2. Cauchy theorem. Types of singularities. Laurent and Taylor series.

Problem 2.1

Find the principal part of the Laurent series of the function

$$\frac{1+2z^2}{z^3+z^5}$$

at $z=0$.

Solution.

$$\frac{1+2z^2}{z^3+z^5} = \frac{1}{z^3}\frac{1+2z^2}{1+z^2} = \frac{1}{z^3}(1+2z^2)(1-z^2+O(z^4)) = \frac{1}{z^3}+\frac{1}{z}+O(z).$$

Problem 2.2

Find the order of the pole and coefficient in front of $\frac{1}{z}$ of Laurent series at $z=0$ for the function

$$\frac{1}{z\left(e^z-1\right)}.$$

Solution.

$$\frac{1}{z\left(e^z-1\right)} = \frac{1}{z^2}\frac{1}{1+z/2+O(z^2)} = \frac{1}{z^2}-\frac{1}{2}\frac{1}{z}+O(1)$$

which has a pole of the 2nd order.

Problem 2.3

Build the Laurent expansion around $z=0$ for the function

$$\frac{1}{z(z-1)}$$

1. for the region $0<|z|<1$.
2. for the region $1<|z|<\infty$.

Solution.

1. For $0<|z|<1$:

$$\frac{1}{z(z-1)} = -\frac{1}{z}\frac{1}{1-z} = -\sum_{n=0}^{\infty} z^{-1+n}$$

2. for the region $1<|z|<\infty$:

$$\frac{1}{z(z-1)} = \frac{1}{z^2}\frac{1}{1-1/z} = \sum_{n=0}^{\infty} z^{-2-n}$$

Problem 2.4

Build the Laurent expansion for the function

$$\frac{z}{z^2+1}$$

around $z = i$. What is the convergence region of the result?

Solution.

$$\frac{z}{z^2+1} = \frac{1}{2(z+i)} + \frac{1}{2(z-i)} = \frac{1}{2(z-i)} - \frac{1}{2}\sum_{n=0}^{\infty}(i/2)^{n+1}(z-i)^n$$

The convergence radius equals the distance from $z = i$ to the nearest singularity $z = -i$ which is equal to 2.

Problem 2.5

Find the singularity type at $z = \pm\pi$ for the function

$$\frac{1}{\sin z} + \frac{2z}{z^2-\pi^2}.$$

Solution. Both terms in the expression are singular but the singuarities cancel in the final result:

$$\left.\frac{1}{\sin z} + \frac{2z}{z^2-\pi^2}\right|_{z\to\pm\pi} = \pm\frac{1}{2\pi} + O(z\mp\pi)$$

hence the points $z = \pm\pi$ are removable singularities.

Problem 2.6

Compute the following integrals along contour $\mathcal{C}$ (unit circle centered at $z = 0$). To this end, i) check that the Laurent series around $z = 0$ converges on $\mathcal{C}$, ii) observe that only one of the terms of the Laurent series contributes to the integral and iii) compute this term and evaluate the integral.

1. $\int_{\mathcal{C}} \frac{ze^z}{\tan z^2} dz$.
2. $\int_{\mathcal{C}} e^{-1/z}\sin\left(\frac{1}{z}\right) dz$.
3. $\int_{\mathcal{C}} \frac{e^z}{z^n} dz \quad$ (for natural $\quad n$).

Solution. In all cases, only term $\propto \frac{1}{z}$ contributes to the integral.

1. The singularity closest to $z = 0$ is located at $z = \pm\pi$ hence the Laurent series converges for $|z| < \pi$. The point $z = 0$ is a simple pole and Laurent series starts from

   $$\frac{ze^z}{\tan z^2} = \frac{1}{z} + O(1),$$

   hence the integral equals $\int_{\mathcal{C}} \frac{dz}{z} = 2\pi i$.
2. There are no singularities apart from $z = 0$ and Laurent series converges for all z. The Laurent series can be found using Taylor series:

   $$e^{-1/z}\sin\left(\frac{1}{z}\right) = (1 - 1/z + ...)\left(1/z - \frac{1}{3}1/z^3 + ...\right).$$

   The coefficient in front of $1/z$ equals 1 hence the integral equals $2\pi i$.

3. There are no singularities apart from $z = 0$ and Laurent series converges for all z. the Laurent series reads:

$$\frac{1}{z^n}\sum_{k=0}^{\infty}\frac{z^k}{k!}.$$

   The coefficient in front of $1/z$ equals $1/(n-1)!$, hence the integral equals $\frac{2\pi i}{(n-1)!}$.

Problem 2.7

Find all the isolated singularities and define their type for the functions

1. $f(z) = \frac{\sin z}{1-\tan z}$.
2. $f(z) = \frac{e^{c/(z-a)}}{e^{z/a}-1}$.

Solution.

1. The numerator is analytic hence the possible singularities are those points where denominator vanishes, $z = \frac{\pi}{4} + \pi n$ with integer n. At all these points numerator does not vanish and the function $f(z)$ has a simple pole.
2. The numerator has an essential singularity at $z = a$ where denominator is finite, hence this point is an essential singularity of $f(z)$. The denominator vanishes linearly at $z = 2\pi ian$ where numerator is finite, hence the function $f(z)$ has simple poles at these points.

Problem 2.8

Find the coefficient in front of $\frac{1}{z}$ of Laurent series at $z = 0$ of the following functions

1. $f(z) = \frac{\sin\frac{1}{z}}{1-z}$.
2. $f(z) = \exp\left(-\exp\left(\frac{1}{z}\right)\right)$.

Solution.

1. Expanding both factors in Taylor series we find

$$f(z) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^k\left(\frac{1}{z}\right)^{2k+1}}{(2k+1)!}z^n.$$

   The term $\propto 1/z$ comes from $n = 2k$ and the resulting coefficient equals $\sin 1$.
2. Using Taylor series

$$\exp(-\exp(t)) = \exp(-1) - t\exp(-1) + O(t^2)$$

   we find that the answer is $-\exp(-1)$.

Problem 2.9

Determine the singularity type of the function $ze^{\frac{1}{z}}e^{-\frac{1}{z^2}}$ at point $z = 0$.

Solution. The point $z = 0$ is an isolated singularity of this function and the associated Laurent series contains arbitrary high negative powers of z, hence this point is an essential singularitty of this function.

Problem 2.10

Function

$$\frac{e^{iz}}{\cos z - 1}$$

can be expanded into Laurent series $\sum_{n=-\infty}^{\infty} c_n z^n$ in the region $2\pi k < |z| < 2\pi(k+1)$ for any integer non–negative k. Find coefficient $c_{-3}^{(k)}$ for such a series for $k = 0$ and $k = 1$.

Solution. The Laurent expansion in the vicinity of $z = 0$ can be computed straightforwardly, via expansion of numerator and denominator:

$$\frac{e^{iz}}{\cos z - 1} = \frac{1 + iz + ...}{-\frac{z^2}{2} + \frac{z^4}{24} + ...} = -\frac{2}{z^2} + ...$$

and the coefficient $c_{-3}^{(0)}$ vanishes. This expansion, however, converges only for $|z| < 2\pi$, since the function is singular at $z = \pm 2\pi$. In order to compute $c_{-3}^{(1)}$, we need to rearrange the function, singling out the problematic part (the one which does not allow the Laurent expansion above to converge). We have to make the singular contribution explicit and hence we write, identically:

$$\frac{e^{iz}}{\cos z - 1} = \left(\frac{e^{iz}}{\cos z - 1} - f_1(z)\right) + f_1(z), \tag{1}$$

where

$$f_1(z) = -\frac{2}{z^2} - \frac{2i}{z} - \frac{2}{(z-2\pi)^2} - \frac{2i}{z-2\pi} - \frac{2}{(z+2\pi)^2} - \frac{2i}{z+2\pi} + \frac{5}{2}.$$

The crucial point about rearrangement as in Eq. (1) is that the first term is regular in the domain $|z| < 4\pi$ (we managed to single out the poles at 0 and $\pm 2\pi$). We now need to expand the function $f_1(z)$. It is important to realize that expansion of the fractions into geometric series over $z/(2\pi)$ would not work, since the convergence domain would not be an appropriate one. Instead, we need to expand over $2\pi/z$ which is smaller than 1 by magnitude in our domain of interest. This gives the series, which converges for $|z| > 2\pi$:

$$f_1(z) = ... - \frac{16i\pi^2}{z^3} - \frac{6}{z^2} - \frac{6i}{z} + \frac{5}{2}.$$

We can read off the requested coefficient directly from this expansion and $c_{-3}^{(1)} = -16i\pi^2$.