A. Video 1.5

1. *Practice with exponential forms of complex numbers*

Evaluate the modulus and argument of:

$$a)\ \ z = 1 + i^{123}, \quad b)\ \ \frac{(1-i)^6}{(1+i\sqrt{3})^5}$$

a) First, observe $\arg i = \pi/2$, and $|i| = 1$. Hence $i = e^{i\pi/2}$ and:

$$i^{123} = (e^{i\pi/2})^{123} = e^{123\pi i/2} = e^{61\pi i+\pi i/2} = e^{61\pi i+\pi i/2} = e^{30(2\pi)i+3\pi i/2} = e^{30(2\pi)i}e^{3\pi i/2} = -i.$$

As a result:

$$z = 1 - i, \quad \arg z = -\frac{\pi}{4}, \quad |z| = \sqrt{2}, \quad z = \sqrt{2}e^{-i\pi/4}.$$

b) Let us start with

$$1 - i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \sqrt{2}e^{-i\pi/4}, \quad 1 + i\sqrt{3} = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) = 2e^{i\pi/3}$$

Hence

$$\frac{(1-i)^6}{(1+i\sqrt{3})^5} = \frac{\sqrt{2}^6 e^{-3\pi i/2}}{2^5 e^{5\pi i/3}} = \frac{e^{i\pi/2}e^{-5\pi i/3}}{4} = \frac{e^{i\pi/2+i\pi/3}}{4} = \frac{1}{4}e^{5\pi i/6} \equiv \frac{1}{4}\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$$

Evaluate:

$$\sin\theta + \sin 2\theta + ... + \sin n\theta$$

$$\sin\theta + \sin 2\theta + ... + \sin n\theta = \operatorname{Im}\left(e^{i\theta} + e^{2i\theta} + ... + e^{in\theta}\right) = \operatorname{Im} e^{i\theta}\left(1 + e^{i\theta} + ... + e^{i(n-1)\theta}\right)$$
$$= \operatorname{Im} e^{i\theta}\frac{e^{in\theta} - 1}{e^{i\theta} - 1}$$

How should we evaluate the imaginary part? The following trick is in order here:

$$e^{i\alpha} - e^{i\beta} = e^{i(\alpha+\beta)/2}\left(e^{i(\alpha-\beta)/2} - e^{-i(\alpha-\beta)/2}\right) = e^{i(\alpha+\beta)/2} 2i \sin\frac{\alpha-\beta}{2}.$$

Let us apply it to the numerator and denominator:

$$e^{i\theta} - 1 = e^{i\theta/2} 2i \sin\frac{\theta}{2}, \quad e^{in\theta} - 1 = e^{in\theta/2} 2i \sin\frac{n\theta}{2}.$$

$$\sin\theta + \sin 2\theta + ... + \sin n\theta = \operatorname{Im} e^{i(n+1)\theta/2}\frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}} = \frac{\sin\frac{n\theta}{2}\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}}$$

B. Solution of equations with complex numbers

Let study how the equations with complex numbers are solved.

$$z^3 = -i$$

To solve the equation it is necessary to switch to exponential form of the complex number $z = |z|e^{\mathrm{arg}z}$, $-i = e^{-i\pi/2+2\pi in}$. It is extremely important to add $2\pi n$ to the argument of the complex number on the r.h.s. of the equation.

$$|z|^3 e^{3\mathrm{arg}z} = e^{-i\pi/2+2\pi in}$$

The complex numbers are equal if their arguments and moduli are the same. Therefore, we have:

$$|z| = 1, \quad \arg z = -\frac{\pi}{6} + \frac{2\pi n}{3}.$$

It is interesting to note that all the roots have the same modulus (1) and therefore, lie on a unit circle. Now let us write down all the roots. Let us start with $n = 0$. We have $z_0 = e^{-i\pi/6}$. Then $n = 1$: $z_1 = e^{i\pi/2}$. Then $z_2 = e^{7\pi/6}$. Let us try $n = 3$. Then we will have the addition of $2\imath n/3 \cdot 3 = 2\pi$ to the initial argument $-\pi/6$.

Therefore, there are only 3 different roots, as there should, since we deal with 2rd order equation.

$$z_0 = e^{i\pi/6}, \quad z_1 = e^{i\pi/2}, \quad z_2 = e^{7\pi/6}.$$

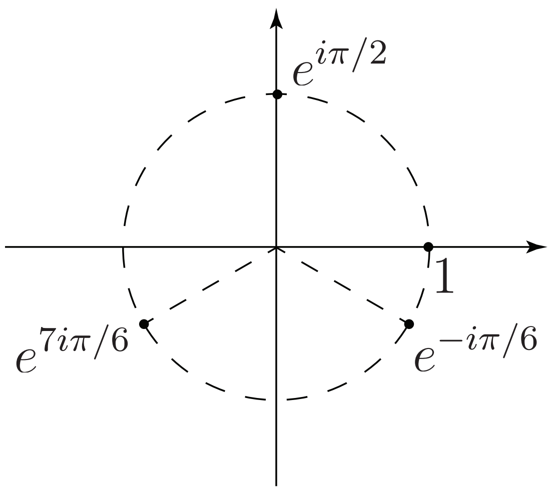

FIG. 1: The roots of the equation $z^3 = -i$.