I. RESIDUE AT INFINITY

In the previous video we proved the residue theorem by shrinking the contour so, nothing left except the infinitesimal circles round the poles. And this way we achieved a great simplification culminating in the formulation of the residue theorem.

But there is one more way to simplify an integral. And it is as important as the previous one.

Suppose we have an integral of the meromorphic function in the complex plane along some closed contour γ. Now, instead of shrinking, let us try to expand the contour. Naturally as we do so we encounter singularities which are positioned outside the contour. Our final goals is to turn the integral into an infinite circle.

But as a result of contours catching the singularities we obtain a more interesting curve. It consists of the infinite circle, a number of infinitesimal circles passed in the clockwise direction round the poles and straight linear pairs of segments stretching from poles to infinity.

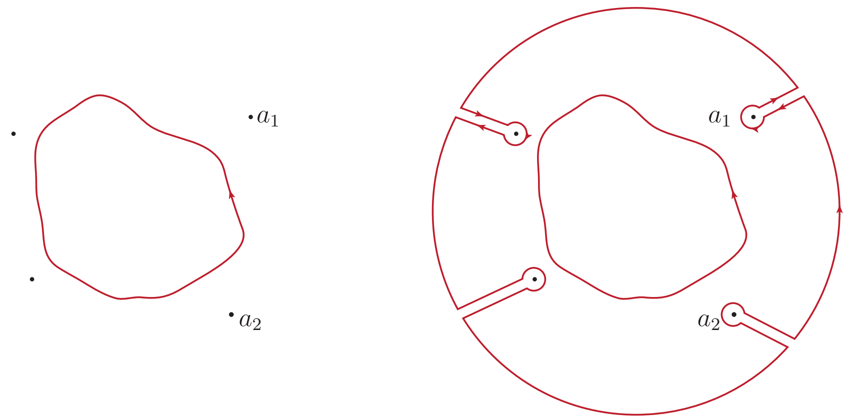

FIG. 1: Towards residue at infinity.

As before, we argue that the integrals along straight infinitely close segments passed in the opposite directions cancel each other. Now the infinitesimal integral round each pole is equal to $2\pi i$ times the residue of the integrand at this pole but with an opposite sign! Indeed, as you see, the orientation is clockwise. As a result, the integral is equal to $-2\pi i$ times the sum of the residues outside the original contour plus the integral over an infinite circle.

And now let us discuss this infinite circular integral. Let us perform the Laurent expansion of our function at $z \to \infty$.

In general,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n. \tag{1}$$

If the expansion has only finite amount of terms with positive powers of z, meaning the expansion starts like this:

$$f(z) = c_n z^n + c_{n-1} z^{n-1} + c_0 + \frac{c_{-1}}{z} + ... \tag{2}$$

then they say that the function has a pole of order n at infinity.

Such an expansion is called the Laurent expansion of the function at infinity.

Now, as we pointed out in the previous video, the integral of each term in the expansion is equal to zero with the exception of the term c_{-1}. This way, the integral along the infinite circle is equal to

$$\int_C f(z)\, dz = 2\pi i c_{-1}. \tag{3}$$

Therefore, we obtained a rather interesting result for an arbitrary closed contour integral. It is equal to the $-2\pi i$ times the sum of the residues outside the contour + $2\pi i$ times coefficient c_{-1} of the Laurent expansion of the function at infinity.

$$\int_\gamma f(z)\, dz = -2\pi i \sum_{a_i \in \text{out}} \operatorname*{res}_{z=a_i} f(z) + 2\pi i c_{-1} \tag{4}$$

For aethetic considerations, this coefficient c_{-1} with minus sign is called the residue of the function at infinity:

$$\operatorname*{res}_{z=\infty} f(z) = -c_{-1}. \tag{5}$$

And this way we obtain a beautiful complementary of the residue theorem. Now it is formulated as follows: the integral of the meromorphic in a complex plane function along a closed contour is found either as a $2\pi i$ times the sum of the residues inside the contour or $2\pi i$ times the sum of the residues outside the contour including the residue at infinity.

Also remember that the residue at infinity is just a clever expression of the integral over an infinite circle.

Now, the theorem proved has a very beautiful consequence:

Since for an arbitrary integral we may write down:

$$\oint = 2\pi i \sum \operatorname*{res}_{inside} f(z) = -2\pi i \sum \operatorname*{res}_{outside} f(z) \tag{6}$$

the sum of the residues of the function in an entire complex plane (including the residue at infinity is zero).

For a specific example, let us reconsider the integral we discussed in the previous video, $\int_{C_4} f(z)dz = 0$ with

$$f(z) = \frac{1}{z(z^2+1)}$$

and contour C_4 being a circle with radius 2, centered at $z = 0$. At a previous approach to this integral, we made a straightforward use of the residue theorem. This required to sum up contribution of the three residues, corresponding to simple poles of $f(z)$ inside C_4, that is $z = 0, \pm i$. Equipped with the notion of residue at infinity, we may observe that the integral can be evaluated in a simpler manner:

$$\int_{C_4} f(z)dz = -2\pi i \mathrm{Res}_{z=\infty} f(z)$$

and residue at infinity can be read off Laurent expansion of $f(z)$ at infinity. What we do, we simply expand our function in terms of inverse powers of z

$$f(z) = \frac{1}{z^3(1+1/z^2)} = \frac{1}{z^3} - \frac{1}{z^5} + ... \tag{7}$$

and we see that there is no term with $1/z$. The function simply decays too fast. Therefore,

$$\mathrm{res}_{z=\infty} f(z) = 0.$$

Hence, we obtain the same answer.