Let us compute, for $a > 0$ and $b > 0$ the following integral:

$$I = \int_0^\infty \frac{\cos 2ax - \cos 2bx}{x^2} dx. \tag{1}$$

First of all, let us make sure that the integral is well defined and study the behavior of the integrand at $x \to 0$. It is a good practice since it reveals valuable information about the structure of the integral as a whole.

Taylor expanding the nominator to second order in x we obtain:

$$\frac{\cos 2ax - \cos 2bx}{x^2} \to 2(b^2 - a^2)\Big|_{x \to 0} \tag{2}$$

Therefore, the integrand is well - defined. Now let us proceed with computation as in previous examples. We expand cosines into sum of exponentials and change the variables in the exponentials with *negative* increments. This way we obtain the integral along the entire real axis:

$$I = \frac{1}{2} \int_{-\infty}^\infty \frac{e^{2iax} - e^{2ibx}}{x^2} dx. \tag{3}$$

However (3) is already ill defined. Indeed, its integrand has a first order pole at $x = 0$:

$$f(z) = \frac{1}{2} \frac{e^{2iaz} - e^{2ibz}}{z^2} \to \frac{i(a-b)}{z}\Big|_{z \to 0} \tag{4}$$

This presents a problem. Apparently, we made a mistake when transforming the integral. We encourage the reader to figure out this mistake independently. The question is, how to tackle the integral.

As we understand, point $x = 0$ is potentially problematic. Therefore, we consider the original integral as a limit of a slightly modified integral:

$$I = \lim_{\varepsilon \to 0} I_\varepsilon \equiv \lim_{\varepsilon \to 0} \int_\varepsilon^\infty \frac{\cos 2ax - \cos 2bx}{x^2} dx. \tag{5}$$

Here ε is an infinitesimal positive parameter. This way we exclude the potentially dangerous point $x = 0$ from our analysis.

Now we use the same algebra and turn our integral into a form suitable for application for Jordan's lemma, but this time however we will not be able to obtain the integral from minus infinity to plus infinity of difference of two exponentials. Rather we'll obtain two separate integrals from - infinity to - epsilon and from + epsilon to + infinity of the same integrand.

$$I_\varepsilon = \int_{-\infty}^{-\varepsilon} \frac{e^{2iax} - e^{2ibx}}{x^2} dx + \int_\varepsilon^\infty \frac{e^{2iax} - e^{2ibx}}{x^2} dx \tag{6}$$

So this way there is no problem with singularity but there is a trade-off now: we have two separate integrals instead of one and in mathematics there is a special name for this combination of integrals. It is called a principal value integration and is denoted as integral with cross sign:

$$⨍ \equiv \Big[\int_{-\infty}^{-\varepsilon} + \int_\varepsilon^\infty \Big]_{\varepsilon \to 0} \tag{7}$$

What it means is that we take the integral of some function along some contour and whenever the contour meets a singularity of the integrand it is split and offset by infinitesimal distance to the left and to the right of the singularity (see Fig. 1).

This procedure is called the integration in the sense of a principal value. Therefore, we reduced our original integral to a more suitable form but over a split contour. However, it is the closed contour that we were aiming at, not the split one.

What shall we do now? Well, it is simple. We close our contour. We'll connect our separate pieces at the origin with infinitesimal upper or lower semicircle - it's up to us to choose, but I choose upper semicircle for the reason which will hopefully be clear to you in a couple of minutes. Then I'll connect to infinite edges of this contour, but this time infinite upper semicircle, keeping in mind the application of Jordan's lemma. Now we'll promote our function into a complex plane and consider a closed contour integral (see Fig. 2).

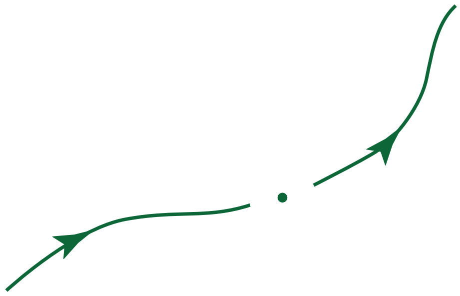

FIG. 1: Towards the principal value integration.

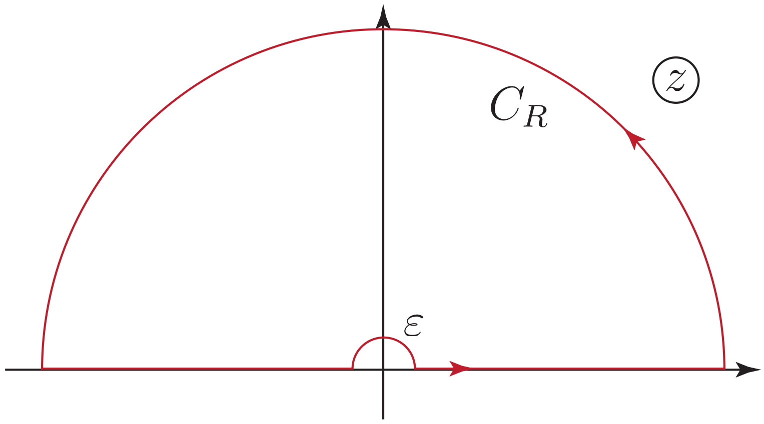

FIG. 2: Towards the evaluation of the integral.

Of course, this integral consists of our principal value integral which is nothing but our initial integral plus the integral along infinitesimal upper semicircle and plus the integral along the upper infinite semicircle C_R:

$$\oint = \fint + \int\limits_{\varepsilon} + \int\limits_{C_R} \tag{8}$$

Due to Jordan's lemma the infinite semicircular integral tends to zero as the radius of the circle tends to infinity, because it's a combination of two exponentials with positive increments a and b. And the preexponential function g(z) which is equal to $1/z^2$ tends to zero uniformly with respect to the argument of the complex number z. Now the closed contour integral itself. from residue theorem,

it should be equal to $2\pi i$ times the sum of the residues inside the contour. But there are no poles inside this contour: our function is analytic in it, so this closed contour integral is in fact, equal to 0.

$$\oint = 0 \tag{9}$$

Miraculously, our principal value integral is now equal to the integral along the infinitesimal semi-circle at the origin.

$$\fint = -\int\limits_{\varepsilon} \tag{10}$$

And this is from my point of view is a charm of complex analysis: we started with a complicated integral from zero to plus infinity and we reduced it to an integral along some infinitesimal circle. And of course, this integral is way easier to compute because we don't need the full function to do this. But just it's Taylor series near the origin. So let us perform the Taylor expansion of our f(z) function.

The first term in the numerator was already obtained by us in (4): $i(a-b)/z$ plus some regular terms. Now, why don't we need these regular terms?

That's because when we integrate along this infinitesimal circle, they will vanish as the radius of the circle tends to zero. This way we only need singular terms, and there is only one of them. And it's what is written in (4).

Now let's plug in this expansion into our integral, and we'll obtain

$$\fint = -i(a-b)\int\limits_{\varepsilon} \frac{dz}{z} \tag{11}$$

Next we introduce the standard parametrization $z = \epsilon e^{i\varphi}$, and therefore dz/z is simply $id\varphi$ and since φ changes from π to 0, this integral is equal to $-i\pi$. Therefore, we obtain:

$$I \equiv \fint = -i(a-b)(-i\pi) = \pi(b-a). \tag{12}$$