Evaluation of integrals

Example 1

Let us now practice with evaluation of integrals via residue theorem.

Compute the following trigonometric integral:

$$\int_0^{2\pi} \frac{d\theta}{5 - 3\sin\theta}$$

The trigonometric integrals with the integration domain spanning the period of the integrand are always taken in a similar manner. Let us first convert this integral into a contour one, over C – a circle centered at 0 of radius 1, oriented positively. This is achieved by substitution $z = e^{i\theta}$. Note that $dz = ie^{i\theta}d\theta = izd\theta$, so that $d\theta = dz/(iz)$. Also, $sin(\theta) = (z - z^{-1})/(2i)$. We obtain:

$$\int_0^{2\pi} \frac{d\theta}{5 - 3\sin\theta} = \oint_C \frac{dz}{iz(5 - \frac{3z-3/z}{2i})} = \oint_C \frac{-2dz}{3z^2 - 10iz - 3}.$$

The singularities of the integrand are simple poles at $z_1 = 3i$ and $z_2 = i/3$. Only one of them, z_2 is inside the integration contour C and has the residue

$$\mathrm{Res}(\frac{-2dz}{3z^2 - 10iz - 3}, i/3) = -i/4.$$

As a result,

$$\int_0^{2\pi} \frac{d\theta}{5 - 3\sin\theta} = 2\pi i(-i/4) = \pi/2.$$

Example 2

Compute the following integral

$$I = \int_{-\infty}^{\infty} \frac{dx}{(x^4 + 1)}. \tag{1}$$

Of course, this integral can be easily computed with the help of ordinary analysis, but our goal here is to practise residue theorem.

Note that the integral at first sight is not equivalent to any closed contour integral. However, the standard trick is to close the contour by a semiarc. The arc is used quite often as the closure of the infinite contour. That is because the modulus of the argument of the function, z, is always large as we move along the arc and we may use the asymptotics of the function instead of the function itself. Therefore we close the contour with upper semicircle (see Fig. 1). And,

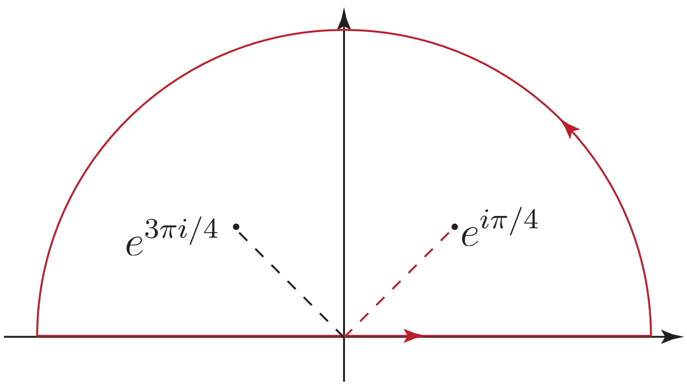

FIG. 1: Towards the computation of integral (1)

naturally, the hope is that the arc integral either vanishes or is reduced to some trivial number. Here, the asymptotics of the integrand at large z is $1/z^4$. It is indeed, quite simple. As we introduce the arc parametrization $z = Re^{i\theta}$ where θ changes from 0 to π, differential becomes large though: $dz = Rie^{i\theta}d\theta$. In the end, the integral behaves as

$$\frac{1}{R^3}\int_0^{\pi} ...d\theta \to 0, \quad R \to \infty.$$

Therefore, instead of original integral we now consider a complex closed contour integral:

$$\oint f(z)\,dz, \quad f(z) = \frac{1}{z^4+1}. \tag{2}$$

We just proved that this integral is, in fact, just our original integral. On the other hand we may compute it using residue theorem:

$$I = \oint f(z)\,dz = 2\pi i\Big(\operatorname*{res}_{z=e^{i\pi/4}} f(z) + \operatorname*{res}_{z=e^{3i\pi/4}} f(z)\Big) \tag{3}$$

Here we included only those residues which corresponds to the pole inside the close contour. Those are the first order poles. And the residues are easily computed by a simplified formula with derivative from the previous section.

$$\operatorname*{res}_{z=z_0} f(z) = \frac{1}{4z_0^3}$$

As a result we obtain the residues:

$$\operatorname*{res}_{z=e^{i\pi/4}} f(z) = \frac{1}{4e^{3\pi i/4}} = -\frac{1}{4}e^{i\pi/4}.$$

$$\operatorname*{res}_{z=e^{3i\pi/4}} f(z) = \frac{1}{4e^{9\pi i/4}} = \frac{1}{4}e^{-i\pi/4}.$$

The sum of two residues can be combined into a sine function $-(i/2)\sin\frac{\pi}{4} = -i/(2\sqrt{2})$. And finally, we the value of the integral:

$$I = \frac{\pi}{\sqrt{2}}$$