I. INTEGRATION (AND SUMMATION) WITH RESIDUES II

Evaluation of residues

So far we realized that for the computation of closed-contour integrals in a complex plane we need to compute the residues at poles.

Recall that basic definition of residues implies that they can be extracted from the respective Laurent series expansions, and one defines the residue as the coefficient c_{-1} of the power expansion.

So this seminar will be dedicated to the polishing of our technique with residues.

And we start with simple examples.

Example 1

Consider the computation of $\mathrm{Res}(\frac{e^z}{z^3}, 0)$. Here $z = 0$ is a third order pole.

Using the power series for e^z, we find the Laurent series:

$$\frac{e^z}{z^3} = \frac{1 + z + \frac{z^2}{2} + \frac{z^3}{6} + ...}{z^3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2}\frac{1}{z} + \frac{1}{6} + ...$$

and the residue equals $\mathrm{Res}(\frac{e^z}{z^3}, 0) = \frac{1}{2}$.

Example 2

Compute the residue:

$$\operatorname*{res}_{z=\infty} e^{a/z} z^n$$

where n is arbitrary positive integer.

This time we need to expand in $1/z$ powers. We start with the exponential:

$$e^{a/z} = \sum_{k=0}^{\infty} \frac{a^k}{z^k k!}$$

To extract $1/z$ term in $z^n e^{a/z}$ we need z^{n+1} term in the expansion of the exponential. It is $a^{n+1}/(n+1)!$

As a result the residue in question is

$$z = \infty \mathrm{res} e^{a/z} z^n = \frac{a^{n+1}}{(n+1)!}$$

Example 3 Find the residues of the function

$$f(z) = \frac{1}{(z-1)^2(z^2+1)}$$

at all final points.

As we see, the denominator has 3 roots $z = 1$ (second order root) and $z = \pm i$ (first order roots).

Hence we have the respective poles of $f(z)$ of the same order.

I. $z = 1$.

We perform the Laurent expansion of $f(z)$ near the point $z = 1$. As usual, we make a change $z = 1 + \varepsilon$ to obtain:

$$f(z) = \frac{1}{\varepsilon^2(2 + 2\varepsilon + \varepsilon^2)} \equiv \frac{1}{\varepsilon^2} \frac{1}{2 + 2\varepsilon + \varepsilon^2} \tag{1}$$

To obtain the residue we need to extract the coefficient in $1/ve$ term. We see that the second fraction in (1) is regular at $\varepsilon = 0$ and it can be easily Taylor expanded in ε: $A + B\varepsilon + ...$. We don't need any higher order terms. The accuracy upt to linear term in ε is enough. Combined with $1/\varepsilon^2$ in front it gives what we need.

Therefore we discard ε^2 term in the denominator:

$$f(z) = \frac{1}{\varepsilon^2} \frac{1}{2 + 2\varepsilon} = \frac{1}{2\varepsilon^2} \frac{1}{1 + \varepsilon} \tag{2}$$

and perform the geometric expansion of the second fraction: $(1+\varepsilon)^{-1} = 1 - \varepsilon + ...$

As a result we obtain the following Laurent expansion:

$$f(z) = \frac{1}{\varepsilon^2}\frac{1}{2+2\varepsilon} = \frac{1}{2\varepsilon^2} - \frac{1}{2\varepsilon} + ... \tag{3}$$

And extract the residue:

$$\operatorname*{res}_{z=1} f(z) = -\frac{1}{2}. \tag{4}$$

II. $z = i$.

This time the necessary change is $z = i + \varepsilon$. For transparency we expand the denominator into simple factors:

$$f(z) = \frac{1}{(z-1)^2(z-i)(z+i)} = \frac{1}{\varepsilon}\frac{1}{(i+\varepsilon-1)^2(2i+\varepsilon)} \tag{5}$$

And again, we have $1/\varepsilon$ term in front of the regular in ε function: $1/[(i-1+\varepsilon)^2(2i+\varepsilon)] = A + B\varepsilon + ...$ That means we need to retain only zero order term in this function, i.e. totally drop ε:

$$f(z) = \frac{1}{\varepsilon}\frac{1}{(i-1)^2 2i} + ... = \frac{1}{4\varepsilon} \tag{6}$$

As a result we read out the residue:

$$\operatorname*{res}_{z=i} f(z) = \frac{1}{4}. \tag{7}$$

III. $z = -i$.

In this case we could in principle do the same expansion but we opt for the different (short) path. As we remember, the sum of all the residues (including the residue at infinity) of a complex function vanishes. But, the reading out the residue is often a very easy task. What is the leading asymptotic behavior of our function at infinity? Obviously, it is:

$$f(z) = \frac{1}{z^4}, \quad z \to \infty. \tag{8}$$

That means $1/z$ term is absent in the expansion. The function simply decays too fast. Therefore the residue at infinity is 0. But from this we immediately conclude that:

$$\operatorname*{res}_{z=-i} f(z) + \operatorname*{res}_{z=i} f(z) + \operatorname*{res}_{z=1} f(z) = 0 \qquad \Rightarrow \operatorname*{res}_{z=-i} f(z) = \frac{1}{4}. \tag{9}$$

Example 4 Find the residues of the function:

$$f(z) = \frac{\cos z}{(z^2+1)^2}. \tag{10}$$

Here we have two second order poles at points $z = \pm i$.

Let us study the case $z = i$. We make an expansion $z = i + \varepsilon$:

$$f(z) = \frac{\cos z}{(z+i)^2(z-i)^2} = \frac{1}{\varepsilon^2}\frac{\cos(i+\varepsilon)}{(2i+\varepsilon)^2} = \frac{1}{\varepsilon^2}\frac{\cosh 1 \cos\varepsilon - i \sinh 1 \sin\varepsilon}{(2i+\varepsilon)^2} \tag{11}$$

As before we have $1/\varepsilon^2$ term in front of the regular in ε function. Hence, we Taylor-expand it in ε up to first order in ε. That implies $\cos\varepsilon = 1$ while $\sin\varepsilon = \varepsilon$:

$$f(i+\varepsilon) = -\frac{1}{4\varepsilon^2}\frac{\cosh 1 - i\varepsilon \sinh 1}{(1 - i\varepsilon/2)^2}. \tag{12}$$

Next, we perform the geometric-type expansion of the denominator:

$$\frac{1}{(1-i\varepsilon/2)^2} = 1 + i\varepsilon + ... \tag{13}$$

And as a result we have:

$$f(i+\varepsilon) = -\frac{1}{4\varepsilon^2}(\cosh 1 - i\varepsilon \sinh 1)(1+i\varepsilon) = -\frac{1}{4\varepsilon^2}(\cosh 1 + i\varepsilon \cosh 1 - i\varepsilon \sinh 1) \tag{14}$$

Finally, we read out the residue:

$$\operatorname*{res}_{z=i} f(z) = -\frac{i}{4}(\cosh 1 - \sinh 1) = -\frac{i}{4e}. \tag{15}$$

In a similar manner, we expand near $z = -i$ to extract the residue. But again, a different, smarter path is possible. Let us try to figure out what the residue at infinity of the function might be. On the one hand, it seems we need to expand function $\cos z$ at large z and collect infinite amount of terms which is not a particular pleasant task.

On the other hand, if we look at function $f(z)$ we immediately notice that it is an even function. It can't have $1/z$ terms in its expansion. Therefore,

$$\operatorname*{res}_{z=\infty} f(z) = 0. \tag{16}$$

But that means, of course, that

$$\operatorname*{res}_{z=-i} f(z) = -\operatorname*{res}_{z=i} f(z) = \frac{i}{4e}. \tag{17}$$

And this complete our first practice with residues. In the next section we will discover a general identity which allows for the computation of the residue of any function without constructing explicit Laurent expansion.