Hw. 2 Problem 1

January 1, 2018

Find the image of the set $U = \{z \in \mathbb{C} : -\frac{\pi}{2} < \text{Re}z < \frac{\pi}{2}\}$ under the function $f(z) = \sin z$.

1. What is the image of the line segment $L_1 = (-\frac{\pi}{2}, \frac{\pi}{2})$ under f?

   Notice that the real function, $\sin x$, is continuous and increasing on L_1, with $\sin(-\frac{\pi}{2}) = -1$ and $\sin(\frac{\pi}{2}) = 1$. Thus, $f(L_1) = (-1, 1)$.

2. What is the image of the imaginary axis $L_2 = \{iy : y \in \mathbb{R}\}$ under f?

   We know that $\sin(iy) = \frac{e^{-y}-e^{y}}{2i}$. For each $y \in \mathbb{R}$, we have that $\sin(iy) \in i\mathbb{R} := \{ia : a \in \mathbb{R}\}$. Consider $g(y) = e^{-y} - e^{y}$. Note that g is continuous on $\mathbb{R}$. Also,

   $$\lim_{y \to \infty} g(y) = -\infty$$

   and

   $$\lim_{y \to -\infty} g(y) = \infty$$

   so $g(\mathbb{R}) = \mathbb{R}$ by the intermediate value theorem (IVT). Hence, $\sin(iy) = 2ig(y)$, so $f(L_2) = L_2$, the imaginary axis itself.

3. What is the image of the vertical line $L_3 = \{-\frac{\pi}{2} + iy : y \in \mathbb{R}\}$ under f?

First, observe that for $y \in \mathbb{R}$,

$$\begin{aligned}\sin(iy - \frac{\pi}{2}) &= \sin(iy)\cos(-\frac{\pi}{2}) + \cos(iy)\sin(-\frac{\pi}{2}) \\ &= -\cos(iy) \\ &= \frac{-e^{-y} - e^{y}}{2}.\end{aligned}$$

Let $h(y) = e^{-y} + e^{y}$. Then $h'(y) = -e^{-y} + e^{y}$ and $h''(y) = h(y) > 0$. Thus, h' is strictly increasing on $\mathbb{R}$ and hence, has at most one zero. Also, $h'(0) = 0$. We conclude that h is strictly decreasing on $(-\infty, 0)$, achieves a global minimum of 2 at 0, and is strictly increasing on $(0, \infty)$. Note that h is unbounded above as $\lim_{y\to\infty} h(y) = \infty$. Hence, $h(\mathbb{R}) = [2, \infty)$. Since $\sin(iy - \frac{\pi}{2}) = -\frac{1}{2}h(y)$, we have $f(L_3) = (-\infty, -1]$.

4. What is the image of the vertical line $L_4 = \{\frac{\pi}{2} + iy : y \in \mathbb{R}\}$ under f?

   Computing $f(L_4)$ involves an argument similar to that for computing $f(L_3)$. We find that $\sin(iy + \frac{\pi}{2}) = \cos(iy)$ and conclude that $f(L_4) = [1, \infty)$.

5. Given your above observations, what do you guess the image of the set U is under f?

   I believe it should be $\mathbb{C} - f(L_3) - f(L_4)$, which is

$$\mathbb{C} - \{z \in \mathbb{C} : \mathrm{Im} z = 0, |z| \geq 1\}.$$

   This claim indeed makes sense. We know that f is entire, so $f(\mathbb{C}) = \mathbb{C}$. Given $S \subseteq \mathbb{C}$, let $\overline{S}$ denote the closure of S, which is S, together with its boundary points. As with the situation in $\mathbb{R}$, $f(\overline{L_1}) = f(\mathbb{R})$. We can somehow view U as a analogue of L_1 for $\mathbb{C}$, and guess that $f(\overline{U}) = f(\mathbb{C}) = \mathbb{C}$. Notice that $U = \overline{U} - L_3 - L_4$, which is what guided my intuition to the above answer for $f(U)$.