A. Practice with regular branches VI

Now it is time to consider a more interesting function. You will encounter such type of functions when you start studying the theory of special functions, like solutions of Bessel or Hypergeometric equations.

let us consider the following function:

$$f(z) = (z^2+1)^{1-\gamma}(iz+1)^{2\gamma}, \quad f(+0) = 1 \tag{1}$$

The branch cut $z \in [-i, i]$.

Find the values $f(-\sqrt{3})$, $res\ f$ at infinity and $df/dz\Big|_{z=0}$. Here we see, that multivaluedness stems from the arbitrary power γ.

First thing you should do is to single out the branch points. To this end find all the roots of the expression inside the power. We expand $z^2+1 = (z+i)(z-i)$. Always turn your expressions into the form of $z\pm$ smth. iz is no good.

Therefore, we end up with the following:

$$f(z) = i^{2\gamma}(z-i)^{1+\gamma}(z+i)^{1-\gamma}, \quad f(+0) = 1 \tag{2}$$

In this form your multivalued function has a very transparent shape and easy to work with. It consists of two complex numbers $z-i$ and $z+i$ which are easy to work with. We just represent them with arrows in the complex plane stemming from branch points $z = \pm i$. Now I leave it up to you to prove that this finite branch cut does make this function single valued. convince yourself both ways, by finding an asymptotics and by making a full rotation round the branch cut.

First thing I'd like to cover is the value of the function at point $-\sqrt{3}$.

We connect the reference point, (+0 to the right of the branch cut) and the destination point with some contour.

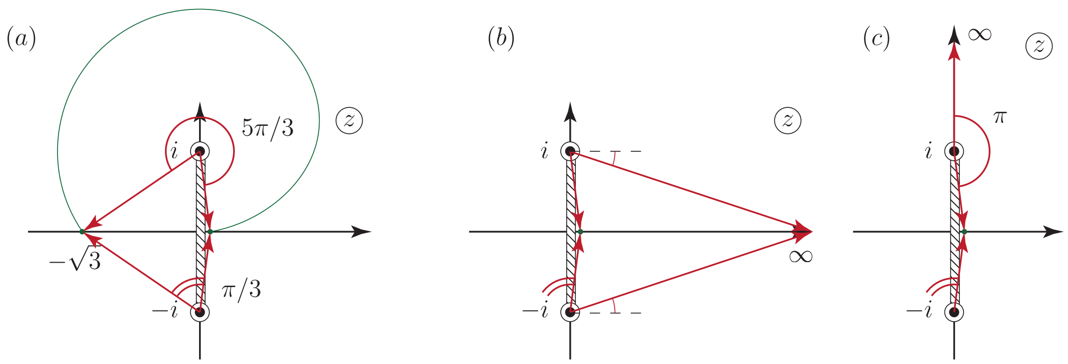

FIG. 1: Towards the practice with $(z^2+1)^{1-\gamma}(iz+1)^{2\gamma}$.

Let us choose the upward contour. Then $\Delta\arg(z-i) = \pi + 2\pi/3 = 5\pi/3$. And the $z+i$ number rotates counterclockwise by $\pi/3$. (see fig. 1(a)) As you see, this multivalued function is not like our original one. It kind of consists of two pieces $i^{2\gamma}(z+i)^{1-\gamma}$ and $(z-i)^{1+\gamma}$. So you have somthething like: $g_1^{1-\gamma}(z)g_2^{1+\gamma}(z)$. So we will treat them like the product of two multivalued functions but never actually separating it.

Now let us proceed to find right the change of the argument of our f function:

$$\Delta\arg f(z) = (1+\gamma)\frac{5\pi}{3} + (1-\gamma)\frac{\pi}{3} = 2\pi + \frac{4\pi}{3}\gamma. \tag{3}$$

Now:

$$f(-\sqrt{3}) = \frac{|g_1(-\sqrt{3})|^{1-\gamma}|g_2(-\sqrt{3})|^{1+\gamma}}{|g_1(0)|^{1-\gamma}|g_2(0)|^{1+\gamma}} f(+0)e^{2\pi i + \frac{4\pi}{3}\gamma i} \tag{4}$$

$$= \frac{|-\sqrt{3}-i|^{1+\gamma}}{|-i|^{1+\gamma}} \frac{|-\sqrt{3}+i|^{1+\gamma}}{|i|^{1+\gamma}} e^{\frac{4\pi}{3}\gamma i} = 4e^{\frac{4\pi i\gamma}{3}}. \tag{5}$$

Now, the derivative. The philosophy of taking the derivative is to express the derivative via the original function itself:

$$\frac{d}{dz}f(z) = i^{2\gamma}(1+\gamma)(z-i)^{\gamma}(z+i)^{1-\gamma} + i^{2\gamma}(1-\gamma)(z-i)^{1+\gamma}(z+i)^{-\gamma} = i^{2\gamma} = (1+\gamma)\frac{f(z)}{z-i} + (1-\gamma)\frac{f(z)}{z+i} \tag{6}$$

And this is how we compute its value at $z = 0$:

$$df(z)/dz = f(0)\left(\frac{1+\gamma}{-i} + \frac{1-\gamma}{i}\right) = 2i\gamma f(+0) \tag{7}$$

And finally, the residue at infinity.

Let me remind you that it is a coefficient of the Laurent expansion $-c_{-1}$. Now, the question is at which direction to approach the infinity. Well, you need to go in such a direction, that rotating angles of the arguments of the constituents of your functions would not depend on the distance.

Suppose you go in horizontal direction. Say, to the right. Then you immediately see, that $z + i$ number rotates in the clockwise direction by the angle $\pi/2$ minus a small angle, which by pythagorian theorem is equal to $\arctan 1/x$ (see fig. 1(b)). And when you start expanding in $1/x$ you will have to expand the arctan which is doable but not very suitable.

If, instead, in this problem you will go into vertical direction (say upward) you will immediately notice that rotation angle of the same complex number is zero, while the rotation of $z - i$ is exactly π in the counterclockwise direction. (see fig. 1(c))

So we opt for a vertical direction. $z = iy$.

$$\Delta \arg f = (1+\gamma)\pi. \tag{8}$$

And

$$f(iy) = |iy - i|^{1+\gamma}|iy + i|^{1-\gamma}e^{\pi i(1+\gamma)} = -(y-1)^{\gamma+1}(y+1)^{1-\gamma}e^{i\pi\gamma} = -y^2\left(1-\frac{1}{y}\right)^{\gamma+1}\left(1+\frac{1}{y}\right)^{-\gamma+1}e^{i\pi\gamma} \tag{9}$$

Since we arelooking for $1/y$ term we need to make an up to $1/y^3$ expansion in each term.

I'm lazy to do so with bare hands, but you should do it. I'll resort to Wolfram mathematica. So it is $4/3\gamma(1-\gamma^2)$ Therefore, the $1/y$ term of our function reads:

$$f(iy) = ... - e^{i\pi\gamma}\frac{4}{3}\gamma(1-\gamma^2)\frac{1}{y} + ... \tag{10}$$

And now we need to return to z notation: $y = z/i$. So we obtain:

$$f(z) = ... - ie^{i\pi\gamma}\frac{4}{3}\gamma(1-\gamma^2)\frac{1}{z} + ... \tag{11}$$

And the residue is:

$$\operatorname*{res}_{z=\infty} f(z) = ie^{i\pi\gamma}\frac{4}{3}\gamma(1-\gamma^2) \tag{12}$$