I. THE DEFINITION OF THE CAUCHY'S PRINCIPAL VALUE OF THE INTEGRAL

Before considering a slightly more interesting example of a principal value integration, let's talk a little bit about this very concept. And it's probably better to start with the simplest possible example: the integration from -1 to 1 of the function $1/x$.

$$I = \int_{-1}^{1} \frac{dx}{x} \tag{1}$$

Taken as it is, this expression is meaningless, because the integration contour passes right through the first order pole of the integrand. But on the other hand, the integrand is an odd function of x and the integration domain is symmetric, so there is a temptation to prescribe a zero value to this expression:

$$I = 0(?) \tag{2}$$

That kind of examples provoked the introduction of the so-called Cauchy's principal value of the integral.

And it is introduced as follows: we simply split the contour and the singularity. Then we insert an infinitesimal separation centered at this singularity so the principal value of this integral is deciphered as the sum of the integrals

$$⨍ \frac{dx}{x} \equiv \lim_{\varepsilon \to 0} \left(\int_{-1}^{-\varepsilon} \frac{dx}{x} + \int_{\varepsilon}^{1} \frac{dx}{x} \right) \tag{3}$$

where epsilon is set to 0 and the end of the calculation.

And indeed one obtains logarithm of epsilon for the first integral and logarithm of one over epsilon for the second integral:

$$⨍ \frac{dx}{x} \equiv \lim_{\varepsilon \to 0} \left(\ln \varepsilon + \ln \frac{1}{\varepsilon} \right) = 0. \tag{4}$$

Summing them up we are already obtained zero and of course setting epsilon to zero we obtain the zero answer. And now let's study a less trivial example.

II. EXAMPLE

Compute the principal value of the integral integral:

$$⨍_{-\infty}^{\infty} \frac{e^{ibx} dx}{x^2 - 1} \quad b < 0; \tag{5}$$

First of all, we see that our integration contour passes through two singularities of the integrand which are simple poles at points $x = \pm 1$.

As a first step let us draw the contour (Fig. **??**(a)).

It is split into three pieces and let us decipher the principle integration sign as the sum of the integrals

$$⨍ = \int_{-\infty}^{-1-\varepsilon} + \int_{-1-\varepsilon}^{1-\varepsilon} + \int_{1+\varepsilon}^{\infty} . \tag{6}$$

Naturally, to employ Residue theorem, we need to close the contour somehow and minding the future application of Jordan's lemma we connect the infinite edges of this contour by a lower infinite semicircle C_R. Next, we connect the adjacent pieces of the contour by two infinitesimal lower semi-circles set at point -1 and 1. The next step is to promote our function into a complex plane, let's denote it

$$f(z) = \frac{e^{ibz}}{z^2 - 1} \tag{7}$$

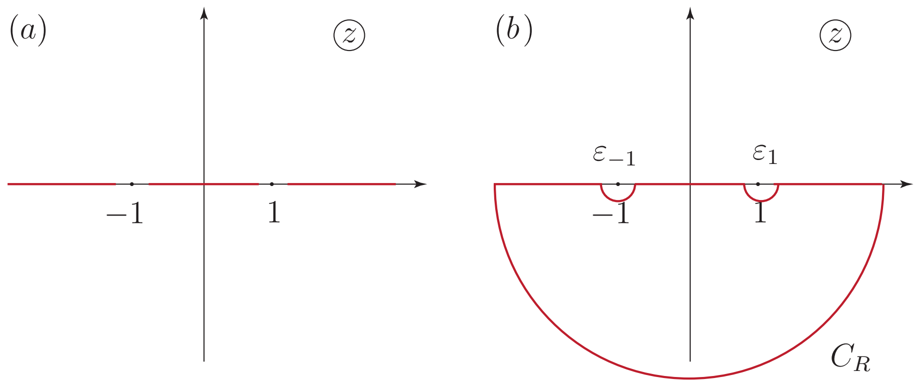

FIG. 1: Towards the calculation of the integral.

and consider a closed contour integral of $f(z)$: it is naturally split into our original principal value integral plus two infinitesimal integrals around semi-circles, centered at point negative one and one and plus the integral along the lower infinite arc:

$$\oint = \fint + \int_{\varepsilon_{-1}} + \int_{\varepsilon_1} + \int_{C_R} . \tag{8}$$

Now the integral along the lower semi-circle vanishes due to Jordan's lemma, because our increment b is negative and our preexponential function which is $1/(z^2-1)$ decays uniformly with respect to the argument of z as z tends to infinity. Next, the closed contour integral itself: the function f is obviously regular inside this contour so the closed counter integral is also equal to zero:

$$\oint = 0. \tag{9}$$

And thus we obtain our principal value integral as the sum with minus sign of two infinitesimal integrals around our semicircles at points 1 and -1.

$$\fint = -\int_{\varepsilon_{-1}} - \int_{\varepsilon_1} \tag{10}$$

And we compute these infinitesimal integrals with the same technique outlined in the previous video: we simply Laurent expand our integrand in the vicinity of -1 and 1.

Let's do the calculation for the left semicircle.

We introduce the change $z = -1 + \varepsilon$ and build a Laurent expansion of our function

$$f(-1+\varepsilon) = \frac{e^{-ib}}{-2\varepsilon} + ... \tag{11}$$

Now the integral is equal to:

$$\int_{\varepsilon_{-1}} = -\frac{1}{2} e^{-ib} \int_{\varepsilon_{-1}} \frac{d\varepsilon}{\varepsilon}; \tag{12}$$

The latter integral is computed by a standard parameterization:

$$\varepsilon = |\varepsilon| e^{i\varphi}, \quad \varphi \in [\pi, 2\pi]. \tag{13}$$

The evaluation is straightforward and we obtain $i\pi$, so the integral along the left semicircle is equal to

$$\int_{\varepsilon_{-1}} = -\frac{i\pi}{2} e^{-ib}. \tag{14}$$

Now the next integral. Again we perform Laurent expansion of our integrand in the vicinity of 1: $z = 1 + \varepsilon$ and

$$f(1+\varepsilon) = \frac{e^{ib}}{2\varepsilon} + ... \tag{15}$$

and we perform the integration:

$$\int_{\varepsilon_1} = \frac{1}{2} e^{ib} \int_{\varepsilon_1} \frac{d\varepsilon}{\varepsilon}. \tag{16}$$

The integral is taken with the same parameterization and naturally yields the same $i\pi$. So the second integral is equal to

$$\int_{\varepsilon_1} = \frac{i\pi}{2} e^{ib}. \tag{17}$$

And as a result, our principal value integral is equal to:

$$⨍ = \frac{i\pi}{2}\left(e^{-ib} - e^{ib}\right). \tag{18}$$

Now this combination of exponentials is reduced to a sine function, and we obtain

$$⨍ = \pi \sin b. \tag{19}$$

And that's it, that completes our initial discussion of the integration methods of complex analysis.