3. Residue theory with applications to computation of complex integrals.

Problem 3.1

Evaluate the following integrals

1. $\int_{-\infty}^{\infty} \frac{x^4}{1+x^6} dx$.
2. $\int_0^{2\pi} \frac{\cos 2\theta}{2+\cos\theta} d\theta$.
3. $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)^2}$ for real $a,\ b$.

Solution.

1. Let us close the contour into upper half–plane. The poles are located at $i, \frac{\sqrt{3}}{2} + \frac{i}{2}, -\frac{\sqrt{3}}{2} + \frac{i}{2}$ and associated residues are $-\frac{i}{6}, \frac{1}{3(\sqrt{3}+i)}, -\frac{1}{3(\sqrt{3}-i)}$. Summing up all of them we find
$$\int_{-\infty}^{\infty} \frac{x^4}{1+x^6} dx = 2\pi i(-i/3) = 2\pi/3.$$
2. Substituting $e^{i\theta} = z$ we may write
$$\int_0^{2\pi} \frac{\cos 2\theta}{2+\cos\theta} d\theta = \text{Re} \oint_{\mathcal{C}} \frac{z^2}{2+\frac{1}{2}(z+1/z)} (-i)\frac{dz}{z} = 2\text{Im} \oint_{\mathcal{C}} \frac{z^2}{4z+z^2+1} dz$$
where $\mathcal{C}$ is a unit circle oriented counter–clockwise. The only singularity inside $\mathcal{C}$ is located at $z = -2+\sqrt{3}$ and the Residue theorem gives:
$$\int_0^{2\pi} \frac{\cos 2\theta}{2+\cos\theta} d\theta = 2\text{Im} \left(\frac{i\left(\sqrt{3}-2\right)^2 \pi}{\sqrt{3}} \right) = \left(\frac{14}{\sqrt{3}} - 8 \right) \pi.$$
3. Its enough to consider positive $a,\ b$ since the integral is an even function of $a,\ b$; the contour can be close to the upper half–plane where the singularities are $ia,\ ib$ with assoiated residues $-\frac{i}{2a(a^2-b^2)^2}, -\frac{i\left(a^2-3b^2\right)}{4b^3(a^2-b^2)^2}$. Summing up two contributions we find
$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)^2} = \frac{\pi(|a|+2|b|)}{2|a||b|^3(|a|+|b|)^2}.$$

Problem 3.2

Evaluate residue at $z = \infty$ of
$$f(z) = z^3 \cos\frac{1}{z-2}.$$

Solution. Expanding at $z \to \infty$, we find $f(z \to \infty) = z^3 - \frac{z}{2} - 2 - \frac{143}{24z} + O\left(\frac{1}{z^2}\right)$. The residue can be read off as minus coefficient in front of $1/z$ hence it equals $143/23$.

Problem 3.3

Evaluate residues of

$$f(z) = \frac{1}{z^3 - z^5}$$

at $z = -1,\ z = 0,\ z = 1$ and $z = \infty$. What is the sum of all residues?

Solution. The residues at $-1, 0, 1, \infty$ are, correspondingly, $-\frac{1}{2}, 1, -\frac{1}{2}, 0$. Their sum equals 0, as you should have expected.

Problem 3.4

Function

$$\frac{e^{iz}}{\cos z - 1}$$

can be expanded into Laurent series $\sum_{n=-\infty}^{\infty} c_n z^n$ in the region $2\pi k < |z| < 2\pi(k+1)$ for any integer non–negative k. Find coefficient $c_{-3}^{(k)}$ for such a series for $k = 0$ and $k = 1$.

Solution. The coefficient $c_{-3}^{(k)}$ can be computed as follows:

$$c_{-3}^{(k)} = \frac{1}{2\pi i} \int_{\mathcal{C}_k} z^2 f(z) dz$$

where $\mathcal{C}_k$ is a simple closed contour lying in the annulus $2\pi k < |z| < 2\pi(k+1)$ (oriented counter–clockwise). The integrals can be computed using residues of $f(z) = \frac{e^{iz}}{\cos z - 1}$:

$$c_{-3}^{(0)} = 2\pi i \sum_{z_0 \in \{0\}} \operatorname{Res}_{z=z_0} z^2 f(z) = 0$$

and

$$c_{-3}^{(1)} = 2\pi i \sum_{z_0 \in \{0, 2\pi, -2\pi\}} \operatorname{Res}_{z=z_0} z^2 f(z) = -16 i \pi^2.$$

Problem 3.5

Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\sin^2 x dx}{x^2(x^2+1)}.$$

Solution. The integral can be written as follows:

$$\int_{-\infty}^{\infty} \frac{\sin^2 x dx}{x^2(x^2+1)} = 2\mathrm{Re} \lim_{\delta \to +0} \left(\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty}\right) f(x) dx$$

with $f(x) = \frac{1 - e^{2ix}}{4x^2(x^2+1)}$. We can proceed as follows:

$$\lim_{\delta \to +0} \left(\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty}\right) f(x) dx = \oint_{\mathcal{C}_1} f(z) dz + \int_{\mathcal{C}_2} f(z) dz$$

where $\mathcal{C}_1$ is a closed contour running $-\infty \to -\delta \to \delta \to \infty \to -\infty$ (so that 0 is lying outside $\mathcal{C}_1$ and the contour is closed in the upper half–plane); $\mathcal{C}_2$ is a counter–clockwise arc $\delta \to -\delta$. Residue theorem gives:

$$\oint_{\mathcal{C}_1} f(z)dz = \frac{1}{4}\left(\frac{1}{e^2} - 1\right)\pi$$

and the second integral can be computed explicitely, using smallness of δ:

$$\int_{\mathcal{C}_2} f(z)dz = \int_{\mathcal{C}_2} (-\frac{i}{2z})dz = \frac{\pi}{2}.$$

As a result, we find:

$$\int_{-\infty}^{\infty} \frac{\sin^2 x dx}{x^2(x^2+1)} = \frac{1}{2}\left(1 + \frac{1}{e^2}\right)\pi.$$

Problem 3.6

Do the following limits exist?

1. $\lim_{R\to\infty} \int_{\mathcal{C}_R} e^{iz}dz$ for $\mathcal{C}_R$ – semicircle of radius $|z| = R$ in the upper half-plane.
2. $\lim_{R\to\infty} \int_{\mathcal{C}_R} e^{iz^2}dz$ for $\mathcal{C}_R$ – the arc $z = Re^{i\phi}$ with $0 \le \phi \le \pi/2$.

Solution.

1. This integral can be computed explicitely:

$$\int_{\mathcal{C}_R} e^{iz}dz = -2\sin R$$

   and the limit does not exist.
2. We can switch to integration over $z^2 = w$:

$$\int_{\mathcal{C}_R} e^{iz^2}dz = \frac{1}{2}\int_{\mathcal{C}'_R} e^{iw}dw/w^{1/2}$$

   where the contour $\mathcal{C}'_R$ is the arc $w = R^2e^{i\phi}$ with $0 \le \phi \le \pi$. The latter integral has a zero limit at $R \to \infty$ by Jordan lemma.

Problem 3.7

Evaluate the following integrals

1. $\int_0^\infty \frac{x-\sin x}{x^3}dx$.
2. $\int_{-\infty}^\infty \frac{e^{-iz}dz}{z^2+9}$.

Solution.

1. Partial integrations yield:

$$\int_0^\infty \frac{x-\sin x}{x^3}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x-\sin x}{x^3}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{1-\cos(x)}{2x^2}dx = \frac{1}{4}\int_{-\infty}^\infty \frac{\sin x}{x}dx.$$

The last integral can be computed in a way similar to the solution of the Problem 3.5 (with the use of Jordan lemma) yielding

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \mathrm{Im} \int_{\mathcal{C}} \frac{e^{iz}}{z} dz$$

where the contour $\mathcal{C}$ is an infinitesemal counter–clockwise arc from δ to $-\delta$, hence $\mathrm{Im} \int_{\mathcal{C}} \frac{e^{iz}}{z} dz = \mathrm{Im} \int_{\mathcal{C}} dz/z = \pi$ and finally

$$\int_0^{\infty} \frac{x - \sin x}{x^3} dx = \pi/4.$$

2. We can close the contour to yhe upper half–plane, with the single pole at $z = i$ contributing:

$$\int_{-\infty}^{\infty} \frac{e^{-iz} dz}{z^2 + 9} = 2\pi i(-\frac{i}{6} e^{-3}) = \frac{\pi}{3} e^{-3}$$

.

Problem 3.8

Evaluate at real k and a:

$$\int_0^{\infty} \frac{x \sin ax}{x^2 + k^2} dx.$$

Solution. Let us start as follows:

$$\int_0^{\infty} \frac{x \sin ax}{x^2 + k^2} dx = \frac{1}{2} \mathrm{Im} \int_{-\infty}^{\infty} \frac{x e^{iax}}{x^2 + k^2} dx.$$

The integral is an odd function of a and even function of k. We will assume that $a > 0,\ k > 0$ and recover the general case in the end. For $a > 0$, the integral can be closed in the upper half–plane. The arc integral vanishes (Jordan's lemma!). The only contributing singularity is a pole at $x = ik$. Hence

$$\int_{-\infty}^{\infty} \frac{x e^{iax}}{x^2 + k^2} dx = \pi i e^{-ak}.$$

As a result

$$\int_0^{\infty} \frac{x \sin ax}{x^2 + k^2} dx = \frac{\pi}{2} e^{-|a||k|} \mathrm{sign}\, k.$$

Problem 3.9

Evaluate the integral:

$$\int_{-\infty}^{\infty} \frac{\cos\left(x - \frac{1}{x}\right)}{1 + x^2} dx.$$

Solution. We start as follows

$$\int_{-\infty}^{\infty} \frac{\cos\left(x - \frac{1}{x}\right)}{1 + x^2} dx = \mathrm{Re} \int_{-\infty}^{\infty} \frac{\exp\left(i(x - \frac{1}{x})\right)}{1 + x^2} dx.$$

The contour can be closed to the upper half–plane: the integral over the arc vanishes. The only contrinuting singularity is a simple pole at $x = i$ and the result reads

$$\int_{-\infty}^{\infty} \frac{\cos\left(x - \frac{1}{x}\right)}{1 + x^2} dx = \mathrm{Re}\left(2\pi i(-\frac{i}{2} e^{-2})\right) = \pi e^{-2}.$$

Problem 3.10

Evaluate the principal value of the following integral

$$\mathrm{PV}\int_0^\infty \frac{xdx}{(x^2+a^2)\sin bx} \text{ for } a>0, b>0.$$

Solution. First, we write

$$\mathrm{PV}\int_0^\infty \frac{xdx}{(x^2+a^2)\sin bx} = \frac{1}{2}\mathrm{PV}\int_{-\infty}^\infty \frac{xdx}{(x^2+a^2)\sin bx}.$$

Let us now complete the contour to the closed one by adding: i) arcs, encircling the poles at $\pm\pi, \pm 2\pi$ etc clockwise and ii) a large counter–clockwise arc in the upper half–plane. We find:

$$\mathrm{PV}\int_{-\Lambda}^\Lambda \frac{xdx}{(x^2+a^2)\sin bx} + I_\epsilon + I_\Lambda = \frac{\pi}{\sinh{(ab)}}$$

with sufficiently large Λ which we may take to equal $\pi(n+1/2)$ for integer n (having in mind $n\to\infty$ in the end). Here I_ϵ equals the sum of integrals over $2*n$ small semi-circles, encircling the poles. The sum of the such integrals around πn and around $-\pi n$ equals zero, hence $I_\epsilon = 0$. Also, I_Λ is the integral along a large semi–circle in the upper half–plane which also vanishes by Jordan's Lemma. Indeed, note that $\frac{1}{\sin(bx)} = e^{ibx}\frac{2i}{e^{2ibx}-1}$ and $|\frac{2i}{e^{2ibx}-1}|$ is limited on C_Λ. Finally, we find

$$\mathrm{PV}\int_0^\infty \frac{xdx}{(x^2+a^2)\sin bx} = \frac{\pi}{2\sinh{(ab)}}.$$