Analysis of a Complex Kind

Week 7

Lecture 4: Finding Residues

Petra Bonfert-Taylor

The Residue Theorem

Recall:

- f has an isolated singularity at z_0 if f is analytic in the punctured disk $\{0 < |z - z_0| < r\}$.
- In that case, f has a Laurent series representation

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

  in this punctured disk. The representation is unique.
- The residue of f at z_0 is $\operatorname{Res}(f, z_0) = a_{-1}$, the coefficient of the term $\dfrac{1}{z - z_0}$.

Residues at Removable Singularities

$$f(z)=\sum_{k=-\infty}^{\infty} a_k(z-z_0)^k, \quad 0<|z-z_0|<r.$$

Recall: z_0 is a *removable singularity* if $a_k=0$ for all $k<0$.
In particular: $a_{-1}=0$ in that case, so that $\operatorname{Res}(f,z_0)=0$.
Example:

$$\begin{aligned} f(z)=\frac{\sin z}{z} &= \frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1} \\ &= \frac{1}{z}\left(z-\frac{1}{3!}z^3+\frac{1}{5!}z^5-\frac{1}{7!}z^7+-\cdots\right) \\ &= 1-\frac{1}{3!}z^2+\frac{1}{5!}z^4-\frac{1}{7!}z^6+-\cdots \end{aligned}$$

Thus $\operatorname{Res}(f,0)=0$.

Residues at Simple Poles

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k, \quad 0 < |z - z_0| < r.$$

Recall: z_0 is a *simple pole* if $a_{-1} \neq 0$ and $a_k = 0$ for all $k \leq -1$. So

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots .$$

How do we isolate a_{-1}? Idea:

$$(z - z_0)f(z) = a_{-1} + a_0(z - z_0) + a_1(z - z_0)^2 + \cdots ,$$

so that

$$\mathrm{Res}(f, z_0) = a_{-1} = \lim_{z \to z_0} \Big((z - z_0) f(z) \Big).$$

Residues at Simple Poles: Example

$$\operatorname{Res}(f, z_0) = a_{-1} = \lim_{z\to z_0}\Big((z - z_0)f(z)\Big).$$

Example: $f(z) = \dfrac{1}{z^2+1}$ has a simple pole at $z_0 = i$ (and another one at $-i$).

$$\begin{aligned}
\operatorname{Res}\left(\frac{1}{z^2+1}, i\right) &= \lim_{z\to i}\left((z-i)\frac{1}{z^2+1}\right)\\
&= \lim_{z\to i}\left((z-i)\frac{1}{(z-i)(z+i)}\right)\\
&= \lim_{z\to i}\frac{1}{z+i} = \frac{1}{2i} = -\frac{i}{2}.
\end{aligned}$$

Residues at Double Poles

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k, \quad 0 < |z - z_0| < r.$$

Recall: z_0 is a *double pole* if $a_{-2} \neq 0$ and $a_k = 0$ for all $k \leq -3$. So

$$f(z) = \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots .$$

How do we isolate a_{-1}? Idea:

$$(z - z_0)^2 f(z) = a_{-2} + a_{-1}(z - z_0) + a_0(z - z_0)^2 + \cdots ,$$

so that

$$\frac{d}{dz}\Big((z - z_0)^2 f(z)\Big) = a_{-1} + 2a_0(z - z_0) + \cdots$$

Hence

$$\operatorname{Res}(f, z_0) = a_{-1} = \lim_{z \to z_0} \frac{d}{dz}\Big((z - z_0)^2 f(z)\Big).$$

Residues at Double Poles: Example

$$\text{Res}(f, z_0) = a_{-1} = \lim_{z \to z_0} \frac{d}{dz}\Big((z - z_0)^2 f(z)\Big).$$

Example: $f(z) = \dfrac{1}{(z-1)^2(z-3)}$ has a double pole at $z_0 = 1$ (and a simple one at 3).

$$\begin{aligned}
\text{Res}\left(\frac{1}{(z-1)^2(z-3)}, 1\right) &= \lim_{z\to 1} \frac{d}{dz}\left((z-1)^2 \frac{1}{(z-1)^2(z-3)}\right) \\
&= \lim_{z\to 1} \frac{d}{dz}\left(\frac{1}{(z-3)}\right) \\
&= \lim_{z\to 1} \frac{-1}{(z-3)^2} = -\frac{1}{4}.
\end{aligned}$$

Residues at Poles of Order n

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k, \quad 0 < |z - z_0| < r.$$

Recall: z_0 is a *pole of order n* if $a_{-n} \neq 0$ and $a_k = 0$ for all $k \leq -(n+1)$.

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots .$$

Then

$$\mathrm{Res}(f, z_0) = a_{-1} = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \Big((z - z_0)^n f(z) \Big).$$

More On Residues

Remark

If $f(z) = \dfrac{g(z)}{h(z)}$, *where* g *and* h *are analytic near* z_0, *and* h *has a simple zero at* z_0, *then*

$$\operatorname{Res}(f(z), z_0) = \frac{g(z_0)}{h'(z_0)}.$$

Example: $f(z) = \dfrac{1}{(z-1)^2(z-3)}$, choose $g(z) = \dfrac{1}{(z-1)^2}$ and $h(z) = (z-3)$.
Then g and h are analytic near $z_0 = 3$, and h has a simple zero at $z_0 = 3$. Thus

$$\operatorname{Res}(f, z_0) = \frac{g(3)}{h'(3)} = \frac{\frac{1}{(3-1)^2}}{1} = \frac{1}{4}.$$