Analysis of a Complex Kind

Week 5

Lecture 2: Complex Integration - Examples and First Facts

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The Path Integral

Recall: Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth curve, and let f be complex-valued and continuous on γ. Then

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

More Examples

1. Let $\gamma(t) = 1 - t(1-i)$, $0 \leq t \leq 1$, and let $f(z) = \operatorname{Re} z$. Then

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_0^1 \operatorname{Re}(1 - t(1-i))(-1)(1-i)dt \\
&= (i-1)\int_0^1 (1-t)dt \\
&= (i-1)\left(t - \frac{1}{2}t^2\right)\Big|_0^1 \\
&= (i-1)\left(1-\frac{1}{2}\right) = \frac{i-1}{2}.
\end{aligned}$$

More Examples

2. Let $\gamma(t) = re^{it}$, $0 \leq t \leq 2\pi$. Then $\gamma'(t) = rie^{it}$. Let $f(z) = \overline{z}$. Then

$$\begin{aligned}
\int_\gamma f(z)dz = \int_\gamma \overline{z}dz &= \int_0^{2\pi} \overline{\gamma(t)}\gamma'(t)dt \\
&= \int_0^{2\pi} re^{-it}rie^{it}dt \\
&= r^2 i \int_0^{2\pi} dt \\
&= 2\pi i r^2 = (2i) \cdot \text{area}(B_r(0)).
\end{aligned}$$

Recall: Integration by Substitution

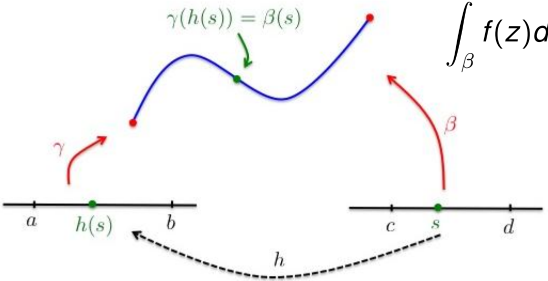
Let $[a, b]$ and $[c, d]$ be intervals in $\mathbb{R}$ and let $h : [c, d] \to [a, b]$ be smooth. Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function. Then

$$\int_{h(c)}^{h(d)} f(t)\, dt = \int_c^d f(h(s))h'(s)\, ds.$$

Example:

$$\begin{aligned}
\int_2^4 s^2(s^3+1)^4 ds &= \frac{1}{3}\int_{h(2)}^{h(4)} t^4 dt \qquad t = h(s) = s^3+1,\, h'(s) = 3s^2 \\
&= \frac{1}{3}\int_9^{65} t^4 dt \\
&= \frac{1}{3}\, \frac{t^5}{5}\Big|_9^{65} \\
&= \frac{1}{15}(65^5 - 9^5).
\end{aligned}$$

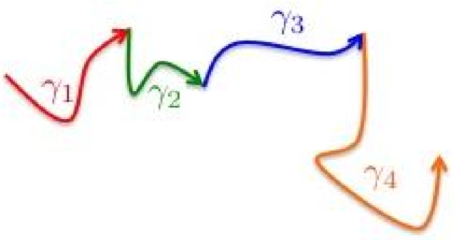
1. Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth curve, and let $\beta : [c, d] \to \mathbb{C}$ be another smooth parametrization of the same curve, given by $\beta(s) = \gamma(h(s))$, where $h : [c, d] \to [a, b]$ is a smooth bijection.
   Let f be a complex-valued function, defined on γ. Then



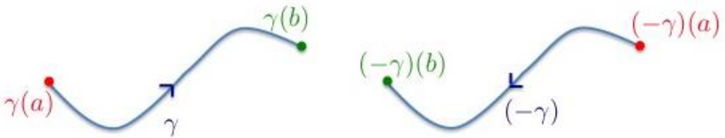
$$\begin{aligned}\int_\beta f(z)dz &= \int_c^d f(\beta(s))\beta'(s)ds \\ &= \int_c^d f(\gamma(h(s)))\gamma'(h(s))h'(s)ds \\ &= \int_a^b f(\gamma(t))\gamma'(t)dt = \int_\gamma f(z)dz.\end{aligned}$$

Therefore, the complex path integral is independent of the parametrization.

2. Let $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$ be a piecewise smooth curve (i.e. γ_{j+1} starts where γ_j ends).
Then
$$\int_{\gamma} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \cdots + \int_{\gamma_n} f(z)dz.$$



3. If $\gamma : [a, b] \to \mathbb{C}$ is a curve, then a curve $(-\gamma) : [a, b] \to \mathbb{C}$ is defined by
$$(-\gamma)(t) = \gamma(a + b - t).$$



Note that $(-\gamma)'(t) = \gamma'(a + b - t)(-1)$. If f is continuous and complex-valued on γ, then:

$$\begin{aligned}
\int_{(-\gamma)} f(z)dz &= \int_a^b f((-\gamma)(s))(-\gamma)'(s)ds = -\int_a^b f(\gamma(a+b-s))\gamma'(a+b-s)ds \\
&= \int_b^a f(\gamma(t))\gamma'(t)dt = -\int_a^b f(\gamma(t))\gamma'(t)dt \\
&= -\int_\gamma f(z)dz.
\end{aligned}$$

Fact

If γ is a curve, c a complex constant and f, g are continuous and complex-valued on γ, then

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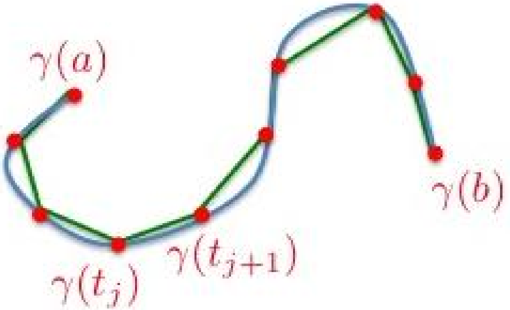
- $\displaystyle\int_\gamma (f(z) + g(z))dz = \int_\gamma f(z)dz + \int_\gamma g(z)dz.$
- $\displaystyle\int_\gamma (cf)(z)dz = c\int_\gamma f(z)dz.$
- $\displaystyle\int_{-\gamma} f(z)dz = -\int_\gamma f(z)dz.$

Given a curve $\gamma : [a, b] \to \mathbb{C}$, how can we find its length?



Let $a = t_0 < t_1 < \cdots < t_n = b$. Then

$$\text{length}(\gamma) \approx \sum_{j=0}^{n} |\gamma(t_{j+1}) - \gamma(t_j)|.$$

If the limit exists as $n \to \infty$, this is the length of γ.

How do you calculate the length of a curve $\gamma : [a, b] \to \mathbb{C}$?

$$\sum_{j=0}^{n} |\gamma(t_{j+1}) - \gamma(t_j)| = \sum_{j=0}^{n} \frac{|\gamma(t_{j+1}) - \gamma(t_j)|}{t_{j+1} - t_j}(t_{j+1} - t_j) \to \int_a^b |\gamma'(t)| dt.$$

Thus:

$$\text{length}(\gamma) = \int_a^b |\gamma'(t)| dt.$$

Examples

- Let $\gamma(t) = Re^{it}$, $0 \leq t \leq 2\pi$, for some $R > 0$. Then $\gamma'(t) = Rie^{it}$, and so

$$\text{length}(\gamma) = \int_0^{2\pi} |Rie^{it}|dt = \int_0^{2\pi} Rdt = 2\pi R.$$

- Let $\gamma(t) = t + it$, $0 \leq t \leq 1$. Then $\gamma'(t) = 1 + i$, and so

$$\text{length}(\gamma) = \int_0^{1} |1 + i|dt = \int_0^{1} \sqrt{2}dt = \sqrt{2}.$$

Integration With Respect To Arc Length

Definition

Let γ be a smooth curve, and let f be a complex-valued and continuous function on γ. Then

$$\int_\gamma f(z)|dz| = \int_a^b f(\gamma(t))|\gamma'(t)|dt$$

is the integral of f over γ with respect to arc length.

Examples:

- $\text{length}(\gamma) = \int_\gamma |dz|$.
- $\int_{|z|=1} z|dz| = \int_0^{2\pi} e^{it} \cdot 1\, dt = -ie^{it}\Big|_0^{2\pi} = 0.$

Note: Piecewise smooth curves are allowed as well (break up the integral into a sum over smooth pieces).

Theorem

If γ is a curve and f is continuous on γ then

$$\left|\int_\gamma f(z)dz\right| \leq \int_\gamma |f(z)||dz|.$$

In particular, if $|f(z)| \leq M$ on γ, then

$$\left|\int_\gamma f(z)dz\right| \leq M \cdot length(\gamma).$$

More Examples

- $\displaystyle\int_{|z|=1} \frac{1}{z}|dz| = \int_0^{2\pi} e^{-it}dt = ie^{-it}\Big|_0^{2\pi} = 0 \qquad (\gamma(t) = e^{it}, 0 \leq t \leq 2\pi).$

- Let $\gamma(t) = t + it$, $0 \leq t \leq 1$. We'd like to find an upper bound for $\displaystyle\int_\gamma z^2 dz$.

  We'll first use the second part of the theorem: $\displaystyle\left|\int_\gamma f(z)dz\right| \leq M \cdot \text{length}(\gamma)$.

  For us, $f(z) = z^2$, and we have that $|f(z)| = |z|^2 \leq (\sqrt{2})^2 = 2$ on γ, so $M = 2$. Also, recall that length$(\gamma) = \sqrt{2}$. Thus

$$\left|\int_\gamma z^2 dz\right| \leq 2\sqrt{2}.$$

We'd like to find a better estimate for $\int_\gamma z^2 dz$, using the first part of the theorem:

$$\left|\int_\gamma f(z)dz\right| \le \int_\gamma |f(z)||dz|.$$

Thus

$$\begin{aligned}
\left|\int_\gamma z^2 dz\right| \le \int_\gamma |z|^2 |dz| = \int_0^1 |\gamma(t)|^2 |\gamma'(t)| dt &= \int_0^1 |t + it|^2 \sqrt{2} dt \\
&= \int_0^1 2t^2 \sqrt{2} dt \\
&= \frac{2\sqrt{2}}{3} t^3 \bigg|_0^1 = \frac{2}{3}\sqrt{2}.
\end{aligned}$$

Note: We calculated $\int_\gamma z^2 dz = \frac{2}{3}(-1 + i)$ during the last lecture!

Next...

Next up:

The Fundamental Theorem of Calculus for Analytic Functions.