Consider the integral

$$I = \int_0^\infty \frac{\cos x}{x^2 + a^2} dx \tag{1}$$

There are two issues here. First, the contour doesn't go from $-\infty$ to ∞. Second, there is cos instead of exponential in the integrand meaning we have a combination of exponentials instead of a single exponential. As a result we won't be able to employ Jordan's lemma. As we will see, we will resolve these two issues simultaneously.

First we need to extend the contour to minus infinity. To do this let us split the cos term into a sum of two exponentials and change the variable form $-x$ to x in the second integral. The last step is done to make exponentials to have the increment of the same sign. We obtain:

$$I = \frac{1}{2}\int\limits_0^\infty \frac{e^{ix}}{x^2 + a^2} dx + \frac{1}{2}\int\limits_0^{-\infty} \frac{e^{ix}}{x^2 + a^2}(-dx) \tag{2}$$

Interchanging the limits of integration in the last integral we absorb the minus sign and obtain:

$$I = \frac{1}{2}\int\limits_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + a^2} dx. \tag{3}$$

Therefore, we achieved our goal. The integral now spans the entire real axis and contains a single exponential with positive increment $\lambda = 1$. This way, keeping in mind future application of Jordan's lemma, we close the contour with an upper semi circle (see Fig. 1). We promote our integrand in to a complex plane $e^{iz} f(z) = e^{iz}/[2(z^2 + a^2)]$ and

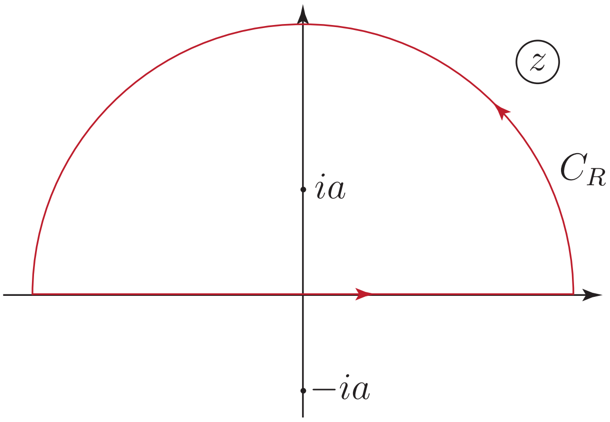

FIG. 1: Towards the application of Jordan's lemma.

consider a closed contour integral:

$$\oint e^{iz} f(z)\, dz = \int\limits_{-\infty}^{\infty} + \int_{C_R} . \tag{4}$$

The integral over upper semicircle C_R vanishes due to Jordan's lemma, since $f(z)$ decays uniformly with respect to $\arg z$ as $z \to \infty$.

Therefore, the closed contour integral is simply equal to the original integral I. We compute the former using residue theorem. Only one first order pole is positioned inside the integration contour $z = ia$. Hence:

$$\oint e^{iz} f(z)\, dz \equiv I = 2\pi i \operatorname*{res}_{z=ia} \frac{e^{iz}}{2(z^2 + a^2)} = 2\pi i \frac{e^{-a}}{4ia} = \frac{\pi}{2a} e^{-a} \tag{5}$$

Here, we assume, of course, that $a > 0$.