I. Even integral lattices

Exercise I-1. Lattice D_n.
Let $n \geq 3$. We put

$$D_n = \{v = (x_1, \ldots, x_n) \in \mathbb{Z}^n \mid x_1 + \cdots + x_n \in 2\mathbb{Z}\}.$$

1) Show that D_n (with the euclidean scalar product) is an even integral lattice.

2) Find all roots of D_n, i.e. $v \in D_n$ such that $(v, v) = 2$. How many roots does D_n have?

3) Describe the dual lattice $D_n^* = \{u \in \mathbb{Q}^n \mid \forall\, v \in D_n : (u, v) \in \mathbb{Z}\}$.

4) Find the finite group D_n^*/D_n which is called *discriminant group*.

5) $\det D_n =?$

6) We put

$$E_8 = \langle D_8, h = (\frac{1}{2}, \ldots, \frac{1}{2})\rangle \subset \mathbb{Q}^8.$$

Prove that E_8 is an even integral quadratic lattice of **determinant one**.

7*) **Very important fact.** Let L be a non degenerate integral quadratic lattice. Then $L \subset L^*$ and $[L^* : L] = |\det L|$. (See, for example, W. Ebeling [Eb]). *Who can give a pure algebraic proof of this fact based only on the elementary divisors theorem?*

Exercise I-2. A Jacobi theta-series. Let $u = (1, -1, 0, \ldots, 0) \in D_m$ be a root. We put

$$\vartheta_{D_m,u}(\tau, z) = \sum_{v \in D_m} \exp\left(\pi i((v, v)\tau + 2(v, u)z)\right) = \sum_{n,l \in \mathbb{Z}} a(n, l) q^n r^l$$

where $q = e^{2\pi i\tau}$, $r = e^{2\pi iz}$.

1) Find several Fourier coefficients. For example, $a(0, 0)$, $a(1, 0)$, $a(1, 1)$, $a(2, 0)$, $a(2, 1)$, $a(2, 2)$, $a(3, 0)$.

2*) *Who can write a program (Pari, Mathematica,...) which gives this formal power series in q and r up to some powers $O(q^n)$?*

3) Find a number theoretical description of $a(n, 0)$.

4) Find functional equations for this Jacobi theta-series

$$\begin{aligned}
\vartheta_{D_m,u}(\tau + b, z) &=? \qquad \forall\, b \in \mathbb{Z},\\
\vartheta_{D_m,u}(\tau, z + \mu) &=? \qquad \forall\, \mu \in \mathbb{Z},\\
\vartheta_{D_m,u}(\tau, z + \lambda\tau) &=? \qquad \forall\, \lambda \in \mathbb{Z}.
\end{aligned}$$

(We assume absolute and uniform convergence of the theta-series.)

5*) *Who can give a short proof about convergence of this theta-series and present it for all participants of the course?*