I. 1.7 APPLICATIONS OF CAUCHY-RIEMANN CONDITIONS

No let us consider some neat examples of application of Cauchy - Riemann conditions to get used to them and realize their importance.

Let us see if they do work on example of simplest functions.

Example 1 Let us check the function $f(z) = z^2$. We have:

$$f = u + iv = x^2 - y^2 + 2ixy \quad \Rightarrow u = x^2 - y^2, \quad v = 2xy; \tag{1}$$

As a result we obtain:

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = 2x$$
$$\frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial x} = 2y$$

And we see, that Cauchy-Riemann conditions are satisfied.

In fact, Cauchy-Riemann conditions are so powerful that they allow to restore the full function from its real or imaginary parts (up to some additive constant).

Example 2

Given the function:

$$u(x, y) = x^3 + 6x^2y - 3xy^2 - 2y^3$$

The additional condition $f(0) = 0$. Find the function $f = u + iv$.

We start from the first pair of Cauchy-Riemannn conditions:

$$\frac{\partial u}{\partial x} = 3x^2 + 12xy - 3y^2 = \frac{\partial v}{\partial y}$$

Integrating we restore v up to an arbitrary function of x:

$$v = \int \frac{\partial u}{\partial x} dy + \psi(x) = \int (3x^2 + 12xy - 3y^2) dy + \psi(x) = 3x^2y + 6xy^2 - y^3 + \psi(x)$$

The second pair of Cauchy-Riemannn conditions yields:

$$-\frac{\partial v}{\partial x} = -6xy - 6y^2 - \psi'(x) = \frac{\partial u}{dy} = 6x^2 - 6xy - 6y^2 \;\Rightarrow\; \psi'(x) = -6x^2 \Rightarrow\; \psi(x) = -2x^3 + \text{const}.$$

Taking into account initial condition $f(0) = 0$ we conclude that const $= 0$ and obtain:

$$v = 3x^2y + 6xy^2 - y^3 - 2x^3 \tag{2}$$

Now we may restore f as the function of z using relations $x = (z + z^*)/2$ and $y = (z - z^*)/(2i)$:

$$v = \left(-1 - \frac{i}{2}\right) z^3 - \left(1 - \frac{i}{2}\right) (z^*)^3$$
$$u = \left(\frac{1}{2} - i\right) z^3 + \left(\frac{1}{2} + i\right) (z^*)^3$$

As a result:

$$f = u + iv = (1 - 2i)z^3. \tag{3}$$

Example 3

Given the modulus of the analytic function:

$$|f| = e^{r^2 \cos 2\varphi} \tag{4}$$

find the full function f.

First, we notice that the function is given in polar coordinates.

Second, we recall that Cauchy-Riemann conditions are not written for the modulus of the function, so we need to resolve this obstacle. The key consideration comes from the observation that if f is analytic then $w = \ln f$ is also analytic. Hence, if we decompose the function:

$$f = |f|e^{\mathrm{arg}} \Rightarrow w = \ln f = \ln |f| + i\mathrm{arg}\, f \tag{5}$$

Hence:

$$w = u + iv, \qquad u = \ln |f|, \quad v = \arg f. \tag{6}$$

And we may write down Cauchy-Riemann conditions for u and v. We switch to cartesian coordinates:

$$u = r^2 \cos 2\varphi = r^2(\cos^2 \varphi - \sin^2 \varphi) = x^2 - y^2. \tag{7}$$

And then we proceed along similar to Example 2 lines:

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \;\Rightarrow\; v = 2xy + \psi(x)$$

Next:

$$-\frac{\partial v}{\partial x} = -2y - \psi'(x) = \frac{\partial u}{dy} = -2y \;\Rightarrow\; \psi'(x) = 0 \;\Rightarrow\; \psi(x) = \mathrm{const}$$

Hence,

$$w = u + iv = x^2 - y^2 + 2xyi + i\mathrm{const} = z^2 + i\mathrm{const} \tag{8}$$

As a result our initial function:

$$f = e^w = e^{i\mathrm{const}} e^{z^2} \tag{9}$$

is restored up to some multiplicative constant.