Def. A stochastic process (random field) is a collection of random variables $\{Y_t\}$, each of which is uniquely associated with some $t \in T$.

T: index set
- (nonempty)

Ex. $\xi \sim N(0,1)$ $T = \{1, \dots, 5\}$

$y_t = \xi + t$

Ex. $\xi \sim Bin(n,p)$, $T = [0, 2\pi]$, $Y_t = \xi \sin(t)$

Ex. $\underbrace{y_t}_{\text{predicted } t\text{-th output}} = a \underbrace{x_t}_{t\text{-th observation (data pt)}} + \xi$ $\quad$ $\underbrace{y_t}_{\text{actual data}} = a x_t + \underbrace{\xi_t}_{t\text{-th error}}$

Ex. Partial differential eq.

$$\frac{\partial y}{\partial t} + y \frac{\partial y}{\partial x} = 0 \qquad y : \overbrace{\mathbb{R}_x \times \mathbb{R}_{t \geq 0}}^{\mathcal{D}} \longrightarrow \mathbb{R}$$

$$y(x, t=0; \xi) = \xi \sin(x)$$

Then $\{y(x,t;\xi)\}_{(x,t) \in \mathcal{D}}$ is a stochastic process.

$\mathcal{D}$: index set

Let $\mathcal{L}^2(\mathbb{R}; d\xi) = \{$ stochastic processes $u(x,t,\xi)$ for $\xi : \Omega \to \mathbb{R}$ s.t. $\underline{\mathrm{var}}_{\xi}(u\underline{(x,t;\xi)}) < \infty$ at every $(x,t) \in \mathcal{D}\}$.

A polynomial chaos expansion writes

* Wiener (Gaussian ξ)

$$\boxed{u(x,t;\xi) = \sum_{k=0}^{\infty} u_i(x,t)\,\Psi_i(\xi)}.$$

- "Writes?" $\longrightarrow$ What does that mean? It means

for $u \in \mathcal{L}^2(\mathbb{R}; d\xi)$,

$$\lim_{M\to\infty} \sum_{k=0}^{M} u_i(x,t)\,\Psi_i(\xi) = u(x,t;\xi).$$

family of polynomials orthog. w.r.t. the PDF of ξ

deterministic coefficient functions

$u_i : \mathcal{D} \longrightarrow \mathbb{R}$

$$\int \Psi_i(s)\Psi_j(s)\,w(s) \overset{h_i \delta_{ij}}{=}$$

- The "random part" ξ is separate from the deterministic part $(x,t) \in \mathcal{D}$.

Polynomial chaos: Ψ_i are orthogonal

Note:

When I say the Ψ_k are orthog. w.r.t. the PDF of ξ (call ω), that means

$$\int \Psi_i(\underline{s})\Psi_j(\underline{s})\, \underline{\omega(s)}\, ds = h_i \delta_{ij}$$

for some scalars $h_i > 0$.

This is useful, because it helps us:

(1) estimate the $u(x,t;\xi)$ without sampling

(2) give us the estimate of $u(x,t;\xi)$ directly (i.e. all its distribution info), not just its mean or variance

Downsides:

(1) convergence is only always guaranteed for an arbitrary u if ξ is Gaussian

(2) If ξ is not Gaussian, then the expansion is called generalized polynomial chaos, and it works, but

Xiu's PhD thesis, Brown, 2001

- you need some (very reasonable) restrictions on $u(x,t;\xi)$
- might be slow (i.e. need big M).

- Fourier series (spectral method)

Ex. $\frac{\partial y}{\partial t} + y\frac{\partial y}{\partial x} = 0$ (Burgers' eqn.) $(*)$

$y : \mathcal{D} \longrightarrow \mathbb{R}$
$\mathcal{D}$ = subset of $\mathbb{R}_x \times \mathbb{R}_{t\geq 0}$

initial condition: $y(x, t=0; \xi) = \xi \sin(x)$.

Goal: Find $y(x,t;\xi)$ with $\xi \sim N(0,1)$. (w/ var. assump. on soln.)

(Weiner ~ Hermite expansion)

① Write PCE $y(x,t;\xi) = \sum_{i=0}^{\infty} y_i(x,t)\,\Psi_i(\xi)$

where $\{\Psi_i\}$ are orthogonal w.r.t. the PDF of ξ,

$\longrightarrow$ that means the Ψ_i are 1D Hermite.

② (Ghanem & Spanos) 1960s:

Truncated to $(M+1)$ total terms

$$y^{(M)}(x,t;\xi) = \sum_{i=0}^{M} y_i^{(M)}(x,t)\,\Psi_i(\xi).$$

since $y^{(M)} \xrightarrow[M\to\infty]{} y$.

③ Substitute $y^{(M)}$ back into the original PDE $(*)$.

$$\sum_{i=0}^{M} \frac{\partial y_i^{(M)}}{\partial t}\Psi_i(\xi) + \left(\sum_{i=0}^{M} y_i^{(M)}\Psi_i(\xi)\right)\left(\sum_{j=0}^{M}\frac{\partial y_j^{(M)}}{\partial x}\Psi_j(\xi)\right) = 0$$

$$\sum_{i=0}^{M} \frac{\partial y_i^{(M)}}{\partial t}(x,t)\, \psi_i(\xi) + \underbrace{\sum_{i=0}^{M}\sum_{j=0}^{M} y_i^{(M)}(x,t)\, \frac{\partial y_j^{(M)}}{\partial x}\, \psi_i(\xi)\psi_j(\xi)}_{=\,0}$$

Orthogonal $\{\psi_i\}$ will come to the rescue!

④ Multiply through by ψ_K for a fixed but arbitrary $K \in \{0,\dots,M\}$ and integrate.

(a)
$$\sum_{i=0}^{M} \frac{\partial y_i^{(M)}}{\partial t}\, \psi_i(\xi)\psi_K(\xi)$$

$$+ \sum_{i=0}^{M}\sum_{j=0}^{M} y_i^{(M)}\, \frac{\partial y_j^{(M)}}{\partial x}\, \psi_i(\xi)\psi_j(\xi)\psi_K(\xi) = 0.$$

(b) Integrate w.r.t. w = PDF of $\underline{\xi}$

$$\sum_{i=0}^{M} \frac{\partial y_i^{(M)}}{\partial t} \int \underline{\psi_i}(\xi)\underline{\psi_K}(\xi)\, d\xi$$

$$+ \sum_{i=0}^{M}\sum_{j=0}^{M} y_i^{(M)}\, \frac{\partial y_j^{(M)}}{\partial x} \int \psi_i(\xi)\psi_j(\xi)\psi_K(\xi)\, d\xi = 0.$$

⑤ $$\frac{\partial y_K^{(M)}}{\partial t}\, \langle \psi_K, \psi_K \rangle_\xi$$ $\underline{K} = 0,\dots,M$

triple product

$$+ \sum_{i=0}^{M}\sum_{j=0}^{M} y_i^{(M)}\, \frac{\partial y_j^{(M)}}{\partial x}\, \langle \underline{\psi_i, \psi_j, \psi_K} \rangle_\xi = 0$$

$$\underbrace{\frac{\partial y_k^{(M)}}{\partial t} \langle \psi_k, \psi_k \rangle_\xi}_{} + \sum_{i=0}^{M} \sum_{j=0}^{M} y_i^{(M)} \frac{\partial y_j^{(M)}}{\partial x} \underbrace{\langle \psi_i, \psi_j, \psi_k \rangle}_{= 0}$$

for $k = 0, \ldots, M$

Szegö ("Orthonogonal Polynomials")

~1930s-40s

$$\langle \psi_i, \psi_k \rangle = \delta_{ik} \underbrace{(i!)}_{h_i}$$

$$\langle \psi_i, \psi_j, \psi_k \rangle = \begin{cases} 0 & \text{if } i+j+k \text{ is odd } \underline{\text{OR}} \\ & \max(i,j,k) > s \\ \dfrac{i!\,j!\,k!}{(s-i)!\,(s-j)!\,(s-k)!} & \end{cases}$$

where $s = \dfrac{i+j+k}{2}$.

For $M=1$, w/ these double/triple prod. formulas,

$$\begin{cases} \dfrac{\partial y_0}{\partial t} + y_0 \dfrac{\partial y_0}{\partial x} + y_1 \dfrac{\partial y_1}{\partial x} = 0 & (K=0) \\ \dfrac{\partial y_1}{\partial t} + y_1 \dfrac{\partial y_0}{\partial x} + y_0 \dfrac{\partial y_1}{\partial x} = 0 & (K=1) \end{cases}$$

For $M=2$,

$$\frac{\partial y_0}{\partial t} + y_0 \frac{\partial y_0}{\partial x} + y_1 \frac{\partial y_1}{\partial x} + y_2 \frac{\partial y_2}{\partial x} = 0 \quad (K=0)$$

more formula $= 0$ $(K=1)$

more formula $= 0$ $(K=2)$

- Solve PDE system for coefficient functions $y_0^{(M)}, \ldots, y_M^{(M)}$
- $y^{(M)}(x,t;\xi) \approx \sum_{K=0}^{M} y_K^{(M)}(x,t)\, \Psi_K(\xi)$
- $\mathbb{E}[y^{(M)}(x,t;\xi)] = \underline{\sum_{K=0}^{M} \underline{y_K^{(M)}(x,t)}\, \mathbb{E}_\xi(\Psi_K(\xi))}$

$$\mathbb{E}_\xi(\psi_K(\xi)) = \begin{cases} 0 & \text{if } K > 0 \\ 1 & \text{if } K = 0 \end{cases}$$

any orthog. poly with ξ centered

that's okay.

$$\mathbb{E}_\xi(\xi) = 0$$

$$\mathbb{E}[y^{(M)}(x,t;\xi)] = y_0^{(M)}(x,t)$$

$$y(x,t;\xi) = y_0^{(M)}(x,t)\underbrace{\psi_0(\xi)}_{1} + y_1^{(M)}(x,t)\psi_1(\xi) + \dots$$

$$\mathrm{var}(y^{(M)}(x,t;\xi)) = \mathrm{var}\left(\sum_{K=0}^{M} y_K^{(M)}\psi_K(\xi)\right)$$

$$= \mathbb{E}\left(\left(\sum_{K=0}^{M} y_K^{(M)}\psi_K(\xi)\right)^2\right) - \left[y_0^{(M)}\right]^2$$

$$= \sum_{K=0}^{M}\sum_{j=0}^{M} y_K^{(M)} y_j^{(M)} \langle \psi_K, \psi_j \rangle - \left[y_0^{(M)}\right]^2$$

$$= \sum_{K=0}^{M}\sum_{j=0}^{M} y_K^{(M)} y_j^{(M)} (K!\,\delta_{Kj}) - \left[y_0^{(M)}\right]^2$$

only surviving terms occur when $j = K$

variance of output

$$= \sum_{K=0}^{M} K!\left[y_K^{(M)}\right]^2 - \left[y_0^{(M)}\right]^2 = \sum_{K=1}^{M} K!\left[y_K^{(M)}\right]^2$$