Week 6

Games without chance

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Nim &
How to Win

Impartial Games, reversible moves

Examples, MEX

ALL Impartial games are equivalent to NIM

$*n$ nim heap of size n

$*1 + *2 + *3 = 0$

1st player lose

$*n$ all by itself

1st player lose $n = 0$

1st player win $n \neq 0$

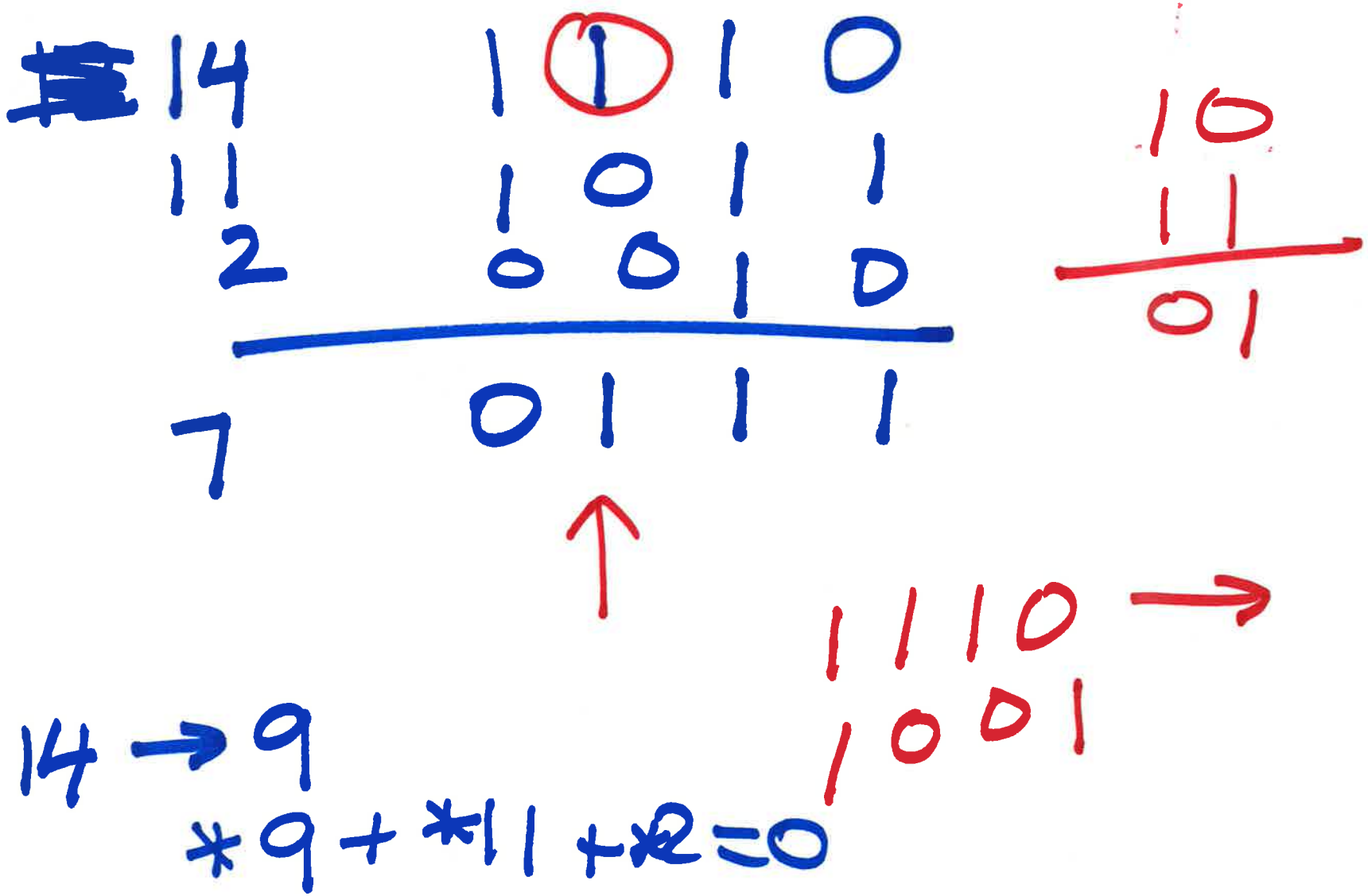
ITOT

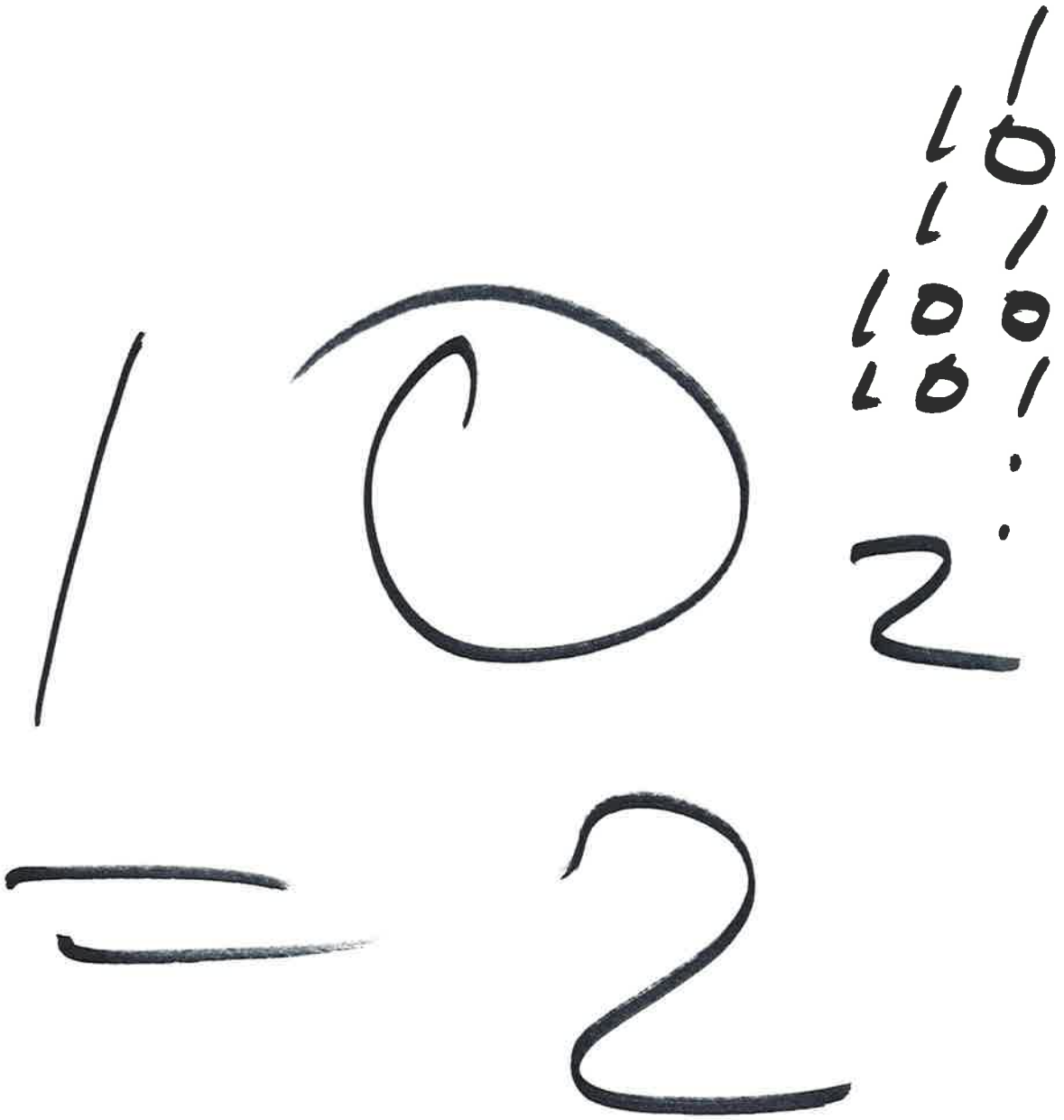
$*a + *b + \cdots + *c = *d$

d is computable from $\begin{cases} d = \\ \text{nim} \\ \text{sum} \end{cases}$ $a, b, c \ldots$

$*1 + *2 + *3 = *0$

$*2 + *3 + *7 = *6$


14
11
2
7
1 1 1 0
1 0 1 1
0 0 1 0
0 1 1 1
1 0
1 1
0 1
1 1 1 0 →
1 0 0 1
14 → 9
*9 + *11 + *2 = 0

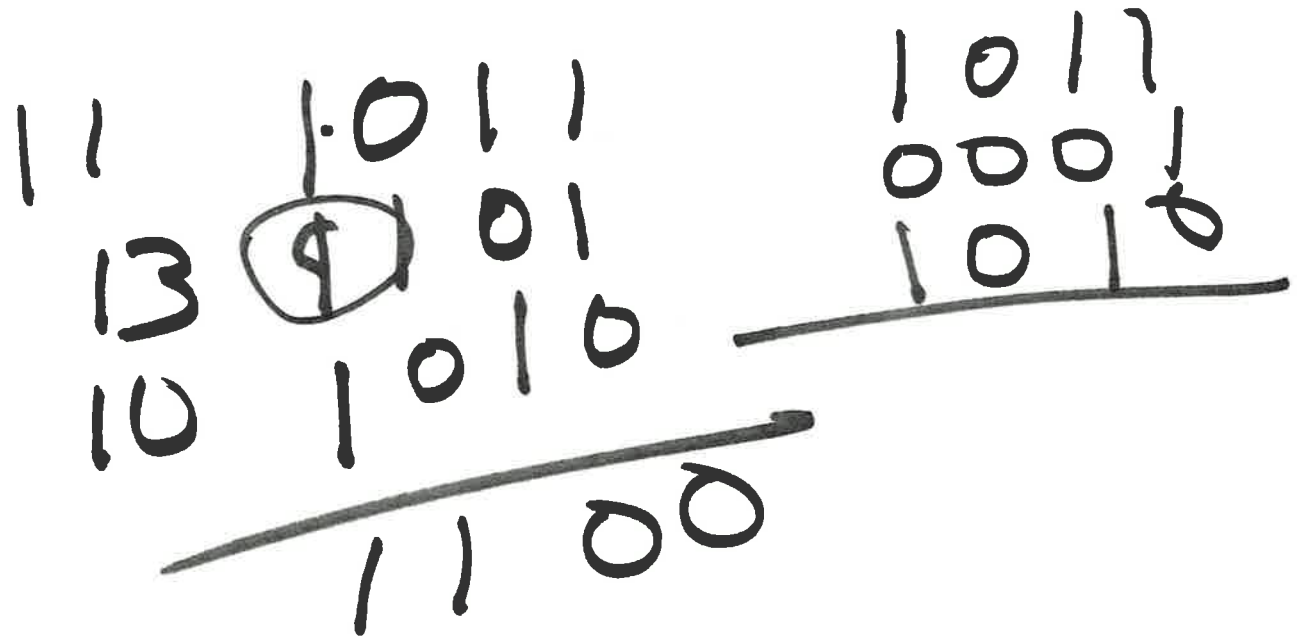


PROBLEM

$*11 + *13 + *10 = ?$

what's the (or a)
best 1st move?

13 → 1



Thm If

$G = \{*a, *b, \ldots, *c\}$

then $G = *d$ where

d = minimal excluded natural number in $\{a, b, \ldots, c\}$

$$G = \{*0, *1, *2, *5, *7\}$$

$$= \{*0, *1, *2\}$$

$$= *3$$

$$3 = MEX(0, 1, 2)$$

MEX(0,3,2,1,17) = 4

MEX(2,3,17,1,6,12) = 0

MEX(1,4,70,3,8,17) = 5

...

MEX(0,1,2,3,4,5,6,7,9)

G is impatial if

(1) Every left move is also a right move and visa versa

(2) Every move is impartial

A subtraction game

$\{1, 2, 5\}$

START WITH A heap of size n. A move (for either player) is remove 1, 2 or 5

n	0	1	2	3	4	5	6
$G(n)$	0	1	2	0	1	2	?

SUBTRACT 1, 2 or 5

Subtraction Games

$\{a, b, \ldots, c\}$ ...

remove a, b, ... or c coins

(1st player with no moves loses)

$G(n) = d$ if

Game with n coins is $= *d$

Suppose G is impartial

then by induction

$$G = \{*a, *b, \ldots *c\}$$

in which case

$G = *d$ where

d = Minimal excluded of $\{a, b, \ldots, c\}$

Induction

if $G = 0$

then $G = *0$

THM: Grundy

If G is impartial then $G = *n$ for some n

n.. Grundy value of G

$$*2 + *3$$

$$\{*0, *1\} + \{*0, *1, *2\}$$

$$= \{*0 + *3, *1 + *3, *0 + *2,$$
$$*1 + *2, *2 + *2\}$$

$$= \{*3, *2, *2, *3, *0\}$$

$$= *1 \quad \text{by minimal exclusion}$$

.4 Ocal Game

remove 1 coin & spit into 2 non-empty piles

0 1 2 3

O O 6 ?

Quiz for week 6

1. Find the nim sum of 1, 5 and 11
2. Find the MEX of $\{0, 1, 2, 3, 8, 4, 11, 5\}$
3. For the subtraction game $\{1, 2, 5\}$, find $G(7)$
4. Find $G(4)$ for the subtraction game $\{1, 2, 3\}$

FIN

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