I. GENERAL FORMULA FOR THE RESIDUE

As you practised with computation of residues, you probably noticed that sometimes, the Laurent expansion can become a bit tedious.

That is why I'll derive for you now the general formula for the residue which avoids the necessity of building a Laurent expansion.

It is extremely helpful in many cases I'll show how it works right after the derivation. We'll also learn an, I'd say, lighter version of it, which will be encountered quite often.

As a start let us write down the expression for the Laurent expansion of the function in the vicinity of nth order pole:

$$f(z) = \frac{c_{-n}}{(z-a)^n} + \frac{c_{-n+1}}{(z-a)^{n-1}} + \frac{c_{-n+2}}{(z-a)^{n-2}} + \frac{c_{-n+3}}{(z-a)^{n-3}} ... + \frac{c_{-1}}{z-a} + c_0 + c_1(z-a) + ...$$

Next we multiply both parts by $(z-a)^n$ to obtain

$$f(z)(z-n)^n = c_{-n} + c_{-n+1}(z-a) + c_{-n+2}(z-a)^2 + c_{-n+3}(z-a)^3 ... + c_{-1}(z-a)^n + c_0(z-a)^n + c_1(z-a)^{n+1} + ...$$

And now we start taking derivatives from both parts, one after another.

After the first differentiation the term c_{-n} which is a constant part in the r.h.s. disappears. The rest of the terms lower the power of (z-a) factor by one and acquire prefactors:

$$[f(z)(z-n)^n]' = c_{-n+1} + 2c_{-n+2}(z-a) + 3c_{-n+2}(z-a)^2 ... + (n-1)c_{-1}(z-a)^{n-2} + nc_0(z-a)^{n-1} + ...$$

Now terms c_{-n+1} is constant. And after the second differentiation it will disappear:

$$[f(z)(z-n)^n]'' = 2c_{-n+2} + 3\cdot 2c_{-n+2}(z-a) ... + (n-1)(n-2)c_{-1}(z-a)^{n-3} + n(n-1)c_0(z-a)^{n-2} + ...$$

and all other powers are diminished by one more unity. Look at term c_{-1}. It acquired the factor $n(n-1)$ after second differentiation. Now make a guess, what would be this factor after $n-1$ differentiations?

Of course $(n-1)!$.

Generally, after the second differentiation I hope the pattern becomes more or less clear. Each differentiation removes frontal terms $c_{-n},\ c_{-n+1},\ c_{n-2}$ and so on. After $n-1$ differentiations all the terms from c_{-n} to c_{-2} will be gone. Just check it yourself. And the power of factor $z-a$ at the coefficient c_{-1} weill be diminished by (n-1) units. That means, this power will simply disappear.

So, after such a differentiation our r.h.s. will look as

$$[f(z)(z-n)^n]^{(n-1)} = (n-1)!c_{-1} + n!c_0(z-a) + ...$$

On the r.h.s. under the dots I concealed senior powers of $(z-a)$. Then we simply take the limit $z=a$ on both sides. Then all the powers of $z-a$ will disappear in the r.h.s. and we obtain:

$$[f(z)(z-n)^n]^{(n-1)}\Big|_{z=a} = (n-1)!c_{-1}.$$

And this is the general formula for the residue of the function with the pole $z=a$ of nth order:

$$\operatorname*{res}_{z=a} f(z) = \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}} f(z)(z-a)^n\Big|_{z=a} \tag{1}$$

Now let us see how it is applied to particular cases.

Example 1.

$$f(z) = \frac{\cos z}{(z-1)^2}$$

Here, we have a second order pole $z=1$. According to our formula for the residue we obtain:

$$\operatorname*{res}_{z=1} f(z) = \frac{d}{dz} f(z)(z-1)^2\Big|_{z=1} = \cos' z\Big|_{z=1} = -\sin 1. \tag{2}$$

Example 2. Let us study our previous example and see if general formula (1) is more effective.

$$f(z) = \frac{1}{(z-1)^2(z^2+1)}$$

$z = 1$ is the second order pole, hence:

$$\operatorname*{res}_{z=1} f(z) = \frac{d}{dz} f(z)(z-1)^2\Big|_{z=1} = \Big(\frac{1}{z^2+1}\Big)'\Big|_{z=1} = -\frac{2z}{(z^2+1)^2}\Big|_{z=1} = -\frac{1}{2}; \tag{3}$$

Now, let us address the residue at point $z = i$. It is a first order pole (which is also called *simple pole*). For this situation we write down the simplified formula:

$$\operatorname*{res}_{z=a} f(z) = f(z)(z-a)\Big|_{z=a} \qquad \textit{simple pole}. \tag{4}$$

We expand the denominator $f(z)$ into factors and perform the cancellation:

$$\operatorname*{res}_{z=i} f(z) = \frac{(z-i)}{(z-i)(z+i)(z-1)^2}\Big|_{z=i} = \frac{1}{2i(i-1)^2} = \frac{1}{4}. \tag{5}$$

And similarly, we obtain the residue at point $z = -i$.

We see that sometimes general formula (1) can be much faster and effective than the Laurent expansion.

Example 3.

$$f(z) = \frac{1}{(z^2+1)^3}$$

Here, $z = \pm i$ are 3rd order poles. From the general residue formula we obtain:

$$\operatorname*{res}_{z=i} f(z) = \frac{1}{2!}\frac{d^2}{dz^2}\frac{1}{(z+i)^3}\Big|_{z=i} = \frac{1}{2}\frac{3\cdot 4}{(z+i)^5}\Big|_{z=i} = \frac{3}{16i}. \tag{6}$$

And similarly, for the point $z = -i$.

And let us discuss an additional formula which is very helpful once we deal with a first order pole. If a function has a first order pole in some point $z = a$ then it can always be represented as a ratio of two functions where the function in denominator has the first order root, while the function in nominator doesn't vanish at this point:

$$f(z) = \frac{h(z)}{g(z)}, \quad h(a) \neq 0.$$

Then we make a Taylor expansion of the function in the denominator restricting it to only first nonzero term: $g(z) = g'(a)(z-a)$. Then in the vicinity of $z = a$ the first term of the Laurent expansion of our original function looks as follows:

$$f(z) = \frac{h(a)}{g'(a)(z-a)}, \quad h(a) \neq 0.$$

Here, we restricted the expansion of the function in nominator by its first term as well, setting $h(z) = h(a)$. But then, we have precisely the expression of the type $\sim 1/(z-a)$. And the coefficient in front of it is our residue. Therefore, for simple poles we have the following nice formula

$$\mathrm{res} f = \frac{h(a)}{g'(a)}. \qquad \textit{first order pole}. \tag{7}$$

Let us see how suitable the formula can be in a typical situation.

Example 4.

$$f(z) = \frac{1}{z^3+1}. \tag{8}$$

Find the residues at all finite points.

Solving the equation $z^3 + 1 = 0$ we find that the function has three first order poles:

$$z_i = \begin{cases} e^{i\pi/3} \\ -1 \\ e^{-i\pi/3} \end{cases} \tag{9}$$

According to formula (7) the residue of the function at each of these points read:

$$\underset{z=z_i}{\mathrm{res}}\, f(z) = \frac{1}{3z_i^2} \tag{10}$$

This way, with a single computation we obtained three residues simultaneously:

$$\underset{z=z_i}{\mathrm{res}}\, f(z) = \begin{cases} \frac{1}{3}e^{-2\pi i/3}, & z = e^{i\pi/3} \\ \frac{1}{3}, & z = -1, \\ \frac{1}{3}e^{2\pi i/3}, & z = e^{-i\pi/3} \end{cases} \tag{11}$$