Analysis of a Complex Kind

Week 7

Lecture 1: Laurent Series

Petra Bonfert-Taylor

Review of Taylor Series

Recall: If $f : U \to \mathbb{C}$ is analytic and $\{|z - z_0| < R\} \subset U$ then f has a power series representation

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \quad \text{where } a_k = \frac{f^{(k)}(z_0)}{k!}, k \geq 0.$$

- What if f is not differentiable at some point?
- Example: $f(z) = \frac{z}{z^2+4}$ is not differentiable at $z = \pm 2i$ (undefined there).
- $f(z) = \operatorname{Log} z$ not continuous on $(-\infty, 0]$, so not differentiable there.

Theorem (Laurent Series Expansion)

If $f : U \to \mathbb{C}$ is analytic and $\{r < |z - z_0| < R\} \subset U$ then f has a Laurent series expansion

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k = \cdots \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots,$$

that converges at each point of the annulus and converges absolutely and uniformly in each sub annulus $\{s \leq |z - z_0| \leq t\}$, where $r < s < t < R$.

The Coefficients a_k

Note: The coefficients a_k are uniquely determined by f. How do we find them?

Example: $f(z) = \dfrac{1}{(z-1)(z-2)}$ is analytic in $\mathbb{C} \setminus \{1, 2\}$.

Let's find the Laurent series in the annulus $\{1 < |z| < 2\}$.Trick:

$$
\begin{aligned}
\frac{1}{(z-1)(z-2)} &= \frac{(z-1)-(z-2)}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1} \\
&= \frac{-1}{2}\frac{1}{1-\frac{z}{2}} - \frac{1}{z(1-\frac{1}{z})} \\
&= \frac{-1}{2}\sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^k - \frac{1}{z}\sum_{k=0}^{\infty}\frac{1}{z^k} \\
&= \sum_{k=0}^{\infty}\frac{-1}{2^{k+1}}z^k + \sum_{k=1}^{\infty}\frac{-1}{z^k} = \sum_{k=0}^{\infty}\frac{-1}{2^{k+1}}z^k + \sum_{k=-\infty}^{-1}(-1)z^k.
\end{aligned}
$$

$$\frac{1}{(z-1)(z-2)} = \sum_{k=0}^{\infty} \frac{-1}{2^{k+1}} z^k + \sum_{k=-\infty}^{-1} (-1) z^k \quad \text{in } \{1 < |z| < 2\}.$$

What if we choose a different annulus? f is also analytic in $\{2 < |z| < \infty\}$!

$$\begin{aligned}
\frac{1}{(z-1)(z-2)} &= \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{z(1-\frac{2}{z})} - \frac{1}{z(1-\frac{1}{z})} \\
&= \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{2}{z}\right)^k - \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^k \\
&= \sum_{k=1}^{\infty}\frac{2^{k-1}}{z^k} - \sum_{k=1}^{\infty}\frac{1}{z^k} = \sum_{k=-\infty}^{-1}(2^{-k-1}-1)z^k.
\end{aligned}$$

$f(z) = \frac{1}{(z-1)(z-2)}$

- $$\frac{1}{(z-1)(z-2)} = \sum_{k=0}^{\infty} \frac{-1}{2^{k+1}} z^k + \sum_{k=-\infty}^{-1} (-1) z^k \quad \text{in } \{1 < |z| < 2\}.$$
- $$\frac{1}{(z-1)(z-2)} = \sum_{k=1}^{\infty} \frac{2^{k-1}}{z^k} - \sum_{k=1}^{\infty} \frac{1}{z^k} = \sum_{k=-\infty}^{-1} (2^{-k-1} - 1) z^k \quad \text{in } \{2 < |z| < \infty\}.$$

What if we choose yet another annulus? f is also analytic in $\{0 < |z-1| < 1\}$!

$$\frac{1}{z-2} = \frac{1}{(z-1)-1} = -\frac{1}{1-(z-1)} = -\sum_{k=0}^{\infty} (z-1)^k \quad \text{in } \{0 < |z-1| < 1\}.$$

So

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1} = -\sum_{k=0}^{\infty} (z-1)^k - \frac{1}{z-1} = \sum_{k=-1}^{\infty} (-1)(z-1)^k \quad \text{in } \{0 < |z-1| < 1\}.$$

Another example: $\dfrac{\sin z}{z^4}$ is analytic in $\mathbb{C} \setminus \{0\}$. What is its Laurent series, centered at 0? Recall:

$$\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \pm \cdots$$

So

$$\frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!}\frac{1}{z} + \frac{1}{5!}z - \frac{1}{7!}z^3 \pm \cdots$$

Thus $a_{-3} = 1, a_{-2} = 0, a_{-1} = -\dfrac{1}{3!}, a_0 = 0, a_1 = \dfrac{1}{5!}, a_2 = 0, a_3 = -\dfrac{1}{7!}, \ldots$.

Calculating a_k

Recall: For a Taylor series,

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \quad |z - z_0| < R,$$

the a_k can be calculated via $a_k = \dfrac{f^{(k)}(z_0)}{k!}$. How about for a Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k, \quad r < |z - z_0| < R?$$

f may not be defined at z_0, so we need a new approach! Back to Taylor series for a second:

$$a_k = \frac{f^{(k)}(z_0)}{k!} \overset{Cauchy}{=} \frac{1}{2\pi i} \int_{|z-z_0|=s} \frac{f(z)}{(z - z_0)^{k+1}} dz$$

for any s between 0 and R. One can show a similar fact for Laurent series:

The Coefficients a_k

Theorem

If f is analytic in $\{r < |z - z_0| < R\}$, then

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k,$$

where

$$a_k = \frac{1}{2\pi i} \int_{|z-z_0|=s} \frac{f(z)}{(z - z_0)^{k+1}} dz$$

for any s between r and R and all $k \in \mathbb{Z}$.

Note: This does not seem all that useful for finding actual values of a_k, but it is useful to estimate a_k. We'll use this when calculating integrals later.