A. Integral III

In this quick example I'd like to discuss with you one more neat example of an integral. It's quite simple but the reason I'd like to address it is because it has some peculiarity which will make you much more cautious in all your future examples of integrals. The integral is as follows:

$$I = \int_{-\infty}^{\infty} \frac{dx}{x+ia} \tag{1}$$

Formally, this integral is divergent. It can be made convergent if we understand it as the symmetric limit:

$$I = \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x+ia}$$

and we will treat it as this from now on.

Let us employ our closure technique. We complement our contour with an upper semicircle (see Fig. 1).

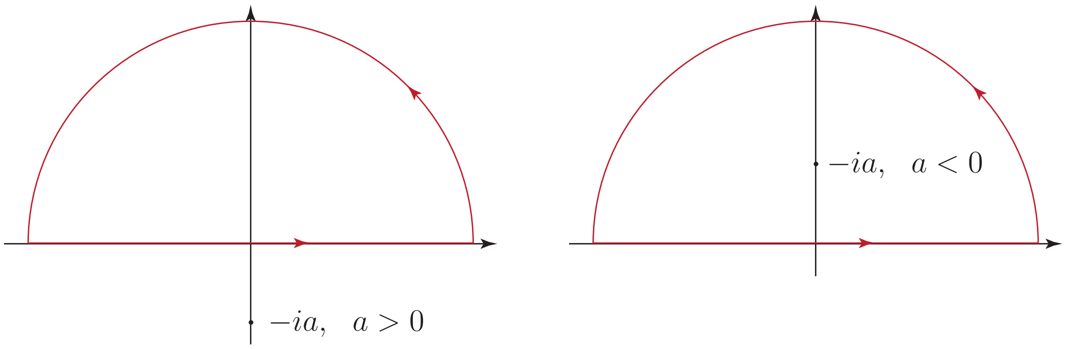

FIG. 1: Towards the computation of integral (1)

Let us prove quickly that this arc integral doesn't change anything, as usual. The parametrization is $z = Re^{i\theta}$. The function's asympotic is $1/R$. Be very careful, when estimating this arc integrals. The integrand may decay, but don't forget to keep in mind additional large factor R coming from differential. This way we have:

$$\int_{\text{arc}} \frac{dz}{z} = \int\limits_0^{\pi} \frac{Rie^{i\theta}d\theta}{Re^{i\theta}} = \int_0^{\pi} d\theta = i\pi.$$

So this time this arc integral doesn't vanish, but is reduced to a simple number.

Therefore, our new closed contour integral reads:

$$\oint = I + i\pi$$

The closed contour integral, as you probably already see by now depends on the sign of a. Let us consider 2 cases.

Suppose $a > 0$, then there are no residues inside the contour, the function is regular. And the closed contour integral is equal to zero. Therefore, the initial integral is equal to:

$$I = -i\pi$$

Suppose now, $a < 0$. Then, there is a residue inside the contour. It is a first order pole and the residue is equal to 1. Then our closed contour integral, according to Cauchy residue theorem is equal to $2\pi i$. As a result, the initial integral is equal to

$$I = 2\pi i - \pi i = \pi i.$$

As a result our Integral is, in fact, nothing but a sign function, up to some constant term.

Therefore, we obtained the integral representation of the sign function:

$$\mathrm{sign} a = -\frac{1}{\pi i} \int\limits_{-\infty}^{\infty} \frac{dx}{z + ia}.$$

Why this result may be interesting? Because sign is a super non-analytic function of a variable a and it is impossible to operate with this function in a complex plane as is.

What we have just obtained is an integral representation of this function in terms of much more regular functions, namely meromorphic functions in a complex plane.