1. Algebra of complex numbers. Integration and differentiation of functions of complex variables.

Problem 1.1

Provide a geometric description of the following sets in the complex plane and derive it geometrically and algebraically. Reduce the equations for the boundaries to the canonical form.

1. $2 < |z - i| < 4$.
2. $|z - 4i| + |z + 4i| = 10$.
3. $\operatorname{Im} \frac{1}{z} = 1$.

Solution.

1. An annulus centered at $z = i$ with radii 2 and 4.

   Geometrically: distance from z to i is between 2 and 4.

   Algebraically:

$$2 < \sqrt{x^2 + (y-1)^2} < 4 \iff 2^2 < x^2 + (y-1)^2 < 4^2.$$

2. An ellipse with foci $\pm 4i$ and semi–major axis 5.

   Geometrically: sum of the distances to $\pm 4i$ is constant and equals 10.

   Algebraically:

$$\sqrt{x^2 + (y-4)^2} + \sqrt{x^2 + (y+4)^2} = 10$$

   and:

$$2\sqrt{x^2 + (y-4)^2}\sqrt{x^2 + (y+4)^2} + 2x^2 + 2y^2 = 68 \iff (x/3)^2 + (y/5)^2 = 1.$$

3. A circle of radius $1/2$ centered at $-i/2$.

   Geometrically: inversion $z = 1/w$ of a line $\operatorname{Im} w = 1$ is a circle.

   Algebraically:

$$\operatorname{Im} \frac{1}{z} = \operatorname{Im} \frac{1}{x + iy} = -\frac{y}{x^2 + y^2} = 1 \iff x^2 + (y + 1/2)^2 = (1/2)^2.$$

Problem 1.2

Let ε be arbitrary n-th rooth of unity (not equal to 1). Prove the following equality

$$1 + 2\varepsilon + 3\varepsilon^2 + ... + n\varepsilon^{n-1} = \frac{n}{\varepsilon - 1}.$$

Solution. We can sum the series as follows:

$$1 + 2\varepsilon + 3\varepsilon^2 + ... + n\varepsilon^{n-1} = \partial_\varepsilon \left(\varepsilon + \varepsilon^2 + \varepsilon^3 + ... + \varepsilon^n\right)$$

which using sum of a geometric series gives

$$\partial_\varepsilon \left(\frac{1 - \varepsilon^{n+1}}{1 - \varepsilon} - 1\right) = \frac{(n(\varepsilon - 1) - 1)\varepsilon^n + 1}{(\varepsilon - 1)^2}.$$

Replacing $\varepsilon^n \to 1$ we come to the desired result.

Problem 1.3

Determine the images

1. of a line $\operatorname{Im} z = 1$ under the map $z \to w(z) = z^3 + 3z - i$.
2. of a circle $|z - i| = 1$ under the map $z \to w(z) = \frac{1}{z-2i}$.

Solution.

1. We can write $z = x + i$ with real x to find
$$w = x^3 + i(1 + 3x^2),$$
hence the image is described by the equation
$$\operatorname{Im} w = 1 + 3|\operatorname{Re} w|^{2/3}.$$
2. We can write $z = i + e^{i\phi}$ with $0 \le \phi < 2\pi$ to find
$$w = \frac{\cos(\phi)}{2 - 2\sin(\phi)} + \frac{1}{2}i.$$
The real part of this expression covers the whole real line, hence the image is described by the equation
$$\operatorname{Im} w = \frac{1}{2}.$$

Problem 1.4

Do the following functions of $z = x + iy$ satisfy Cauchy–Riemann conditions?

1. $w(z) = x^2 + y^2$.
2. $w(z) = x^2 - y^2 + 2ixy$.
3. $w(z) = \frac{1}{x+iy}$.

Solution.

1. No: $\partial_x u = 2x \neq \partial_y v = 0$.
2. Yes: $w(z = x + iy) = z^2$.
3. Yes: $w(z = x + iy) = \frac{1}{z}$.

Problem 1.5

Recover an analytic function $f(z = x + iy)$ satisfying the following equations

1. $|f| = e^{r^3 \cos 3\varphi}$ with $z = re^{i\varphi}$.
2. $\arg f = xy$.

Solution.

1. Observe that

$$ff^* = |f|^2 = e^{2r^3 \cos 3\varphi} = e^{z^3 + (z^*)^3} = e^{z^3} e^{(z^*)^3}$$

hence $f = e^{z^3}$.

We can do it different way. We introduce analytic $w = \ln f$

$$w = \ln |f| + i\arg f = u + iv = r^3 \cos 3\varphi + iv. \tag{1}$$

Next,

$$u = r^3 \cos 3\varphi = r^3(4\cos^3 \varphi - 3\cos\varphi) = 4x^3 - 3(x^2 + y^2)x = x^3 - 3xy^2. \tag{2}$$

Next, we right down Cauchy-Riemann conditions.

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 = \frac{\partial v}{\partial y}, \quad \Rightarrow \quad v = 3x^2y - y^3 + \theta(x) \tag{3}$$

$$\frac{\partial u}{\partial y} = -6xy = -\frac{\partial v}{\partial x} = -6xy - \theta'(x) \quad \Rightarrow \theta(x) = C. \tag{4}$$

Hence:

$$w = u + iv = x^3 - 3xy^2 + i(3x^2y - y^3) + iC = z^3 + iC \quad \Rightarrow f = e^w = e^{z^3 + iC}. \tag{5}$$

2. Observe that

$$f/f^* = e^{2i\arg f} = e^{2ixy} = e^{\frac{z^2 - (z^*)^2}{2}} = e^{z^2/2}/e^{(z^*)^2/2}$$

hence $f = e^{z^2/2}$.

Or, differently $w = \ln f = \ln |f| + i\arg f = u + iv$.

Hence $v = xy$. What our lectures we already know that the corresponding function is $w = \frac{z^2}{2} + C$. And we obtain the same answer.

Problem 1.6

Find all harmonic functions of the following form

1. $u = \varphi(x^2 - y^2)$.
2. $u = \varphi\left(\frac{y}{x}\right)$.

Solution.

1. Cauchy–Riemann conditions yield:

$$\partial^2_{xy} v = 2\varphi'\left(x^2 - y^2\right) + 4x^2\varphi''\left(x^2 - y^2\right) = 2\varphi'\left(x^2 - y^2\right) - 4y^2\varphi''\left(x^2 - y^2\right)$$

which gives $\varphi''(t) = 0$ with $t = x^2 - y^2$. Hence, $u = c_0 + c_1(x^2 - y^2)$ and $v = 2xyc_1$ and as a result:

$$f(z) = c_0 + c_1 z^2$$

with real $c_{0,1}$.

2. Cauchy–Riemann conditions yield:

$$\partial^2_{xy} v = \frac{y^2\varphi''\left(\frac{y}{x}\right)}{x^4} + \frac{2y\varphi'\left(\frac{y}{x}\right)}{x^3} = -\frac{\varphi''\left(\frac{y}{x}\right)}{x^2}$$

which gives $(t + 1/t)\varphi''(t) + 2\varphi'(t) = 0$ with $t = y/x$. Hence, $u = c_0 + c_1 \arctan \frac{y}{x}$ and as a result:

$$f(z) = c_0 + ic_2 - ic_1 \ln z$$

with real $c_{0,1,2}$.

Problem 1.7

Calculate the integral along the unit circle $\mathcal{C}$, centered at $z = 0$

1. $\int_{\mathcal{C}} z dz$.
2. $\int_{\mathcal{C}} z^* dz$.

Solution. We can parametrize $\mathcal{C}$ using polar coordinates $z = e^{i\phi}$ to find:

1. $\int_{\mathcal{C}} z dz = \int_0^{2\pi} i e^{2i\phi} d\phi = 0$.
2. $\int_{\mathcal{C}} z^* dz = \int_0^{2\pi} i d\phi = 2\pi i$.

Problem 1.8

Calculate the integral

$$\int_{\mathcal{C}} \frac{ydx - xdy}{x^2 + y^2}$$

1. along the unit circle $\mathcal{C}$, centered at $z = 0$, counter–clockwise.
2. along the unit circle $\mathcal{C}$, centered at $z = 2$, counter–clockwise.

Solution.

1. using polar coordinates $x = \cos\phi,\ y = \sin\phi$ we find

$$\int_{\mathcal{C}} \frac{ydx - xdy}{x^2 + y^2} = -\int d\phi = -2\pi.$$

2. We may use the Green's theorem to convert the contour integral into an area one over the unit disk $\mathcal{D}$, centered at $z = 2$:

$$\int_{\mathcal{C}} \frac{ydx - xdy}{x^2 + y^2} = \int_{\mathcal{D}} \left(\partial_x \frac{-x}{x^2 + y^2} - \partial_y \frac{y}{x^2 + y^2} \right) dxdy = 0.$$

   Note that this approach would fail for the previous integral. Why?

NB. The integrand in these examples can also be written as follows:

$$\frac{ydx - xdy}{x^2 + y^2} = -\mathrm{Im}\, \frac{dz}{z}$$

and the line integral we computed can be understood as an imagnary part of a complex integral. In what follows, we will learn more handy ways to evaluate these integrals.

Problem 1.9

Consider a function of a natural number n defined by the following integral

$$p(n) = \frac{1}{2\pi i} \int_{\mathcal{C}} dz z^{-1-n} \prod_{k=1}^{\infty} \frac{1}{1 - z^k},$$

where $\mathcal{C}$ is a circle of a radius smaller than 1, centered at $z = 0$ and oriented counter–clockwise.

1. Show that $p(n)$ is a natural number.
2. Evaluate $p(1)$ and $p(4)$.

Solution.

1. Assuming $|z| < 1$ and expanding each of the terms of a product into convergent geometric series we find:
$$p(n) = \frac{1}{2\pi i}\int_{\mathcal{C}} dz z^{-1-n}\prod_{k=1}^{\infty}(1+z^k+z^{2k}+z^{3k}+...)$$
which is a sum of an infinite number of terms of the form
$$Z_m = \frac{1}{2\pi i}\int_{\mathcal{C}} dz z^{-1-m}$$
with integer m. Writing $z = \rho e^{i\phi}$ and switching to integration over ϕ we find that Z_m does not depend on ρ and $Z_m = \delta_{m,0}$, hence the $p(n)$ is a sum of a finite number of 1–s and is an integer.
2. As factorized expression for $p(n)$ above suggests, together with $Z_m = \delta_{m,0}$, the function $p(n)$ is a coefficient in front of z^n of the polynomial
$$(1+z+z^2+z^3+...)(1+z^2+z^4+z^6+...)(1+z^3+z^6+z^9+...)...$$
Expanding the brackets, we find $p(1) = 1$ and $p(4) = 5$.

NB. The function $p(n)$ is known as Partition function P and the integral representation you have just explored allows for a relatively simple derivation of the famous Hardy–Ramanujan asymptotic partition formula, but this would require the methods which are slighty outside the present course.

Problem 1.10

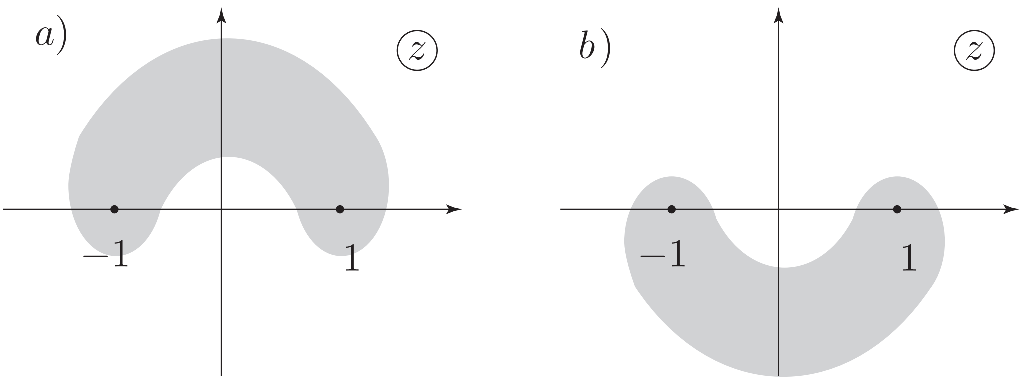

FIG. 1: Region $\mathcal{D}$ for the Problem 1.10.

Consider the function $y(z)$ satisfying $y(1) = 0$ and $y'(z) = \frac{1}{2z}$ in the region $\mathcal{D}$. Evaluate $y(-1)$ for

1. the region $\mathcal{D}$ shown on the Fig. 1a.
2. the region $\mathcal{D}$ shown on the Fig. 1b.

Solution. The differential equation can be integrated to give:
$$y(-1) = \frac{1}{2}\int_{\mathcal{C}}\frac{dz}{z}$$
where $\mathcal{C}$ is arbitrary contour in the region $\mathcal{D}$, starting at $z = 1$ and ending at $z = -1$. The integrals can be evaluated by taking a circular arc for $\mathcal{C}$ and parametrizing it with polar coordinates, $z = e^{i\phi}$.

1. $y(-1) = \frac{1}{2} \int_0^{\pi} i d\phi = i\pi/2$.
2. $y(-1) = \frac{1}{2} \int_0^{-\pi} i d\phi = -i\pi/2$.

NB. The fact the the result does not depend on the choice of the particular contour in $\mathcal{D}$ is not trivial and depends on the function $1/z$ being analytic in this domain. We will discuss this point in more details in the next lecture.