Analysis of a Complex Kind

Week 5

Lecture 1: Complex Integration

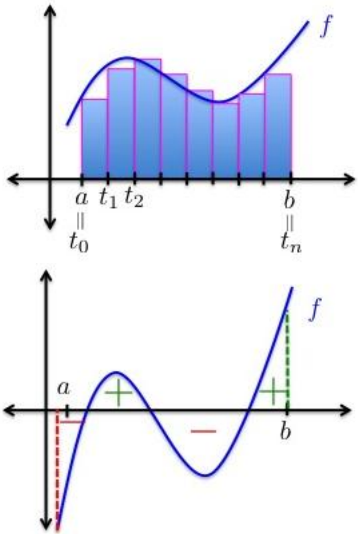
Petra Bonfert-Taylor

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} \sum_{j=0}^{n-1} f(t_j)(t_{j+1} - t_j),$$

where $a = t_0 < t_1 < \cdots < t_n = b$.

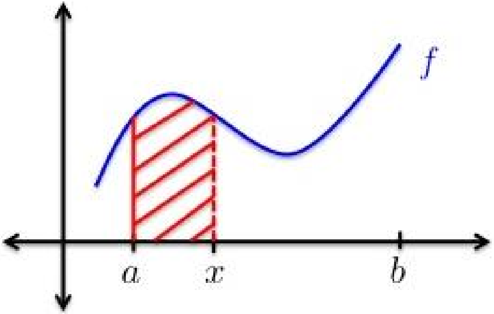
- If $f \geq 0$ on $[a, b]$ then $\int_a^b f(t)\,dt$ is the "area under the curve."
- Otherwise: sum of the areas above the x-axis minus sum of the areas below the x-axis.



The Fundamental Theorem of Calculus

Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous, and define $F(x) = \int_a^x f(t)dt$. Then F is differentiable and $F'(x) = f(x)$ for $x \in [a, b]$.



Antiderivatives

Let $f : [a, b] \to \mathbb{R}$ as above. A function $F : [a, b] \to \mathbb{R}$ that satisfies that $F'(x) = f(x)$ for all $x \in [a, b]$ is called an *antiderivative* of f.

Note: If F and G are both antiderivatives of the same function f, then

$$(G - F)'(x) = G'(x) - F'(x) = f(x) - f(x) = 0 \quad \text{for all } x \in [a, b],$$

and so $G - F$ is constant.

Conclusion: Let G be any antiderivative of f. Then

$$\int_a^b f(t)\, dt = G(b) - G(a).$$

Instead of integrating over an interval $[a, b] \subset \mathbb{R}$ we are in $\mathbb{C}$! What will we integrate over? Curves!

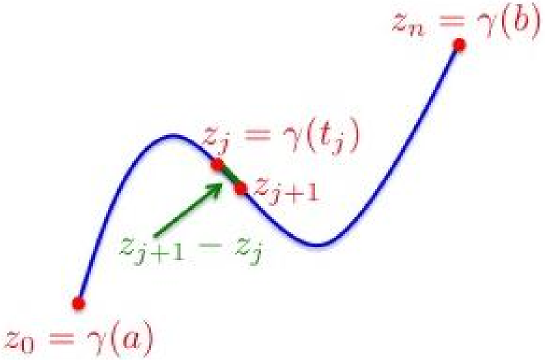
Recall: A curve is a smooth or piecewise smooth function

$$\gamma : [a, b] \to \mathbb{C}, \gamma(t) = x(t) + iy(t).$$

If f is complex-valued on γ, we define

$$\int_\gamma f(z)dz = \lim_{n\to\infty} \sum_{j=0}^{n-1} f(z_j)(z_{j+1} - z_j),$$

where $z_j = \gamma(t_j)$ and $a = t_0 < t_1 < \cdots < t_n = b$.



The Path Integral

$$\int_\gamma f(z)\,dz = \lim_{n\to\infty} \sum_{j=0}^{n-1} f(z_j)(z_{j+1} - z_j),$$

where $z_j = \gamma(t_j)$ and $a = t_0 < t_1 < \cdots < t_n = b$. One can show:
If $\gamma : [a, b] \to \mathbb{C}$ is a smooth curve and f is continuous on γ, then

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

Proof Idea:

$$\begin{aligned}\sum_{j=0}^{n-1} f(z_j)(z_{j+1} - z_j) &= \sum_{j=0}^{n-1} f(\gamma(t_j))\frac{\gamma(t_{j+1}) - \gamma(t_j)}{t_{j+1} - t_j}(t_{j+1} - t_j) \\ &\to \int_a^b f(\gamma(t))\gamma'(t)\,dt \quad \text{as n} \to \infty.\end{aligned}$$

Integrals over Complex-Valued Functions

Note: If $g : [a, b] \to \mathbb{C}$, $g(t) = u(t) + iv(t)$, then

$$\int_a^b g(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt.$$

Examples

- $\displaystyle\int_0^\pi e^{it}dt = \int_0^\pi \cos t\, dt + i\int_0^\pi \sin t\, dt = \sin t|_0^\pi - i\cos t|_0^\pi = 0 - i(-1-1) = 2i.$
- Alternatively: $\displaystyle\int_0^\pi e^{it}dt = -ie^{it}\Big|_0^\pi = -ie^{i\pi} + ie^0 = 2i.$
- $\displaystyle\int_0^1 (t+i)dt = \left(\frac{1}{2}t^2 + it\right)\Big|_0^1 = \frac{1}{2} + i.$

Examples of Path Integrals

1. $\gamma(t) = t + it,\ 0 \leq t \leq 1,\ \gamma'(t) = 1 + i,\ f(z) = z^2$. Then

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_0^1 f(\gamma(t))\gamma'(t)dt = \int_0^1 (t + it)^2(1 + i)dt \\
&= \int_0^1 (t^2 + 2it^2 - t^2)(1 + i)dt = \int_0^1 2it^2 - 2t^2 dt \\
&= -2\int_0^1 t^2 dt + 2i\int_0^1 t^2 dt \\
&= -\frac{2}{3}t^3\Big|_0^1 + \frac{2i}{3}t^3\Big|_0^1 \\
&= -\frac{2}{3} + i\frac{2}{3} = \frac{2}{3}(-1 + i).
\end{aligned}$$

Examples of Path Integrals

2. $\displaystyle\int_{|z|=1} \frac{1}{z}\, dz = ?$

   Let $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$. Then $\gamma'(t) = ie^{it}$, so:

$$\begin{aligned}
\int_{|z|=1} \frac{1}{z} dz &= \int_0^{2\pi} \frac{1}{\gamma(t)} \gamma'(t) dt \\
&= \int_0^{2\pi} \frac{1}{e^{it}} ie^{it} dt \\
&= i \int_0^{2\pi} dt \\
&= it\Big|_0^{2\pi} = 2\pi i.
\end{aligned}$$

Examples of Path Integrals

$$\gamma(t) = e^{it}, \quad 0 \le t \le 2\pi, \quad \gamma'(t) = ie^{it}$$

3. $\displaystyle\int_{|z|=1} z\,dz = ?$

$$\begin{aligned}
\int_{|z|=1} z\,dz &= \int_0^{2\pi} \gamma(t)\gamma'(t)\,dt = \int_0^{2\pi} e^{it} ie^{it}\,dt \\
&= i\int_0^{2\pi} e^{2it}\,dt = \frac{1}{2}e^{2it}\Big|_0^{2\pi} \\
&= \frac{1}{2}(e^{4\pi i} - e^0) = 0.
\end{aligned}$$

Examples of Path Integrals

$$\gamma(t) = e^{it}, \quad 0 \le t \le 2\pi, \quad \gamma'(t) = ie^{it}$$

4. $\displaystyle\int_{|z|=1} \frac{1}{z^2} dz = ?$

$$\begin{aligned} \int_{|z|=1} \frac{1}{z^2} dz &= \int_0^{2\pi} \frac{1}{\gamma^2(t)} \gamma'(t) dt = \int_0^{2\pi} \frac{ie^{it}}{e^{2it}} dt \\ &= \int_0^{2\pi} ie^{-it} dt = -e^{-it}\Big|_0^{2\pi} \\ &= -e^{-2\pi i} + e^0 = 0. \end{aligned}$$

5. In general:

$$\int_{|z|=1} z^m dz = \begin{cases} 2\pi i & \text{, if } m = -1 \\ 0 & \text{, otherwise.} \end{cases}$$

Next: More examples and first facts.