Analysis of a Complex Kind

Week 7

Lecture 5: Evaluating Integrals via the Residue Theorem

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Recall: The Residue Theorem

Theorem (Residue Theorem)

Let D be a simply connected domain, and let f be analytic in D, except for isolated singularities. Let C be a simple closed curve in D (oriented counterclockwise, no singularities of f on C), and let $z_1, \cdots, z_n$ be those isolated singularities of f that lie inside of C. Then

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, z_k).$$

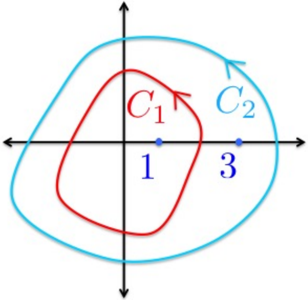
Examples:

1. $\displaystyle\int_{|z|=1} e^{\frac{3}{z}} dz = 2\pi i \, \mathrm{Res}(f, 0)$, where $f(z) = e^{\frac{3}{z}} = \displaystyle\sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{3}{z}\right)^k$. Thus $\mathrm{Res}(f, 0) = 3$, so that
$$\int_{|z|=1} e^{\frac{3}{z}} dz = 6\pi i.$$

2. $$\int_{|z|=2} \tan z dz = 2\pi i \left(\text{Res}(f, \frac{\pi}{2}) + \text{Res}(f, -\frac{\pi}{2})\right), \text{ where } f(z) = \tan z = \frac{\sin z}{\cos z}.$$

   - To find $\text{Res}(f, \frac{\pi}{2})$: Note that $f(z) = \frac{g(z)}{h(z)}$, where $g(z) = \sin(z)$ and $h(z) = \cos(z)$ are analytic near $\frac{\pi}{2}$ and $h(\frac{\pi}{2}) = 0$. Thus $\text{Res}(f, \frac{\pi}{2}) = \frac{g(\frac{\pi}{2})}{h\prime(\frac{\pi}{2})} = \frac{\sin(\frac{\pi}{2})}{-\sin(\frac{\pi}{2})} = -1$.
   - Similarly, $\text{Res}(f, -\frac{\pi}{2}) = \frac{g(-\frac{\pi}{2})}{h\prime(-\frac{\pi}{2})} = \frac{\sin(-\frac{\pi}{2})}{-\sin(-\frac{\pi}{2})} = \frac{-1}{-(-1)} = -1$.

Thus $\int_{|z|=2} \tan z dz = 2\pi i \, (-1 - 1) = -4\pi i$.



3. $$\int_{C_1} \frac{1}{(z-1)^2(z-3)}\,dz = 2\pi i \operatorname{Res}(f,1).$$

$$\begin{aligned}\operatorname{Res}(f,1) &= \lim_{z\to 1} \frac{d}{dz}\left(\frac{(z-1)^2}{(z-1)^2(z-3)}\right)\\ &= \lim_{z\to 1} \frac{d}{dz}\frac{1}{z-3} = \lim_{z\to 1}\frac{-1}{(z-3)^2} = -\frac{1}{4}.\end{aligned}$$

Thus $\displaystyle\int_{C_1} \frac{1}{(z-1)^2(z-3)}\,dz = -\frac{1}{4}2\pi i = -\frac{\pi i}{2}$.

4. $$\int_{C_2} \frac{1}{(z-1)^2(z-3)}\,dz = 2\pi i\,(\operatorname{Res}(f,1)+\operatorname{Res}(f,3)) = 2\pi i\left(-\frac{1}{4}+\frac{1}{4}\right) = 0.$$

$$\operatorname{Res}(f,3) = \lim_{z\to 3}\left(\frac{z-3}{(z-1)^2(z-3)}\right) = \lim_{z\to 3}\frac{1}{(z-1)^2} = \frac{1}{4}.$$

The Residue Theorem an also be used to evaluate real integrals, for example of the following forms:

- $\displaystyle\int_0^{2\pi} R(\cos t, \sin t)dt$, where $R(x, y)$ is a rational function of the real variables x and y.
- $\displaystyle\int_{-\infty}^{\infty} f(x)dx$, where f is a rational function of x.
- $\displaystyle\int_{-\infty}^{\infty} f(x)\cos(\alpha x)dx$, where f is a rational function of x.
- $\displaystyle\int_{-\infty}^{\infty} f(x)\sin(\alpha x)dx$, where f is a rational function of x.

In our last lecture we'll demonstrating an example of the last type.