Analysis of a Complex Kind

Week 6

Lecture 3: The Radius of Convergence of a Power Series

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Recall: Given a power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$, there exists a number R with $0 \leq R \leq \infty$ such that the series converges (absolutely) in $\{|z - z_0| < R\}$ and diverges in $\{|z - z_0| > R\}$.

How do you find R?

Theorem (Ratio Test)

If the sequence $\left\{ \left| \frac{a_k}{a_{k+1}} \right| \right\}$ has a limit as $k \to \infty$ then this limit is the radius of convergence, R, of the power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$.

Note: "∞" is an allowable limit.

Examples

- $\displaystyle\sum_{k=0}^{\infty} z^k$: Here, $a_k = 1$, so $\left|\dfrac{a_k}{a_{k+1}}\right| = 1 \to 1$ as $k \to \infty$. Thus $R = 1$.
- $\displaystyle\sum_{k=0}^{\infty} kz^k$: Here, $a_k = k$, so $\left|\dfrac{a_k}{a_{k+1}}\right| = \dfrac{k}{k+1} \to 1$ as $k \to \infty$. Thus $R = 1$.
- $\displaystyle\sum_{k=0}^{\infty} \frac{z^k}{k!}$: Here, $a_k = \dfrac{1}{k!}$, so $\left|\dfrac{a_k}{a_{k+1}}\right| = \dfrac{(k+1)!}{k!} = k+1 \to \infty$ as $k \to \infty$. Thus $R = \infty$.
- $\displaystyle\sum_{k=0}^{\infty} \frac{z^k}{k^k}$: Here, $a_k = \dfrac{1}{k^k}$, so $\left|\dfrac{a_k}{a_{k+1}}\right| = \dfrac{(k+1)^{k+1}}{k^k} \to ???$ as $k \to \infty$.

The Root Test

The last example (where $a_k = \dfrac{1}{k^k}$) is easier to treat using the following:

Theorem (Root Test)

If the sequence $\left\{ \sqrt[k]{|a_k|} \right\}$ *has a limit as* $k \to \infty$ *then* $R = \dfrac{1}{\lim\limits_{k\to\infty} \left\{ \sqrt[k]{|a_k|} \right\}}$.

Note:

- If $\lim\limits_{k\to\infty} \left\{ \sqrt[k]{|a_k|} \right\} = 0$ then $R = \infty$.
- If $\lim\limits_{k\to\infty} \left\{ \sqrt[k]{|a_k|} \right\} = \infty$ then $R = 0$.

Examples

- $\displaystyle\sum_{k=0}^{\infty} \frac{z^k}{k^k}$: Here, $a_k = \dfrac{1}{k^k}$, so $\sqrt[k]{|a_k|} = \dfrac{1}{k} \to 0$ as $k \to \infty$. Thus $R = \infty$.
- $\displaystyle\sum_{k=0}^{\infty} kz^k$: Here, $a_k = k$, so $\sqrt[k]{|a_k|} = \sqrt[k]{k} \to 1$ as $k \to \infty$. Thus $R = 1$.
- $\displaystyle\sum_{k=0}^{\infty} 2^k z^k$: Here, $a_k = 2^k$, so $\sqrt[k]{|a_k|} = 2 \to 2$ as $k \to \infty$. Thus $R = \dfrac{1}{2}$.
- $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} z^{2k}$: Here, $a_{2k} = \dfrac{(-1)^k}{2^k}$, and $a_{2k+1} = 0$, so $\sqrt[2k]{|a_{2k}|} = \dfrac{1}{2^{1/2}}$ for $k \geq 1$ and $\sqrt[2k+1]{|a_{2k+1}|} = 0$, and so the sequence $\sqrt[k]{|a_k|}$ $(k \geq 1)$ is

$$0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, \cdots,$$

and this sequence does not have a limit. Note: $\left|\dfrac{a_k}{a_{k+1}}\right|$ has no limit either.

The Cauchy Hadamard Criterion

Thus for the series $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} z^{2k}$, neither the root test nor the ratio test “work”.

Yet, letting $w = z^2$, the series becomes $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} w^k$, and $\displaystyle\sqrt[k]{\left|\frac{(-1)^k}{2^k}\right|} = \frac{1}{2} \to \frac{1}{2}$ as $k \to \infty$, so the new series converges for $|w| < 2$. Thus the original series converges for $|z|^2 < 2$, i.e. for $|z| < \sqrt{2}$. Is there a formula that finds this?

Fact (Cauchy-Hadamard)

The radius of convergence of the power series $\displaystyle\sum_{k=0}^{\infty} a_k (z - z_0)^k$ *equals*

$$R = \frac{1}{\limsup\limits_{k\to\infty} \sqrt[k]{|a_k|}}.$$

Analytic Functions And Power Series

Recall: $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ is analytic in $\{|z-z_0|<R\}$, where R is the radius of convergence of the power series. Now:

Theorem

Let $f : U \to \mathbb{C}$ be analytic and let $\{|z-z_0|<r\} \subset U$. Then in this disk, f has a power series representation

$$f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k, \quad |z-z_0|<r, \quad \text{where } a_k = \frac{f^{(k)}(z_0)}{k!}, k \geq 0.$$

The radius of convergence of this power series is $R \geq r$.

Examples

$$\boxed{f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \quad |z - z_0| < r, \quad \text{where } a_k = \frac{f^{(k)}(z_0)}{k!}, k \geq 0.}$$

- $f(z) = e^z$, then $f^{(k)}(z) = e^z$. Letting $z_0 = 0$, we have $f^{(k)}(z_0) = e^0 = 1$ for all k. Thus $a_k = \dfrac{1}{k!}$ for all k, and so

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \quad \text{for all } z \in \mathbb{C}.$$

- $f(z) = e^z$ as above, but now let $z_0 = 1$. Then $f^{(k)}(z_0) = e^1 = e$ for all k. Thus $a_k = \dfrac{e}{k!}$ for all k, and so

$$e^z = \sum_{k=0}^{\infty} \frac{e}{k!}(z-1)^k \quad \text{for all } z \in \mathbb{C}.$$

- $f(z) = \sin z$ is analytic in $\mathbb{C}$. Let $z_0 = 0$. Then

$$
\begin{aligned}
f(z) &= \sin z, \quad f(0) = 0 \\
f'(z) &= \cos z, \quad f'(0) = 1 \\
f''(z) &= -\sin z, \quad f''(0) = 0 \\
f^{(3)}(z) &= -\cos z, \quad f^{(3)}(0) = -1 \\
f^{(4)}(z) &= \sin z, \quad f^{(4)}(0) = 0 \cdots
\end{aligned}
$$

Thus

$$
\begin{aligned}
\sin z &= 0 + \frac{1}{1!} z + \frac{0}{2!} z^2 + \frac{-1}{3!} z^3 + \frac{0}{4!} z^4 + \frac{1}{5!} z^5 + \cdots \\
&= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - + \cdots \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1}.
\end{aligned}
$$

- $f(z) = \cos z$ is analytic in $\mathbb{C}$. Let $z_0 = 0$. We could do the same analysis as for $\sin z$, or ...:

$$\begin{aligned}
\cos z = \frac{d}{dz} \sin z &= \frac{d}{dz} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (2k+1) z^{2k} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k} \\
&= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + - \cdots
\end{aligned}$$

A Corollary

Note: The theorem implies that an analytic function is entirely determined in a disk by all of its derivatives $f^{(k)}(z_0)$ at the center z_0 of the disk.

Corollary

If f and g are analytic in $\{|z - z_0| < r\}$ and if $f^{(k)}(z_0) = g^{(k)}(z_0)$ for all k, then $f(z) = g(z)$ for all z in $\{|z - z_0| < r\}$.

Next up: The Riemann Zeta Function and the Riemann Hypothesis.