Analysis of a Complex Kind

Week 5

Lecture 4: Cauchy's Theorem and Integral Formula

Petra Bonfert-Taylor

Cauchy's Theorem

Theorem (Cauchy's Theorem for Simply Connected Domains)

Let D be a simply connected domain in $\mathbb{C}$, and let f be analytic in D. Let $\gamma : [a, b] \to D$ be a piecewise smooth, closed curve in D (i.e. $\gamma(b) = \gamma(a)$). Then

$$\int_\gamma f(z)dz = 0.$$

Example: $f(z) = e^{\left(z^3\right)}$ is analytic in $\mathbb{C}$, and $\mathbb{C}$ is simply connected. Therefore, $\displaystyle\int_\gamma f(z)dz = 0$ for any closed, piecewise smooth curve in $\mathbb{C}$.

Proof idea: Since D has no holes, γ can be deformed continuously to a point in D. Show that the integral does not change along the way by using the Cauchy Theorem in a disk.

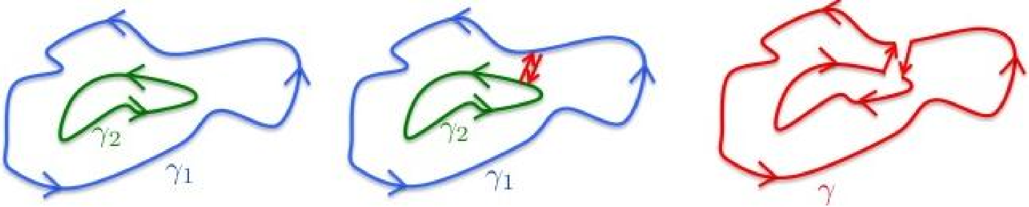
A First Conclusion

Corollary

Let γ_1 and γ_2 be two simple closed curves (i.e. neither of the curves intersects itself), oriented counterclockwise, where γ_2 is inside γ_1. If f is analytic in a domain D that contains both curves as well as the region between them, then

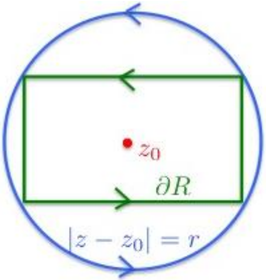
$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

Proof idea: Here is a neat trick: Form a "joint curve" γ as in the picture below. As f is analytic in a simply connected region, containing γ, we have $\int_\gamma f(z)dz = 0$.



- Let R be a rectangle, centered at z_0. Claim: $\displaystyle\int_{\partial R} \frac{1}{z - z_0} dz = 2\pi i$.

  Proof: We calculate that $\displaystyle\int_{|z-z_0|=r} \frac{1}{z - z_0} dz = 2\pi i$. Since f is analytic "between" the two curves, the integrals must agree.



Examples

- $\displaystyle\int_{|z|=1} \frac{1}{z^2+2z}\,dz = ?$

$$\frac{1}{z^2+2z} = \frac{1}{z(z+2)} = \frac{1}{2}\frac{(z+2)-z}{z(z+2)} = \frac{1}{2}\left(\frac{1}{z} - \frac{1}{z+2}\right).$$

Thus

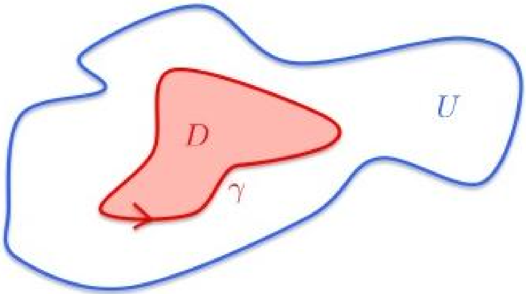
$$\begin{aligned}\int_{|z|=1} \frac{1}{z^2+2z}\,dz &= \frac{1}{2}\left(\int_{|z|=1} \frac{1}{z}\,dz - \int_{|z|=1} \frac{1}{z+2}\,dz\right)\\ &= \frac{1}{2}(2\pi i - 0) = \pi i\end{aligned}$$

since the function $z \mapsto \dfrac{1}{z+2}$ is analytic in the simply connected domain $B_{1.5}(0)$, which in turn contains the curve we're integrating over.

Theorem (Cauchy Integral Formula)

Let D be a simply connected domain, bounded by a piecewise smooth curve γ, and let f be analytic in a set U that contains the closure of D (i.e. D and γ). Then

$$f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}dz \quad \textit{for all } w \in D.$$



The Proof of the Cauchy Integral Formula

The proof of the Cauchy Integral Formula goes as follows:

- Let $w \in D$, pick $\varepsilon > 0$ such that $\overline{B_\varepsilon(w)} \subset D$.
- Using Cauchy's theorem, we see that

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}dz = \frac{1}{2\pi i}\int_{\partial B_\varepsilon(w)} \frac{f(z)}{z-w}dz,$$

  since the integrand is analytic in a region containing these two curves and the area in between them.
- It is easily seen that

$$\frac{1}{2\pi i}\int_{\partial B_\varepsilon(w)} \frac{f(z)}{z-w}dz = \frac{1}{2\pi}\int_0^{2\pi} f(w+\varepsilon e^{it})dt.$$

- This is true for any (small) $\varepsilon > 0$, and as $\varepsilon \to 0$, the right-hand side approaches $f(w)$.

Examples

$$\boxed{f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}dz \quad \text{for all } w \in D.}$$

- $\displaystyle\int_{|z|=2} \frac{z^2}{z-1}dz = ?$

  Here, we have $f(z) = z^2$ and $w = 1$, and so

$$\frac{1}{2\pi i}\int_{|z|=2} \frac{f(z)}{z-1}dz = f(1) = 1.$$

  Hence

$$\int_{|z|=2} \frac{z^2}{z-1}dz = 2\pi i.$$

Examples

$$\boxed{f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}dz \quad \text{for all } w \in D.}$$

- $\displaystyle\int_{|z|=1} \frac{z^2}{z-2}dz = ?$

  First thought: $f(z) = z^2$ and $w = 2\ldots$ However: $w = 2$ is not *inside* the curve!

  But: The function $z \mapsto \dfrac{z^2}{z-2}$ is analytic in $B_{1.5}(0)$, and this implies (using Cauchy's theorem) that

$$\int_{|z|=1} \frac{z^2}{z-2}dz = 0.$$

$$\boxed{f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}dz \quad \text{for all } w \in D.}$$

- $\displaystyle\int_{|z|=1} \frac{\mathrm{Log}(z+e)}{z}dz = ?$
  Here, $f(z) = \mathrm{Log}(z+e)$, and $w = 0$. The function f is analytic in $\{\mathrm{Re}\, z > -e\}$, which contains the curve we're integrating over along with its inside. Also, w is inside the curve. Thus

$$\int_{|z|=1} \frac{\mathrm{Log}(z+e)}{z}dz = 2\pi i\,\mathrm{Log}(0+e) = 2\pi i.$$

Analyticity of the Derivative

Here is an amazing consequence of the Cauchy Integral Formula:

Theorem

If f is analytic in an open set U, then f' is also analytic in U.

Idea of proof:

- We first use the Cauchy Integral Formula to show that for any $w \in U$, the derivative $f'(w)$ can be found via

$$f'(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-w)^2} dz,$$

  where γ is the boundary of a small disk, centered at w; small enough so that it fits into U.

- We next show that the right-hand side defines an analytic function in w, and therefore f' must be analytic.

The Cauchy Integral Formula for Derivatives

Repeated application of the previous theorem shows that an analytic function has infinitely many derivatives!!

Continuing along the same lines as the previous proof yields the following extension of the Cauchy Integral Formula:

Theorem (Cauchy Integral Formula for Derivatives)

Let D be a simply connected domain, bounded by a piecewise smooth curve γ, and let f be analytic in a set U that contains the closure of D (i.e. D and γ). Then

$$f^{(k)}(w) = \frac{k!}{2\pi i} \int_\gamma \frac{f(z)}{(z-w)^{k+1}} dz \quad \textit{for all } w \in D,\ k \geq 0.$$

Here, $f^{(k)}$ denotes the kth derivative of f.

Examples

$$\boxed{f^{(k)}(w) = \frac{k!}{2\pi i}\int_\gamma \frac{f(z)}{(z-w)^{k+1}}dz \quad \text{for all } w \in D,\ k \geq 0.}$$

- $\displaystyle\int_{|z|=2\pi} \frac{z^2 \sin z}{(z-\pi)^3} dz = ?$

  Here, we have $f(z) = z^2 \sin z$, $w = \pi$, and $k = 2$. We thus need to find $f''(\pi)$!

$$\begin{aligned} f'(z) &= 2z \sin z + z^2 \cos z \\ f''(z) &= 2 \sin z + 2z \cos z + 2z \cos z - z^2 \sin z, \quad \text{so } f''(\pi) = -4\pi. \end{aligned}$$

  Thus

$$\int_{|z|=2\pi} \frac{z^2 \sin z}{(z-\pi)^3} dz = \frac{2\pi i}{2!} f^{(2)}(\pi) = \frac{2\pi i(-4\pi)}{2} = -4\pi^2 i.$$

Examples

$$\boxed{f^{(k)}(w) = \frac{k!}{2\pi i}\int_\gamma \frac{f(z)}{(z-w)^{k+1}}dz \quad \text{for all } w \in D,\ k \geq 0.}$$

- $\displaystyle\int_{|z|=2} \frac{e^z}{(z+1)^2}dz = ?$

  Here, we have $f(z) = e^z$, $w = -1$, and $k = 1$. We thus need to find $f'(-1)$!

$$f'(z) \quad = \quad e^z, \quad \text{so } f'(-1) = e^{-1} = \frac{1}{e}.$$

  Thus

$$\int_{|z|=2} \frac{e^z}{(z+1)^2}dz = \frac{2\pi i}{1!}f'(-1) = \frac{2\pi i}{e}.$$

Next...

Next up:

Amazing consequences of
Cauchy's Theorem and Integral Formula!