Hw. 2 Problem 2

January 1, 2018

Let $u(x, y) = x^2 - y^2 - y$. Find a real-valued function $v(x, y)$ such that $v(0, 0) = 1$ and together, u and v satisfy the Cauchy-Riemann equations in the entire complex plane.

1. Find partial derivatives $u_x(x, y)$ and $u_y(x, y)$.

$$u_x(x, y) = 2x$$
$$u_y(x, y) = -2y - 1$$

2. Using these partial derivatives and the Cauchy-Riemann equations, give equations for the partial derivatives $v_x(x, y)$ and $v_y(x, y)$.

$$v_x = -u_y = 2y + 1$$
$$v_y = u_x = 2x$$

3. Find functions $v(x, y)$ that satisfy the equation for the partial derivative with respect to x.

   $v(x, y) = \int v_x dx = \int 2y + 1 \; dx = 2xy + x + g(y)$, where $g(y)$ is independent of x.

4. Find functions $v(x, y)$ that satisfy the equation for the partial derivative with respect to y.

   $v(x, y) = \int v_y dy = \int 2x \; dy = 2xy + h(x)$, where $h(x)$ is independent of y.

5. Now find a function $v(x, y)$ that satisfies both equations for the partial derivatives at the same time.

   We have $v(x, y) = 2xy + h(x) = 2xy + x + g(y)$, so $h(x) = x + g(y)$. Thus, $g(y) = c$ in a constant. Hence,

   $$v(x, y) = 2xy + x + c, c \in \mathbb{C}.$$

6. Finally, check whether the function you found in the previous step satisfies $v(0, 0) = 1$. If not, modify the function so that it does.

   We want $1 = v(0, 0) = 0 + 0 + c$, giving $c = 1$. The desired function is hence, $2xy + x + 1$.