I. CAUCHY INTEGRAL THEOREM

In the previous lecture, we have introduced the concept of analytic function and the concept of complex integration along a curve. These properties, related to differentiation and to integration respectively turn out to be closely related. This relation is one of the most fundamental properties of analytic functions which renders complex analysis so useful in applications. It can be stated in the form of the Cauchy integral theorem. It reads as follows.

Let U be an open subset of the complex plane C which is simply connected. Consider analytic function $f(z)$: $U \to C$ and let γ be a path in U with coinciding start and end points. (fig. (1)) Then

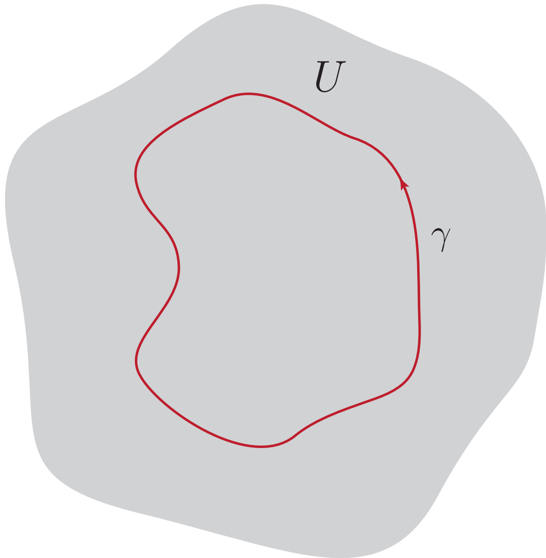

FIG. 1: Towards Cauchy theorem

$$\oint_\gamma f(z)\, dz = 0. \tag{1}$$

The key to proof of this beautiful equality is geometric interpretation of the complex numbers. Let us start with writing $f(z) = u + iv$ and $z = x + iy$. The integral becomes:

$$\oint_\gamma f(z)\, dz = \oint_\gamma (u+iv)(dx+idy) = \oint_\gamma (udx - vdy) + i \oint_\gamma (vdx + udy). \tag{2}$$

The real and imaginary parts of the right-hand-side are defined by real contour integrals of the form which should be familiar to those of you who have studied multivariate real calculus. The Green's theorem allows to rearrange a contour integral of this type – along a contour γ – to an area integral covering the domain D, bounded by this contour:

$$\oint_\gamma (L\, dx + M\, dy) = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) dx\, dy. \tag{3}$$

This theorem is valid as functions L and M entering the left-hand-side have continuous partial derivatives in the region D. Applying this relation to the real and imaginary parts of $\oint_\gamma f(z)\, dz$ (Green's theorem is applicable thanks to analyticity of the function $f(z)$ in the domain bounded by γ) we find:

$$\oint_\gamma f(z)\, dz = \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + i \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right). \tag{4}$$

We now see that equations in the brackets are identically zero due to Cauchy-Riemann conditions! This completes the proof.

A curious remark is in order. The proof we discussed relays on the Green's theorem. In fact, this particular form of the Green's theorem did appear for the first time in Cauchy's treatment and the first proof was actually published by Riemann in his dissertation on complex analysis. Is not is interesting that this fact of multivariate real analysis was first rigorously proven in the study of complex analysis?

II. CAUCHY INTEGRAL FORMULA

As we have learnt in the previous video, an integral of a holomorphic function along a closed contour vanishes, $\oint_\gamma f(z)\,dz = 0$. Since not all interesting functions are holomorphic, it is important to understand possible generalizations of this observation. It turns out that a minimal modification of the integrand allows to formulate another statement, known as Cauchy's integral formula which is probably a central statement in complex analysis.

Consider the closed disk D defined as

$$D = \left\{z : |z - z_0| \leq r\right\}$$

and let γ be the circle, oriented counterclockwise, which forms the boundary of D (fig. (2)). Then for every a inside D, one has

$$f(a) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-a}\,dz.$$

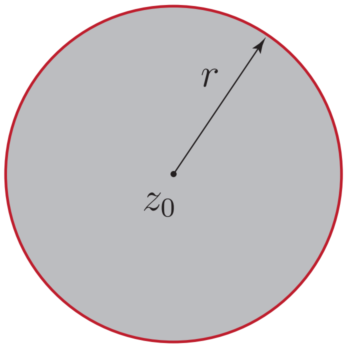

FIG. 2: Towards Cauchy formula

This equation expresses the remarkable fact that a holomorphic function defined on a disk is completely determined by its values on its boundary. Like the Cauchy's integral theorem this statement only requires the function f to be complex differentiable.

The proof is relatively simple and apart from Cauchy's integral theorem, we will use the value of the following integral:

$$\oint_C \frac{1}{z-a}\,dz$$

where C is an arbitrary circle centered at a. This can be calculated directly via substitution $z(t) = a + \epsilon e^{i\phi}$ where $0 \leq \phi \leq 2\pi$ and ϵ is the radius of the circle. Indeed:

$$\oint_C \frac{1}{z-a}\,dz = \int_0^{2\pi} \frac{1}{\epsilon e^{i\phi}} i\epsilon e^{i\phi} d\phi = 2\pi i.$$

Let us now compute

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-a}\,dz - f(a).$$

Using Cauchy's integral theorem, we can deform the contour γ to a small circle (of radius ϵ) surrounding the point a. In the limit of $\epsilon \to 0$ (note that the value of the integral is actually independent on ϵ):

$$
\begin{aligned}
\left|\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz - f(a)\right| &= \left|\frac{1}{2\pi i}\oint_C \frac{f(z)-f(a)}{z-a}\,dz\right| \\
&= \left|\frac{1}{2\pi i}\int_0^{2\pi}\left(\frac{f\big(a+\epsilon e^{i\phi}\big)-f(a)}{\epsilon e^{i\phi}}\cdot \epsilon e^{i\phi} i\right)\,d\phi\right| \\
&\le \frac{1}{2\pi}\int_0^{2\pi}\frac{\big|f\big(a+\epsilon e^{i\phi}\big)-f(a)\big|}{\epsilon}\,\epsilon\,d\phi \\
&\le \max_{|z-a|=\epsilon}|f(z)-f(a)| \xrightarrow[\epsilon\to 0]{} 0.
\end{aligned}
$$

This integral formula has many applications. First, it implies that a function which is holomorphic in a disk (in fact, any open set) is an analytic function, meaning that it can be represented as a power series. We will soon use it to construct Taylor series expansions in the complex plane. The formula is also used to prove the residue theorem, which we will use extensively in the coming lectures to compute real integrals.

Let us consider a useful generalization of the Cauchy's integral formula, applicable to functions, analytical in annulus (contrary to disk). Let C and C' be two concentric circles (so that C' is inside of C). Let $f(z)$ be a function, analytic at C, C' and between. We will need an expression for $f(a)$ where a is a point between the circles. Let us start with the following integral (interpreted counterclockwise, as usual)

$$
\frac{1}{2\pi i}\oint_C \frac{f(z)dz}{z-a}. \tag{5}
$$

The contour C can be deformed into combination of C' and a infinitesimal circle, surrounding the point a counter-clockwise. Recalling that an integral of $\frac{f(z)}{z-a}$ over an infinitesimal circle surrounding point a equals $2\pi i$, we find:

$$
\frac{1}{2\pi i}\oint_C \frac{f(z)dz}{z-a} = \frac{1}{2\pi i}\oint_{C'} \frac{f(z)dz}{z-a} + f(a). \tag{6}
$$

As a result, we find:

$$
f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{z-a} - \frac{1}{2\pi i}\oint_{C'} \frac{f(z)dz}{z-a}. \tag{7}
$$

III. TAYLOR SERIES IN THE COMPLEX PLANE

Approximation of a function $f(x)$ by an infinite sum of the most elementary functions – powers of x – is one of the most important applications in real analysis. Recall that the concept of analyticity in real analysis is formulated precisely in terms of such polynomial approximations. A function that is equal to its Taylor series in an open interval is called as an analytic function in that interval. In the real case, analyticity is not a simple consequence of differentiability. First, note that in real analysis there are a number of functions with only finitely many derivatives. For example, $f(x) = x^2\sin(1/x)$ has only one derivative at $x = 0$, and $f'(x)$ is not even continuous, let alone differentiable. Moreover, there are also real-valued functions that are infinitely–differentiable yet do not have a convergent Taylor series. These kinds of pathologies do not occur in the complex world. Recall that complex analyticity requires existence of a first complex derivative only. We will see now a good reason for this: in the complex world, one derivative is as good as an infinite number of derivatives; differentiability at one point translates to differentiability on a neighborhood of that point. We will also wee that if $f(z)$ is complex analytic at $z = a$, then it's Taylor series automatically converges to $f(z)$ with some positive radius of convergence.

This amazing fact is a relatively simple consequence of Cauchy's integral formula. Consider a function $f(z)$, analytic in the neighbourhood of a point $z = a$. Let C be a circle, centered at a and such that $f(z)$ is analytic on and inside the circle C. Observe now that from

$$
f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{z-a} \tag{8}
$$

it follows that

$$
f'(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{(z-a)^2} \tag{9}
$$

and similarly, all higher derivatives can be evaluated (it follows now that a complex analytic function is in fact infinitely differentiable). Let us now consider $z = a + h$ to be a point inside C. Cauchy's integral theorem yields:

$$f(a+h) = \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{z-a-h} = \frac{1}{2\pi i}\oint_C f(z)\left(\frac{1}{z-a} + \frac{h}{(z-a)^2} + ... + \frac{h^n}{(z-a)^{n+1}} + r_n(z)\right)dz \tag{10}$$

with

$$r_n(z) = \frac{h^{n+1}}{(z-a)^{n+1}(z-a-h)}. \tag{11}$$

The series above can be more compactly rewritten as follows:

$$f(a+h) = f(a) + hf'(a) + ... + \frac{h^n f^{(n)}}{n!} + R_n(h) \tag{12}$$

with residual

$$R_n(h) = \frac{1}{2\pi i}\oint_C f(z)\frac{h^{n+1}}{(z-a)^{n+1}(z-a-h)}dz. \tag{13}$$

The function $\frac{f(z)}{z-a-h}$ in the right–hand–side of the last expression is a continuous function on a circle C and hence is limited from above, that is, there exists such M that

$$|\frac{f(z)}{z-a-h}| \leq M \tag{14}$$

for all $z \in C$. The residual can thus be bounded as follows:

$$|\frac{1}{2\pi i}\oint_C f(z)\frac{h^{n+1}}{(z-a)^{n+1}(z-a-h)}dz| \leq \frac{1}{2\pi}M2\pi R\left(\frac{|h|}{R}\right)^{n+1}, \tag{15}$$

where R is a radius of the circle C. The final expression thus tends to zero for $a + h$ inside C ($|h| < R$). We thus conclude that a series

$$f(z) = f(a) + (z-a)f'(a) + ... + \frac{(z-a)^n f^{(n)}}{n!} + ... \tag{16}$$

converges for all z such that the function is analytic in a circle with radius $|z - a|$, centered at a. We thus see that the convergence radius is precisely the distance to the singularity of $f(z)$, closest to the expansion point a.

Let us consider an example. Consider the function and associated Taylor expansion

$$(1 - 2zh + h^2)^{-1/2} = 1 + hP_1(z) + h^2P_2(z) + ... \tag{17}$$

What is the set of h for which this series converges? Observe that this expansion goes around the point $h = 0$. In order to understand the convergence domain, we need to find the singularities of the left–hand–side. These are the points of vanishing denominator and are determined by equation $1 - 2zh + h^2 = 0$ that is $h = z \pm \sqrt{z^2 - 1}$. We thus conclude that the series in Eq. (17) converges for $|h| < \min(|z + \sqrt{z^2-1}|, |z - \sqrt{z^2-1}|)$.

IV. LAURENT SERIES

Now we consider an important generalization of the Taylor series. As a motivation, imagine a function $f(z)$, analytic at all points in the complex plane with exception of the singular point z_2 and we want to construct a power series centered at z_1 which would be valid at the point x – see the Figure (3). We may try to expand the function $f(z)$ into a Taylor series around a point z_1. However, from the geometry of the points z_1, z_2, x it is clear that such a Taylor series would have a convergence radius equal to $|z_1 - z_2|$ and will not converge at x. Do we have any other option? It turns out that a power series expansion around z_1 converging at the point x can be constructed, but with a price of introducing a possibility of negative powers. Let us be more precise and slightly more general.

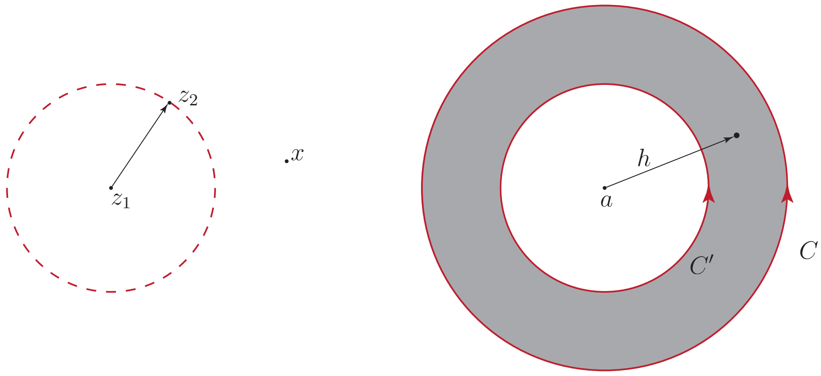

FIG. 3: Towards the derivation of Laurent series

Let C and C' be two concentric circles centered at point a (so that C' is inside of C). Let $f(z)$ be a function, analytic at C, C' and between. Let $a+h$ be some point in the annulus. As we have discussed previously we may write:

$$f(a+h) = \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{z-a-h} - \frac{1}{2\pi i}\oint_{C'} \frac{f(z)dz}{z-a-h}. \tag{18}$$

The first integral can be treated in a way completely aligned with the one used in the derivation of the Taylor series. If you will recall this derivation, you will realize that the fact that residual term tends to zero relies on inequality $|z| > |a+h|$ for all $z \in C$. This inequality fails for the second integral: indeed, $|z| < |a+h|$ for $z \in C'$ and we have to treat the second term in a different manner. However, a slight rearrangement helps:

$$\frac{1}{2\pi i}\oint_{C'} \frac{f(z)dz}{z-a-h} = -\frac{1}{2\pi i}\oint_{C'} \frac{f(z)dz}{h-(z-a)} = \frac{1}{2\pi i}\oint_{C'} f(z)\left(\frac{1}{h} + \frac{z-a}{h^2} + ... + \frac{(z-a)^n}{h^{n+1}} - r_n(z)\right)dz \tag{19}$$

with

$$r_n(z) = -\frac{(z-a)^{n+1}}{h^{n+1}(z-a-h)}.$$

Similarly to the proof of Taylor's theorem we may find that the contribution of residual terms tends to zero at $n \to \infty$. As a result, we find:

$$f(a+h) = a_0 + a_1 h + a_2 h^2 + ... + \frac{b_1}{h} + \frac{b_2}{h^2} + ..., \tag{20}$$

with

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{(z-a)^{n+1}}, \quad b_n = \frac{1}{2\pi i}\oint_{C'} (z-a)^{n-1} f(z)dz. \tag{21}$$

This result is known as Laurent's theorem.

Can we rewrite the coefficients in front of non-negative powers of h, the integrals defining a_n, in terms of derivatives of the function $f(z)$ at the point $z=a$? In general, no: the Laurent's theorem is exactly to deal with a situation when the function is not differentiable at $z=a$.

Let us consider a specific example, the function $f(z) = e^{\frac{x}{2}(z-\frac{1}{z})}$. This function is analytic everywhere apart from $z=0$ and hence it allows for a Laurent's expansion

$$f(z) = a_0 + a_1 z + a_2 z^2 + ... + \frac{b_1}{z} + \frac{b_2}{z^2} + ...,$$

with

$$a_n = \frac{1}{2\pi i}\oint_C e^{\frac{x}{2}(z-\frac{1}{z})}\frac{dz}{z^{n+1}}, \quad b_n = \frac{1}{2\pi i}\oint_{C'} e^{\frac{x}{2}(z-\frac{1}{z})} z^{n-1} dz$$

where C, C' are arbitrary circles centered at $z = 0$. We may consider the circles of radius 1 and parametrize them as $z = e^{i\phi}$ to find

$$a_n = \frac{1}{2\pi i} \int_0^{2\pi} e^{ix\sin\phi} e^{-in\phi} i d\phi. \tag{22}$$

We now observe that the integral $\int_0^{2\pi} \sin(n\phi - x\sin\phi)d\phi$ vanishes, since it is supposed to change sign under $\phi \to 2\pi - \phi$. As a result:

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} \cos(n\phi - x\sin\phi)d\phi.$$

Similarly, we may find that

$$b_n = (-1)^n \frac{1}{2\pi} \int_0^{2\pi} \cos(n\phi - x\sin\phi)d\phi. \tag{23}$$

V. LAURENT EXPANSION, PRACTICE

Example 1
Given function:

$$f(z) = \frac{1}{z^2 - z - 2} \tag{24}$$

build its Laurent expansion centered at point $z = 0$ such that point $z = 3i/2$ belongs to the annulus of convergence.
Solution
First we expand the function into simple fractions:

$$f(z) = \frac{1}{z^2 - z - 2} = \frac{1}{(z-2)(z+1)} = \frac{1}{3}\Big(\frac{1}{z-2} - \frac{1}{z+1}\Big) \tag{25}$$

Then we expand each fraction into geometric series. For the first fraction, the radius of convergence is the closest singularity to the origin $R_1 = 2$. Point $3i/2$ belongs to the convergence radius and we perform the geometric Taylor expansion:

$$\frac{1}{z-2} = -\frac{1}{2}\frac{1}{1-\frac{z}{2}} = -\frac{1}{2}\sum_{n=0}^{\infty}\Big(\frac{z}{2}\Big)^n \tag{26}$$

For the second fraction, the radius of convergence is $R_2 = 1$ and point $z = 3i/2$ lies outside convergence disk. Hence, we expand in powers $1/z$:

$$\frac{1}{z+1} = \frac{1}{z}\frac{1}{1+\frac{1}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n z^{-n} \tag{27}$$

As a result we obtain the expansion in positive and negative powers of z:

$$f(z) = -\frac{1}{6}\sum_{n=0}^{\infty}\Big(\frac{z}{2}\Big)^n - \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n z^{-n-1} \tag{28}$$

Example 2
Given function:

$$f(z) = \frac{1}{\sin^3 z} \tag{29}$$

find the principal part of its Laurent expansion centered at its pole.
Solution The pole is positioned at points $z = \pi n, \quad n \in Z$.

The second method is as follows. We perform a change $z = \pi n + \varepsilon$ and Taylor expand in ε:

$$f(\pi n + \varepsilon) = \frac{1}{\sin^3(\pi n + \varepsilon)} = \frac{(-1)^n}{\sin^3 \varepsilon} = \frac{(-1)^n}{\left(\varepsilon - \frac{\varepsilon^3}{6} + ...\right)^3} = \frac{(-1)^n}{\varepsilon^3\left(1 - \frac{\varepsilon^2}{6} + ...\right)^3} = \frac{(-1)^n}{\varepsilon^3}\left(1 + \frac{\varepsilon^2}{2} + ...\right)$$
$$= \frac{(-1)^n}{\varepsilon^3} + \frac{(-1)^n}{2\varepsilon} + ...$$

Obviously, we missed all the regular terms, but fortunately we were hunting just for the singular part of the expansion, so we are fine.

Therefore, the Laurent expansion reads:

$$f(z) = \frac{(-1)^n}{(z - \pi n)^3} + \frac{(-1)^n}{2(z - \pi n)} + ... \tag{30}$$

And the radius of convergence is obviously the distance betweent the expansion point and the closest singularity, which is π. Hence $R = \pi$.

VI. LAURENT EXPANSION, PRACTICE

Example 1

Given function:

$$f(z) = \frac{1}{z^2 - z - 2} \tag{31}$$

build its Laurent expansion centered at point $z = 0$ such that point $z = 3i/2$ belongs to the annulus of convergence.

Solution

First we expand the function into simple fractions:

$$f(z) = \frac{1}{z^2 - z - 2} = \frac{1}{(z-2)(z+1)} = \frac{1}{3}\left(\frac{1}{z-2} - \frac{1}{z+1}\right) \tag{32}$$

Then we expand each fraction into geometric series. For the first fraction, the radius of convergence is the closest singularity to the origin $R_1 = 2$. Point $3i/2$ belongs to the convergence radius and we perform the geometric Taylor expansion:

$$\frac{1}{z-2} = -\frac{1}{2}\frac{1}{1 - \frac{z}{2}} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n \tag{33}$$

For the second fraction, the radius of convergence is $R_2 = 1$ and point $z = 3i/2$ lies outside convergence disk. Hence, we expand in powers $1/z$:

$$\frac{1}{z+1} = \frac{1}{z}\frac{1}{1 + \frac{1}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n z^{-n} \tag{34}$$

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$$f(z) = -\frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n z^{-n-1} \tag{35}$$

Example 2

Given function:

$$f(z) = \frac{1}{\sin^3 z} \tag{36}$$

find the principal part of its Laurent expansion centered at its pole.

Solution The pole is positioned at points $z = \pi n, \quad n \in Z$.

The second method is as follows. We perform a change $z = \pi n + \varepsilon$ and Taylor expand in ε:

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Obviously, we missed all the regular terms, but fortunately we were hunting just for the singular part of the expansion, so we are fine.

Therefore, the Laurent expansion reads:

$$f(z) = \frac{(-1)^n}{(z - \pi n)^3} + \frac{(-1)^n}{2(z - \pi n)} + ... \tag{37}$$

And the radius of convergence is obviously the distance betweent the expansion point and the closest singularity, which is π. Hence $R = \pi$.

VII. TYPES OF SINGULARITIES

Up to now, we have been discussing the complex functions in the domain of their analyticity. The points of non-analyticity also deserve special attention and we are now equipped to classify them. In complex analysis, there are several classes of singularities. These include the isolated singularities, the nonisolated singularities and the branch points.

An isolated singularity of a function $f(z)$ is a point z_0 such that $f(z)$ is analytic on the punctured disc $0 < |z - z_0| < r$ but is undefined at $z = z_0$. An example of an isolated singularity is $z = i$ for the function $z/(z - i)$. It turns out that all isolated singularities can be classified by the form of a Laurent series in the punctured disk. Three possibilities for Laurent Series for $0 < |z - z_0| < R$ exist, corresponding to three types of isolated singularities:

1. $c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + ...$

   This Laurent series does not contain negative powers. Clearly, it defines a function for which a point $z = z_0$ is actually a point of analyticity, that is why the point z_0 is called removable singularity. Example: a function $f(z) = \sin z/z$, At the first sight, the function is undefined at $z = 0$, but we can attribute $f(0) = 1$ to recover a function, analytic at $z = 0$ (thus removing an apparent singularity).

2. $c_{-n}(z - z_0)^{-n} + ...c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + ...$

   This Laurent series has negative powers, but stops at finite number of negative powers. In this case the point z_0 is called a pole of the order n. The case of $n = 1$, is termed a simple pole. Example: the function $z/(z - i)^2$ has a pole of order 2 at $z = i$.

3. $... + c_{-1}(z - z_0)^{-1} + ...c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + ...$ This Laurent series has terms of arbitrary large negative power. Such a point z_0 is called essential singularity. Example: consider Laurent series of $e^{1/z}$ at $z = 0$ which has arbitrary large negative powers of z and hence $z = 0$ is an essential singularity of this function.

The distinction between a pole and an essential singularity can be formulated in a different (but equivalent) way. For a pole z_0, $|f(z)|$ goes to infinity at $z \to z_0$. For an essential singularity, $|f(z)|$ is neither bounded at $z = z_0$ nor it has an infinity as a limit at $z \to z_0$. For an example of the function $e^{1/z}$, think of approaching $z = 0$ from the right along the real axis: you get increasingly large function values, tending towards infinity as you get closer to 0. But if you approach $z = 0$ from the left along the real axis, the exponent becomes more and more negative, and the exponential tends towards 0, not infinity. As a result, the limit at $z \to 0$ does not exist.

This completes our discussion of isolated singularities. Non-isolated singularities do not admit such a nice classification, but the name suggests that these are singularities that are not isolated.

The typical example would be $f(z) = \tan \frac{1}{z}$. It has a non-isolated singularity at point $z = 0$. Indeed, this function has first order pole at points

$$z_n = \frac{1}{\frac{\pi}{2} + \pi n} \tag{38}$$

Since $\lim_{n\to\infty} = 0$, in any however small neighborhood of $z = 0$ we will find infinite amount of singularities. Hence $z = 0$ is non-isolated singularity.

Finally, branch points are generally the result of a multi-valued behaviour of the function, such as $\sqrt{z}$ or $\ln(z)$, which are defined within a certain limited domain so that the function can be made single-valued within the domain. When the cut is really needed, the function will have distinctly different values on each side of the branch cut. We will discuss this type of behaviour extensively later in the course.