I. VIDEO 1.8

II. 1.8 CONFORMAL MAPPINGS

The last chapter of this lecture is dedicated to one more application of analytic functions. Namely, the conformal mappings.

Let us consider some complex function $f(z)$ in some domain such that its derivative df/dz never vanishes in this domain. As we pointed out earlier complex function f can be considered as a mapping between complex domain of its argument to complex domain of its values. Therefore, let us draw two complex planes z and f.

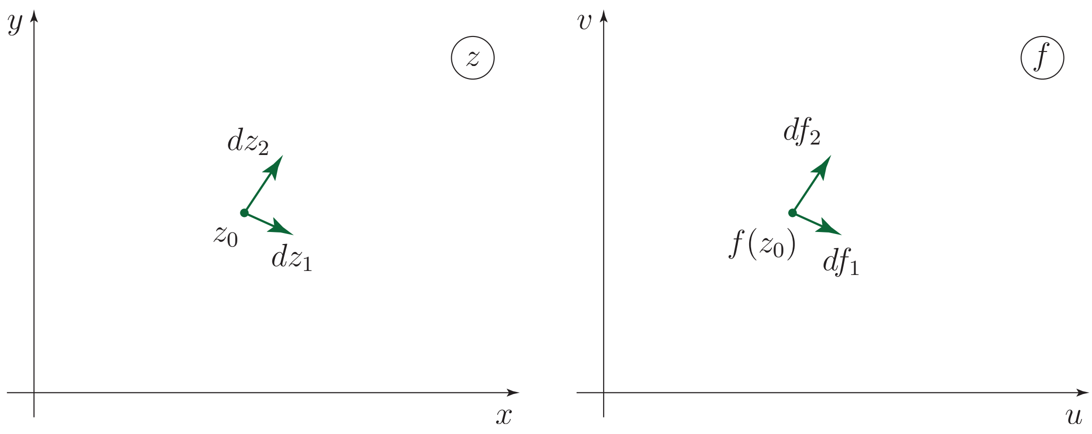

FIG. 1:

Then let us take some point z_0 and draw two infinitesimal vectors dz_1 and dz_2 from this point in complex plane z.

Obviously point z_0 is mapped onto $f(z_0)$ in the complex plane f. Analogously, points z_0+dz_1 and z_0+dz_2 are also mapped onto corresponding points in complex plane f: $f(z_0+dz_1)$ and $f(z_0+dz_2)$. These can be simplified using Taylor expansion of function f:

$$f(z_0+dz_1) = f(z_0) + \frac{df}{dz}\Big|_{z_0} dz_1,$$
$$f(z_0+dz_2) = f(z_0) + \frac{df}{dz}\Big|_{z_0} dz_2.$$

This way we obtain the image of vectors dz_1 and dz_2. Those are vectors

$$df_1 = \frac{df}{dz}\Big|_{z_0} dz_1,$$
$$df_2 = \frac{df}{dz}\Big|_{z_0} dz_2.$$

Now let us characterize the initial vectors by their moduli and its arguments:

$$dz_1 = |dz_1|e^{i\alpha_1}, \quad dz_2 = |dz_2|e^{i\alpha_2}$$

Then we introduce the same representation for the derivative:

$$\frac{df}{dz}\Big|_{z=z_0} = \Big|\frac{df}{dz}\Big|e^{i\gamma}$$

As a result the mapped vectors df_1 and df_2 read:

$$df_1 = \Big|\frac{df}{dz}\Big|e^{i(\gamma+\alpha_1)}|dz_1|, \quad df_2 = \Big|\frac{df}{dz}\Big|e^{i(\gamma+\alpha_2)}|dz_2|$$

This way we see, that the mapping by a holomorphic function turns the infinitesimal vectors at a particular point by the same angle γ given by the argument of the derivative of the function at this point. Therefore, the angle between the vectors is retained by the mapping, while the lengths of the vectors is multiplied by the same number given by the modulus of the derivative of the function.

Also, I leave it up to you to prove that an infinitesimal circle in complex plane z is turned into infinitesimal circle in complex plane f. The mapping which satisfies these two conditions is called conformal.

We may say that any holomorphic function with non-zero derivative in some domain implements a conformal mapping of this domain. The condition is, in fact, a little bit more strict, but it is irrelevant for our introductory discussion. What I'd like you to learn, however, is how to build the images of contours under conformal mapping. It is an essential skill in complex analysis because whenever you deal with an integral in a complex plane, and you make a change of a complex variable, you need to understand how your integration contour is transformed.

So, let us study a simple example.

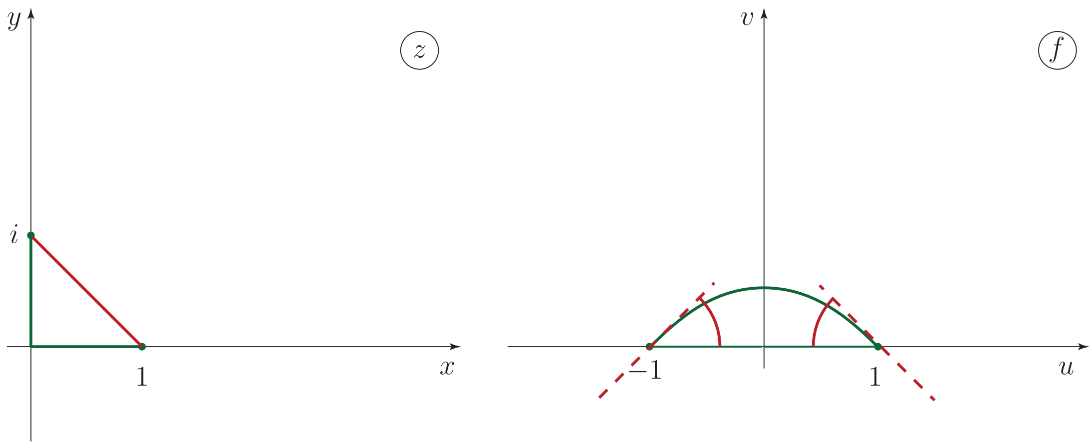

FIG. 2:

We consider the right triangle with apexes at the origin and points 1 and i. Let us see, how it is transformed by a conformal mapping $f = z^2$.

First of all, let us again redraw the complex plane z and the complex lane $w = u + iv$. Our function realises the mapping $u = x^2 - y^2$ and $v = 2xy$.

Then, let us parameterize the first side of our triangle $x = t,\ y = 0,\ t \in [0, 1]$ then, its image becomes $u = t^2,\ \ v = 0$, that is, a unit segment starting at the origin going to the right along u-axis.

The parametrization for the next side, $x = 0,\ y = t,\ $. We obtain $u = -t^2,\ v = 0$, that is the unit segment starting at the origin but going to the left along u-axis.

Finally, the segment connecting 1 and i. It is the part of a line with equation $y = 1 - x$. We parameterize $x = t$ and y as $1 - t$ to obtain:

$$u = x^2 - y^2 = (x - y)(x + y) = (2t - 1), \quad v = 2xy = 2t(1 - t) \tag{1}$$

To obtain the curve $v(u)$, we express t via $t = (1/2)(u + 1)$ and, as a result we get:

$$v = \frac{1}{2}(u + 1)(1 - u) = \frac{1}{2} - \frac{1}{2}u^2. \tag{2}$$

And we see, that we obtained a parabolic curve connecting the edge points -1 and 1. This is how the conforming map is done. Now if you compute the tangents of the parabola at $u = \pm 1$ you will see that they are equal to ± 1. That means the respective angles are $\pm\pi/4$ which precisely corresponds to the acute angles of our initial right triangle in z plane.

And indeed, our mapping preserves the angles between lines as it should. However, the $\pi/2$ angle is destroyed, it is turned into π. Try to understand why this happened.

The last issue I'd like to touch in this week is the integration along the curve in the complex plane. Certainly if we can define the sum of complex numbers, the limit and convergence, then, obviously, we can define the notion of the integral.

Suppose we have some contour starting at point z_1 and ending at z_2 in a complex plane and suppose we have some function $f(z)$.

Then we can split the contour into small linear segments Δz_i and compose the sum:

$$I = \sum_i f(z_i)\Delta z_i \tag{3}$$

Then shrinking the step of our partition we obtain a well defined limit which is called an integral of a complex function along the contour:

$$I = \int_{z_0}^{z_1} f(z)dz. \tag{4}$$

The integral can be split into two real 2D curve integrals in a natural way. If $f = u + iv$ and $dz = dx + idy$ than

$$I = \int_\gamma (udx - vdy) + i \int_\gamma (udy + vdx). \tag{5}$$

For example, we may consider an integral along the upper right quarter circle of the function $f(z) = z$. The integral can be rewritten as

$$I = \int_\gamma (xdx - ydy) + i \int_\gamma (xdy + ydx) = \frac{1}{2} \int_\gamma d(x^2 - y^2) + i \int_\gamma d(xy) = \frac{1}{2} \int_\gamma d(x^2 - y^2 + 2ixy) \tag{6}$$

$$= \frac{1}{2} \int_\gamma dz^2 = \frac{1}{2} z_1^2 - \frac{1}{2} z_0^2 = -\frac{1}{2} - \frac{1}{2} = -1 \tag{7}$$

As we see, the two dimensional integral can be reduced to a simple antiderivative but taken in a complex plane. We understand that the value of the integral doesn't depend here on the integration path, but on the position of the beginning and end points of the contour.

There is a fundamental reason for this in complex analysis, it is hidden in the analyticity of the omplex function and we will essentially dedicate the rest of course to the exploration of this unique property. But upto now I think that is it for this week.