A. Residue theorem

In the previous video I outlined a certain technique of tackling the integrals. Before we continue with practicing this technique let us pinpoint the crucial step in what we did. We managed to reduce the computation of original integral to evaluating an integral over a small loop surrounding a singularity at $z = i$. Moreover, everything that mattered at this stage was the behaviour of the integrand in the vicinity of this pole and was entirely defined by the behavior of the integrand near this pole. If you think a little what we actually did, we Laurent expanded the integrand near the pole.

That is the main reason we studied attentively Laurent expansions of functions during our entire previous week.

Now, time has come to introduce the central definition of our course. The residue of a function $f(z)$ at the point $z = z_0$ which is an isolated singularity of $f(z)$ is defined as coefficient c_{-1} of an associated Laurent series:

$$f(z) = ... + \frac{c_{-1}}{(z - z_0)} + c_0 + ... \tag{1}$$

Symbolically, this is denoted as follows:

$$\mathrm{Res}_{z=z_0} f(z) = c_{-1}. \tag{2}$$

Before we state the residue theorem, let us also introduce one more important concept, the one of meromorphic function. A function that is analytic on all of domain D except for a set of isolated points, which are poles of the function, is called meromorphic function on D.

Residue theorem

Residue theorem concerns the integral I of a meromorphic function over a positively oriented non-self-intersecting closed curve γ. It states that

$$\oint_\gamma f(z)\, dz = 2\pi i \sum \mathrm{Res}_{z=z_k} f(z), \qquad \text{Residue theorem} \tag{3}$$

with the sum running over singular points of $f(z)$ inside γ, a_k. (see Fig. 1(a))

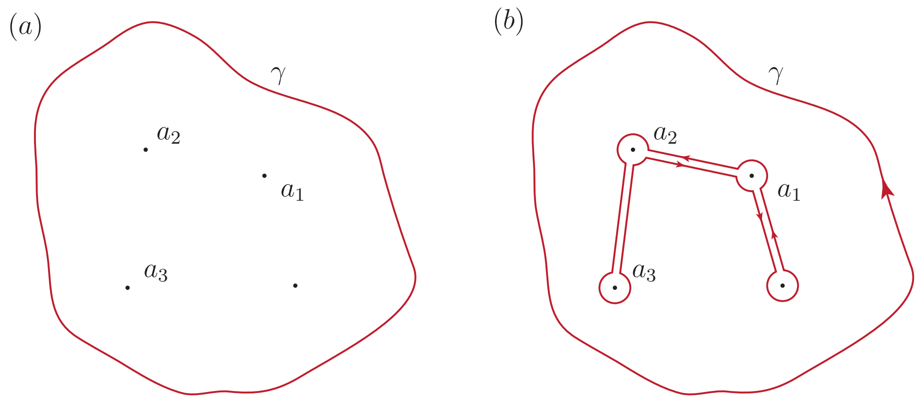

FIG. 1: Towards the residue theorem

Now let me remind you what is the positive orientation of the contour with respect to some domain. We say that the contour circles around some domain in a positive direction if this domain stays to the left as we move along the contour.

For example, as we move along the circle in the counterclockwise direction, we may say that it is a positive orientation of the contour with respect to the inner disk. But if we move in the clockwise direction along the same contour we may say that the contour is positively oriented with respect to the exterior of the disk.

The proof of the residue theorem is relatively straightforward.

But before addressing it let us prove a relatively simple preliminary identity. Let us consider an integral along a circle centered at some point a passed in the counterclockwise direction (see Fig 2) Then, we have the following identity:

$$\oint_\gamma (z-a)^n\, dz = \begin{cases} 0, & n \neq -1, \\ 2\pi i, & n = -1. \end{cases} \tag{4}$$

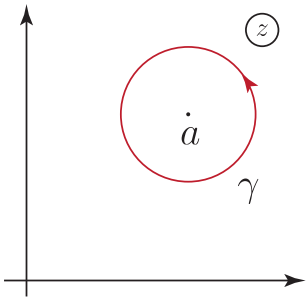

FIG. 2: Towards the residue theorem

To prove it we introduce a parametrization $z = a + re^{i\varphi}$ where r is the radius of a circle. Then we obtain:

$$\oint_\gamma (z-a)^n \, dz = r^{n+1} \int_0^{2\pi} e^{i(n+1)\varphi} i d\varphi \tag{5}$$

And, of course, the last integral vanishes due to periodicity of anti derivative with the only exception of the case $n = -1$ when it is equal to $2\pi i$.

Now, back to the proof of the residue theorem. As the first step, we can deform the contour to a combination of infinitesimal circles surrounding all singularities a_k inside the integration contour and straight lines. This deformation formes a dumbbell-like shape and doesn't change the value of the integral, since no singularities are crossed (see Fig. 1(b)).

Next, the combination of the integrals along the pairs of straight segments disappears (since they are infinitely close to each other and passed in opposite directions).

Therefore, in the end we are left with integrals along the infinitesimal circles, or rather the sum of such integrals.

$$\oint_\gamma f(z) \, dz = \sum_k \oint_{\gamma_k} f(z) \, dz, \tag{6}$$

where γ_k is an infinitesimal circle surrounding a_k. Let us compute one of these integrals, assuming that Laurent series in the vicinity of a_k has the following form:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - a_k)^n. \tag{7}$$

Using identity (4) we immediately see, that only c_{-1} term survives. Therefore, each integral along the respective infinitesimal circle is given by $2\pi i$ times the corresponding coefficient c_{-1} of the Laurent expansion, that is, the residue.

Thus, the residue theorem (3) is proven.

Finally, let us formulate a general approach, generalizing our exercise with computation of $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ above. In order to evaluate real integrals, the residue theorem is used in the following manner: the integrand is extended to the complex plane and its residues are computed (which is usually easy), and a part of the real axis is extended to a closed curve by attaching a half-circle in the upper or lower half-plane, forming a semicircle (sometimes we close the contour with rectangular shape, it depends on the integrand). The integral over this curve can then be computed using the residue theorem. Often, the half-circle part of the integral will tend towards zero as the radius of the half-circle grows, leaving only the real-axis part of the integral, the one we were originally interested in.

Before proceeding with practice in evaluation of integrals along the real line, let us start with a simple exercise in contour integration.

Example 1

Let us consider the function

$$f(z) = \frac{1}{z(z^2+1)}$$

and compute $\int_{C_i} f(z) dz$ along the contours, shown on the Fig. 3. Note that all singularities of $f(z)$ are simple poles and the residues can be read off the following expansion:

$$f(z) = \frac{1}{z} + \frac{-1/2}{z+i} + \frac{-1/2}{z-i}.$$

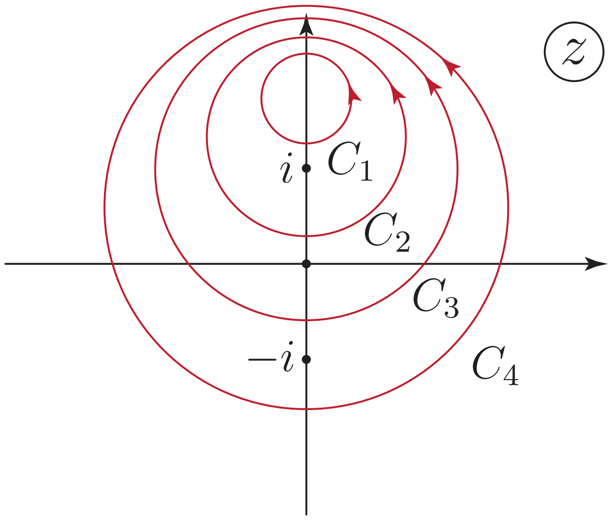

FIG. 3: Integration contours C_i.

- $\int_{C_1} f(z)dz = 0$, as there are no residues inside C_1.
- $\int_{C_2} f(z)dz = -\pi i$, as the only residue inside C_2 is $-1/2$.
- $\int_{C_3} f(z)dz = \pi i$, as there are two residues inside C_3: $1, -1/2$.
- $\int_{C_4} f(z)dz = 0$, as there are three residues inside C_3: $1, -1/2, -1/2$ and their sum vanishes.