Analysis of a Complex Kind

Week 6

Lecture 5: The Prime Number Theorem

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The Power of Complex Analysis

Complex analysis is an extremely powerful field. This is demonstrated for example, by the ability to prove a deep theorem in number theory, the Prime Number Theorem, using complex analysis.

We'll describe this in more detail in this final lecture.

The Prime Counting Function

Let $\pi(x)$ = number of primes less than or equal to x. This function is called the *prime counting function*. Example:

$$
\begin{aligned}
\pi(1) &= 0 \\
\pi(2) &= 1 \\
\pi(3) &= 2 \\
\pi(4) &= 2 \\
\pi(5) &= 3 \\
\pi(6) &= 3 \\
\pi(7) &= \pi(8) = \pi(9) = \pi(10) = 4 \\
\pi(11) &= \pi(12) = 5 \cdots
\end{aligned}
$$

It seems impossible to find an explicit formula for $\pi(x)$. One thus studies the asymptotic behavior of $\pi(x)$ as x becomes very large.

The Prime Number Theorem

$\pi(x) =$ number of primes less than or equal to x.

Theorem (Prime Number Theorem)

$$\pi(x) \sim \frac{x}{\ln x} \quad \textit{as } x \to \infty.$$

Note: The symbol "$\sim$" means that the quotient of the two quantities approaches 1 as $x \to \infty$, i.e.

$$\frac{\pi(x)}{x/\ln x} \to 1 \quad \text{as } x \to \infty.$$

A Brief History Of $\pi(x) \sim x/\ln x$

- Euler (around 1740) discovered the connection between the zeta function $\zeta(s)$ (for real values of s) and the distribution of prime numbers.
- 60 years later, Legendre and Gauss conjectured the prime number theorem, after numerical calculations.
- Another 60 years later, Tchebychev showed that there are constants A, B (with $0 < A < B$) such that $A\dfrac{x}{\ln x} \leq \pi(x) \leq B\dfrac{x}{\ln x}$.
- In 1859 Riemann published his seminal paper "On the Number of Primes Less Than a Given Magnitude". In this paper he constructed the analytic continuation of the zeta function and introduced revolutionary ideas, connection its zeros to the distribution of prime numbers.
- Hadamard and de la Vallée Poussin used these ideas and independently proved the Prime Number Theorem in 1896.
- The main step in their proofs is to establish that $\zeta(s)$ has no zeros on $\{\mathrm{Re}\, s = 1\}$.

How is $\zeta(s)$ Related To Prime Numbers?

Euler discovered:

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}},$$

where the (infinite!) product is over all primes.
Here is why this is true:

$$\begin{aligned}
\zeta(s) &= \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \cdots \\
&= \left(1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{8^s} + \cdots\right)\left(1 + \frac{1}{3^s} + \frac{1}{9^s} + \cdots\right)\left(1 + \frac{1}{5^s} + \frac{1}{25^s} + \cdots\right)\cdots \\
&= \prod_p \sum_{k=0}^{\infty} \frac{1}{p^{ks}} \\
&= \prod_p \frac{1}{1 - \frac{1}{p^s}}.
\end{aligned}$$

$$\zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}}$$

Note:

- This product formula shows that $\zeta(s) \neq 0$ for $\operatorname{Re} s > 1$.
- The key step in the proof of the prime number theorem is that ζ has no zeros on $\{\operatorname{Re} s = 1\}$.
- The details of the proof of the prime number theorem go beyond the scope of this course.
- The prime number theorem says that $\pi(x) \sim \dfrac{x}{\ln x}$, but it doesn't have any information about the difference $\pi(x) - \dfrac{x}{\ln x}$!
- However, the prime number theorem can also be written as $\pi(x) \sim \operatorname{Li}(x)$, where $\operatorname{Li}(x) = \displaystyle\int_2^x \frac{1}{\ln t}\,dt$ is the (offset) logarithmic integral function.

The Riemann Hypothesis and Prime Numbers

- The proofs of the prime number theorem by Hadamard and de la Vallée Poussin actually show that $\pi(x) = \mathrm{Li}(x) +$ error term, where the error term grows to infinity at a controlled rate.
- Von Koch (in 1901) was able to give best possible bounds on the error term, assuming the Riemann hypothesis is true. Schoenfeld (in 1976) made this precise and proved that the Riemann hypothesis is equivalent to

$$|\pi(x) - \mathrm{li}(x)| < \frac{\sqrt{x}\ln x}{8\pi},$$

  where $\mathrm{li}(x) = \displaystyle\int_0^x \frac{1}{\ln t}dt$ is the (un-offset) logarithmic integral function, related to $\mathrm{Li}(x)$ via $\mathrm{Li}(x) = \mathrm{li}(x) - \mathrm{li}(2)$.

The veracity of the Riemann Hypothesis would therefore imply further results about the distribution of prime numbers, in particular, they'd be distributed beautifully regularly about there "expected" locations.

Final Remarks

In this course, we have learned about

- Complex numbers, algebra, geometry and topology in the complex plane, complex functions;
- Complex dynamics, Julia sets of quadratic polynomials, the Mandelbrot set, conjecture of local connectedness of its boundary;
- Complex differentiation, the Cauchy-Riemann equations, analytic functions;
- Conformal mappings, inverse functions, Möbius transformations, the Riemann mapping theorem;
- Complex integration, Cauchy theory and consequences (such as Liouville's theorem, the maximum principle, etc.);
- Complex series, power (Taylor) series, the Riemann zeta function and its relation to prime numbers, and the Riemann Hypothesis.