Analysis of a Complex Kind

Week 7

Bonus Lecture 6: Evaluating an Improper Integral via the Residue Theorem

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An Improper Integral

Goal

Evaluate $\displaystyle\int_0^\infty \frac{\cos x}{1+x^2}\,dx.$

Note: $\displaystyle\int_0^\infty \cdots dx$ means $\displaystyle\lim_{R\to\infty}\int_0^R \cdots dx$, so we need to consider $\displaystyle\int_0^R \frac{\cos x}{1+x^2}\,dx$ and then let $R \to \infty$.

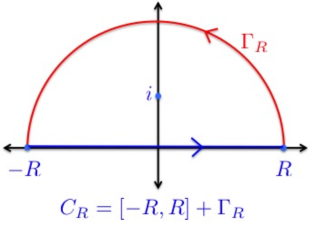
Idea:

$$\begin{aligned}
\int_0^R \frac{\cos x}{1+x^2}\,dx &= \frac{1}{2}\int_{-R}^R \frac{\cos x}{1+x^2}\,dx \\
&= \frac{1}{2}\int_{-R}^R \frac{\cos x + i\sin x}{1+x^2}\,dx \\
&= \frac{1}{2}\int_{-R}^R \frac{e^{ix}}{1+x^2}\,dx.
\end{aligned}$$

So far:

$$\int_0^R \frac{\cos x}{1+x^2}dx = \frac{1}{2}\int_{-R}^R \frac{e^{ix}}{1+x^2}dx.$$

Idea:



$$\begin{aligned}
\frac{1}{2}\int_{-R}^R \frac{e^{ix}}{1+x^2}dx &= \frac{1}{2}\int_{[-R,R]} \frac{e^{iz}}{1+z^2}dz \\
&= \frac{1}{2}\int_{C_R} \frac{e^{iz}}{1+z^2}dz - \frac{1}{2}\int_{\Gamma_R} \frac{e^{iz}}{1+z^2}dz \\
&= \frac{1}{2}\cdot 2\pi i \operatorname{Res}\left(\frac{e^{iz}}{1+z^2}, i\right) - \frac{1}{2}\int_{\Gamma_R} \frac{e^{iz}}{1+z^2}dz.
\end{aligned}$$

We thus need to find the residue of $f(z) = \dfrac{e^{iz}}{1+z^2}$ at $z_0 = i$ and estimate the integral over Γ_R.

Finding the Residue

1. Finding the residue of $f(z) = \dfrac{e^{iz}}{1+z^2}$ at $z_0 = i$:
   - f has a simple pole at $z = i$.
   - Thus $\text{Res}(f, i) = \lim_{z\to i}(z-i)f(z) = \lim_{z\to i}\dfrac{(z-i)e^{iz}}{1+z^2} = \lim_{z\to i}\dfrac{e^{iz}}{z+i} = \dfrac{e^{-1}}{2i}$.
   - Hence $\dfrac{1}{2}\displaystyle\int_{C_R}\dfrac{e^{iz}}{1+z^2}dz = \dfrac{1}{2}2\pi i\dfrac{1}{2ie} = \dfrac{\pi}{2e}$.
2. Estimating $\dfrac{1}{2}\displaystyle\int_{\Gamma_R}\dfrac{e^{iz}}{1+z^2}dz$:
   - We are only interested in what happens as $R \to \infty$.
   - Want to show: $\dfrac{1}{2}\displaystyle\int_{\Gamma_R}\dfrac{e^{iz}}{1+z^2}dz \to 0$ as $R \to \infty$.
   - Therefore, it suffices to show that $\left|\displaystyle\int_{\Gamma_R}\dfrac{e^{iz}}{1+z^2}dz\right| \leq \text{const}(R)$, where the constant goes to zero as $R \to \infty$.

Estimating the Integral

2. continued. To show: $\left|\int_{\Gamma_R} \frac{e^{iz}}{1+z^2} dz\right| \leq \text{const}(R)$, where $\text{const}(R) \to 0$.
   - Recall: $\left|\int_{\Gamma_R} f(z) dz\right| \leq \text{length}(\Gamma_R) \cdot \max_{z \in \Gamma_R} |f(z)|$.
   - Thus $\left|\int_{\Gamma_R} \frac{e^{iz}}{1+z^2} dz\right| \leq \text{length}(\Gamma_R) \cdot \max_{z \in \Gamma_R} \left|\frac{e^{iz}}{1+z^2}\right|$.
   - $\left|\frac{e^{iz}}{1+z^2}\right| = \frac{e^{\text{Re}(iz)}}{|1+z^2|} = \frac{e^{-y}}{|1+z^2|} \leq \frac{e^{-y}}{R^2-1} \leq \frac{1}{R^2-1}$ for $z \in \Gamma_R$, since $|z| = R$ and $y \geq 0$ on Γ_R.
   - So $\left|\int_{\Gamma_R} \frac{e^{iz}}{1+z^2} dz\right| \leq \pi R \cdot \frac{1}{R^2-1} \to 0$ as $R \to \infty$.
   - Thus $\int_{\Gamma_R} \frac{e^{iz}}{1+z^2} dz \to 0$ as $R \to \infty$

Conclusion

To find: $\displaystyle\int_0^\infty \frac{\cos x}{1+x^2}dx$. Here is what we have:

1. $$\int_0^\infty \frac{\cos x}{1+x^2}dx = \lim_{R\to\infty}\int_0^R \frac{\cos x}{1+x^2}dx$$
2. $$\int_0^R \frac{\cos x}{1+x^2}dx = \frac{1}{2}\int_{C_R}\frac{e^{iz}}{1+z^2}dz - \frac{1}{2}\int_{\Gamma_R}\frac{e^{iz}}{1+z^2}dz$$
3. $$\frac{1}{2}\int_{C_R}\frac{e^{iz}}{1+z^2}dz = \frac{1}{2}\cdot 2\pi i\,\mathrm{Res}\left(\frac{e^{iz}}{1+z^2}, i\right) = \frac{\pi}{2e}$$
4. $\displaystyle\int_{\Gamma_R}\frac{e^{iz}}{1+z^2}dz \to 0$ as $R\to\infty$.

Hence $\displaystyle\int_0^\infty \frac{\cos x}{1+x^2}dx = \frac{\pi}{2e}$.