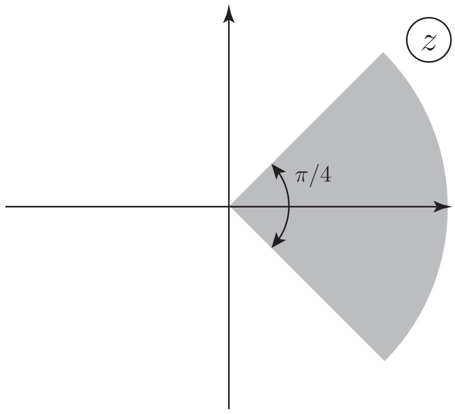

FIG. 1: Initial domain Ω of the definition of the function $f(z)$.

Practice with analytic continuation: contour deformation.

Problem 1

In this lecture, you practiced a lot with constructing regular branches of roots and logarithms. By now, it must be clear to you that this procedure has a lot in common with analytical continuation. Indeed, in constructing a regular branch of root or logarithm our goal was to construct an analytical function which represents one of the possible branches of the multivalued inverse to a given function (power or exponential). In this episode, we turn to another facet of analytical continuation, which is a bit more advanced but equally important. Let us start with the following example. Consider the following integral, depending on parameter z:

$$f(z) = \int_0^\infty \frac{e^{-t^2}}{(t - e^{i\pi/4}z)(t - e^{-i\pi/4}z)} dt. \tag{1}$$

The exponential in the numerator ensures the convergence of the integral. The denominator can be problematic though, depending on the parameter z. For example, at real z the denominator of the integrand does not vanish and the function $f(z)$ is well defined by the Eq. (1). It is clear though that for $z = \rho e^{-i\pi/4}$ or $z = \rho e^{i\pi/4}$ for real positive ρ the integrand has a pole on the integration contour and using the definition in Eq. (1) is problematic at these rays in the complex plane of z. Graphically, it looks as the Fig. (1) shows. The integral in Eq. (1) converges in the domain Ω and defines analytic function there which we will call $F(z)$. As we have discussed, the integral in Eq. (1) loses its meaning at the boundary of domain Ω so the function $F(z)$ seems to be not defined at the rays. At the same time, outside the shaded region the integral in Eq. (1) is well defined. Let us try to understand this situation from the point of view of analytical continuation. To summarize, our starting point is the function $F(z)$, defined by an integral $f(z)$ and analytic in the domain Ω. Can we analytically continue it to a larger domain?

As we will see, the answer is yes. The key to this problem is the contour deformation. Let us switch to the complex plane of the integration variable t, see Fig. (2). At real positive z the poles of the integrand are located at the rays going to infinity at angles $\pm\pi/4$ to the real positive axis. As the argument z approaches the boundary of the domain Ω, the poles rotate in the complex plane of t and this the integration contour: this is precisely what limits the region where the function $F(z)$ can be defined by the integral $f(z)$. However, we can deform the integration contour. Consider, for example two deformations (a) and (b) shown on the Fig. (2) and two associated functions (here $\alpha = a,\ b$)

$$f_\alpha(z) = \int_\alpha \frac{e^{-t^2}}{(t - e^{i\pi/4}z)(t - e^{-i\pi/4}z)} dt. \tag{2}$$

Let us call Ω_+ an 'upper' (Im$z > 0$) piece of Ω and Ω_- – the 'lower' (Im$z < 0$) piece of Ω. Observe that $f_a(z) \equiv f(z) = F(z)$ for $z \in \Omega_+$ and $f_b(z) \equiv f(z) = F(z)$ for $z \in \Omega_-$. However, the functions defined by the integrals $f_{a,b}(z)$ have a different domain of analyticity over their argument z. Indeed, examination of when the poles hit the contours, we find the analyticity domains as shown on the Fig. (3). Thus, the functions $f_\alpha(z)$ coincide with the function $F(z)$ in the domains $\Omega_\pm$ respectively but are analytic beyond these domains. As a result, we can construct an analytic continuation of the function $F(z)$ to the whole right complex semiplane Re$z > 0$ as follows:

$$F(z) = f_a(z),\ \text{Im}z > 0 \quad \text{and} \quad F(z) = f_b(z),\ \text{Im}z < 0.$$

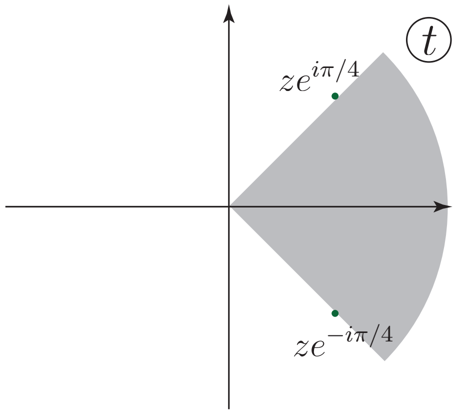

FIG. 2: Complex plane of the integration variable t.

Several comments are in order. To better understand our result, let us get some specific results for the analytical continuation we have just defined. Let us see how to evaluate $F(e^{i\pi/4})$. Indeed, the original definition in Eq. (1) fails at this point due to upparent singularity in the integrand. We know however tha analytical continuation of the function $F(z)$ to the 1st quadrant is given by $f_a(z)$ which, evaluated at required point leads to:

$$F(e^{i\pi/4}) = f_a(e^{i\pi/4}) = \int_\alpha \frac{e^{-t^2}}{(t-i)(t-1)} dt.$$

In this Equation, the integrand is not singular at the integration contour and the integral converges. Evaluating it numerically, we may find $F(e^{i\pi/4}) \approx 0.11 - 1.71i$. We see that the analytically continued function is well defined at the boundary Ω even though the original integral losses its meaning there.

Let us now come back to where we started with functions $f_\alpha(z)$ and wonder: where did the contours a and b come from? In our reasoning above the seemed to fall from nowhere. Let us see how one may come to these contours. To this end, let us come back to the Fig. 2. This image is valid for real z. Rotating the argument z away from the real axis, we observe that the poles shown on this Figure rotate at the same rate and in the same direction as z itself. Consider, for example, rotation of z from the real axis counterclockwise. As soon as the lower pole hits the integration contour we may imagine that the corresponding pole 'picks up' the original integration contour and the continues to drag it upwards at further rotation of z counterclockwise. This is how you may reach an idea that the contour a is a reasonable one for analytical continuation of the original function to the full 1st quadrant. Now consider rotation of z from the real axis clockwise. As soon as the upper pole hits the integration contour we may imagine that the corresponding pole 'picks up' the original integration contour and the continues to drag it downwards at further rotation of z counterclockwise. We thus find that the contour b is a reasonable one for analytical continuation of the original function to the full 4th quadrant. You will find this heuristics very useful in practical analytical continuation, which you will start to manage in your homework for this Lecture.

Problem 2

Let us consider another example of analytic continuation with contour deformation. Consider the function $f(z)$ defined via the following integral:

$$f(z) = \int_i^{-\infty} \frac{e^{zt} dt}{\sqrt{t^2+1}}$$

where the branch of square root is specified by its arithmetic value at $t > 0$ and a cut, joining two branch points $\pm i$.

Due to convergence issues, this definition is valid only for the domain $\mathrm{Re} z > 0$. Let us now explore the analytic continuation of this function to a larger domain. Our goal will be to establish multi-valued character of this analytical continuation. To keep the discussion simple, we will analytically continue $f(z)$ for z evolving along the contour shown on the Fig. 4, left. The rotation of the variable z should be accompanied by rotation of the integration contour to ensure convergence of the integral in the process. The possible integration contours in the complex plane of t associated to points $1, 2, 3, 4$ in the complex plane of z are shown on the Fig. 4, right. Let us compute the change the

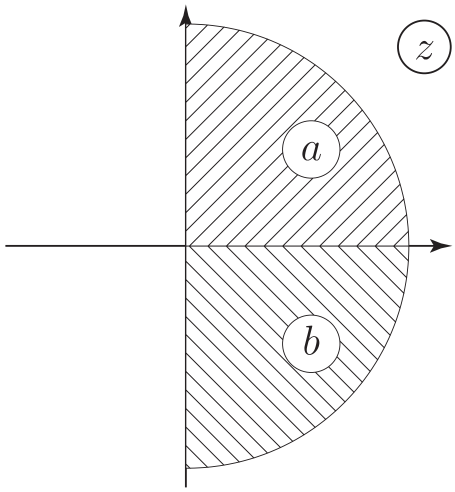

FIG. 3: Complex plane of z with analyticity domains of functions $f_\alpha(z)$.

function $f(z)$ accumulates after a full rotation: $\delta(x) = f(e^{2\pi i}x) - f(x)$ for $x > 0$. This change is equal to a difference of two contour integrals:

$$\delta(x) = \int_{C_4} \frac{e^{xt}dt}{\sqrt{t^2+1}} - \int_{C_1} \frac{e^{xt}dt}{\sqrt{t^2+1}} = \oint_C \frac{e^{xt}dt}{\sqrt{t^2+1}}$$

where the contour C embraces the cut clockwise. The last integral can be computed explicitly in terms of special functions of hyper–geometric family. We will not proceed in this direction and instead will evaluate the function $\delta(x)$ only for small x. In this limit, one can expand the exponential to derive

$$\delta(x \ll 1) \approx \oint_C \frac{(1+xt)dt}{\sqrt{t^2+1}} = \oint_C \frac{dt}{\sqrt{t^2+1}}$$

The resulting contour integral can be computed expanding the contour to infinite radius. The branch of the square root we have chosen expands at $t \to \infty$ as follows:

$$\frac{1}{\sqrt{t^2+1}} = \frac{1}{t}$$

giving

$$\oint_C \frac{dt}{\sqrt{t^2+1}} = -2\pi i.$$

and finally:

$$\Delta f = -2\pi i.$$

Multivalued functions.

As we have seen in a number of exercises, analytic continuation of a function f_0 from certain disk D_0 centered at z_0 along different paths ending up at the same point z can give different values. Can we still make sense of an analytically continued function in such situation? Suppose, function f_0 initially defined in region D_0 can be analytically continued along arbitrary path γ to arbitrary point z in a larger region D. Then any attempt to define an analytic function on D fails if $f_\gamma(z)$ is not solely a function of z but attains different values depending on the choice of the path γ. We have two options to resolve this problem. A somewhat naive solution relies on the concept of multiple–valued functions. In this approach, the values of a complex function are sets of complex numbers. The analytic continuation of a function

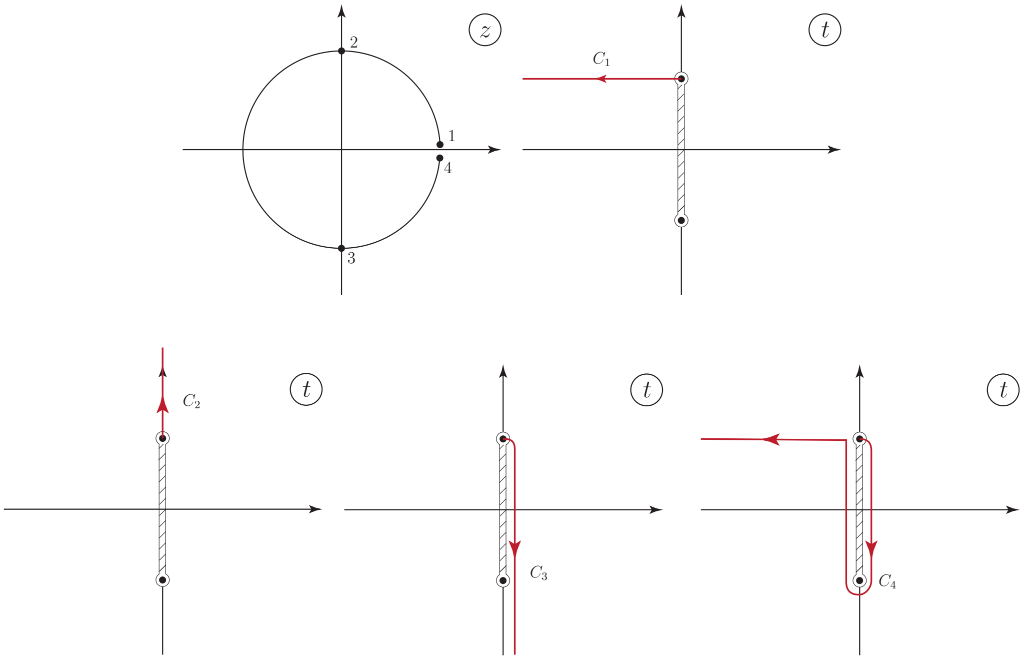

FIG. 4: Illustration for Problem 2.

f_0 defined in region D_0 onto a domain D then yields the multiple–valued function $F(z) = \{f_\gamma(z) : \gamma \in \Gamma D(z)\}$, where $\Gamma D(z)$ stands for the set of all paths in D with fixed initial point z_0 and variable terminal point z. The description of the set on the right–hand side is not very convenient, since it involves the paths γ as parameters. In fact the set $F(z)$ cannot be too large: it is either finite or countable. Let us consider an example.

Consider the function element f_0, D_0 with D_0 being a unit disk centered at $z = 1$ and

$$f_0(z) = \sum_{k=0}^{\infty} C_k^{1/2} (z-1)^k.$$

This series converges for $z \in D_0$ and defines analytic function inside D_0. Luckily, we know that this function defines a square root function in this domain and with our experience with regular branches we can analytically continue the pair f_0, D_0 to an arbitrary point z_1 in C $\{0\}$ along an arbitrary path γ. The result of such analytical continuation is $f_1(z_1) = r^{1/2} \exp(i\phi/2)$ where $r = |z_1|$ and $\phi = \arg_\gamma z_1$ denotes the continuous branch of the argument function along γ with $\arg_\gamma z_0 = 0$ at the initial point $z_0 = 1$ of γ. The value of ϕ depends on the path γ, but all possible values differ only by an integer multiple of 2π from the principal value $\mathrm{Arg} z_1$. Consequently, however the path γ has been chosen, $f_1(z_1)$ must always attain one of the two values

$$|z_1|^{1/2} \exp(i\mathrm{Arg} z_1/2), \quad -|z_1|^{1/2} \exp(i\mathrm{Arg} z_1/2),$$

Thus the multiple–valued square root function SQRT, obtained by analytic continuation of (f_0, D_0) onto $C \backslash \{0\}$, attains two values at every point $z = re^{i\phi}$ where $r \neq 0$:

$$\mathrm{SQRT}(z) = \left\{ (-1)^k \sqrt{r} e^{i\phi/2}, \quad k = 0, 1 \right\}$$

The value corresponding to $k = 0$ is called the principal value (principal branch or main branch) of the square root and denoted by $\mathrm{Sqrt}(z)$.

Notwithstanding their name, the multi–valued functions are not ordinary functions and must be handled with care. That even simple operations involving multiple–valued functions may be problematic becomes evident when one tries to define the sum $\sqrt{z} + \sqrt{z}$ and then compares the result with $2\sqrt{z}$. A better alternative to multiple-valued functions rests on a brilliant idea of Bernhard Riemann. He observed that several copies of the complex plane carrying the branches of a multiple–valued function can be glued together in a clever way creating a new domain, called a Riemann surface. The Riemann surface comprises the different branches of the multiple–valued function. This gives rise to a single–valued analytic function on a Riemann surface.

Riemann surfaces: introduction.

In the end of this week let us consider one more fundamental and beautiful concept, namely the Riemann surface of a multivalued functions. Riemann surfaces make the very concept of a multivalued function geometrically more transparent.

Let us consider a function $\ln z$ with a branch cut from zero to plus infinity. And let us recall how the value of the function evolves as we rotate in the counterclockwise direction round the branch point $z = 0$. We start from the upper bank of the branch cut at point x. The argument of the function grows steadily from zero to 2π until we reach the lower bank of the branch cut where the value of the function becomes $\ln x + 2\pi i$ and the argument is rotated by 2π. That is how things look if we use the definition of the regular branch coinciding with the arithmetic value of the log on the upper bank.

We may consider another definition: we may put the value of the function on the upper bank to be equal to $\ln x + 2\pi i$. That is a different regular branch. To distinguish it we will plot its values on a different complex plane with the same branch cut. Then after the full rotation round the branch point we arrive at the value $\ln x + 4\pi i$ on the lower bank.

Then, we may consider the third definition of the log, namely the one that assumes the value of $\ln x + 4\pi i$ on the upper bank of the branch cut and take the third complex plane.

And so on.

This way, we may say that to describe the full spectra of the values of our log we need an infinite amount of complex planes. And as you notice: the value of the first branch on the lower bank of a cut on the plane always coincides with the values of the second branch on the upper bank of a branch cut on the second plane and so on.

And here comes Riemann's idea:

Let us put the second complex plane under the first one and glue the lower bank of the first branch cut with the upper bank of the branch cut on the second plane. We can do it, because the values of two regular branches of the function are identical. Then we position the third plane under the second and glue the lower bank of the branch cut of the second plane with the upper bank of the branch cut of the third plane.

And we may proceed in chain to infinity, gluing together complex planes.

We obtain an infinite tower of glued planes. It is a 2D manifold with infinite amount of sheets (they are called now Riemann sheets).

And each sheet accommodates the values of a particular regular branch of our log. So in the end all values of all regular branches of the log function are allocated. The multivaluedness is gone for good.

The log is single valued now for any point on this manyfold, but at the price of turning the complex plane into a nontrivial topological object.

So, that was the Riemann's idea. He suggested turning complex plane into 2 dimensional manyfold imbedded into 3D space so it will be able to host all the values of all regular branches of the initially multivalued function rendering it single valued.

And this idea happened to be quite influental. It is used extensively in asymptotic complex analysis. If you read the book by Handelsman on asymptotics, here it is, the contour defining the Schlaefli representation of the Bessel function. It is a non self intersecting 8 lying on two Riemann sheets of the square root function.

This way Riemann surfaces entered the world of mathematical physics and became its habitat. All its little children and beasts resettled from complex plane on to Riemann surfaces.

Riemann surface: definition

The construction of the Riemann surface of log–function which we have just discussed is not only helpful to get an intuitive understanding, it also very practical and we will exercise it in what follows. This approach is also not completely satisfactory because it relies too much on visual persuasion. Let us now put these ideas on a slightly more solid basis. Although we can not afford developing self-contained rigorous exposition here, we will try at least to give you a flavor of it.

Let us consider the disk chain method of exploring the analytic functions. This technique can be compared with exploring a landscape. At the beginning we stand somewhere (at a point z_0) and can see the function just in a small region (the disk of convergence of its Taylor series at z_0). Moving to the boundary of that region, the horizon changes and we may get access to a new part of the landscape which could not be seen before. Continuing this process, we walk around in the plane and discover more and more of the territory where the function lives. Its possible that after coming back to the point z_0 we started we do not recover the same Taylor series as we started from The set of all possible *locally* analytic functions obtained in this manner defines a *globally* analytic function (which is not a function in a traditional sense but rather a set of functions defined on a set of patches of a complex plane).

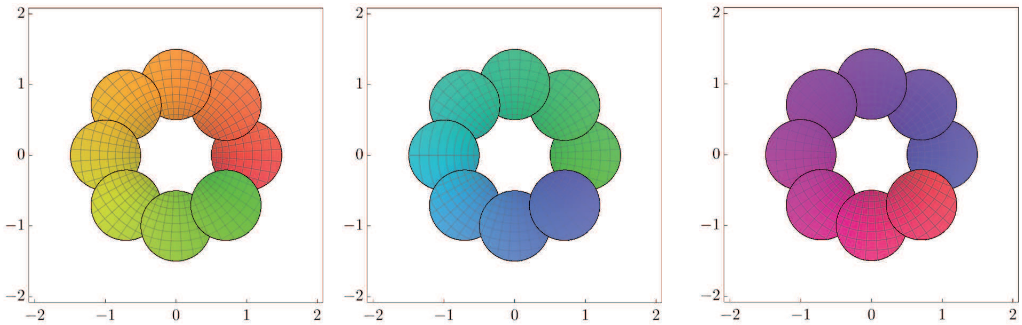

FIG. 5: Disk chains for cubic root.

The fact that for the *globally* analytic function two different locally analytic functions may correspond to the same patch of a complex plane hints that usage of a single complex number z to identify our position on a complex plane is not sufficient to recover the function's behaviour. There exist many possible locally analytic functions which can be obtained at given point z via all possible analytic continuations from all other points. One has to promote z by adding an additional label g: $z \to (z, g)$ to specifies a single of all possible continuations reachable at z. The label g belongs to a countable set and the Riemann surface is defined as the set of all possible pairs

$$S(f) = (z, g) \in C \times F : z \in C, g \in F_z. \quad (3)$$

Note that the set of possible values of g depends – in principle – on z.

At this stage, we are still very close to the language of multi–valued functions: attaching the label g is morally equivalent to specifying a branch of multi-valued function. The crucial step to render the concept of the Riemann surface really useful is to consider the set $S(f)$ as a domain of a function f, This is possible since each point (z, g) of $S(f)$ is equipped with g which by definition contains all the information to evaluate f at the point z. Moreover, the projection $(z, g) \to z$ allows to generalize all operations of complex analysis from the usual complex plane z to the Riemann surface. This allows to define the concept of analyticity on the Riemann surface and it turns out that a *globally* analytic function becomes analytic function on a Riemann surface.

Let us see how this terminology applies to our previous example of *log*–function. In this case, at each z the possible values of $\log z$ are enumerated with integer k:

$$g(z) = \mathrm{Log} z + 2\pi i k$$

Thus, the Riemann surface of Log is comprised of the points (z, k) with complex z and integer k. All points (z, k) with fixed k form a sheet S_k. Every sheet S_k is a copy of the punctured complex plane, slit along a certain line which allows to connect two adjacent sheets [FIG]. These connection lines denote the points in individual sheets where the coordinate g in (z, g) switches between its discrete values.

Riemann surface: further examples – I

Our previous example of a Riemann surface was the one of a multi–valued log function. It contained infinite amount of sheets but had a very simple shape. We will now study the Riemann surfaces stemming from algebraic functions. The simplest example is a multi–valued function $z^{1/3}$. A natural expectation is that it will have 3 sheets. Let us see how this emerges from the concept of a *globally* analytic function. Consider the branch of a cubic root specified by its arithmetic value on a real axis and extended to the domain $|z - 1| < 1/2$. Building the disk chains we can reach each point in the complex plane and find a corresponding analytic continuation of this function. As we may expect, there exist several possible analytic continuations at each point, depending on the path chosen, which are shown on the Fig. 5.

This Figure shows that 3 distinct *locally* analytic function can be derived at each z and this list is exhaustive. In order to enumerate these possibilities at each z, we introduce three sheets, specifying lines in the complex plane which correspond to switch between these sheets. The conventional choice is to make these lines aligned among different sheets whenever possible. In the present case, these lines can be chosen to be R_- on each sheet. The resulting phase portraits are shown on the Fig. 6. On the formal language, each of the sheets corresponds to certain value of g among $g = 0, 1, 2$. Since g has to change at the lines R_-, we need to specify the correspondence rule between the sheets. This

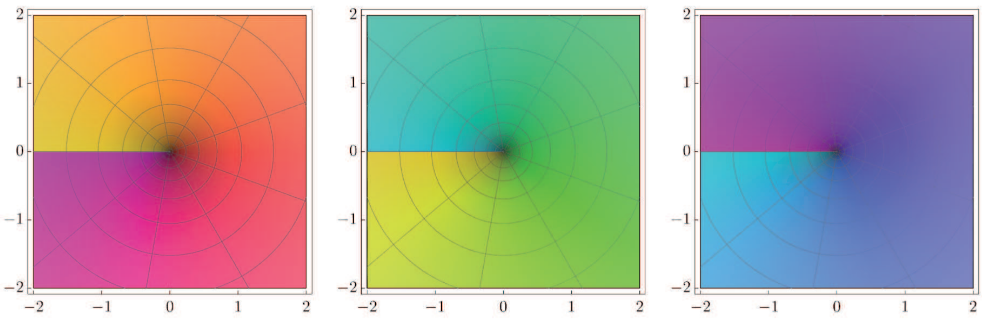

FIG. 6: Phase portraits for cubic root on three sheets.

FIG. 7: 3D model of Riemann Surface for cubic root.

correspondence rule has to ensure the analyticity of $z^{1/3}$ on the resulting Riemann surface and hence it is uniquely determined by the phase portraits shown on the 6. These phase portraits imply the following rules:

$$U_1 \leftrightarrow L_2, \quad U_2 \leftrightarrow L_3, \quad U_3 \leftrightarrow L_1$$

where U_k and L_k denote the upper and lower banks of the line R_- on the k–th sheet.

Having established the topological structure of the Riemann surface of $z^{1/3}$, let us take a look at the wooden model of this object, constructed by Prof. Richard P. Baker 7. Although correct, this model has somewhat unpleasing sharp kinks joining the sheets. We may smooth them and render the following 3D model of the same surface **??**. Please note that in this model, the exact shape of a piece of the surface, joining the sheets, has no deep meaning and serves just to render an aesthetically pleasing image.

Riemann surface: further examples – II

In this section, we consider a more interesting example of Riemann surface of the function $f(z) = \sqrt{1-z^2}$. Consider the branch specified by $f(2) = i\sqrt{3}$. We will build disk chains starting from $|z-2| < 1/2$ we can reach each point in the complex plane and find a corresponding analytic continuation of this function. The situation is a bit richer than for cubic root: consider chains shown on the Fig. 8 where the function comes back to its original value and chains on the Fig. 9 where the function restores its value only after double winding.

In this case, it is clear that 2 distinct locally analytic functions can be constructed at each point (with exception of branch points $z = \pm 1$). However the ‘connection rules’ allow for two distinct Riemann surfaces. Let us first describe these surfaces using phase portraits.

First option is shown on the Fig. 10. These phase portraits imply the following rules:

$$U_{1,-1} \leftrightarrow L_{2,-1}, \quad L_{1,1} \leftrightarrow U_{2,1}$$

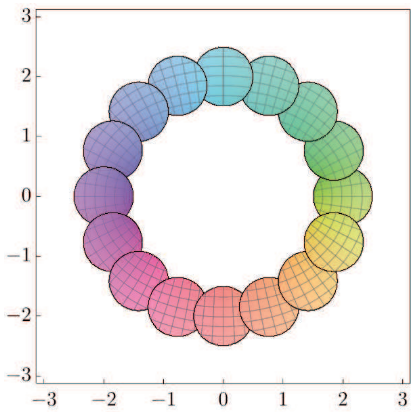

FIG. 8: Disk chains for $\sqrt{1-z^2}$, path i).

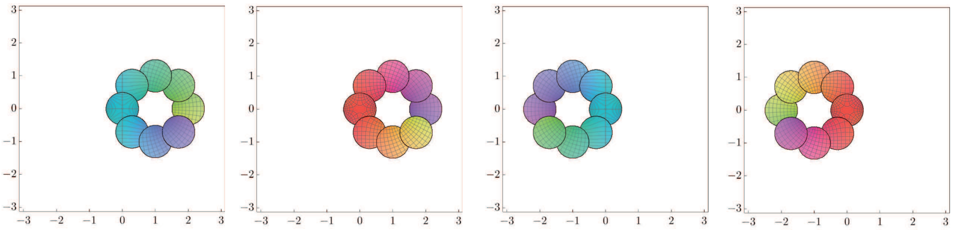

FIG. 9: Disk chains for $\sqrt{1-z^2}$, paths ii) and iii).

where $U_{k,\pm 1}$ and $L_{k,\pm 1}$ denote the upper and lower banks of the cuts emerging from ± 1 at the sheet k.

Second option is shown on the Fig. 11. These phase portraits imply the following simple rule:

$$U_1 \leftrightarrow L_2, \quad U_2 \leftrightarrow L_1$$

where U_k and L_k denote the upper and lower banks of the cut $(-1,1)$ at the sheet k.

Having established the topological structure of the Riemann surface, let us render the following 3D model of the same surfaces 15. Please note that in this model, the exact shape of a piece of the surface, joining the sheets, has no deep meaning and serves just to render an aesthetically pleasing image.

Integration over Riemann surface

Now let us consider the Riemann surface of $\ln(1-z^2)$. This function allows for semi infinite branch cuts only. And the procedure of its creation is very similar to the previous one. But unlike double valued function $\sqrt{1-z^2}$ the log will have infinite amount of sheets.

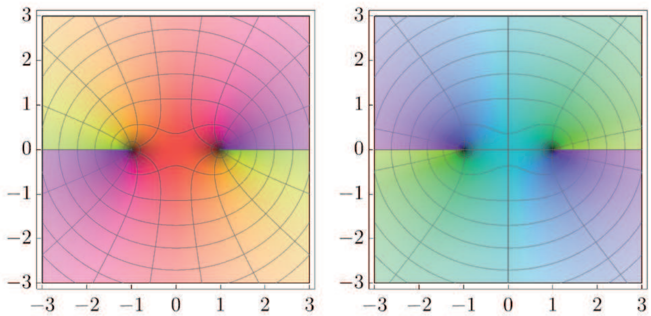

FIG. 10: Phase portraits for square root on two sheets, option i).

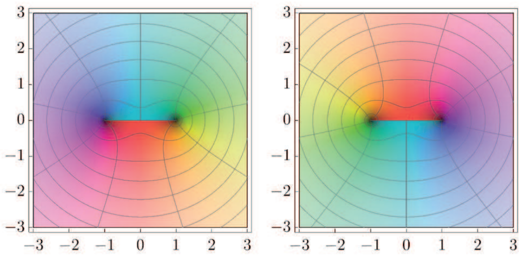

FIG. 11: Phase portraits for square root on two sheets, option ii).

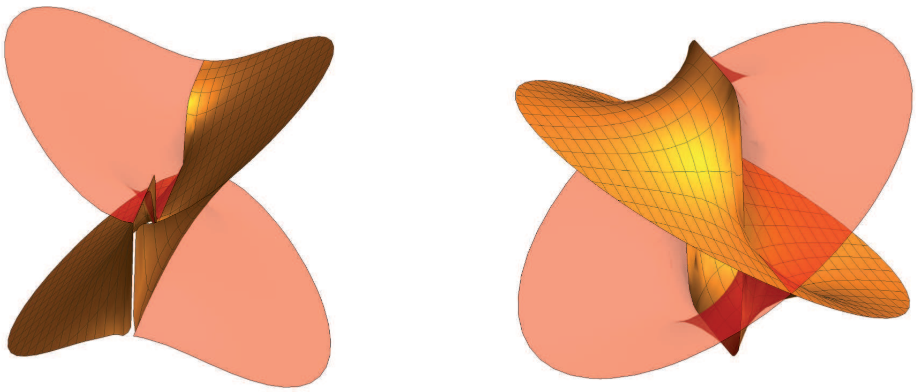

FIG. 12: 3D model of Riemann Surface for square root: two options.

Well, this Riemann surface allows for quite an amazing phenomenon. We may draw an 8-shaped contour without self intersections which lives on two Riemann sheets. We may also integrate functions on this contour.

Let us consider the integral of the function $\int\limits_{8-\mathrm{shape}} \ln(1-z^2)dz = \int_1^{-1} \ln(1-x^2)dx + \int_{-1}^{1}[\ln(1-x^2)+2\pi i]dx = 4\pi i.$

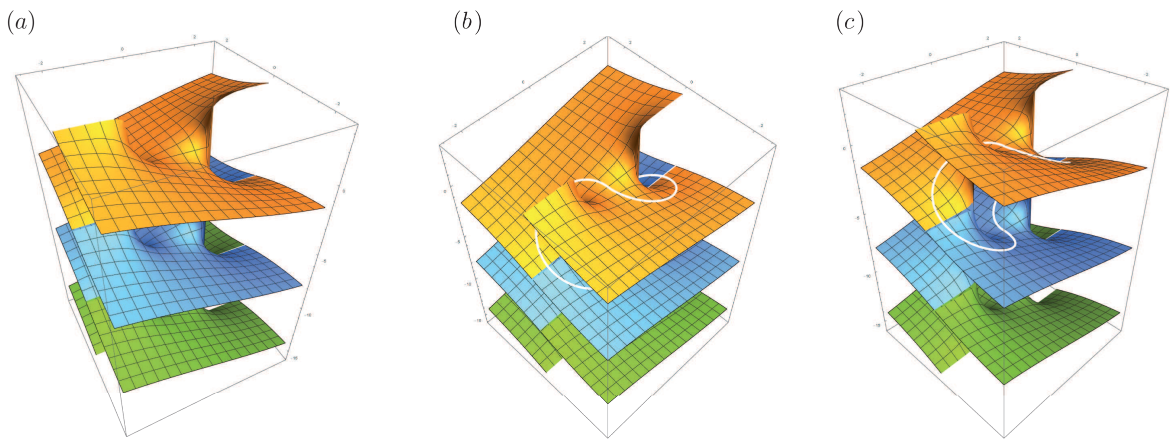

FIG. 13: (a) 3D model of Riemann Surface for $\ln(1-z^2)$; (b,c) The eight-shaped integration contour

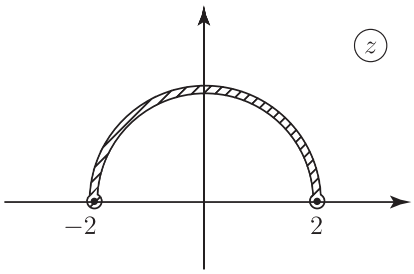

FIG. 14: The branch cut for $f(z)$

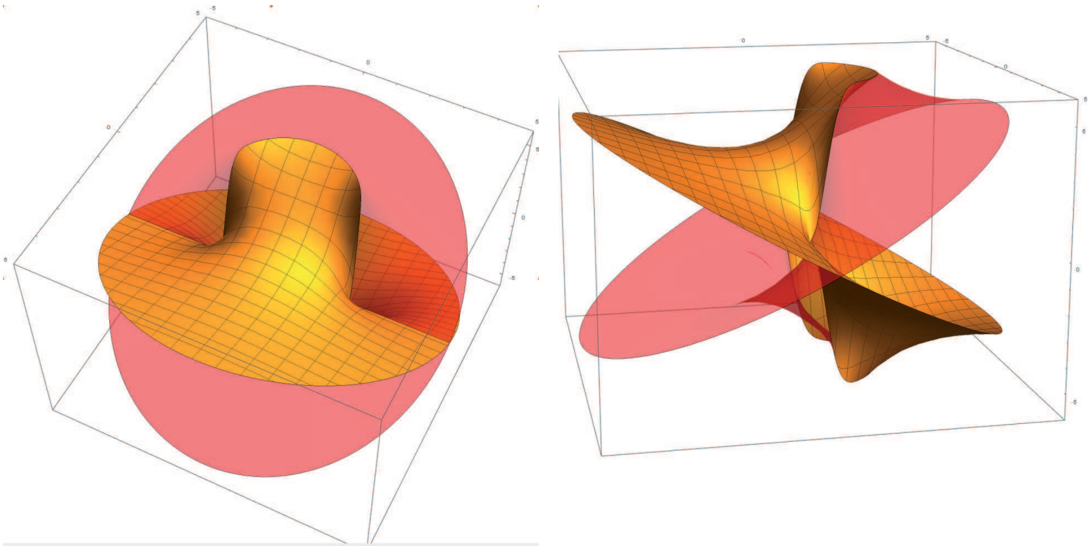

FIG. 15: Riemann surface for $\sqrt{4-z^2}$

Visualization of a Riemann surface

Here I'd like to discuss briefly, how the Riemann surface is drawn using some standard math packages.

Let us construct a Riemann surface of function $f(z) = \sqrt{4-z^2}$ with a branch cut in the form of upper semicircle connection points -2 and 2.

First of all we understand that it consists of 2 sheets and unavoidably has self intersections.

The surfaces with finite branch cuts with self intersections are most easily visualised as inclined sheets with cuts in the middle.

So our plan is as follows. We draw an inclined disk and cut it with a knife along the upper semicirlce. At the cut, the function has a gap, and as a result we bend the the edges of the cut in opposite directions. And then we draw a second sheet with opposite inclination and glue it with the first one via the cut.

The question is how to draw such surfaces. We follow the recepee of 3 mathematicians from Berlin University and put the reference in the supplement to this video.

We represent them as altitudes in z-directions as functions of coordinates (x, y), $\quad u(x, y)$.

It turns out that harmonic functions make the nice looking smooth surfaces. Therefore, we will seek the surfaces as the solutions of Laplace equation with appropriate boundary conditions.

First of all we draw two sheets such that their projections on xy plane are disks of radius R.

Let us consider the first disk. First of all it should be inclined with respect to xy plane. So the boundary condition is $u(x, y) = \alpha y$ as $x^2+y^2 = R^2$. For the other sheet we will have $u(x, y) = -\alpha y$ as $x^2+y^2 = R^2$. Then we make a cut with a knife along the curve $y = \sqrt{4-x^2}$ on the sheet. We bend the upper bank of the cut upwards (by h) and the lower bank downwards (by h). So let us split the sheet into two halves ($y > 0$ and $y < 0$). The boundary condition for the upper half is $u(x, y) = h$ as $y = \sqrt{4-x^2}$ and $u(x, y) = 0, x \in [-R, -2] \cup [2, R]$ while for the lower half it is $u(x, y) = -h$ as $y = \sqrt{4-x^2}$ and $u(x, y) = 0, x \in [-R, -2] \cup [2, R]$.

And we solve Laplace equation $\Delta u(x, y) = 0$