Analysis of a Complex Kind

Week 7

Lecture 3: The Residue Theorem

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Motivation

Recall: f has an isolated singularity at z_0 if f is analytic in $\{0 < |z - z_0| < r\}$ for some $r > 0$. In that case, f has a Laurent series expansion

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k, \quad 0 < |z - z_0| < r.$$

Observe: If $0 < \rho < r$ then

$$\int_{|z-z_0|=\rho} f(z)dz = \sum_{k=-\infty}^{\infty} a_k \int_{|z-z_0|=\rho} (z - z_0)^k dz.$$

What is $\displaystyle\int_{|z-z_0|=\rho} (z - z_0)^k dz$?

Motivation

What is $\displaystyle\int_{|z-z_0|=\rho} (z-z_0)^k dz$?

- For $k \neq -1$, the function $h(z) = (z-z_0)^k$ has a primitive, namely $H(z) = \dfrac{1}{k+1}(z-z_0)^{k+1}$. Therefore, $\displaystyle\int_{|z-z_0|=\rho} (z-z_0)^k dz = 0$ for $k \neq -1$.
- For $k = -1$, the integral is $\displaystyle\int_{|z-z_0|=\rho} \frac{1}{z-z_0} dz$. We can use the Cauchy Integral Formula (or compute this directly) and find $\displaystyle\int_{|z-z_0|=\rho} (z-z_0)^k dz = 2\pi i$ for $k = -1$.

Hence

$$\int_{|z-z_0|=\rho} f(z)dz = \sum_{k=-\infty}^{\infty} a_k \int_{|z-z_0|=\rho} (z-z_0)^k dz = 2\pi i a_{-1}.$$

Therefore, a_{-1} gets special attention!

The Residue

Definition

If f has an isolated singularity at z_0 with Laurent series expansion

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k, \quad 0 < |z - z_0| < r,$$

then the *residue of f at* z_0 is $\operatorname{Res}(f, z_0) = a_{-1}$.

Examples:

- $f(z) = \dfrac{1}{(z-1)(z-2)} = \dfrac{-1}{z-1} + \sum_{k=0}^{\infty}(-1)(z-1)^k$ in $0 < |z-1| < 1$.
  Therefore, $\operatorname{Res}(f, 1) = -1$.
- $f(z) = \dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \sum_{n=0}^{\infty}(-1)^n(z-2)^n$ in $0 < |z-2| < 1$.
  Therefore, $\operatorname{Res}(f, 2) = 1$.

More examples:

- $f(z) = \dfrac{\sin z}{z^4} = \dfrac{1}{z^3} + \left(-\dfrac{1}{3!}\right) \cdot \dfrac{1}{z} + \dfrac{1}{5!}z - \dfrac{1}{7!}z^3 + - \cdots$ in $0 < |z| < \infty$.
  Therefore, $\operatorname{Res}(f, 0) = \dfrac{-1}{3!} = \dfrac{-1}{6}$.

- $f(z) = \cos\left(\dfrac{1}{z}\right) = 1 - \dfrac{1}{z^2} \cdot \dfrac{1}{2!} + \dfrac{1}{z^4} \cdot \dfrac{1}{4!} - \dfrac{1}{z^6}\dfrac{1}{6!} + - \cdots$. Therefore, $\operatorname{Res}(f, 0) = 0$.

- $f(z) = \sin\left(\dfrac{1}{z}\right) = \dfrac{1}{z} - \dfrac{1}{z^3} \cdot \dfrac{1}{3!} + \dfrac{1}{z^5} \cdot \dfrac{1}{5!} - + \cdots$. Therefore, $\operatorname{Res}(f, 0) = 1$.

- $f(z) = \dfrac{\cos z - 1}{z^2} = -\dfrac{1}{2!} + \dfrac{z^2}{4!} - + \cdots$. Therefore, $\operatorname{Res}(f, 0) = 0$.

More Examples

More examples:

- $$f(z) = \frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)} = \frac{1}{2i}\frac{(z+i)-(z-i)}{(z-i)(z+i)}$$
  $$= \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right) = \frac{1}{2i}\cdot\frac{1}{z-i} + \text{(analytic function near } i\text{)}.$$
  Therefore, $\mathrm{Res}(f, i) = \frac{1}{2i} = -\frac{1}{2}i$.
- Similarly, $f(z) = \frac{1}{z^2+1} = \frac{-1}{2i}\frac{1}{z+i} + \text{(analytic function near } -i\text{)}$.
  Therefore, $\mathrm{Res}(f, -i) = \frac{-1}{2i} = \frac{1}{2}i$.

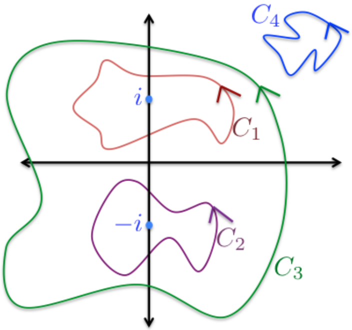
The Residue Theorem

Theorem (Residue Theorem)

Let D be a simply connected domain, and let f be analytic in D, except for isolated singularities. Let C be a simple closed curve in D (oriented counterclockwise), and let $z_1, \cdots, z_n$ be those isolated singularities of f that lie inside of C. Then

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f, z_k).$$

Example: $f(z) = \dfrac{1}{z^2+1}$ is analytic in $D = \mathbb{C}$, except for isolated singularities at $z = \pm i$.



- $\displaystyle\int_{C_1} f(z)dz = 2\pi i \operatorname{Res}(f, i) = 2\pi i(-\frac{1}{2}i) = \pi$
- $\displaystyle\int_{C_2} f(z)dz = 2\pi i \operatorname{Res}(f, -i) = 2\pi i(\frac{1}{2}i) = -\pi$
- $\displaystyle\int_{C_3} f(z)dz = 2\pi i(\operatorname{Res}(f, i) + \operatorname{Res}(f, -i))$
  $\displaystyle = 2\pi i(-\tfrac{1}{2}i + \tfrac{1}{2}i) = 0$
- $\displaystyle\int_{C_4} f(z)dz = 0.$

What's Next?

We just learned:

Theorem (Residue Theorem)

Let D be a simply connected domain, and let f be analytic in D, except for isolated singularities. Let C be a simple closed curve in D (oriented counterclockwise), and let $z_1, \cdots, z_n$ be those isolated singularities of f that lie inside of C. Then

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, z_k).$$

In order to be able to fully take advantage of this powerful theorem, we'll need strategies and techniques that'll help us to calculate residues!