Analysis of a Complex Kind

Week 6

Lecture 2: Power Series

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Power Series (Taylor Series)

Definition

A power series (also called Taylor series), centered at $z_0 \in \mathbb{C}$, is a series of the form

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k.$$

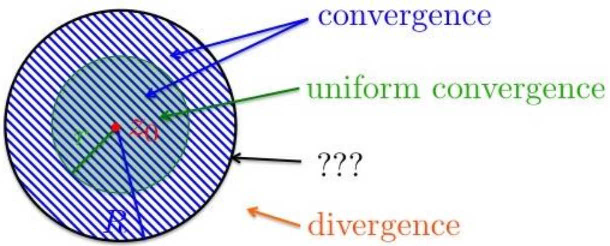
Examples:

- $\displaystyle\sum_{k=0}^{\infty} z^k$ is a power series with $a_k = 1$, $z_0 = 0$. It converges for $|z| < 1$.
- $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} z^{2k} = 1 - \frac{z^2}{2} + \frac{z^4}{4} - \frac{z^6}{8} + - \cdots = \sum_{k=0}^{\infty} \left(\frac{-z^2}{2}\right)^k = \sum_{k=0}^{\infty} w^k$,
  where $w = \frac{-z^2}{2}$. This series converges when $|w| < 1$, and diverges when $|w| \geq 1$. Therefore, the original series converges when $|z| < \sqrt{2}$ and diverges when $|z| \geq \sqrt{2}$.

For what values of z does a power series converge?

Theorem

Let $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ be a power series. Then there exists a number R, with $0 \le R \le \infty$, such that the series converges absolutely in $\{|z-z_0|<R\}$ and diverges in $\{|z-z_0|>R\}$. Furthermore, the convergence is uniform in $\{|z-z_0| \le r\}$ for each $r<R$.



We call R the *radius of convergence* of the power series.

Examples

- $\displaystyle\sum_{k=0}^{\infty} z^k$ has radius of convergence 1.
- $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} z^{2k}$ has radius of convergence $\sqrt{2}$.
- $\displaystyle\sum_{k=0}^{\infty} k^k z^k$??? Pick an arbitrary $z \in \mathbb{C} \setminus \{0\}$. Observe that $\left|k^k z^k\right| = (k|z|)^k \geq 2^k$ as soon as $k \geq \dfrac{2}{|z|}$, thus the series does not converge for any $z \neq 0$. The radius of convergence of this power series is 0!
- $\displaystyle\sum_{k=0}^{\infty} \frac{z^k}{k^k}$??? Pick an arbitrary $z \in \mathbb{C}$. Observe that $\left|\dfrac{z^k}{k^k}\right| = \left(\dfrac{|z|}{k}\right)^k \leq \left(\dfrac{1}{2}\right)^k$ as soon as $k \geq 2|z|$. Thus the series converges absolutely for all $z \in \mathbb{C}$, and so $R = \infty$!

Analyticity of Power Series

Theorem

Suppose that $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ *is a power series of radius of convergence* $R > 0$.

Then

$$f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k \text{ is analytic in } \{|z-z_0| < R\}.$$

Furthermore, the series can be differentiated term by term, i.e.

$$f'(z) = \sum_{k=1}^{\infty} a_k \cdot k(z-z_0)^{k-1}, \quad f''(z) = \sum_{k=2}^{\infty} a_k \cdot k(k-1)(z-z_0)^{k-2}, \quad \ldots$$

In particular, $f^{(k)}(z_0) = a_k \cdot k!$, *i.e.* $a_k = \dfrac{f^{(k)}(z_0)}{k!}$ *for* $k \geq 0$.

Example

Recall that $\sum_{k=0}^{\infty} z^k$ has radius of convergence 1, and so by the theorem,

$$f(z) = \sum_{k=0}^{\infty} z^k \text{ is analytic in } \{|z| < 1|\}.$$

Taking the derivative and differentiating term by term (as in the theorem), we find

$$f'(z) = \sum_{k=1}^{\infty} kz^{k-1} \quad (= \sum_{k=0}^{\infty}(k+1)z^k).$$

But we also know that $f(z) = \dfrac{1}{1-z}$, and so $f'(z) = \dfrac{1}{(1-z)^2}$. Thus

$$\sum_{k=0}^{\infty}(k+1)z^k = \frac{1}{(1-z)^2}.$$

Integration of Power Series

Note: Power series can similarly be integrated term by term:

Fact

If $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ *has radius of convergence* R*, then for any* w *with* $|w - z_0| < R$ *we have that*

$$\int_{z_0}^{w} \sum_{k=0}^{\infty} a_k(z - z_0)^k dz = \sum_{k=0}^{\infty} a_k \int_{z_0}^{w} (z - z_0)^k dz = \sum_{k=0}^{\infty} a_k \frac{1}{k+1}(w - z_0)^{k+1}.$$

Here, the integral is taken over any curve in the disk $\{|z - z_0| < R\}$ *from* z_0 *to* w*.*

Example

Let's again look at the power series $\sum_{k=0}^{\infty} z^k$, which has $R = 1$. Then for any w with $|w| < 1$ we thus have

$$\int_0^w \sum_{k=0}^{\infty} z^k dz = \sum_{k=0}^{\infty} \int_0^w z^k dz = \sum_{k=0}^{\infty} \frac{1}{k+1} w^{k+1} = \sum_{k=1}^{\infty} \frac{w^k}{k}.$$

We also know that $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$, hence

$$\int_0^w \sum_{k=0}^{\infty} z^k dz = \int_0^w \frac{1}{1-z} dz = -\operatorname{Log}(1-z)|_0^w = -\operatorname{Log}(1-w).$$

Here, we used that $\operatorname{Log} z$ is analytic in $\mathbb{C} \setminus (-\infty, 0]$, hence $-\operatorname{Log}(1-z)$ is analytic in $\mathbb{C} \setminus [1, \infty)$, in particular in $\{|z| < 1\}$, where it is a primitive of $\frac{1}{1-z}$.

We have shown:

$$\int_0^w \sum_{k=0}^{\infty} z^k dz = \sum_{k=1}^{\infty} \frac{w^k}{k} \qquad \text{and} \qquad \int_0^w \sum_{k=0}^{\infty} z^k dz = -\operatorname{Log}(1-w),$$

hence

$$\sum_{k=1}^{\infty} \frac{w^k}{k} = -\operatorname{Log}(1-w) \quad \text{for } |w| < 1.$$

Letting $z = 1 - w$ this becomes

$$\operatorname{Log} z = -\sum_{k=1}^{\infty} \frac{(1-z)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(z-1)^k \quad \text{for } |z-1| < 1.$$

Question

How do you find R, the radius of convergence of a power series?