Hw. 3 Problem 2

January 22, 2018

Let $f(z) = \sin z$ and consider the domain

$$D = \{z \in \mathbb{C} : 0 < \mathrm{Re}z < \pi, 0 < \mathrm{Im}z < 1\}$$

(an open rectangle). Find the maximum of $|f(z)|$ on $\overline{D}(= D \cup \partial D)$ as well as the z-value(s) at which $|f|$ attains this maximum value.

Let $g(z) := |f(z)|^2$ with $z = x + iy$. Then $g(z) = \cos^2(x)\cosh^2(y) + \sin^2(x)\sinh(y) - \cos^2(x)\sinh^2(y) + \cos^2(x)\sinh^2(y) = \cos^2(x) + \sinh^2(y)$.

Notice that $\sinh(y) \geq 0$ is strictly increasing on $(0, 1)$. Thus, g is nonconstant on D, so f is nonconstant on D. By the maximal principle, the maximum of $|f|$ over $\overline{D}$ must occur on ∂D. For each $x \in \mathbb{R}$, $g(x + i) > g(x)$. Thus, $|f(z)|$ can be maximum only if $\mathrm{Im}z = 1$. We know that $\cos^2(x) \in [0, 1]$ for all $x \in \mathbb{R}$. Also, $\cos^2(x) = 1$ if and only if $x = n\pi$ with $n \in \mathbb{Z}$. We hence have that the maximum of $|f(z)|$ on $\overline{D}$ occurs at $z = i$ and $z = \pi + i$. This value is $\sqrt{1 + \sinh^2(1)} = \sqrt{1 + \frac{1}{4}(e - \frac{1}{e})^2}$.