Note that $\{a_n\}, \{b_n\}$ are linear recurences. Thus, there are integers $k, l > 0$, polynomials $p_1(n), p_2(n), \ldots, p_k(n), r_1(n), r_2(n), \ldots, r_l(n)$, and complex numbers $\kappa_1, \kappa_2, \ldots, \kappa_k, \lambda_1, \lambda_2, \ldots, \lambda_l$ such that $a_n = \sum_{i=1}^{k} \kappa_i^n p_i(n)$ and $b_n = \sum_{i=1}^{l} \lambda_i^n r_i(n)$ for each $n \geq 0$. Now we have

$$a_n b_n = \sum_{i=1}^{k} \sum_{j=1}^{l} (\kappa_i \lambda_j)^n \, \sigma_{ij}(n),$$

where $\sigma_{ij}(n) = p_i(n) r_j(n)$ is a polynomial. Thus, $\{a_n b_n\}$ is a linear recurence and $C(q)$ is rational. $\square$