

COM SCI M238 (Quantum Programming)

October 18, 2022

1 9.22 0th

- first quantum computer working in 2016
- rn = 127 qubits
- google projected to have 1mil qubits by 2029
- largest simulation of a quantum computer by a classical computer: nasa, 70 (perfect) qubits, half a year
 - only need 37 for practical uses
- 1% err rate
 - current err corrections reqs ~1000 qubits to support 1 perfect qubit
- moores law of quantum computing
 - err rate improves linearly per qubit
- quantum volume for the decade: 100 qubits * 1000 ops = 100k ops
 - need to push decoherence time
- good problem: small input, lots of calculation, small output, easily verifiable
- double slit experiment
 - numbers of photons that come thru when either slit is covered are not additive
 - complex α_1 and α_2 for probability for either, can be negative, amplitude leq 1
 - eg $\alpha_1 = 1/\sqrt{2}$ and $\alpha_2 = -1/\sqrt{2}$
 - probability = $|\alpha|^2 = 1/2$
 - probability when both are uncovered = $|\alpha_1 + \alpha_2|^2 = 0$
- borns rule: when measured, a state with amplitude α is observed with probability $|\alpha|^2$

2 9.27 1t

	classical	quantum
software	boolean algebra	linear algebra
hardware	classical mechanics	quantum mechanics

2.1 4 postulates that define the interface between us and the qubits

1. state space rule

- $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- $\left(\frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \right)$

2. composition rule

- tensor product

3. step rule

- unitary matrix
- $U|\psi\rangle = |\varphi\rangle$
- $|\psi\rangle$ and $|\varphi\rangle$ are unit vectors of size 2^n
- U is $2^n \times 2^n$ but can be programmed in polynomial amount of code

4. measurement rule

- bit $\xrightarrow{\text{load}}$ quantum $\xrightarrow{\text{compute}}$ quantum $\xrightarrow{\text{measure}}$ bit

2.2 from classical computing to probabilistic computing to quantum computing

- classical

$$\text{- step: } \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

- rows: from 00, 01, 10, 11

- cols: to 00, 01, 10, 11

- probabilistic: model of the world with uncertainties

- state is a vector, taking a step = multiply by probability matrix

$$\text{- step: } \begin{bmatrix} 0 & 0 & 0 & 1/4 \\ 1 & 2/3 & 1/3 & 1/4 \\ 0 & 1/3 & 1/3 & 1/4 \\ 0 & 0 & 1/3 & 1/4 \end{bmatrix}$$

$$\text{- tensor product: } \begin{bmatrix} p \\ 1-p \end{bmatrix} \otimes \begin{bmatrix} q \\ 1-p \end{bmatrix} \rightarrow \begin{bmatrix} pq \\ p(1-q) \\ (1-p)q \\ (1-p)(1-q) \end{bmatrix}$$

$$\text{- "not gate": } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} q \\ p \end{bmatrix}$$

$$\text{- fair flip: } \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(p+q) \\ \frac{1}{2}(p+q) \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

$$\text{- } \begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix} \text{ can never equal } \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{bmatrix}$$

$$\text{- } \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = (ac)(bd) \neq (ad)(bc) = 0 \cdot 0 = 0$$

- comparison between probabilistic and quantum

	probabilistic	quantum
value	real	complex
state	vector of probabilities $\sum p_i = 1$	vector of amplitudes $\sum a ^2 = 1$
step	stochastic matrix	unitary matrix

– quantum

– fair flip: $H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

– $H|0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

– $H|1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

– distinguishing $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$:

– $H \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = |0\rangle$

– $H \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = |1\rangle$

– $f: \{0,1\} \rightarrow \{0,1\}$

2.3 encoding a function to be invertible

- encoding of $f: \{0,1\}^2 \rightarrow \{0,1\}^2$
 - $U_f(x, b) = (x, b \oplus f(x))$ invertible
 - $(U_f \circ U_f)(x, b) = U_f(U_f(x, b)) = U_f(x, b \oplus f(x)) = (x, b \oplus f(x) \oplus f(x)) = (x, b)$

3 9.29 1th

3.1 complex numbers

- $a + ib$
- $\overline{a + ib} = a - ib$
- $e^{i\theta} = \cos \theta + i \sin \theta$
- $\overline{e^{i\theta}} = \cos \theta - i \sin \theta = e^{-i\theta}$

3.2 hilbert space

- complex vector space w inner product

- $\left\langle \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \right\rangle = \overline{\alpha_1} \beta_1 + \overline{\alpha_2} \beta_2 = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}^* \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} \overline{\alpha_1} & \overline{\alpha_2} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$
- write $|\psi\rangle = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$ and $\langle\varphi| = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$
- bra-ket notation: $\langle\psi| = \begin{bmatrix} \overline{\alpha_1} & \overline{\alpha_2} \end{bmatrix}$
- inner product: $\langle\psi|\varphi\rangle$
- $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
- $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$
- $H|0\rangle = |+\rangle$
- $H|1\rangle = |-\rangle$
- $\langle 0|1\rangle = 1^* \cdot 0 + 0^* \cdot 1 = 0$
- $\langle +|-\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{2}(1 \cdot 1 + 1 \cdot (-1)) = 0$
- outer product: $|\psi\rangle\langle\varphi| = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \begin{bmatrix} \overline{\beta_1} & \overline{\beta_2} \end{bmatrix} = \begin{bmatrix} \alpha_1 \overline{\beta_1} & \alpha_1 \overline{\beta_2} \\ \alpha_2 \overline{\beta_1} & \alpha_2 \overline{\beta_2} \end{bmatrix}$
- a matrix U is unitary iff $UU^* = I$ (equivalent to $U^*U = I$)

3.3 partial measurement

- start state: $|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$
- measure qubit 1 $\rightarrow 0$
- new state: $\frac{\alpha_{00}|00\rangle + \alpha_{01}|01\rangle}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}} = |0\rangle \otimes \frac{\alpha_{00}|0\rangle + \alpha_{01}|1\rangle}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}}$

3.4 generalization of tensor product

- outer product: $|\psi\rangle\langle\varphi| = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \otimes \begin{bmatrix} \overline{\beta_1} & \overline{\beta_2} \end{bmatrix} = \begin{bmatrix} \alpha_1 \overline{\beta_1} & \alpha_1 \overline{\beta_2} \\ \alpha_2 \overline{\beta_1} & \alpha_2 \overline{\beta_2} \end{bmatrix}$
- $\begin{bmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{bmatrix} \otimes B = \begin{bmatrix} \alpha_{00}B & \alpha_{01}B \\ \alpha_{10}B & \alpha_{11}B \end{bmatrix}$
- $|00\rangle = |0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
- $\underbrace{|101\rangle}_{\text{5 in decimal}} = |1\rangle \otimes |0\rangle \otimes |1\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$
- inner product \rightsquigarrow matrix product
- outer product \rightsquigarrow matrix product + tensor product

- \otimes associative: $A \otimes (B + C) = A \otimes B + A \otimes C$
- $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$
- “floating scalar rule”: $(\alpha A) \otimes B = A \otimes (\alpha B) = \alpha(A \otimes B)$
- $|\psi\rangle \cdot \langle\varphi| \cdot |\gamma\rangle = |\psi\rangle \cdot \langle\varphi|\gamma\rangle = \langle\varphi|\gamma\rangle \cdot |\psi\rangle$

4 10.4 2t

- todo: wire drawing
- $|0\rangle \otimes |0\rangle \otimes |0\rangle = |000\rangle$
- $(I \otimes CNOT)(I \otimes X \otimes I)(H \otimes I \otimes I) \underbrace{|000\rangle}_{\text{vector}}$
- it is more error-prone to do I than X or H
 - qiskit etc will put single operations that cancel out instead
- todo: drawing - cnot but ctrl qbit on top
- $(I \otimes S)(CNOT \otimes I)(I \otimes S)|000\rangle$
 - where S is swap
- bob creates $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and sends one of the qubits to alice
- alice has 2 bits ab
 - if $a = 1$, alice applies Z to A
 - $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
 - if $b = 1$, alice applies X to A
 - send A to bob
- bob
 - $CNOT(A, B)$
 - apply H to A
 - measure A, B

ab	alice 1	alice 2	bob 1	bob 2	bob measure
00	$\frac{1}{\sqrt{2}}(00\rangle + 11\rangle)$	$\frac{1}{\sqrt{2}}(00\rangle + 11\rangle)$	$ +0\rangle$	$ 00\rangle$	00
01	$\frac{1}{\sqrt{2}}(00\rangle + 11\rangle)$	$\frac{1}{\sqrt{2}}(10\rangle + 01\rangle)$	$ +1\rangle$	$ 01\rangle$	01
10	$\frac{1}{\sqrt{2}}(00\rangle - 11\rangle)$	$\frac{1}{\sqrt{2}}(00\rangle - 11\rangle)$	$ -0\rangle$	$ 10\rangle$	10
11	$\frac{1}{\sqrt{2}}(00\rangle - 11\rangle)$	$\frac{1}{\sqrt{2}}(10\rangle - 01\rangle)$	$- -1\rangle$	$- 11\rangle$	11

4.1 quantum teleportation

- $\underbrace{|01\rangle}_{\text{Alice}} \underbrace{|0\rangle}_{\text{Bob}}$, last 2 are a bell pair
- alice has $\alpha|0\rangle + \beta|1\rangle$

- start state: $(\alpha |0\rangle + \beta |1\rangle)_A \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{BC}$
- alice
 - todo: drawing
 - $CNOT(A, B)$
 - $H(A)$
 - measure
 - $\frac{1}{\sqrt{2}}(\alpha |000\rangle + \alpha |011\rangle + \beta |100\rangle + \beta |111\rangle) \rightarrow \frac{1}{\sqrt{2}}(\alpha |000\rangle + \alpha |011\rangle + \beta |110\rangle + \beta |101\rangle)$
 - $\rightarrow \frac{1}{2}(\alpha |001\rangle + \alpha |101\rangle + \alpha |011\rangle + \alpha |111\rangle + \beta |010\rangle - \beta |110\rangle + \beta |000\rangle - \beta |100\rangle)$
 - $\rightarrow \frac{1}{2}(|00\rangle \otimes (\alpha |0\rangle + \beta |1\rangle) + |01\rangle \otimes (\beta |0\rangle + \alpha |1\rangle) + |10\rangle \otimes (\alpha |0\rangle - \beta |1\rangle) + |11\rangle \otimes (-\beta |0\rangle + \alpha |1\rangle))$
 - $\frac{1}{4}$ probability of sending any of the 4
 - $00 \rightarrow \alpha |0\rangle + \beta |1\rangle$
 - $01 \rightarrow \beta |0\rangle + \alpha |1\rangle$
 - $10 \rightarrow \alpha |0\rangle - \beta |1\rangle$
 - $11 \rightarrow -\beta |0\rangle + \alpha |1\rangle$
- bob
 - $b = 1 \Rightarrow X(C)$
 - $00 \rightarrow \alpha |0\rangle + \beta |1\rangle$
 - $01 \rightarrow \alpha |0\rangle + \beta |1\rangle$
 - $10 \rightarrow \alpha |0\rangle - \beta |1\rangle$
 - $11 \rightarrow \alpha |0\rangle - \beta |1\rangle$
 - $a = 1 \Rightarrow Z(C)$
 - $00 \rightarrow \alpha |0\rangle + \beta |1\rangle$
 - $01 \rightarrow \alpha |0\rangle + \beta |1\rangle$
 - $10 \rightarrow \alpha |0\rangle + \beta |1\rangle$
 - $11 \rightarrow \alpha |0\rangle + \beta |1\rangle$
- alice's qubit destroyed when measured

4.2 no cloning theorem

- no quantum operation maps $|\psi\rangle$ to $|\psi\rangle$
- suppose $U|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle$
- pick $|\psi_1\rangle, |\psi_2\rangle$ such that $\langle\psi_1|\psi_2\rangle \neq 0$ and $\langle\psi_1|\psi_2\rangle \neq 1$
- lemma: $\langle(v_1 \otimes v_2)|(w_1 \otimes w_2)\rangle = \langle v_1|w_1\rangle \cdot \langle v_2|w_2\rangle$
- $\langle\psi_1|\psi_2\rangle = \langle\psi_1|\psi_2\rangle \cdot \langle 0|0\rangle = \langle\psi_1 0|\psi_2 0\rangle = \langle U(|\psi_1 0\rangle), U(|\psi_2 0\rangle)\rangle = \langle\psi_1 \psi_1|\psi_2 \psi_2\rangle = \langle\psi_1|\psi_2\rangle \cdot \langle\psi_1|\psi_2\rangle$

4.3 universality

- $NAND(x_1, x_2) = CCNOT(x_1, x_2, 1)$
 - $CCNOT$ can simulate all of boolean logic
- $\{CCNOT, H\}$ is universal for all real unitaries
- $\left\{CCNOT, H, S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}\right\}$ is universal for all unitaries

- quantum computers implement $\{CNOT, H, T\}$
 - $S = T^2$

5 10.6 2th

5.1 deutsch-jozsa problem

- input: a function $f: \{0, 1\}^n \rightarrow \{0, 1\}$
- assumption: either f is constant or f is balanced
- probabilistically, just guess balanced
- 2^n inputs, try one after another, know it's constant after $2^{n-1} + 1$ tries
- $n = 1$
 - $f: \{0, 1\} \rightarrow \{0, 1\}$
 - $U_f: \text{qubit}^{\otimes 2} \rightarrow \text{qubit}^{\otimes 2}$
 - make unitary
 - $U_f(|x\rangle \otimes |b\rangle) = |x\rangle \otimes |b \oplus f(x)\rangle$

input	f_0	f_1	f_2	f_3
0	0	0	1	1
1	0	1	0	1

- $U_{f_0}(|0\rangle \otimes |b\rangle) = |0\rangle \otimes |b \oplus f(0)\rangle = |0b\rangle$
- $U_{f_0}(|1\rangle \otimes |b\rangle) = |1\rangle \otimes |b \oplus f(1)\rangle = |1b\rangle$
- U_{f_0} is the 4×4 identity matrix
- f is constant
 - $f(0) = 0 \wedge f(1) = 0 \vee f(0) = 1 \wedge f(1) = 1$
 - $f(0) \oplus f(1) = 0$
- f is balanced
 - $f(0) \oplus f(1) = 1$
- idea 1: use superposition of $|0\rangle, |1\rangle$
- observation: U_f moves $f(0), f(1)$ to the exponent so we can do addition
- idea 2: H will move $f(0) \oplus f(1)$ “back down”
- deutsch algorithm
- todo: drawing
- $\text{measure}_0((H \otimes I)U_f(H \otimes H)|01\rangle) = \begin{cases} 0 & f \text{ constant} \\ 1 & f \text{ balanced} \end{cases}$
- lemma 1: $\forall a \in \{0, 1\} : |0 \oplus a\rangle - |1 \oplus a\rangle = (-1)^a(|0\rangle - |1\rangle)$
- lemma 2: $\forall x \in \{0, 1\}^n : U_f(|x-\rangle) = (-1)^{f(x)}|x-\rangle$
 - $U_f(|x-\rangle) = \frac{1}{\sqrt{2}}(U_f(|x0\rangle) - U_f(|x1\rangle)) = \frac{1}{\sqrt{2}}(|x\rangle \otimes |0 \oplus f(x)\rangle - |x\rangle \otimes |1 \oplus f(x)\rangle)$
 - $= |x\rangle \otimes \frac{1}{\sqrt{2}}(|0 \oplus f(x) - |1 \oplus f(x)\rangle) = |x\rangle \otimes \frac{1}{\sqrt{2}}(-1)^{f(x)}(|0\rangle - |1\rangle) = (-1)^{f(x)}|x-\rangle$
- lemma 3: $\forall a \in \{0, 1\} : H\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}(-1)^a|1\rangle\right) = |a\rangle$

- $(H \otimes I)U_f(H \otimes H)|01\rangle = (H \otimes I)U_f(|+-\rangle) = (H \otimes I)U_f\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |-\rangle$
- $= (H \otimes I)\frac{1}{\sqrt{2}}U_f\left(\sum_{x \in \{0,1\}} |x-\rangle\right) = (H \otimes I)\frac{1}{\sqrt{2}}\sum_{x \in \{0,1\}} (-1)^{f(x)} |x-\rangle$
- $= (H \otimes I)\frac{1}{\sqrt{2}}((-1)^f(0)|0\rangle + (-1)^{f(1)}|1\rangle) \otimes |-\rangle = (H \otimes I)\frac{1}{\sqrt{2}}(-1)^{f(0)}(|0\rangle + (-1)^{f(0) \oplus f(1)}|1\rangle) \otimes |-\rangle$
- $= (-1)^{f(0)}|f(0) \oplus f(1)\rangle \otimes |-\rangle$
- norm: $|(-1)^{f(0)}|^2 = 1$

5.2 deutsch-jozsa algorithm

- $n > 1$
- todo: drawing
- $\text{measure}_{0:n}((H^{\otimes n} \otimes I)U_f H^{\otimes(n+1)}(|0\rangle^{\otimes n} \otimes |1\rangle))$ gives an n -bit bitstring
- lemma 4 (todo: 5???: $\forall x \in \{0,1\} : H(|x\rangle) = \frac{1}{\sqrt{2}}\sum_{y \in \{0,1\}} (-1)^{x \cdot y} |y\rangle$)
 - $\text{RHS} = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x|1\rangle)$, then see lemma 3
- lemma 5: $H^{\otimes n}(|x\rangle) = \frac{1}{\sqrt{2^n}}\sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle$
 - $H^{\otimes n}(|0\rangle) = H(|x_1\rangle) \otimes \dots \otimes H(|x_n\rangle) = \frac{1}{\sqrt{2}}\sum_{y_1 \in \{0,1\}} |y_1\rangle \otimes \dots \otimes \frac{1}{\sqrt{2}}\sum_{y_n \in \{0,1\}} |y_n\rangle$
 - $= \frac{1}{\sqrt{2^n}}\sum_{y_1 \in \{0,1\}} \dots \sum_{y_n \in \{0,1\}} (-1)^{x_1 y_1} \dots (-1)^{x_n y_n} |y_1 \dots y_n\rangle = \frac{1}{\sqrt{2^n}}\sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle$
- $(H^{\otimes n} \otimes I)U_f H^{\otimes(n+1)}(|0\rangle^{\otimes n} \otimes |1\rangle) = (H^{\otimes n} \otimes I)U_f\left(\frac{1}{\sqrt{2^n}}\sum_{x \in \{0,1\}^n} |x\rangle\right) \otimes |-\rangle$
- $= (H^{\otimes n} \otimes I)\frac{1}{\sqrt{2^n}}\left(\sum_{x \in \{0,1\}^n} (-1)^{f(x)} |x\rangle\right) \otimes |-\rangle$
- $= \frac{1}{\sqrt{2^n}}\left(\sum_{x \in \{0,1\}^n} \frac{1}{\sqrt{2^n}}\sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle\right) \otimes |-\rangle$
- $= \frac{1}{2^n}\left(\sum_x \sum_y (-1)^{f(x) \oplus x \cdot y} |y\rangle\right) \otimes |-\rangle$
- for $|y\rangle = |0\rangle^{\otimes n} |-\rangle$, only final state $= \frac{1}{2^n}(\sum_x (-1)^{f(x)} |0\rangle^{\otimes n}) \otimes |-\rangle$

6 10.11 3t

6.1 bernstein-vazirani problem

- input: a function $f: \{0,1\}^n \rightarrow \{0,1\}$
- assumption: $f(x) = (a \cdot x) \oplus b$
- output: a, b
- $f(0 \dots 0) = b$
- $f(0 \dots 01) = a_n \oplus b$
- $a = 0 \dots 0 \Rightarrow f(x) = b$: f is constant
- $a \neq 0 \dots 0$: f is balanced
 - every input has a sister input (e.g. flipping the last bit) that flips the output
- ideas
 - superposition
 - U_f will move something to the exponent (of -1)
 - $H^{\otimes n}$ will move the exponent back down

- goal: final state $|a\rangle$
- circuit
 - todo: drawing
 - $\text{measure}_{1:n}((H^{\otimes n} \otimes I)U_f H^{\otimes(n+1)}(|0 \dots 01\rangle))$
- $\frac{1}{2^n} \sum_{x \in \{0,1\}^n} \sum_{y \in \{0,1\}^n} (-1)^{(x \cdot y) \oplus f(x)} |y\rangle = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} \sum_{y \in \{0,1\}^n} (-1)^{(x \cdot y) \oplus (a \cdot x) \oplus b} |y\rangle$
 - $= \frac{1}{2^n} \sum_{x \in \{0,1\}^n} \sum_{y \in \{0,1\}^n} (-1)^{(x \cdot (y \oplus a)) \oplus b} |y\rangle = \frac{(-1)^b}{2^n} \sum_{x \in \{0,1\}^n} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot (y \oplus a)} |y\rangle$
 - amplitude of $|a\rangle = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{x \cdot (a \oplus a)} = (-1)^b$
 - then original expression $= (-1)^b |a\rangle$

6.2 another problem

- input: $f: \{0,1\}^2 \rightarrow \{0,1\}$
- assumption: f is 1 on a single input
- output: the single input
- circuit
 - goal: $|cd\rangle$
 - todo: drawing
 - $\text{measure}_{1:n}((V \otimes I)U_f H^{\otimes 3} |0 \dots 01\rangle)$
 - need to find V
 - let
 - $\varphi_{00} = \frac{1}{2} (-|00\rangle + |01\rangle + |10\rangle + |11\rangle)$
 - $\varphi_{01} = \frac{1}{2} (|00\rangle - |01\rangle + |10\rangle + |11\rangle)$
 - $\varphi_{10} = \frac{1}{2} (|00\rangle + |01\rangle - |10\rangle + |11\rangle)$
 - $\varphi_{11} = \frac{1}{2} (|00\rangle + |01\rangle + |10\rangle - |11\rangle)$
 - $(V \otimes I)U_f H^{\otimes 3} |001\rangle = (V \otimes I)U_f \frac{1}{2} (|00-\rangle + |01-\rangle + |10-\rangle + |11-\rangle)$
 - $= (V \otimes I) \frac{1}{2} ((-1)^{f(00)} |00-\rangle + (-1)^{f(01)} |01-\rangle + (-1)^{f(10)} |10-\rangle + (-1)^{f(11)} |11-\rangle)$
 - $= (V \otimes I)(\varphi_{cd} \otimes |-\rangle)$
 - want $V(\varphi_{cd}) = |cd\rangle$
 - $V = \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$
 - $\left(= \begin{bmatrix} \varphi_{00} & \varphi_{01} & \varphi_{10} & \varphi_{11} \end{bmatrix}^{-1} \right)$

7 10.13 3th

7.1 simon's problem

- input: $f: \{0,1\}^n \rightarrow \{0,1\}^n$
- assumption: $\exists s \in \{0,1\}^n \forall x, y : f(x) = f(y) \Leftrightarrow (x + y) \in \{0^n, s\}$
- output: s

- $n = 3, s = 110$

x	$f(x)$
000	101
001	010
010	000
011	110
100	000
101	110
110	101
111	010

7.2 simon's alg

- repeat $f \rightarrow$ quantum generate \rightarrow equations \rightarrow classical solve $\rightarrow s$ until the chance of success is high
- $y_1 \cdot s = 0, \dots, y_{n-1} \cdot s = 0$
- ideally the y_i 's are linearly independent
- $P(y_i$'s are linearly independent) $> \frac{1}{4}$

7.3 events

- E_1 : y_1 is not 0
- for $k = 2, \dots, n-1$:
 - E_k : y_k is not in the span of y_1, \dots, y_{k-1}
- E : y_1, \dots, y_{n-1} are linearly independent
- sample from a space of size 2^{n-1}
- $P(E_1) = P(E_1 \wedge \dots \wedge E_n) = P(E_1) \cdot P(E_2 \wedge \dots \wedge E_{n-1} \mid E_1) = P(1) \cdot \prod_{k=2}^n P(E_k \mid E_1 \wedge \dots \wedge E_{k-1})$
 - $P(E_1) = 1 - 1/2^{n-1}$
 - $\prod_k = 1 - 2^{k-1}/2^{n-1}$
 - $= (1 - 1/2^{n-1}) \dots (1 - 1/2) > 1/4$
 - for all $k = 1, \dots, n-1$: $(1 - 1/2^{n-1}) \dots (1 - 1/2^{n-(k-1)}) > (1 - 1/2^{n-k})$
 - for $k+1$
 - lhs: $(1 - 1/2^{n-1}) \dots (1 - 1/2^{n-k}) > (1 - 1/2^{n-k})^2 = \left(\frac{2^{n-k}-1}{2^{n-k}}\right)^2 = \frac{2^{2n-2k}-2^{n-(k-1)}+1}{2^{2n-2k}}$
 - rhs: $1 - 1/2^{n-(k+1)} = \frac{2^{n-(k+1)}-1}{2^{n-(k+1)}} = \frac{2^{2n+2k}-2^{n-(k-1)}}{2^{2n+2k}}$
- run the whole thing $4m$ times
- $P(\text{failure}) < \left(1 - \frac{1}{4}\right)^{4m} < e^{-m}$

7.4 circuit

- todo: drawing
- $U_f |x\rangle \otimes |b\rangle = |x\rangle \otimes |b \oplus f(x)\rangle$

- $\text{measure}_{1:n}(H^{\otimes n} \otimes I^{\otimes n})U_f(H^{\otimes n} \otimes I^{\otimes n})|0^n\rangle \otimes |0^n\rangle$
- $= (H^{\otimes n} \otimes I^{\otimes n})U_f\left(\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle\right) \otimes |0^n\rangle$
- $= (H^{\otimes n} \otimes I^{\otimes n})\frac{1}{\sqrt{2^n}} \left(\sum_{x \in \{0,1\}^n} |x\rangle\right) \otimes |f(x)\rangle$
- $= \frac{1}{2^n} \sum_{x \in \{0,1\}^n} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle \otimes |f(x)\rangle$
- $= \sum_{y \in \{0,1\}^n} |y\rangle \left(\frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{x \cdot y} |f(x)\rangle\right)$
- $s = 0$
 - $\left|\frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{x \cdot y} |f(x)\rangle\right|^2$
 - f is injective so $|f(x)\rangle$ is an orthonormal basis \rightarrow use pythagorean theorem
 - $= \frac{1}{2^{2n}} \sum_{x \in \{0,1\}^n} |(-1)^{x \cdot y} |f(x)\rangle|^2$
 - $= \frac{1}{2^n}$ uniform distribution
- $s \neq 0$
 - y is drawn from a set A of size 2^{n-1}
 - for $z \in A$, we have $x_z, x'_z \in \{0,1\}^n$ where $f(x_z) = f(x'_z) = z$ and $x_z \oplus x'_z = s$
 - $\left|\frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{x \cdot y} |f(x)\rangle\right|^2 = \left|\frac{1}{2^n} \sum_{z \in A} ((-1)^{x_z \cdot y} + (-1)^{x'_z \cdot y}) |z\rangle\right|^2$
 - $= \left|\frac{1}{2^n} \sum_{z \in A} (-1)^{x_z \cdot y} (1 + (-1)^{s \cdot y}) |z\rangle\right|^2$
 - todo: $|z\rangle$ s are orthogonal???
 - $= \begin{cases} 2^{1-n} & s \cdot y = 0 \\ 0 & s \cdot y = 1 \end{cases}$ uniform distribution

8 10.18 4t

8.1 grover's problem

- input: $f: \{0,1\}^n \rightarrow \{0,1\}$
- output: $\begin{cases} 1 & \exists x \in \{0,1\}^n : f(x) = 1 \\ 0 & \text{otherwise} \end{cases}$
- $Z_f |x\rangle = (-1)^{f(x)} |x\rangle$
- $Z_0 |x\rangle = \begin{cases} -|x\rangle & x = 0^n \\ |x\rangle & x \neq 0^n \end{cases}$
 - todo: drawing
 - NAND into 1 qubit, use Z gate, uncompute NAND
- $G = -H^{\otimes n} Z_0 H^{\otimes n} Z_f$

8.2 grover's algorithm

- x : n qubits, initially $|0^n\rangle$
- 1. apply $H^{\otimes n}$ to x
- 2. repeat (apply G to x) $O(\sqrt{2^n})$ times
- 3. measure x and output the result

8.3 example $n = 2$

$$\begin{aligned}
& - f: \{0,1\}^2 \rightarrow \{0,1\} \\
& - f(00) = f(01) = f(10) = 0 \\
& - f(11) = 1 \\
& - H^{\otimes 2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle) = \frac{1}{2} ((|00\rangle + |01\rangle + |10\rangle + |11\rangle) \\
& \quad + (|00\rangle - |01\rangle + |10\rangle - |11\rangle) \\
& \quad + (|00\rangle + |01\rangle - |10\rangle - |11\rangle) \\
& \quad - (|00\rangle - |01\rangle - |10\rangle + |11\rangle)) = |00\rangle + |01\rangle + |10\rangle - |11\rangle \\
& - GH^{\otimes 2}|00\rangle = G \frac{1}{2} (|00\rangle + |01\rangle + |10\rangle + |11\rangle) \\
& \quad = -H^{\otimes 2} Z_0 H^{\otimes 2} Z_f \frac{1}{2} (|00\rangle + |01\rangle + |10\rangle + |11\rangle) \\
& \quad = -H^{\otimes 2} Z_0 H^{\otimes 2} \frac{1}{2} (|00\rangle + |01\rangle + |10\rangle - |11\rangle) \\
& \quad = -H^{\otimes 2} Z_0 \frac{1}{2} (|00\rangle + |01\rangle + |10\rangle - |11\rangle) \\
& \quad = -H^{\otimes 2} \frac{1}{2} (-|00\rangle + |01\rangle + |10\rangle - |11\rangle) \\
& \quad = -H^{\otimes 2} \frac{1}{2} (-2|00\rangle + (|00\rangle + |01\rangle + |10\rangle - |11\rangle)) \\
& \quad = -\frac{1}{2} \left(-2 \cdot \frac{1}{2} (|00\rangle + |01\rangle + |10\rangle + |11\rangle) + (|00\rangle + |01\rangle + |10\rangle - |11\rangle) \right) \\
& \quad = |11\rangle
\end{aligned}$$

8.4 example

$$\begin{aligned}
& - \text{notation} \\
& \quad - A = \{x \in \{0,1\}^n : f(x) = 1\} \\
& \quad - B = \{x \in \{0,1\}^n : f(x) = 0\} \\
& \quad - N = 2^n, a = |A|, b = |B| \\
& \quad - |A\rangle = \frac{1}{\sqrt{a}} \sum_{x \in A} |x\rangle \\
& \quad - |B\rangle = \frac{1}{\sqrt{b}} \sum_{x \in B} |x\rangle \\
& - |A\rangle \text{ and } |B\rangle \text{ are orthogonal} \\
& - \text{lemma} \\
& \quad - G|A\rangle = \left(1 - \frac{2a}{N}\right) |A\rangle - \frac{2\sqrt{ab}}{N} |B\rangle \\
& \quad - G|B\rangle = \frac{2\sqrt{ab}}{N} |A\rangle - \left(1 - \frac{2b}{N}\right) |B\rangle \\
& \quad - \text{that is, } \text{span}\{|A\rangle, |B\rangle\} \text{ is closed under } G \\
& \quad - |h\rangle = H^{\otimes n} |0^n\rangle = \frac{1}{\sqrt{N}} \sum_{x \in \{0,1\}^n} |x\rangle = \frac{\sqrt{a}}{\sqrt{N}} |A\rangle + \frac{\sqrt{b}}{\sqrt{N}} |B\rangle \\
& \quad - Z_0 = \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I - 2|0^n\rangle\langle 0^n|
\end{aligned}$$

- $H^{\otimes n} Z_0 H^{\otimes n} = H^{\otimes n} (I - 2|0^n\rangle\langle 0^n|) H^{\otimes n}$

$$= H^{\otimes n} I H^{\otimes n} - 2H^{\otimes n} |0^n\rangle\langle 0^n| H^{\otimes n}$$

$$= I - 2|h\rangle\langle h|$$
- $G|A\rangle = -H^{\otimes n} Z_0 H^{\otimes n} Z_f |A\rangle$

$$= (I - 2|h\rangle\langle h|)(-Z_f)|A\rangle$$

$$= (I - |h\rangle\langle h|)|A\rangle$$

$$= |A\rangle - 2|h\rangle\langle h|A\rangle$$

$$= |A\rangle - 2\langle h|A\rangle|h\rangle \in \text{span}\{|A\rangle, |B\rangle\}$$

$$= |A\rangle - 2 \cdot \frac{\sqrt{a}}{\sqrt{N}} \cdot \left(\frac{\sqrt{a}}{\sqrt{N}}|A\rangle + \frac{\sqrt{b}}{\sqrt{N}}|B\rangle \right)$$

$$= \left(1 - \frac{2a}{N}\right)|A\rangle - \frac{2\sqrt{ab}}{\sqrt{N}}|B\rangle$$
- $M = G_{\{|B\rangle, |A\rangle\}} = \begin{bmatrix} -\left(1 - \frac{2b}{N}\right) & -\frac{2\sqrt{ab}}{N} \\ \frac{2\sqrt{ab}}{N} & 1 - \frac{2a}{N} \end{bmatrix}$
- $\left(\frac{\sqrt{a}}{\sqrt{N}}\right)^2 + \left(\frac{\sqrt{b}}{\sqrt{N}}\right)^2 = 1$
- let θ be such that $\sin \theta = \frac{\sqrt{a}}{\sqrt{N}}$ and $\cos \theta = \frac{\sqrt{b}}{\sqrt{N}}$
- $R_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \Rightarrow R_\theta^2 = M$
- after k iterations: $\sin((2k+1)\theta)|A\rangle + \cos((2k+1)\theta)|B\rangle$
- want $\sin((2k+1)\theta) \approx 1$
 - $(2k+1)\theta \approx \frac{\pi}{2}$
 - $2k+1 \approx \frac{\pi}{2\theta}$
 - $k \approx \frac{\pi}{4\theta} - \frac{1}{2}$
 - for $a=1$, $\theta \approx \sin \theta = \frac{1}{\sqrt{N}}$
 - $k \approx \frac{\pi\sqrt{N}}{4} - \frac{1}{2} \approx \sqrt{N}$
- previous example: $\sin \theta = \frac{\sqrt{1}}{\sqrt{4}} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6} \Rightarrow (2+1)\theta = \frac{\pi}{2}$