Analysis of a Complex Kind

Week 6

Lecture 1: Infinite Series of Complex Numbers

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Infinite Series

Definition

An infinite series

$$\sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + \cdots + a_n + a_{n+1} + \cdots$$

(with $a_k \in \mathbb{C}$) converges to S if the sequence of partial sums $\{S_n\}$, given by

$$S_n = \sum_{k=0}^{n} a_k = a_0 + a_1 + \cdots + a_n$$

converges to S.

Example

Example: Consider $\sum_{k=0}^{\infty} z^k$, for some $z \in \mathbb{C}$. We have that

$$S_n = 1 + z + z^2 + \cdots + z^n.$$

Can we find a closed formula for S_n in order to help us find the limit as $n \to \infty$?
Trick:

$$\begin{aligned} S_n &= 1 + z + z^2 + \cdots + z^n, \quad \text{so} \\ z \cdot S_n &= \phantom{1 +{}} z + z^2 + \cdots + z^n + z^{n+1}, \quad \text{thus} \\ S_n - zS_n &= 1 - z^{n+1}. \end{aligned}$$

Hence $S_n = \frac{1 - z^{n+1}}{1 - z}$ for $z \neq 1$, and since $z^{n+1} \to 0$ as $n \to \infty$ as long as $|z| < 1$ we have that

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1 - z} \quad \text{for } |z| < 1.$$

What happens for $|z| \geq 1$?

Theorem

If a series $\displaystyle\sum_{k=0}^{\infty} a_k$ *converges then* $a_k \to 0$ *as* $k \to \infty$.

In our example: If $|z| \geq 1$, then $|z|^k \not\to 0$ as $k \to \infty$, thus $\displaystyle\sum_{k=0}^{\infty} z^k$ does not converge for $|z| \geq 1$. We say the series *diverges* for $|z| \geq 1$.

Let's now analyze the real and imaginary parts of the equation $\displaystyle\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$ for $|z| < 1$:

Writing $z = re^{i\theta}$ we have $z^k = r^k e^{ik\theta} = r^k \cos(k\theta) + ir^k \sin(k\theta)$. Thus

$$\sum_{k=0}^{\infty} z^k = \sum_{k=0}^{\infty} r^k \cos(k\theta) + i \sum_{k=0}^{\infty} r^k \sin(k\theta).$$

Furthermore,

$$\begin{aligned}
\frac{1}{1-z} = \frac{1}{1-re^{i\theta}} &= \frac{1-re^{-i\theta}}{(1-re^{i\theta})(1-re^{-i\theta})} \\
&= \frac{1-r\cos\theta + ir\sin\theta}{1-re^{i\theta}-re^{-i\theta}+r^2} = \frac{1-r\cos\theta+ir\sin\theta}{1-2r\cos\theta+r^2}.
\end{aligned}$$

Thus

$$\sum_{k=0}^{\infty} r^k \cos(k\theta) = \frac{1-r\cos\theta}{1-2r\cos\theta+r^2} \quad \text{and} \quad \sum_{k=0}^{\infty} r^k \sin(k\theta) = \frac{r\sin\theta}{1-2r\cos\theta+r^2}.$$

Another Example

Next, consider $\sum_{k=1}^{\infty} \frac{i^k}{k}$. Does this series converge?

- We note that $\sum_{k=1}^{\infty} \left| \frac{i^k}{k} \right| = \sum_{k=1}^{\infty} \frac{1}{k}$ is the harmonic series, which is known to diverge. One way to see this:

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{\geq 1/2} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{\geq 1/2} + \underbrace{\left(\frac{1}{9} + \cdots + \frac{1}{16}\right)}_{\geq 1/2} + \cdots$$

- But does the series itself (without the absolute values) converge? Let's split it up into real and imaginary parts.

Does $\sum_{k=1}^{\infty} \frac{i^k}{k}$ Converge?

Note: When k is even (i.e. k is of the form $k = 2n$), then $i^k = i^{2n} = (-1)^n$ is real. When k is odd (i.e. k is of the form $k = 2n+1$), then $i^k = i^{2n+1} = i(-1)^n$ is purely imaginary. Thus

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{i^k}{k} &= \sum_{n=1}^{\infty} \frac{i^{2n}}{2n} + \sum_{n=0}^{\infty} \frac{i^{2n+1}}{2n+1} \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} + i \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.
\end{aligned}$$

But

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - + \cdots$$

is the alternating harmonic series, which converges.

Absolute Convergence

Definition

A series $\sum_{k=0}^{\infty} a_k$ converges absolutely if the series $\sum_{k=0}^{\infty} |a_k|$ converges.

Examples:

- $\sum_{k=0}^{\infty} z^k$ converges and converges absolutely for $|z| < 1$.
- $\sum_{k=1}^{\infty} \frac{i^k}{k}$ converges, but not absolutely.

Theorem

If $\sum_{k=0}^{\infty} a_k$ *converges absolutely, then it also converges, and* $\left|\sum_{k=0}^{\infty} a_k\right| \leq \sum_{k=0}^{\infty} |a_k|$.

Example

Here is an example: If $|z| < 1$, then the series $\sum_{k=0}^{\infty} z^k$ converges absolutely, so

$$\left|\sum_{k=0}^{\infty} z^k\right| \leq \sum_{k=0}^{\infty} |z|^k.$$

But the left-hand side equals $\left|\frac{1}{1-z}\right|$, and the right-hand side equals $\frac{1}{1-|z|}$, so that

$$\left|\frac{1}{1-z}\right| \leq \frac{1}{1-|z|}.$$

Next up: Power Series.