A. Video 1.3

1. Trigonometric interpretation

Trigonometric interpretation is naturally connected to the geometric one. Its construction is achieved by switching the coordinate system to the polar one and represent z by $|z|$ and φ. The polar angle φ counted from the positive direction of the real axis is called argument of a complex number z.

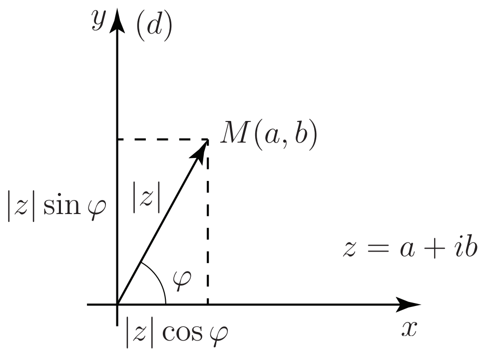

FIG. 1: Trigonometric form of a complex number

$$\varphi = \arg z. \tag{1}$$

A sort of ambiguity is present here, as this angle is defined up to addition of integer factors of 2π: this fact will be very important in what follows later. Now, to summarize our discussion of trigonometric representation, we wrap it up by the following equation:

$$z = a + ib = |z|\cos\varphi + i|z|\sin\varphi = |z|(\cos\varphi + i\sin\varphi). \tag{2}$$

Thus, a complex number is uniquely recovered from its modulus and argument.

Complex conjugation keeps the modulus intact and an argument can be considered as changing sign:

$$\begin{aligned} z^* = a - ib = |z|\cos\varphi - i|z|\sin\varphi = |z|(\cos\varphi - i\sin\varphi) \\ = |z|(\cos(-\varphi) + i\sin(-\varphi)). \end{aligned}$$

Let us inspect the effect of multiplication and division on the modulus and argument.

$$\begin{gathered} z_1 = |z_1|(\cos\varphi_1 + i\sin\varphi_1), \quad z_2 = |z_2|(\cos\varphi_2 + i\sin\varphi_2), \\ z_3 = z_1 z_2 = |z_1||z_2|(\cos\varphi_1 + i\sin\varphi_1)(\cos\varphi_2 + i\sin\varphi_2) = \\ |z_1||z_2|(\cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2 + i(\cos\varphi_1\sin\varphi_2 + \cos\varphi_2\sin\varphi_1)) \\ = |z_1||z_2|(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)) = |z|(\cos\varphi + i\sin\varphi). \end{gathered}$$

Quite nicely, the modulus multiply under the product and arguments – add up:

$$|z_1 z_2| = |z_1||z_2|, \quad \arg(z_1 z_2) = \arg z_1 + \arg z_2 \tag{3}$$

For division operation, we find:

$$\begin{gathered} z_1 = |z_1|(\cos\varphi_1 + i\sin\varphi_1), \quad z_2 = |z_2|(\cos\varphi_2 + i\sin\varphi_2), \\ z_3 = z_1/z_2 = \frac{z_1 z_2^*}{|z_2|^2} = \frac{|z_1||z_2|}{|z_2|^2}(\cos\varphi_1 + i\sin\varphi_1)(\cos\varphi_2 - i\sin\varphi_2) = \\ \frac{|z_1|}{|z_2|}(\cos\varphi_1\cos\varphi_2 + \sin\varphi_1\sin\varphi_2 + i(\sin\varphi_1\cos\varphi_2 - \sin\varphi_2\cos\varphi_1)) \\ = \frac{|z_1|}{|z_2|}(\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)) = |z|(\cos\varphi + i\sin\varphi). \end{gathered}$$

Thus, the modulus of the result of division is just a result of division of modulus and the arguments should be subtracted:

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \quad \arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 \tag{4}$$

This allows for a very simple interpretation of the power function. Indeed, raising the complex number z to the power n multiplies the argument by a factor of n times and the modulus (a real number) is simply raised to the same n-th power.

Consider a number of unit modulus:

$$z_0 = \cos\theta + i\sin\theta, \quad |z_0| = 1 \tag{5}$$

Then:

$$z_0^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \tag{6}$$

Let us consider an example. Compute argument and modulus of the complex number $z = -1 - i$ and write down the trigonometric representation. Reduce the argument to the interval $(-\pi, \pi]$.

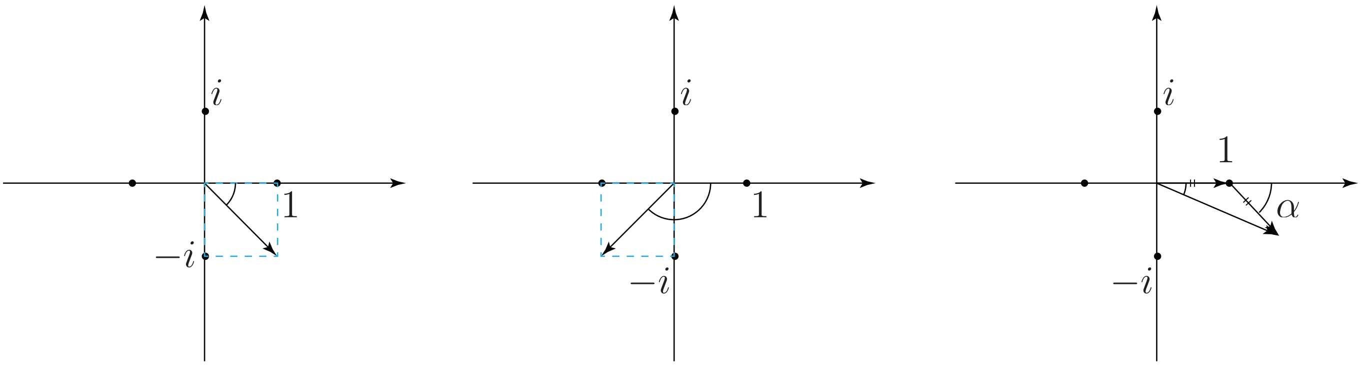

FIG. 2: Computation of arguments of complex numbers

We find $|z| = \sqrt{2}$

$$z = |z|(\cos\theta + i\sin\theta) = \sqrt{2}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) \tag{7}$$

and hence:

$$\cos\theta = -\frac{1}{\sqrt{2}}, \rightarrow \theta = \pm\frac{3\pi}{4}, \quad \sin\theta = -\frac{1}{\sqrt{2}} \Rightarrow \theta = -\frac{3\pi}{4} \tag{8}$$

So, $\arg z = -3\pi/4$, which can be also easily read off the Fig. 2(b)

$$z = \sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right) \quad \text{trignometric form} \tag{9}$$