Consider a term in the expansion of $(x+y)^n$ of the form $t_1 t_2 \dots t_n$ where $t_i \in \{x, y\}$ for each $1 \le i \le n$. Let $k = |\{1 \le i \le n : t_i = x\}|$. Then $t_1 t_2 \dots t_n = q^m x^k y^{n-k}$, where

$$m = \sum_{\substack{1 \le i \le n, \\ t_i = x}} \sum_{\substack{1 \le j < i, \\ t_j = y}} 1.$$

We construct a Young diagram fitting in a $k \times (n-k)$ rectangle as follows:

Define indices $n \ge i_1 > i_2 > \dots > i_k \ge 1$, where $x = t_{i_1} = t_{i_2} = \dots = t_{i_k}$. Note that for each i_j, the number of y's occurring before position i_j is $i_j - 1 - (k - j) = i_j + j - k - 1$. This quantity non-strictly decreases as j increases. Let the j-th row of our Young diagram consist of $i_j + j - k - 1$ columns, so the entire diagram has m grids.

We now have a bijection between the terms in $(x+y)^n$ equal to $q^m x^k y^{n-k}$ and the Young diagrams of size m fitting in $k \times (n-k)$ rectangle. The number of such diagrams is the coefficient of q^m in $\begin{bmatrix} n \\ k \end{bmatrix}_q$. $\square$