Analysis of a Complex Kind

Week 7

Lecture 2: Isolated Singularities of Analytic Functions

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Definition

A point z_0 is an *isolated singularity* of f if f is analytic in a punctured disk $\{0 < |z - z_0| < r\}$ centered at z_0.

- $f(z) = \dfrac{1}{z}$ has an isolated singularity at $z_0 = 0$.
- $f(z) = \dfrac{1}{\sin z}$ has isolated singularities at $z_0 = 0, \pm\pi, \pm 2\pi, \ldots$.
- $f(z) = \sqrt{z}$ and $f(z) = \operatorname{Log} z$ do not have isolated singularities at $z_0 = 0$ since these functions cannot be defined to be analytic in any punctured disk around 0.
- $f(z) = \dfrac{1}{z - 2}$ has an isolated singularity at $z_0 = 2$.

Laurent Series

By Laurent's Theorem, if f has an isolated singularity at z_0 (so f is analytic in the annulus $\{0 < |z - z_0| < r\}$ for some $r > 0$) then f has a Laurent series expansion there:

$$\begin{aligned} f(z) &= \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \\ &= \underbrace{\cdots \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{z - z_0}}_{\text{principal part}} + \underbrace{a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots}_{\text{analytic}} \end{aligned}$$

Three fundamentally different things can happen that influence how f behaves near z_0:

Three Types of Isolated Singularities

$$f(z) = \cdots \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots, \quad 0 < |z-z_0| < r$$

Examples:

- $f(z) = \frac{\cos z - 1}{z^2} = \frac{1}{z^2}\left(-\frac{z^2}{2!} + \frac{z^4}{4!} - + \cdots\right) = -\frac{1}{2!} + \frac{z^2}{4!} - + \cdots$ No negative powers of z!
- $f(z) = \frac{\cos z}{z^4} = \frac{1}{z^4}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - + \cdots\right) = \frac{1}{z^4} - \frac{1}{2!}\frac{1}{z^2} + \frac{1}{4!} - \frac{z^2}{6!} + - \cdots$ Finitely many negative powers of z!
- $f(z) = \cos\left(\frac{1}{z}\right) = 1 - \frac{1}{2!}\frac{1}{z^2} + \frac{1}{4!}\frac{1}{z^4} - \frac{1}{6!}\frac{1}{z^6} + - \cdots$ Infinitely many negative powers of z!

We classify singularities based upon these differences:

Definition

Suppose z_0 is an isolated singularity of an analytic function f with Laurent series $\sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$, $0 < |z-z_0| < r$. Then the singularity z_0 is

- *removable* if $a_k = 0$ for all $k < 0$.
- a *pole* if there exists $N > 0$ so that $a_{-N} \neq 0$ but $a_k = 0$ for all $k < -N$. The index N is the *order* of the pole.
- *essential* if $a_k \neq 0$ for infinitely many $k < 0$.

Types of Singularities

The following table illustrates this definition:

z_0 is a ...	Laurent series in $0 < \lvert z - z_0 \rvert < r$
Removable singularity	$a_0 + a_1(z - z_0) + \cdots$
Pole of order N	$\dfrac{a_{-N}}{(z - z_0)^N} + \cdots + \dfrac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots$
Simple pole	$\dfrac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots$
Essential singularity	$\cdots + \dfrac{a_{-2}}{(z - z_0)^2} + \dfrac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots$

Removable Singularities

Recall: z_0 *is a removable singularity of* f *if its Laurent series, centered at* z_0 *satisfies that* $a_k = 0$ *for all* $k < 0$.

Example: $f(z) = \dfrac{\sin z}{z} = 1 - \dfrac{z^2}{3!} + \dfrac{z^4}{5!} - + \cdots, 0 < |z| < \infty$.

The Laurent series looks like a Taylor series! Taylor series are analytic within their region of convergence. Thus, if we define $f(z)$ to have the value 1 at $z_0 = 0$, then f becomes analytic in $\mathbb{C}$:

$$f(z) = \begin{cases} \dfrac{\sin z}{z}, & z \neq 0 \\ 1 & z = 0 \end{cases} \text{ is analytic in } \mathbb{C}.$$

We have removed the singularity.

Theorem (Riemann's Theorem)

Let z_0 *be an isolated singularity of* f. *Then* z_0 *is a removable singularity if and only if* f *is bounded near* z_0.

Poles

Recall: *z_0 is a pole of order N of f if its Laurent series, centered at z_0 satisfies that $a_{-N} \neq 0$ and $a_k = 0$ for all $k < -N$.*

Example: $f(z) = \frac{\sin z}{z^5} = \frac{1}{z^4} - \frac{1}{3!}\frac{1}{z^2} + \frac{1}{5!} - \frac{1}{7!}z^2 + - \cdots$ has a pole of order 4 at 0.

Theorem

Let z_0 be an isolated singularity of f. Then z_0 is a pole if and only if $|f(z)| \to \infty$ as $z \to z_0$.

Note: If $f(z)$ has a pole at z_0 then $\frac{1}{f(z)}$ has a removable singularity at z_0 (and vice versa).

Recall: *z_0 is an essential singularity of f if its Laurent series, centered at z_0 satisfies that $a_k \neq 0$ for infinitely many $k < 0$.*

Example: $f(z) = e^{\frac{1}{z}} = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{z^k} = 1 + \frac{1}{z} + \frac{1}{2!} \frac{1}{z^2} + \frac{1}{3!} \frac{1}{z^3} + \cdots$ has an essential singularity at $z_0 = 0$.

Note that if $z = x \in \mathbb{R}$, then

$f(z) = e^{1/x} \to \infty$ as $x \to 0$ from the right and $f(z) = e^{1/x} \to 0$ as $x \to 0$ from the left.

Also, if $z = ix \in i\mathbb{R}$ then

$$f(z) = e^{1/ix} = e^{-i/x} \text{ lies on the unit circle for all } x.$$

It appears that f does not have a limit as $z \to z_0$.

Theorem (Casorati-Weierstraß)

Suppose that z_0 is an essential singularity of f. Then for every $w_0 \in \mathbb{C}$ there exists a sequence $\{z_n\}$ with $z_n \to z_0$ such that $f(z_n) \to w_0$ as $n \to \infty$.

Casorati-Weierstraß

Casorati-Weierstraß: Suppose that z_0 is an essential singularity of f. Then for every $w_0 \in \mathbb{C}$ there exists a sequence $\{z_n\}$ with $z_n \to z_0$ such that $f(z_n) \to w_0$.

Example:

- Let $f(z) = e^{\frac{1}{z}}$. Then f has an essential singularity at 0. Let's pick a point $w_0 \in \mathbb{C}$, say, $w_0 = 1 + \sqrt{3}i$.
- By Casorati-Weierstraß there must exist $z_n \in \mathbb{C} \setminus \{0\}$ such that $e^{\frac{1}{z_n}} \to 1 + \sqrt{3}i$ as $n \to \infty$.
- How do we find z_n?
- Idea: We can find z_n such that $e^{\frac{1}{z_n}} = 1 + \sqrt{3}i$, namely $\frac{1}{z_n} = \log(1 + \sqrt{3}i)$.
- Recall: $\log(z) = \ln(|z|) + i \arg(z)$.
- So $\log(1 + \sqrt{3}i) = \ln 2 + i\frac{\pi}{3} + 2n\pi i$. Pick $z_n = \dfrac{1}{\ln 2 + i\frac{\pi}{3} + 2n\pi i}$.

Casorati-Weierstraß: Essential Singularity of $e^{\frac{1}{z}}$ at $z_0 = 0$.

$$z_n = \frac{1}{\ln 2 + i\frac{\pi}{3} + 2n\pi i}.$$

Then $z_n \to 0$ as $n \to \infty$. Furthermore:

$$\begin{aligned} e^{\frac{1}{z_n}} &= e^{\ln(2) + i\frac{\pi}{3} + 2n\pi i} \\ &= 2e^{i\frac{\pi}{3}} \\ &= 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\ &= 1 + \sqrt{3}i = w_0 \end{aligned}$$

for all n.
We thus found z_n with $z_n \to 0$ such that $f(z_n) = w_0$ for all n.

Picard's Theorem

We just observed a much stronger result that is true (but much harder to prove) for essential singularities:

Theorem (Picard)

Suppose that z_0 is an essential singularity of f. Then for every $w_0 \in \mathbb{C}$ with at most one exception there exists a sequence $\{z_n\}$ with $z_n \to z_0$ such that $f(z_n) = w_0$.

Example: $f(z) = e^{1/z}$ has an essential singularity at $z_0 = 0$. Also, $f(z) \neq 0$ for all z, and so by Picard's theorem, for every $w_0 \neq 0$ there must exist infinitely many z_n with $z_n \to 0$ such that $f(z_n) = w_0$.

Pick $w_0 = 1$ for example. Then $f(z) = w_0$ if $e^{1/z} = 1$, that is $\dfrac{1}{z} = 2n\pi i$ for some $n \in \mathbb{Z}$. Now let $z_n = \dfrac{1}{2n\pi i}$. Then $z_n \to 0$ as $n \to \infty$, and $f(z_n) = 1$ for all n.