Analysis of a Complex Kind

Week 6

Lecture 4: The Riemann Zeta Function And The Riemann Hypothesis

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The Riemann Zeta Function

- The zeta function was first introduced by Leonhard Euler (1707-1783), who used it in the study of prime numbers.
- In particular, he used its properties to show that $\sum_{p \text{ prime}} \frac{1}{p}$ diverges.
- This shows in particular, that there are infinitely many primes, but also some information about their distribution.
- Bernhard Riemann (1826-1866) used this function (a century after Euler) to obtain results on the asymptotic distribution of prime numbers.
- In today's lecture we'll study the Riemann zeta function (along with the Riemann hypothesis), and in the next (and last) lecture we'll study it's relation to prime numbers.

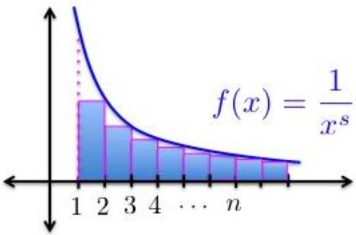
Recall:

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges (harmonic series),}$$

but

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{converges for all } s > 1.$$

This can be seen as follows:



$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^s} \leq 1 + \int_1^{\infty} \frac{1}{x^s} dx &= 1 + \frac{1}{1-s} \frac{1}{x^{s-1}} \Big|_1^{\infty} \\
&= 1 - \frac{1}{1-s} \\
&= \frac{s}{s-1} \quad (s > 1).
\end{aligned}$$

The Zeta Function

We now consider $s \in \mathbb{C}$ instead of $s \in \mathbb{R}$!

Definition

For $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$, the zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

- It is traditional to call the complex variable "s" instead of "z"!
- What is n^s for $s \in \mathbb{C}$? Note that for real s, we have that $n^s = e^{\ln n^s} = e^{s \ln n}$, so we define

$$n^s = e^{s \log n} = e^{s \ln n} \quad \text{for } s \in \mathbb{C}.$$

Convergence of $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$

Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s}$ converge for $\operatorname{Re} s > 1$?

Since $n^s = e^{s \ln n}$, we have that $|n^s| = \left| e^{s \ln n} \right| = e^{\operatorname{Re} s \ln n} = n^{\operatorname{Re} s}$. Thus

$$\sum_{n=1}^{\infty} \left| \frac{1}{n^s} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{\operatorname{Re} s}},$$

and since $\operatorname{Re} s > 1$, the series on the right converges. Thus $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges absolutely in $\{\operatorname{Re} s > 1\}$.

In fact, the convergence is uniform in $\{\operatorname{Re} s \geq r\}$ for any $r > 1$, and this can be used to show that $\zeta(s)$ is analytic in $\{\operatorname{Re} s > 1\}$.

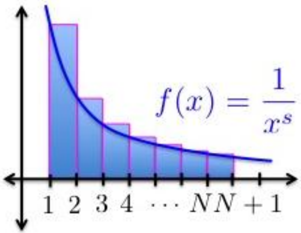
One can now show the following (this theorem goes back to Riemann):

Theorem

The zeta function has an analytic continuation into $\mathbb{C} \setminus \{1\}$, and this continuation satisfies that $\zeta(s) \to \infty$ as $s \to 1$.

Slightly easier to construct is an extension to the right half plane $\{Re s > 0\}$, minus the point 1, and we outline this construction here:

Motivation in $\mathbb{R}$:



$$\sum_{n=1}^{N} \frac{1}{n^s} = \int_1^{N+1} \frac{1}{x^s} dx + \sum_{n=1}^{N} \delta_n(s),$$

where

$$\delta_n(s) = \frac{1}{n^s} - \int_n^{n+1} \frac{1}{x^s} dx.$$

Analytic Continuation of the Zeta Function

$$\underbrace{\sum_{n=1}^{N} \frac{1}{n^s}}_{\to\zeta(s)} = \underbrace{\int_1^{N+1} \frac{1}{x^s} dx}_{\to 1/(s-1)} + \sum_{n=1}^{N} \delta_n(s), \quad \text{where} \quad \delta_n(s) = \frac{1}{n^s} - \int_n^{n+1} \frac{1}{x^s} dx.$$

Observe that $\sum_{n=1}^{N} \delta_n(s)$ is analytic in $\{\mathrm{Re}\, s > 0\}$. One can show that $\sum_{n=1}^{N} \delta_n(s)$ converges, as $N \to \infty$, to an analytic function $H(s)$ in $\{\mathrm{Re}\, s > 0\}$. Thus

$$\zeta(s) = \frac{1}{s-1} + H(s) \quad \text{holds for } \mathrm{Re}\, s > 1,$$

where $H(s)$ is analytic in $\{\mathrm{Re}\, s > 0\}$.

Analytic Continuation of the Zeta Function

$$\zeta(s) = \frac{1}{s-1} + H(s) \quad \text{holds for } \operatorname{Re} s > 1, \tag{*}$$

where

- $H(s)$ is analytic in $\{\operatorname{Re} s > 0\}$.
- $s \mapsto \dfrac{1}{s-1}$ is analytic in $\{\operatorname{Re} s > 0\} \setminus \{1\}$

We can therefore use (*) to define the zeta function in all of $\{\operatorname{Re} s > 0\} \setminus \{1\}$.

This definition agrees with the original definition in $\{\operatorname{Re} s > 1\}$.

Riemann was actually able to extend the zeta function to an analytic function in all of $\mathbb{C} \setminus \{1\}$.

The Zeros of the Zeta Function

Of much interest are the zeros of the zeta function, i.e. those $s \in \mathbb{C}$, for which $\zeta(s) = 0$.
One can show:

Theorem

The only zeros of the zeta function outside of the strip $\{0 \leq \operatorname{Re} s \leq 1\}$ are at the negative even integers, $-2, -4, -6, \ldots$.

- The zeros $-2, -4, -6, \ldots$ are often called the "trivial zeros", and the region to be studied remains the strip $\{0 \leq \operatorname{Re} s \leq 1\}$.
- A key result is that zeta has no zeros on the line $\{\operatorname{Re} s = 1\}$, this is an essential fact in the proof of the prime number theorem (see next lecture).
- From the fact that zeta has no zeros on $\{\operatorname{Re} s = 1\}$ it can easily be deduced that it has no zeros on $\{\operatorname{Re} s = 0\}$ either, via a functional equation.

The Riemann Hypothesis

In his seminal paper in which he proved the analytic continuation of the zeta function to $\mathbb{C} \setminus \{1\}$, Riemann initiated important insights into the distribution of prime numbers. In this paper, he expressed his belief in the veracity of the following:

Conjecture (Riemann Hypothesis)

In the strip $\{0 < \operatorname{Re} s < 1\}$, *all zeros of* ζ *are on the line* $\{\operatorname{Re} s = \frac{1}{2}\}$.

Much research has been done in attempts to prove this conjecture:

- $\zeta(s)$ has infinitely many zeros in $\{0 < \operatorname{Re} s < 1\}$.
- The asymptotic distribution of the zeros of ζ in $\{0 < \operatorname{Re} s < 1\}$ is known.
- At least one third of the zeros in $\{0 < \operatorname{Re} s < 1\}$ lie on the critical line $\{\operatorname{Re} s = \frac{1}{2}\}$.
- Trillions of zeros of zeta have been calculated - so far all of them lie on the critical line.

Conjecture (Riemann Hypothesis)

In the strip $\{0 < \operatorname{Re} s < 1\}$*, all zeros of* ζ *are on the line* $\{\operatorname{Re} s = \frac{1}{2}\}$*.*

- Numerical evidence and much research point to the validity of this conjecture, but it is to this day unproved and remains one of the most famous unsolved problems in mathematics.
- The Riemann Hypothesis is on the list of seven “Millennium Prize Problems” (declared by the Clay Mathematics Institute in 2000). Only one of these has been solved so far (as of summer 2013) - the so-called Poincaré Conjecture (by Grigori Perelman).
- The Riemann Hypothesis has strong implications on the distribution of prime numbers and on the growth of many other important arithmetic functions. It would greatly sharpen many number-theoretic results.

Next (last lecture): The Prime Number Theorem.