# II. Definition of Jacobi forms

**Exercise II-1. Operator $U_d : J_{k,m} \to J_{k,d^2m}$.**
Consider a holomorphic Jacobi form $\varphi(\tau, z) \in J_{k,m}$. Let $d \in \mathbb{N}$. Using only the definition of Jacobi forms, prove that

$$\psi(\tau, z) = \varphi(\tau, dz) \in J_{k,\,md^2}.$$

**Exercise II-2. The action of $SL_2(\mathbb{R})$.**

For any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, $\tau \in \mathbb{H}$ we put

$$M\langle\tau\rangle = \frac{a\tau + b}{c\tau + d}, \qquad j(M, \tau) = c\tau + d.$$

We note that $j(M, \tau) \neq 0$. It is easy to see that

$$M \cdot \begin{pmatrix} \tau & \bar{\tau} \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} M\langle\tau\rangle & M\langle\bar{\tau}\rangle \\ 1 & 1 \end{pmatrix} \begin{pmatrix} j(M, \tau) & 0 \\ 0 & j(M, \bar{\tau}) \end{pmatrix}.$$

and

$$\det \begin{pmatrix} \tau & \bar{\tau} \\ 1 & 1 \end{pmatrix} = \tau - \bar{\tau} = 2iy \quad (\tau = x + iy).$$

Let $M_1, M_2 \in SL_2(\mathbb{R})$. Using the associativity of the matrix product $M_1 \cdot M_2 \cdot \begin{pmatrix} \tau & \bar{\tau} \\ 1 & 1 \end{pmatrix}$, to prove that

1) $(M_1 \cdot M_2)\langle\tau\rangle = M_1\langle M_2\langle\tau\rangle\rangle$;

2) $j(M_1 M_2, \tau) = j(M_1, M_2\langle\tau\rangle) j(M_2, \tau)$;

3) $\operatorname{Im}(M\langle\tau\rangle) = \frac{\operatorname{Im}(\tau)}{|j(M,\tau)|^2}$.

**Exercise II-3. The symplectic group $Sp_n$.**
1) Prove that the condition $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{R})$, $A, B, C, D \in M_n(\mathbb{R})$ is equivalent to the following relations

$$A^t D - B^t C = E_n, \quad A^t B = B^t A, \quad C^t D = D^t C$$

or

$${}^tAD - {}^tCB = E_n, \quad {}^tAC = {}^tCA, \quad {}^tBD = {}^tDB$$

where ${}^tX$ denotes the transposition of $X$.

2) In particular, check that

$$M^{-1} = \begin{pmatrix} {}^tD & -{}^tB \\ -{}^tC & {}^tA \end{pmatrix},$$

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in Sp_n(\mathbb{R}) \Leftrightarrow D \in GL_n(\mathbb{R}) \text{ and } A = {}^tD^{-1},$$

$$\begin{pmatrix} E_n & B \\ 0 & E_n \end{pmatrix} \in Sp_n(\mathbb{R}) \Leftrightarrow {}^tB = B.$$

3) Prove that $Sp_1(\mathbb{Z}) \simeq SL_2(\mathbb{Z})$.

**Exercise II-4. A generalisation of Exercise II-2.**

Let $Z = X + iY \in \mathbb{H}_n$ be an element of the Siegel upper half-plane of genus $n$ ($X, Y$ are symmetric real matrices of order $n$ and $Y > 0$). For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{R})$ we put

$$M\langle Z\rangle = (AZ + B)(CZ + D)^{-1}, \quad j(M, Z) = CZ + D.$$

We assume that $J(M, Z) = \det(j(M, Z)) = \det(CZ+D) \neq 0$. (See Exercise II-5 below.) Then using the method of Exercise II-3 to prove that

1) $(M_1 \cdot M_2)\langle Z\rangle = M_1\langle M_2\langle Z\rangle\rangle$;

2) $J(M_1M_2, Z) = J(M_1, M_2\langle Z\rangle)J(M_2, Z)$;

3) $\mathrm{Im}(M\langle Z\rangle) = {}^tj(M, \bar{Z})^{-1}\mathrm{Im}(Z)j(M, Z)^{-1}$.

4) Let $F : \mathbb{H}_n \to \mathbb{C}$. We put

$$F|_kM(Z) := \det(CZ + D)^{-k}F(M\langle Z\rangle), \quad M \in Sp_n(\mathbb{R}).$$

Show that

$$F|_k(M_1M_2) = (F|_kM_1)|_kM_2 \quad \forall\, M_1, M_2 \in Sp_n(\mathbb{R}).$$

**Exercise II-5. The Jacobi modular group.**

1) Prove that the Jacobi modular group $\Gamma^J$ is a semidirect product of two its subgroups (see 2)–3))

$$\Gamma^J = \{M = \begin{pmatrix} a & 0 & b & * \\ * & 1 & * & * \\ c & 0 & d & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \in Sp_2(\mathbb{Z})\} \cong SL_2(\mathbb{Z}) \ltimes H(\mathbb{Z}).$$

2) We put

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma^J(\mathbb{R}), \quad [\begin{pmatrix} p \\ q \end{pmatrix}, r] = \begin{pmatrix} 1 & 0 & 0 & p \\ -q & 1 & p & r \\ 0 & 0 & 1 & q \\ 0 & 0 & 0 & 1 \end{pmatrix} \in H(\mathbb{R}).$$

Find

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \langle Z \rangle \quad \text{and} \quad [\begin{pmatrix} p \\ q \end{pmatrix}, r]\langle Z \rangle.$$

3) Show that for any $M \in SL_2(\mathbb{R})$

$$[M][\begin{pmatrix} p \\ q \end{pmatrix}, r][M]^{-1} = [M \cdot \begin{pmatrix} p \\ q \end{pmatrix}, r].$$

4) We denote by $\Gamma^J(\mathbb{R})$ the group of real points of the Jacobi group. Let $Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in \mathbb{H}_2$. Show that $\tau, \omega \in \mathbb{H}_1$ are points of usual upper half-plane. Find $J(M, Z) = \det(CZ + D)$ for $M \in \Gamma^J(\mathbb{R})$.

**Exercise II-6$^*$. Non vanishing automorphic factor $J(M, Z)$.**

1) Let $M \in Sp_n(\mathbb{R})$. Prove that $J(M, iE_n) = \det(iC + D) \neq 0$.

2) It is clear that

$$\begin{pmatrix} y^{\frac{1}{2}} & xy^{-\frac{1}{2}} \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} \langle i \rangle = x + iy.$$

Find in the literature the proof of the existence of square root from any positive definite matrix. (Let $Y > 0$. Then there exists a symmetric matrix $Y^{\frac{1}{2}}$ such that $Y = (Y^{\frac{1}{2}})^2$.) Using this fact, to generalise the identity above. Then proof that $J(M, Z)$ does not vanish.