A. Introduction (Video 4.1)

We start the theory of multivalued functions.

For the first time multivalued functions were apparently systematically addressed by the French mathematician Augustin Louis Cauchy in his memoir on Calculus of complex numbers in 1846. There he introduced such concepts as branching and branch cuts. Young Riemann was mostly unaware of Cauchy's work and rederived most his results but with less rigor and deeper level of insight. In 1851 under the supervision of Carl Gauss he wrote his thesis on "The foundation of the theory of function of a single complex variable" where he introduced the concept of what is now called Riemann surface.

We will build our introduction in baby steps in the most elementary way. By the end of this week you will master the technique of separation of regular branches and have some experience with Riemann surfaces and analytic continuation.

To give you the idea how important the concept of a multivalued function is let me show you some simple example.

Consider the simplest possible function $f(x) = \sqrt{x}$. Even from junior high - school years you remember that this function has two values. Indeed, square root of 4 is either 2 or minus 2. In high school we resolved this two value problem by the introduction of the notion of the arithmetic root. We just decided to call arithmetic root the positive root and in most cases considered this only single value of this function. And this worked surprisingly well! In real analysis this definition appears to be consistent. Whenever we encountered the square root the silent agreement was to understand it as a positive number.

But things change drastically once you transcend the real axis and rise into the complex plane.

So let us see how the square root is going to look like when the positive number under it (say x_0) gradually moves in to a complex plane. We rotate the vector representing the initial number by φ. The new complex number will be (see fig. 1)(a)

$$z = x_0 e^{i\varphi}. \tag{1}$$

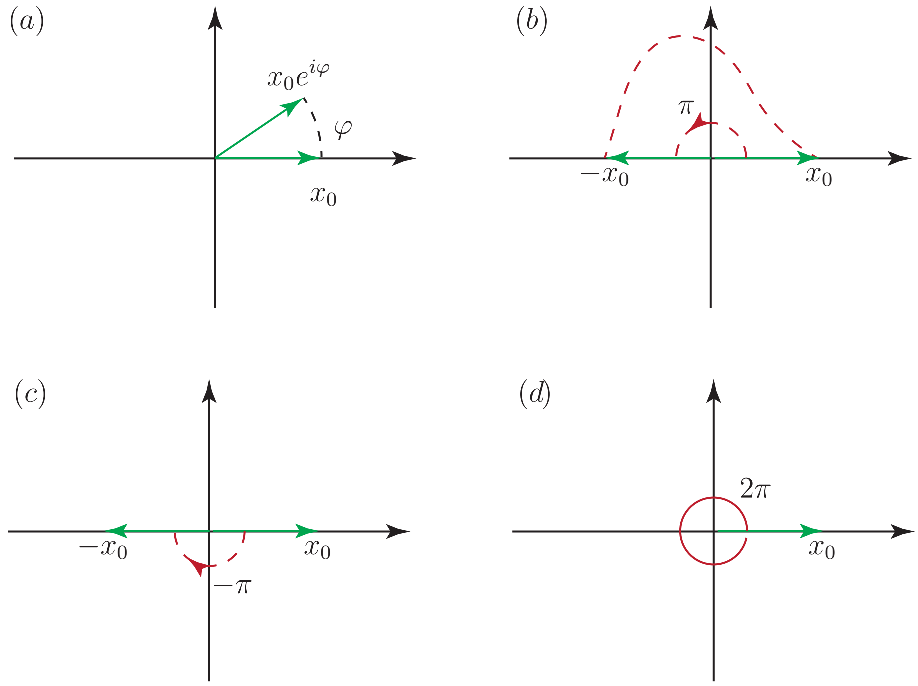

FIG. 1: Towards the definition of the square root.

How will we define the square root of this number now? Well, obviously, the most natural way is this:

$$\sqrt{z} = \sqrt{x} e^{i\varphi/2}. \tag{2}$$

This operation is called an analytical continuation of the function $\sqrt{x}$ into the complex plane along the curve. As you see, it requires three components: the initial point (or reference point) x_0, the contour, and the definition of the function itself.

Once we have agreed to take the arithmetic value for the square root of the positive number we managed to ascribe a single value to the $\sqrt{z}$, as we go along the contour.

Now, by making a pi rotation we end up with the value of the $\sqrt{-x} = \sqrt{x}e^{i\pi/2} = i\sqrt{x}$. So everything seems well defined.

But what if we try to get to the same point but along a different path (see fig. 1)(b). Say the differently curved contour in the upper complex plane. All right, the change of the argument is the same and doesn't seem to depend on the shape of the contour. The modulus is also well defined. So we again arrive at the same result. So it seems the analytical continuation is invariant under the contour deformation.

But, what if we choose the contour in the lower complex plane and arrive at point -x say via lower semicircle. Then, the change of the argument is $-\pi$ and we obtain a different result for $\sqrt{-x}$: $\sqrt{x}e^{-i\pi/2} = -i\sqrt{x}$, (see fig. 1)(c).

So it seems that we have a problem. The issue becomes even worse if I make a full counterclockwise rotation from point x_0 to arrive back in the same point. Then, the value of the function changes continuously from $\sqrt{x_0}$ to $-\sqrt{x_0}$ (see fig. 1)(d). We tried try to avoid the problem of two values of the square root. Instead, it is just grinning at us again. As the complex number under the square root makes a full rotation the value of the square root interpolates between its two values.

Let us try to understand the origin of the problem. In fact, it is almost clear. The culprit is the change of the argument of the complex number under the square root by 2 pi.

Indeed, the difference between $-x_0$ when approached from below and $-x_0$ when approached from above is that in the first case we treated the number as $x_0e^{i\pi}$ while in the second one as $x_0e^{-i\pi}$. 2pi.

In the case of a full rotation, it is the same.

So what shall we do?

Before we go any further let us make one more interesting observation. Suppose we make an analytical continuation along two full circles instead of just one. Then the complex number will turn by 4π and will turn into $x_0e^{4\pi i}$ (see fig. 2(a)).

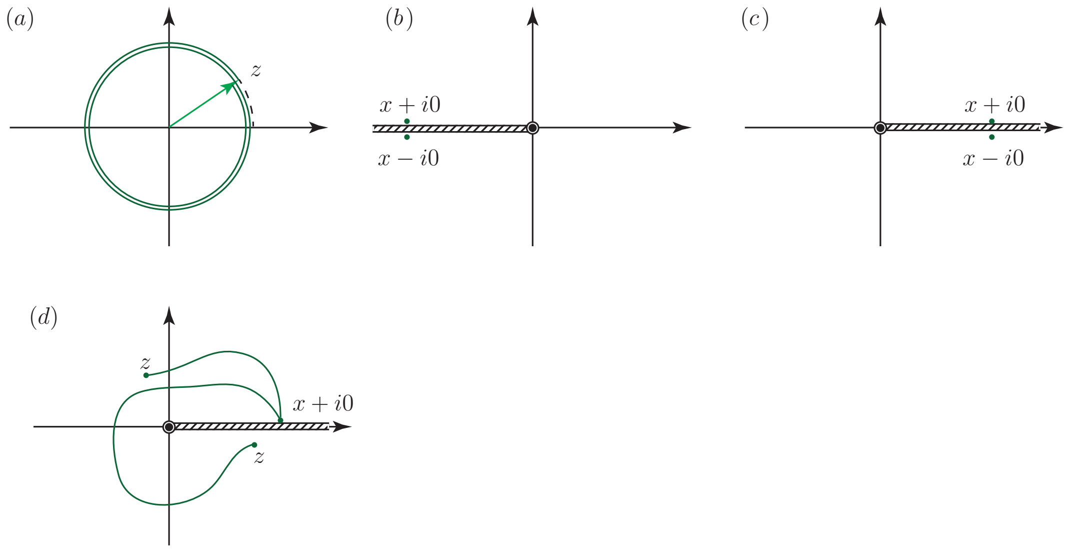

FIG. 2: Towards the definition of the square root.

Huh-huh! The square root then yields $\sqrt{x_0}e^{2\pi i}$. So after two full rotations we returned to the original value of our function.

Obviously, this property doesn't depend on a particular point at which we explore the value of the square root. We may start at any complex z_0, perform a full rotation $z \to ze^{2\pi i}$ and arrive at the value $f(z_0) = -\sqrt{z_0}$, that is, a second value of the square root. By making one more rotation we return to the initial value.

Now we understand that as in the real calculus, at each point the square root function can't take more than two values.

How can we cure this problem? The solution proposed by Cauchy was crude, but effective. We simply forbid a full rotation of the complex numbers under the root sign. But how do we achieve this?

Well, we have to modify the structure of the complex plane itself. We make a cut. The origin of this cut should always be a zero of the complex number under the root and tend to infinity in any direction. The cut is not necessarily a straight line. But at least in the beginning, for pedagogical reasons, we will stick to straight cuts.

Now let us see what is going to happen with our beloved square root of z once we make a branch cut along, say, negative real semi axis (see fig. 2(b)).

And we see how the problem is solved. The analytical continuation along the upper and lower arc lead not only to different answers but also to different points!

One of them lies slightly above the branch cut and the other one slightly below. They are separated by infinitely thin cut. To distinguish them we introduces new notation: $x + i0$ for the upper point and $x - i0$ for the lower one.

But, wait a little, now the real negative numbers lie exactly inside the branch cut. They are inaccessible. So the real negative numbers are forbidden for this choice of the branch cut. Our square root function is ill defined there. This is the gruesome reality of the multivalued functions.

On the bright sight you have a complete freedom of choice of the direction of the branch cut. Suppose your problem requires the ability to work with real negative numbers. Then just choose another direction of the cut. Why not trying the cut along the positive real semiaxis or some other direction? If you draw the branch cut along the positive semi axis (often you need exactly that kind of cut) the strictly positive numbers are inside the branch cut and no longer available. But the square root on the upper and lower banks of the branch cut makes perfect sense. (see fig. 2(c))

Suppose we decide that on the upper bank of the branch cut, our square root function is equal to the arithmetic root. So $f(x + i0) = \sqrt{x}$. Then on the lower bank at almost the same point we have $f(x - i0) = \sqrt{xe^{2\pi i}} = \sqrt{x}e^{i\pi} = -\sqrt{x}$.

Here, let us elaborate a little bit further. We decided that our square root assumes the arithmetic value on the upper bank of the branch cut along the real positive semi axis. By making different rotations and building different contours we can now determine the unique value of the square root in the entire complex plane, albeit spoiled by the branch cut (see fig. 2(d)).

This way our square root function became single valued. We reduce the set of its values by half. This portion of the square root function has a special name in complex analysis. It is called a regular branch of the square root.

We could use another definition. Let the square root assume negative value on the upper bank of the branch cut.

$$f(x + i0) = -\sqrt{x} \tag{3}$$

Then connecting the upper bank with other points with some contours in the complex plane and making respective analytical continuations along these contours we determine the unique $f(z)$ for the entire plane. At each point its value would differ in sign comparing to the previous definition of $f(z)$. This way we covered a different portion of the multivalued function.

And they say we define a second regular branch of the square root.

You see that on the upper and lower banks of the branch cut the square root undergoes a jump. Its value changes instantly from positive value to the negative one. This is yet another reason we need a branch cut.

So let me summarize the principal points of our introductory remarks.

First, after analytically continuing the square root along the curve we realized that even this simplest function can't be well defined in the entire complex plane. It inevitably starts branching. We also understood that the problem is connected to the complex number under the square root changing its argument by 2pi.

Second, we came to the conclusion that this problem can be solved in a rather radical fashion, by cutting a complex plane. And we introduced the notion of the branch cut.

And third, we discussed the notion if the branch cut.

In the next slide we will learn how to work with different branches of multivalued functions and study way more interesting functions.