I. INTEGRATION WITH RESIDUES I

In this lecture, we will practice with evaluation of real integrals. This is a very beautiful application of complex analysis: we will find out that many of the real integrals which may seem rather non-trivial from a real point of view, can be easily evaluated by promoting the integrands from real to complex functions. Before turning to general considerations, let us consider some simple example.

$$I = \int_{-\infty}^{\infty} f(x)dx, \quad f(x) = \frac{1}{1+x^2}. \tag{1}$$

Let us start with computing this integral via purely real methods, a very straightforward approach:

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{\infty} d\arctan x = \frac{\pi}{2} - (-\frac{\pi}{2}) = \pi. \tag{2}$$

Now let me show you how the real integrals are taken using complex analysis.

As is always with powerful methods, when applied to simple situations, they may seem even more involved. The advantage is however, that powerful methods allows to solve cases previously untractable.

First of all, complex analysis likes to operate with closed contours. Indeed, all Cauchy relations are formulated for such contours.

Now we will treat our real integral as an integral taken along a straight line contour positioned in a complex plane. To employ powerful Cauchy theorems we need to close the contour. Well, how do we do that?

In complex analysis we will do this in almost every example. The general prescription is that we should close the contour with some simple curve. The hope is that the integral along this additional curve either disappears or is easily computed and is given by some simple value.

In our case how do we connect two infinite ends of the straight line? Well, for example, by an lower or upper infinite semicircle. Therefore, instead of initial integral we will consider a close contour integral (see Fig).

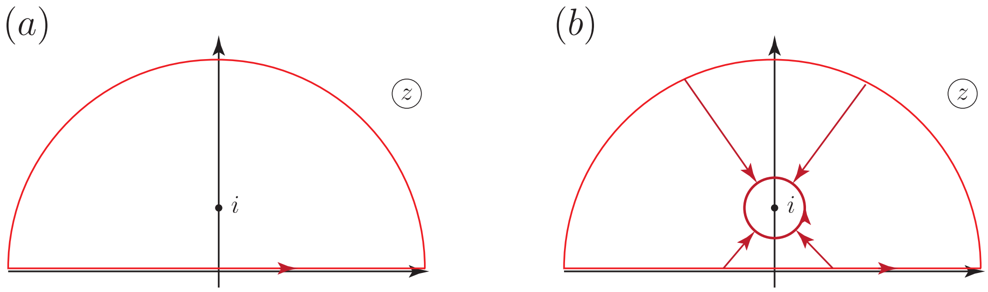

FIG. 1: Integration contour for evaluation of I.

And remember the main consequence of Cauchy integral theorem. The closed contour integral retains its value under any deformation of the contour as long as this deformation doesn't touch the singularities of the integrand. Here our integrand has singularities at points $z = \pm i$. One of them is positioned inside the contour. So we shrink the contour to such extent that it becomes an infinitesimal circle round the pole $z = i$. And, as it turns out, this integral is extremely easily computable with the help of a suitable parametrization.

But of course, this is not our original integral. We modified it with the additional ark. However, the elegance of this ark-trick is that the integrals along the infinite semicircle are usually quite straightforward to evaluate. The principal observation is that as we move along the circle, the complex number z remains large at all points of the circle. Therefore, instead of original complicated integrand we may use its asymptotics for large z. And the asymptotics is always much simpler than the original function. Like in our case. It is simply equal to $1/z^2$.

Now, we integrate $1/z^2 dz$ along some contour. The result is the difference of antiderivatives $1/z$ at the endpoints of the arc. But those are zeroes! So the integral along the big arc simple vanishes. So, despite the fact that we modified our original integral, turning it into the closed contour integral, in reality we didn't change its value. The closed contour integral is equal to our original integral.

With this trick we reduced the task to the computation of an infinitesimal circular integral round the pole $z = i$. We introduce a parametrization $z = i + \varepsilon e^{i\varphi}$. Next we expand our integrand as $1/(z-i)(z+i)$. As we did many times before, $dz/(z-i) = id\varphi$. As a result, we have the integral:

$$\int_0^{2\pi} \frac{id\varphi}{2i + \varepsilon e^{i\varphi}} \tag{3}$$

We discard ε in the denominator and obtain $1/2 \int d\varphi = \pi$. The same answer.

Although in this example you may find that a purely real approach is simpler than the complex one, you can't but notice a certain geometrical elegance of the complex analysis approach.

And in our next video I'll give you powerful theorems which will essentially automate the whole procedure of tackling such integrals.