I. INTEGRATION WITH JORDAN'S LEMMA

We will now consider a very important class of integrals of the following form:

$$\int_{-\infty}^{\infty} e^{i\lambda z} f(z) dz$$

for $f(z)$ holomorphic on a half-plane. The importance of such integrals is in the fact, that any Fourier transform is given by this type of an integral.

We already understand that the crucial ingredient in evaluation of this kind of integral via residue theorem is the completion of the contour by an arc (in the upper or lower half–plane of z). For integration via residues to be practical, it is desirable for the integral over this arc to vanish. From what we have learn't from our previous example, this would require for the function $f(z)$ to satisfy

$$\lim_{|z|\to\infty} f(z)z = 0$$

uniformly in $0 \leq \arg z \leq \pi$. Amazingly, it turns out that this requirement can be relaxed to

$$\lim_{|z|\to\infty} f(z) = 0$$

uniformly in $0 \leq \arg z \leq \pi$. This relaxation is possible thanks to oscillatory nature of the integrand, namely, the exponential function which helps the convergence. The precise statement is known as Jordan's lemma.

Jordan's lemma

Consider positive a contour C_R – semicircle of radius R in the upper half–plane, centered at $z = 0$. Let the function $f(z)$ satisfy $\lim_{|z|\to\infty} f(z) = 0$ uniformly in $0 \leq \arg z \leq \pi$, then for $\lambda > 0$:

$$\lim_{R\to\infty} \int_{C_R} e^{i\lambda z} f(z) dz = 0.$$

The proof is simple and demonstrates an important technique of estimation of oscillatory integrals, so let us follow it.

First of all, let us grasp what the uniform convergence of f means. It means that for any radius R, $\max\{|f(z)|\}\Big|_{z\in C_R} < \varepsilon_R$ and $\varepsilon_R \to 0$ as $R \to \infty$

For arbitrary positive ϵ let us choose R_0 such that $|f(z)| < \epsilon/\pi$ for all $|z| > R_0$. Then for $\rho > \rho_0$ we have:

$$\left|\int_{C_\rho} e^{imz} f(z) dz\right| = \left|\int_0^\pi e^{im\rho(cos\phi + i\sin\phi)} f(\rho e^{i\phi})\rho e^{i\phi} i d\phi\right| < \frac{\epsilon}{\pi}\rho \int_0^\pi e^{-m\rho\sin\phi} d\phi = \frac{2\epsilon}{\pi}\rho \int_0^{\pi/2} e^{-m\rho\sin\phi} d\phi.$$

Next, using for $0 \leq \phi \leq \pi/2$ the inequality $\frac{\sin\phi}{\phi} < \frac{\sin\pi/2}{\pi/2} = 2/\pi$ we arrive to

$$\left|\int_{C_\rho} e^{imz} f(z) dz\right| < \frac{2\epsilon}{\pi}\rho \int_0^{\pi/2} e^{-2m\rho\phi/\pi} d\phi = \frac{\epsilon}{m}\left(1 - e^{-m\rho}\right) < \frac{\epsilon}{m}.$$

Hence, at appropriately large ρ the integral can be made arbitrary small and the Jordan's lemma is proven.

Obviously, if you compute the integral along the lower semicircle, the statement stays the same, but this time the integral vanishes for negative λ.

Let us now consider examples of oscillatory integrals.

Let us compute, for real a the following integral:

$$I(a) = \int_{-\infty}^{\infty} \frac{e^{iax}}{x+i} dx.$$

We need to complete the integration line to the closed contour. The proper way to do it depends on the sign of a.

For $a > 0$ we should close the contour in the upper half–plane (see Fig. 1(a)).

Note that the integrand is analytic inside the closed contour and hence the contour integral vanishes. The integral over the arc tends zero by Jordan's lemma and hence the original integral also vanishes.

$$I(a) = 0, \quad a > 0$$

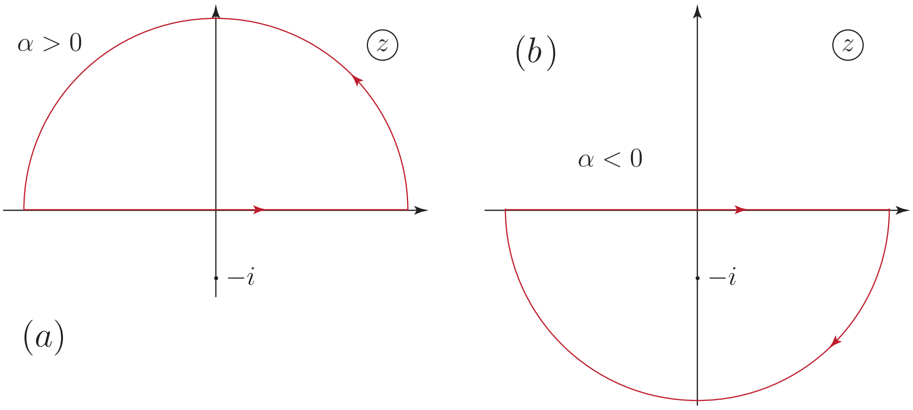

FIG. 1: Towards the application of Jordan's lemma.

For $a < 0$ the contour should be closed in the lower half–plane. (see Fig. 1(b)) In this case, the integral over the arc vanishes again and the original integral equals

$$2\pi i \times (\text{residues of } \frac{e^{iax}}{x+i}) \text{ inside the integration contour}$$

which equals $I(a) = 2\pi i e^{a}, \quad a < 0.$