We first show the existence. That is, for each postive integer m, $\exists \{a_k\}_{k\geq 1}$ such that $m = \sum_{k=1}^{\infty} a_k f_k$, with the given restrictions. Since $1 = f_1$, the result holds for $m = 1$. Suppose $m > 1$ and such a representation exists for each $1 \leq r \leq m$. Let n be chosen so that $f_n \leq m < f_{n+1}$. If $m = f_n$, then the result holds with

$$a_k = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}.$$

Suppose $m > f_n$. Note that $1 \leq m - f_n < f_{n-1}$. By the Induction Hypothesis, $m - f_n = \sum_{k=1}^{\infty} b_k f_k$, with the desired restrictions on $\{b_k\}_{k\geq 1}$. Since $m - f_n < f_{n-1}$, we have $b_k = 0$ for $k \geq n-1$. Hence, if we let $a_k = \begin{cases} 1, & k = n \\ b_k, & k \neq n \end{cases}$, then we have a desired representation for m.

For $n \geq 0$, define

$$g_n = \begin{cases} 0, & n = 0 \\ f_1, & n = 1 \\ g_{n-2} + f_n, & n \geq 2 \end{cases}.$$

We show that for each n, $g_{n+1} = f_{n+1}$. This is clear for $n < 2$. Suppose $n \geq 2$ so that for each $0 \leq m < n$, $g_m + 1 = f_{m+1}$. It holds that $g_n + 1 = g_{n-2} + 1 + f_n = f_{n-1} + f_n = f_{n+1}$.

Suppose $\{a_k\}_{k\geq 1}$ is a sequence of 0's and 1's so that for all $k \geq 1$, $\{a_k, a_{k+1}\} \neq \{1\}$, and there is some $r \geq 0$ such that $a_r = 1$ if $r > 0$, and $a_k = 0$ if $k > r$. Define $S(\{a_k\}_{k\geq 1}) := \sum_{k=0}^{\infty} a_k f_k$. I claim that $S(\{a_k\}_{k\geq 1}) \leq g_r$. We proceed with induction on r. If $r = 0$, then all a_k are 0 and $S(\{a_k\}_{k\geq 1}) = 0 = g_0$. If $r = 1$, then $S(\{a_k\}_{k\geq 1}) = f_1 = g_1$. Fix some $r_0 > 1$ and assume $S(\{a_k\}_{k\geq 1}) \leq g_r$ whenever $r < r_0$. Consider $r = r_0$. Define b_k to be 0 if $k = r$ and a_k otherwise. Since $a_{r-1} = 0$, we have $b_k = 0$ for each $k \geq r-1$. Thus, $S(\{a_k\}_{k\geq 1}) = S(\{b_k\}_{k\geq 1}) + f_r \leq g_{r-2} + f_r = g_r$.

Lastly, we show that any positive integer m can be written **uniquely** in the given form. It is clear that $1 = f_1$ gives the unique representation for 1. Let $m > 1$ and assume desired representations are unique for each $t = 1, 2, \ldots, m-1$. Suppose $m = S(\{a_k\}_{k\geq 1}) = S(\{b_k\}_{k\geq 1})$ with the desired restrictions for $\{a_k\}_{k\geq 1}, \{b_k\}_{k\geq 1}$. Let $p, q \geq 1$ be chosen so that $1 = a_p = b_q$, $a_k = 0$ for $k > p$, and $b_k = 0$ for $k > q$. Assume wlog that $p \leq q$. If follows that $m - f_p = \sum_{k=1}^{p-1} a_k f_k = \sum_{k=1}^{p-1} b_k f_k$. By the Induction Hypothesis, $a_k = b_k$ for $k < p$, and hence, $a_k = b_k$ for all k, proving the uniqueness. $\square$