ECE 113 (Digital Signal Processing)

November 30, 2022

1 9.26 1m

- syllabus

1.1 ece102: signals & systems

- signals were modeled as functions of time
  - $x(t) : t \in \mathbb{R}$ (continuous time)
  - $x : \mathbb{R} \to \mathbb{R}, \mathbb{C}$
  - sampling $t = nT_s$
    - T_s = sampling interval
    - n = integer
    - $t - x$ plot $\xrightarrow{\text{sampling}}$ $n - x$ plot
    - $x(t)$ bandlimited ($|X(f)| \leq f_{\max}$) then if sampling rate is $f_s > 2f_{\max}$ (nyquist rate) then it is possible to recover the original signal
    - sampling at lower rate then there will be aliasing
- systems
  - $x(t) \to \boxed{S} \to y(t)$
  - $x[n] \to \boxed{S} \to y[n]$

1.2 three ways to write out signals

- mathematical expression
  - $x[n] = 3\cos(2n), n \in \mathbb{Z}$
- tabular list of significant samples
  - $x[n] = \{-1, 8, 0, 5, -2\}, n \in \mathbb{Z}$
    - underspecified signal: loss of indices
    - typically put an arrow underneath the 0 index
    - convention: any sample not listed has 0 amplitude
- plotting

1.3 signal operations

- arithmetic
  - $g[n] = x[n] + A$
  - $g[n] = Bx[n]$
  - signal addition: $g[n] = x_1[n] + x_2[n]$
  - signal multiplication: $g[n] = x_1[n] \cdot x_2[n]$
- time shifting
  - ece102

- $x(t) \to x(t-\tau)$ delays signal by shifting it right
- $x(t) \to x(t+\tau)$ advances signal by shifting left

- ece113
  - $g[n] = x[n-k], k > 0$

- time scaling
  - ece102
    - $x(at), a > 1$: contraction
    - $x(at), 0 < a < 1$: dilation
  - ece113
    - $g[n] = x[2n]$ (downsampling)
    - $g[n] = x\left[\frac{1}{2}n\right]$ (upsampling)
    - $g[n] = x[-n]$ (time reversal)

1.4 basic signals

- $\delta(t)$ from ece102 $\to$ hard concept to grasp
- $\delta[t]$ from ece113 $\to$:)
  - $\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$
  - $\delta[n]$ does not have complexities of $\delta(t)$
- sampling property of δ
  - $x[n] * \delta[n] = x[0] \cdot \delta[0] = x[0]$
  - $x[n] * \delta[n-k] = x[k] \cdot \delta[0] = x[k]$

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2.1 sampling properties hold in discrete domain as well

- $\sum_{n=-\infty}^{\infty} x[n] \cdot \delta[n-k] = \sum_n x[k]\delta[n-k] = x[k]\sum_n \delta[n-k] = x[k]$

2.2 systems

- unit step system $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$
  - $u[n] - u[n-1] = \delta[n]$
  - $\delta[n]$ is the first derivative of $u[n]$
  - ece102: $\delta(t) = \frac{du(t)}{dt}$
  - $u[n] = \sum_{k=0}^{\infty} \delta[n-k]$
- unit ramp signal $r[t] = \begin{cases} n & n \geq 0 \\ 0 & n < 0 \end{cases}$

  - 102: $r(t) = tu(t)$
  - $r[n] = n \cdot u[n] = \sum_{k=1}^{\infty} u[n-k]$

2.3 how wan se express $x[n]$ using basic signals?

- generic signal representation using the “canonical basis” of $\{\delta[n-k] \forall k\}$
- $x[n] = \{1, 2, (0), 1, 1\}$
- $x[n] = \delta[n+2] + 2\delta[n+1] + \delta[n-1] - \delta[n-2]$

2.4 sinusoidal signals

- $x(t) = A\cos(\omega_0 t + B)$, $\omega_0 = 2\pi f_0$
- $x[n] = A\cos(\Omega_0 n + \theta)$, $\Omega_0 = 2\pi F_0 \in [-\pi, \pi)$ where F_0 is the normalized frequency
- $x[n] = A\cos(2\pi f_0 n T_0 + \theta) = A\cos\left(2\pi \frac{f_0}{f_s} n + \theta\right)$
- $F_n := \frac{f_0}{f_s} < 1$

2.5 complex exponential

- $x[n] = Ae^{j(2\pi F_0 n + \theta)}$

2.6 additional signal properties

1. duration of signal
2. real or complex
3. periodicity: a signal is periodic if $\exists N \in \mathbb{Z} : x[n+N] = x[n] \; \forall n$

- $x[n]$ is also periodic with $2N, 3N$, etc.
- $x[n + kN] = x[n] \; \forall k, n$
- fundamental period: smallest positive N such that $x[n+N] = x[n]$
- e.g. $x[n] = \cos(2\pi F_0 n) \to \exists k \in \mathbb{Z} : 2\pi F_0 N = 2\pi k \to F_0 N \in \mathbb{Z}, N = \frac{k}{F_0}$
- is $x[n] = \cos(0.2n)$ periodic?
  - no
  - $2\pi F_0 = 0.2 \to \frac{k}{F_0} = 10k\pi \notin \mathbb{Z}$

3 10.3 2m

3.1 recap

- recall: a signal is periodic if $x[n+N] = x[n]$ for all n and some $N \in \mathbb{Z}$
- is $\cos(0.2n)$ periodic?
  - no due to irrational period
- is the sum of x_1 and x_2 periodic if they are periodic signals?

3.2 periodicity

- $x_1[n] = 2\cos(0.4\pi n), x_2[n] = 1.5\cos(0.48\pi n)$
- $F_1 = \frac{0.4\pi}{2\pi} = 0.2$
- $F_2 = \frac{0.48\pi}{2\pi} = 0.24$
- $N_1 = \frac{K_1}{F_1} = \frac{K_1}{0.2} = 5K_1 = 5$
- $N_2 = \frac{K_2}{F_2} = \frac{K_2}{24/100} = \frac{100}{24}K_2 = \frac{25}{6}K_2 = 25$
- fact: sum of two periodic signals is itself a periodic signal
- $x_3[n] := x_1[n] + x_2[n]$ is periodic with period $N_1 N_2$
  - $x_3[n + N_1N_2] = x_1[n + N_1N_2] + x_2[n + N_1N_2] = x_1[n] + x_2[n] = x_3[n]$
  - e.g. $N_1 = 5, N_2 = 25 \Rightarrow 2\cos(0.4\pi n) + 1.5\cos(0.48\pi n)$ has a period of $5 \cdot 25 = 125$
  - fundamental period: $\mathrm{lcm}(5, 25) = 25$
- e.g. $x[n]$ is periodic w $N = 12$
- are the following periodic
  - $y[n] = x[-n]$
    - yes: $y[n + 12] = x[-(n + 12)] = x[-n - 12] = y[n]$
  - $y[n] = x[n + 1]$
    - yes: $y[n + 12] = x[(n + 12) + 1] = x[n + 1] = y[n]$
  - $y[n] = x[3n]$
    - yes: $y[n + 4] = x[3(n + 4)] = x[3n + 12] = x[3n] = y[n]$
  - $y[n] = x^2[n]$
    - yes: $y[n + 12] = (x[n + 12])^2 = (x[n])^2 = y[n]$

3.3 even and odd signals

- even: $x[n] = x[-n]$
- odd: $x[n] = -x[-n]$ or $-x[n] = x[-n]$
- any signal can be decomposed into odd and even components
  - $x[n] = x_{\text{even}}[n] + x_{\text{odd}}[n]$
- $x_{\text{even}}[n] = \frac{x[n]+x[-n]}{2}$
- $x_{\text{odd}}[n] = \frac{x[n]-x[-n]}{2}$

3.4 complex signals

- $x[n] = x_R[n] + jx_I[n] = |x[n]|\, e^{j\angle x[n]}$
- $x^*[n] = x_R[n] - jx_I[n] = |x[n]|\, e^{-j\angle x[n]}$
- "even symmetry" for complex
  - $x[-n] = x^*[n]$
  - $x^*[-n] = x[n]$
- "odd antisymmetry" for complex
  - $x[-n] = -x^*[n]$
  - $x^*[-n] = -x[n]$

3.5 decompose of even / odd for complex signals

- $x[n] = x_e[n] + x_o[n]$
- $x_e[n] = \frac{x[n]+x^*[-n]}{2}$
- $x_o[n] = \frac{x[n]-x^*[-n]}{2}$

3.6 energy + power of a signal

- energy: $E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$
  - if $E_x < +\infty$: **finite energy signal**
  - e.g. $x[n] = \cos(2\pi 0.1n + 0.3\pi)$ is not finite energy
- power
  - consider a periodic signal with period N
  - its power is expressed as $P_x = \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2$
  - consider a non-periodic signal
  - its power is expressed as $P_x = \lim_{M\to\infty} \frac{1}{2M+1}\sum_{n=-M}^{M} |x[n]|^2$
- e.g. calculate the energy of $x[n] = (0.5)^n u[n]$
  - $E_x = \sum_{n=-\infty}^{\infty} |(0.5)^n u[n]|^2 = \sum_{n=-\infty}^{\infty}(0.5)^{2n}u^2[n] = \sum_{n=0}^{\infty}(0.5)^{2n} = \frac{1}{1-0.25} = \frac{4}{3}$

3.7 systems

- $x[n] \to \boxed{S} \to y[n]$
- properties
  - linearity
    - $y_1[n] = S\{x_1[n]\}$
    - $y_2[n] = S\{x_2[n]\}$
    - $x_3[n] = ax_1[n] + bx_2[n]$
    - $y_3[n] = S\{x_3[n]\}$
    - linear $\Rightarrow y_3[n] = ay_1[n] + by_2[n]$
    - $S\left\{\sum_{k=1}^{N} a_k x_k[n]\right\} = \sum_{k=1}^{N} a_k S\{x_k[n]\}$
  - time-invariance
    - $y[n] = S\{x[n]\}$
    - time invariance $\Rightarrow y[n-k] = S\{x[n-k]\}$
  - causality
    - if current output $y[n]$ depends only on the current and past input samples, the system is causal
    - e.g. $y[n] = x[n] - nx[n-2]$: causal
    - e.g. $y[n] = (n+1)x[n]$: causal (also memory-less)
    - e.g. $y[n] = x[n] + x[n+2]$: not causal
  - bounded input, bounded output (bibo) stability
    - if $|x[n]| < B$ for all n, then $|y[n]| < C$ for all n, B, C are constants

    - e.g. $y[n] = 3x[n] + 2x[n-1]$
        - suppose $|x[n]| < B$
        - stable: $\left|y[n]\right| = |3x[n] + 2x[n-1]| \le 3|x[n]| + 2|x[n-1]| < 5B$
    - e.g. $y[n] = x^2[n]$
        - suppose $|x[n]| < B$
        - stable: $\left|y[n]\right| = \left|x^2[n]\right| = |x[n]|^2 < B^2$
    - e.g. $y[n] = nx[n-1]$
        - not stable: as $n \to \infty$, $y \to \infty$
- e.g. $y[n] = 3x[n] + 2x[n-1]$
  - time invariant: $x_1[n] = x[n-k] \Rightarrow y_1[n] = 3x[n-k] + 2x[n-k-1] = y[n-k]$
- e.g. $y[n] = x^2[n]$
  - not linear
  - time invariant: $x_1[n] = x[n-k] \Rightarrow y_1[n] = x_1^2[n] = x^2[n-k] = y[n-k]$
- e.g. $y[n] = nx[n-1]$
  - not time invariant: $x_1[n] = x[n-k] \Rightarrow y_1[n] = nx_1[n-1] = nx[n-1-k] \ne (n-k)x[n-1-k] = y[n-k]$
- e.g. $y[n] = x[n/4]$
  - linear
    - $y_1[n] = x_1[n/4]$
    - $y_2[n] = x_2[n/4]$
    - $x_3[n] = ax_1[n] + bx_2[n]$
    - $y_3[n/4] = x_3[n/4] = ax_1[n/4] + bx_3[n/4] = ay_1[n] + by_2[n]$
  - not time invariant
  - not causal
    - $n = -4 \Rightarrow y[-4] = x[-1]$
  - stable
    - suppose $|x[n]| < B$
    - $\left|y[n]\right| = |x[n/4]| < B$

4 10.5 2w

4.1 hw q4

- suppose $x[n]$ periodic with period N, is the following periodic
- $y[n] = x[1-2n]$
  - yes, $y[n+N] = x[1-2n-2N] = x[1-2n] = y[n]$
- $y[n] = x[n] + (-1)^n x[0]$
  - yes, $y[n+2N] = x[n+2N] + (-1)^{n+2N}x[0] = x[n] + (-1)^n x[0] = y[n]$
  - fundamental period: $\begin{cases} \mathrm{lcm}(N,2) & x[0] \ne 0 \\ N \end{cases}$

4.2 constant coefficient difference equation

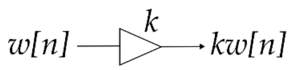
- $x[n] \rightarrow \boxed{S} \rightarrow y[n] = S\{x[n]\}$
- $\sum_{k=0}^{M} a_k y[n-k] = \sum_{k=0}^{N} b_k x[n-k]$
- $a_0, \ldots, a_M, b_0, \ldots, b_N$: constant coefficients
- M, N: order of the equation
    - often, $\max\{M, N\}$ to get the system order
- e.g. $y[n] = \frac{1}{N} \sum_{k=0}^{N-1} x[k]$
    - $a_0 = 1$
    - $b_0, \ldots, b_{N-1} = \frac{1}{N}$
    - order: $N - 1$
    - length N moving average
- e.g. exponential smoother
    - $y[n] = (1-\alpha)y[n-1] + \alpha x[n]$
    - $0 < \alpha < 1$
    - $a_0 = 1, a_1 = \alpha - 1$
    - $M = 1$
    - $b_0 = \alpha$
    - $N = 0$
    - because $\{a_1, \ldots, a_M\} \neq \{0\}$, this system is recursive, or, equivalently, has feedback
- if a system has feedback, then we need to know the *initial condition* of the system
    - initial condition is the time when input is applied
- e.g. $y[n] = y[n-1] + x[n]$
    - suppose $x[n] = u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$
    - $y[0] = y[-1] + x[0], y[-1] = y[-2] + x[-1], \ldots$
    - initial condition: $y[-1] = 2$
    - $y[-1] = 2, y[0] = y[-1] + x[0] = 2 + 1 = 3, y[1] = y[0] + x[1] = 3 + 1 = 4, \ldots$
    - $y[-1] = 3 \Longrightarrow y[0] = 4, y[1] = 5, \ldots$
- "relaxed system": a system is relaxed if $y[n] = 0$ for $n \rightarrow \infty$
- are relaxed systems described by a constant coefficient difference equation lti?

4.3 visual method to look at ccds

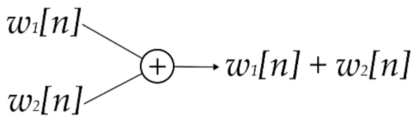
- $\sum_{k=0}^{M} a_k y[n-k] = \sum_{k=0}^{N} b_k x[n-k]$
- e.g. delay element
    - "unit delay"

$$w[n] \longrightarrow \boxed{D} \longrightarrow w[n - 1]$$

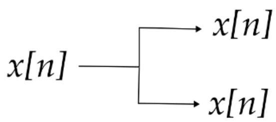
- e.g. multiplier
  - constant multiplier



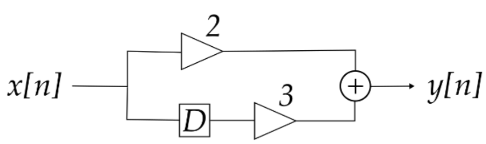
- e.g. adder



- e.g. branch



- block diagram for $y[n] = 2x[n] + 3x[n-1]$



4.4 impulse response

- $\delta[n] \rightarrow \boxed{S} \rightarrow h[n] := S\{\delta[n]\}$
- recall: when a system is lti, then we have an expression for the output with respect to an arbitrary $x[n]$
- $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$
- $y[n] = S\{x[n]\} = S\{\sum_{-\infty}^{\infty} x[k]\delta[n-k]\} = \sum_{k=-\infty}^{\infty} S\{\delta[n-k]\} = \sum_{-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$

4.5 convolution properties

- commutative: $x_1[n] * x_2[n] = x_2[n] * x_1[n]$
- distributive: $x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$
- associative: $x_1[n] * (x_2[n] * x_3[n]) = (x_1[n] * x_2[n]) * x_3[n]$
- identity: $x[n] * \delta[n] = x[n]$
  - $x[n] * \delta[n-m] = x[n-m]$

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5.1 simple discrete-time convolution

- $h[n] = \{(4)_{N_2=7}, 3, 2, 1\}$

- $x[n] = \{(-3)_{N_1=5}, 7, 4\}$
- $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$
- what length will the output $y[n]$ be? where is the support?
  - $N_1 \leq k \leq N_1 + 2$
  - $N_2 \leq n-k \leq N_2 + 3$
  - $n - N_2 - 3 \leq k \leq n - N_2$
  - $y[n] = \sum_{k=K_1}^{K_2} x[k]h[n-k]$
  - $K_1 = \max\{N_1, n - N_2 - 3\} = \max\{5, n-10\}$
  - $K_2 = \min\{N_1 + 2, n - N_2\} = \min\{7, n-7\}$
  - $n - 10 \leq 7 \Rightarrow n \leq 17$
  - $n - 7 \geq 5 \Rightarrow n \geq 12$
  - support: $\{12, 13, 14, 15, 16, 17\}$
- generalization
  - $x[n]$ starts at N_1 with length L_x
  - $h[n]$ starts at N_2 with length L_h
  - $K_1 = \max\{N_1, n - N_2 - L_h\}$
  - $K_2 = \min\{N_1 + L_x, n - N_2\}$
  - $n - N_2 - L_h \leq N_1 + L_x \Rightarrow n \leq N_1 + N_2 + L_x + L_h$
  - $n - N_2 \geq N_1 \Rightarrow n \geq N_1 + N_2$
  - nonzero samples of convolution: $N_1 + N_2 \leq n \leq N_1 + N_2 + L_h + L_x$
  - number of nonzeros: $L_h + L_x + 1$

5.2 graphical way of computing convolution

- $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$
- sketch $x[k]$
- sketch $h[-k]$
- sketch $h[n-k]$
- sample by sample multiplication of $x[k] \cdot h[n-k]$ for every n that overlaps
- e.g.
  - $x[n] = u[n] - u[n-7]$
  - $h[n] = (0.9)^n u[n]$
  - $y = x * h$
  - $y = 0$ when $n < N_1 + N_2 = 0$
  - $0 \leq n \leq 6$
    - $y[n] = \sum_{k=0}^{n} x[k] \cdot h[n-k] = \sum_{k=0}^{n} 1 \cdot (0.9)^{n-k} = (0.9)^n \sum_{k=0}^{n} (0.9)^{-k} = (0.9)^n \frac{1-(0.9)^{-(n+1)}}{1-(0.9)^{-1}}$
  - $n > 6$

5.3 discrete time fourier series

- consider a discrete-time periodic signal
  - $x[n] = x[n+N]$

- infinite duration signal, which we represent as a vector of N samples
- $x = \begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix} \in \mathbb{R}^n, \mathbb{C}^n$
- key learning aims for 113
  - decompose a signal $x[n]$ into a set of "basis" signals
- $x[n] = \sum c_k \underbrace{y_k[n]}_{\text{"basis"}}$
  - $c_k \in \mathbb{R}, \mathbb{C}$ are scalar coefficients

5.4 review: inner product

- $\langle x, y \rangle = \sum_{k=0}^{N-1} x[k]y[k]$ for $x, y \in \mathbb{R}^N$
  - "dot product"
- $\langle x, x \rangle = \sum_{k=0}^{N-1} x[k] \cdot x[k] = \sum_{k=0}^{N-1} x[k]^2 = |x|_2^2$
  - "squared norm"
- $|x|_2 = \sqrt{\langle x, x \rangle}$
  - "norm" of a vector

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6.1 discrete time fourier series

- $x[n] = \sum_k c_k y_k[n]$
- $x = \sum_k c_k y_k$
- inner product: $\langle x, y \rangle = \sum_{k=0}^{N-1} x_k \cdot y_k$
- $\langle x, x \rangle = \sum_{k=0}^{N-1} x_k^2 = |x|_2^2$
- $|x|_2 = \sqrt{\langle x, x \rangle}$
- $\langle x, y \rangle = |x|_2 |y|_2 \cos \alpha$
- $\cos \alpha = \frac{\langle x,y \rangle}{|x|_2 |y|_2} = \frac{\langle x,y \rangle}{|x||y|}$
- $\langle x, y \rangle = 0$ when x and y are orthogonal
- find a set $\{y_k\}_k$ of basis vectors that are *orthonormal*
  - $\langle y_k, y_m \rangle = \begin{cases} 0 & k \neq m \\ 1 & k = m \end{cases}$
  - e.g. $y_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $y_1 q l \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \end{bmatrix}$ are orthonormal
- synthesis: given a basis and coefficients, synthesize vectors
  - $c_0 = \sqrt{2}, c_1 = \sqrt{2} \cdot 5$

- $x = \sum c_k y_k = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$
- analysis: given a basis and a signal x, what is the representation of x in this basis?
  - find c_0, c_1 given x, y_0, y_1
  - $c_0 = \langle x, y_0 \rangle = \sum x[k] y_0[k]$
  - $c_1 = \langle x, y_1 \rangle = \sum x[k] y_1[k]$

6.2 periodic signals

- suppose $\tilde{x}[n]$ is a periodic signal
- goal is to represent $\tilde{x}[n] = \sum_k \tilde{c}_k \phi_k[n]$
- $\phi_k[n] = e^{j\Omega_k n}$ fourier basis
- Ω_k is the angular frequency of the k-th basis vector
- $\tilde{x}[n] = \sum_k \tilde{c}_k e^{j\Omega_k n}$
- three open questions
  - what is Ω_k?
  - how many basis signals / vectors are needed?
  - what are the coefficients $\tilde{c}_k$?

6.3 what is Ω_k

- suppose $\tilde{x}$ is periodic with period N
- what is the periodicity of ϕ_k?
- ϕ_k has period N
- therefore $\phi_k[n] = \phi_k[n+N] \Rightarrow e^{j\Omega_k n} = e^{j\Omega_k(n+N)} = e^{j\Omega_k n} e^{j\Omega_k N}$
- $e^{j\Omega_k N} = 1$
- $\Omega_k N = 2\pi k \Rightarrow \Omega_k = \frac{2\pi}{N} \cdot k$
- $\phi_k(n) = e^{j\frac{2\pi}{N}kn}$

6.4 how many $\phi_k[n]$ are there

- suppose we have a finite set of N basis vectors $\{\phi_0, \ldots, \phi_{N-1}\}$
- is this sufficient?
- $\phi_{k+N}[n] = e^{j\frac{2\pi}{N}(k+N)n} = e^{j\frac{2\pi}{N}kn} e^{j2\pi n} = e^{j\frac{2\pi}{N}kn} = \phi_k[n]$
- $\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{c}_k \phi_k[n]$
- find the basis vectors for $\tilde{x}[n] = \cos(0.2\pi n)$
  - $\Omega_0 = 0.2\pi$
  - $F_0 = \frac{0.2\pi}{2\pi} = 0.1$
  - $N = 10 \Rightarrow \{\phi_0, \ldots, \phi_9\}$
  - $\tilde{x}[n] = \sum_{k=0}^{9} \tilde{c}_k e^{j0.2\pi kn}$

- how to compute $\tilde{c}_k$

$$\begin{aligned}
\left\langle \tilde{x}[n], e^{j\frac{2\pi}{N}kn} \right\rangle &= \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn} \\
&= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \tilde{c}_m e^{j\frac{2\pi}{N}mn} e^{-j\frac{2\pi}{N}kn} \\
&= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \tilde{c}_m e^{j\frac{2\pi}{N}(m-k)n} \\
&= \tilde{c}_k N
\end{aligned}$$

$$\tilde{c}_k = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn}$$

6.5 discrete time fourier series (dtfs) summary

- synthesis equation: $\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{c}_k e^{j\frac{2\pi}{N}kn}$
- analysis equation: $\tilde{c}_k = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn}$

7 10.19 4w

7.1 discrete time fourier series (dtfs)

- $\tilde{x}[n] = \tilde{x}[n+N] = \tilde{x}[n+kN]$
- $\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{c}_k e^{j\frac{2\pi}{N}kn}$
- $\tilde{c}_k = \left\langle \tilde{x}[n], \frac{1}{N} e^{j\frac{2\pi}{N}kn} \right\rangle$
- $\left\langle x[n], y[n] \right\rangle := \sum_{n=0}^{N-1} x[n] y^*[n]$
- $\tilde{c}_k = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn}$
- side note: pulse train signal
  - $\tilde{x}[n] = \begin{cases} 1 & -L \le n \le L \\ 0 & L < n < N - L \end{cases}$
  - periodic with N
  - homework or exam: calculate dtfs $\tilde{x}[n] = \sum_{n=0}^{N-1} \tilde{c}_k e^{j\frac{2\pi}{N}kn}$

7.2 properties of dtfs

- periodicity: coefficients are perioddic with N
  - $\tilde{c}_k = \tilde{c}_{k+rN}, r \in \mathbb{Z}$
- linearity
  - $\tilde{x}_1[n], \tilde{x}_2[n]$ periodic with same period N
  - $\tilde{x}_1[n] \xrightarrow{\text{dtfs}} \tilde{c}_k$
  - $\tilde{x}_2[n] \xrightarrow{\text{dtfs}} \tilde{d}_k$

  - $\alpha\tilde{x}_1[n] + \beta\tilde{x}_2[n] \xrightarrow{\text{dtfs}} \alpha\tilde{c}_k + \beta\tilde{d}_k$
  - proof
    - $\tilde{x}_1[n] = \sum_{n=0}^{N-1} \tilde{c}_k e^{j\frac{2\pi}{N}kn}$
    - $\tilde{x}_2[n] = \sum_{n=0}^{N-1} \tilde{c}_k e^{j\frac{2\pi}{N}kn}$
    - $\alpha\tilde{x}_1[n] + \beta\tilde{x}_2[n] = \sum_{n=0}^{N-1}(\alpha\tilde{c}_k + \beta\tilde{d}_k)e^{j\frac{2\pi}{N}kn}$
- if $x[n]$ is real-valued, then the dtfs $\tilde{c}_k$ for a signal is a conjugate symmetric sequence
  - $\tilde{c}_k^* = \tilde{c}_{-k}$
  - $|\tilde{c}_k| = |\tilde{c}_{-k}|$
  - proof
    - $\tilde{x}[n] = \tilde{x}^*[n]$
    - $\tilde{c}_k^* = \left(\frac{1}{N}\sum_{n=0}^{N-1}\tilde{x}[n]e^{-j\frac{2\pi}{N}kn}\right)^* = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{x}[n]e^{j\frac{2\pi}{N}kn} = \tilde{c}_{-k}$
  - $\tilde{c}_k = |\tilde{c}_k|\, e^{j\angle\tilde{c}_k}$
- dtfs of even and odd signals
  - $\tilde{x}[n]$ is real and even $\Rightarrow$ then $\tilde{x}^*[n] = \tilde{x}[n]$ and $\tilde{x}[n] = \tilde{x}[-n]$
  - $\tilde{c}_{-k} = \frac{1}{N}\sum_{n=0}^{N-1}\tilde{x}[n]e^{-j\frac{2\pi}{N}kn} = \frac{1}{N}\sum_{n'=0}^{-(N-1)}\tilde{x}[-n']e^{-j\frac{2\pi}{N}kn'} = \frac{1}{N}\sum_{n'=0}^{-(N-1)}\tilde{x}[n']e^{-j\frac{2\pi}{N}kn'} = \tilde{c}_k$
  - $\tilde{c}_k^* = \tilde{c}_{-k}$ real signal
  - $\tilde{c}_k = \tilde{c}_{-k}$ even signal
  - real even signal $\longrightarrow$ real dtfs
- shifting in time
  - $\tilde{x}[n] \longrightarrow \tilde{c}_k$
  - $\tilde{x}[n-m] \longrightarrow e^{-j\frac{2\pi}{N}km}\tilde{c}_k$
  - proof
    - $\tilde{y}[n] = \tilde{x}[n-m]$
    - $\tilde{y}[n] = \sum_{k=0}^{N-1}\tilde{d}_k e^{j\frac{2\pi}{N}kn}$
    - $\tilde{d}_k = \frac{1}{N}\sum_{n=0}^{N-1}\tilde{y}[n]e^{-j\frac{2\pi}{N}kn} = \frac{1}{N}\sum_{n=0}^{N-1}\tilde{x}[n-m]e^{-j\frac{2\pi}{N}kn} = \frac{1}{N}\sum_{n'=-m}^{N-1-m}\tilde{x}[n']e^{-j\frac{2\pi}{N}k(n'+m)}$
      - $= e^{-j\frac{2\pi}{N}km}\frac{1}{N}\sum_{n'=-m}^{N-1-m}\tilde{x}[n]e^{-j\frac{2\pi}{N}kn'} = e^{-j\frac{2\pi}{N}km}\tilde{c}_k$

8 10.24 5m

8.1 periodic convolution

- $\tilde{x}[n]$ with period N
- $\tilde{h}[n]$ with period N
- $\tilde{x}[n] \otimes \tilde{h}[n] = \sum_{k=0}^{N-1}\tilde{x}[k]\tilde{h}[n-k]$
- "linear convolution"
  - $\tilde{x}[n] * \tilde{h}[n] = \sum_{k=-\infty}^{\infty}\tilde{x}[k]\tilde{h}[n-k] \neq \tilde{x}[n] \otimes \tilde{h}[n]$
- finding $\tilde{y}[n] = \tilde{x}[n] \otimes \tilde{h}[n]$
  - $\tilde{x}[n] = \{\dots, 0, 1, 2, 3, 4, \dots\}, 0 \leq n \leq 4$
  - $\tilde{h}[n] = \{\dots, 3, 3, -3, 2, -1, \dots\}, 0 \leq n \leq 4$
  - $\tilde{y}[n] = \tilde{x}[n] \otimes \tilde{h}[n] = \sum_{k=0}^{4}\tilde{x}[k]\tilde{h}[n-k]$

- $\tilde{y}[0] = \sum_{k=0}^{4} \tilde{x}[k]\tilde{h}[0-k] = \tilde{x}[0]\tilde{y}[0] + \tilde{x}[1]\tilde{y}[-1] + \cdots$
    - $\tilde{h}[k] = \tilde{h}[N+k]$
- $\tilde{y} \xrightarrow{\text{dtfs}} \tilde{e}_k$
- $\tilde{x} \xrightarrow{\text{dtfs}} \tilde{c}_k$
- $\tilde{h} \xrightarrow{\text{dtfs}} \tilde{d}_k$
- proposition: $\tilde{e}_k = N \cdot \tilde{c}_k \cdot \tilde{d}_k$
    - proof
        - $\tilde{y}[n] = \sum_{k=0}^{N-1} \tilde{x}[k]\tilde{h}[n-k] = \sum_{k=0}^{N-1} \tilde{e}_k e^{j\frac{2\pi}{N}kn}$
        - $\tilde{e}_k = \frac{1}{N}\sum_{n=0}^{N-1}\left(\sum_{m=0}^{N-1}\tilde{x}[m]\tilde{h}[n-m]\right)e^{-j\frac{2\pi}{N}kn} = \frac{1}{N}\sum_{m=0}^{N-1}\tilde{x}[m]\sum_{n=0}^{N-1}\tilde{h}[n-m]e^{-j\frac{2\pi}{N}kn}$
        - $= \frac{1}{N}\sum_{m=0}^{N-1}\tilde{x}[m]\tilde{d}_k \cdot N \cdot e^{-j\frac{2\pi}{N}km} = \sum_{m=0}^{N-1}\tilde{x}[m]\tilde{d}_k e^{-j\frac{2\pi}{N}km} = \tilde{d}_k\sum_{m=0}^{N-1}\tilde{x}[m]e^{-j\frac{2\pi}{N}km} = \tilde{d}_k\tilde{c}_k N$

8.2 discrete time fourier transform (dtft)

- consider a non-periodic signal $x[n]$ with finite length
    - $x[n] \neq 0$ for $|n| \leq M$
    - there exists $2M+1$ non-zero samples
- create a periodic extension of x
    - $\tilde{x}[n] = \sum_{k=-\infty}^{\infty} x[n + k(2M+1)]$
- $\tilde{x}[n]$ is periodic so can be analyzed using dtfs
    - $\tilde{x}[n] = \sum_{k=-M}^{M} \tilde{c}_k e^{j\frac{2\pi}{2M+1}kn}$(synthesis)
- since $\tilde{x}[n]$ is periodic there exists an analysis component as well
    - $\tilde{c}_k = \frac{1}{2M+1}\sum_{n=-M}^{M}\tilde{x}[n]e^{-j\frac{2\pi}{2M+1}kn}$
    - $\Omega_0 = \frac{2\pi}{2M+1}$
    - $\Omega_k = k\Omega_0$
    - $k \in [-M, M]$
    - $\Omega_k \in [-\pi, \pi]$
    - $\tilde{c}_k = \frac{1}{2M+1}\sum_{n=-M}^{M} x[n]e^{-j\frac{2\pi}{2M+1}kn}$
    - $(2M+1)\tilde{c}_k = \sum_{n=-M}^{M} x[n]e^{-j\frac{2\pi}{2M+1}kn}$
- as $M \to \infty$
    - $\Omega_0 \to \Delta\Omega$
    - $k\Omega = k\frac{2\pi}{2M+1} := \Omega \in \mathbb{R}$
    - $\Omega \in [-\pi, \pi]$
    - $\lim_{M\to\infty}(2M+1)\tilde{c}_k := X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$ (dtft)
- $X(\Omega)$ is complex
- $X(\Omega) = \Re\{X(\Omega)\} + j\Im\{X(\Omega)\}$
- $X(\Omega) = |X(\Omega)|\, e^{j\angle X(\Omega)}$
- visualizing $X(\Omega)$ requires two plots
- ece102:
    - $x(t) \xrightarrow{\mathcal{F}} X(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$, $\omega \in (-\infty, \infty)$

- ece113:
  - $x[n] \xrightarrow{\mathcal{F}} X(\Omega) := \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$, $\Omega \in [-\pi, \pi]$ $X(\Omega)$ is periodic with 2π
- $$\begin{aligned} X(\Omega + 2\pi) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j(\Omega+2\pi)n} \\ &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}e^{-j2\pi n} \\ &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = X(\Omega) \end{aligned}$$
- $X(\Omega) \xrightarrow{\mathcal{F}^{-1}} x[n]$
  - $x[n] = \lim_{M\to\infty} \tilde{x}[n] = \lim_{M\to\infty} \sum_{k=-M}^{M} \tilde{c}_k e^{j\frac{2\pi}{2M+1}kn}$
  - $\lim_{M\to\infty} \sum_{k=-M}^{M} c_k \frac{2M+1}{2\pi} \Delta\Omega e^{j\Omega n}$
  - $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}\, d\Omega$ (idtft)
- $\delta[n] \xrightarrow{\mathcal{F}} \sum_{n=-\infty}^{\infty} d[n]e^{-j\Omega n} = e^{-j\Omega 0} = 1$
- $x[n] = \delta[n - n_0]$
  - $X(\Omega) = \sum_{n=-\infty}^{\infty} = \sum_{n=-\infty}^{\infty} \delta[n-n_0]e^{-j\Omega n} = e^{-j\Omega n_0}$
- $x[n] = a^n u[n]$, $|a| < 1$
  - existence of dtft
    - $x[n]$ has to be absolutely summable
      - $\sum_{n=-\infty}^{\infty} |x[n]| < +\infty$
    - or square summable
      - $\sum_{n=-\infty}^{\infty} |x[n]|^2$
  - $$\begin{aligned} X(\Omega) &= \sum_{n=-\infty}^{\infty} a^n u[n]e^{-j\Omega n} \\ &= \sum_{n=0}^{\infty} a^n e^{-j\Omega n} \\ &= \sum_{n=0}^{\infty} (ae^{-j\Omega})^n \\ &= \frac{1}{1 - ae^{-j\Omega}} \end{aligned}$$

9 10.26 5w

9.1 dtft

- analysis equation: $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$ (continuous)
  - $\Omega \in [-\pi, \pi]$
- synthesis equation (idtft): $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}\, d\Omega$
- $x[n] = \begin{cases} 1 & -L \le n \le L \\ 0 & \text{otherwise} \end{cases}$

- $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$
$$= \sum_{n=-L}^{L} e^{-j\Omega n}$$
$$= \frac{e^{j\Omega L} - e^{-j\Omega(L+1)}}{1 - e^{-j\Omega}}$$
$$= \frac{e^{-j\Omega/2}(e^{j\Omega(L+1/2)} - e^{-j\Omega(L+1/2)})}{e^{-j\Omega/2}(e^{j\Omega/2} - e^{-j\Omega/2})}$$
$$= \frac{\sin(\Omega(L+1/2))}{\sin(\Omega/2)}$$

- $X(\Omega) = \begin{cases} 1 & -\Omega_c < \Omega < \Omega_c \\ 0 & \text{otherwise} \end{cases}$
  - $x[n] = \frac{1}{2\pi} \int_{-\Omega_c}^{\Omega_c} 1 \cdot e^{j\Omega n}\, d\Omega$
$$= \frac{1}{2\pi} \left(\frac{1}{j\Omega} e^{j\Omega n} \right) \Big|_{-\Omega_c}^{\Omega_c}$$
$$= \frac{1}{\pi} \cdot \frac{1}{n} \left(\frac{e^{j\Omega_c n} - e^{-j\Omega_c n}}{2j} \right)$$
$$= \frac{1}{\pi n} \sin(\Omega_c n)$$
$$= \frac{\Omega_c}{\pi} \operatorname{sinc}(\Omega_c n)$$

10 11.7 7m

10.1 dtft → dft

- periodic signal: dtfs, coefficients are discrete
- non-periodic signal: dtftc, continuous transform domain
- discrete-time signal $x[n] \xrightarrow{\text{dtft}} X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$
  - $\Omega \in \mathbb{R}$
  - $X(\Omega) = X(\Omega + 2\pi)$
  - problem: Ω is continuous
- non-periodic signal with finite duration (length N)
  - $x[n]$ given for $n = 0, \ldots, N-1$
  - $x[n] = 0$ for $n < 0$ and $n \geq N$
- first goal: find the periodic extension of $x[m]$
  - $\tilde{x}[n] = \sum_{m=-\infty}^{\infty} x[n - mN]$
  - $\tilde{x}[n] \xrightarrow{\text{dtfs}} \tilde{c}_k$
  - $\tilde{c}_k := \frac{1}{N} \sum_{n=0}^{N-1} \tilde{x}[n]e^{-j\frac{2\pi}{N}kn}$
- second goal: extract one period of $\tilde{c}_k$; in particular, c_k is periodic with N

  - $\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{c}_k e^{j\frac{2\pi}{N}kn}$
- define $x[n] = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi}{N}kn}$ where $c_k = \begin{cases} \tilde{c}_k & k = 0, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases}$
- rename $c_k N := X[k]$ for $k = 0, \ldots, N-1$
  - dft: $x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}$
  - inverse dft: $X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$
- N samples map to N coefficients

10.2 dft as a sampling of the dtft

- dtft: $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} = \sum_{n=0}^{N-1} x[n] e^{-j\Omega n}$
- dft: $X[k] = X(\Omega)\Big|_{\Omega = \frac{2\pi}{N}k}$

10.3 dft in linear algebraic view

- $\mathbf{x} = [x[0], x[1], \ldots, x[N-1]]$
- $\mathbf{X} = [X[0], X[1], \ldots, X[N-1]]$
- $\mathbf{X} = W\mathbf{x}$
  - $X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$
  - $\omega_N := e^{-j\frac{2\pi}{N}}$
  - dft: $X[k] = \sum_{n=0}^{N-1} x[n] \omega_N^{kn}$
  - idft: $x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \omega_N^{-kn}$
- $W = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega_N^1 & \cdots & \omega_N^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_N^{N-1} & \cdots & \omega_N^{(N-1)^2} \end{bmatrix}$
- $\mathbf{x} = W^{-1}\mathbf{x}$
- $W^{-1} = \frac{1}{N}\overline{W}$

10.4 from a basis perspective

- $\mathbf{x} = W^{-1}\mathbf{X} = \frac{1}{N} \sum X[i] W^{-1}[:, i]$

10.5 example

- $x[n] = u[n] - u[n-10]$
- $X[k] = \sum_{n=0}^{9} e^{-j\frac{2\pi}{10}kn} = \frac{1-e^{-j\frac{2\pi}{10}k\cdot 10}}{1-e^{-j\frac{2\pi}{10}k}} = \frac{1-e^{-j2\pi k}}{1-e^{-j\frac{2\pi}{10}k}}$
- $X[0] = \lim_{k\to 0} \frac{1-e^{-j2\pi k}}{1-e^{-j\frac{2\pi}{10}k}} = 10$
- $X[\geq 1] = 0$
- dtft has waves: $\frac{\sin(5\Omega)}{\sin(0.5\Omega)} e^{-j4.5\Omega}$

11 11.9 7w

11.1 dft review

- $x[n]$ length N signal
- $x[n] = 0$ for $n < 0, n \geq N$
- $x[n] \xrightarrow{\text{dft}} X[k]$
- we arrived at the dft through the following steps
  - $x[N$ samples$] \longrightarrow$ periodic extension $\longrightarrow$
  - $\tilde{x}[n] \longrightarrow dtfs \longrightarrow$
  - $\tilde{c}_k \longrightarrow X[k] := N \cdot c_k, k = 0, 1, \ldots, N-1$
  - dft: $X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$
  - idft: $x[n] = \frac{1}{N} \sum_{k=0}^{n-1} X[k] e^{j\frac{2\pi}{N}kn}$
- dft is practical, dtft is idealized
- recall dtft $x[n] \xrightarrow{\text{dtft}} X(\Omega), \Omega \in \mathbb{R}$
- relationship between dft and dtft: $X[k] = X(\Omega)\Big|_{\Omega=\frac{2\pi}{N}k}$
- e.g.:
  - $u[n] - u[n-10]$
  - dtft looks like sinc
  - dft has 10 at $n = 0$ and 0 everywhere else = lossy

11.2 sample spacing puzzle

- zero padding
- $x[n] \longrightarrow g[n] = \begin{cases} x[n] & n \in [0, N-1] \\ 0 & n \in [N, N+M-1] \end{cases}$
- $G[k] = X(\Omega)\Big|_{\Omega=\frac{2\pi}{N+M}k}$

11.3 dft properties

- $x[n] \longrightarrow X[k]$
- time shifting property
  - $x[n-m] \longrightarrow e^{-j\omega km} X[k]$
    - wrong because sampling window doesnt change
    - solution: cyclic shift, periodic extension
  - $g[n] = x[(n-m) \bmod N]$
  - $x[(n-m) \bmod N] \longrightarrow e^{-j\omega km} X[k]$
    - where $\omega = \frac{2\pi}{N}$
- linearity
  - $x_1[n] \longrightarrow X_1[k]$
  - $x_2[n] \longrightarrow X_2[k]$
  - $\alpha x_1[n] + \beta x_2[n] \longrightarrow \alpha X_1[k] + \beta X_2[k]$

11.4 determine the dtft

- $x[n] = (0.7)^n \cos(0.2\pi n)u[n]$
  - $x_1[n] = (0.7)^n u[n] \longrightarrow X_1(\Omega) = \frac{1}{1-0.7e^{-j\Omega}}$
  - $X(\Omega) = \text{DTFT}(\cos(0.2\pi n)x_1[n])$

$$\begin{aligned} &= \left(\frac{1}{2}e^{j0.2\pi n} + \frac{1}{2}e^{-j0.2\pi n}\right)x_1[n] \\ &= \frac{1}{2}e^{j0.2\pi n}x_1[n] + \frac{1}{2}e^{-j0.2\pi n}x_1[n] \\ &= \frac{1}{2}X_1(\Omega - 0.2\pi) + \frac{1}{2}X_1(\Omega + 0.2\pi) \\ &= \frac{1}{2}\cdot\frac{1}{1-0.7e^{-j(\Omega-0.2\pi)}} + \frac{1}{2}\cdot\frac{1}{1-0.7e^{-j(\Omega+0.2\pi)}} \end{aligned}$$

- $x[n] = nx[n-1]$
  - time shift: $x[n-1] \longrightarrow e^{-j\Omega\cdot 1}X(\Omega)$
  - differentiation in freq: $nx[n-1] \longrightarrow j\frac{d}{d\Omega}\left[e^{-j\Omega}X(\Omega)\right]$
- $x_2[n] = e^{j\frac{\pi\Omega}{2}}(x[n] * x[n])$

12 11.14 8m

12.1 dft conv thm

- conv thm for dtft
  - $x[n] \longrightarrow h[n] \longrightarrow y[n] = x[n] * h[n]$
  - $\overset{\text{dtft}}{\leftrightarrow} X(\Omega) \longrightarrow H(\Omega) \longrightarrow Y(\Omega) = X(\Omega)H(\Omega), \Omega \in \mathbb{R}$
- in practice, we compute dft, so does the property hold?
  - $X[k] = \text{DFT}(x[n]), k = 0, 1, \ldots, N_x - 1$
  - $H[k] = \text{DFT}(h[n]), k = 0, 1, \ldots, N_h - 1$
  - $Y[k] = \text{DFT}(y[n]), k = 0, 1, \ldots, N_{y-1}$
- $X[k] \cdot H[k] \overset{?}{=} Y[k]$
  - $N_y = N_x + N_h - 1$
  - no
- try to derive an analog of conv thm
  - recall that we got dft from "periodic" extension of a finite signal
  - assumption: $x[n], h[n]$ both have length N
  - $x[n] \longrightarrow \tilde{x}[n] \xrightarrow{\text{dtfs}} \tilde{c}_k \longrightarrow c_k \mid X(k) := c_k \cdot N$
  - $h[n] \longrightarrow \tilde{h}[n] \xrightarrow{\text{dtfs}} \tilde{d}_k \longrightarrow d_k \mid H(k) := d_k \cdot N$
- periodic convolution
  - $\tilde{y}[n] = \tilde{x}[n] \otimes \tilde{h}[n] = \sum_{k=0}^{N-1} \tilde{x}[k]\tilde{h}[n-k]$

  - $\tilde{y}[n] \xrightarrow{\text{dtfs}} \tilde{e}_k$
  - recall $\tilde{e}_k = N\tilde{c}_k\tilde{d}_k$ proven in an earlier lecture
  - taking N samples of $\tilde{y}[n]$ such that $y[n] = \tilde{y}[n], n = 0, 1, \ldots, N-1$
  - $y[n] = \tilde{y}[n] = \sum_{n=0}^{N-1} \tilde{x}[k]\tilde{h}[n-k], n = 0, 1, \ldots, N-1$
  - $y[n] = \sum_{n=0}^{N-1} x[k]\tilde{h}[n-k], n = 0, 1, \ldots, N-1$
  - $y[n] = \sum_{k=0}^{N-1} x[k]h[(n-k) \bmod N]$
  - circular convolution: applies to two finite duration signals of length N
  - $y[n] = x[n] \circ h[n]$

- example
  - $x[n] = \{1, 2, 3, -4, 6\}$
  - $h[n] = \{5, 4, 3, 2, 1\}$
  - $y[0] = \sum_{k=0}^{N-1} x[k]h[(-k) \bmod N] = 5 \cdot 1 + 2 \cdot 1 + 3 \cdot 2 + (-4) \cdot 3 + 6 \cdot 4 = 25$
- $y[n] = \sum_{k=0}^{N-1} x[k]h[(n-k) \bmod N]$
- $\xrightarrow{N\text{-point DFT}} Y[k] = X[k]H[k]$
- $y[n] = \text{IDFT}(Y[k]) = \text{IDFT}(X[k] \cdot H[k])$
- recall: $\tilde{e}_k = N\tilde{d}_k\tilde{c}_k$
  - $y[n] = \tilde{y}[n] \to \tilde{e}_k, n = 0, \ldots, N-1$
  - $x[n] = \tilde{x}[n] \to \tilde{c}_k, n = 0, \ldots, N-1$
  - $h[n] = \tilde{h}[n] \to \tilde{d}_k, n = 0, \ldots, N-1$
  - $\tilde{e}_k = N\tilde{c}_k\tilde{d}_k \forall\, k$
  - dft relationship:
    - $X[k] = Nc_k$
    - $H[k] = Nd_k$
    - $Y[k] = Ne_k$
    - $c_k = \frac{X[k]}{N}$
    - $d_k = \frac{H[k]}{N}$
    - $e_k = \frac{Y[k]}{N}$
- generalize this insight to not same length sequences
  - $y_l[n] = x[n] * h[n]$
  - $x[n]$ length N_x
  - $h[n]$ length N_h
  - $y_l[n]$ length $N_y = N_x + N_h - 1$
  - trick: zero pad x and h to have the same length
  - $x_p[n] = \begin{cases} x[n] & n = 0, \ldots, N_x - 1 \\ 0 & n = N_x, \ldots, N_y - 1 \end{cases}$
  - $h_p[n] = \begin{cases} h[n] & n = 0, \ldots, N_h - 1 \\ 0 & n = N_h, \ldots, N_y - 1 \end{cases}$
  - $y^c[n] = x_p[n] \circ h_p[n] = \sum_{k=0}^{N_y-1} x_p[k]h_p[(n-k) \bmod N_y]$
  - $y_l[n] = \sum_{k=0}^{N_y-1} x[k] \cdot h[n-k]$
  - $X[k] = \text{DFT}(x_p[n])$
  - $H[k] = \text{DFT}(h_p[n])$

  - $Y[k] = H[k] \cdot X[k] = \text{DFT}(x_p(n) \circ h_p(n)) = \text{DFT}(x[n] * h[n])$
- summary
  - $y[n] = x[n] * h[n] = \text{IDFT}(X[k] \cdot H[k])$ assuming $X[k]$ is DFT of zero padded x: $X(\Omega)\Big|_{\Omega=\frac{2\pi}{N_y}k}$

12.2 fast fourier theorem (fft) + algo analysis

- what is the runtime of a dft?
  - $x[n] \to N$ values
  - $X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}, k = 0, 1, \ldots, N-1$
  - for 1 coeffieicnt: N complex multiplications, $N-1$ complex additions
  - repeating for N coefficients
  - $N \cdot (N + (N-1))$ operations $= \mathcal{O}(N^2)$
- radix-2 fft
  - preliminaries
    - N-dft has $\Theta(N^2)$ complexity
    - $\omega_N = e^{-j\frac{2\pi}{N}}, X[k] = \sum_{n=0}^{N-1} x[n]\omega_N^{kn}$
    - properties of ω_N:
      - $\omega_N^2 = \omega_{N/2}$
        - $e^{-j\frac{2\pi}{N}\cdot 2} = e^{-j\frac{2\pi}{N/2}}$
      - $\omega_N^{2+N/2} = -\omega_N^2$
        - $e^{-j\frac{2\pi}{N}(2+N/2)} = e^{-j\frac{2\pi}{N}\cdot 2} \cdot e^{-j\pi} = -\omega_N^2$
      - $\omega_N^{2(N/2)} = (-1)^2$
        - $e^{-j\frac{2\pi}{N}\cdot 2(N/2)} = (e^{-j\pi})^2 = (-1)^2$
  - assumption: $N = 2^p$
  - even samples $= \{x[0], x[2], x[4], \ldots, x[N-2]\}$
  - odd samples $= \{x[1], x[3], [5], \ldots, x[N-1]\}$
  - $$\begin{aligned} X[k] &= \sum_{n=\text{even}} x[n]\omega_N^{kn} + \sum_{n=\text{odd}} x[n]\omega_N^{kn} \\ &= \sum_{m=0}^{N/2-1} x[2m]\omega_N^{k\cdot 2m} + \sum_{m=0}^{N/2-1} x[2m+1]\omega_N^{k(2m+1)} \\ &= \sum_{m=0}^{N/2-1} x[2m]\omega_{N/2}^{km} + \omega_N^k \sum_{m=0}^{N/2-1} x[2m+1]\omega_{N/2}^{km} \\ &= X_e[k] + \omega_N^k X_o[k] \end{aligned}$$
  - $X_e[k]$ has length $N/2$
  - observe that $X_e[k] = X_e[k+N/2], X_o[k] = X_o[k+N/2]$
  - now:
    - $X[k] = X_e[k] + \omega_N^k X_o[k]$
    - $X[k+N/2] = X_e[k+N/2] + \omega_N^{k+N/2} X_o[k+N/2]$
    - $= X_e[k] - \omega_N^k X_o[k]$
  - complexity: $\Theta((N/2)^2) \to 2\Theta(N^2/4) + \Theta(N/2) + \Theta(N)$

13 11.16 8w

13.1 warm up problem

- suppose we have $\{y[0], y[1]\}$, compute the 2-point dft
  - $Y[k] = y[0]e^0 + y[1]e^{-j\frac{2\pi}{N}k}$
  - $Y[0] = y[0] + y[1]$
  - $Y[1] = y[0] - y[1]$

13.2 radix-2 fft

- $X[k] = \sum_{m=0}^{N/2-1} x[2m]\omega_N^{k\cdot 2m} + \sum_{m=0}^{N/2-1} x[2m+1]\omega_N^{k(2m+1)}$
- $X[k] = X_e[k] - \omega_N^k X_o[k]$
- $X_e[k] \to \Theta((N/2)^2) = \Theta(N^2/4)$
- $X_o[k] \to \Theta((N/2)^2) = \Theta(N^2/4)$
- $X[k + N/2] = X_e[k + N/2] - \omega_N^{k+N/2} X_o[k + N/2]$
- complexity: $2 \cdot N^2/4 + N/2$ multiplications + N additions = $\Theta(N^2 + N) = \Theta(N^2)$
- continue splitting
  - $X_e[k] = X_{ee}[k] + \omega_{N/2}^k X_{eo}[k]$
  - $X_e[k + N/4] = X_{ee}[k] - \omega_{n/2}^k X_{eo}[k]$
  - $X_o[k] = X_{oe}[k] + \omega_{N/2}^k X_{oo}[k]$
  - $X_o[k + N/4] = X_{oe}[k] - \omega_{N/2}^k X_{oo}[k]$
- can continue splitting p times
- $\Theta(N)$ complexity in each state
- total complexity: $\Theta(N \log N)$
- $N \log N \ll N^2$

13.3 application of fft (compression)

- 1 hour song at 44kHz
  - humans can only hear 20 - 20000 Hz
  - very few sounds are above 10000 Hz
  - song is usually on a major scale, few dominant notes
    - just look at fft and keep top 100 frequencies by magnitude
  - $\to$ 100 floating points
- 2d fft to compress image
  - $x[n_1, n_2] \to X[k_1, k_2]$
  - bright disc in the center and everything else is dark
  - lossy compression

13.4 z-transform

- motivation
  - dtft of a step function $u[n]$
  - doesnt exist
  - recall that a signal must be absolutely summable to have a dtft
  - intuition (102)
  - fourier transform of an exponential function (e.g. e^3): kill / damp it by multiplying by $e^{-3} \rightarrow$ laplace transform
  - z-transform helps with analysis of exploding signals
  - z-transform helps solve diff eqs
- next phase of sig + sys
  - $y[n] = ax[n-1] + x[n-2] \longrightarrow H(\Omega)$
  - reverse the idea
    - given what $H(\Omega)$ looks like
    - design the system engineering accordingly

14 11.21 9m

14.1 z-transform

- $x[n] \xrightarrow{\mathbb{Z}} X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}, z = re^{j\Omega} \in \mathbb{C}$
- $X(z) = \mathbb{Z}\{x[n]\}$
- observe: z-transform is a superset of dtft
- $X(z)\big|_{z=re^{j\Omega}} = \sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-j\Omega n}$
  - $r = 1 \Rightarrow X(\Omega)$ (dtft)
- dtft is obtained from z-transform for setting $z = e^{j\Omega}$ if $X(z)$ exists for $e^{j\Omega}$
- example
  - $x[n] = \left\{\dot{1}, 5, 7, 9, 13\right\}$: finite duration, causal
  - $X(z) = \displaystyle\sum_{n=-\infty}^{\infty} x[n]z^{-n}$
    $= 1z^0 + 5z^{-1} + 7z^{-2} + 9z^{-3} + 13z^{-4}$
  - $X(z)$ exists for $z \neq 0 + j \cdot 0$ or $|z| \neq 0$
  - concept: region of convergence: values of z for which $X(z)$ "exists"
    - "exists" means it is not infinity
- example
  - $x[n] = \left\{3, 1, \dot{2}, 4, 5\right\}$: non-causal, finite duration
  - $\mathbb{Z}\{x[n]\} = X(z) = 3z^2 + z + 2 + 4z^{-1} + 5z^{-2}$
  - region of convergence: $|z| \neq 0$ and $|z| \nrightarrow \infty$
- example

  - try to compute $\mathbb{Z}\{\delta[n]\}$
  - $\delta[n] \xrightarrow{\mathbb{Z}} \delta[0]z^{-0} = 1$
  - roc: entire z-plane

- example
  - $\mathbb{Z}\{\delta[n-k]\} = z^{-k}$
  - roc: $|z| \neq 0$ if $k > 0$, $|z| \nrightarrow \infty$ if $k < 0$, entire z-plane if $k = 0$

- example
  - $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$
  - $\mathbb{Z}\{u[n]\} = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$
  - roc: $|z| > 1$
  - summary: dtft, which is $X(z)\big|_{z=e^{j\Omega}}$ is not in the roc; this is consistent with our understanding that dtft of $u[n]$ does not exist

- example
  - compute the z-transform of $x[n] = a^n u[n]$: infinite duration, causal
  - $X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$
  - roc: $\left|az^{-1}\right| < 1$ or $|z| > |a|$
  - dtft exists if $|a| < 1$

- observation: z-transform are fractional polynomials
  - $X(z)$ is defined by $z_1, z_2, \ldots, z_M$ zeroes and $p_1, p_2, \ldots, p_N$ poles (roc does not contain poles)

- examples
  - $x[n] = \begin{cases} 1 & 0 \leq n \leq N-1 \\ 0 & n < 0 \vee n > N \end{cases}$
  - $X(z) = \sum_{n=0}^{N-1} z^{-n} = \frac{1-z^{-N}}{1-z^{-1}} = \frac{z^N - 1}{z^{N-1}(z-1)}$
  - roc: $|z| \neq 0$
    - note: $z = 1$ works

- example
  - $x[n] = -a^n u[-n-1] = \begin{cases} -a^n & n < 0 \\ 0 & n \geq 0 \end{cases}$
  - $X(z) = \sum_{n=-\infty}^{\infty} -a^n u[-n-1] z^{-n} = \sum_{n=-\infty}^{-1} -a^n z^{-n} = -\sum_{m=1}^{\infty} a^{-m} z^m = -\frac{a^{-1}z}{1-a^{-1}z} = \frac{z}{z-a}$
  - roc: $\left|a^{-1}z\right| < 1$ or $|z| < |a|$
  - z-transform is uniquely defined by $X(z)$ *and* roc

- example
  - $x[n] = 3^{-n}u[n] - 2^{-n}u[-n-1]$
  - $x_1[n] = 3^{-n}u[n]$
  - $x_2[n] = 2^{-n}u[-n-1]$
  - $X(z) = X_1(z) - X_2(z) = \frac{z}{z-3^{-1}} - \frac{z}{z-2^{-1}}$
  - roc: $|z| \in (3^{-1}, \infty) \cap [0, 2^{-1}) = (3^{-1}, 2^{-1})$

- summary
  - roc is circularly shaped in z-transform
  - roc cannot have any poles
  - causal signals are outside the circle
  - anticausal signals are inside the circle
  - signals that decompose into causal and anticausal signals have a ring shaped roc

15 11.23 9w

15.1 z-transform properties

- recall: $x[n] \xrightarrow{\mathbb{Z}} X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$ where $z \in \mathbb{C}$
  - $z = e^{j\Omega} \longrightarrow$ dtft
- linearity
  - if $x_1[n] \xrightarrow{\mathbb{Z}} X_1(z)$ and $x_2[n] \xrightarrow{\mathbb{Z}} X_2(z)$
  - then $\alpha x_1[n] + \beta x_2[n] \xrightarrow{\mathbb{Z}} \alpha X_1(z) + \beta X_2(z)$
  - roc: recalculate poles
- time shifting
  - if $x[n] \xrightarrow{\mathbb{Z}} X(z)$
  - then $x[n-k] \xrightarrow{\mathbb{Z}} X(z)z^{-k}$
  - proof
    - $\mathbb{Z}\{x[n-k]\} = \sum_{n=-\infty}^{\infty} x[n-k]z^{-n} = \sum_{m=-\infty}^{\infty} x[m]z^{-(m+k)} = z^{-k}\sum_{m=-\infty}^{\infty} x[m]z^{-m} = z^{-k}X(z)$
  - new roc $\neq$ roc
  - $k > 0$: roc excludes $|z| = 0$
  - $k < 0$: roc excludes $|z| \to \infty$
- convolution property
  - if $x_1[n] \xrightarrow{\mathbb{Z}} X_1(z)$ and $x_2[n] \xrightarrow{\mathbb{Z}} X_2(z)$
  - then $x_1[n] * x_2[n] \xrightarrow{\mathbb{Z}} X_1(z)X_2(z)$
  - proof
    - $x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]$
    - $$\begin{aligned}\mathbb{Z}\left\{\sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]\right\} &= \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]\right)z^{-n} \\ &= \sum_{k=-\infty}^{\infty} x_1[k]\sum_{n=-\infty}^{\infty} x_2[n-k]z^{-n} \\ &= x_1[k]\sum_{k=-\infty}^{\infty} X_2(z)z^{-k} \\ &= X_2(z)\sum_{k=-\infty}^{\infty} x_1[k]z^{-k} \\ &= X_1(z)X_2(z)\end{aligned}$$

– combined roc depends on the poles (pay attention to zero-pole cancellations)

15.2 inverse z-transform

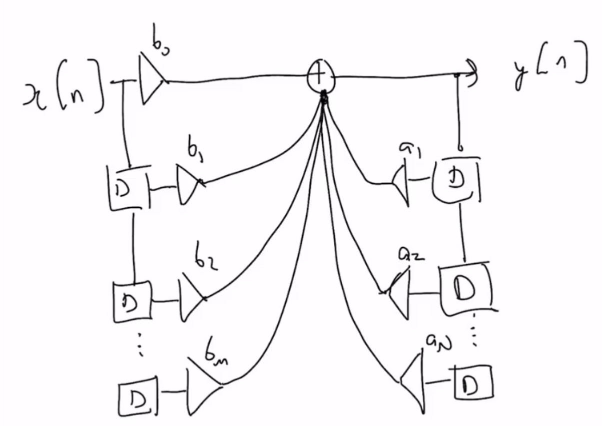
- $X(z) = \frac{B(z)}{(z-p_1)(z-p_2)\cdots(z-p_N)}$
- recall
  - $a^n u[n] \xrightarrow{\mathbb{Z}} \frac{z}{z-a}, |z| > |a|$
  - $-a^n u[-n-1] \xrightarrow{\mathbb{Z}} \frac{z}{z-a}, |z| < |a|$
- $X(z) = \frac{k_1 z}{z-p_1} + \frac{k_2 z}{z-p_2} + \cdots + \frac{k_N z}{p-p_N}$
- example
  - $X(z) = \frac{(z-1)(z+2)}{(z-1/2)(z-2)}$
  - $\frac{X(z)}{z} = \frac{(z-1)(z+2)}{z(z-1/2)(z-2)} = \frac{k_1}{z} + \frac{k_2}{z-1/2} + \frac{k_3}{z-2}$
  - $k_1 = \left(\frac{k_1}{z} + \frac{k_2}{z-1/2} + \frac{k_3}{z-2}\right) \cdot (z-0)\Big|_{z=0} = (z-0)\frac{X(z)}{z} = X(z)\Big|_{z=0} = \frac{(0-1)(0+2)}{(0-1/2)(0-2)} = -2$
  - $k_2 = \frac{(1/2-1)(1/2+2)}{(1/2)(1/2-2)} = \frac{5}{3}$
  - $k_3 = \frac{(2-1)(2+2)}{2(2-1/2)} = \frac{4}{3}$
  - $X(z) = -2 + \frac{(5/3)z}{z-1/2} + \frac{(4/3)z}{z-2}$
  - $X(z) \xrightarrow{\mathbb{Z}^{-1}} x[n] = -2\delta[n] + \frac{5}{3}x_1[n] + \frac{4}{3}x_2[n]$
  - roc $|z| > 2$: $x_1[n] = 2^{-n}u[n], x_2[n] = 2^n u[n]$
  - roc $|z| < 2^{-1}$: $x_1[n] = -2^n u[-n-1], x_2[n] = -2^{-n}u[-n-1]$
  - roc $2^{-1} < |z| < 2$: $x_1[n] = 2^{-n}u[n], x_2[n] = -2^{-n}u[-n-1]$
- summary
  - find poles $\{p_i\}, i = 1, \ldots, N$
  - find coefficients for partial fraction $k_i = X(z)(z-p_i)\Big|_{z=p_i}$
  - based on roc, digure out which causality of canonical z-transform pairs to use

15.3 using z-transform to analyze / design dtlti systems

- $H(z) = \mathbb{Z}\{h[n]\}$
- $H(z) = \frac{Y(z)}{X(z)}$: easier to obtain
- recall: dtlti is defined by a constant coefficient difference equation w suitable initial conditions
- $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$
  - z-tranfrom on both sides: $\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k X(z) z^{-k}$
  - $\frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k Z^{-k}} = H(z)$
- given $H(z)$, say smth abt the system
  - causal? $h[n]$ only defined for $n \geq 0$
    - check that $|z| \to \infty$ is defined on the z-plane
    - $\lim_{z\to\infty} H(z) = \lim_{z\to\infty} \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} < \infty$
    - for causality: $N \geq M$
  - stable? $h[n]$ bounded or dtft bounded / exists
    - check that unit circle is in roc

15.4 design of dtlti

- $H(z) = \frac{b_0+b_1z^{-1}+\cdots+b_Mz^{-M}}{1+a_11^{-1}+\cdots+a_Nz^{-N}}$
- $y[n] = b_0x[n] + b_1x[n-1] + \cdots + b_Mx[n-M] - a_1y[n-1] - \cdots - a_Ny[n-N]$



- steps
  - given $H(z)$ in pole / zero notation
  - given $H(z)$ in polynomial
  - constant coefficient diff eq
  - system circuit / design
- gap: how to go from high level idea, e.g. "filter out frequencies from 30 - 50Hz", to $H(z)$

16 11.28 10m

16.1 review

- recall: $H(z) = \frac{(z-z_1)\cdots(z-z_N)}{(z-p_1)\cdots(z-p_M)}$ (poles and zeroes)
- $H(z) = \frac{\sum_{k=0}^{M} b_kz^{-k}}{\sum_{k=0}^{N} a_kz^{-k}}$ (polynomial coefficients / constant coefficient difference equation → block diagram)

16.2 design question

- low pass filter with pas_and of 20000 Hz
- stable system: dtft exists → roc includes unit circle
- $H(z) = \frac{\prod_{i=0}^{M}(z-z_i)}{\prod_{i=0}^{N}(z-p_i)}$
- $H(e^{j\Omega}) = \frac{\prod_{i=0}^{M}(e^{j\Omega}-z_i)}{\prod_{i=0}^{N}(e^{j\Omega}-p_i)}$
- $\left|H(e^{j\Omega})\right| = \frac{\prod_{i=0}^{M}(e^{j\Omega}-z_i)}{\prod_{i=0}^{N}(e^{j\Omega}-p_i)}$
  - Ω can express in Hz if we want to
- key insight

  - a frequency in Hz that is near a zero is attenuated
  - a frequency near a pole is amplified

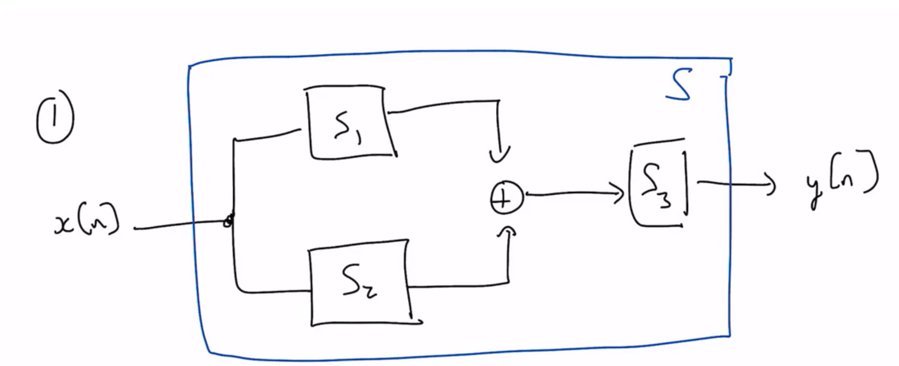
16.3 fir filter

- "finite impulse response"
- $H(e^{j\omega}) = \sum_{k=0}^{M} b_k z^{-n}$
- $y[n] = x[n] + b_1 x[n-1] + b_2 x[n-2]$: ~convolution
- easy to do optimization to find b_k's
- find frequency / phase response
- with enough b_k's, we can make pretty much any filter
- problem: requires large M

16.4 iir filter

- $H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$
- add feedback
- complex designs with fewer coefficients
- nonlinear effects (phase unpredictable)

16.5 practice



- "detective / sleuth" question
  - evaluate $H(z)$
  - S_1: has a pole at 0.5, a zero at 0, and $H_1(1) = -3$
  - S_2: $y[n] - 0.7y[n-1] = 2.5x[n] - 1.6x[n-1]$
  - S_3: $h_3[n] = \frac{5}{4}(0.8)^n u[n-1]$
  - $H(z) = (H_1(z) + H_2(z)) \cdot H_3(z)$
  - $H_1(z) = A \cdot \frac{z}{z-0.5} = -\frac{3z}{2(z-0.5)}$
    - roc: $|z| > 0.5$ (check if H goes to infinity as $z \to \infty$)
  - $H_2(z) = \frac{2.5-1.6z^{-1}}{1-0.7z^{-1}} = \frac{2.5z-1.6}{z-0.7}$
    - roc: $|z| > 0.7$
  - $H_3(z) = \mathbb{Z}\{h_3[n]\} = \mathbb{Z}\left\{(0.8)^{n-1}u[n-1]\right\} = z^{-1}\left(\frac{z}{z-0.8}\right) = \frac{1}{z-0.8}$

  - roc: $|z| > 0.8$
- $H(z) = \frac{(z-1)(z-0.8)}{(z-0.5)(z-0.7)(z-0.8)} = \frac{z-1}{(z-0.5)(z-0.7)}$
  - roc: $|z| > 0.7$ because of zero-pole cancellation