I. RIEMANN SPHERE

As you probably start to feel, the infinity plays a special role in complex analysis, quite unlike the role in real calculus. Though the variety of infinitely distant points in a plane form a circle (Fig. 1(a)), sometimes we expand our function in the vicinity of infinity, or introduce a pole and residue at infinity.

In other words, the infinity sometimes behaves like a regular point in complex analysis. Riemann was the first who suggested a nice geometrical interpretation of this observation. Let us put a sphere (now known as a Riemann sphere) on a complex plane. Then every point of the complex plane can be projected on a sphere with the help of so-called stereographic projection. We connect the point on a plane with the north pole of a sphere. The intersection with sphere gives us the image of the point on a plane on a sphere. It is obvious that every point on a complex plane has a unique projection on a sphere (see Fig. 1(b)).

The reverse is almost true with only one exception: the north pole of a sphere. It doesn't have a distinct counterpart in a complex plane. On the contrary, it seems that every infinitely distant point in a plane is projected onto a north pole of a Riemann sphere. If you use your imagination, you will easily understand that the circle of a large radius on a complex plane is projected on to infinitesimal circle near the north pole on a Riemann sphere (Fig. 1(c)).

To make the correspondence completely 1 to 1, we introduce a formally infinitely distant point on a complex plane $z = \infty$ which corresponds to the north pole of a Riemann sphere. Number $z = \infty$ doesn't take part in arithmetic operations as ordinary complex numbers. However, they say that the sequence z_n convergence to infinity if for any positive M we can find number n_0 starting from which $|z_n| > M$. This terminology is justified because the stereographic projection of our sequence indeed converges to the north pole.

A complex plane with addition of an infinitely distant point is called the extended complex plane. It is equivalent to the sphere, or like topologiest say the two objects are diffeomorphic manifolds. The extended complex plane is therefore compact.

As a function on the complex plane is understood as a mapping between two complex planes, we now understand the function in extended complex plane as the mapping between two Riemann spheres (Fig. 1(d)). In particular, the notion of an infinite limit of the function is no longer unusual. It simply means that the corresponding value of a function is on a north pole on a sphere of its values.

It is not hard to prove that circle on a sphere are projected as circles on a Riemann sphere (Fig. 1(e)). While it is obvious that the line in complex plane becomes a circle on a sphere passing through the north pole. Indeed, to build a projection of a line simply draw a plane via the north pole of a sphere and the line in a plane. The intersection of the obtained this way plane with a sphere is the projection of a line. But on the other hand, the intersection of a plane and a sphere is always a circle (Fig. 1(f).

The neighborhood of infinity is understood as the exterior of a circle of radius $|z| > R$. Geometrically, this is easy to understand in terms of stereographic projection. If we have the circle of radius R then the region $|z| > R$ is projected on to an interior of a circle surrounding the north pole of a sphere.

All the definition of the limits, connectedness are carried over a Riemann surface without any change.

The Riemann sphere is a very useful geometrical concept, which is suitable to work with when we deal with infinities in a complex plane. Though we won't use it much in our course, in differential geometry and algebraic topology it has many beautiful applications.

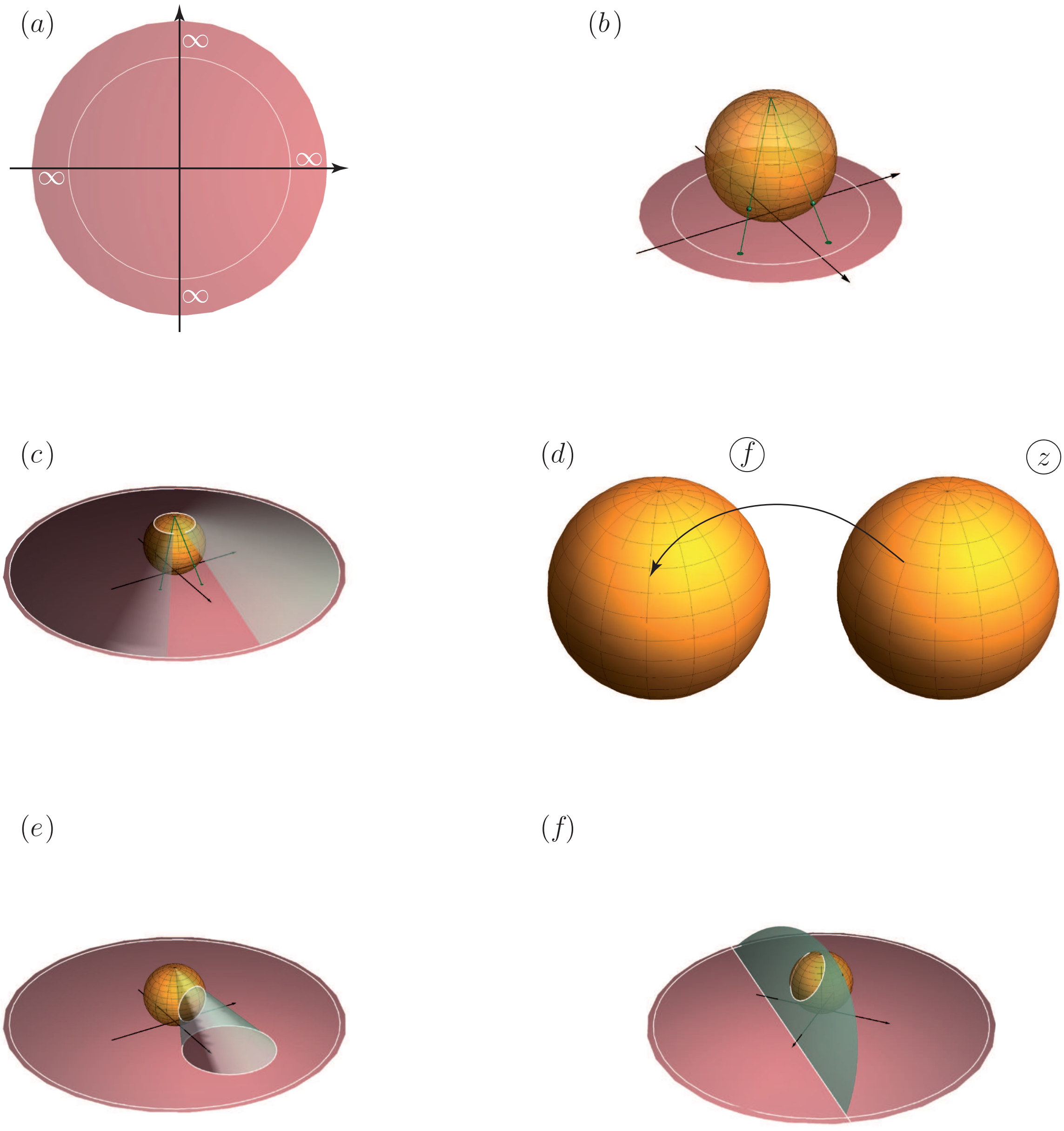

FIG. 1: Riemann sphere.