# ECE 113 (Digital Signal Processing)

October 14, 2022

# 1 9.26 1m

- syllabus

## 1.1 ece102: signals & systems

- signals were modeled as functions of time
  - $x(t) : t \in \mathbb{R}$ (continuous time)
  - $x : \mathbb{R} \to \mathbb{R}, \mathbb{C}$
  - sampling $t = nT_s$
    - $T_s$ = sampling interval
    - $n$ = integer
    - $t - x$ plot $\xrightarrow{\text{sampling}}$ $n - x$ plot
    - $x(t)$ bandlimited ($|X(f)| \leq f_{\max}$) then if sampling rate is $f_s > 2f_{\max}$ (nyquist rate) then it is possible to recover the original signal
    - sampling at lower rate then there will be aliasing
- sytems
  - $x(t) \to \boxed{S} \to y(t)$
  - $x[n] \to \boxed{S} \to y[n]$

## 1.2 three ways tor write out signals

- mathematical expression
  - $x[n] = 3\cos(2n), n \in \mathbb{Z}$
- tabular list of significant samples
  - $x[n] = \{-1, 8, 0, 5, -2\}, n \in \mathbb{Z}$
    - underspecified signal: loss of indices
    - typically put an arrow underneath the 0 index
    - convention: any sample not listed has 0 amplitude
- plotting

## 1.3 signal operations

- arithmetic
  - $g[n] = x[n] + A$
  - $g[n] = Bx[n]$
  - signal addition: $g[n] = x_1[n] + x_2[n]$
  - signal multiplication: $g[n] = x_1[n] \cdot x_2[n]$
- time shifting
  - ece102

    - $x(t) \to x(t-\tau)$ delays signal by shifting it right
    - $x(t) \to x(t+\tau)$ advances signal by shifting left
  - ece113
    - $g[n] = x[n-k], k > 0$
- time scaling
  - ece102
    - $x(at), a > 1$: contraction
    - $x(at), 0 < a < 1$: dilation
  - ece113
    - $g[n] = x[2n]$ (downsampling)
    - $g[n] = x\left[\frac{1}{2}n\right]$ (upsampling)
    - $g[n] = x[-n]$ (time reversal)

### 1.4 basic signals

- $\delta(t)$ from ece102 $\to$ hard concept to grasp
- $\delta[t]$ from ece113 $\to$ : )
  - $\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$
  - $\delta[n]$ does not have complexities of $\delta(t)$
- sampling property of $\delta$
  - $x[n] * \delta[n] = x[0] \cdot \delta[0] = x[0]$
  - $x[n] * \delta[n-k] = x[k] \cdot \delta[0] = x[k]$

## 2 9.28 1w

### 2.1 sampling properties hold in discrete domain as well

- $\sum_{n=-\infty}^{\infty} x[n] \cdot \delta[n-k] = \sum_n x[k]\delta[n-k] = x[k] \sum_n \delta[n-k] = x[k]$

### 2.2 systems

- unit step system $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$
  - $u[n] - u[n-1] = \delta[n]$
  - $\delta[n]$ is the first derivative of $u[n]$
  - ece102: $\delta(t) = \frac{du(t)}{dt}$
  - $u[n] = \sum_{k=0}^{\infty} \delta[n-k]$
- unit ramp signal $r[t] = \begin{cases} n & n \geq 0 \\ 0 & n < 0 \end{cases}$

  - 102: $r(t) = tu(t)$
  - $r[n] = n \cdot u[n] = \sum_{k=1}^{\infty} u[n-k]$

### 2.3 how wan se express $x[n]$ using basic signals?

- generic signal representation using the "canonical basis" of $\{\delta[n-k] \forall k\}$
- $x[n] = \{1, 2, (0), 1, 1\}$
- $x[n] = \delta[n+2] + 2\delta[n+1] + \delta[n-1] - \delta[n-2]$

### 2.4 sinusoidal signals

- $x(t) = A\cos(\omega_0 t + B)$, $\omega_0 = 2\pi f_0$
- $x[n] = A\cos(\Omega_0 n + \theta)$, $\Omega_0 = 2\pi F_0 \in [-\pi, \pi)$ where $F_0$ is the normalized frequency
- $x[n] A\cos(2\pi f_0 n T_0 + \theta) = A\cos\left(2\pi \frac{f_0}{f_s} n + \theta\right)$
- $F_n := \frac{f_0}{f_s} < 1$

### 2.5 complex exponential

- $x[n] = Ae^{j(2\pi F_0 n + \theta)}$

### 2.6 additional signal properties

1. duration of signal
2. real or complex
3. periodicity: a signal is periodic if $\exists N \in \mathbb{Z} : x[n+N] = x[n] \; \forall n$

- $x[n]$ is also periodic with $2N, 3N$, etc.
- $x[n+kN] = x[n] \; \forall k, n$
- fundamental period: smallestpositive $N$ such that $x[n+N] = x[n]$
- e.g. $x[n] = \cos(2\pi F_0 n) \rightarrow \exists k \in \mathbb{Z} : 2\pi F_0 N = 2\pi k \rightarrow F_0 N \in \mathbb{Z}, N = \frac{k}{F_0}$
- is $x[n] = \cos(0.2n)$ periodic?
  - no
  - $2\pi F_0 = 0.2 \rightarrow \frac{k}{F_0} = 10k\pi \notin \mathbb{Z}$

## 3 10.3 2m

### 3.1 recap

- recall: a signal is periodic if $x[n+N] = x[n]$ for all $n$ and some $N \in \mathbb{Z}$
- is $\cos(0.2n)$ periodic?
  - no due to irrational period
- is the sum of $x_1$ and $x_2$ periodic if they are periodic signals?

### 3.2 periodicity

- $x_1[n] = 2\cos(0.4\pi n), x_2[n] = 1.5\cos(0.48\pi n)$
- $F_1 = \frac{0.4\pi}{2\pi} = 0.2$
- $F_2 = \frac{0.48\pi}{2\pi} = 0.24$
- $N_1 = \frac{K_1}{F_1} = \frac{K_1}{0.2} = 5K_1 = 5$
- $N_2 = \frac{K_2}{F_2} = \frac{K_2}{24/100} = \frac{100}{24}K_2 = \frac{25}{6}K_2 = 25$
- fact: sum of two periodic signals is itself a periodic signal
- $x_3[n] := x_1[n] + x_2[n]$ is periodic with period $N_1N_2$
  - $x_3[n + N_1N_2] = x_1[n + N_1N_2] + x_2[n + N_1N_2] = x_1[n] + x_2[n] = x_3[n]$
  - e.g. $N_1 = 5, N_2 = 25 \Rightarrow 2\cos(0.4\pi n) + 1.5\cos(0.48\pi n)$ has a period of $5 \cdot 25 = 125$
  - fundamental period: $\mathrm{lcm}(5, 25) = 25$
- e.g. $x[n]$ is periodic w $N = 12$
- are the following periodic
  - $y[n] = x[-n]$
    - yes: $y[n + 12] = x[-(n + 12)] = x[-n - 12] = y[n]$
  - $y[n] = x[n + 1]$
    - yes: $y[n + 12] = x[(n + 12) + 1] = x[n + 1] = y[n]$
  - $y[n] = x[3n]$
    - yes: $y[n + 4] = x[3(n + 4)] = x[3n + 12] = x[3n] = y[n]$
  - $y[n] = x^2[n]$
    - yes: $y[n + 12] = (x[n + 12])^2 = (x[n])^2 = y[n]$

### 3.3 even and odd signals

- even: $x[n] = x[-n]$
- odd: $x[n] = -x[-n]$ or $-x[n] = x[-n]$
- any signal can be decomposed into odd and even components
  - $x[n] = x_{\text{even}}[n] + x_{\text{odd}}[n]$
- $x_{\text{even}}[n] = \frac{x[n]+x[-n]}{2}$
- $x_{\text{odd}}[n] = \frac{x[n]-x[-n]}{2}$

### 3.4 complex signals

- $x[n] = x_R[n] + jx_I[n] = |x[n]|\, e^{j\angle x[n]}$
- $x^*[n] = x_R[n] - jx_I[n] = |x[n]|\, e^{-j\angle x[n]}$
- "even symmetry" for complex
  - $x[-n] = x^*[n]$
  - $x^*[-n] = x[n]$
- "odd antisymmetry" for complex
  - $x[-n] = -x^*[n]$
  - $x^*[-n] = -x[n]$

### 3.5 decompose of even / odd for complex signals

- $x[n] = x_e[n] + x_o[n]$
- $x_e[n] = \frac{x[n]+x^*[-n]}{2}$
- $x_o[n] = \frac{x[n]-x^*[-n]}{2}$

### 3.6 energy + power of a signal

- energy: $E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$
  - if $E_x < +\infty$: **finite energy signal**
  - e.g. $x[n] = \cos(2\pi 0.1n + 0.3\pi)$ is not finite energy
- power
  - consider a periodic signal with period $N$
  - its power is expressed as $P_x = \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2$
  - consider a non-periodic signal
  - its power is expressed as $P_x = \lim_{M\to\infty} \frac{1}{2M+1}\sum_{n=-M}^{M} |x[n]|^2$
- e.g. calculate the energy of $x[n] = (0.5)^n u[n]$
  - $E_x = \sum_{n=-\infty}^{\infty} |(0.5)^n u[n]|^2 = \sum_{n=-\infty}^{\infty}(0.5)^{2n}u^2[n] = \sum_{n=0}^{\infty}(0.5)^{2n} = \frac{1}{1-0.25} = \frac{4}{3}$

### 3.7 systems

- $x[n] \to \boxed{S} \to y[n]$
- properties
  - linearity
    - $y_1[n] = S\{x_1[n]\}$
    - $y_2[n] = S\{x_2[n]\}$
    - $x_3[n] = ax_1[n] + bx_2[n]$
    - $y_3[n] = S\{x_3[n]\}$
    - linear $\Rightarrow y_3[n] = ay_1[n] + by_2[n]$
    - $S\left\{\sum_{k=1}^{N} a_k x_k[n]\right\} = \sum_{k=1}^{N} a_k S\{x_k[n]\}$
  - time-invariance
    - $y[n] = S\{x[n]\}$
    - time invariance $\Rightarrow y[n-k] = S\{x[n-k]\}$
  - causality
    - if current output $y[n]$ depends only on the current and past input samples, the system is causal
    - e.g. $y[n] = x[n] - nx[n-2]$: causal
    - e.g. $y[n] = (n+1)x[n]$: causal (also memory-less)
    - e.g. $y[n] = x[n] + x[n+2]$: not causal
  - bounded input, bounded output (bibo) stability
    - if $|x[n]| < B$ for all $n$, then $|y[n]| < C$ for all $n$, $B, C$ are constants

    - e.g. $y[n] = 3x[n] + 2x[n-1]$
        - suppose $|x[n]| < B$
        - stable: $\left|y[n]\right| = |3x[n] + 2x[n-1]| \leq 3\,|x[n]| + 2\,|x[n-1]| < 5B$
    - e.g. $y[n] = x^2[n]$
        - suppose $|x[n]| < B$
        - stable: $\left|y[n]\right| = \left|x^2[n]\right| = |x[n]|^2 < B^2$
    - e.g. $y[n] = nx[n-1]$
        - not stable: as $n \to \infty$, $y \to \infty$
- e.g. $y[n] = 3x[n] + 2x[n-1]$
    - time invariant: $x_1[n] = x[n-k] \Rightarrow y_1[n] = 3x[n-k] + 2x[n-k-1] = y[n-k]$
- e.g. $y[n] = x^2[n]$
    - not linear
    - time invariant: $x_1[n] = x[n-k] \Rightarrow y_1[n] = x_1^2[n] = x^2[n-k] = y[n-k]$
- e.g. $y[n] = nx[n-1]$
    - not time invariant: $x_1[n] = x[n-k] \Rightarrow y_1[n] = nx_1[n-1] = nx[n-1-k] \neq (n-k)x[n-1-k] = y[n-k]$
- e.g. $y[n] = x[n/4]$
    - linear
        - $y_1[n] = x_1[n/4]$
        - $y_2[n] = x_2[n/4]$
        - $x_3[n] = ax_1[n] + bx_2[n]$
        - $y_3[n/4] = x_3[n/4] = ax_1[n/4] + bx_3[n/4] = ay_1[n] + by_2[n]$
    - not time invariant
    - not causal
        - $n = -4 \Rightarrow y[-4] = x[-1]$
    - stable
        - suppose $|x[n]| < B$
        - $\left|y[n]\right| = |x[n/4]| < B$

# 4 10.5 2w

## 4.1 hw q4

- suppose $x[n]$ periodic with period $N$, is the following periodic
- $y[n] = x[1-2n]$
    - yes, $y[n+N] = x[1-2n-2N] = x[1-2n] = y[n]$
- $y[n] = x[n] + (-1)^n x[0]$
    - yes, $y[n+2N] = x[n+2N] + (-1)^{n+2N} x[0] = x[n] + (-1)^n x[0] = y[n]$
    - fundamental period: $\begin{cases} \mathrm{lcm}(N,2) & x[0] \neq 0 \\ N & \end{cases}$

### 4.2 constant coefficient difference equation

- $x[n] \to \boxed{S} \to y[n] = S\{x[n]\}$
- todo

## 5 10.10 3m

### 5.1 simple discrete-time convolution

- $h[n] = \{(4)_{N_2=7}, 3, 2, 1\}$
- $x[n] = \{(-3)_{N_1=5}, 7, 4\}$
- $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$
- what length will the output $y[n]$ be? where is the support?
  - $N_1 \leq k \leq N_1 + 2$
  - $N_2 \leq n - k \leq N_2 + 3$
  - $n - N_2 - 3 \leq k \leq n - N_2$
  - $y[n] = \sum_{k=K_1}^{K_2} x[k]h[n-k]$
  - $K_1 = \max\{N_1, n - N_2 - 3\} = \max\{5, n - 10\}$
  - $K_2 = \min\{N_1 + 2, n - N_2\} = \min\{7, n - 7\}$
  - $n - 10 \leq 7 \Rightarrow n \leq 17$
  - $n - 7 \geq 5 \Rightarrow n \geq 12$
  - support: $\{12, 13, 14, 15, 16, 17\}$
- generalization
  - $x[n]$ starts at $N_1$ with length $L_x$
  - $h[n]$ starts at $N_2$ with length $L_h$
  - $K_1 = \max\{N_1, n - N_2 - L_h\}$
  - $K_2 = \min\{N_1 + L_x, n - N_2\}$
  - $n - N_2 - L_h \leq N_1 + L_x \Rightarrow n \leq N_1 + N_2 + L_x + L_h$
  - $n - N_2 \geq N_1 \Rightarrow n \geq N_1 + N_2$
  - nonzero samples of convolution: $N_1 + N_2 \leq n \leq N_1 + N_2 + L_h + L_x$
  - number of nonzeros: $L_h + L_x + 1$

### 5.2 graphical way of computing convolution

- $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$
- sketch $x[k]$
- sketch $h[-k]$
- sketch $h[n-k]$
- sample by sample multiplication of $x[k] \cdot h[n-k]$ for every $n$ that overlaps
- e.g.
  - $x[n] = u[n] - u[n-7]$
  - $h[n] = (0.9)^n u[n]$
  - $y = x * h$

  - $y = 0$ when $n < N_1 + N_2 = 0$
  - $0 \leq n \leq 6$
    - $y[n] = \sum_{k=0}^{n} x[k] \cdot h[n-k] = \sum_{k=0}^{n} 1 \cdot (0.9)^{n-k} = (0.9)^n \sum_{k=0}^{n} (0.9)^{-k} = (0.9)^n \frac{1-(0.9)^{-(n+1)}}{1-(0.9)^{-1}}$
  - $n > 6$

### 5.3 discrete time fourier series

- consider a discrete-time periodic signal
  - $x[n] = x[n+N]$
- infinite duration signal, which we represent as a vector of $N$ samples
- $x = \begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix} \in \mathbb{R}^n, \mathbb{C}^n$
- key learning aims for 113
  - decompose a signal $x[n]$ into a set of "basis" signals
- $x[n] = \sum c_k \underbrace{y_k[n]}_{\text{"basis"}}$
  - $c_k \in \mathbb{R}, \mathbb{C}$ are scalar coefficients

### 5.4 review: inner product

- $\langle x, y \rangle = \sum_{k=0}^{N-1} x[k] y[k]$ for $x, y \in \mathbb{R}^N$
  - "dot product"
- $\langle x, x \rangle = \sum_{k=0}^{N-1} x[k] \cdot x[k] = \sum_{k=0}^{N-1} x[k]^2 = |x|_2^2$
  - "squared norm"
- $|x|_2 = \sqrt{\langle x, x \rangle}$
  - "norm" of a vector

## 6 10.12 3w

### 6.1 discrete time fourier series

- $x[n] = \sum_k c_k y_k[n]$
- $x = \sum_k c_k y_k$
- inner product: $\langle x, y \rangle = \sum_{k=0}^{N-1} x_k \cdot y_k$
- $\langle x, x \rangle = \sum_{k=0}^{N-1} x_k^2 = |x|_2^2$
- $|x|_2 = \sqrt{\langle x, x \rangle}$
- $\langle x, y \rangle = |x|_2 \, |y|_2 \cos \alpha$
- $\cos \alpha = \frac{\langle x, y \rangle}{|x|_2 |y|_2} = \frac{\langle x, y \rangle}{|x| |y|}$

- $\langle x, y \rangle = 0$ when $x$ and $y$ are orthogonal
- find a set $\{y_k\}_k$ of basis vectors that are *orthonormal*
  - $\langle y_k, y_m \rangle = \begin{cases} 0 & k \neq m \\ 1 & k = m \end{cases}$
  - e.g. $y_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $y_1 ql \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \end{bmatrix}$ are orthonormal
- synthesis: given a basis and coefficients, synthesize vectors
  - $c_0 = \sqrt{2}, c_1 = \sqrt{2} \cdot 5$
  - $x = \sum c_k y_k = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$
- analysis: given a basis and a signal $x$, what is the representation of $x$ in this basis?
  - find $c_0, c_1$ given $x, y_0, y_1$
  - $c_0 = \langle x, y_0 \rangle = \sum x[k] y_0[k]$
  - $c_1 = \langle x, y_1 \rangle = \sum x[k] y_1[k]$

## 6.2 periodic signals

- suppoer $\tilde{x}[n]$ is a periodic signal
- goal is to represent $\tilde{x}[n] = \sum_k \tilde{c}_k \phi_k[n]$
- $\phi_k[n] = e^{j\Omega_k n}$ fourier basis
- $\Omega_k$ si the angular frequency of the $k$-th basis vector
- $\tilde{x}[n] = \sum_k \tilde{c}_k e^{j\Omega_k n}$
- three open questions
  - what is $\Omega_k$?
  - how many basis signals / vectors are needed?
  - what are the coefficients $\tilde{c}_k$?

## 6.3 what is $\Omega_k$

- suppose $\tilde{x}$ is periodic with period $N$
- what is the periodicity of $\phi_k$?
- $\phi_k$ has period $N$
- therefore $\phi_k[n] = \phi_k[n+N] \Rightarrow e^{j\Omega_k n} = e^{j\Omega_k (n+N)} = e^{j\Omega_k n} e^{j\Omega_k N}$
- $e^{j\Omega_k N} = 1$
- $\Omega_k N = 2\pi k \Rightarrow \Omega_k = \frac{2\pi}{N} \cdot k$
- $\phi_k(n) = e^{j\frac{2\pi}{N} kn}$

## 6.4 how many $\phi_k[n]$ are there

- suppose we have a finite set of $N$ basis vectors $\{\phi_0, \ldots, \phi_{N-1}\}$
- is this sufficient?
- $\phi_{k+N}[n] = e^{j\frac{2\pi}{N}(k+N)n} = e^{j\frac{2\pi}{N} kn} e^{j2\pi n} = e^{j\frac{2\pi}{N} kn} = \phi_k[n]$

- $\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{c}_k \phi_k[n]$
- find the basis vectors for $\tilde{x}[n] = \cos(0.2\pi n)$
  - $\Omega_0 = 0.2\pi$
  - $F_0 = \frac{0.2\pi}{2\pi} = 0.1$
  - $N = 10 \Rightarrow \{\phi_0, \ldots, \phi_9\}$
  - $\tilde{x}[n] = \sum_{k=0}^{9} \tilde{c}_k e^{j0.2\pi kn}$
- how to compute $\tilde{c}_k$

$$\begin{aligned}\left\langle \tilde{x}[n], e^{j\frac{2\pi}{N}kn} \right\rangle &= \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn} \\ &= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \tilde{c}_m e^{j\frac{2\pi}{N}mn} e^{-j\frac{2\pi}{N}kn} \\ &= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \tilde{c}_m e^{j\frac{2\pi}{N}(m-k)n} \\ &= \tilde{c}_k N\end{aligned}$$

$$\tilde{c}_k = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn}$$

## 6.5 discrete time fourier series (dtfs) summary

- synthesis equation: $\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{c}_k e^{j\frac{2\pi}{N}kn}$
- analysis equation: $\tilde{c}_k = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn}$