and θ_∞ (which is angle that $r\to\infty$).

7-H3. Use r_{max} and r_{min} to find r_0 + ϵ, and then find E for the new orbit. How much kinetic energy is required at A? b) Times from area of ellipse and Keplers Law compared to known time for circular orbit. c) Get $v_\perp$ from L, then v_r from $v_\perp$+

7-H4. For parabola $\epsilon = 1$. $r(\theta_1) = R_0$, $r(\theta_2) = R_0$, $\theta_1 - \theta_2$ implies $\theta_1, \theta_2 = \frac{\pi}{2}, \frac{3\pi}{2}$. Get times from areas

7-H5. Use energy diagram approach with U_{eff} [eq. 9.10]. r_{max} and r_{min} are the turning points of the radial motion. [Arithmetic <u>is</u> messy.]

7-H6. What particular value of L makes $U_{eff} = 0$? This then determines $\dot{\theta}(r)$. Convert time integration to a radial integration.

7-H7. Differentiate, making use of $\dot{L} = 0$, the force law, the explicit form of L, and the expression for $\dot{\vec{r}}$ in polar coordinates.

SPECIAL PRE-THANKSGIVING UNIT TEST SCHEDULE

M	T	W	T	F
8 Gone	9 Today	10 2-6 (224)	11 Veterens Day X	12 X
15 2-10 (224)	16 Grade 2-10 (208)	17 2-6 (224)	18 2-10 (224)	19 X
22 2-10 (224)	23 Grade 2-10 (208)	24 No Tests	25 Turkey	26 Leftovers

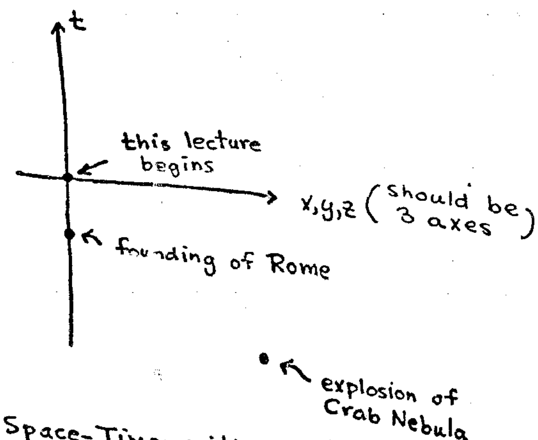
Room 208

1

Prologue to Relativity

(1) Space – Time + Four-Vectors

Space- Time is a 4-dimensional Space. Points in space-time represent <u>events</u>. They are labelled by their location (3 space coord.) + their time (1 time coord.)



Graph of Space-Time with Three Significant Events labelled

2

Displacements between events are labelled by <u>four</u> coordinate differences. They are the prototypes of four-vectors. We denote four-vectors by underlining:

$$\underline{A} \Leftrightarrow (A_t, \vec{A}) \Leftrightarrow (A_t, A_x, A_y, A_z).$$

The position of an event in space-time is an "honorary four-vector" [displacement from an (arbitrary) center of (space-time) coords.]

$$\underline{r} \Leftrightarrow (t, \vec{r}) \Leftrightarrow (t, x, y, z)$$

3

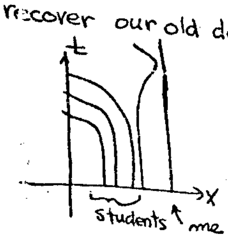
The history of a particle is given by giving a path in space-time:

$$\underline{r} = \underline{r}(\sigma)$$ ← arb. parameter, monotone increasing along path

$$\Leftrightarrow \quad t = t(\sigma), \quad \vec{r} = \vec{r}(\sigma)$$

$$\Leftrightarrow \quad t = t(\sigma), \quad x = x(\sigma), \; y = y(\sigma), \; z = z(\sigma).$$

[We can, of course, choose $\sigma = t$, in which case we recover our old description.]



← space-time diagram showing motions of some bodies in a typical Physics 12a lecture

For example, for a particle moving in a straight line with constant velocity $\vec{V}$, we can write

4

$$t = \sigma, \quad \vec{r} = \vec{V}\sigma$$

or $\quad t = \sinh\sigma, \quad \vec{r} = \vec{V}\sinh\sigma$

or $\quad t = \sigma(\sigma^2+1) \quad \vec{r} = \vec{V}\sigma(\sigma^2+1)$

etc. ad. infinitum.

Other things than particle motions can be described this way. Example: A laser pulse moving in the x-direction

$$t = \sigma \qquad \vec{r} = c\sigma\vec{i}$$

$c \equiv$ Velocity of light $\approx 3 \times 10^8$ m/sec.

To describe a system of N particles, we must give the motion of all the particles:

$$\underline{r}_a = \underline{r}_a(\sigma_a) \qquad a = 1 \cdots N$$

↑ different parameter for each path (why not?)

(2) Invariances

An invariance (transformation) is an invertible transformation that turns physically allowed motions (i.e. soln's of the eqs. of motion), and only physically allowed motions, into physically allowed motions.

Example: Space Translations

$$\vec{r}_a(t) \to \vec{r}_a{}'(t) = \vec{r}_a(t) + \vec{a}$$

some constant vector

For every $\vec{r}_a(t)$ there is a $\vec{r}_a{}'(t)$
" " $\vec{r}_a{}'(t)$ " " " $\vec{r}_a(t)$

$\vec{r}_a{}'(t)$ is a sol'n of the Eqs. of motion if and only if $\vec{r}_a(t)$ is a sol'n.

General Properties: (1) The inverse of an invariance is an invariance. (2) The product of two invariances is an invariance. ["Product" of two transformations ≡ result of applying the " " in succession.]

Note: We have taken <u>active</u> viewpoint (system transformed, observer fixed). <u>Passive</u> viewpoint (system fixed, observer (inverse) transformed) also OK. Either is good — but confusing them in the middle of an argument is disastrous! Be careful!

Our example is an instance of a "space-time" invariance. We can think of the transformation as acting on space-time

$$\vec{r} \to \vec{r}' = \vec{r} + \vec{a}$$

$$t \to t' = t$$

Particle motions are "carried along"

A Catalog of Space-Time Invariances of Isolated Systems (ca. 1850)

Arbitrary products of:

(1) Space Translations:

$$\vec{r} \to \vec{r} + \vec{a} \qquad t \to t$$

arb. const. vector

(2) Time Translations:

$$\vec{r} \to \vec{r} \qquad t \to t + T$$

arb. const.

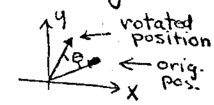
(3) Rotations:

$$\vec{r} \to \underset{\sim}{R}\,\vec{r} \qquad t \to t$$

rotation about some axis by some angle (a linear operator)

For example, rotation about the z-axis by $\angle\theta$

$t \to t$
$z \to z$
$x \to x\cos\theta - y\sin\theta$
$y \to x\sin\theta + y\cos\theta$



(4) Galilean Transformations

$$\vec{r} \to \vec{r} + \vec{V}t \qquad t \to t$$

arb. const. vector

Notes: (A) (1) can be replaced by the apparenty weaker statement of inv. under x-translations only

$$x \to x + a, \quad y \to y, \quad z \to z, \quad t \to t$$

This plus rot. inv. (3) implies all of (1). Likewise (4) can be replaced by

$$x \to x + Vt, \quad y \to y, \quad z \to z, \quad t \to t.$$

(B). The catalog is not complete. It does not contain, for example, mirror reflection in the x-y plane:

$$x \to x, \quad y \to y, \quad z \to -z, \quad t \to t.$$

Unlike the catalog items, this can not be continuously connected to the identity

$$x \to x, \; y \to y, \; z \to z, \; t \to t$$

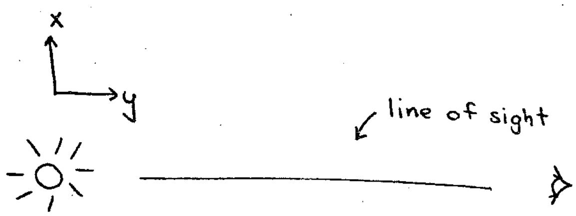
We shall avoid such "disconnected" transformations. (Policy of cowardice).

(C) The catalog is "metaphysics"

["Met..."

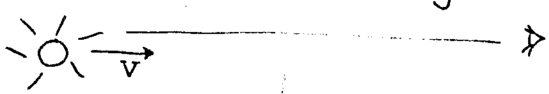
The Theory of Relativity

A Crisis With Galilean Invariance



velocity of light emitted by body at rest is $c\,\vec{j}$ ← speed of light

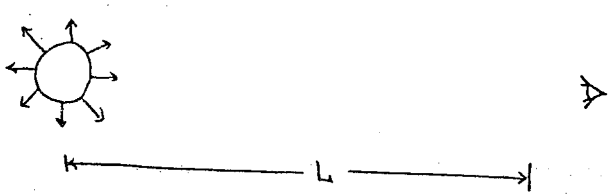
What is velocity of light emitted by body moving $\vec{V}$? ← with velocity



Answer: $\vec{V} + c\,\vec{j}$, by Galilean Inv.

Note: No theory of light is needed.

To check: Consider exploding sphere of luminous gas



Explosion at $t=0$. Assume $|\vec{v}|$ constant over sphere, $|\vec{v}| \ll c$, radius of sphere $\ll L$. Light from front of sphere moves with speed $c + |\vec{v}|$ arrives at

$$t = \frac{L}{c+|\vec{v}|} = \frac{L}{c}\,\frac{1}{1+\frac{|\vec{v}|}{c}} \approx \frac{L}{c}\left(1 - \frac{|\vec{v}|}{c}\right)$$

Light from top moves with speed

$$\sqrt{c^2 + |\vec{v}|^2} = c\sqrt{1 + \frac{|\vec{v}|^2}{c^2}} \approx c$$ ← not correct

Arrives at

$$t = \frac{L}{c}$$

Thus, even if explosion is very short, signal is spread out over $\Delta t = \frac{L}{c}\,\frac{|\vec{v}|}{c}$

Can we check? Yes! (slides)

Data for Crab Nebula Supernova

Distance from Earth $\approx$ 5000 lt. yrs.
Radius of Cloud $\approx$ 5 lt. yrs.
Explosion observed by Chinese $\approx 10^3$ yrs ago
Average rate of expansion $\approx 5 \times 10^{-3}$ lt. yrs./yr
(Presumably greater at time of explosion)

$$\frac{L}{c}\,\frac{|\vec{v}|}{c} = 5000 \times 5 \times 10^{-3} = 25 \text{ yrs.}$$

Nonsense! New star dimmed to half its brightness in $\approx$ 2 months, disappeared completely in $\approx$ 2 yrs.

Also: Supernovae Observed in Distant Galaxies $> 10^6$ lt. yrs. away
Should lead to spread out of $>$ 1000 yrs.
They dim in $\approx$ 2 months also.

All evidence indicates speed of light independent of speed of source!!

What Is to Be Done?

Two Possibilities:

(1) Abandon Galilean Inv. + restrict the catalog to (1), (2) + (3).

(2) Try to find transformations that replace Galilean Inv. These should (a) reduce to Gal. Inv. in some "appropriate" limit. (b) be consistent with the independence of the speed of light of the condition of its source.

Let's try (2). [What can we lose?]

We still assume (1), (2), + (3) are OK, but replace

$$x \to x + Vt,\; y \to y,\; z \to z,\; t \to t$$

by

$$x \to x'(x,t),\; y \to y,\; z \to z,\; t \to t'(x,t)$$

where x' + t' are functions we are to find.
(Irrelevance of y+z pure simplifying assumption —we might have to investigate relaxing this if things don't work.)

NOTE

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$$x \to x'(x,t) \qquad t \to t'(x,t)$$

[y+z supressed]. To restrict these functions we need facts about the laws of nature.

Fact 1: Newton's 1st Law. Implies transf. must turn strt. lines in space-time into strt. lines. Therefore, must be (possibly inhomogeneous) linear transf.

$$\begin{aligned} x &\to Ax + Bt + C \\ t &\to Dx + Et + F \end{aligned} \qquad A, B, C, D, E, F \text{ constants.}$$

We can always set $C=F=0$ by multiplying by space+time translations. Thus, we might as well restrict ourselves to this case

$$\begin{aligned} x &\to Ax + Bt \\ t &\to Dx + Et \end{aligned}$$

END OF STEP ONE IN ARGUMENT

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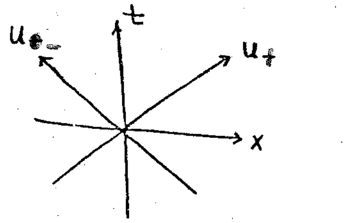
Fact 2: Speed of Light independent of condition of source.

Convenient to work in "natural units", i.e. $c=1$. Thus if we measure time in

$\left\{\begin{array}{l} \text{yrs.} \\ \text{sec.} \\ \text{nanosec} = 10^{-9}\,\text{sec} \end{array}\right\}$ we measure distance in

$\left\{\begin{array}{l} \text{lt. yrs.} \\ \text{lt.sec.} \approx 3\times 10^5\,\text{km} \\ \text{lt. nanosec.} \approx \text{ft.} \end{array}\right\}$.

In these units the paths of light pulses emitted in the x-direction at $x=0$, $t=0$ are 45° lines



Introduce new coords.:

$$u_\pm = t \pm x$$

In these coords., the paths are the axes

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The transf. is still linear

$$\begin{aligned} u_+ &\to \alpha u_+ + \beta u_- \\ u_- &\to \gamma u_+ + \delta u_- \end{aligned} \qquad \alpha, \beta, \gamma, \delta \text{ consts.}$$

Two possibilities:

(1) $\beta = \gamma = 0$ or (2) $\alpha = \delta = 0$

Possibility (2) not continuously connected to identity — ignore it.

$$u_+ \to \alpha u_+ \qquad u_- \to \delta u_-$$

$\alpha \neq 0$ $\delta \neq 0$ for invertibility

Restrict ourselves to $\alpha > 0$, $\delta > 0$ (Other possibilities not cont. conn. to identity, as before).

END OF STEP TWO IN ARGUMENT

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Rot. inv. says that x can be turned into $-x$, i.e. u_+ and u_- interchanged. Thus if

$$u_+ \to \alpha u_+ \ , \ u_- \to \delta u_-$$

is an invarianse, so is

$$u_+ \to \delta u_+ \ , \ u_- \to \alpha u_- .$$

So is their product:

$$u_+ \to \alpha\delta\, u_+ \ , \ u_- \to \alpha\delta\, u_- .$$

So are repeated products:

$$u_\pm \to (\alpha\delta)^n u_\pm \qquad n = 1, 2, 3 \ldots$$

Written out in terms of x, y, z, t

$$x \to (\alpha\delta)^n x \ , \ y \to y, \ z \to z, \ t \to (\alpha\delta)^n t$$

Fact 3: This is not an invariance unless $\alpha\delta = 1$. [There are no long thin planets]

$\therefore \quad \alpha\delta = 1$

END OF STEP THREE AND END OF ARGUMENT

17

It is convenient to define ϕ by

$$\alpha \equiv e^{\phi} \qquad \delta = e^{-\phi}.$$

Our transformation than becomes

$$u_\pm \to e^{\pm\phi} u_\pm \qquad y,z \to y,z$$

Equivalently

$$t = \tfrac{1}{2}(u_+ + u_-) \to \tfrac{1}{2}(e^{\phi}u_+ + e^{-\phi}u_-)$$
$$= \tfrac{1}{2}(e^{\phi} + e^{-\phi})t + \tfrac{1}{2}(e^{\phi} - e^{-\phi})x$$
$$t \to (\cosh\phi)t + (\sinh\phi)x$$

Likewise

$$x \to (\cosh\phi)x + (\sinh\phi)t$$
$$y \to y \qquad z \to z$$

This is called "Lorentz transformation in the x-direction with rapidity ϕ."
[Note formal similarity to rotation on 7]

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Strange as they look, Lorentz transfs. are our <u>only hope</u> if we wish to find a replacement for the discredited Galililean tranfs. Will they work? Only one way to tell — explore the consequences of Lorentz invariance.

> GEORGE (*facing away, out front, emotionless*): Meeting a friend in a corridor, Wittgenstein said: 'Tell me, why do people always say it was *natural* for men to assume that the sun went round the earth rather than that the earth was rotating?' His friend said, 'Well, obviously, because it just *looks* as if the sun is going round the earth.' To which the philosopher replied, 'Well, what would it have looked like if it had looked as if the earth was rotating?'
>
> — Tom Stoppard, <u>Jumpers</u>

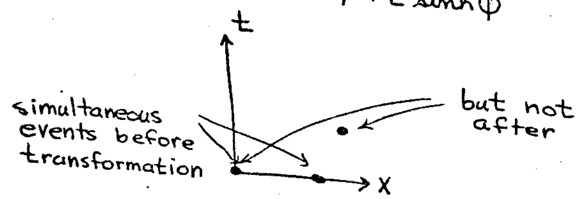
[I may stop here if time is short]

19

Properties of Lorentz Invariance:

(1) Simultaneity is not invariant!

$$t \to t\cosh\phi + x\sinh\phi$$
$$x \to x\cosh\phi + t\sinh\phi$$



(2) What does a Lorentz transformation do to a particle at rest?

$$x=y=z=0 \qquad t=\sigma \qquad \text{(before transf.)}$$
$$t \to t' = t\cosh\phi + x\sinh\phi = \sigma\cosh\phi$$
$$x \to x' = x\cosh\phi + t\sinh\phi = \sigma\sinh\phi$$
$$y \to y' = y = 0$$
$$z \to z' = z = 0$$
$$\vec{v} = v\,\vec{i} \qquad v = \frac{x'}{t'} = \frac{\sinh\phi}{\cosh\phi} = \tanh\phi$$

[In unnatural units, $\frac{v}{c} = \tanh\phi$]

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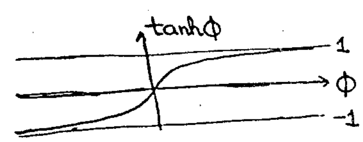


Note no matter what ϕ is, $v<1$
[In unnatural units, $v<c$]

In passive way of thinking, this is transf. that tells us how things are described by an observer travelling with velocity $-\vec{i}\,v$.
note minus sign.

(3) Going from rapidity to velocity

$$v = \tanh\phi$$
$$\cosh\phi = \frac{1}{\sqrt{1-(\tanh\phi)^2}} = \frac{1}{\sqrt{1-v^2}}$$
$$\sinh\phi = \frac{\tanh\phi}{\sqrt{1-(\tanh\phi)^2}} = \frac{v}{\sqrt{1-v^2}}$$

standard formulas — easy to prove from defs. of sinh, cosh, tanh.

(4) The Galilean Limit

Choose scale of time so all observations are made in $|t|<1$. For a typical "classical" experiment, this means $|x|\ll 1$. Consider a Lorentz transf with $v=\tanh\phi \ll 1$. Expand the Lorentz transf. keeping only terms of zeroth and first order in the small quantities:

$$t \to t\cosh\phi + x\sinh\phi$$
$$= t\,\frac{1}{\sqrt{1-v^2}} + \frac{x\,v}{\sqrt{1-v^2}}$$
$$\approx t$$

$$x \to x\cosh\phi + t\sinh\phi$$
$$= x\,\frac{1}{\sqrt{1-v^2}} + t\,\frac{v}{\sqrt{1-v^2}}$$
$$\approx x + vt$$

Galilean Invariance!

[This gives a precise definition of the "appropriate limit" on 12]

1

Where We Are

(Provisional) catalog of space-time invariances (that can be continuously connected to the identity transformation): Products of
(1) Space translations (2) Time translations
(3) Rotations (4) Lorentz transformations

Lorentz transformation in x-direction with rapidity ϕ:

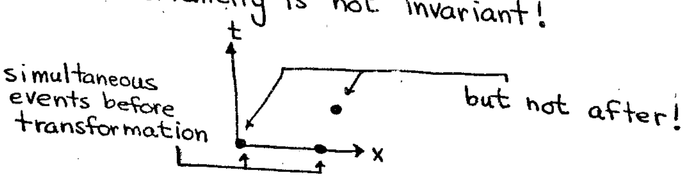
$$\left.\begin{array}{l} x \to x' = x\cosh\phi + t\sinh\phi \\ t \to t' = t\cosh\phi + x\sinh\phi \\ y \to y' = y \qquad z \to z' = z \end{array}\right\} \text{ in natural units } (c=1)$$

Our task: To explore the consequences of the assumption of Lorentz invariance until either (1) we understand the world or (2) we reach a contradiction with experiment. (not with "natural" ideas — "What would it have looked like if it had looked as if the earth were rotating?")

2

Consequences of Lorentz Invariance
(The remainder of this unit)

(1) Simultaneity is not invariant!



In passive viewpoint, simultaneous events according to a stationary observer are not simultaneous to a moving observer.

3

(2) The Galilean Limit

Let us consider an experiment only involving objects with velocities $\ll 1$. Choose scale of time so all observations take place for $|t|<1$. Then $|x| \ll 1$. Consider a Lorentz transf with $\phi \ll 1$, and expand neglecting terms of second order or higher in small quantities.

$$\cosh\phi = \tfrac{1}{2}(e^{\phi}+e^{-\phi}) \approx 1$$
$$\sinh\phi = \tfrac{1}{2}(e^{\phi}-e^{-\phi}) \approx \phi$$

$$x \to x\cosh\phi + t\sinh\phi \approx x + t\phi$$
$$t \to t\cosh\phi + x\sinh\phi \approx t + x\phi \approx t$$
$$y \to y \qquad z \to z$$

Reduces to Galilean transformation, if we identify ϕ with v* for small ϕ!

[* in unnatural units, v/c]

Very important "Every successful physical theory swallows its predecessor alive."
— J. Blish

4

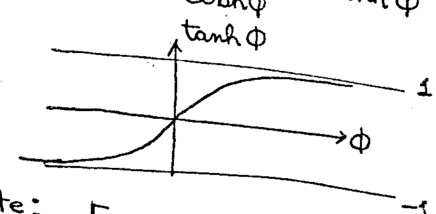
(3) What does a Lorentz transf do to a particle at rest?

Before transf. $x=y=z=0 \qquad t=\sigma$

$$x \to x' = x\cosh\phi + t\sinh\phi = \sigma\sinh\phi$$
$$y \to y' = y = 0$$
$$z \to z' = z = 0$$
$$t \to t' = t\cosh\phi + x\sinh\phi = \sigma\cosh\phi$$

Gives particle velocity $v\vec{i}$, where

$$v = \frac{x'}{t'} = \frac{\sinh\phi}{\cosh\phi} = \tanh\phi$$



Note: For any ϕ, $v<1$. [In unnatural units $v<c$]

Warning: This is not a proof that motion faster than light is impossible.

⑤

$$\cosh\phi = \frac{1}{\sqrt{1-(\tanh\phi)^2}} = \frac{1}{\sqrt{1-v^2}}$$

$$\sinh\phi = \frac{\tanh\phi}{\sqrt{1-(\tanh\phi)^2}} = \frac{v}{\sqrt{1-v^2}}$$

standard formulas; easy to prove from defs. of cosh, sinh, + tanh.

$$\boxed{\begin{array}{ll} x \to x' = \dfrac{x}{\sqrt{1-v^2}} + \dfrac{vt}{\sqrt{1-v^2}} & y \to y' = y \\ t \to t' = \dfrac{t}{\sqrt{1-v^2}} + \dfrac{vx}{\sqrt{1-v^2}} & z \to z' = z \end{array}}$$

This expresses the transf directly in terms of $v = \tanh\phi$ — sometimes useful.
As a rule, though, the algebra is simpler if you use ϕ throughout the computation + change to v (if necessary) only at the end.

⑥

(4) Lorentz Transf. from a passive viewpoint.

Every invariance transf. can be viewed either <u>actively</u>, as a transf. of the system, or <u>passively</u>, as a change of the $\left\{\begin{array}{l}\text{coordinates}\\ \text{observer}\end{array}\right\}$.

Example: Galililean transf.

Active: $\vec{r}\,' = \vec{r} + \vec{v}t$

$\vec{r}$ — old position in old coord.
$\vec{r}\,'$ — new position in old coord

Passive: $\vec{r} = \vec{r}\,' + \vec{v}t$

$\vec{r}$ — old position in old coord.
$\vec{r}\,'$ — old position in new coord. (those appropriate to an observer moving with velocity $\vec{v}$)

Note: (1) Interchange of $\vec{r}$ and $\vec{r}\,'$, old and new. — inversion of transformation
(2) Change of meaning of $\vec{r}\,'$

⑦

Thus x', y', z', t', the coords. appropriate to an observer moving with velocity $v\vec{i}$, $v = \tanh\phi$, are given by

$$x = x'\cosh\phi + t'\sinh\phi \qquad y = y'$$
$$t = t'\cosh\phi + x'\sinh\phi \qquad z = z'$$

We can easily draw these coords on a space-time diagram if we have three points

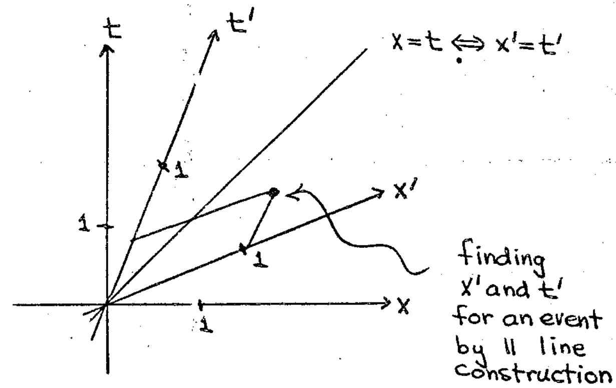
$$\left.\begin{array}{lll} x'=0,\ t'=0 & \Longleftrightarrow & x=0,\ t=0 \\ x'=1,\ t'=0 & \Longleftrightarrow & x=\cosh\phi,\ t=\sinh\phi \\ x'=0,\ t'=1 & \Longleftrightarrow & x=\sinh\phi,\ t=\cosh\phi \end{array}\right\}$$

See Figure on ⑧

These fix new coord axes + scale along axes. New coord of arbitrary event then obtained graphically by ∥ line construction.

This sort of space-time diagram showing two coord. systems is frequently very useful.

⑧



Lorentz transf. looks formally like a rotation, but it is <u>not</u> a rotation:

(a) The 45° line doesn't budge

(b) The new axes are <u>not</u> ⊥ (*)

[In fact, they make equal ∠'s with the 45° line: $(\cosh\phi\,\vec{j} + \sinh\phi\,\vec{i})\cdot(\vec{i}+\vec{j})$
$= \cosh\phi + \sinh\phi = (\sinh\phi\,\vec{j} + \cosh\phi\,\vec{i})\cdot(\vec{i}+\vec{j})$]

(c) Unit pt. along new axis <u>not</u> unit length from origin. (*)

(*) Later we will redefine geometry of space-time to cancel the <u>not</u>'s.

(9)

(5) Time Dilation

A clock emits 1 tick/second when it is at rest. What is the time interval between ticks when it is moving ~~at~~ with constant velocity $\vec{v}$?

Choose $\vec{v} = |\vec{v}|\,\vec{i}$. Assume tick occurs at $x=t=0$. Adapt x', t' coords so clock is at rest in these coords. Next tick occurs at $t'=1$ sec, $x'=0$. $\Rightarrow t = \cosh\phi$ (see (7)).

Alternative argument: Parameterize clock at rest by $x=0$, $t=\sigma$. Ticks occur at $\sigma = 0, 1, 2 \cdots$. By Lorentz inv., for moving clock, $t = (\cosh\phi)\sigma$ (see (4)), ticks still occur at $\sigma = 0, 1, 2 \cdots$. Thus, once again interval between ticks is

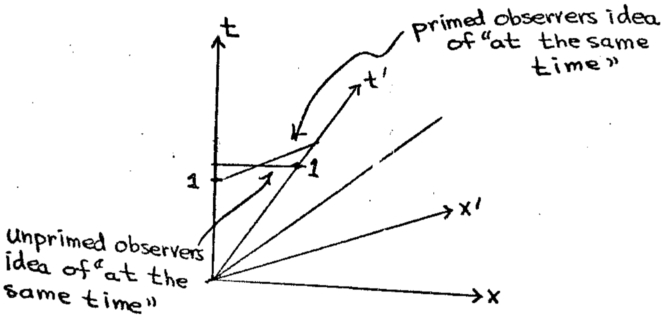
$$\Delta t = \cosh\phi = \frac{1}{\sqrt{1-v^2}}$$

(Steadily) Moving Clocks Run Slow!

[Non-steadily-moving (accelerating) clocks next lecture]

(10)

Thus, given two observers moving with different velocities, each observes the other's clock to run slow.



"Although our clocks both showed noon when we passed, when* your clock showed 1PM, my clock showed 1:10. Your clock runs slow!"

— Either observer

*i.e., at the same time

(11)

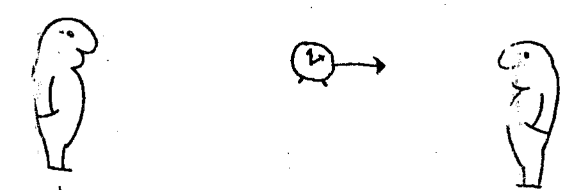
Of course, "observes" here is not a coordinate-independent concept. Here is a " " question: If the stationary observer watches the moving clock through a telescope, what times does his clock show when he sees the moving clock show 0, 1 sec, 2 sec., etc.?

Clock shows 0 at	$t=t_0$, $x=x_0$ (Assume $x_0>0$)
Signal received at	$t=t_0+x_0$, $x=0$
Clock shows 1 at	$t=t_0+\frac{1}{\sqrt{1-v^2}}$, $x=x_0+\frac{v}{\sqrt{1-v^2}}$
Signal received at	$t=t_0+x_0+\frac{1+v}{\sqrt{1-v^2}}$, $x=0$

Time between receipt of sucessive signals

$$\Delta t = \frac{1+v}{\sqrt{1-v^2}} = \sqrt{\frac{(1+v)^2}{(1+v)(1-v)}} = \sqrt{\frac{1+v}{1-v}}$$

This is $\begin{Bmatrix} >1 \\ <1 \end{Bmatrix}$ if v is $\begin{Bmatrix} >0 \\ <0 \end{Bmatrix}$



"I see the clock run slow"   "I see the clock run fast"

(12)

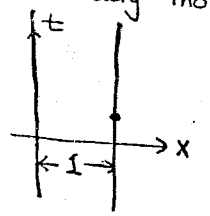
[More about this next lecture — Doppler effect].

Moral: When we say an observer "observes" something, this is just a loose way of saying certain coordinates are assigned to events in a coord. system where the observer is at rest. This is not the same as giving the actual raw measurements made by even an idealized experimenter.

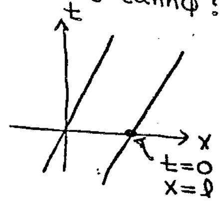
Warning: This does not mean coordinates are bad. Coordinates are good. Just don't confuse them with raw measurements.

(6) Lorentz Contraction

A measuring rod (aligned along the x-axis) at rest has length 1. What is its length if it is in steady motion with $\vec{v} = \vec{i}\tanh\phi$?



space-time paths of ends of rod at rest



same for transformed rod. What is ℓ?

Apply backwards* Lorentz transf to event shown (* i.e., rapidity $-\phi$)

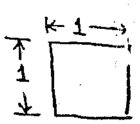
$$x \to x\cosh(-\phi) + t\sinh(-\phi)$$
$$= \ell\cosh\phi + 0$$

but this must lie on line $x = 1$.

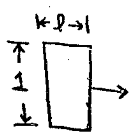
$$\therefore\ \ell = \frac{1}{\cosh\phi} = \sqrt{1-v^2}$$

Moving rods shrink!

Notes: (1) No contraction if rod is aligned ⊥ to direction of motion ($y \to y$, $z \to z$)



top view of unit cube at rest



same in motion

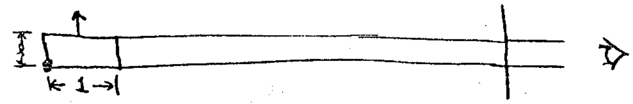
$\ell = \sqrt{1-v^2}$

(2) Thus, given two observers moving with different velocities, each observes the other to be contracted.

(3) Once again, "observes" here means "assigns coordinates such that". What is actually seen? This is complicated in general, but easy to work out for a special case:

[I may skip this if time is short]

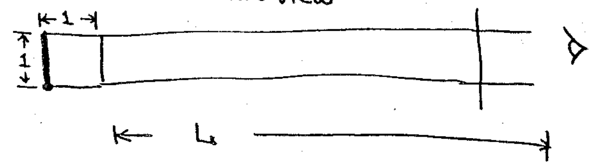
top view



$\ell = \sqrt{1-v^2}$

"picture plane"

side view



Assume: Unit cube moves ⊥ to one face, ⊥ to line of sight. Also $L \gg 1$, so we can obtain image by drawing ∥ lines to picture plane.

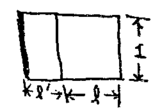
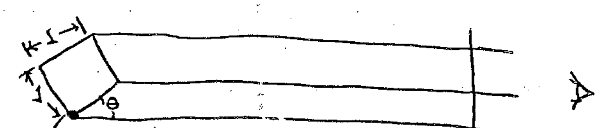


image in picture plane

$\ell = \sqrt{1-v^2}$

$\ell' = v$ (because light from back edge that reaches picture plane at same time as light from front face had to be emitted one time unit earlier)

This is just like the image of a stationary tilted cube. Top view



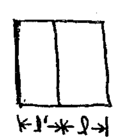


image in picture plane

$\ell = \cos\theta$

$\ell' = \sin\theta$

same as above if we define θ by

$\cos\theta = \sqrt{1-v^2}$ $\quad \sin\theta = v$

This is also true (with different formula for θ) for motion in arb. direction (rel. to line of sight) so long as $L \gg 1$.

(17)

(7) Invariant Dot Product

Lorentz transf along x-direction

$$(t+x) \to e^{\phi}(t+x)$$
$$(t-x) \to e^{-\phi}(t-x)$$

$$\therefore \quad t^2-x^2 = (t+x)(t-x) \to t^2-x^2$$

Since $y \to y \quad z \to z$

$$t^2-x^2-y^2-z^2 \to t^2-x^2-y^2-z^2$$

This object is also invariant under rotations

$$t^2-\vec{r}^{\,2} \to t^2-\vec{r}^{\,2}$$

$\therefore$ it is invariant under Lorentz transformations in any direction. Of course, it is not invariant under space and time translations.

However, given two events

$$\underline{r}_1 \Leftrightarrow (t_1, \vec{r}_1) \qquad \underline{r}_2 \Leftrightarrow (t_2, \vec{r}_2)$$

This is invariant under everything

$$(t_1-t_2)^2 - (\vec{r}_1-\vec{r}_2)^2$$

(18)

Since a four-vector (by definition) is an object that transforms like a coordinate difference (space-time displacement) for any four-vector $\underline{a} \Leftrightarrow (a_t, \vec{a})$, this is invariant:

$$(a_t)^2 - \vec{a}\cdot\vec{a} \equiv \underline{a}\cdot\underline{a} \qquad (\text{def. of } \underline{a}\cdot\underline{a})$$

We define, for any two four-vectors, $\underline{a}$ and $\underline{b}$, $\underline{a}\cdot\underline{b}$ by

$$\underline{a}\cdot\underline{b} \equiv \tfrac{1}{4}\left[(\underline{a}+\underline{b})\cdot(\underline{a}+\underline{b}) - (\underline{a}-\underline{b})\cdot(\underline{a}-\underline{b})\right]$$
$$= a_t b_t - \vec{a}\cdot\vec{b}$$
$$= a_t b_t - a_x b_x - a_y b_y - a_z b_z$$

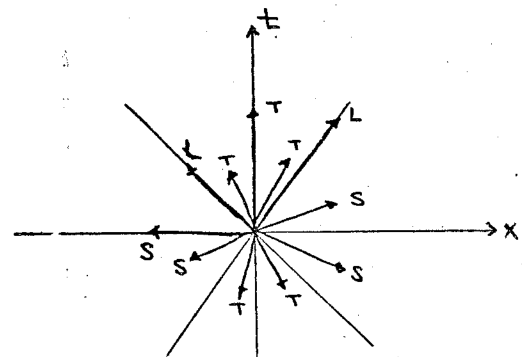
This is "invariant dot product". Only the minus signs distinguish it from the ordinary dot product. (Lorentz transfs. are like rotations, but they are <u>not</u> rotations)

(19)

(8) Timelike, Lightlike, and Spacelike Vectors

We say a vector is $\left\{\begin{array}{l}\text{timelike}\\ \text{lightlike}\\ \text{spacelike}\end{array}\right\}$ if $\left\{\begin{array}{l}\underline{a}\cdot\underline{a}>0\\ \underline{a}\cdot\underline{a}=0\\ \underline{a}\cdot\underline{a}<0\end{array}\right\}$. For example, one such vector is $\left\{\begin{array}{l}a_t=1, \vec{a}=0\\ a_t=1, \vec{a}=\vec{i}\\ a_t=0, \vec{a}=\vec{i}\end{array}\right\}$.

Because $\underline{a}\cdot\underline{a}$ is invariant, so is this classification.

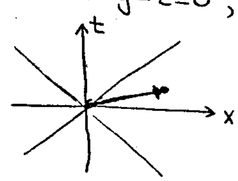


Some random vectors of the three types
$(a_y=a_z=0) \qquad a\cdot a = (a_t)^2 - (a_x)^2$

(20)

(9) The Causal Structure of Space-Time or Why I Can't Travel Faster Than Light

If I could travel faster than light, I could start a journey at $x=y=z=t=0$ and arrive at $y=z=0$, $x=x_0$, $t=t_0$, with $x_0>t_0$



$\Leftarrow$ This journey assumed possible

By Lorentz transformation, I obtain an equally possible journey that arrives at

$$x_0' = x_0 \cosh\phi + t_0 \sinh\phi$$
$$t_0' = x_0 \sinh\phi + t_0 \cosh\phi$$

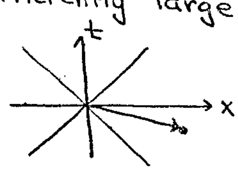
As $\phi \to -\infty$

$$\cosh\phi \to \tfrac{1}{2}e^{-\phi} \qquad \sinh\phi \to -\tfrac{1}{2}e^{-\phi}$$

$$t_0' \to (t_0 - x_0)e^{-\phi}/2 \to -\infty$$

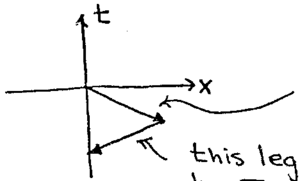
note (−) sign

For sufficiently large $-\phi$, t_0' becomes <0!



$\Leftarrow$ This journey deduced possible

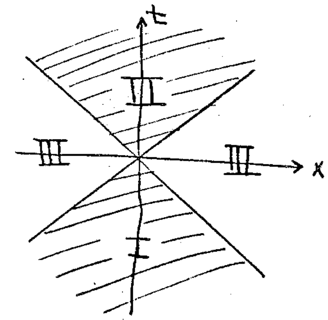
I.e., I can travel backwards in time.
I can even travel to the past of a stationary observer



This is usually believed to be impossible (grandfather paradox). ∴ The original assumption is impossible.

Phrased more abstractly:
If we assume cause can not be later in time than effect, then Lorentz invariance implies cause and effect can not be connected by a space-like four-displacement.

Causal structure of space-time



Think of this as a slice of a surfa of rotation; the 45° lines are a slice of a cone $|\vec{x}|=t$ (the light cone)

Events in region I (includes boundary) can affect events at origin

Events in region II (includes boundary) can be affected by events at origin

Events in region III (excludes boundary) can neither affect nor be affected by events at origin.

1

Consequences of Lorentz Invariance (Con't.)

(10) Proper Time, Four-Velocity, and "Relativistic Addition of Velocities"

Consider free-particle motion, i.e. motion with constant velocity, $\vec{V}$,

$$\frac{d\vec{r}}{dt} = \vec{V}.$$

Define "proper time", s, as time shown by a clock moving with the particle. This is a Lorentz-invariant parameter.
By last lecture,

$$\frac{dt}{ds} = \frac{1}{\sqrt{1-|\vec{V}|^2}} \quad \left(\begin{array}{c}\text{time-dilation}\\ \text{formula}\end{array}\right)$$

$$\therefore \quad \frac{d\vec{r}}{ds} = \frac{dt}{ds}\frac{d\vec{r}}{dt} = \frac{1}{\sqrt{1-|\vec{V}|^2}}\,\vec{V}$$

2

Define $\underline{u}$ by

$$\underline{u} \equiv \frac{d\underline{r}}{ds} \Leftrightarrow \frac{d}{ds}(t, \vec{r}) = \frac{1}{\sqrt{1-|\vec{V}|^2}}(1, \vec{V})$$

The four-vector $\underline{u}$ is called "the four-velocity". The formula above gives the components of $\underline{u}$ in any coord. system.

Notes: (1) $\underline{u}\cdot\underline{u} = \frac{1}{1-|\vec{V}|^2}(1-\vec{V}\cdot\vec{V}) = 1$

$\underline{u}$ is a "unit time-like four-vector".

(2) $$\frac{d\underline{r}}{ds} = \underline{u}$$

$$\therefore \quad \underline{r} = \underline{u}s + \underline{r}_0$$

$\underline{r}_0$: space-time pt. where clock is set to zero

This is "four-vector form" of motion of isolated particle.

3

An application of $\underline{u}$: Two free particles have velocities $\vec{V}_1$ and $\vec{V}_2$. What is $|\vec{V}_2'|$, where $\vec{V}_2'$ is the velocity of particle 2 in a coord. system where particle 1 is at rest? (Note: No sense in asking for direction of $\vec{V}_2'$ unless we specify more about coords — we can always make space rotations.)

In given coords

$$\underline{u}_1 \Leftrightarrow \frac{1}{\sqrt{1-|\vec{V}_1|^2}}(1, \vec{V}_1)$$

$$\underline{u}_2 \Leftrightarrow \frac{1}{\sqrt{1-|\vec{V}_2|^2}}(1, \vec{V}_2)$$

$$\underline{u}_1\cdot\underline{u}_2 = \frac{1}{\sqrt{1-|\vec{V}_1|^2}}\,\frac{1}{\sqrt{1-|\vec{V}_2|^2}}(1-\vec{V}_1\cdot\vec{V}_2)$$

In coords where 1 is at rest

$$\underline{u}_1 \Leftrightarrow (1, \vec{0}) \qquad \underline{u}_2 = \frac{1}{\sqrt{1-|\vec{V}_2'|^2}}(1, \vec{V}_2')$$

$$\underline{u}_1\cdot\underline{u}_2 = \frac{1}{\sqrt{1-|\vec{V}_2'|^2}}$$

4

But $\underline{u}_1\cdot\underline{u}_2$ is Lorentz-invariant!

$$\therefore \quad \frac{1}{\sqrt{1-|\vec{V}_2'|^2}} = \frac{1-\vec{V}_1\cdot\vec{V}_2}{\sqrt{1-|\vec{V}_1|^2}\,\sqrt{1-|\vec{V}_2|^2}}$$

$$1-|\vec{V}_2'|^2 = \frac{(1-|\vec{V}_1|^2)(1-|\vec{V}_2|^2)}{(1-\vec{V}_1\cdot\vec{V}_2)^2}$$

This is sometime called "the relativistic formula for addition of velocities". (Reason for name on 5).

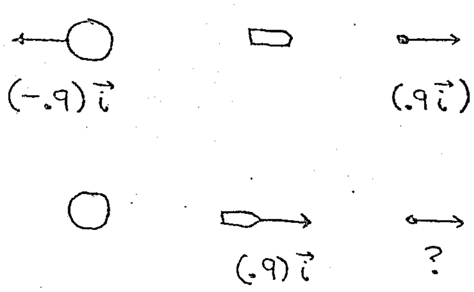
Notes: (1) No need to do explicit Lorentz transf. (2) Answer unchanged by exchange of 1 and 2.

This formula deuglifies a lot if $\vec{V}_1$ and $\vec{V}_2$ are $\parallel$. Let $\vec{V}_1 = V_1\vec{i}$, $\vec{V}_2 = V_2\vec{i}$, $\vec{V}_2' = V_2'\vec{i}$; then

$$|V_2'| = \left|\frac{V_2 - V_1}{1 - V_2 V_1}\right|$$

(For proof see appendix.)

Numerical Example: Spaceship moving with $\vec{v} = (.9)\vec{i}$ (rel. to earth) sends out lifeboat with $\vec{v} = (.9)\vec{i}$ (according to captain of spaceship). What is speed of lifeboat according to observer on earth?



Situation in coord. where ship is at rest

Situation in coord. where earth is at rest

Let 1 be earth, 2 lifeboat.

$$|V_2'| = \frac{V_2 - V_1}{1 - V_1 V_2} = \frac{.9 + .9}{1 + (.9)^2} = \frac{1.8}{1.81} \approx 0.994.$$

Note very different from Galilean answer (1.8). [Reason for name "relativistic addition of velocities"] However, if instead of .9 we had .1 we would obtain

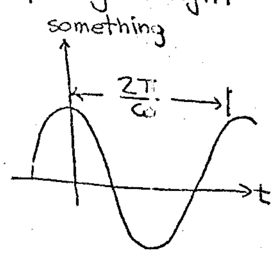
$$|V_2'| = \frac{.1 + .1}{1 + (.1)^2} = \frac{.2}{1.01} = 0.198$$

Not so different.

(11) Doppler Effect.

An atom at rest, if "excited" emits light of frequency ω_0. What is ω, the frequency of light emitted by an excited atom moving with velocity $\vec{v}$?

Preliminary: (Very little) about the frequency of light



(To find out what "something" is, take Physics 12b)

Parameterize path of light pulse

$$\frac{d\vec{r}}{dt} = \vec{e} \qquad |\vec{e}| = 1$$

by parameter σ, chosen so peaks of "something" occur at $\sigma = 0, 2\pi, 4\pi$, etc.

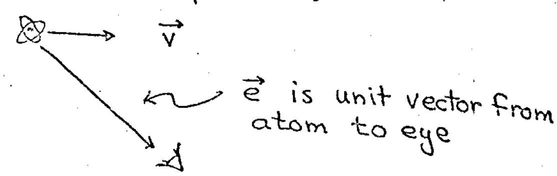
$$\frac{dt}{d\sigma} = \omega \qquad \frac{d\vec{r}}{d\sigma} = \frac{dt}{d\sigma}\frac{d\vec{r}}{dt} = \omega\vec{e}$$

Define $\underline{k}$, a four-vector, by

$$\underline{k} \equiv \frac{d\underline{r}}{d\sigma} \Leftrightarrow \omega(1, \vec{e})$$

The parameter σ is analagous to s
The four-vector $\underline{k}$ " " " $\underline{u}$
(The analogy is not perfect: $\underline{k}\cdot\underline{k} = \omega^2(1^2 - \vec{e}\cdot\vec{e}) = 0.$)

Now for the problem:



In coordinates of lab (those in which eye is at rest), four-velocity of atom is given by

$$\underline{u} \Leftrightarrow \frac{1}{\sqrt{1-|\vec{v}|^2}}(1, \vec{v})$$

Likewise

$$\underline{k} \Leftrightarrow \omega(1, \vec{e})$$

$$\therefore \quad \underline{k}\cdot\underline{u} = \frac{\omega}{\sqrt{1-|\vec{v}|^2}}(1 - \vec{e}\cdot\vec{v})$$

In coordinates where atom is at rest

$$\underline{u} \Leftrightarrow (1, \vec{0})$$

$$\underline{k} \Leftrightarrow \omega_0(1, \vec{e}') \qquad [\vec{e}' \text{ will be irrelevant}]$$

$$\underline{k}\cdot\underline{u} = \omega_0$$

But $\underline{k}\cdot\underline{u}$ is Lorentz-Invariant!

$$\therefore \quad \omega_0 = \frac{\omega}{\sqrt{1-|\vec{v}|^2}}(1 - \vec{e}\cdot\vec{v})$$

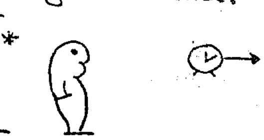
$$\omega = \frac{\omega_0\sqrt{1-|\vec{v}|^2}}{1 - \vec{e}\cdot\vec{v}}$$

Notes: (1) This is a doable experiment + it has been done! (The results agree)
(2) Results simplify if $\vec{e}$ is ∥ to $\vec{v}$. Take $\vec{v} = v\vec{e}$, then

$$\omega = \omega_0 \frac{\sqrt{1-v^2}}{1-v} = \omega_0\sqrt{\frac{1+v}{1-v}}$$

Same formula as for time dilation.* Why not — ticking clock or oscillating atom, it shouldn't make any difference.

[*



]

(9)

(12) The Clock Effect or "The Twin Paradox"

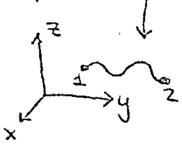
This is the most disquieting effect in relativistic kinematics. It is also the first effect we will deal with that involves accelerated motion. Thus I will go through the argument slowly, developing the concepts in a way that paralells the development of something we all understand — length.

> "Here we are at last!" said the raven. "What a long way it is! In half the time I could have gone to Paradise and seen my cousin—him, you remember, who never came back to Noah! Dear! dear! it is almost winter."
> "Winter!" I cried; "it seems but half a day since we left home!"
> "That is because we have traveled so fast," answered the raven. "

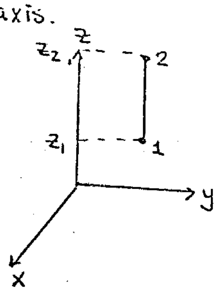
—George Macdonald: Lilith (ca. 1880)

(10)

The LENGTH of a path in space is both a mathematical concept (to be developed on these slides) and a physical one (what we measure when we lay a tape measure along the path). We can not prove they are equal, but we can make it plausible.



Step 1: Straight line segment ∥ to the z-axis.

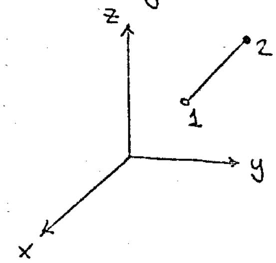


$$\ell = z_2 - z_1 \equiv \Delta z$$

Mathematics and Physics Agree

(11)

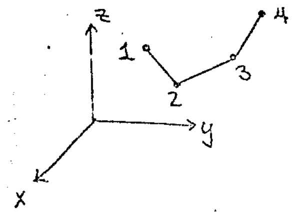
Step 2: Straight line segment in arbitrary orientation



$$\ell = \sqrt{(\vec{r}_2 - \vec{r}_1)\cdot(\vec{r}_2 - \vec{r}_1)} \equiv \sqrt{(\Delta\vec{r})\cdot(\Delta\vec{r})}$$

Reason:
· Rotational Invariance

Step 3: Broken line

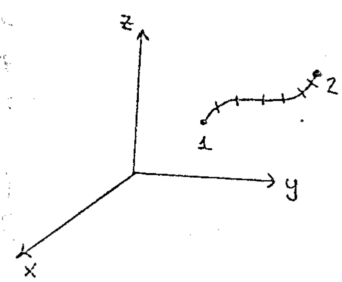


$$\ell = \sqrt{(\vec{r}_2 - \vec{r}_1)\cdot(\vec{r}_2 - \vec{r}_1)} + \sqrt{(\vec{r}_3 - \vec{r}_2)\cdot(\vec{r}_3 - \vec{r}_2)} + \sqrt{(\vec{r}_4 - \vec{r}_3)\cdot(\vec{r}_4 - \vec{r}_3)} \equiv \sum_{\text{segments}} \sqrt{(\Delta\vec{r})\cdot(\Delta\vec{r})}$$

Reason: Definition (but a plausible one: tape-measure length is additive—an experimental fact.)

(12)

Step 4: Arbitrary (Piece-wise differentiable) Path



Curve given by $\vec{r} = \vec{r}(\sigma)$

σ some arbitrary parameter such that $\sigma_2 \geq \sigma \geq \sigma_1$

$\vec{r}(\sigma_1) = \vec{r}_1$, $\vec{r}(\sigma_2) = \vec{r}_2$

Approximate curve by broken line (say, defined by points shown). For broken line

$$\ell = \sum_{\text{segments}} \sqrt{(\Delta\vec{r})\cdot(\Delta\vec{r})}$$

$$= \sum \Delta\sigma \sqrt{\left(\frac{\Delta\vec{r}}{\Delta\sigma}\right)\cdot\left(\frac{\Delta\vec{r}}{\Delta\sigma}\right)}$$

Take limit $\Delta\sigma \to 0$ [Archimedes]

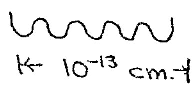
$$\ell = \int_{\sigma_1}^{\sigma_2} d\sigma \sqrt{\frac{d\vec{r}}{d\sigma}\cdot\frac{d\vec{r}}{d\sigma}}$$

Reason: Just a definition again (but a plausible one — why shouldn't things be continuous?).

13

Real and Ideal Tape Measures

A real tape measure may have problems if the path bends too much



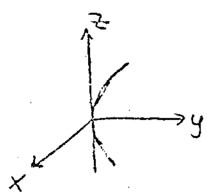
A good measure of bending is "curvature"

$$K^2 = \frac{d^2\vec{r}}{d\ell^2} \cdot \frac{d^2\vec{r}}{d\ell^2}$$

This is (1) Rotationally invariant

(2) Zero if the path is a straight line

(3) In coords. where the path is tangent to the z-axis, measures bending away from the axis



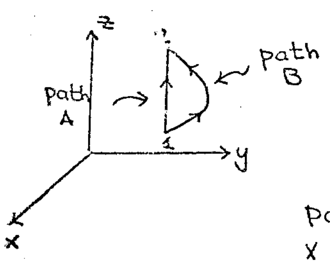
$$\frac{d\vec{r}}{d\ell} \cdot \frac{d\vec{r}}{d\ell} = 1$$

$$\therefore \frac{d^2\vec{r}}{d\ell^2} \cdot \frac{d\vec{r}}{d\ell} = 0$$

$$\vec{r} = \vec{0} + \frac{d\vec{r}}{d\ell}\ell + \frac{1}{2}\frac{d^2\vec{r}}{d\ell^2}\ell^2 + \cdots$$

14

A Theorem of Geometry



A straight line (parallell to the z-axis) is the shortest path connecting two points (with the same x and y coordinates)

Choose $\sigma = z$

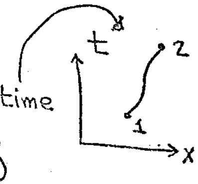
$$\ell_A = \int_{z_1}^{z_2} dz = z_2 - z_1$$

$$\ell_B = \int_{z_1}^{z_2} dz \sqrt{1 + \left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2} > \ell_A$$

By rotational invariance, we can drop the parenthetical qualifiers.

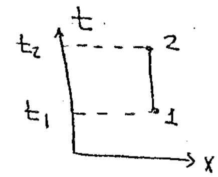
15

The PROPER TIME



along a path in space-time is both a mathematical concept (to be defined) and a physical one (the time measured by a clock travelling along the path). We can not <u>prove</u> that they are equal, but we can make it plausible.

Step 1: Straight line segment ∥ to the t-axis.

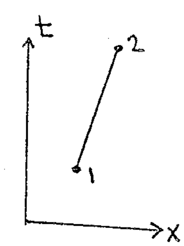


$$s = t_2 - t_1 \equiv \Delta t$$

Mathematics and Physics Agree

16

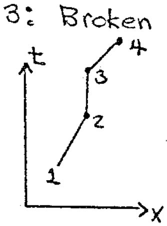
Step 2: Straight-Line Segment in Arbitrary orientation ($|\vec{v}| < 1$)



$$s = \sqrt{(\underline{r}_2 - \underline{r}_1)\cdot(\underline{r}_2 - \underline{r}_1)} \equiv \sqrt{\Delta\underline{r} \cdot \Delta\underline{r}}$$

Reason: Lorentz Invariance

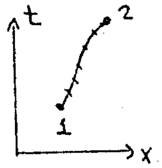
Step 3: Broken Line



$$s = \sqrt{(\underline{r}_2 - \underline{r}_1)\cdot(\underline{r}_2 - \underline{r}_1)} + \sqrt{(\underline{r}_3 - \underline{r}_2)\cdot(\underline{r}_3 - \underline{r}_2)} + \sqrt{(\underline{r}_4 - \underline{r}_3)\cdot(\underline{r}_4 - \underline{r}_3)} \equiv \sum_{\text{segments}} \sqrt{\Delta\underline{r} \cdot \Delta\underline{r}}$$

Reason: Definition (but a plausible one: clock time is additive — an experimental fact).

Step 4: Arbitrary (Piece-wise differentiable) Path



Path given by

$\underline{r} = \underline{r}(\sigma)$

σ some arbitrary parameter such that $\sigma_2 \geq \sigma \geq \sigma_1$

$\underline{r}(\sigma_1) = \underline{r}_1$, $\underline{r}(\sigma_2) = \underline{r}_2$

Approximate path by broken line (say, defined by points shown). For broken line

$$s = \sum_{\text{segments}} \sqrt{(\Delta \underline{r})\cdot(\Delta \underline{r})}$$

$$= \sum (\Delta\sigma) \sqrt{\frac{\Delta \underline{r}}{\Delta\sigma} \cdot \frac{\Delta \underline{r}}{\Delta\sigma}}$$

Take limit $\Delta\sigma \to 0$

$$s = \int_{\sigma_1}^{\sigma_2} d\sigma \sqrt{\frac{d\underline{r}}{d\sigma} \cdot \frac{d\underline{r}}{d\sigma}}$$

Reason: Just a definition again (but a plausible one—why shouldn't things be continuous?)

Real and Ideal Clocks

A real clock may have problems if it accelerates too much.

A good measure is "proper acceleration"

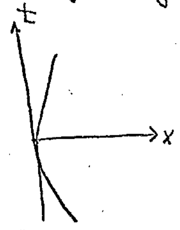
$$a^2 = -\frac{d^2 \underline{r}}{ds^2} \cdot \frac{d^2 \underline{r}}{ds^2}$$

to cancel minus sign in dot product

This is (1) Lorentz invariant

(2) Vanishes if the path is a straight line

(3) In coordinates where clock is momentarily at rest, measures acceleration (bending away from t-axis)

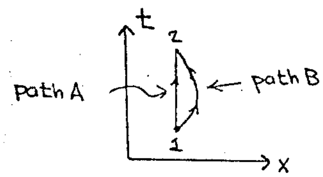


$$\frac{d\underline{r}}{ds} \cdot \frac{d\underline{r}}{ds} = 1$$

$$\therefore \frac{d^2\underline{r}}{ds^2} \cdot \frac{d\underline{r}}{ds} = 0$$

$$\underline{r} = \underline{0} + \frac{d\underline{r}}{ds} s + \frac{1}{2} \frac{d^2\underline{r}}{ds^2} s^2 + \cdots$$

A Theorem of Relativity



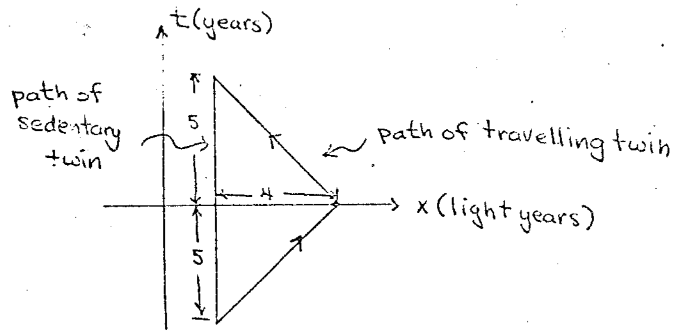
An unaccelerated clock (at rest) shows the longest time of any clock that travels between two space-time points (with the same space coordinates)

Choose $\sigma = t$

$$s_A = \int_{t_1}^{t_2} dt = t_2 - t_1$$

$$s_B = \int_{t_1}^{t_2} dt \sqrt{1 - \left|\frac{d\vec{r}}{dt}\right|^2} < s_A$$

By Lorentz invariance, we can drop the parenthetical qualifiers.

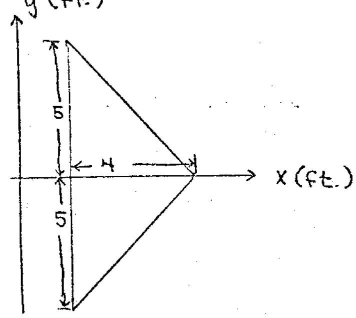


Age of sedentary twin $= 5+5 = 10$ yrs.

Age of travelling twin $= 2\sqrt{5^2 - 4^2}$

$= 2\sqrt{25-16} = 6$ yrs.

The Twin "Paradox"



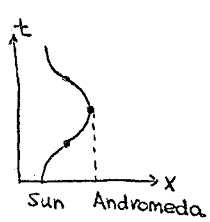
Length of straight line = 5+5 = 10 ft.
Length of broken line $= 2\sqrt{5^2+4^2}$
$= 2\sqrt{41} \approx 12.8$ ft.

The Length "Paradox"

Remarks: (1) This effect has been measured experimentally.

(2) Solution for ~~pmp~~ motion in the x-direction, $x(0) = v(0) = 0$, with constant proper acceleration = 1 lt. yr/(yr)^2 ≈ 10 m/(sec)^2.

s (yrs.)	t (yrs.)	x (lt. yrs.)
1	1.17	0.34
2	3.62	2.76
3	10.0	9.07
4	27.3	26.3
5	74.2	73.2
10	11,000	11,000
20	242,000,000	242,000,000



$\Leftarrow$ Full journey is 4 segments like this

Appendix to (4)

$$|v_2'|^2 = 1 - \frac{(1-v_1^2)(1-v_2^2)}{(1-v_1v_2)^2}$$

$$= \frac{1 - 2v_1v_2 + v_1^2v_2^2 - (1 - v_1^2 - v_2^2 + v_1^2v_2^2)}{(1-v_1v_2)^2}$$

$$= \frac{v_2^2 + v_1^2 - 2v_1v_2}{(1-v_1v_2)^2}$$

$$= \left(\frac{v_2 - v_1}{1 - v_1v_2}\right)^2$$

A Communication from the Holy Office:

K & K is hereby removed from the Index Librorum Prohibitorum; the faithful are allowed to read the relevant chapters if they wish. These are Chapters 11, 12, and 14.1-14.3, on relativistic kinematics, and Chapter 13 and the remainder of Chapter 14, on relativistic dynamics, the subject of the lectures of the week of November 30.

You should be warned of the following points, which, though not error in the strict sense, are liable to cause confusion in the minds of the ill-instructed:

(1) K & K use unnatural units, thus contaminating their equations with senseless factors of c.

(2) K & K define the invariant dot product to have the opposite sign to that used in the lectures. Further, for any four-vector, $\underline{a}$, they define a_4 to be ia_t. Thus,

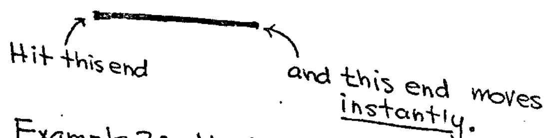
$$-\underline{a}\cdot\underline{a} = |\vec{a}|^2 - (a_t)^2 = |\vec{a}|^2 + (a_4)^2.$$

This has the advantage of making $-\underline{a}\cdot\underline{a}$ the sum of four squares; thus increasing the formal similarity between relativistic vector algebra and conventional vector algebra; it has the disadvantage of introducing unnecessary imaginary numbers into all equations.

Relativistic Dynamics

Many concepts of non-relativistic mechanics are inconsistent with relativistic causality (no transfer of information faster than speed of light).

Example 1: Rigid Rod



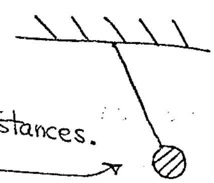
Example 2: Newton's Law of Gravity

$\overset{\bullet}{1}$ $\qquad$ $\overset{\bullet}{2}$ $\qquad \vec{F}_{12} = -\vec{F}_{21} = \dfrac{Gm_1m_2(\vec{r}_2-\vec{r}_1)}{|\vec{r}_2-\vec{r}_1|^3}$

Move particle 1 and particle 2 responds (by changing its acceleration) instantly.

Digression

This does not mean these concepts should be discarded in everyday circumstances.
Example: Pendulum.



[Time it takes impulse to get from bob to support] = [length of rod] / [speed of sound in rod] << [period of pendulum]

So it is a very good approx. to treat rod as rigid. Nevertheless, [speed of sound in rod] << [speed of light], so we are nowhere near a conflict with relativity.

Likewise, for gravity, in a typical case

$$\frac{\text{time for light to go from sun to earth}}{\text{period of earth's revolution}} = \frac{8 \text{ min.}}{1 \text{ yr}} \approx 1.5 \times 10^{-5}$$

so even if gravity propagates at, say, $\frac{1}{10}$ the speed of light, we might as well treat it as instantaneous.

Thus, we have two options:

(1) Develop detailed dynamical theories consistent with relativistic causality.

(2) Restrict ourselves to problems where speed of propagation of influence from one particle to another is irrelevant:

(a) Asymptotic Conservation Laws
(b) Particle subject to external force

We will follow (2) here; (1) is part of 12b.

Asymptotic Conservation Laws: Preliminaries

We consider processes where in the far past + far future we can consider the system as composed of a set of isolated particles.

Thus, for each particle, as $t \to -\infty$

$$\vec{r}_i(t) = \vec{V}_i^{\,in} t + \vec{r}_{io}^{\,in} \qquad i = 1, 2 \ldots N^{in}$$

and as $t \to +\infty$

$$\vec{r}_i(t) = \vec{V}_i^{\,out} t + \vec{r}_{io}^{\,out} \qquad i = 1, 2 \ldots N^{out}$$

(1) We do not assume $N^{in} = N^{out}$. Number (and characters) of incoming particles can be different from number (and characters) of outgoing particles.

(2) We do not assume "particles" are pointlike, structureless, etc. "Particles" can be complex systems as seen by a nearsighted experimenter.

(5)

Asymptotic Cons. Laws: Momentum

Old Asymptotic Mom. Cons:

$$\sum_{\substack{\text{incoming}\\ \text{particles}}} \vec{p}_i = \sum_{\substack{\text{outgoing}\\ \text{particles}}} \vec{p}_i, \qquad (1)$$

where

$$\vec{p}_i = m_i \vec{v}_i, \qquad (2)$$

and m_i is a constant depending on the particle. ("Mass is the quantity that makes mom. cons. work." — Lecture I)

We will attempt to generalize this by replacing (2) above by

$$\vec{p}_i = \underbrace{m_i(|\vec{v}_i|)}_{\text{some function of } |\vec{v}_i|} \vec{v}_i \qquad (2')$$

We shall see that the form of $m_i(|\vec{v}_i|)$ is essentially completely determined by Lorentz invariance.

(6)

To exploit Lorentz inv., it is convenient to rewrite (2′) in terms of the four-velocity $\underline{u}$:

$$u_t = \frac{1}{\sqrt{1-|\vec{v}|^2}} \qquad \vec{u} = \frac{\vec{v}}{\sqrt{1-|\vec{v}|^2}}$$

$$\vec{p}_i = m_i(|\vec{v}_i|)\, \vec{v}_i \equiv f_i(u_t)\, \vec{u}_i$$

$$f_i = m_i \sqrt{1-|\vec{v}_i|^2}$$

Consider Lorentz transf. in x-direction

$$u_t \to u_t \cosh\phi + u_x \sinh\phi$$
$$u_x \to u_t \sinh\phi + u_x \cosh\phi$$
$$u_y, u_z \to u_y, u_z$$

By Lorentz invariance, we can thus obtain many new conservation laws from our assumed one. E.g. we can replace p_{iy} by

$$p_{iy}' \equiv f_i(u_t \cosh\phi + u_x \sinh\phi)\, u_{iy}$$

etc.

(7)

This is a disaster! (In a problem where there are only two outgoing particles, six conservation laws are enough to fix all components of $\vec{v}_1^{\,out}$ and $\vec{v}_2^{\,out}$ in terms of ~~[illegible]~~ incoming velocities. This is contrary to experiment).

Only one way out

$$f_i = \text{const.} \equiv m_{oi}$$

$$\vec{p}_i = m_{oi}\, \vec{u} = \frac{m_{oi}}{\sqrt{1-|\vec{v}_i|^2}}\, \vec{v}_i = m_i \vec{v}_i$$

$$m_i = \frac{m_{oi}}{\sqrt{1-|\vec{v}_i|^2}}$$

m_{oi} is mass as measured by mom. cons. if $|\vec{v}| \ll 1$. It is called "rest mass".

Note: All we have shown is that if there exists an asymptotic mom. cons. law then it must be of this form. We have not proved (we can not prove) that there is such a law. This is for experiment. (Experiment says there is.)

(8)

Asymptotic Conservation Laws: Energy

$$\sum_{\substack{\text{incoming}\\ \text{particles}}} \vec{p}_i = \sum_{\substack{\text{outgoing}\\ \text{particles}}} \vec{p}_i$$

$$\vec{p}_i = m_{oi}\, \vec{u}_i$$

Define the four-vector $\underline{p}_i$ ("the four-momentum") by

$$\underline{p}_i = m_{oi}\, \underline{u}$$

Mom. Cons. + Lorentz Inv. implies

$$\sum_{\substack{\text{incoming}\\ \text{particles}}} p_{ti} = \sum_{\substack{\text{outgoing}\\ \text{particles}}} p_{ti}$$

A new(?) asymptotic conservation law! (But only one — it's safe). What is it?

(9)

[Drop i for notational simplicity]

$$p_t = m_o u_t = \frac{m_o}{\sqrt{1-|\vec{v}|^2}} = m$$

Looks like cons. of mass. But m is velocity dependent (unlike non-relativistic mass)

$$m = \frac{m_o}{\sqrt{1-|\vec{v}|^2}} \approx m_o + \tfrac{1}{2} m_o |\vec{v}|^2 \qquad |\vec{v}| \ll 1$$

Thus $\underline{\text{if}}$ $\displaystyle\sum_{\substack{\text{incoming}\\ \text{particles}}} m_{oi} = \sum_{\substack{\text{outgoing}\\ \text{particles}}} m_{oi}$

then $\displaystyle\sum_{\text{in}} \tfrac{1}{2} m_{oi} |\vec{v}_i|^2 \approx \sum_{\text{out}} \tfrac{1}{2} m_{oi} |\vec{v}_i|^2$

$|\vec{v}_i| \ll 1$

These are elastic collisions and this is asymptotic cons. of kinetic energy. If $\sum_{in} m_{oi} \neq \sum_{out} m_{oi}$, these are inelastic collisions. Thus, we can just as well call

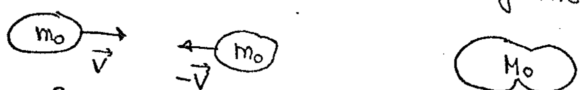
$$p_t = E \quad (\text{"total energy"})$$

(10)

Thus, $E = m$

[in unnatural units, $E = mc^2$]

In non-relativistic mechanics, you can hide E. (See end of 2nd energy lecture). In relativistic mechanics, you can $\underline{\text{not}}$ hide E: $E = m$, and m is known from mom. cons. experiments. $\underline{\text{However}}$, E can be "as good as hidden". Example: Two blobs of putty collide and stick together



far past — far future

$E \approx 2(m_o + \tfrac{1}{2} m_o |\vec{v}|^2) \quad |\vec{v}| \ll 1$ — $E = M_o$

$\vec{p} = 0$ — $\vec{p} = 0$

$\therefore\ M_o = 2m_o + m_o |\vec{v}|^2$

As opposed to the pre-relativity prediction $M_o = 2m_o$

However, fractional error is $\dfrac{m_o |\vec{v}|^2}{2m_o} = \dfrac{|\vec{v}|^2}{2} \approx \tfrac{1}{2} \times 10^{-14}$ for $|\vec{v}| = 30$ m/sec

(11)

Summary of relations between $E, \vec{p}, \underline{p}, \vec{v}$, $m + m_o$ for an isolated system:

$$E = \frac{m_o}{\sqrt{1-|\vec{v}|^2}} \qquad \vec{p} = m\vec{v} = \frac{m_o \vec{v}}{\sqrt{1-|\vec{v}|^2}}$$

$$E = m$$

$$\underline{p} = m_o \underline{u} \Leftrightarrow (E, \vec{p})$$

$$\underline{p} \cdot \underline{p} = m_o^2\, \underline{u} \cdot \underline{u} = m_o^2 = E^2 - |\vec{p}|^2$$

$$E = \sqrt{|\vec{p}|^2 + m_o^2}$$

$$\vec{v} = \vec{p}/E$$

[Some of these in unnatural units:

$$E = \frac{m_o c^2}{\sqrt{1-|\vec{v}|^2/c^2}} \qquad \vec{p} = \frac{m_o \vec{v}}{\sqrt{1-|\vec{v}|^2/c^2}}$$

$$E = mc^2$$

$$E = \sqrt{|\vec{p}|^2 c^2 + m_o^2 c^4}$$

$$\vec{v} = \vec{p} c^2 / E \quad]$$

We sometimes define K ("relativistic kinetic energy") by $E = m_o + K$ [$E = m_o c^2 + K$].

(12)

Particles of Zero Rest Mass

In pre-relativistic physics, if we consider the limit $m \to 0$, $\vec{p}$ fixed, we get nonsense:

$$\vec{v} = \frac{\vec{p}}{m} \to \infty \qquad E = \text{const.} + \frac{|\vec{p}|^2}{2m_o} \to \infty$$

In relativistic physics, if we consider the limit $m_o \to 0$, $\vec{p}$ fixed, we don't get nonsense:

$$E = \sqrt{|\vec{p}|^2 + m_o^2} \longrightarrow |\vec{p}|$$

$$|\vec{v}| = \frac{|\vec{p}|}{E} = 1$$

A particle of zero rest mass travels with the speed of light! [Of course, we can't parameterize the path of such a particle by proper time, so $\underline{u} = \frac{d\underline{r}}{ds}$ is ill-defined — but everything else is OK.] The neutrino seems to be such a particle.

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Max Planck's Mind-Blowing Discovery (1900)

The energy of a light pulse (of given frequency, ω) is not a continuous variable. Energy comes in integral multiples of a fundamental unit (the "quantum"), given by

$$E = \hbar\,\omega$$

Where $\hbar$ = (Planck's constant)/2π

$$\approx 1.05 \times 10^{-27} \text{ erg. sec}$$

This is a very small number in everyday terms. E.g. one quantum of red light is enough energy to lift 1g. 2.5×10^{-15} cm in earth's gravity.

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Another Zero-Rest-Mass Particle: The Photon

With every light pulse there is associated a 4-vector $\underline{k}$

$$\underline{k} \Leftrightarrow (\omega, \omega\vec{e})$$

$\vec{e}$: unit vector in direction of propagation

For minimal pulse,

$$\hbar\omega = E = p_t \quad \text{(time component of } \underline{p}\text{)}$$

∴, by Lorentz inv.

$$\hbar\,\underline{k} = \underline{p}$$

$$\hbar\omega\vec{e} = \vec{p}$$

momentum carried by light pulse

Note: Pulse acts like a rest-mass-zero particle, $E = |\vec{p}|$. This particle is called "the photon".

Pulses composed of N photons (with same ω, $\vec{e}$) also act like zero-rest-mass systems

$$E = N\hbar\omega, \quad \vec{p} = N\hbar\omega\vec{e}$$

15

Applications of Four-Momentum Conservation

(1) Two-body Decay

$$1 \longrightarrow 2 + 3$$

$$\underline{p}_1 = \underline{p}_2 + \underline{p}_3$$

Work in coord. system where initial particle is at rest:

$$\underline{p}_1 \Leftrightarrow (m_{01}, \vec{0})$$

$$\underline{p}_2 \Leftrightarrow \left(\sqrt{m_{02}^2 + |\vec{p}_2|^2},\ \vec{p}_2\right)$$

$$\underline{p}_3 \Leftrightarrow \left(\sqrt{m_{03}^2 + |\vec{p}_2|^2},\ -\vec{p}_2\right)$$

Note: m_{01} must be $\geq (m_{02} + m_{03})$

$$m_{01} = \sqrt{(m_{02})^2 + |\vec{p}_2|^2} + \sqrt{(m_{03})^2 + |\vec{p}_2|^2}$$

Can be solved for $|\vec{p}_2|$, $E_2 = \sqrt{(m_{02})^2 + |\vec{p}_2|^2}$, E_3, etc. However, we can avoid this tedious algebra with a little cunning:

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$$m_{01} E_2 = \underline{p}_1 \cdot \underline{p}_2$$

$$\underline{p}_1 - \underline{p}_2 = \underline{p}_3$$

$$(\underline{p}_1 - \underline{p}_2) \cdot (\underline{p}_1 - \underline{p}_2) = \underline{p}_1 \cdot \underline{p}_1 - 2\,\underline{p}_1 \cdot \underline{p}_2 + \underline{p}_2 \cdot \underline{p}_2$$
$$= \underline{p}_3 \cdot \underline{p}_3$$

But $\underline{p}_1 \cdot \underline{p}_1 = m_{01}^2$ etc.

$$2\,\underline{p}_1 \cdot \underline{p}_2 = 2 m_{01} E_2 = \underline{p}_1 \cdot \underline{p}_1 + \underline{p}_2 \cdot \underline{p}_2 - \underline{p}_3 \cdot \underline{p}_3$$
$$= (m_{01})^2 + (m_{02})^2 - (m_{03})^2$$

Likewise

$$2 m_{01} E_3 = (m_{01})^2 + (m_{03})^2 - (m_{02})^2$$

Numerical Example: π^- decay

$$\pi^- \to \mu^- + \nu$$

Rest mass of $\begin{Bmatrix} \pi^- \\ \mu^- \\ \nu \end{Bmatrix} = \begin{Bmatrix} 140 \\ 106 \\ 0 \end{Bmatrix}$ MeV.

[1 MeV = 1.6×10^{-6} erg]

$\therefore$ In coords. where π^- is at rest

$$2 m_{0\pi} E_\nu = (m_{0\pi})^2 - (m_{0\mu})^2$$

$$E_\nu = \frac{(m_{0\pi} + m_{0\mu})(m_{0\pi} - m_{0\mu})}{2 m_{0\pi}}$$

$$= \frac{(246)(34)}{280} \text{ MeV} \approx 30 \text{ MeV}$$

$$E_\mu = m_{0\pi} - E_\nu = 110 \text{ MeV}$$

$$|\vec{p}_\mu| = |\vec{p}_\nu| = E_\nu = 30 \text{ MeV}$$

$$|\vec{v}_\mu| = \frac{|\vec{p}_\mu|}{|E_\mu|} = \frac{30}{110} \approx 0.27$$

$$|\vec{v}_\nu| = 1$$

(2) Relativistic Rocket

Rocket at rest emits fuel particle carrying energy ΔE, momentum $\Delta\vec{p}$. $|\Delta\vec{p}| \leq |\Delta E| << M_0$, initial rest mass of rocket. What is final rest mass, M_0', and velocity, $\vec{v}$, of rocket?

Initial $E = M_0$ Initial $\vec{p} = 0$

Final $E = M_0 - \Delta E$ Final $\vec{p} = -\Delta\vec{p}$

$$M_0' = \sqrt{(M_0 - \Delta E)^2 - (\Delta\vec{p})^2} \approx M_0 - \Delta E$$

(neglecting terms of 2nd order in small quantities $\Delta E, \Delta\vec{p}$)

$$\vec{v} = \frac{-\Delta\vec{p}}{M_0 - \Delta E} \approx -\frac{\Delta\vec{p}}{M_0}$$

$\left|\frac{\Delta\vec{p}}{\Delta E}\right|$ = speed of fuel particle $\equiv U$

$$|\Delta\vec{v}| = -\Delta M_0 \, U/M_0$$

Assume this takes place in proper-time interval Δs

$$\left|\frac{\Delta\vec{v}}{\Delta s}\right| = -\frac{\Delta M_0}{\Delta s} U/M_0$$

But $\left|\frac{\Delta\vec{v}}{\Delta s}\right|$ in $\lim_{\Delta s \to 0}$ and in frame where $\vec{v} = 0$ is proper acceleration, a. $\left[a^2 = -\frac{du}{ds} \cdot \frac{du}{ds}\right]$

$$\therefore \boxed{a = -\frac{dM_0}{ds} \frac{U}{M_0}} \Longleftarrow \text{Relativistic Rocket Eq.}$$

If a = const. the soln to this Eq. is

$$M_0(s) = M_0(0)\, e^{-as/U}$$

Numerical Example: End of last lecture.

$a = 1$ lt yr./(yr)^2 $U = 1$ (most favorable case)

$s = 10$ yrs.

$$M_0(10) = M_0(0)\, e^{-10} = 4.5 \times 10^{-5}\, M_0(0)$$

$$M_0(0) = M_0(10) \times 2.2 \times 10^4$$

Discouragingly large payload/fuel ratio [Especially since the rocket has to decelerate for 10 yrs before it can refuel.]

For next week: read K+K 6.1–6.7.

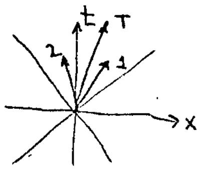
1

Applications of Four-Momentum Conservation (Continued)

(3) Two particles scattering (possibly inelastic)

$$1+2 \longrightarrow 3+4$$

$$\underline{p}_1+\underline{p}_2=\underline{p}_3+\underline{p}_4 \equiv \underbrace{\underline{p}_T}_{\text{total four-mom.}}$$



Choose coordinates such that time axis is aligned with $\underline{p}_T$:

$$\underline{p}_T \Leftrightarrow (E_T, \vec{0})$$

This is relativistic analog of going to center-of-mass frame.

$$\underline{p}_T=\underline{p}_1+\underline{p}_2 \qquad \underline{p}_T=\underline{p}_3+\underline{p}_4$$

Each of these Eqs. is identical to that encountered in decay of a particle at rest into two others. Thus, without further ado, we can simply transpose our analysis of application (1):

2

$$\underline{p}_1 \Leftrightarrow \left(\sqrt{(m_{01})^2+|\vec{p}_1|^2}, \vec{p}_1\right)$$

$$\underline{p}_2 \Leftrightarrow \left(\sqrt{(m_{02})^2+|\vec{p}_2|^2}, -\vec{p}_1\right)$$

$$E_T=\sqrt{(m_{01})^2+|\vec{p}_1|^2}+\sqrt{(m_{02})^2+|\vec{p}_1|^2}$$

fixes E_T in terms of $|\vec{p}_1|$ (or vice versa)

$$\underline{p}_3 \Leftrightarrow \left(\sqrt{(m_{03})^2+|\vec{p}_3|^2}, \vec{p}_3\right)$$

$$\underline{p}_4 \Leftrightarrow \left(\sqrt{(m_{04})^2+|\vec{p}_3|^2}, -\vec{p}_3\right)$$

$$E_T=\sqrt{(m_{03})^2+|\vec{p}_3|^2}+\sqrt{(m_{04})^2+|\vec{p}_3|^2}$$

fixes $|\vec{p}_3|$ in terms of E_T

[These can be solved as in (1), but I won't bother to do this explicitly here]

Further, we can always choose space axes so $\vec{p}_1$ lies along x-axis + $\vec{p}_3$ is in x-y plane. Only remaining variable is θ, the $\angle$ between $\vec{p}_1$ + $\vec{p}_3$

$$\vec{p}_1\cdot\vec{p}_3=|\vec{p}_1||\vec{p}_3|\cos\theta$$

θ is analog to non-relativistic center-of-mass scattering angle.

3

Specific Example:

$$e^- + e^+ \longrightarrow \gamma + \gamma$$

electron, positron, photons

$m_{0e^+}=m_{0e^-}=1$ (choice of mass scale)

$m_{0\gamma}=0$

$$\underline{p}_{e^-} \Leftrightarrow \left(\sqrt{p^2+1}, p\vec{i}\right)$$

$$\underline{p}_{e^+} \Leftrightarrow \left(\sqrt{p^2+1}, -p\vec{i}\right)$$

$$\underline{p}_{\gamma} \Leftrightarrow \left(\sqrt{p^2+1}, \sqrt{p^2+1}\,\vec{e}\right)$$

$$\underline{p}_{\gamma'} \Leftrightarrow \left(\sqrt{p^2+1}, -\sqrt{p^2+1}\,\vec{e}\right)$$

Where $\vec{e}$ is unit vector in x-y plane

$$\vec{e}\cdot\vec{i}=\cos\theta$$

This describes the process in our "standard" coord. system. Description in other coord. systems can be obtained by Lorentz transf. (or by invariant-dot-product tricks)

4

For example: A positron of energy E collides with an electron at rest; two photons emerge. What are the maximum + minimum allowed energies for one of the photons?

In coord. system where electron is at rest

$$\underline{p}_{e^-} \Leftrightarrow (1, \vec{0})$$

$$E=\underline{p}_{e^+}\cdot\underline{p}_{e^-}=p^2+1+p^2=2p^2+1$$

This Eq. gives "c-o-m" variable p in terms of given data, E,

$$p=\sqrt{\frac{E-1}{2}}$$

$$E_\gamma=\underline{p}_{e^-}\cdot\underline{p}_\gamma=p^2+1-p\sqrt{p^2+1}\cos\theta$$

$E_\pm$ obtained when $\cos\theta=\mp 1$

$$E_\pm=p^2+1\pm p\sqrt{p^2+1}$$

$$=\frac{E+1}{2}\pm\sqrt{\frac{(E-1)}{2}\frac{(E+1)}{2}}$$

$$E_\pm=\frac{1}{2}\left(E+1\pm\sqrt{E^2-1}\right) \Leftarrow \text{The answer}$$

Consistency Check: If $E=1$ (positron at rest)

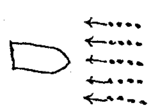
$$E_\pm=1 \qquad \text{OK}$$

5

(4) Relativistic Ramjet

Spaceship (Rest mass M_0) moves through region of space containing n particles of rest mass m_0 at rest per unit volume. Spaceship contains Little Wonder® engine — sweeps up all particles encountered in cross-sectional area A + converts them to particle of rest mass m_0' ($< m_0$). M_0 does not change. What is proper acceleration, a, if spaceship has velocity $v\vec{i}$?

Work in coord where ship is (momentarily) at rest:



partial view of dust particles (note Lorentz contraction)

Step 1: Assume ship acquires velocity $U\vec{i}$ ($\ll 1$) in transforming one particle. Then in interval Δs

$$|\Delta\vec{v}| = U \underbrace{A v \Delta s}_{\substack{\text{volume}\\ \text{swept}\\ \text{out}}} \underbrace{n/\sqrt{1-v^2}}_{\substack{\text{particles}\\ \text{per unit}\\ \text{volume}}}$$

6

$$a = \lim_{\Delta s \to 0} \left|\frac{\Delta\vec{v}}{\Delta s}\right| = UAvn/\sqrt{1-v^2}$$

Step 2: Compute U

Before: $\quad p_{\text{ship}} \Leftrightarrow (M_0, \vec{0})$

$$p_{\text{particle}} \Leftrightarrow \left(\frac{m_0}{\sqrt{1-v^2}}, -\frac{m_0}{\sqrt{1-v^2}}\, v\vec{i}\right)$$

After: $\quad p_{\text{ship}} \Leftrightarrow \left(\frac{M_0}{\sqrt{1-U^2}}, \frac{M_0}{\sqrt{1-U^2}}\, U\vec{i}\right)$

$$\approx (M_0, M_0 U\vec{i}) \qquad [U \ll 1]$$

$$p_{\text{particle}} \Leftrightarrow \left(\frac{m_0'}{\sqrt{1-(v')^2}}, \frac{-m_0'}{\sqrt{1-(v')^2}}\, v'\vec{i}\right)$$

Conservation:

$$\cancel{M_0} + \frac{m_0}{\sqrt{1-v^2}} = \cancel{M_0} + \frac{m_0'}{\sqrt{1-(v')^2}} \qquad \text{(fixes } v'\text{)}$$

$$-\frac{m_0 v}{\sqrt{1-v^2}} = M_0 U - \frac{m_0' v'}{\sqrt{1-(v')^2}} \qquad \text{(fixes } U\text{)}$$

7

After some trivial algebra (see Appendix I) we find

$$a = -\frac{Anv}{(1-v^2)M_0}\left[m_0 v - \sqrt{m_0^2 - m_0'^2 + m_0'^2 v^2}\,\right]$$

The only amusing thing about this ugly formula is that it approaches a finite limit as $v \to 1$. (see App. II)

$$\lim_{v\to 1} a = \frac{An}{2M_0}\left[m_0 - \frac{(m_0')^2}{m_0}\right]$$

After a while (in constant medium) ship develops constant proper acceleration!

(Maybe this is the engine of our twin-paradox ship)

8

External Forces

We consider particle under action of external force. For simplicity, we restrict ourselves to the case where the force does not change m_0.

$$\vec{p} = \frac{m_0 \vec{v}}{\sqrt{1-|\vec{v}|^2}} = m_0 \frac{d\vec{r}}{ds}$$

$$E = \frac{m_0}{\sqrt{1-|\vec{v}|^2}} = m_0 \frac{dt}{ds}$$

Just definitions of $\vec{p}$ + E (but our analysis of asymptotic conservation laws indicates that these might be interesting objects to study). (Warning: Do not confuse E with non-relativistic $E = K + V$ — it has no V term — is much more like K alone [plus a constant].)

Newton's Second Law:

$$\frac{d\vec{p}}{dt} = \vec{F}$$

Just a definition (as always).

Amusing result (Work-Energy Theorem):

$$E^2 = \vec{p}\cdot\vec{p} + (m_0)^2$$

$$2E\frac{dE}{dt} = 2\vec{p}\cdot\frac{d\vec{p}}{dt} = 2\vec{p}\cdot\vec{F}$$

$$\frac{dE}{dt} = \frac{\vec{p}}{E}\cdot\vec{F} = \vec{v}\cdot\vec{F}$$

$$\boxed{E(t_2)-E(t_1) = \int_{t_1}^{t_2} dt\, \frac{d\vec{r}}{dt}\cdot\vec{F}}$$

To get some concrete examples, we need some law of force. I will choose

$$\frac{d\vec{p}}{dt} = \vec{F} = \underset{\text{"electric charge"}}{q}\,(\underset{\text{"electric field"}}{\vec{E}(\vec{r})} + \vec{v}\times\underset{\text{"magnetic field"}}{\vec{B}(\vec{r})})$$

This happens to be the true force law for a particle in an external electric + magnetic field. (If you don't believe me, it's just something I made up to generate problems.)

Example 1: Constant Magnetic Field

$$\vec{E}=0 \qquad \vec{B}=B_0\vec{k} \qquad \vec{F}=q\,\vec{v}\times B_0\vec{k}$$

$$\frac{dE}{dt} = \vec{v}\cdot\vec{F} = 0$$

$$E = \frac{m_0}{\sqrt{1-|\vec{v}|^2}} \text{ is const.} \Rightarrow |\vec{v}| \text{ is const.}$$

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}\left(\frac{m_0}{\sqrt{1-|\vec{v}|^2}}\vec{v}\right) = qB_0\vec{v}\times\vec{k}$$

$$\frac{d\vec{v}}{dt} = \omega\,\vec{v}\times\vec{k} \qquad \left[\omega \equiv \frac{qB_0\sqrt{1-|\vec{v}|^2}}{m_0}\right]$$

$$\frac{dv_z}{dt} = 0 \Rightarrow v_z = v_z(0)$$

$$\frac{dv_x}{dt} = \omega v_y \qquad \frac{dv_y}{dt} = -\omega v_x$$

If we choose x-axis so $v_y(0)=0$,

$$v_x(t) = v_x(0)\cos\omega t \qquad v_y(t) = -v_x(0)\sin\omega t$$

If we choose center of coords. so $\vec{r}(0)=\vec{0}$,

$$z = v_z(0)t$$
$$x = +\frac{v_x(0)}{\omega}\sin\omega t$$
$$y = \frac{v_x(0)}{\omega}[\cos\omega t - 1]$$

Particle moves in helix. (Circle if $v_z(0)=0$)

Example 2: Constant Electric Field

$$\vec{B}=0 \qquad \vec{E}=E_0\vec{k} \qquad \vec{F}=qE_0\vec{k}$$

$$\frac{d\vec{p}}{dt} = m_0\frac{d\vec{r}}{ds} = qE_0\vec{k}$$

$$\frac{dp_{x,y}}{dt} = 0 \qquad p_x(t)=p_x(0) \qquad p_y(t)=p_y(0)$$

$$x(s) = \frac{p_x(0)}{m_0}s + x(0) \qquad y(s) = \frac{p_y(0)}{m_0}s + y(0)$$

Note: p_x is constant — not v_x

$$\frac{dp_z}{dt} = qE_0 \Rightarrow p_z = p_z(0) + qE_0 t$$

$$\frac{dp_z}{ds} = \frac{dp_z}{dt}\frac{dt}{ds} = \frac{dp_z}{dt}\frac{E}{m_0} = \frac{qE_0}{m_0}E$$

$$\frac{dE}{dt} = \vec{v}\cdot\vec{F} = \frac{dz}{dt}qE_0$$

$$\frac{dE}{ds} = \frac{dz}{ds}qE_0 = \frac{qE_0}{m_0}p_z$$

We could integrat from her but it' a mess I do a trick instead

Just like equations for v_x, v_y, except (−) sign is missing. Choose $s=0$ such that $p_z(s=0)=0$.

(13)

$$E = E(0)\cosh \frac{qE_0}{m_0}s \qquad P_z = E(0)\sinh \frac{qE_0}{m_0}s$$

$$E = m_0 \frac{dt}{ds} \qquad P_z = m_0 \frac{dz}{ds}$$

$$t = \frac{E(0)}{qE_0} \sinh\left(\frac{qE_0}{m_0}s\right) + t(0)$$

$$z = \frac{E(0)}{qE_0}\left[\cosh\left(\frac{qE_0}{m_0}s\right) - 1\right] + z(0)$$

Particle describes hyperbola in z-t plane.

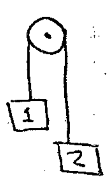
(14)

Some Comments About Relativity + Constant Gravitational Field

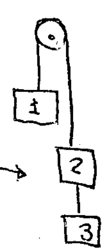
(1) What is weight?

Assume following properties of balance:

(1) With every sealed box we can associate a number, W, the weight, such that if $W_1 > W_2$, box 1 goes down and box 2 goes up →



(2) Weight is additive: Weight of two boxes is sum of individual weights. (This is same as single box of weight $W_2 + W_3$)



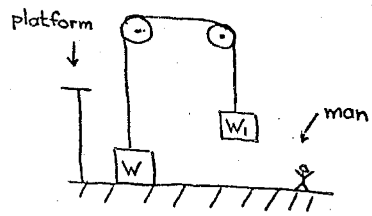
Note: Assumptions are plausible but <u>strong</u> — boxes can contain particles with relativistic velocities. (But so do real boxes — protons + neutrons inside a nucleus).

(15)

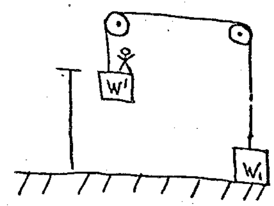
I will now argue that whatever weight is, it must be <u>conserved</u>. I.e., not possible to arrange apparatus inside box that changes W to W′ (W′≠W) and back again.

Assume the contrary. $W > W'$. Choose W_1 such that $W > W_1 > W'$.

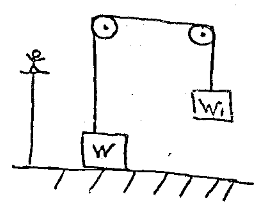
Three-stage perpetual motion machine:



Stage 1: Assume weight of man < $W_1 - W'$

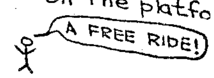


Stage 2: W changes to W′ and man rides to platform.



Stage 3: W′ changes back to W + man gets off.

We are back where we started <u>except</u> the man is on the platform!



(16)

Thus: (1) W must be conserved

(2) In non-relativistic limit,

$$W = mg$$

Only one candidate:

$$\frac{W}{g} = \text{relatistic mass} = \text{energy} \; !$$

(2) The gravitational red shift (A very sloppy argument)

Consider any object moving in the z-direction in earth's gravity

$$\frac{dE}{dt} = -g\frac{dz}{dt}E \Leftarrow \text{Work-energy theorem + Weight = energy}/g$$

$$\frac{dE}{dz} = -gE$$

$$E(z) = e^{-g(z-z_0)}E(z_0)$$

Let object be photon, $E = \hbar\omega$

$$\omega(z) = e^{-g(z-z_0)}\omega(z_0)$$

"Gravitational Red Shift"

~~This~~ [In unnatural units

$$\omega(z) = e^{-g(z-z_0)/c^2}\,\omega(z_0)\;]$$

This is <u>very</u> small effect — for example for $z - z_0 = 10\text{ m.} \approx 3.3\times10^{-8}$ lt. yr.

$g \approx 1$ lt.-yr./(yr)^2,

$$e^{-g(z-z_0)} \approx 1 - 3.3\times10^{-8}$$

Nevertheless, it has been measured — for $z - z_0 = 10$ m! — by Pound + Rebka. Theory + experiment agree to within experimental error [~10%].

Appendix I

$$\frac{m_0}{\sqrt{1-v^2}} = \frac{m_0'}{\sqrt{1-(v')^2}}$$

$$-M_0 U = \frac{m_0 v}{\sqrt{1-v^2}} - \frac{m_0' v'}{\sqrt{1-v'^2}}$$

$$= \frac{m_0}{\sqrt{1-v^2}}\,(v - v')$$

$$(1-(v')^2) = \frac{(m_0')^2}{(m_0)^2}\,(1-v^2)$$

$$v' = \sqrt{1-\left(\frac{m_0'}{m_0}\right)^2(1-v^2)}$$

$$U = -\frac{1}{M_0}\,\frac{m_0}{\sqrt{1-v^2}}\left[v - \sqrt{1-\left(\frac{m_0'}{m_0}\right)^2(1-v^2)}\right)$$

$$= -\frac{1}{M_0}\,\frac{1}{\sqrt{1-v^2}}\left[m_0 v - \sqrt{m_0^2 - m_0'^2 + m_0'^2 v^2}\right]$$

Appendix 2

Problem is to compute

$$\lim_{v\to1} \frac{m_0 v - \sqrt{m_0^2 - m_0'^2 + m_0'^2 v^2}}{1-v^2}$$

Both numerator + denominator vanish at $v=1$. Use L'Hopital's rule:

$$\frac{d}{dv}(1-v^2)\Big|_{v=1} = -2$$

$$\frac{d}{dv}\left(m_0 v - \sqrt{m_0^2 - m_0'^2 + m_0'^2 v^2}\right)\Big|_{v=1}$$

$$= m_0 - \frac{(m_0')^2}{m_0}$$

Thus, desired limit is

$$-\frac{1}{2}\left[m_0 - \frac{(m_0')^2}{m_0}\right]$$

HOMEWORK COMMENTARY

9-H1 $\underline{P} = \underline{P}_1 + \underline{P}_2$ ← Known from info. given $M^2 = \underline{P}\cdot\underline{P}$

Check: For small E (with small momentum transfer) the pirate's rest mass should increase by approximately the Doppler shifted energy of the light pulse in his frame.

9-H2 $E = mc^2$ etc.

9-H3 Decay, as in lecture.

9-H4 $F = \dfrac{\Delta P}{\Delta t}$, $\Delta E = F\Delta x$ for constant force
WORK-ENERGY THEOREM

9-H5 $\vec{P}_\gamma = -\vec{P}_e$, then find E_γ ($\gamma \equiv$ photon) To transform to the zero-momentum frame there are two ways:

1) Lorentz transform $(E, \vec{P})$ just like $(t, \vec{x})$

2) You desire a 'frame where $\underline{P}_e'^{in} = (m_0, \vec{0})$. Then for any four-vector $\underline{P}' = (E', \vec{P}')$, $\underline{P}'\cdot\underline{P}_e'^{in} = m_0 E'$. But by Lorentz invariance you know $\underline{P}'\cdot\underline{P}_e'^{in} = \underline{P}\cdot\underline{P}_e^{in}$ which you can caculate from the info in the z-m (unprimed) fr
Once you have E' you can get $\vec{P}'$.

9-H6 For the photon-electron collision it is <u>much</u> easie
to transform the "in" conditions to the z-m frame, switch directions of particles, and transform back t
the lab frame. (As in 9-H5)

9-H7 $M_{N^{*0}} \cong 1514$ (MeV), E_γ required $\cong 749$ (MeV)
Remember that the N^* is at rest in the z-m frame.

9-H8 The energy flux varies as $1/R^2$ (same total energy over $4\pi R^2$ are
Momentum flux follows from $E = Pc$ for photons.

Erratum

I screwed up the computation at the end of last lecture: (1) I confused lt.-yrs. + lt.-sec. (Dumb!) Gravitational red shift is ≈ one part in 10^{16} per meter. (2) Pound + Rebka used a 22 meter tower. (3) They obtained 1% experimental error. [All in all, an even more amazing experiment then I said it was.]

Notice

There will be only __11__ units in the course. Lectures on these units will be complete before the holidays, and sections on unit 11 will be held the first week of reading period. (Because there are fewer units than planned, the grading formula will be based on 180 points, with the expected levels scaled accordingly.)

→ Tests 1–5 must be completed before the holidays. Copies of those tests with solutions will be on reserve in the S.C. Library during reading period, for use in the library only. __Do not copy the tests__.

Tests 6–11 will still be available during the first two weeks of reading period. Then they will also be placed on reserve.

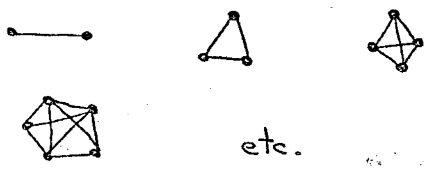
Information about the final exam, Special meetings during reading period, review sections, and unit test hours will be announced Thursday Dec 16 and posted outside Physics Focus.

The final exam will be $\boxed{\text{Thursday Jan 27}}$

①

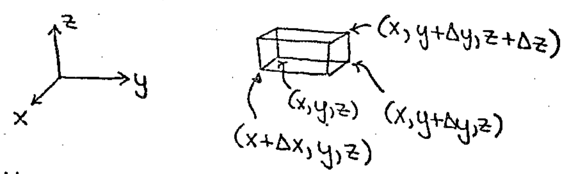
The idea of a rigid body

A rigid body is an assembly of N point particles with forces of constraint acting between them such that the distance between *any* two particles is fixed.



etc.

Sometimes we will replace the discrete particles by a continuous distribution of mass, $\mu(\vec{r})$



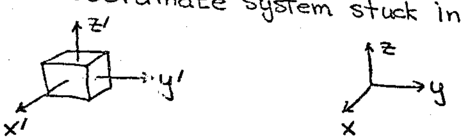
Mass contained in small box is $\mu(x,y,z)\,\Delta x\,\Delta y\,\Delta z$

($\mu=0$ outside the body.)

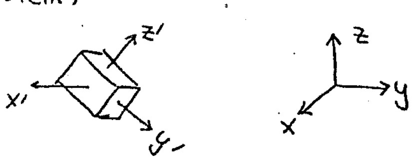
②

We could describe the position of a rigid body by giving the position of every particle in the body, $\vec{r}_a$, $a=1\cdots N$. This is horribly overcomplete.

Here is a better way: Imagine a coordinate system stuck in the body



As the body moves, this coordinate system moves with it. (It is called "body-fixed" system)



Since the coordinates of a particle in the body are always the same in the body-fixed system, to describe the position of a particle, we need only to describe the body-fixed system.

③

This requires 6 numbers.

3 to tell where the center of coord. is

2 to " the orientation of the z' axis

1 " " " " " " y' axis

$$3+2+1=6$$

Another way of saying the same thing:

$$\underset{\substack{\uparrow\\ \text{position of}\\ a\text{th particle}\\ \text{in space-}\\ \text{fixed coord}}}{\vec{r}_a} = \underset{\substack{\uparrow\\ \text{some}\\ \text{vector}}}{\vec{a}} + \underset{\substack{\uparrow\\ \text{some}\\ \text{rotation}}}{\underset{\sim}{R}}\ \vec{r}\,'_a \longleftarrow \begin{array}{l}\text{position in}\\ \text{body-fixed}\\ \text{coord.}\\ \text{(or, equivalently,}\\ \text{position of}\\ \text{particle when}\\ \text{two coord. systems}\\ \text{coincide)}\end{array}$$

6 numbers:

3 for the translation $\vec{a}$

3 for the rotation $\underset{\sim}{R}$

(2 for the axis of rotation

1 for " $\angle$ " ")

④

To keep track of 6 numbers we need 6 equations of motion: We have them:

3 for the components of total momentum, $\vec{P}$,

3 " " " " " angular momentum, $\vec{L}$.

Total Momentum Reviewed:

$$\vec{P} = \sum_a \vec{p}_a = \sum_a m_a \frac{d\vec{r}_a}{dt} = M\frac{d\vec{R}}{dt}$$

where $M=\sum_a m_a \qquad \vec{R}=\frac{1}{M}\sum_a m_a\vec{r}_a$

$$\boxed{\frac{d\vec{P}}{dt} = \sum_a \vec{F}_a^{\,ext} \equiv \vec{F}^{\,ext}} \Longleftarrow \text{3 Eqs. of motion}$$

Angular Momentum Reviewed:

$$\vec{L} = \sum_a \vec{r}_a\times\vec{p}_a = \sum_a \vec{r}_a\times m_a\frac{d\vec{r}_a}{dt}$$

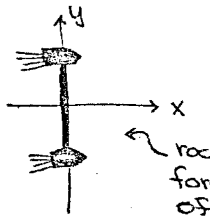
$$\boxed{\frac{d\vec{L}}{dt} = \sum_a \vec{r}_a\times\vec{F}_a^{\,ext} \equiv \underbrace{\vec{\tau}^{\,ext}}_{\text{"external torque"}}} \Longleftarrow \text{3 Eqs. of motion}$$

Some facts about $\vec{L}$:

(1) $\vec{L}$ is a vector – there are three components to keep track of. (Trivial, but easy to forget.)

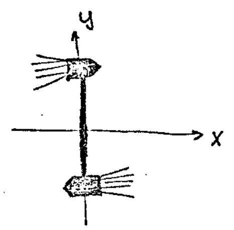
(2) $\vec{L}$ depends on the origin of coordinates. (Ditto)

(3) $\vec{L}$ – Eq. + $\vec{P}$ – Eq. are independent for any system more complicated than a single particle



$$\vec{F}^{ext} = \dot{\vec{P}} \neq \vec{0}$$
$$\vec{\tau}^{ext} = \dot{\vec{L}} = \vec{0}$$

rockets apply equal forces to two ends of rod



$$\vec{F}^{ext} = \dot{\vec{P}} = \vec{0}$$
$$\vec{\tau}^{ext} = \dot{\vec{L}} \neq \vec{0}$$

(4) A sometimes-useful formula

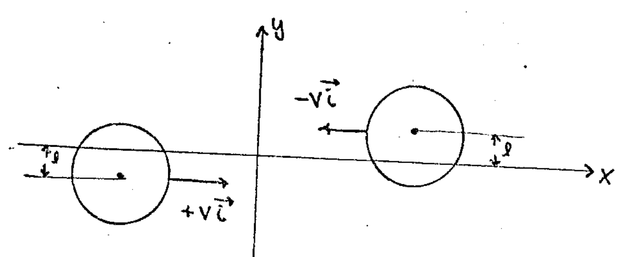
$$\vec{L} = \vec{R} \times \vec{P} \quad + \quad \sum_a m_a (\vec{r}_a - \vec{R}) \times \frac{d}{dt}(\vec{r}_a - \vec{R})$$

"$\vec{L}$ of center-of-mass *about center of coords." "$\vec{L}$ about center of mass"

[Proved in early lecture – proof reviewed in Appendix]

This formula is useful in computing $\vec{L}$.

Example: Two uniform discs of equal mass m collide in c-o-m system. After collision, they stick together. What happens?



Momentum Balance:

Before: $\vec{P} = mv\vec{i} - mv\vec{i} = \vec{0}$

After: $\vec{P} = \vec{0}$

C-o-m stays fixed at $x = y = 0$ (no surprise)

Angular Momentum Balance:

Before: Add up two discs independently

For left disc: $\vec{L}$ about c-o-m of left disc $= \vec{0}$

$\vec{R} \times \vec{P} = (x\vec{i} - \ell\vec{j}) \times mv\vec{i} = m\ell v\vec{k}$

of left disc only

For right disc, likewise:

$\vec{R} \times \vec{P} = (x'\vec{i} + \ell\vec{j}) \times (-mv\vec{i}) = m\ell v\vec{k}$

of right disc only

$\vec{L} = 2m\ell v\vec{k}$

After: $\vec{L} = 2m\ell v\vec{k}$

Combined system must rotate. (Shortly we will develop machinery to compute how fast.)

(Demonstrate)

Keeping track of all three components of $\vec{L}$ is complicated. To make life simple we will concentrate (for this week) on rigid bodies that are constrained (by a pivot or axle, for example) to always rotate about some fixed axis. By convention we will take this to be the z-axis.

Important fact: The pivot (or axle) exerts zero τ_z on the body.

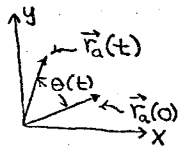
$$\vec{\tau} = \sum_a \vec{r}_a \times \vec{F}_a$$
$$\tau_z = \sum_a (x_a F_{ya} - y_a F_{xa})$$

but the pivot is at $x = y = 0$ (the z-axis).

Note: The pivot can contribute to τ_x, τ_y, and all three components of $\vec{F}$. It can be neglected only in the L_z equation. But, since we need only keep track of one angle, all we need is one equation.

Rotation about a fixed axis
(Mainly a review)

$$x_a(t) = x_a(0)\cos\theta(t) - y_a(0)\sin\theta(t)$$
$$y_a(t) = y_a(0)\cos\theta(t) + x_a(0)\sin\theta(t)$$
$$z_a(t) = z_a(0)$$



$$\dot{x}_a(t) = \dot{\theta}(t)\,[x_a(0)(-\sin\theta) - y_a(0)\cos\theta]$$
$$= \dot{\theta}(t)\,[-y_a(t)]$$
$$\dot{y}_a(t) = \dot{\theta}(t)\,x_a(t)$$
$$\dot{z}_a(t) = 0$$

[Side remark: This is the vector Eq.
$$\frac{d\vec{r}_a}{dt} = \dot{\theta}(t)\,\vec{k}\times\vec{r}_a$$
written out in components.]

$$L_z = \sum_a m_a[x_a\dot{y}_a - y_a\dot{x}_a]$$
$$= \sum_a m_a[(x_a)^2 + (y_a)^2]\,\dot{\theta}$$
← note: independent of $\theta(t)$

[Side remark: For a single particle with $z=0$, this is our old polar coord. result $L_z = mr^2\dot{\theta}$.]

Define
$$I \equiv \sum_a m_a[(x_a)^2 + (y_a)^2]$$
"Moment of Inertia" (about z-axis)

$$\boxed{L_z = I\dot{\theta}}$$

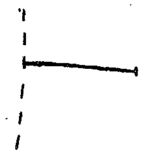
For a continuous distribution of matter
$$I = \iiint dx\,dy\,dz\,\mu(x,y,z)\,(x^2+y^2)$$
[$\int$'s run over volume occupied by body]

For rotation about a general fixed axis
$$I = \sum_a m_a(\rho_a)^2$$
where ρ_a is ~~a~~ perpendicular distance to axis from particle a.

(Remedial Demonstration)

Moments of Inertia of Some Simple Bodies

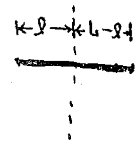
(1) Rigid thin uniform rod, length L, mass M; axis $\perp$ to rod, passing through one end



$$I = \int_0^L \underset{\substack{\uparrow \\ \text{distance of small segment squared}}}{x^2}\ \underbrace{\frac{M}{L}dx}_{\substack{\uparrow \\ \text{mass of small segment}}}$$

$$I = \frac{M}{L}\,\frac{L^3}{3} = \frac{1}{3}ML^2$$

(2) Same, with axis passing through rod a distance ℓ from one end $(\ell \leq L)$



$$I = \frac{1}{3}M\frac{\ell}{L}\ell^2 \leftarrow \text{from left side}$$
$$+\frac{1}{3}M\frac{(L-\ell)}{L}(L-\ell)^2 \leftarrow \text{from right side}$$
$$I = \frac{1}{3}\frac{M}{L}[\ell^3 + (L-\ell)^3]$$

Digression (Parallel axis theorem)

$$I = \sum_a m_a(x_a^2 + y_a^2)$$

Let I_0 be the moment of inertia about an axis $\parallel$ to the z axis and passing through the center of mass

$$I_0 = \sum_a m_a[(x_a - X)^2 + (y_a - Y)^2]$$
$$= I - 2X\sum_a m_a x_a - 2Y\sum_a m_a y_a + \sum_a m_a(X^2 + Y^2)$$

But $\sum_a m_a = M$ $\quad \sum_a m_a x_a = XM$ $\quad \sum_a m_a y_a = YM$

$$I_0 = I - 2MX^2 - 2MY^2 + MX^2 + MY^2$$

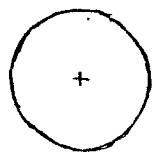
$$\boxed{I = I_0 + MX^2 + MY^2} \Leftarrow \text{very useful}$$

Check for rod (1): $I_0 = \frac{1}{3}\frac{M}{L}\left[\left(\frac{L}{2}\right)^3 + \left(\frac{L}{2}\right)^3\right] = \frac{M}{12}$

$M(X^2+Y^2) = \frac{1}{4}ML^2$

$$ML^2\left(\frac{1}{4} + \frac{1}{12}\right) = ML^2\left(\frac{3}{12} + \frac{1}{12}\right) = \frac{1}{3}ML^2 \quad \underline{\text{OK}}$$

(3) Thin uniform hoop, mass M, radius R; axis $\perp$ to plane of hoop, passing through center



$$I = MR^2$$

(all pts. at same x^2+y^2)

(4) Thin uniform circular disc, radius R; axis as before.

Break it up into concentric hoops of thickness dr, mass $\frac{M}{\pi R^2} 2\pi r\,dr$

$$I = \int_0^R \frac{2M}{R^2} r\,dr\, r^2$$

$$= \frac{2M}{R^2} \frac{R^4}{4} = \frac{M}{2} R^2$$

(This + $\parallel$ axis theorem is what you need for Homework problem ①)

(5) Thin uniform spherical shell, mass M, radius R; axis passing through center

$$I = \sum_a m_a (x_a^2 + y_a^2)$$

$$= \sum_a m_a (y_a^2 + z_a^2)$$

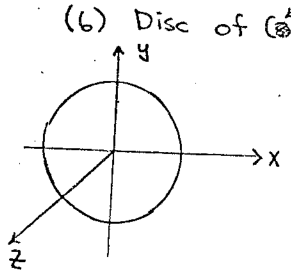
$$= \sum_a m_a (x_a^2 + z_a^2)$$

$$3I = \sum_a m_a (2x_a^2 + 2y_a^2 + 2z_a^2)$$

$$= 2MR^2$$

$$\therefore\ I = \frac{2}{3} MR^2 \qquad \text{(cute trick)}$$

(6) Disc of (4); axis a diameter of disc



$$I = \sum_a m_a (x_a^2 + z_a^2)$$

$$= \sum_a m_a (x_a^2)$$

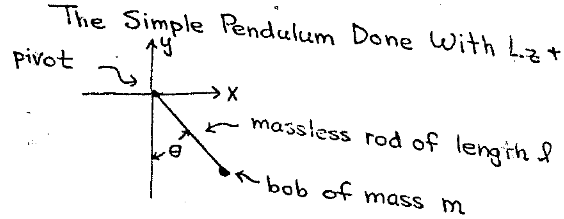
$$= \sum_a m_a (y_a^2)$$

$$2I = \sum_a m_a [(x_a)^2 + (y_a)^2]$$

$$= \frac{M}{2} R^2$$

$$\therefore\ I = \frac{M}{4} R^2 \qquad \text{(Same cute trick)}$$

The Simple Pendulum Done With $L_z + \tau_z$



I (about z-axis) $= m\ell^2$

$\vec{F}$ (of gravity on bob) $= -mg\,\vec{j}$

$$L_z = I\dot{\theta} = m\ell^2\dot{\theta}$$

$$\tau_z = xF_y - yF_x = -x\,mg = -\ell\sin\theta\, mg$$

Note: No need to consider force exerted by pivot (small but real advantage over our first $-\vec{F} = m\vec{a}-$ treatment).

$$\frac{d}{dt} L_z = \tau_z$$

$$m\ell^2\ddot{\theta} = -mg\ell\sin\theta \qquad \text{[same Eq. as before — of course]}$$

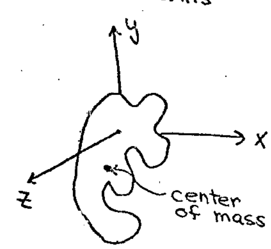
Linear approx: $\sin\theta \approx \theta$

$$\ddot{\theta} = -\frac{g}{\ell}\theta$$

Simple harmonic motion with $\omega = \sqrt{\frac{g}{\ell}}$

The Physical Pendulum

Arbitrary rigid body free to swing about the z-axis



Define θ to be angle between $\vec{R}$ (position of c.o.m.) and y-axis

$$X = \ell\sin\theta$$

where ℓ is $\perp$ distance from c-o-m to z-axis

$$\tau_z = \sum_a x_a F_{ay} - y_a F_{ax} = \sum_a -x_a m_a g$$

$$= -gMX = -gM\ell\sin\theta$$

$$L_z = I\dot{\theta}$$

$$\frac{dL_z}{dt} = I\ddot{\theta} = \tau_z = -gM\ell\sin\theta$$

Linear approx: $\sin\theta \approx \theta$

$$\ddot{\theta} = -\frac{gM\ell}{I}\theta$$

Simple Harmonic Motion with $\omega = \sqrt{\frac{gM\ell}{I}}$

(17)

Of course, I may be incredibly difficult to compute. However by ∥ axis theorem, we know I for any placement of the pivot if we know $I_0 = I$ when pivot passes through c-o-m.

$$I = I_0 + M\ell^2 \qquad \text{(see (12))}$$

Define k ("radius of gyration") by

$$\sqrt{I_0/M} = k$$

(For example, for uniform rod, $k = \sqrt{\tfrac{1}{12}}\, L$)

$$I = M(k^2+\ell^2)$$

$$\omega = \sqrt{\frac{g\ell}{k^2+\ell^2}}$$

(For example, for uniform rod of length L,

$$\omega = \sqrt{\frac{g\ell}{(L^2/12)+\ell^2}}$$

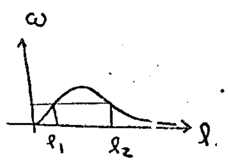
where ℓ is distance from center of rod to pivot point.)

(Demonstrate)

(18)

Kater's Trick

$$\omega = \sqrt{\frac{g\ell}{k^2+\ell^2}}$$



Note there are two values of ℓ, $\ell_1 + \ell_2$, that give the same ω. If we know ℓ_1 and ℓ_2, (experimentally) we can solve for k

$$\frac{g\ell_1}{k^2+\ell_1^2} = \frac{g\ell_2}{k^2+\ell_2^2}$$

$$\cancel{g}(k^2+\ell_2^2)\ell_1 = \cancel{g}(k^2+\ell_1^2)\ell_2$$

$$k^2(\ell_1-\ell_2) = \ell_1\ell_2(\ell_1-\ell_2)$$

$$k^2 = \ell_1\ell_2$$

$$\omega^2 = \frac{g\ell_1}{\ell_1\ell_2+\ell_1^2} = \frac{g}{\ell_1+\ell_2}$$

Kater's trick: (1) Pick ℓ_1 at random (2) Measure ω
(3) Find ℓ_2 [on the other side of the c-o-m].
(4) Measure $\ell_1+\ell_2$ (distance between pivot pts.)
Gives very "clean" measurement of g. (Non-uniformity of rod + precise location of c-o-m no problem.)

(19)

Appendix

Preliminary identities

$$\sum_a m_a \vec{r}_a = M\vec{R} = \sum_a m_a \vec{R}$$

$$\therefore \quad \sum_a m_a(\vec{r}_a - \vec{R}) = \vec{0}$$

$$\frac{d}{dt}\vec{0} = \vec{0} \Rightarrow \sum_a m_a \frac{d}{dt}(\vec{r}_a - \vec{R}) = \vec{0}$$

Now,

$$\vec{L} = \sum_a m_a \vec{r}_a \times \frac{d}{dt}\vec{r}_a$$

$$= \sum_a m_a[(\vec{r}_a-\vec{R})+\vec{R}] \times \frac{d}{dt}[(\vec{r}_a-\vec{R})+\vec{R}]$$

$$= \sum_a m_a(\vec{r}_a-\vec{R}) \times \frac{d}{dt}(\vec{r}_a-\vec{R})$$

$$+ \sum_a m_a \vec{R} \times \frac{d}{dt}(\vec{r}_a-\vec{R})$$

$$+ \sum_a m_a(\vec{r}_a-\vec{R}) \times \frac{d\vec{R}}{dt}$$

$$+ \sum_a m_a \vec{R} \times \frac{d\vec{R}}{dt}$$

$=\vec{0}$ by identity

QED

For next week: (1) Read K+K, Chapter 7. (2) Review the diagonalization theorem for symmetric linear operators in the notes on coupled oscillators; we will use it in Tuesday's lecture.

Fixed-Axis Rotation (Con't.) (1)

Angular Velocity + Linear Momentum

$$\left.\begin{array}{l}\dot{x}_a = -\dot{\theta}\, y_a \\ \dot{y}_a = \dot{\theta}\, x_a \\ \dot{z}_a = 0\end{array}\right\} \text{rotation about } z\text{-axis}$$

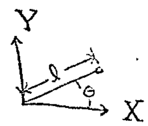
$$P_x = \sum_a m_a \dot{x}_a = -\dot{\theta} \sum_a m_a y_a = -\dot{\theta} M Y$$

y-coord. of c.o.m. → Y

Likewise

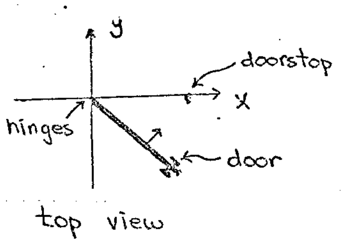
$$P_y = +\dot{\theta} M X \qquad P_z = 0$$

In terms of ℓ, the $\perp$ distance from z-axis to c.o.m.,



$$\boxed{P_x = -M\dot{\theta}\ell \sin\theta \qquad P_y = M\dot{\theta}\ell\cos\theta \qquad P_z = 0}$$

Application: Door, Doorstop, + Hinges (2)



The door swings about the z-axis until it hits the doorstop at $x = x_0, y = 0$. The stop exerts a force on the door $= F(t)\vec{j}$. What is $\vec{F}'(t)$, the force the pivot (hinges) exerts on the door?

Notation: Door has mass M, moment of inertia I, center of mass distance ℓ from z-axis [For uniform door of width L, $I = \frac{1}{3}ML^2$, $\ell = \frac{1}{2}L$]

Force is exerted only when θ is very close to zero

$$\therefore \quad P_x \approx 0 \qquad P_y \approx M\dot{\theta}\ell = \frac{M L_z \ell}{I} \qquad [L_z = I\dot{\theta}]$$

$$\frac{d\vec{P}}{dt} = F(t)\vec{j} + \vec{F}'(t) = \vec{j}\,\frac{M\ell}{I}\frac{dL_z}{dt}$$
$$= \vec{j}\,\frac{M\ell}{I}\tau_z$$

(3)

$$\therefore \quad \vec{F}'(t) = \vec{j}\left[\frac{M\ell}{I}\tau_z - F(t)\right]$$

In general, $\tau_z = xF_y - yF_x$

In case at hand, $\tau_z = x_0 F(t)$

[Note: No contribution from pivot.]

$$\therefore \quad \vec{F}'(t) = \vec{j}\left[\frac{M\ell x_0}{I} - 1\right]F(t)$$

$\vec{F}'(t) = 0$ if and only if

$$x_0 = \frac{I}{M\ell} \qquad \left[= \tfrac{2}{3}L \text{ for uniform door}\right]$$

Good place to put stop if you want to minimize force on hinges $[= -\vec{F}']$.

Same physics applies to tennis racquet hitting ball – hit at wrong place + you sprain your wrist.

(4)

Angular Velocity + Kinetic Energy

$$\dot{x}_a = -y_a\dot{\theta} \qquad \dot{y}_a = x_a\dot{\theta} \qquad \dot{z}_a = 0$$

$$K = \frac{1}{2}\sum_a m_a(\dot{x}_a^2 + \dot{y}_a^2 + \dot{z}_a^2)$$
$$= \frac{1}{2}\dot{\theta}^2 \sum_a m_a(y_a^2 + x_a^2 + 0)$$

$$\boxed{K = \frac{1}{2}I\dot{\theta}^2}$$

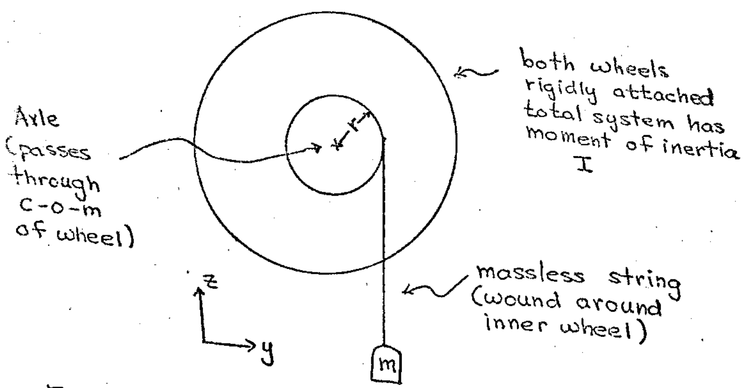
Analogies

Particle Constrained to Move along z-axis	Rigid Body Constrained to Rotate about z-axis
$P_z = m\dot{z}$	$L_z = I\dot{\theta}$
$\dot{P}_z = F_z$	$\dot{L}_z = \tau_z$
(constraint gives no F_z)	(constraint gives no τ_z)
$K = \frac{1}{2}m\dot{z}^2$	$K = \frac{1}{2}I\dot{\theta}^2$
$= \frac{1}{2m}P_z^2$	$= \frac{1}{2I}L_z^2$
z, P_z, m	θ, L_z, I

(Demonstrat...)

5

Application of E conservation: Wheel + Weight



Everything is at rest when z (of weight) is z_0.
What is angular velocity of wheel for general z?

$$E = \underbrace{\tfrac{1}{2} I \dot\theta^2}_{K \text{ for wheel}} + \underbrace{\tfrac{1}{2} m \dot z^2}_{K \text{ for weight}} + \underbrace{mgz}_{\text{gravitational potential energy of weight}}$$

Note: No need to include potential energy of wheel — c.o.m. does not move.

6

Constraint: $\underbrace{r\dot\theta}_{\text{rate wheel takes up string}} = \underbrace{+\dot z}_{\text{rate weight rises}}$

$$E = \tfrac{1}{2}(I + mr^2)\dot\theta^2 + mgz$$

$$\underbrace{mgz_0}_{\text{initial } E} = \underbrace{\tfrac{1}{2}(I+mr^2)\dot\theta^2 + mgz}_{\text{final } E} \qquad \text{(Energy Cons.)}$$

$$\boxed{\dot\theta = \sqrt{\frac{2mg(z_0 - z)}{I + mr^2}}} \Leftarrow \text{the answer}$$

For demonstration apparatus

$$mr^2 \ll I = \left(\begin{array}{c}\text{some number}\\ \text{depending on}\\ \text{mass distr.}\end{array}\right) \times \underbrace{M}_{\text{mass of wheel}} \times \underbrace{R^2}_{\text{radius of wheel}}$$

$$\therefore \quad \dot\theta \approx \sqrt{\frac{2mg(z_0 - z)}{I}}$$

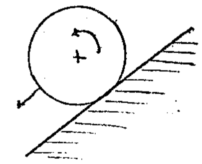
(Demonstrate)

7

Motion where Axis of Rotation has Fixed Direction (But not Fixed Location)

We now consider more general motions, in which a rigid body rotates about an axis fixed in the body, and always ∥ to the z-axis, but the body can otherwise move.

E.g. a drum rolling on an inclined plane:



In equations, we consider motions of the form

$$x_a(t) = x_a(0)\cos\theta(t) - y_a(0)\sin\theta(t) + A_x(t)$$
$$y_a(t) = x_a(0)\sin\theta(t) + y_a(0)\cos\theta(t) + A_y(t)$$
$$z_a(t) = z_a(0) + A_z(t)$$

where $\vec A$ is some vector.

8

This is hardly more complicated than fixed-axis rotation. The relevant Eqs. of motion are

$$\boxed{\begin{array}{l}\dot{\vec P} = M\ddot{\vec R} = \vec F^{ext}\\ \dot L_z = \tau_z^{ext}\end{array}}$$

Our task is to compute L_z in terms of $\vec R$ and θ. From last lecture,

$$\vec L = \vec R \times \vec P + \sum_a m_a(\vec r_a - \vec R) \times \frac{d}{dt}(\vec r_a - \vec R)$$

Also,

$$\vec R = \frac{1}{M}\sum_a m_a \vec r_a = X\vec i + Y\vec j + Z\vec k$$

$$\therefore \quad X(t) = X(0)\cos\theta(t) - Y(0)\sin\theta(t) + A_x(t)$$

etc. for Y and Z

$$\therefore \quad [x_a - X](t) = [x_a - X](0)\cos\theta(t) - [y_a - Y(0)]\sin\theta(t)$$

etc. for other coord.

$\vec{A}$ has disappeared! $\vec{r}^a - \vec{R}$ obeys the <u>same</u> Eq. $\vec{r}^a$ did for fixed-axis rotation. (Demonstrate). Thus, without further ado,

$$\vec{k}\cdot\sum_a m_a(\vec{r}_a-\vec{R})\times\frac{d}{dt}(\vec{r}_a-\vec{R}) = I_0\dot{\theta}$$

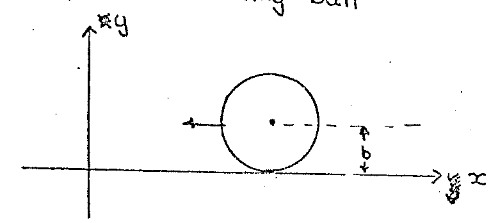
moment of inertia about c.o.m (<u>not</u> axis of rotation)

$$L_z = \vec{k}\cdot(\vec{R}\times\vec{P}) + I_0\dot{\theta}$$

$$\boxed{L_z = X P_y - Y P_x + I_0\dot{\theta}}$$

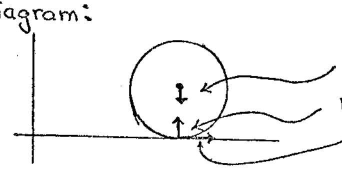
Note: This is L_z about <u>any</u> origin <u>fixed</u> in space

Example 1: Bowling Ball



Uniform spherical bowling bowl is released with velocity $V(0)\vec{i}$ and no rotation at $t=0$. Eventually friction between ball + alley (x-axis) causes ball to roll without slipping. What is $\vec{v}$ when this happens?

Force diagram:



(Note that in computing torque, gravity acts as if it were applied at c.o.m. [See physical pendulum discussion of Tuesday])

Constraint: $Y=b$, $0=M\ddot{Y}=F_y^{ext}=N-Mg$

$\therefore$ $N=Mg$

Computation of $\tau_z = xF_y - yF_x$

$$\tau_z = X(-Mg) + XN - 0f = 0$$

[Note: Critical that origin of coord. lies on alley.]

$\therefore$ L_z is constant!

$$L_z = \underset{\overset{\shortparallel}{0}}{X}P_y - YP_x + I_0\dot{\theta} = -bMV_x + I_0\dot{\theta}$$

Initially $L_z = -bMV_x(0)$

Finally, rolls without slipping $\Rightarrow$ $V_x + b\dot{\theta} = 0$

$$-bMV_x(0) = -bMV_x - I_0 V_x/b$$

$$V_x = \frac{V_x(0)}{1+(I_0/b^2M)}$$

For uniform sphere, $I_0 = \frac{2}{5}b^2M$

$\therefore$ $V_x = \frac{5}{7}V_x(0)$

[If ball were replaced by thin hoop, $I_0 = b^2M$ $\Rightarrow$ $V_x = \frac{1}{2}V_x(0)$]

Digression: On a possibly confusing point:

Friction plays an important role in insuring that the ball rolls without slipping. However, in contrast to all previous cases of friction + moving bodies, here friction does <u>zero</u> work on the body. (while it is rolling without slipping)

Two explanations:

(1) Friction serves only to enforce a constraint:

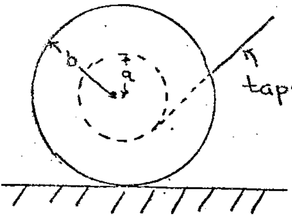
$$V_x + b\dot{\theta} = 0$$

Constraint forces to zero work.

(2) $\frac{dW}{dt} = \vec{v}\cdot\vec{F}$ (Work-energy theorem)

but friction is applied at the point of contact of ball + alley, which has $\vec{v}=0$.

Example 2: Spool + Tape



Spool is at rest + tape is wound around inner cylinder. Force is applied to tape. In what direction does c-o-m of spool accelerate?

Assume: (1) c-o-m on axis of rotation (2) friction insures spool rolls without slipping.

If origin of coord. is placed on tabletop, L_z is computed just as for bowling ball rolling without slipping

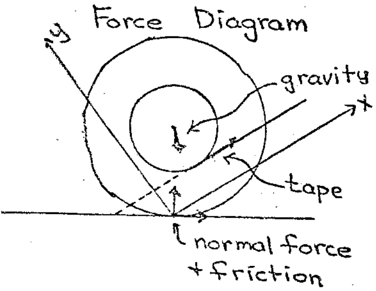
$$L_z = -\left(bM + \frac{I_0}{b}\right) V_x$$

$$\dot{L}_z = \tau_z$$

Thus if τ_z is $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$ the c-o-m accelerates to the $\left\{\begin{matrix}\text{left}\\ \text{right}\end{matrix}\right\}$.

Computation of τ_z is facilitated by placing origin of coords directly under c-o-m, and tilting x+y coords. (Rotation about z-axis does not affect τ_z)

Force Diagram



All forces except tape force give zero τ_z!
(Normal force + friction are at $x=y=0$; for gravity $\vec{r}$ is $\parallel$ to $\vec{F}$ — $\vec{r}\times\vec{F}=0$)

$\vec{F}$ tape = (positive number) $\vec{i}$

$$\tau_z = x F_y - y F_x$$

τ_z is $\left\{\begin{matrix}\text{pos}\\ \text{neg}\end{matrix}\right\}$ if dotted line passes to the $\left\{\begin{matrix}\text{right}\\ \text{left}\end{matrix}\right\}$ of the origin; in this case the c-o-m accelerates to the $\left\{\begin{matrix}\text{left}\\ \text{right}\end{matrix}\right\}$.

In particular, this is true when the tape is $\left\{\begin{matrix}\text{vertical}\\ \text{horizontal}\end{matrix}\right\}$

(Demonstrate)

Kinetic Energy

$$K = \tfrac{1}{2} M |\dot{\vec{R}}|^2 + \tfrac{1}{2} \sum_a m_a \left|\tfrac{d}{dt}(\vec{r}_a - \vec{R})\right|^2$$

(Old homework problem—derivation almost identical to that in Appendix to last lecture)

For the case at hand,

$$K = \tfrac{1}{2} M |\dot{\vec{R}}|^2 + \tfrac{1}{2} I_0 \dot{\theta}^2$$

[same reasoning as on (4) for fixed-axis rotation]

Example 1: Back to the bowling-ball problem: How much energy did the bowling ball lose?

Initially $K = \tfrac{1}{2} M |V_x(0)|^2$

Finally $V_x = \tfrac{5}{7} V_x(0)$ $\quad \dot{\theta} = -V_x/b$

$$K = \tfrac{1}{2} M |V_x|^2 + \tfrac{1}{2} \frac{I_0}{b^2} |V_x|^2$$

but $I_0/b^2 = \tfrac{2}{5} M$

$$K = \tfrac{1}{2} M \left(\tfrac{7}{5} |V_x|^2\right) = \tfrac{1}{2} M \times \tfrac{7}{5} \times \left(\tfrac{5}{7} V_x(0)\right)^2$$

$$= \tfrac{5}{7}\, \tfrac{1}{2} M |V_x(0)|^2$$

The ball lost $\tfrac{2}{7}$ of its initial energy.

Example 2: Two identical bowling balls are released from rest on two inclined planes of identical inclination. Plane I is covered with ice; ball I slides without rolling. Plane II is covered with sandpaper; ball II rolls without slipping. How do the motions of the two c-o-m's differ?

I is just old particle on frictionless plane $\Rightarrow$ constant acceleration (depending on $\angle$ of plane). Like all 1-d problems, dynamics is determined totally by form of

$$E = \underbrace{\tfrac{1}{2} M |\dot{\vec{R}}|^2}_{K} + \underbrace{Mgz}_{\text{gravitational potential}}$$

For II we have just shown that

$$K = \tfrac{1}{2} M |\dot{\vec{R}}|^2 \times \tfrac{7}{5}$$

potential is same. Thus ~~II~~ II acts as if it were subject to same $\vec{F}$ as I, but had 7/5 the mass.

$\therefore$ acceleration of II $= \tfrac{5}{7}$ (acceleration of I)

17

Prologue to General Rigid-Body Motion

Rotations about a fixed axis have a simple property:

If you first rotate about the z-axis by $\angle\theta_1$,

And if you then rotate about the z-axis by $\angle\theta_2$,

Then the result is the same as rotating about the z axis by $\angle(\theta_1+\theta_2)$.

There is no such "addition formula" for rotations about different axes. There can not be because the result depends on the order in which the rotations are performed. (Demonstrate) Addition is independent of the order of the terms.

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This lack of simplicity makes general rigid-body motion

(1) complicated (a bad thing)
(2) unintuitive (a neutral thing)
(3) surprising (a good thing)

Next week is last chance to take Units 1-5 tests.

SPECIAL HOURS NEXT WEEK

Monday	2-10	224 SC
Tuesday	3-6	224 SC
Wednesday	2-10	224SC (We may have to move to Lecture Hall A at 7PM)
Thursday	2-10	224 SC

Have a happy holiday with a clear conscience!

HOMEWORK COMMENTARY

10-H1 Setup was described completely in lecture.

10-H2 $I_{TOTAL} = \sum I_{parts} \rightarrow$ Integral over dimension divided into little dx or dr.
I_{disk}, I_{shell} given in lecture; $I_{\substack{cylinder\\shell}}$ is obvious.

10-H3 Neither the string nor the weight leads to a component of torque in the z-direction so L_z is conserved. At the lowest point of the motion, $\vec{v}$ is horizontal (by definition) as well as $\perp$ to $\vec{r}$.

10-H4 In part d) it slows down to $\boxed{2\ rad/sec}$: Check.

10-H5 a) $I = \frac{5}{12}MR^2$ You can break it up into strips, and use the parallel axis theorem to find their moment of inertia about the pivot. Then add them up (integrate over x).
c) $\vec{a}_{cm} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$ but the c of m moves in a circle ($\dot{r} = \ddot{r} = 0$) leaving a radial acceleration depending on its angular speed and a tangential acceleration depending on its angular accel
d) $\vec{F} = M\vec{a}_{cm}$

10-H6 The gravitational PE depends on ~~position~~ height of the center of mass. Assume rod comes to rest when writing down the conservation laws, then solve for the unknown quantities.
Check: a) $\frac{m}{M} = 1/3$ c) $\frac{m}{M} = \frac{3}{4}$

(1)

Spin Angular Mom. + Spin Torque

$$\frac{d\vec{P}}{dt} = \vec{F}^{ext}$$

$$\frac{d\vec{L}}{dt} = \sum_a \vec{r}_a \times \vec{F}_a^{ext} = \vec{\tau}^{ext}$$

$$\vec{L} = \vec{R}\times\vec{P} + \sum_a m_a(\vec{r}_a - \vec{R})\times\frac{d}{dt}(\vec{r}_a - \vec{R})$$

$$\vec{L} = \vec{L}_o + \vec{L}_s$$

("orbital $\vec{L}$") ("spin $\vec{L}$")

$$\vec{\tau}^{ext} = \sum_a \vec{R}\times\vec{F}_a^{ext} + \sum_a(\vec{r}_a - \vec{R})\times\vec{F}_a^{ext}$$

$$\vec{\tau}^{ext} = \vec{\tau}_o + \vec{\tau}_s$$

("orbital torque") ("spin torque")

(2)

$$\frac{d}{dt}\vec{L}_o = \frac{d}{dt}(\vec{R}\times\vec{P}) = R\times\frac{d\vec{P}}{dt} = \vec{R}\times\vec{F}^{ext}$$

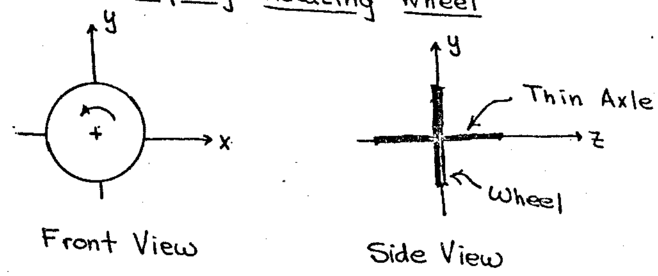
$$= \sum_a \vec{R}\times\vec{F}_a^{ext} = \vec{\tau}_o$$

$$\therefore \boxed{\frac{d}{dt}\vec{L}_s = \vec{\tau}_s} \Longleftarrow \text{very useful}$$

Note: This is trivial if $\vec{R}$ is constant. (Just choose $\vec{R}$ to be at origin of coords.) It is non-trivial if $\vec{R}$ is not constant, as is always the case if $\vec{F}^{ext} \neq \vec{0}$.

(3)

Rapidly Rotating Wheel



Front View Side View

We restrict ourselves to motion where the wheel rotates rapidly about the axle with angular velocity ω. "Rapidly" means all other motions about the c.o.m. take place on a time scale $\gg \frac{2\pi}{\omega}$. Thus we may safely approximate

$$\vec{L}_s = I_o\omega\vec{e}$$

where $\vec{e}$ is a unit vector directed along the axis of the wheel. [In the figure, $\vec{e} = \vec{k}$.]

(4)

Properties of the Wheel

(1) How does ω change when forces are applied to axle?

$$\frac{d}{dt}\vec{L}_s\cdot\vec{L}_s = 2\vec{L}_s\cdot\frac{d\vec{L}_s}{dt} = 2I_o\omega\vec{e}\cdot\frac{d\vec{L}_s}{dt}$$

$$= 2I_o\omega\vec{e}\cdot\vec{\tau}_s$$

Choose coords as shown in the figure on (3).

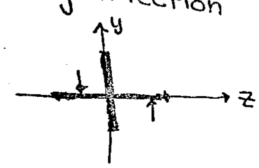
$$\vec{e}\cdot\vec{\tau}_s = \tau_{sz} = \sum_a x_a F_{ya} - y_a F_{xa} = 0$$

because every pt. on the axle has $x = y = 0$

$$\therefore \boxed{|\vec{L}| \text{ is const.} \Rightarrow \omega \text{ is const.}}$$

(2) How does $\vec{e}$ change when forces are applied to axle?

Choose coords as before, let $\vec{F}$ be in y-direction $\vec{\tau}_s = \vec{r}\times\vec{F}$ is in x-direction.



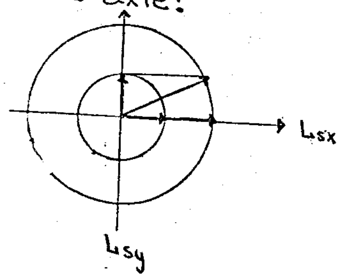
⑤

Since $\vec{L}_s = I_0\omega\vec{e}$ and $\frac{d\vec{L}_s}{dt} = \vec{\tau}_s$,

e_x changes. Wheel moves in plane ⊥ to applied force! (Feels funny)

Note: For given $\Delta\vec{L}_s$, $\Delta\vec{e} = \frac{\Delta\vec{L}_s}{I_0\omega}$.

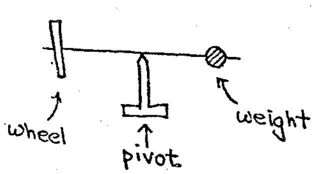
The faster the wheel is spinning, the harder it is to change the direction of the axle!



The larger $|\vec{L}_s|$ is, the smaller the change in direction made by a given $\Delta\vec{L}_{sy}$

⑥

(3) Counterbalanced wheel

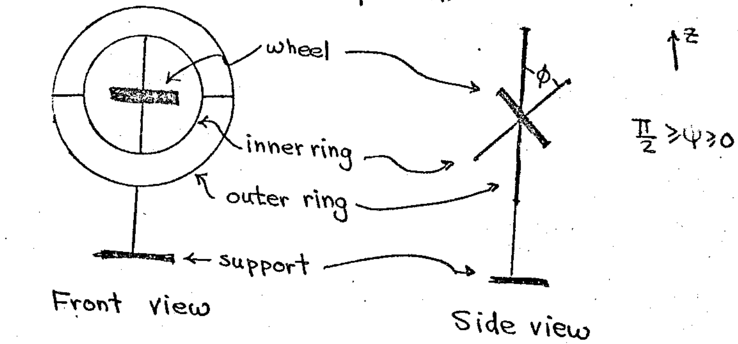


Weight is moved until c-o-m is at pivot.

Since all external forces are applied at the pivot, $\vec{r}-\vec{R}=0$, $\vec{\tau}_s=\vec{0}$, and no matter what you do to the support, $\vec{e}$ does not change! Of course, since in practice there are small residual torques (friction, air currents, effects of non-zero diameter of axle), the larger ω, the better this works. (Demonstrate)

⑦

(4) Wheel in "Cardan suspension"

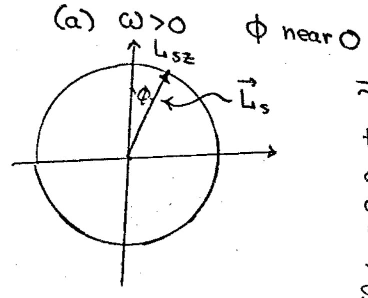


Suppose we turn the outer ring about the z-axis in a positive direction. This leads to a $\vec{\tau}_s$ on the wheel. Two obvious properties for ϕ near zero. (1) $\tau_{sz} \geq 0$. (2) $\tau_{sz}=0$ when $\phi=0$, ∴ is an increasing function of ϕ near $\phi=0$. (Demonstrate the obvious)

⑧

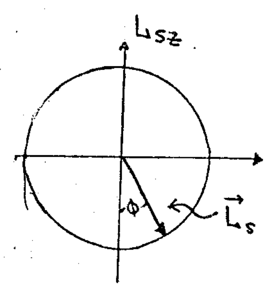
Two situations

(a) $\omega>0$ ϕ near 0



$\vec{\tau}_s$ tends to move axle to the vertical. Effect diminishes in magnitude as process proceeds. Motion with axis vertical + outer ring spinning in same direction as wheel is stable.

(b) $\omega<0$ ϕ near 0



$\vec{\tau}_s$ tends to move axle away from vertical. Effect increases in magnitude as process proceeds. Motion with axis vertical + outer ring spinning in opposite direction to wheel is unstable.

(Demonstrate)

(9)

The Angular Velocity Vector

So far we have done very special cases: (1) fixed axis (2) axis of fixed direction (3) fixed $|\vec{L}|$. To do more general problems, we need a more general formalism.

We will begin by studying rigid body motion with one point ($\vec{r}=0$) fixed. To go to the most general case, we merely have to replace $\vec{r}_a$ by $\vec{r}_a - \vec{R}$, and $\vec{L}$ by $\vec{L}_s$.

Memory of fixed axis rotation (about z-axis):

$$\dot{x}_a = -y_a\dot{\theta} \qquad \dot{y}_a = +x_a\dot{\theta} \qquad \dot{z}_a = 0$$

If we define

$$\vec{\omega} \equiv \dot{\theta}\vec{k}$$

then

$$\boxed{\dot{\vec{r}}_a = \vec{\omega} \times \vec{r}_a}$$

(10)

Geometrical Interpretation: To get $\vec{r}_a(t+\Delta t)$, rotate $\vec{r}_a(t)$ about the axis $\frac{\vec{\omega}}{|\vec{\omega}|}$ by an angle $\Delta\theta = |\vec{\omega}|\Delta t$.

But this is the most general small rotation, if we drop the restriction $\vec{\omega} \parallel$ to $\vec{k}$.

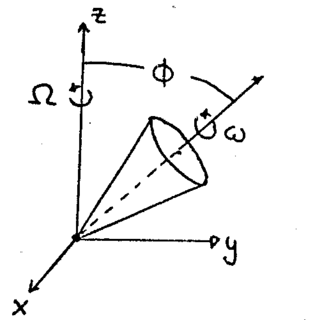
Thus the most general rigid body motion (with $\vec{r}=0$ fixed) is described by

$$\boxed{\dot{\vec{r}}_a(t) = \vec{\omega}(t) \times \vec{r}_a(t)}$$

with $\vec{\omega}$ ("the angular velocity vector") an arbitrary function of time.

Warning: Given $\vec{\omega}(t)$, these diff. eqs. determine $\vec{r}_a(t_2)$ in terms of $\vec{r}_a(t_1)$. However, if the axis is not fixed, the answer is <u>not</u> simply a function of $\int_{t_1}^{t_2} dt\, \vec{\omega}(t)$. In particular, if the $\int = 0$, this does <u>not</u> imply $\vec{r}_a(t_2) = \vec{r}_a(t_1)$. (Demonstrate)

(11)



$\vec{\omega}$ is useful for describing complex motions.

Example: Steady precession of top. Top rotates with angular velocity ω about axis of symmetry $\vec{k}'$. At the same time, entire apparatus is on turntable rotating about z axis with angular velocity Ω. What is $\vec{\omega}$? $[\vec{k}\cdot\vec{k}' = \cos\phi]$

$$\vec{\omega} = \underset{\substack{\uparrow \\ \text{turntable}}}{\Omega\vec{k}} + \underset{\substack{\uparrow \\ \text{spin of top}}}{\omega\vec{k}'}$$

[Why add? You <u>always</u> add infinitesimal changes].

This is not the end of the story; $\vec{k}'$ is carried around by turntable

$$\vec{k}' = \cos\phi\,\vec{k} + \sin\phi\,[\cos\omega t\ \vec{i} + \sin\omega t\ \vec{j}]$$

[t=0 chosen when $\vec{k}'$ is in x-z plane

$$\vec{\omega} = [(\Omega + \omega\cos\phi)\vec{k} + \omega\sin\phi\,[\cos\omega t\,\vec{i} + \sin\omega t\,\vec{j})]$$

(12)

$\vec{\omega}$, $\vec{L}$ and the moment of inertia tensor

$$\vec{L} = \sum_a m_a \vec{r}_a \times \dot{\vec{r}}_a$$

$$\dot{\vec{r}}_a = \vec{\omega} \times \vec{r}_a$$

$$\therefore, \quad \vec{L} = \sum_a m_a \vec{r}_a \times (\vec{\omega} \times \vec{r}_a)$$

Vector identity: For any two vectors $\vec{A}$ and $\vec{B}$

$$\vec{A} \times (\vec{B} \times \vec{A}) = \vec{B}(\vec{A}\cdot\vec{A}) - \vec{A}(\vec{A}\cdot\vec{B})$$

(Proof in appendix)

$$\therefore, \quad \vec{L} = \sum_a m_a \vec{\omega}(\vec{r}_a\cdot\vec{r}_a) - \vec{r}_a(\vec{r}_a\cdot\vec{\omega})$$

Let us write this out in component form

$$L_x = \sum_a \{ m_a \omega_x (\cancel{x_a^2} + y_a^2 + z_a^2) - \cancel{x_a x_a \omega_x} - x_a y_a \omega_y - x_a z_a \omega_z \}$$

etc for L_y + L_z

This is a set of linear Eqs. giving one vector as a function of another, i.e., a linear operator eq. (see small vibrations lecture)

$$\vec{L} = \underset{\sim}{I}\ \vec{\omega}$$

$\uparrow$ "moment of inertia tensor"

$\underset{\sim}{I}$ is completely determined by nine numbers

$$I_{xx} \equiv \vec{i}\cdot\underset{\sim}{I}\,\vec{i} = \sum_a m_a(y_a^2+z_a^2)$$
$$I_{yy} \equiv \vec{j}\cdot\underset{\sim}{I}\,\vec{j} = \sum_a m_a(x_a^2+z_a^2)$$
$$I_{zz} \equiv \vec{k}\cdot\underset{\sim}{I}\,\vec{k} = \sum_a m_a(x_a^2+y_a^2)$$
$$I_{xy} \equiv \vec{i}\cdot\underset{\sim}{I}\,\vec{j} = \sum_a m_a(-x_a y_a) = I_{yx}$$

etc.

Note only six numbers are really needed — $\underset{\sim}{I}$ is symmetric linear op.

Written out in full

$$\left.\begin{aligned} L_x &= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\ L_y &= I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\ L_z &= I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \end{aligned}\right\}\ \text{Ugh!}$$

If we had to work with these Eqs. we would go bananas. Fortunately, we are rescued by

*** The Great Diagonalization Theorem ***

For any symmetric linear op on a 3d ~~vector~~ space we can find three orthogonal unit eigenvectors. We denote these by $\vec{i}'$, $\vec{j}'$, and $\vec{k}'$. The directions they define are called "principal axes." The associated eigenvalues are denoted by I_1, I_2, I_3 and are called "principal moments of inertia"

$$\underset{\sim}{I}\,\vec{i}' = I_1\vec{i}' \qquad \underset{\sim}{I}\,\vec{j}' = I_2\vec{j}' \qquad \underset{\sim}{I}\,\vec{k}' = I_3\vec{k}'$$

If we write $\vec{L} = L_{x'}\vec{i}' + L_{y'}\vec{j}' + L_{z'}\vec{k}'$

and $\vec{\omega} = \omega_{x'}\vec{i}' + \omega_{y'}\vec{j}' + \omega_{z'}\vec{k}'$

then

$$\boxed{\begin{aligned} L_{x'} &= I_1\omega_{x'} \\ L_{y'} &= I_2\omega_{y'} \\ L_{z'} &= I_3\omega_{z'} \end{aligned}} \Leftarrow \text{not so ugh!}$$

Remarks

(1) Once we have found the principal axes we can compute the principal moments from (13)

$$I_1 = I_{x'x'} = \sum_a m_a(y_a'^2+z_a'^2) \quad \text{etc.}$$
$$I_{x'y'} = 0 = \sum_a m_a(x_a' y_a') \quad \text{etc.}$$

(This last is a check that we really have principal axes)

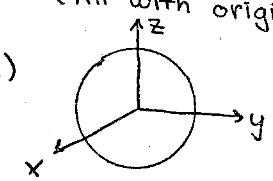
(2) Warning: $\vec{L}$ and $\vec{\omega}$ are $\parallel$ only if $I_1=I_2=I_3$, in general.

(3) As the body rotates, $\vec{i}'$, $\vec{j}'$, and $\vec{k}'$ rotate with it (sad but true).

Principal Moments of Some Simple Bodies

(All with origin at c.o.m.)

(1)



Uniform sphere, radius R, mass M.

Axes as given are obviously principal axes. For every mass pt. at (x,y,z), there is another at $(x,-y,z)$, so

$$I_{xy} = -\sum_a m_a(x_a y_a) = 0, \text{ etc.}$$

By ~~eg~~ study of fixed axis rotation, if $\vec{\omega} = \vec{k}\,\dot{\theta}$, $L_z = \frac{2}{5}MR^2\dot{\theta}$

$\therefore\ I_3 = \frac{2}{5}MR^2$

By symmetry, $I_1=I_2=I_3$

$$\therefore\ \boxed{I_1=I_2=I_3=\tfrac{2}{5}MR^2}$$

(17)

(2) Uniform Rectangular Parallelopiped, $|x| \le \frac{a}{2}$, $|y| \le \frac{b}{2}$, $|z| \le \frac{c}{2}$, mass M.

Coord. axes are principal axes, as before.

$$I_3 = \underset{\substack{\uparrow \\ \text{mass density}}}{\frac{M}{abc}} \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy \int_{-c/2}^{c/2} dz\, (x^2+y^2)$$

$$= \frac{M}{abc} \left\{ cb \left[\tfrac{1}{3}\left(\tfrac{a}{2}\right)^3 - \tfrac{1}{3}\left(-\tfrac{a}{2}\right)^3 \right] + ac \left[\tfrac{1}{3}\left(\tfrac{b}{2}\right)^3 - \tfrac{1}{3}\left(-\tfrac{b}{2}\right)^3 \right] \right\}$$

$$= \frac{M}{abc} \frac{1}{12} \left[cba^3 + acb^3 \right]$$

$$= \frac{M}{12}(a^2+b^2)$$

Likewise for I_2, I_1

$$\boxed{\begin{aligned} I_1 &= \tfrac{M}{12}(b^2+c^2) \\ I_2 &= \tfrac{M}{12}(a^2+c^2) \\ I_3 &= \tfrac{M}{12}(a^2+b^2) \end{aligned}}$$

(18)

Limiting cases of (2):

(3) Uniform cube of side a

$a=b=c$

$$\boxed{I_1 = I_2 = I_3 = \frac{M}{6}a^2} \quad *$$

(4) Rectangular plate of sides a + b

$c=0$

$$\boxed{I_1 = \frac{M}{12}b^2 \qquad I_2 = \frac{M}{12}a^2 \quad I_3 = \frac{M}{12}(a^2+b^2)}$$

(5) Uniform rod of length a

$b=c=0$

$$\boxed{I_1 = 0 \qquad I_2 = I_3 = \frac{M}{12}a^2}$$

Note: If $a = \sqrt{\frac{12}{5}}\, R$ cube of side a + sphere of radius R (with equal masses) are dynamically <u>indistinguishable</u>. Under equal $\vec{p}^{\,ext}$, $\vec{\tau}^{\,ext}$, they execute <u>the same motions</u>.

(19)

Appendix: Choose coords. so $\vec{A} = A_x\vec{i}$

$\vec{B} = B_x\vec{i} + B_y\vec{j} + B_z\vec{k}$

$$\vec{A} \times (\vec{B} \times \vec{A}) = A_x^2\, \vec{i} \times \left[(B_x\vec{i} + B_y\vec{j} + B_z\vec{k}) \times \vec{i} \right]$$

$$= A_x^2\, \vec{i} \times \left[-B_y\vec{k} + B_z\vec{j} \right]$$

$$= A_x^2 \left[B_y(+\vec{j}) + B_z\vec{k} \right]$$

$$\vec{B}(\vec{A}\cdot\vec{A}) - \vec{A}(\vec{A}\cdot\vec{B}) = (B_x\vec{i} + B_y\vec{j} + B_z\vec{k})(A_x)^2 - A_x\vec{i}(A_xB_x)$$

$$= (B_y\vec{j} + B_z\vec{k})(A_x)^2$$

(1)

Fundamental Eqs. Reviewed

(1) For motion of a rigid body with one point ($\vec{r}=\vec{0}$) fixed

$$\frac{d\vec{r}_a}{dt} = \vec{\omega}\times\vec{r}_a .$$

(2) For any rigid body there exist three orthogonal axes, the principal axes, fixed in the body, such that if $\vec{i}\,'$, $\vec{j}\,'$, $\vec{k}\,'$ are unit vectors along these axes, and if

$$\vec{\omega} = \omega_{x'}\vec{i}\,' + \omega_{y'}\vec{j}\,' + \omega_{z'}\vec{k}\,'$$

then

$$\vec{L} = I_1\omega_{x'}\vec{i}\,' + I_2\omega_{y'}\vec{j}\,' + I_3\omega_{z'}\vec{k}\,'$$

The consts. $I_{1,2,3}$ are called principal moments.

(3) $$\frac{d\vec{L}}{dt} = \vec{\tau}$$

(4) If there is no fixed pt. these Eqs. are still true, with the replacements

$\vec{r}_a \to \vec{r}_a - \vec{R}$ $\quad \vec{L}\to\vec{L}_s \quad \vec{\tau}\to\vec{\tau}_s$

(2)

The Euler Equations

$$\vec{L} = I_1\omega_{x'}\vec{i}\,' + I_2\omega_{y'}\vec{j}\,' + I_3\omega_{z'}\vec{k}\,'$$

$$\dot{\vec{L}} = I_1\dot{\omega}_{x'}\vec{i}\,' + I_2\dot{\omega}_{y'}\vec{j}\,' + I_3\dot{\omega}_{z'}\vec{k}\,' + I_1\omega_{x'}\dot{\vec{i}}\,' + I_2\omega_{y'}\dot{\vec{j}}\,' + I_3\omega_{z'}\dot{\vec{k}}\,'$$

But $\vec{i}\,'$, $\vec{j}\,'$, $\vec{k}\,'$ are vectors fixed in the body (like $\vec{r}_a$). Thus

$$\frac{d\vec{i}\,'}{dt} = \vec{\omega}\times\vec{i}\,' = (\omega_{x'}\vec{i}\,' + \omega_{y'}\vec{j}\,' + \omega_{z'}\vec{k}\,')\times\vec{i}\,'$$

$$= -\omega_{y'}\vec{k}\,' + \omega_{z'}\vec{j}\,'$$

Likewise

$$\frac{d\vec{j}\,'}{dt} = -\omega_{z'}\vec{i}\,' + \omega_{x'}\vec{k}\,'$$

$$\frac{d\vec{k}\,'}{dt} = -\omega_{x'}\vec{j}\,' + \omega_{y'}\vec{i}\,'$$

[I obtained these by cycling:

$i \to j \to k \to i \qquad x \to y \to z \to x$]

(3)

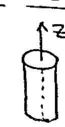
Gathering together terms

$$\frac{d\vec{L}}{dt} = \vec{i}\,'\left[I_1\dot{\omega}_{x'} + \omega_{y'}\omega_{z'}(I_3-I_2)\right] + \vec{j}\,'\left[I_2\dot{\omega}_{y'} + \omega_{z'}\omega_{x'}(I_1-I_3)\right] + \vec{k}\,'\left[I_3\dot{\omega}_{z'} + \omega_{x'}\omega_{y'}(I_2-I_1)\right]$$

This set of three Eqs. (one for each component of $\vec{L}$) is called "the Euler eqs."

(4)

Torque-Free Motion for $I_1 = I_2$

(e.g.  ,  , or anything else with an axis of rotational symmetry.)

Let $I_1 = I_2$ $\qquad \vec{\tau} = \frac{d\vec{L}}{dt} = \vec{0}$.

From the $\vec{k}\,'$ part of the Euler Eqs.:

$$I_3\dot{\omega}_{z'} = 0 \Rightarrow \omega_{z'} = \text{const.}$$

From the $\vec{i}\,'$ part of the Euler Eqs.:

$$I_1\dot{\omega}_{x'} + \omega_{y'}\omega_{z'}(I_3-I_2) = 0$$

$$\dot{\omega}_{x'} = -\Omega\,\omega_{y'}$$

where $\Omega = \dfrac{(I_3-I_2)\,\omega_{z'}}{I_1}$ (a const.)

Likewise

$$I_2\dot{\omega}_{y'} + \omega_{z'}\omega_{x'}(I_1-I_3) = 0$$

$$\Rightarrow \dot{\omega}_{y'} = \Omega\,\omega_{x'}$$

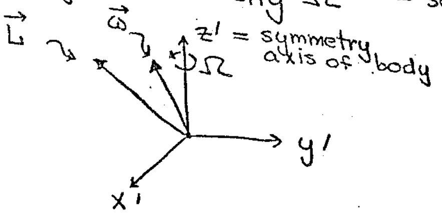
$$\dot{\omega}_{x'} = -\Omega\omega_{y'} \qquad \dot{\omega}_{y'} = \Omega\omega_{x'} \qquad \dot{\omega}_{z'} = 0$$

These are of the same form as the Eqs.

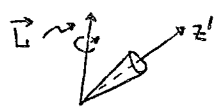
$$\dot{x}_a = -\dot{\theta}\, y_a \qquad \dot{y}_a = \dot{\theta}\, x_a \qquad \dot{z}_a = 0$$

that describe rotation about the z-axis

Thus, in <u>body-fixed</u> <u>coords.</u>, $\vec{\omega}$ rotates about the z'-axis with constant angular velocity Ω — so does $\vec{L}$, of course.



In <u>space-fixed</u> coords., $\vec{L}$ is const. $\left(\frac{d\vec{L}}{dt} = \vec{0}\right)$.
Thus, the symmetry axis rotates about $\vec{L}$, as does $\vec{\omega}$.
(Demonstrate)



<u>Torque-free Motion with $I_1 \neq I_2 \neq I_3$.</u>

The general case is too difficult for us. However, it is easy to analyze the case where $\vec{\omega}$ is almost ∥ to one of the principal axes (say, z')

$$|\omega_{z'}| \gg |\omega_{x'}| \qquad |\omega_{z'}| \gg |\omega_{y'}|$$

$$I_3 \dot{\omega}_{z'} = (I_1 - I_2)\omega_{x'}\omega_{y'} \approx 0$$

(dropping terms of 2nd order in small quantities)

$\therefore$, $\omega_z' = \text{const.}$

$$I_2 \dot{\omega}_{y'} = (I_3 - I_1)\omega_{x'}\omega_{z'}$$
$$I_1 \dot{\omega}_{x'} = (I_2 - I_3)\omega_{y'}\omega_{z'}$$

$$\therefore, \quad I_1 \ddot{\omega}_{x'} = (I_2 - I_3)\dot{\omega}_{y'}\omega_{z'} = \frac{(I_2 - I_3)(I_3 - I_1)(\omega_z')^2}{I_2}\,\omega_{x'}$$

This is identical with

$$m\ddot{x} = -kx$$

If we identify x with $\omega_{x'}$, m with I_1, and k with

$$\frac{(I_3 - I_2)(I_3 - I_1)\,\omega_{z'}^2}{I_2}$$

Three cases

(a) I_3 is the largest principal moment
$I_3 - I_2 > 0, \quad I_3 - I_1 > 0 \Rightarrow k > 0$
<u>stability</u>

(b) I_3 is the smallest principal moment
$I_3 - I_2 < 0, \quad I_3 - I_1 < 0 \Rightarrow k > 0$
<u>stability</u>

(c) I_3 is the middle principal moment
$I_3 - I_2 \lessgtr 0, \quad I_3 - I_1 \gtrless 0 \Rightarrow k < 0$
<u>instability</u>

A note on the demonstration:
The easiest way to study torque free motion is to study motion about the c.o.m. for a freely falling body. Gravity acts like a force concentrated at the c.o.m. + thus makes zero contribution to

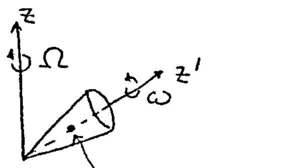
$$\vec{\tau}_s = \vec{F} \times (\vec{r} - \vec{R})$$

(9)

The Gyroscope

(1) Exact theory of steady precession

(Demonstrate)



$\vec{k}\cdot\vec{k}' = \cos\phi$

ϕ, Ω, ω constants

$I_1 = I_2$

c.o.m. distance l from the origin.

Question: Is this motion a solution of

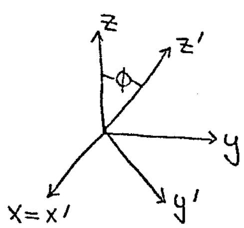
$$\frac{d\vec{L}}{dt} = \vec{\tau}^{ext} = l\vec{k}' \times (-Mg\vec{k}) \quad ?$$

If so, what relations (if any) are there between Ω, ω, and ϕ?

Step 1: Since the situation at any time is just a rotation of the situation at $t=0$, we need only check Eqs. of motion at $t=0$.

(10)

Step 2: Establish coords at $t=0$ and find relations between $\vec{i}, \vec{j}, \vec{k}$, and $\vec{i}', \vec{j}', \vec{k}'$



$$\vec{k}' = \vec{k}\cos\phi + \vec{j}\sin\phi$$
$$\vec{j}' = -\vec{k}\sin\phi + \vec{j}\cos\phi$$
$$\vec{i}' = \vec{i}$$

$$\vec{k} = \vec{k}'\cos\phi - \vec{j}'\sin\phi$$
$$\vec{j} = \vec{k}'\sin\phi + \vec{j}'\cos\phi$$

Step 3: Compute $\vec{\omega}$

$$\vec{\omega} = \Omega\vec{k} + \omega\vec{k}' \quad \text{(from last lecture)}$$
$$= (\omega + \Omega\cos\phi)\vec{k}' - \Omega\sin\phi\,\vec{j}'$$

Step 4: Compute $\vec{L}$

$$\vec{L} = I_3(\omega + \Omega\cos\phi)\vec{k}' - I_2\Omega\sin\phi\,\vec{j}'$$
$$= [I_3\cos\phi(\omega+\Omega\cos\phi) + I_2\Omega(\sin\phi)^2]\vec{k}$$
$$+ [I_3\sin\phi(\omega+\Omega\cos\phi) - I_2\Omega\cos\phi\sin\phi]\vec{j}$$

(11)

Step 5: Compute $\dot{\vec{L}}$.

Since the whole shebang is rotating about the z-axis

$$\dot{\vec{L}} = \Omega\vec{k} \times \vec{L}$$
$$= -\vec{i}\,[I_3\sin\phi(\omega+\Omega\cos\phi) - I_2\Omega\cos\phi\sin\phi]\Omega$$

Step 6: Set $\dot{\vec{L}} = \vec{\tau}$

$$\vec{\tau} = -Mgl\,\vec{k}' \times \vec{k}$$
$$= -Mgl(\vec{k}\cos\phi + \vec{j}\sin\phi) \times \vec{k}$$
$$= -\vec{i}\,Mgl\sin\phi$$

Thus it works if

$$\boxed{Mgl = \Omega I_3\omega + (I_3 - I_2)\Omega^2\cos\phi}$$

the answer

(12)

This expression simplifies for a "fast gyroscope" $\omega \to \infty$

$$Mgl \approx \Omega I_3 \omega$$
$$\Omega \approx \frac{Mgl}{\omega I_3}$$

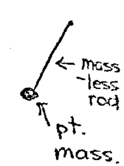
This could also have been obtained by "rapidly rotating wheel" method – see K+K.

Note: Ω inversely proportional to ω, independent of ϕ. (Demonstrate)

Somewhat surprisingly, the gyroscope also "works" if $\omega = 0$.

$$Mgl = (I_3 - I_2)\Omega^2\cos\phi$$

But is it really surprising? Consider spherical pendulum.



$I_3 = 0 \qquad I_2 = Ml^2 \qquad I_3 - I_2 < 0$

Spherical pendulum can rotate in a horizonal circle if $\cos\phi < 0$. Obvious!

Moral: Two things that look <u>very different</u> (pendulum, gyroscope) may be two different limiting cases of <u>the same thing</u>.

(13)

The Gyroscope

(2) Approximate Theory of Nutation

Steady precession is not the only motion of a gyroscope. The general case is a mess (see K+K). However, there is one case that is easy: a fast gyroscope.

For a fast gyroscope the only effect of gravity (in steady precession) is to make the gyro precess slowly:

$$\Omega = \frac{Mg\ell}{\omega I_3}$$

Thus, it is reasonable that for time scales $<< 2\pi/\Omega$, we can ignore gravity. We then have a problem we have solved (see ⑤) — the symmetry axis rotates about $\vec{L}$ with an angular frequency proportional to ω. [Note consistency: This is fast on a scale of $2\pi/\Omega$.]

THIS IS THE LAST LECTURE FOR THE TERM.

DURING READING PERIOD, PROFESSOR SKOCPOL WILL BE IN LECTURE HALL C ON TUESDAYS AND THURSDAYS FROM 11–12 TO ANSWER QUESTIONS AND PROVIDE INDIVIDUAL AND SMALL GROUP INSTRUCTION.

UNIT TESTS WILL BE GIVEN AT THE USUAL TIMES (MON 2-10, WED 3-6, THURS 2-10) IN 224: [UNITS 6-11 for two weeks, 10-11 the last week]

SECTIONS WILL MEET AT THE REGULAR TIMES FOR UNIT 11 AND REVIEW SESSIONS.

ALL LAB REPORTS ARE DUE BEFORE THE BEGINNING OF FINAL EXAMS (i.e. BY THURS JAN 20)

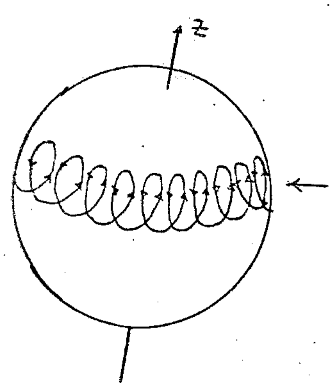
If you want to check your prefinal point total come to Physics Focus on Jan 25 or 26.

If you want to be notified of your final grade, enclose a post card with your exam

MEETING OF PEOPLE CURIOUS ABOUT PHYSICS AS CONCENTRATION: THURS JAN 13, 10 AM

(14)

Superposing these two motions, we have a picture where the symmetry axis rotates rapidly about $\vec{L}$ which in turn precesses slowly about $\vec{k}$. This is called "nutation"



Curve traced out on unit sphere by sym. axis

(Demonstrate)