IV. Zeros of Jacobi forms

Exercise IV-1. A generalisation of the main theorem.

Let $\varphi : \mathbb{C} \to \mathbb{C}$ be a holomorphic function ($\varphi \not\equiv 0$) satisfying a more complicated equation

$$(E_m^-): \quad \varphi(z + \lambda\tau + \mu) = (-1)^{m(\lambda+\mu)} e^{-\pi i m(\lambda^2\tau + 2\lambda z)} \varphi(z)$$

for any $\lambda, \mu \in \mathbb{Z}$. (Here we put $m/2$ instead of m under the exponent.) Show that φ has exactly m zeros in any fundamental domain $\mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$.

Exercise IV-2. Jacobi forms of odd weights.

1) Consider a weak Jacobi form $\varphi \in J_{2k+1,m}^{(weak)}$ of odd weight. Prove that φ has at least four zeros in a fundamental parallelogram. (See III.)

2) Show that $\varphi \in J_{2k+1,1}^{(weak)} = \{0\}$.

Exercise IV-3. Taylor expansion of Jacobi forms of odd weights.

1) Analyse the Taylor expansion in z of $\varphi(\tau, z) \in J_{2k+1,m}$.

2) Prove that

$$\dim J_{2k+1,2} = 0 \quad \text{for} \quad 1 \le 2k+1 \le 9$$

and

$$\dim J_{11,2} \le 1.$$

Exercise IV-4. The first non vanishing Taylor coefficient.

Let $\phi(\tau, z) \in J_{k,m}^{w}$ be a weak Jacobi form. Consider its Taylor expansion in z

$$\phi(\tau, z) = f_k(\tau) + \sum_{n \ge 1} f_{k+n}(\tau) z^n$$

where $\phi(\tau, 0) = f_k(\tau) \in M_k(\mathrm{SL}_2(\mathbb{Z}))$. Assume that

$$f_k(\tau) \equiv f_{k+1}(\tau) \equiv \cdots \equiv f_{k+d-1}(\tau) \equiv 0.$$

1) Using the definition of Jacobi forms, prove that

$$f_{k+d}(\tau) \in M_{k+d}(\mathrm{SL}_2(\mathbb{Z})).$$

2) Let $\phi(\tau, z) \in J_{k,m}$ and $\phi(\tau, 0) \equiv 0$. Is it true that $f_{k+d}(\tau)$ is a cusp form?

Exercise IV-5. Weak Jacobi forms of weight 0.

Prove that

$$1)\ \dim J_{2k,1}^{(weak)} = 0 \quad \text{if } 2k < -2.$$

$$2)\ \dim J_{0,1}^{(weak)} \leq 1.$$

$$3)\ \dim J_{0,2}^{(weak)} \leq 2.$$