$$
\begin{aligned}
\lim_{n\to\infty} \frac{c_{n+1}}{c_n} &= \frac{\frac{1}{n+2}\binom{2n+2}{n+1}}{\frac{1}{n+1}\binom{2n}{n}} \\
&= \lim_{n\to\infty} \frac{n+1}{n+2} \frac{\frac{(2n+2)!}{(n+1)!^2}}{\frac{(2n)!}{n!^2}} \\
&= \lim_{n\to\infty} \frac{n+1}{n+2} \frac{(2n+2)!}{(2n)!} \frac{n!^2}{(n+1)!^2} \\
&= \lim_{n\to\infty} (2n+1)(2n+2) \frac{n+1}{(n+2)(n+1)^2} \\
&= \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(n+1)(n+2)} \\
&= 2 * 2 = \boxed{4}
\end{aligned}
$$