A. Video 1.4

B. Exponential representation of a complex number, Euler's identity

Let us now turn to the exponential form of complex numbers, which somewhat naturally follows from the trigonometric one but is also more coincise and more powerful. It is based on a famous Euler's identity. First, consider the complex exponential:

$$f(\theta) = e^{i\theta}. \tag{1}$$

How should this equation be interpreted? In the real domain, the Taylor series of exponential function converges for an arbitrary x. It turns out that the Taylor series can be equally well applied for the complex arguments of the exponential function. Indeed, recall the definition of the radius of convergence concering the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{2}$$

As soon as

$$\lim_{n\to\infty} \left| \frac{a_{n+1} z^{n+1}}{a_n z^n} \right| = q < 1 \tag{3}$$

the series is majorated by a geometric series $\sum\limits_{n=0}^{\infty} q^n$ and converges. For the case of complex exponential:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{4}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1} z^{n+1}}{a_n z^n} \right| = \lim_{n\to\infty} \left| \frac{z}{n+1} \right| = 0 \tag{5}$$

for arbitrary complex z. This implies that the Taylor series for e^z converges for all complex z. Now, let us use this definition for our function $f(\theta)$:

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{i^n \theta^n}{n!} = 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + ... \tag{6}$$

Observe that the powers of i repeat itself $i^0 = i^4 = i^{4k} = 1$. Lets inspect the first terms of the series:

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{i^n \theta^n}{n!} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + ... \tag{7}$$

Splitting the real and imaginary parts:

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + ...\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + ...\right) \tag{8}$$

we see that the real part reproduces the Taylor series $\cos\theta$, while imaginary part combines to $\sin\theta$. We thus arrive to the famous Euler's idenity:

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{9}$$

This gives the new, exponential form of writing complex numbers:

$$z = |z|(\cos\theta + i\sin\theta) = |z|e^{i\theta} \tag{10}$$

This representation makes most of the properties we discussed above almost trivial. Indeed:

$$z = z_1 z_2 = |z_1|e^{i\theta_1}|z_2|e^{i\theta_2} = |z_1||z_2|e^{i(\theta_1+\theta_2)},$$
$$z = z_1/z_2 = \frac{|z_1|e^{i\theta_1}}{|z_2|e^{i\theta_2}} = \frac{|z_1|}{|z_2|}e^{i(\theta_1-\theta_2)},$$

Consider Euler's identity for θ and for $-\theta$ we write:

$$e^{i\theta} = \cos\theta + i\sin\theta,$$
$$e^{-i\theta} = \cos\theta - i\sin\theta,$$

which yields the following handy representations of trigonometric functions:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2},$$
$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$