A. Introduction (Video 1.1)

We will start with some very simple definitions and operations. Of course, we will not spend too much time of them and those of you who are already familiar with this material please stay with us for a while. Apart from algebraic aspects of complex numbers, we will exercise in geometrical interpretations which will be later very useful for construction of regular branches of multi-valued functions and computation of complicated integrals in the following lectures.

1. Definition of a complex number and algebraic operations

$$z = x + iy, \quad x, y \in R, \quad i^2 = -1$$

where $x = \operatorname{Re} z$, and $y = \operatorname{Im} z$.

It is easy to construct a multiplication rule:

$$z_3 = z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = \underbrace{x_1 x_2 - y_1 y_2}_{\operatorname{Re} z_3} + i \underbrace{(x_1 y_2 + x_2 y_1)}_{\operatorname{Im} z_3} \tag{1}$$

It immediately follows that complex numbers inherit commutativity and distributivity from real numbers. A bit less trivial fact which makes the definition above very useful is that complex number can also be divided! For division of complex number z by a real number x_0 it is of course trivial, as division can be substituted by multiplication by an inverse number:

$$z/x_0 = \frac{1}{x_0}(x + iy) = \frac{x}{x_0} + i\frac{y}{x_0}. \tag{2}$$

Before turning to division of complex number by complex number, let us define the conjugation operation:

$$z = x + iy, \quad z^* = x - iy. \tag{3}$$

Note that the product zz^* is a real number:

$$zz^* = (x + iy)(x - iy) = x^2 + y^2 \tag{4}$$

The quantity $|z|$ defined as

$$|z| = \sqrt{x^2 + y^2} \tag{5}$$

is called modulus of a complex number z. As a result, we see $zz^* = |z|^2$.

Now, let us come back to division of complex numbers. To this end, we have to be able to find solutions to the following equation:

$$zz_1 = z_2, \tag{6}$$

for complex number $z = x + iy$, considering $z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2$ as given complex number. This equation has a formal solution

$$z = z_2/z_1 = z_2 z_1^{-1} \tag{7}$$

But what is actually z_2/z_1? To understand it, let us multiply our equation by z_1^*:

$$z|z_1|^2 = z_2 z_1^* \tag{8}$$

As a result

$$z = \frac{z_2 z_1^*}{|z_1|^2} \tag{9}$$

We find that inversion should be defined as follows:

$$z_1^{-1} = \frac{z_1^*}{|z_1|^2} \tag{10}$$

Let us consider a little exercise and compute the fraction $\frac{z_1}{z_2}$ where

$$z_1 = 2 - 5i, \quad z_2 = -3 + 4i.$$

We find:

$$\frac{z_1}{z_2} = \frac{(2-5i)(-3-4i)}{25} = -\frac{26}{25} + \frac{7}{25}i$$

2. *Geometric interpretation*

It is often convenient to think of a complex number as a 2D vector, see Fig. 1. Here $z = a + ib$ and two coordinates are equal to a and b. Then complex number -1 is represented by a unit vector oriented to the left and complex number i – by a unit vector forming $\pi/2$ angle with the x-axis. Naturally, sum or difference of complex numbers can

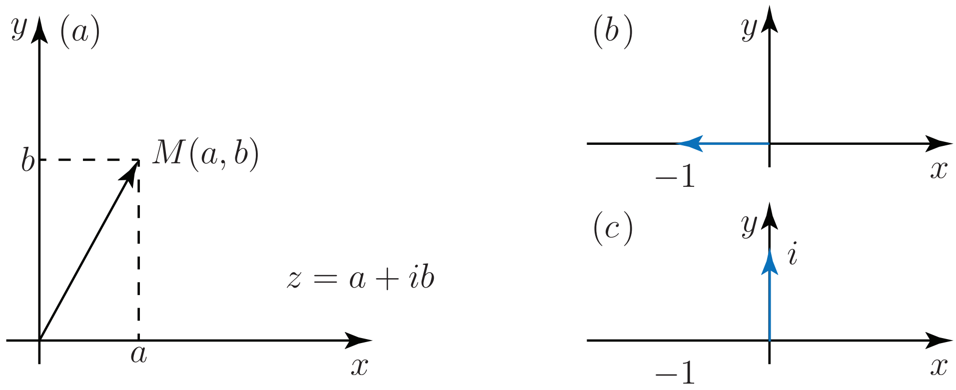

FIG. 1: Geometric interpretation of complex numbers.

be interpreted as sum or difference of associated vectors in the complex plane.

3. *Example of geometric interpretation*

Let us now practice with determination of loci of certain complex equations. Find the loci of the following equation: $|z+1| = 1$. Let us start with drawing a vector to a given point z, and will think of $z+1$ as $z-(-1)$. That is, let us think of $z+1$ as a vector pointing from -1 to z. According to the definition of our set, the length of this vector equals one. Thus, the logic of our equation is a unit circle centered at -1. This problem of course allows for an analytic solution which is less elegant though:

$$|z+1| = |x + iy + 1| = \sqrt{(x+1)^2 + y^2} = 1 \ \Rightarrow (x+1)^2 + y^2 = 1$$

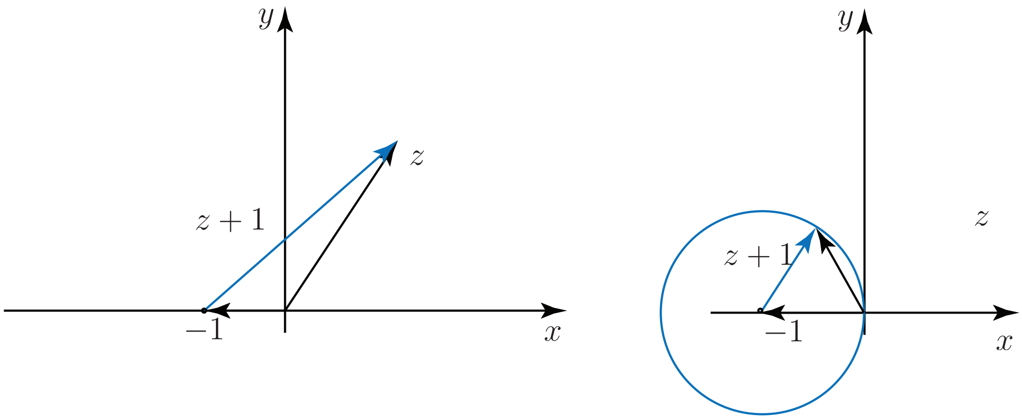

FIG. 2: The loci of equation $|z+1| = 1$.