Analysis of a Complex Kind

Week 5

Lecture 5: Consequences of Cauchy's Theorem and Integral Formula

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Cauchy's Theorem and Integral Formula

Recall:

Theorem (Cauchy's Theorem for Simply Connected Domains)

Let D be a simply connected domain, and let f be analytic in D. Let γ be a piecewise smooth, closed curve in D (i.e. $\gamma(b) = \gamma(a)$). Then

$$\int_\gamma f(z)dz = 0.$$

and

Theorem (Cauchy Integral Formula for Derivatives)

Let D be a simply connected domain, bounded by a piecewise smooth curve γ, and let f be analytic in a set U that contains the closure of D (i.e. D and γ). Then

$$f^{(k)}(w) = \frac{k!}{2\pi i}\int_\gamma \frac{f(z)}{(z-w)^{k+1}}dz \quad \textit{for all } w \in D,\ k \geq 0.$$

Cauchy's Estimate. (Recall: $f^{(k)}(w) = \frac{k!}{2\pi i}\int_\gamma \frac{f(z)}{(z-w)^{k+1}}dz$)

Theorem (Cauchy's Estimate)

Suppose that f is analytic in an open set that contains $\overline{B_r(z_0)}$, and that $|f(z)| \leq m$ holds on $\partial B_r(z_0)$ for some constant m. Then for all $k \geq 0$,

$$|f^{(k)}(z_0)| \leq \frac{k!m}{r^k}.$$

Proof: By the Cauchy Integral Formula we have that

$$\begin{aligned} |f^{(k)}(z_0)| &= \frac{k!}{2\pi}\left|\int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{k+1}}dz\right| \leq \frac{k!}{2\pi}\int_{|z-z_0|=r} \frac{|f(z)|}{|z-z_0|^{k+1}}|dz| \\ &\leq \frac{k!m}{2\pi r^{k+1}} \cdot 2\pi r = \frac{k!m}{r^k}. \end{aligned}$$

Liouville's Theorem

$$\boxed{|f^{(k)}(z_0)| \leq \frac{k!m}{r^k}}$$

Theorem (Liouville)

Let f be analytic in the complex plane (thus f is an entire function). If f is bounded then f must be constant.

Proof: Suppose that $|f(z)| \leq m$ for all $z \in \mathbb{C}$. Pick $z_0 \in \mathbb{C}$. Since $\mathbb{C}$ contains $\overline{B_r(z_0)}$ for any $r > 0$, we obtain from Cauchy's estimate:

$$|f'(z_0)| \leq \frac{m}{r}$$

for any $r > 0$. Letting $r \to \infty$ we find that $f'(z_0) = 0$. Since z_0 was arbitrary, $f'(z) = 0$ for all z, hence f is constant.

Example

Example

Suppose that f is an entire function, $f = u + iv$, and suppose that $u(z) \leq 0$ for all $z \in \mathbb{C}$. Then f must be constant.

Proof: Consider the function $g(z) = e^{f(z)}$. Then g is an entire function as well. Furthermore,

$$|g(z)| = e^{\mathrm{Re}\, f(z)} = e^{u(z)} \leq e^0 = 1 .$$

Thus g is an entire and bounded function, and so Liouville's theorem implies that g is constant. This now implies that f is constant (look at g').

Use Liouville to Prove Fundamental Theorem of Algebra

Theorem (Fundamental Theorem of Algebra)

Any polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ (with $a_0, \ldots, a_n \in \mathbb{C}$, $n \geq 1$ and $a_n \neq 0$) has a zero in $\mathbb{C}$, i.e. there exists $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.

Proof: Suppose to the contrary that there exists a polynomial p as in the theorem that has no zeros. Then $f(z) = \dfrac{1}{p(z)}$ is an entire function! Goal: Apply Liouville's theorem to f!

$$p(z) = z^n \left(a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right), \text{ so}$$

$$|p(z)| \geq |z|^n \left(|a_n| - \frac{|a_{n-1}|}{|z|} - \cdots - \frac{|a_0|}{|z|^n} \right) \xrightarrow{|z| \to \infty} \infty.$$

Thus $|f(z)| \to 0$ as $|z| \to \infty$, and so f is bounded in $\mathbb{C}$. By Liouville, f is constant, and so p is constant. This is a contradiction.

Factoring of Polynomials

Consequence of the Fundamental Theorem of Algebra: Polynomials can be factored in $\mathbb{C}$:

$$p(z) = a_n(z - z_1)(z - z_2) \cdots (z - z_n),$$

where $z_1, z_2, \ldots, z_n \in \mathbb{C}$ are the zeros of p (and not necessarily distinct).

Example: $p(x) = x^2 + 1$ has no zeros in $\mathbb{R}$, thus cannot be factored in $\mathbb{R}$. However, in $\mathbb{C}$, $p(z) = z^2 + 1$ has the two zeros i and $-i$ and thus factors as

$$p(z) = (z - i)(z + i).$$

Another consequence of the Cauchy Integral Formula is the following powerful result (we'll skip the proof):

Theorem (Maximum Principle)

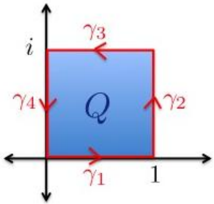
Let f be analytic in a domain D and suppose there exists a point $z_0 \in D$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in D$. Then f is constant in D.

Consequence

If $D \subset \mathbb{C}$ is a bounded domain, and if $f : \overline{D} \to \mathbb{C}$ is continuous in $\overline{D}$ and analytic in D, then $|f|$ reaches its maximum on ∂D.

Example

Let $f(z) = z^2 - 2z$. What is $\max |f(z)|$ on the square $Q = \{z = x + iy \,:\, 0 \le x, y \le 1\}$?



Since f is analytic inside Q and continuous on Q, we know that the maximum of $|f|$ occurs on ∂Q.

- On γ_1: $0 \le x \le 1$, $y = 0$, so
$$|f(z)| = |f(x)| = |x^2 - 2x| = |x(x-2)|.$$
The maximum on γ_1 occurs at $x = 1$, so $|f(z)| \le |f(1)| = 1$ on γ_1.
- On γ_2: $0 \le y \le 1$, $x = 1$, so
$$|f(z)| = |f(1 + iy)| = |1 - y^2 + 2iy - 2 - 2iy| = |-1 - y^2| = y^2 + 1.$$
The maximum on γ_2 occurs at $y = 1$, so $|f(z)| \le |f(1+i)| = 2$ on γ_2.
- On γ_3, γ_4: Similarly, one sees that $|f(z)| \le |f(i)| = |-1 - 2i| = \sqrt{5}$ on γ_3 and γ_4.

Thus $|f(z)| \le |f(i)| = \sqrt{5}$ on Q.

Next...

Next up: Series representations of analytic functions.