III. Special values of Jacobi forms

Exercise III-1. Jacobi forms of odd weights.

1) Let $\varphi(\tau, z)$ be a weak Jacobi form of weight k and index m. Prove that

$$\varphi(\tau, -z) = (-1)^k \varphi(\tau, z).$$

2) Assume that k is odd. Prove that for any $\tau \in \mathbb{H}$

$$\varphi(\tau, 0) = \varphi(\tau, \frac{1}{2}) = \varphi(\tau, \frac{\tau}{2}) = \varphi(\tau, \frac{\tau+1}{2}) = 0.$$

Exercise III-2. The modular group Γ_X.

Let $X = \begin{pmatrix} p \\ -q \end{pmatrix}$ where $p, q \in \mathbb{Q}$. We define

$$\Gamma_X = \{M \in SL_2(\mathbb{Z}) : MX \equiv X \mod \mathbb{Z}\}.$$

1) Let N be the minimal positive integer, such that $N \cdot X \in \mathbb{Z}^2$ (this number is called the level of X). Show that Γ_X contains the principle congruence subgroup

$$\Gamma(N) = \{M \in SL_2(\mathbb{Z}) : M \equiv E_2 \mod N\}.$$

2) Find Γ_X for $X = \begin{pmatrix} 1/N \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1/N \end{pmatrix}$ for any natural N and for

$$X = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}, \quad \begin{pmatrix} 1/3 \\ -1/3 \end{pmatrix}, \quad \begin{pmatrix} 1/4 \\ -1/4 \end{pmatrix}.$$

3) Prove that for any $m \in \mathbb{N}$

$$\chi_X(M) = \exp(2\pi i m \det(MX, X))$$

is a character of Γ_X. Find its oder.

4*) Let $p = \frac{r}{N}$, $q = \frac{s}{N}$ where $(r, N) = (s, N) = 1$. Try to find (at least for some series of (p, q)) the index $[\Gamma_X : \Gamma(N)]$.

Exercise III-3. A generalisation of the main theorem.

We proved in the lecture that

$$\varphi_X(\tau) = e^{2\pi i m(q^2\tau + pq)} \varphi(\tau, q\tau + p)$$

is a modular form with respect to Γ_X. Analyse $\varphi|_{k,m}[-X, 0]$ and prove that the special value $\varphi_X(\tau)$ of Jacobi form φ is modular with respect to a group larger than Γ_X.