$$\check{T}(\check{t},\check{x}_1,\check{x}_2)=e^{-\lambda^2\check{t}}(A\cos(k\check{x}_1)+B\sin(k\check{x}_1))[C\cos(\sqrt{\lambda^2-k^2}\,\check{x}_2)+D\sin(\sqrt{\lambda^2-k^2}\,\check{x}_2)]$$

$T=-\Delta T\check{T}+T_0$

$$\frac{\partial T}{\partial x_1}\Big|_{x_1=0}=0,\quad \frac{\partial T}{\partial x_1}\Big|_{x_1=W}=0,\quad T(t,x_1,x_2=0)=T(t,x_1,x_2=W)=T_0$$

$$\xrightarrow{\text{scaling}} -\Delta T\check{T}(\check{t},\check{x}_1,\check{x}_2=0)+T_0$$
$$=-\Delta T\check{T}(\check{t},\check{x}_1,\check{x}_2=1)+T_0=T_0$$

$$\Rightarrow \frac{\partial \check{T}}{\partial \check{x}_1}\Big|_{\check{x}_1=0}=0 \quad ①$$

$$\frac{\partial \check{T}}{\partial \check{x}_1}\Big|_{\check{x}_1=1}=0 \quad ②$$

$$\Rightarrow \check{T}(\check{t},\check{x}_1,\check{x}_2=0)=0 \quad ③$$
$$\check{T}(\check{t},\check{x}_1,\check{x}_2=1)=0 \quad ④$$

$$\frac{\partial \check{T}}{\partial \check{x}_1}=e^{-\lambda^2\check{t}}[-Ak\sin(k\check{x}_1)+Bk\cos(k\check{x}_1)][C\cos(\sqrt{\lambda^2-k^2}\,\check{x}_2)+D\sin(\sqrt{\lambda^2-k^2}\,\check{x}_2)]$$

$$① \Rightarrow A\cdot(-k)\sin(0)+Bk\cos(0)=0$$
$$B=0$$

lumped

$$\check{T}(\check{t},\check{x}_1,\check{x}_2)=e^{-\lambda^2\check{t}}(\cancel{A}\cos(k\check{x}_1))[C\cos(\sqrt{\lambda^2-k^2}\,\check{x}_2)+D\sin(\sqrt{\lambda^2-k^2}\,\check{x}_2)]$$

$$\frac{\partial \check{T}}{\partial \check{x}_1}=e^{-\lambda^2\check{t}}[-k\sin(k\check{x}_1)][C\cos(\sqrt{\lambda^2-k^2}\,\check{x}_2)+D\sin(\sqrt{\lambda^2-k^2}\,\check{x}_2)]$$

$$② \Rightarrow -k\sin(k)=0$$
$$k_m=m\pi \quad [-\infty,\infty]$$

$$\therefore \check{T}(\check{t},\check{x}_1,\check{x}_2)=e^{-\lambda^2\check{t}}[\cos(m\pi\check{x}_1)][C\cos(\sqrt{\lambda^2-k^2}\,\check{x}_2)+D\sin(\sqrt{\lambda^2-k^2}\,\check{x}_2)]$$

$$③ \Rightarrow C\cos(0)+D\sin(0)=0$$
$$C=0$$

$$\therefore \check{T}(\check{t},\check{x}_1,\check{x}_2)=\sum_{-\infty}^{\infty}e^{-\lambda^2\check{t}}A_m\cos(m\pi\check{x}_1)[\sin(\sqrt{\lambda^2-k^2}\,\check{x}_2)]$$

$$④ \Rightarrow \sin(\sqrt{\lambda^2-k^2})=0$$

$$n\pi=\sqrt{\lambda^2-k^2} \quad n=[0,\infty]$$
$$n\pi=\sqrt{\lambda^2-m^2\pi^2}$$
$$\lambda^2=(n^2+m^2)\pi^2$$

$$\therefore \check{T}=\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}e^{-(n^2+m^2)\pi^2\check{t}}A_{mn}\cos(m\pi\check{x}_1)\sin(n\pi\check{x}_2)$$

$$\tilde{T} = \sum_{n=0}^{\infty}\sum_{m=\infty}^{\infty} e^{-(n^2+m^2)\pi^2 \tilde{t}} A_{mn} \cos(m\pi \tilde{x}_1)\sin(n\pi \tilde{x}_2)$$

initial condition $T(t=0, x_1, x_2) = \frac{T_1 - T_0}{W^4}\left(\frac{1}{W^2}x_1^4 - 2\frac{1}{W}x_1^3 + x_1^2 + W^2\right)(x_2^2 - Wx_2) + T_0$

$$= (T_1 - T_0)\left(\frac{1}{W^4}x_1^4 - 2\frac{1}{W^3}x_1^3 + \frac{1}{W^2}x_1^2 + 1\right)\left(\frac{1}{W^2}x_2^2 - \frac{1}{W}x_2\right) + T_0$$

let $\tilde{T} = \frac{T - T_0}{T_1 - T_0}$, $x_1 = W\tilde{x}_1$, $x_2 = W\tilde{x}_2$, $t = \tilde{t}\, t_s$

Scaling analysis:

$$\tilde{T}(\tilde{t}=0, \tilde{x}_1, \tilde{x}_2) = (\tilde{x}_1^4 - 2\tilde{x}_1^3 + \tilde{x}_1^2 + 1)(\tilde{x}_2^2 - \tilde{x}_2)$$

$$\tilde{T}(\tilde{t}, \tilde{x}_1, \tilde{x}_2) = \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} e^{-(n^2+m^2)\pi^2\tilde{t}} A_{mn}\cos(m\pi\tilde{x}_1)\sin(n\pi\tilde{x}_2)$$

$$\tilde{T}(\tilde{t}=0, \tilde{x}_1, \tilde{x}_2) = \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} A_{mn}\cos(m\pi\tilde{x}_1)\sin(n\pi\tilde{x}_2) = (\tilde{x}_1^4 - 2\tilde{x}_1^3 + \tilde{x}_1^2 + 1)(\tilde{x}_2^2 - \tilde{x}_2)$$

let $A_{mn} = A_m A_n$

$$\sum_{m=-\infty}^{\infty} A_m \cos(m\pi\tilde{x}_1) = (\tilde{x}_1^4 - 2\tilde{x}_1^3 + \tilde{x}_1^2 + 1) \quad ①$$

$$\sum_{n=0}^{\infty} A_n \sin(n\pi\tilde{x}_2) = (\tilde{x}_2^2 - \tilde{x}_2) \quad ②$$

$① \Rightarrow$

$$\sum_{m=-\infty}^{\infty} A_m \cos(m\pi\tilde{x}_1) = (\tilde{x}_1^4 - 2\tilde{x}_1^3 + \tilde{x}_1^2 + 1)$$

$$\sum_{m=\infty}^{\infty} A_m \int_0^1 \cos(m\pi\tilde{x}_1)\cos(m'\pi\tilde{x}_1)\,d\tilde{x}_1 = \int_0^1 \cos(m'\pi\tilde{x}_1)(\tilde{x}_1^4 - 2\tilde{x}_1^3 + \tilde{x}_1^2 + 1)\,d\tilde{x}_1$$

$$A_m = \frac{\int_0^1 \cos(m\pi\tilde{x}_1)(\tilde{x}_1^4 - 2\tilde{x}_1^3 + \tilde{x}_1^2 + 1)\,d\tilde{x}_1}{\int_0^1 \cos^2(m\pi\tilde{x}_1)\,d\tilde{x}_1}$$

See next page for $A_n$

(2) $\Longrightarrow$ $$\sum_{n=0}^{\infty} A_n \sin(n\pi \tilde{X}_2) = (\tilde{X}_2^2 - \tilde{X}_2)$$

$$\sum_{n=0}^{\infty} A_n \int_0^1 \sin(n\pi \tilde{X}_2)\sin(n'\pi \tilde{X}_2)\,d\tilde{X}_2 = \int_0^1 (\tilde{X}_2^2 - \tilde{X}_2)\sin(n'\pi \tilde{X}_2)\,d\tilde{X}_2$$

↑

only when $n = n'$, the integral is non-zero

$$A_n = \frac{\int_0^1 \sin(n\pi \tilde{X}_2)(\tilde{X}_2^2 - \tilde{X}_2)\,d\tilde{X}_2}{\int_0^1 \sin^2(n\pi \tilde{X}_2)\,d\tilde{X}_2}$$

$$A_{mn} = \frac{\int_0^1 \cos(m\pi \tilde{X}_1)(\tilde{X}_1^4 - 2\tilde{X}_1^3 + \tilde{X}_1^2 + 1)\,d\tilde{X}_1 \int_0^1 \sin(n\pi \tilde{X}_2)(\tilde{X}_2^2 - \tilde{X}_2)\,d\tilde{X}_2}{\int_0^1 \cos^2(m\pi \tilde{X}_1)\,d\tilde{X}_1 \int_0^1 \sin^2(n\pi \tilde{X}_2)\,d\tilde{X}_2}$$

$$\tilde{T} = \sum_{n=0}^{\infty}\sum_{m=\infty}^{\infty} A_{mn} \cdot e^{-(n^2+m^2)\pi^2 \tilde{t}} \cos(m\pi \tilde{X}_1)\sin(n\pi \tilde{X}_2)$$