A. Practice with regular branches I

As a first non-trivial example, let us consider function $f(z) = \sqrt{1 - z^2}$ with different configurations of branch cuts. It is a composite function of the type $\sqrt{g(z)}$ with $g(z) = 1 - z^2$. The branch points are the zeroes of the $g(z)$, i.e. points $z = \pm 1$.

Therefore, let us draw two branch from each of the branch point to infinity. Let the left and right branch cut go in the downward direction and the regular branch of function f be fixated by a natural condition $f(0) = 1$. The assignment is to find the value of the regular branch at points $z = 2$, $z = -2$ and find the residue of f at infinity.

The most important point of the method is a geometrical interpretation of the function under the root. We see that it is represented by a product of two simple complex numbers $g(z) = (1 - z)(1 + z)$. Therefore, for arbitrary z number $1 - z$ is represented by the arrow with the origin at z and endpoint at 1. While number $1 + z$ is an arrow with the origin at -1 and the end point at z. So we see, that wherever we position z these two arrows follows its direction rotating round our branch points, (see fig. 1(a)).

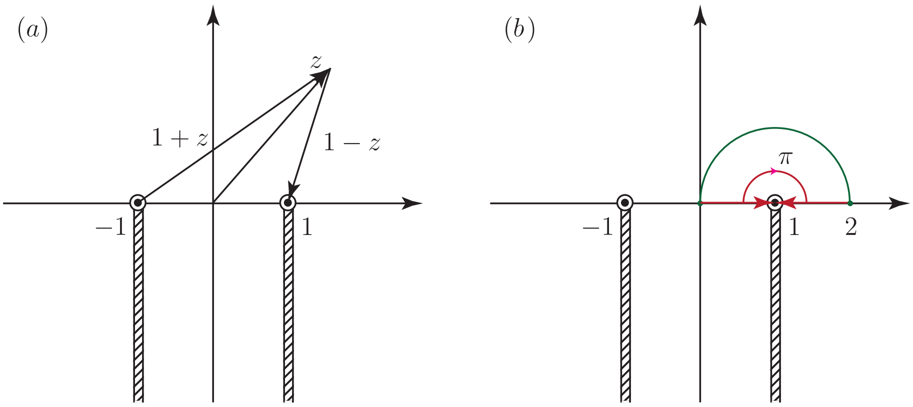

FIG. 1: Towards the practice with $\sqrt{1 - z^2}$.

Now we want to compute the value of f at point $z = 2$ According to the scheme we position a reference point $z_0 = 0$ in the complex plane and draw a contour connecting 0 and 2. Let it be an upper circle. And now we trace the changes of the arguments of the constituents of our g-function. Arrow $1 - z$ rotates by an angle π in the clockwise direction. Don't get confused by the fact that it rotates round its head rather than the origin. Convince yourself that if you put yourself in a system of reference connected to the origin, then the head rotates by pi in the same clockwise direction (see fig. 1(b)). Therefore, the change of the argument is of $1 - z$ is

$$\Delta \arg (1 - z) = -\pi.$$

At the same time arrow $1 + z$ just sways but doens't rotate at all once its end trevel from 0 to 2. Therefore the change of its argument is 0. Hence,

$$\Delta \arg (1 + z) = 0.$$

Since g-function is the product of these two numbers, the change of the argument of g is a sum of the changes of the arguments of its constituents, therefore:

$$\Delta \arg g = \Delta \arg (1 - z) + \Delta \arg (1 + z) = -\pi + 0 = -\pi.$$

And then we write down the formula for the value of our regular branch:

$$f(2) = \sqrt{\left| \frac{g(2)}{g(0)} \right|} e^{\frac{i}{2} \Delta \arg g} f(0) = \sqrt{3} e^{-i\pi/2} \cdot 1 = -i\sqrt{3}.$$

In the analogous fashion let us compute the value of the function at point -2. The contour connecting the point 0 and -2 is again an upper semi circle. As we travel from z_0 to z the arrow representing number $1 + z$ rotates by π in a counterclockwise direction, so its change of the argument is:

$$\Delta \arg (1 + z) = \pi.$$

while the arrow representing $1-z$ just sways and doesn't rotate.

$$\Delta\arg(1-z) = 0.$$

As a result, the change of the argument of g is now

$$\Delta\arg g = \Delta\arg(1-z) + \Delta\arg(1+z) = 0 + \pi = \pi.$$

Therefore, the change of the argument of g differs in sign from the previous one. As a result, we find for the regular branch:

$$f(-2) = \sqrt{\left|\frac{g(-2)}{g(0)}\right|} e^{\frac{i}{2}\Delta\arg g} f(0) = \sqrt{3}e^{i\pi/2} \cdot 1 = i\sqrt{3}.$$

In the next slide we find the residue at infinity.

B. Practice with regular branches II

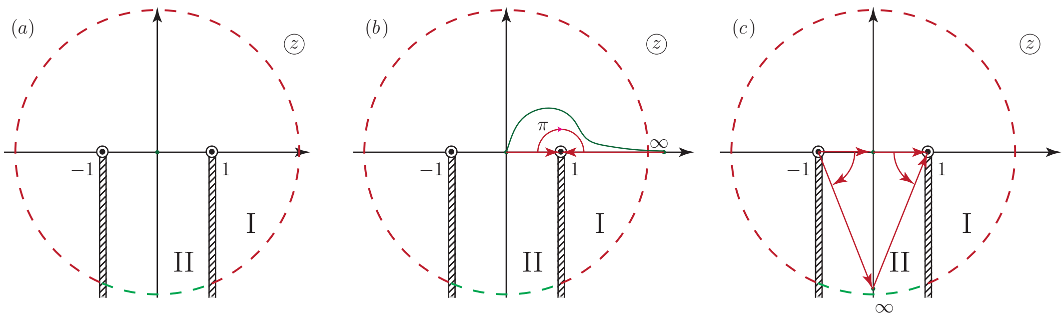

FIG. 2: Towards the practice with $\sqrt{1-z^2}$.

Now let us find the residue of our function at infinity. To this end we need to Laurent expand it at $z \to \infty$. But then, which direction to choose as we go to infinity?

We see, that infinitely distant circle is split by branch cuts going to infinity into several unconnected arcs. (see fig. 2(a)) That means, the function will have a different Laurent expansion depending on whether we go to infinity in the different sectors cut out by the branch cuts.

So the residue at infinity is ill defined and the question about its value is senseless.

Instead, let us now answer a different but also extremely important question. We will find the first two terms of Laurent expansion of our function in all the arcs.

Our first target would be the outer arc. To find the Laurent expansion there, we need to specify a particular direction. The answer is is independent of your choice as long as this direction belongs to the arc. So, choose the most suitable one. In our situation the direction along the positive axis seems reasonable. (see fig. 2(b)) We just put $z = x$ to return to z in the final formula.

This time our destination point is positioned far to the right in the complex plane. As before, we connect the reference point with the final point and trace the angles of the rotating arrows. The arrow representing number $1-z$ rotates by pi in the clockwise direction, while the arrow representing $1+z$ just sways and doesn't turn. As before, that means the change of the argument of g is $-\pi$.

$$f(x) = \sqrt{\left|\frac{g(x)}{g(0)}\right|} e^{\frac{i}{2}\Delta\arg g} f(0) = \sqrt{x^2-1}e^{-i\pi/2} \cdot 1$$

And now we may use the real analysis to make a Taylor expansion of the function under the square root. We have:

$$\sqrt{x^2-1} = x\sqrt{1-\frac{1}{x^2}} \approx x - \frac{1}{2x}.$$

Therefore, we completed our task. The Laurent expansion on the right real semiaxis looks as follows:

$$f(x) = -i\Big(z - \frac{1}{2z}\Big) = -iz + \frac{i}{2z}$$

Now changing x to z we obtain the expansion in the entire outer arc of a complex plane.

Now let us build the Laurent expansion in the arc between the branch cuts.

This time, however, we need to be more careful with the changes of the argument. Let us choose the destination point on an imaginary axis: $z = -iy$ (see. fig. 2(c)). As our destination point moves along the axis, the change of the argument of $1 - z$ arrow is $\pi/2$ in the counterclockwise direction but not quite:

$$\Delta\arg(1-z) = \frac{\pi}{2} - \delta\varphi$$

where $\delta\varphi = \arctan(1/y)$ On the other hand, the rotation of the arrow $1 + z$ is also almost $\pi/2$:

$$\Delta\arg(1+z) = -\pi/2 + \delta\varphi.$$

Therefore, the total change of the argument of g function is:

$$\Delta\arg g = 0$$

Then we obtain:

$$f(-iy) = \sqrt{\left|\frac{g(-iy)}{g(0)}\right|} f(0) = \sqrt{1+y^2} \approx y + \frac{1}{2y} = iz - \frac{i}{2z}.$$

And we see, that the expansion does look different in the different regions of the complex semi plane.

This is always the case when the infinitely distant circle is split with the branch cuts of a multivalued function.

However, if branch cuts doesn't stretch to infinity, the Laurent expansion will have a universal appearance at infinity.

Therefore, let us study the different configuration of branch cuts for the same function on the next slide.

C. Practice with regular branches III

This time we address the same function $f(z) = \sqrt{1-z^2}$ but with a finite branch cut. we see that there exists a possibility to connect both branch points with a single branch cut, a segment [-1,1].

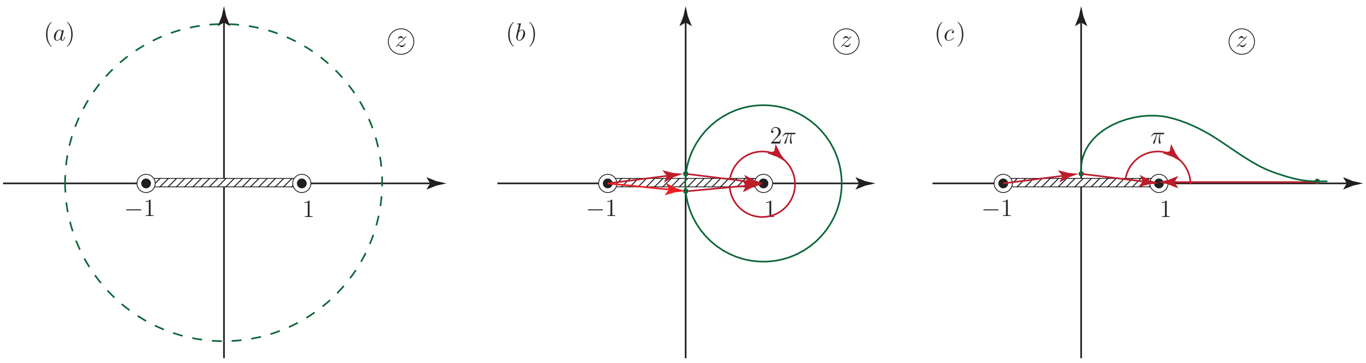

FIG. 3: Towards the practice with $\sqrt{1-z^2}$.

Now, the issue with finite branch cuts is that they are always dangerous. Despite the fact that they prevent rotations round each individual branch point, one can always draw a sufficiently large contour that circumvents the branch cut. And there always exists a question of whether the function returns to its original value after such a rotation (see fig. 3(a)).

Let us see, how this can be checked. Let us draw a contour circumventing a branch cut and trace the change of the argument of function $g(z)$. Both arrows, $1 - z$ and $1 + z$ makes full counterclockwise turns. As a result, the change of the arguments

$$(1-z) = 2\pi, \quad (1+z) = 2\pi.$$

And the change of the argument of g is $2\pi + 2\pi = 4\pi$. Therefore, the change of the argument of f function is going to be 2π and f returns to its initial value after such a rotation. As a result the rotation round 2 branch points simultaneously doesn't change the value of the function. Therefore, such a branch cut makes our function single valued. Finite branch cuts are sometimes quite crucial. We will see in our next week, that they help to compute real integrals along the finite segments of the real axis.

Is it possible that a finite branch cut connecting all the branch points of the function doesn't make it single valued?

Absolutely. Consider a function $f_1(z) = \sqrt[3]{(1-z^2)}$ with the same branch cut. Then after full rotation round the same contour the argument of g will again change by 4π but the change of the argument of the function f_1 is going to be $4\pi/3$. And after circling such a path the value of the function will change $f_1 \to f_1 e^{4\pi i/3}$.

There is another way to check if a finite branch cut makes a function. The main idea is to deform the contour circumventing the branch cut to a very large one. Then, to check the behavior of the function as we move along the contour it is enough to find its asymptotics.

For example function $f(z)$ at large z behaves as

$$f(z) \to \sqrt{-z^2} = \sqrt{-1}z$$

The $\sqrt{-1}$ should not concern us here, being some number fixated a a choice of a regular branch. The z dependance is what of importance. And we see, that on a stretched contour $f(z)$ behaves as a single valued function z. That is the main reason why it return to its original value as we circle round the contour. On the other hand the asymptotics of a function $f_1(z)$ is:

$$f_1(z) \to \sqrt[3]{-z^2} = \sqrt[3]{-1}z^{2/3}$$

Though it becomes a very simple but still multivalued function at large z. That is why it changes its values as we make a full turn.

So you now have two ways to check if a branch cut renders a function single valued. Use them at your convenience.

Now let us fixate the regular branch of our function by the condition that it coincides with an arithmetic root when we on the upper bank of a branch cut.

$$f(x + i0) > 0, \quad x \in [-1, 1].$$

The assignment is to find the value of the function at each point on the lower bank of a branch cut and to find the residue at infinity.

Let us first find the value of function at point $x - i0$. We choose a symmetric point on the upper bank $x + i0$ where we know the value of our regular branch. Let us draw now some contour connecting two points. Say a right circle. Then, the change of the argument of $1 - z$ is 2π in the clockwise direction (see fig. 3(b)):

$$\Delta\arg(1 - z) = -2\pi.$$

Number $1 + z$ is turned by an infinitesimal angle so

$$\Delta\arg(1 + z) = 0.$$

Therefore, the change of the argument of g is

$$\Delta\arg g = -2\pi.$$

As a result,

$$f(x - i0) = \sqrt{\left|\frac{g(x + i0)}{g(x - i0)}\right|} e^{-i\pi} f(x + i0) = -f(x + i0).$$

Now, the residue at infinity. Accordingly to what we said before, now, since the branch cut doesn't stretch to infinity, in any direction of the complex plane, the Laurent expansion of our function is going to look identical. So, for simplicity, let us choose the horizontal direction along the real axis to the right $z = x$.

Now we choose the reference point somewhere on the upper bank of a branch cut $x_0 + i0$ and draw a contour connecting our reference and destination points. Let us find the rotation angles of our arrows. Arrow $1 - z$ rotates clockwise by π:

$$\Delta\arg(1 - z) = -\pi.$$

Arrow $1 + z$ just sways but in the end return to its position, $\Delta\arg(1 + z) = 0$. Therefore,

$$\Delta\arg g = -\pi.$$

$$f(x) = \sqrt{\left|\frac{g(x)}{g(x_0 + i0)}\right|} e^{-i\pi/2} \sqrt{g(x_0 + i0)}$$

Now, since $g(x_0 + i0) > 0$ we may remove the modulus sign in the denominator and cancel the corresponding expressions in the denominator and nominator. As a result we obtain:

$$f(x) = -i\sqrt{|g(x)|} = -i\sqrt{|1 - x^2|} = -i\sqrt{x^2 - 1}.$$

Now we use real analysis to expand the square root to obtain:

$$f(x) = -ix\sqrt{1 - \frac{1}{x^2}} = -ix + \frac{i}{2x} + ...$$

Finally, we obtain the Laurent expansion in the entire complex plane by changing $x \to z$:

$$f(x) = -iz + \frac{i}{2z} + ...$$

The residue at infinity:

$$\operatorname*{res}_{z=\infty} f(z) = -c_{-1} = -\frac{i}{2}.$$

This way, we learnt how the regular branch separation works in the complex plane for different geometry of branch cuts and for different tasks, from different determining the values of functions at particular points to Laurent expansions and residues.

More interesting applications and logarithms await, so stay with us, you won't be disappointed.