

ECE 113 (Digital Signal Processing)

November 30, 2022

1 9.26 1m

- syllabus

1.1 ece102: signals & systems

- signals were modeled as functions of time
 - $x(t) : t \in \mathbb{R}$ (continuous time)
 - $x : \mathbb{R} \rightarrow \mathbb{R}, \mathbb{C}$
 - sampling $t = nT_s$
 - T_s = sampling interval
 - n = integer
 - $t - x$ plot $\xrightarrow{\text{sampling}}$ $n - x$ plot
 - $x(t)$ bandlimited ($|X(f)| \leq f_{\max}$) then if sampling rate is $f_s > 2f_{\max}$ (nyquist rate) then it is possible to recover the original signal
 - sampling at lower rate then there will be aliasing
- systems
 - $x(t) \rightarrow \boxed{S} \rightarrow y(t)$
 - $x[n] \rightarrow \boxed{S} \rightarrow y[n]$

1.2 three ways to write out signals

- mathematical expression
 - $x[n] = 3 \cos(2n), n \in \mathbb{Z}$
- tabular list of significant samples
 - $x[n] = \{-1, 8, 0, 5, -2\}, n \in \mathbb{Z}$
 - underspecified signal: loss of indices
 - typically put an arrow underneath the 0 index
 - convention: any sample not listed has 0 amplitude
- plotting

1.3 signal operations

- arithmetic
 - $g[n] = x[n] + A$
 - $g[n] = Bx[n]$
 - signal addition: $g[n] = x_1[n] + x_2[n]$
 - signal multiplication: $g[n] = x_1[n] \cdot x_2[n]$
- time shifting
 - ece102

- $x(t) \rightarrow x(t - \tau)$ delays signal by shifting it right
- $x(t) \rightarrow x(t + \tau)$ advances signal by shifting left
- ece113
 - $g[n] = x[n - k], k > 0$
- time scaling
 - ece102
 - $x(at), a > 1$: contraction
 - $x(at), 0 < a < 1$: dilation
 - ece113
 - $g[n] = x[2n]$ (downsampling)
 - $g[n] = x[\frac{1}{2}n]$ (upsampling)
 - $g[n] = x[-n]$ (time reversal)

1.4 basic signals

- $\delta(t)$ from ece102 \rightarrow hard concept to grasp
- $\delta[t]$ from ece113 \rightarrow :)
 - $\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$
 - $\delta[n]$ does not have complexities of $\delta(t)$
- sampling property of δ
 - $x[n] * \delta[n] = x[0] \cdot \delta[0] = x[0]$
 - $x[n] * \delta[n - k] = x[k] \cdot \delta[0] = x[k]$

2 9.28 1w

2.1 sampling properties hold in discrete domain as well

- $\sum_{n=-\infty}^{\infty} x[n] \cdot \delta[n - k] = \sum_n x[k] \delta[n - k] = x[k] \sum_n \delta[n - k] = x[k]$

2.2 systems

- unit step system $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$
 - $u[n] - u[n - 1] = \delta[n]$
 - $\delta[n]$ is the first derivative of $u[n]$
 - ece102: $\delta(t) = \frac{du(t)}{dt}$
 - $u[n] = \sum_{k=0}^{\infty} \delta[n - k]$
- unit ramp signal $r[t] = \begin{cases} n & n \geq 0 \\ 0 & n < 0 \end{cases}$

- 102: $r(t) = tu(t)$
- $r[n] = n \cdot u[n] = \sum_{k=1}^{\infty} u[n-k]$

2.3 how can we express $x[n]$ using basic signals?

- generic signal representation using the “canonical basis” of $\{\delta[n-k] \forall k\}$
- $x[n] = \{1, 2, (0), 1, 1\}$
- $x[n] = \delta[n+2] + 2\delta[n+1] + \delta[n-1] + \delta[n-2]$

2.4 sinusoidal signals

- $x(t) = A \cos(\omega_0 t + B)$, $\omega_0 = 2\pi f_0$
- $x[n] = A \cos(\Omega_0 n + \theta)$, $\Omega_0 = 2\pi F_0 \in [-\pi, \pi]$ where F_0 is the normalized frequency
- $x[n] = A \cos(2\pi f_0 n T_0 + \theta) = A \cos\left(2\pi \frac{f_0}{f_s} n + \theta\right)$
- $F_n := \frac{f_0}{f_s} < 1$

2.5 complex exponential

- $x[n] = A e^{j(2\pi F_0 n + \theta)}$

2.6 additional signal properties

1. duration of signal
2. real or complex
3. periodicity: a signal is periodic if $\exists N \in \mathbb{Z} : x[n+N] = x[n] \forall n$
 - $x[n]$ is also periodic with $2N, 3N$, etc.
 - $x[n+kN] = x[n] \forall k, n$
 - fundamental period: smallest positive N such that $x[n+N] = x[n]$
 - e.g. $x[n] = \cos(2\pi F_0 n) \rightarrow \exists k \in \mathbb{Z} : 2\pi F_0 N = 2\pi k \rightarrow F_0 N \in \mathbb{Z}, N = \frac{k}{F_0}$
 - is $x[n] = \cos(0.2n)$ periodic?
 - no
 - $2\pi F_0 = 0.2 \rightarrow \frac{k}{F_0} = 10k\pi \notin \mathbb{Z}$

3 10.3 2m

3.1 recap

- recall: a signal is periodic if $x[n+N] = x[n]$ for all n and some $N \in \mathbb{Z}$
- is $\cos(0.2n)$ periodic?
 - no due to irrational period
- is the sum of x_1 and x_2 periodic if they are periodic signals?

3.2 periodicity

- $x_1[n] = 2 \cos(0.4\pi n)$, $x_2[n] = 1.5 \cos(0.48\pi n)$
- $F_1 = \frac{0.4\pi}{2\pi} = 0.2$
- $F_2 = \frac{0.48\pi}{2\pi} = 0.24$
- $N_1 = \frac{K_1}{F_1} = \frac{K_1}{0.2} = 5K_1 = 5$
- $N_2 = \frac{K_2}{F_2} = \frac{K_2}{24/100} = \frac{100}{24}K_2 = \frac{25}{6}K_2 = 25$
- fact: sum of two periodic signals is itself a periodic signal
- $x_3[n] := x_1[n] + x_2[n]$ is periodic with period N_1N_2
 - $x_3[n + N_1N_2] = x_1[n + N_1N_2] + x_2[n + N_1N_2] = x_1[n] + x_2[n] = x_3[n]$
 - e.g. $N_1 = 5, N_2 = 25 \Rightarrow 2 \cos(0.4\pi n) + 1.5 \cos(0.48\pi n)$ has a period of $5 \cdot 25 = 125$
 - fundamental period: $\text{lcm}(5, 25) = 25$
- e.g. $x[n]$ is periodic w $N = 12$
- are the following periodic
 - $y[n] = x[-n]$
 - yes: $y[n + 12] = x[-(n + 12)] = x[-n - 12] = y[n]$
 - $y[n] = x[n + 1]$
 - yes: $y[n + 12] = x[(n + 12) + 1] = x[n + 1] = y[n]$
 - $y[n] = x[3n]$
 - yes: $y[n + 4] = x[3(n + 4)] = x[3n + 12] = x[3n] = y[n]$
 - $y[n] = x^2[n]$
 - yes: $y[n + 12] = (x[n + 12])^2 = (x[n])^2 = y[n]$

3.3 even and odd signals

- even: $x[n] = x[-n]$
- odd: $x[n] = -x[-n]$ or $-x[n] = x[-n]$
- any signal can be decomposed into odd and even components
 - $x[n] = x_{\text{even}}[n] + x_{\text{odd}}[n]$
- $x_{\text{even}}[n] = \frac{x[n] + x[-n]}{2}$
- $x_{\text{odd}}[n] = \frac{x[n] - x[-n]}{2}$

3.4 complex signals

- $x[n] = x_R[n] + jx_I[n] = |x[n]| e^{j\angle x[n]}$
- $x^*[n] = x_R[n] - jx_I[n] = |x[n]| e^{-j\angle x[n]}$
- “even symmetry” for complex
 - $x[-n] = x^*[n]$
 - $x^*[-n] = x[n]$
- “odd antisymmetry” for complex
 - $x[-n] = -x^*[n]$
 - $x^*[-n] = -x[n]$

3.5 decompose of even / odd for complex signals

- $x[n] = x_e[n] + x_o[n]$
- $x_e[n] = \frac{x[n] + x^*[-n]}{2}$
- $x_o[n] = \frac{x[n] - x^*[-n]}{2}$

3.6 energy + power of a signal

- energy: $E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$
 - if $E_x < +\infty$: **finite energy signal**
 - e.g. $x[n] = \cos(2\pi 0.1n + 0.3\pi)$ is not finite energy
- power
 - consider a periodic signal with period N
 - its power is expressed as $P_x = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2$
 - consider a non-periodic signal
 - its power is expressed as $P_x = \lim_{M \rightarrow \infty} \frac{1}{2M+1} \sum_{n=-M}^M |x[n]|^2$
- e.g. calculate the energy of $x[n] = (0.5)^n u[n]$
 - $E_x = \sum_{n=-\infty}^{\infty} |(0.5)^n u[n]|^2 = \sum_{n=-\infty}^{\infty} (0.5)^{2n} u^2[n] = \sum_{n=0}^{\infty} (0.5)^{2n} = \frac{1}{1-0.25} = \frac{4}{3}$

3.7 systems

- $x[n] \rightarrow \boxed{S} \rightarrow y[n]$
- properties
 - linearity
 - $y_1[n] = S\{x_1[n]\}$
 - $y_2[n] = S\{x_2[n]\}$
 - $x_3[n] = ax_1[n] + bx_2[n]$
 - $y_3[n] = S\{x_3[n]\}$
 - linear $\Rightarrow y_3[n] = ay_1[n] + by_2[n]$
 - $S\{\sum_{k=1}^N a_k x_k[n]\} = \sum_{k=1}^N a_k S\{x_k[n]\}$
 - time-invariance
 - $y[n] = S\{x[n]\}$
 - time invariance $\Rightarrow y[n-k] = S\{x[n-k]\}$
 - causality
 - if current output $y[n]$ depends only on the current and past input samples, the system is causal
 - e.g. $y[n] = x[n] - nx[n-2]$: causal
 - e.g. $y[n] = (n+1)x[n]$: causal (also memory-less)
 - e.g. $y[n] = x[n] + x[n+2]$: not causal
 - bounded input, bounded output (bibo) stability
 - if $|x[n]| < B$ for all n , then $|y[n]| < C$ for all n , B, C are constants

- e.g. $y[n] = 3x[n] + 2x[n-1]$
 - suppose $|x[n]| < B$
 - stable: $|y[n]| = |3x[n] + 2x[n-1]| \leq 3|x[n]| + 2|x[n-1]| < 5B$
- e.g. $y[n] = x^2[n]$
 - suppose $|x[n]| < B$
 - stable: $|y[n]| = |x^2[n]| = |x[n]|^2 < B^2$
- e.g. $y[n] = nx[n-1]$
 - not stable: as $n \rightarrow \infty, y \rightarrow \infty$
- e.g. $y[n] = 3x[n] + 2x[n-1]$
 - time invariant: $x_1[n] = x[n-k] \Rightarrow y_1[n] = 3x[n-k] + 2x[n-k-1] = y[n-k]$
- e.g. $y[n] = x^2[n]$
 - not linear
 - time invariant: $x_1[n] = x[n-k] \Rightarrow y_1[n] = x_1^2[n] = x^2[n-k] = y[n-k]$
- e.g. $y[n] = nx[n-1]$
 - not time invariant: $x_1[n] = x[n-k] \Rightarrow y_1[n] = nx_1[n-1] = nx[n-1-k] \neq (n-k)x[n-1-k] = y[n-k]$
- e.g. $y[n] = x[n/4]$
 - linear
 - $y_1[n] = x_1[n/4]$
 - $y_2[n] = x_2[n/4]$
 - $x_3[n] = ax_1[n] + bx_2[n]$
 - $y_3[n/4] = x_3[n/4] = ax_1[n/4] + bx_2[n/4] = ay_1[n] + by_2[n]$
 - not time invariant
 - not causal
 - $n = -4 \Rightarrow y[-4] = x[-1]$
 - stable
 - suppose $|x[n]| < B$
 - $|y[n]| = |x[n/4]| < B$

4 10.5 2w

4.1 hw q4

- suppose $x[n]$ periodic with period N , is the following periodic
- $y[n] = x[1-2n]$
 - yes, $y[n+N] = x[1-2n-2N] = x[1-2n] = y[n]$
- $y[n] = x[n] + (-1)^n x[0]$
 - yes, $y[n+2N] = x[n+2N] + (-1)^{n+2N} x[0] = x[n] + (-1)^n x[0] = y[n]$
 - fundamental period: $\begin{cases} \text{lcm}(N, 2) & x[0] \neq 0 \\ N & \end{cases}$

4.2 constant coefficient difference equation

- $x[n] \rightarrow \boxed{S} \rightarrow y[n] = S \{x[n]\}$
- $\sum_{k=0}^M a_k y[n-k] = \sum_{k=0}^N b_k x[n-k]$
- $a_0, \dots, a_M, b_0, \dots, b_N$: constant coefficients
- M, N : order of the equation
 - often, $\max \{M, N\}$ to get the system order
- e.g. $y[n] = \frac{1}{N} \sum_{k=0}^{N-1} x[k]$
 - $a_0 = 1$
 - $b_0, \dots, b_{N-1} = \frac{1}{N}$
 - order: $N - 1$
 - length N moving average
- e.g. exponential smoother
 - $y[n] = (1 - \alpha)y[n-1] + \alpha x[n]$
 - $0 < \alpha < 1$
 - $a_0 = 1, a_1 = \alpha - 1$
 - $M = 1$
 - $b_0 = \alpha$
 - $N = 0$
 - because $\{a_1, \dots, a_M\} \neq \{0\}$, this system is recursive, or, equivalently, has feedback
- if a system has feedback, then we need to know the *initial condition* of the system
 - initial condition is the time when input is applied
- e.g. $y[n] = y[n-1] + x[n]$
 - suppose $x[n] = u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$
 - $y[0] = y[-1] + x[0], y[-1] = y[-2] + x[-1], \dots$
 - initial condition: $y[-1] = 2$
 - $y[-1] = 2, y[0] = y[-1] + x[0] = 2 + 1 = 3, y[1] = y[0] + x[1] = 3 + 1 = 4, \dots$
 - $y[-1] = 3 \Rightarrow y[0] = 4, y[1] = 5, \dots$
- “relaxed system”: a system is relaxed if $y[n] = 0$ for $n \rightarrow \infty$
- are relaxed systems described by a constant coefficient difference equation lti?

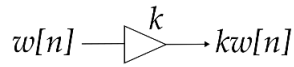
4.3 visual method to look at ccdds

- $\sum_{k=0}^M a_k y[n-k] = \sum_{k=0}^N b_k x[n-k]$
- e.g. delay element
 - “unit delay”

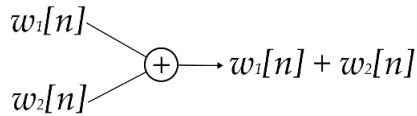
$$w[n] \xrightarrow{\boxed{D}} w[n-1]$$

- e.g. multiplier

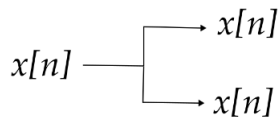
- constant multiplier



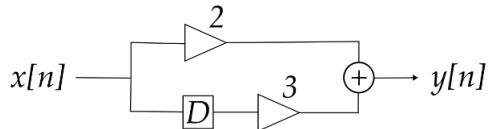
- e.g. adder



- e.g. branch



- block diagram for $y[n] = 2x[n] + 3x[n-1]$



4.4 impulse response

- $\delta[n] \rightarrow \boxed{S} \rightarrow h[n] := S\{\delta[n]\}$
- recall: when a system is lti, then we have an expression for the output with respect to an arbitrary $x[n]$
- $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$
- $y[n] = S\{x[n]\} = S\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\} = \sum_{k=-\infty}^{\infty} S\{\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$

4.5 convolution properties

- commutative: $x_1[n] * x_2[n] = x_2[n] * x_1[n]$
- distributive: $x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$
- associative: $x_1[n] * (x_2[n] * x_3[n]) = (x_1[n] * x_2[n]) * x_3[n]$
- identity: $x[n] * \delta[n] = x[n]$
 - $x[n] * \delta[n-m] = x[n-m]$

5 10.10 3m

5.1 simple discrete-time convolution

- $h[n] = \{(4)_{N_2=7}, 3, 2, 1\}$

- $x[n] = \{(-3)_{N_1=5}, 7, 4\}$
- $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$
- what length will the output $y[n]$ be? where is the support?
 - $N_1 \leq k \leq N_1 + 2$
 - $N_2 \leq n - k \leq N_2 + 3$
 - $n - N_2 - 3 \leq k \leq n - N_2$
 - $y[n] = \sum_{k=K_1}^{K_2} x[k]h[n-k]$
 - $K_1 = \max \{N_1, n - N_2 - 3\} = \max \{5, n - 10\}$
 - $K_2 = \min \{N_1 + 2, n - N_2\} = \min \{7, n - 7\}$
 - $n - 10 \leq 7 \Rightarrow n \leq 17$
 - $n - 7 \geq 5 \Rightarrow n \geq 12$
 - support: $\{12, 13, 14, 15, 16, 17\}$
- generalization
 - $x[n]$ starts at N_1 with length L_x
 - $h[n]$ starts at N_2 with length L_h
 - $K_1 = \max \{N_1, n - N_2 - L_h\}$
 - $K_2 = \min \{N_1 + L_x, n - N_2\}$
 - $n - N_2 - L_h \leq N_1 + L_x \Rightarrow n \leq N_1 + N_2 + L_x + L_h$
 - $n - N_2 \geq N_1 \Rightarrow n \geq N_1 + N_2$
 - nonzero samples of convolution: $N_1 + N_2 \leq n \leq N_1 + N_2 + L_h + L_x$
 - number of nonzeros: $L_h + L_x + 1$

5.2 graphical way of computing convolution

- $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$
- sketch $x[k]$
- sketch $h[-k]$
- sketch $h[n-k]$
- sample by sample multiplication of $x[k] \cdot h[n-k]$ for every n that overlaps
- e.g.
 - $x[n] = u[n] - u[n-7]$
 - $h[n] = (0.9)^n u[n]$
 - $y = x * h$
 - $y = 0$ when $n < N_1 + N_2 = 0$
 - $0 \leq n \leq 6$
 - $y[n] = \sum_{k=0}^n x[k] \cdot h[n-k] = \sum_{k=0}^n 1 \cdot (0.9)^{n-k} = (0.9)^n \sum_{k=0}^n (0.9)^{-k} = (0.9)^n \frac{1-(0.9)^{-(n+1)}}{1-(0.9)^{-1}}$
 - $n > 6$

5.3 discrete time fourier series

- consider a discrete-time periodic signal
 - $x[n] = x[n + N]$

- infinite duration signal, which we represent as a vector of N samples
- $x = \begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix} \in \mathbb{R}^n, \mathbb{C}^n$
- key learning aims for 113
 - decompose a signal $x[n]$ into a set of "basis" signals
- $x[n] = \sum c_k \underbrace{y_k[n]}_{\text{"basis"}}$
- $c_k \in \mathbb{R}, \mathbb{C}$ are scalar coefficients

5.4 review: inner product

- $\langle x, y \rangle = \sum_{k=0}^{N-1} x[k]y[k]$ for $x, y \in \mathbb{R}^N$
 - "dot product"
- $\langle x, x \rangle = \sum_{k=0}^{N-1} x[k] \cdot x[k] = \sum_{k=0}^{N-1} x[k]^2 = |x|_2^2$
 - "squared norm"
- $|x|_2 = \sqrt{\langle x, x \rangle}$
 - "norm" of a vector

6 10.12 3w

6.1 discrete time fourier series

- $x[n] = \sum_k c_k y_k[n]$
- $x = \sum_k c_k y_k$
- inner product: $\langle x, y \rangle = \sum_{k=0}^{N-1} x_k \cdot y_k$
- $\langle x, x \rangle = \sum_{k=0}^{N-1} x_k^2 = |x|_2^2$
- $|x|_2 = \sqrt{\langle x, x \rangle}$
- $\langle x, y \rangle = |x|_2 |y|_2 \cos \alpha$
- $\cos \alpha = \frac{\langle x, y \rangle}{|x|_2 |y|_2} = \frac{\langle x, y \rangle}{|x| |y|}$
- $\langle x, y \rangle = 0$ when x and y are orthogonal
- find a set $\{y_k\}_k$ of basis vectors that are *orthonormal*
 - $\langle y_k, y_m \rangle = \begin{cases} 0 & k \neq m \\ 1 & k = m \end{cases}$
 - e.g. $y_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $y_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \end{bmatrix}$ are orthonormal
- synthesis: given a basis and coefficients, synthesize vectors
 - $c_0 = \sqrt{2}, c_1 = \sqrt{2} \cdot 5$

$$- x = \sum c_k y_k = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$$

- analysis: given a basis and a signal x , what is the representation of x in this basis?
 - find c_0, c_1 given x, y_0, y_1
 - $c_0 = \langle x, y_0 \rangle = \sum x[k] y_0[k]$
 - $c_1 = \langle x, y_1 \rangle = \sum x[k] y_1[k]$

6.2 periodic signals

- suppose $\tilde{x}[n]$ is a periodic signal
- goal is to represent $\tilde{x}[n] = \sum_k \tilde{c}_k \phi_k[n]$
- $\phi_k[n] = e^{j\Omega_k n}$ fourier basis
- Ω_k is the angular frequency of the k -th basis vector
- $\tilde{x}[n] = \sum_k \tilde{c}_k e^{j\Omega_k n}$
- three open questions
 - what is Ω_k ?
 - how many basis signals / vectors are needed?
 - what are the coefficients \tilde{c}_k ?

6.3 what is Ω_k

- suppose \tilde{x} is periodic with period N
- what is the periodicity of ϕ_k ?
- ϕ_k has period N
- therefore $\phi_k[n] = \phi_k[n + N] \Rightarrow e^{j\Omega_k n} = e^{j\Omega_k(n+N)} = e^{j\Omega_k n} e^{j\Omega_k N}$
- $e^{j\Omega_k N} = 1$
- $\Omega_k N = 2\pi k \Rightarrow \Omega_k = \frac{2\pi}{N} \cdot k$
- $\phi_k(n) = e^{j\frac{2\pi}{N} kn}$

6.4 how many $\phi_k[n]$ are there

- suppose we have a finite set of N basis vectors $\{\phi_0, \dots, \phi_{N-1}\}$
- is this sufficient?
- $\phi_{k+N}[n] = e^{j\frac{2\pi}{N}(k+N)n} = e^{j\frac{2\pi}{N}kn} e^{j2\pi n} = e^{j\frac{2\pi}{N}kn} = \phi_k[n]$
- $\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{c}_k \phi_k[n]$
- find the basis vectors for $\tilde{x}[n] = \cos(0.2\pi n)$
 - $\Omega_0 = 0.2\pi$
 - $F_0 = \frac{0.2\pi}{2\pi} = 0.1$
 - $N = 10 \Rightarrow \{\phi_0, \dots, \phi_9\}$
 - $\tilde{x}[n] = \sum_{k=0}^9 \tilde{c}_k e^{j0.2\pi kn}$

- how to compute \tilde{c}_k

$$\begin{aligned}
 \left\langle \tilde{x}[n], e^{j\frac{2\pi}{N}kn} \right\rangle &= \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn} \\
 &= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \tilde{c}_m e^{j\frac{2\pi}{N}mn} e^{-j\frac{2\pi}{N}kn} \\
 &= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \tilde{c}_m e^{j\frac{2\pi}{N}(m-k)n} \\
 &= \tilde{c}_k N \\
 \tilde{c}_k &= \frac{1}{N} \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn}
 \end{aligned}$$

6.5 discrete time fourier series (dtfs) summary

- synthesis equation: $\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{c}_k e^{j\frac{2\pi}{N}kn}$
- analysis equation: $\tilde{c}_k = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn}$

7 10.19 4w

7.1 discrete time fourier series (dtfs)

- $\tilde{x}[n] = \tilde{x}[n + N] = \tilde{x}[n + kN]$
- $\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{c}_k e^{j\frac{2\pi}{N}kn}$
- $\tilde{c}_k = \left\langle \tilde{x}[n], \frac{1}{N} e^{j\frac{2\pi}{N}kn} \right\rangle$
- $\langle x[n], y[n] \rangle := \sum_{n=0}^{N-1} x[n] y^*[n]$
- $\tilde{c}_k = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn}$
- side note: pulse train signal
 - $\tilde{x}[n] = \begin{cases} 1 & -L \leq n \leq L \\ 0 & L < n < N - L \end{cases}$
 - periodic with N
 - homework or exam: calculate dtfs $\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{c}_k e^{j\frac{2\pi}{N}kn}$

7.2 properties of dtfs

- periodicity: coefficients are periododdic with N
 - $\tilde{c}_k = \tilde{c}_{k+rN}, r \in \mathbb{Z}$
- linearity
 - $\tilde{x}_1[n], \tilde{x}_2[n]$ periodic with same period N
 - $\tilde{x}_1[n] \xrightarrow{\text{dtfs}} \tilde{c}_k$
 - $\tilde{x}_2[n] \xrightarrow{\text{dtfs}} \tilde{d}_k$

- $\alpha \tilde{x}_1[n] + \beta \tilde{x}_2[n] \xrightarrow{\text{dtfs}} \alpha \tilde{c}_k + \beta \tilde{d}_k$
- proof
 - $\tilde{x}_1[n] = \sum_{n=0}^{N-1} \tilde{c}_k e^{j \frac{2\pi}{N} kn}$
 - $\tilde{x}_2[n] = \sum_{n=0}^{N-1} \tilde{d}_k e^{j \frac{2\pi}{N} kn}$
 - $\alpha \tilde{x}_1[n] + \beta \tilde{x}_2[n] = \sum_{n=0}^{N-1} (\alpha \tilde{c}_k + \beta \tilde{d}_k) e^{j \frac{2\pi}{N} kn}$
- if $x[n]$ is real-valued, then the dtfs \tilde{c}_k for a signal is a conjugate symmetric sequence
 - $\tilde{c}_k^* = \tilde{c}_{-k}$
 - $|\tilde{c}_k| = |\tilde{c}_{-k}|$
 - proof
 - $\tilde{x}[n] = \tilde{x}^*[n]$
 - $\tilde{c}_k^* = \left(\frac{1}{N} \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j \frac{2\pi}{N} kn} \right)^* = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{x}[n] e^{j \frac{2\pi}{N} kn} = \tilde{c}_{-k}$
 - $\tilde{c}_k = |\tilde{c}_k| e^{j \angle \tilde{c}_k}$
- dtfs of even and odd signals
 - $\tilde{x}[n]$ is real and even \Rightarrow then $\tilde{x}^*[n] = \tilde{x}[n]$ and $\tilde{x}[n] = \tilde{x}[-n]$
 - $\tilde{c}_{-k} = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j \frac{2\pi}{N} kn} = \frac{1}{N} \sum_{n'=0}^{-(N-1)} \tilde{x}[-n'] e^{-j \frac{2\pi}{N} kn'} = \frac{1}{N} \sum_{n'=0}^{-(N-1)} \tilde{x}[n'] e^{-j \frac{2\pi}{N} kn'} = \tilde{c}_k$
 - $\tilde{c}_k^* = \tilde{c}_{-k}$ real signal
 - $\tilde{c}_k = \tilde{c}_{-k}$ even signal
 - real even signal \rightarrow real dtfs
- shifting in time
 - $\tilde{x}[n] \rightarrow \tilde{c}_k$
 - $\tilde{x}[n-m] \rightarrow e^{-j \frac{2\pi}{N} km} \tilde{c}_k$
 - proof
 - $\tilde{y}[n] = \tilde{x}[n-m]$
 - $\tilde{y}[n] = \sum_{k=0}^{N-1} \tilde{d}_k e^{j \frac{2\pi}{N} kn}$
 - $\tilde{d}_k = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{y}[n] e^{-j \frac{2\pi}{N} kn} = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{x}[n-m] e^{-j \frac{2\pi}{N} kn} = \frac{1}{N} \sum_{n'=-m}^{N-1-m} \tilde{x}[n'] e^{-j \frac{2\pi}{N} k(n'+m)}$
 - $= e^{-j \frac{2\pi}{N} km} \frac{1}{N} \sum_{n'=-m}^{N-1-m} \tilde{x}[n'] e^{-j \frac{2\pi}{N} kn'} = e^{-j \frac{2\pi}{N} km} \tilde{c}_k$

8 10.24 5m

8.1 periodic convolution

- $\tilde{x}[n]$ with period N
- $\tilde{h}[n]$ with period N
- $\tilde{x}[n] \otimes \tilde{h}[n] = \sum_{k=0}^{N-1} \tilde{x}[k] \tilde{h}[n-k]$
- “linear convolution”
 - $\tilde{x}[n] * \tilde{h}[n] = \sum_{k=-\infty}^{\infty} \tilde{x}[k] \tilde{h}[n-k] \neq \tilde{x}[n] \otimes \tilde{h}[n]$
- finding $\tilde{y}[n] = \tilde{x}[n] \otimes \tilde{h}[n]$
 - $\tilde{x}[n] = \{\dots, 0, 1, 2, 3, 4, \dots\}, 0 \leq n \leq 4$
 - $\tilde{h}[n] = \{\dots, 3, 3, -3, 2, -1, \dots\}, 0 \leq n \leq 4$
 - $\tilde{y}[n] = \tilde{x}[n] \otimes \tilde{h}[n] = \sum_{k=0}^4 \tilde{x}[k] \tilde{h}[n-k]$

- $\tilde{y}[0] = \sum_{k=0}^4 \tilde{x}[k] \tilde{h}[0-k] = \tilde{x}[0] \tilde{y}[0] + \tilde{x}[1] \tilde{y}[-1] + \dots$
- $\tilde{h}[k] = \tilde{h}[N+k]$
- $\tilde{y} \xrightarrow{\text{dtfs}} \tilde{e}_k$
- $\tilde{x} \xrightarrow{\text{dtfs}} \tilde{c}_k$
- $\tilde{h} \xrightarrow{\text{dtfs}} \tilde{d}_k$
- proposition: $\tilde{e}_k = N \cdot \tilde{c}_k \cdot \tilde{d}_k$
- proof
 - $\tilde{y}[n] = \sum_{k=0}^{N-1} \tilde{x}[k] \tilde{h}[n-k] = \sum_{k=0}^{N-1} \tilde{e}_k e^{j \frac{2\pi}{N} kn}$
 - $\tilde{e}_k = \frac{1}{N} \sum_{n=0}^{N-1} \left(\sum_{m=0}^{N-1} \tilde{x}[m] \tilde{h}[n-m] \right) e^{-j \frac{2\pi}{N} kn} = \frac{1}{N} \sum_{m=0}^{N-1} \tilde{x}[m] \sum_{n=0}^{N-1} \tilde{h}[n-m] e^{-j \frac{2\pi}{N} kn}$
 - $= \frac{1}{N} \sum_{m=0}^{N-1} \tilde{x}[m] \tilde{d}_k \cdot N \cdot e^{-j \frac{2\pi}{N} km} = \sum_{m=0}^{N-1} \tilde{x}[m] \tilde{d}_k e^{-j \frac{2\pi}{N} km} = \tilde{d}_k \sum_{m=0}^{N-1} \tilde{x}[m] e^{-j \frac{2\pi}{N} km} = \tilde{d}_k \tilde{c}_k N$

8.2 discrete time fourier transform (dtft)

- consider a non-periodic signal $x[n]$ with finite length
 - $x[n] \neq 0$ for $|n| \leq M$
 - there exists $2M+1$ non-zero samples
- create a periodic extension of x
 - $\tilde{x}[n] = \sum_{k=-\infty}^{\infty} x[n+k(2M+1)]$
- $\tilde{x}[n]$ is periodic so can be analyzed using dtfs
 - $\tilde{x}[n] = \sum_{k=-M}^M \tilde{c}_k e^{j \frac{2\pi}{2M+1} kn}$ (synthesis)
- since $\tilde{x}[n]$ is periodic there exists an analysis component as well
 - $\tilde{c}_k = \frac{1}{2M+1} \sum_{n=-M}^M \tilde{x}[n] e^{-j \frac{2\pi}{2M+1} kn}$
 - $\Omega_0 = \frac{2\pi}{2M+1}$
 - $\Omega_k = k\Omega_0$
 - $k \in [-M, M]$
 - $\Omega_k \in [-\pi, \pi]$
 - $\tilde{c}_k = \frac{1}{2M+1} \sum_{n=-M}^M x[n] e^{-j \frac{2\pi}{2M+1} kn}$
 - $(2M+1)\tilde{c}_k = \sum_{n=-M}^M x[n] e^{-j \frac{2\pi}{2M+1} kn}$
- as $M \rightarrow \infty$
 - $\Omega_0 \rightarrow \Delta\Omega$
 - $k\Omega_0 = k \frac{2\pi}{2M+1} := \Omega \in \mathbb{R}$
 - $\Omega \in [-\pi, \pi]$
 - $\lim_{M \rightarrow \infty} (2M+1)\tilde{c}_k := X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$ (dtft)
- $X(\Omega)$ is complex
- $X(\Omega) = \Re\{X(\Omega)\} + j\Im\{X(\Omega)\}$
- $X(\Omega) = |X(\Omega)| e^{j\angle X(\Omega)}$
- visualizing $X(\Omega)$ requires two plots
- ece102:
 - $x(t) \xrightarrow{\mathcal{F}} X(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt, \omega \in (-\infty, \infty)$

- ece113:
 - $x[n] \xrightarrow{\mathcal{F}} X(\Omega) := \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$, $\Omega \in [-\pi, \pi]$ $X(\Omega)$ is periodic with 2π
 - $X(\Omega + 2\pi) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\Omega+2\pi)n}$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} e^{-j2\pi n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = X(\Omega)$$
 - $X(\Omega) \xrightarrow{\mathcal{F}^{-1}} x[n]$
 - $x[n] = \lim_{M \rightarrow \infty} \tilde{x}[n] = \lim_{M \rightarrow \infty} \sum_{k=-M}^M \tilde{c}_k e^{j\frac{2\pi}{2M+1}kn}$
 - $\lim_{M \rightarrow \infty} \sum_{k=-M}^M c_k \frac{2M+1}{2\pi} \Delta\Omega e^{j\Omega n}$
 - $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega$ (idftft)
 - $\delta[n] \xrightarrow{\mathcal{F}} \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\Omega n} = e^{-j\Omega 0} = 1$
 - $x[n] = \delta[n - n_0]$
 - $X(\Omega) = \sum_{n=-\infty}^{\infty} \delta[n - n_0]e^{-j\Omega n} = e^{-j\Omega n_0}$
 - $x[n] = a^n u[n]$, $|a| < 1$
 - existence of dtft
 - $x[n]$ has to be absolutely summable
 - $\sum_{n=-\infty}^{\infty} |x[n]| < +\infty$
 - or square summable
 - $\sum_{n=-\infty}^{\infty} |x[n]|^2$
 - $X(\Omega) = \sum_{n=-\infty}^{\infty} a^n u[n]e^{-j\Omega n}$

$$= \sum_{n=0}^{\infty} a^n e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} (ae^{-j\Omega})^n$$

$$= \frac{1}{1 - ae^{-j\Omega}}$$

9 10.26 5w

9.1 dtft

- analysis equation: $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$ (continuous)
 - $\Omega \in [-\pi, \pi]$
- synthesis equation (idftft): $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega$
- $x[n] = \begin{cases} 1 & -L \leq n \leq L \\ 0 & \text{otherwise} \end{cases}$

$$\begin{aligned}
- X(\Omega) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \\
&= \sum_{n=-L}^L e^{-j\Omega n} \\
&= \frac{e^{j\Omega L} - e^{-j\Omega(L+1)}}{1 - e^{-j\Omega}} \\
&= \frac{e^{-j\Omega/2}(e^{j\Omega(L+1/2)} - e^{-j\Omega(L+1/2)})}{e^{-j\Omega/2}(e^{j\Omega/2} - e^{-j\Omega/2})} \\
&= \frac{\sin(\Omega(L+1/2))}{\sin(\Omega/2)} \\
- X(\Omega) &= \begin{cases} 1 & -\Omega_c < \Omega < \Omega_c \\ 0 & \text{otherwise} \end{cases} \\
- x[n] &= \frac{1}{2\pi} \int_{-\Omega_c}^{\Omega_c} 1 \cdot e^{j\Omega n} d\Omega \\
&= \frac{1}{2\pi} \left(\frac{1}{j\Omega} e^{j\Omega n} \right) \Big|_{-\Omega_c}^{\Omega_c} \\
&= \frac{1}{\pi} \cdot \frac{1}{n} \left(\frac{e^{j\Omega_c n} - e^{-j\Omega_c n}}{2j} \right) \\
&= \frac{1}{\pi n} \sin(\Omega_c n) \\
&= \frac{\Omega_c}{\pi} \text{sinc}(\Omega_c n)
\end{aligned}$$

10 11.7 7m

10.1 dtft \rightarrow dft

- periodic signal: dtfs, coefficients are discrete
- non-periodic signal: dtftc, continuous transform domain
- discrete-time signal $x[n] \xrightarrow{\text{dtft}} X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$
 - $\Omega \in \mathbb{R}$
 - $X(\Omega) = X(\Omega + 2\pi)$
 - problem: Ω is continuous
- non-periodic signal with finite duration (length N)
 - $x[n]$ given for $n = 0, \dots, N-1$
 - $x[n] = 0$ for $n < 0$ and $n \geq N$
- first goal: find the periodic extension of $x[m]$
 - $\tilde{x}[n] = \sum_{m=-\infty}^{\infty} x[n - mN]$
 - $\tilde{x}[n] \xrightarrow{\text{dtfs}} \tilde{c}_k$
 - $\tilde{c}_k := \frac{1}{N} \sum_{n=0}^{N-1} \tilde{x}[n]e^{-j\frac{2\pi}{N}kn}$
- second goal: extract one period of \tilde{c}_k ; in particular, c_k is periodic with N

- $\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{c}_k e^{j\frac{2\pi}{N}kn}$
- define $x[n] = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi}{N}kn}$ where $c_k = \begin{cases} \tilde{c}_k & k = 0, \dots, N-1 \\ 0 & \text{otherwise} \end{cases}$
- rename $c_k N := X[k]$ for $k = 0, \dots, N-1$
 - dft: $x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}$
 - inverse dft: $X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$
- N samples map to N coefficients

10.2 dft as a sampling of the dtft

- dtft: $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} = \sum_{n=0}^{N-1} x[n] e^{-j\Omega n}$
- dft: $X[k] = X(\Omega) \Big|_{\Omega=\frac{2\pi}{N}k}$

10.3 dft in linear algebraic view

- $\mathbf{x} = [x[0], x[1], \dots, x[N-1]]$
- $\mathbf{X} = [X[0], X[1], \dots, X[N-1]]$
- $\mathbf{X} = \mathbf{W}\mathbf{x}$
 - $X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$
 - $\omega_N := e^{-j\frac{2\pi}{N}}$
 - dft: $X[k] = \sum_{n=0}^{N-1} x[n] \omega_N^{kn}$
 - idft: $x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \omega_N^{-kn}$
- $\mathbf{W} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & \omega_N^1 & \dots & \omega_N^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_N^{N-1} & \dots & \omega_N^{(N-1)^2} \end{bmatrix}$
- $\mathbf{x} = \mathbf{W}^{-1}\mathbf{X}$
- $\mathbf{W}^{-1} = \frac{1}{N} \overline{\mathbf{W}}$

10.4 from a basis perspective

- $\mathbf{x} = \mathbf{W}^{-1}\mathbf{X} = \frac{1}{N} \sum X[i] \mathbf{W}^{-1}[:, i]$

10.5 example

- $x[n] = u[n] - u[n-10]$
- $X[k] = \sum_{n=0}^9 e^{-j\frac{2\pi}{10}kn} = \frac{1-e^{-j\frac{2\pi}{10}k \cdot 10}}{1-e^{-j\frac{2\pi}{10}k}} = \frac{1-e^{-j2\pi k}}{1-e^{-j\frac{2\pi}{10}k}}$
- $X[0] = \lim_{k \rightarrow 0} \frac{1-e^{-j2\pi k}}{1-e^{-j\frac{2\pi}{10}k}} = 10$
- $X[\geq 1] = 0$
- dtft has waves: $\frac{\sin(5\Omega)}{\sin(0.5\Omega)} e^{-j4.5\Omega}$

11 11.9 7w

11.1 dft review

- $x[n]$ length N signal
- $x[n] = 0$ for $n < 0, n \geq N$
- $x[n] \xrightarrow{\text{dft}} X[k]$
- we arrived at the dft through the following steps
 - $x[N \text{ samples}] \rightarrow$ periodic extension \rightarrow
 - $\tilde{x}[n] \rightarrow \text{dft} \rightarrow$
 - $\tilde{c}_k \rightarrow X[k] := N \cdot c_k, k = 0, 1, \dots, N-1$
 - dft: $X[k] = \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi}{N} kn}$
 - idft: $x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j \frac{2\pi}{N} kn}$
- dft is practical, dtft is idealized
- recall dtft $x[n] \xrightarrow{\text{dtft}} X(\Omega), \Omega \in \mathbb{R}$
- relationship between dft and dtft: $X[k] = X(\Omega) \Big|_{\Omega = \frac{2\pi}{N} k}$
- e.g.:
 - $u[n] - u[n-10]$
 - dtft looks like sinc
 - dft has 10 at $n = 0$ and 0 everywhere else = lossy

11.2 sample spacing puzzle

- zero padding
- $x[n] \rightarrow g[n] = \begin{cases} x[n] & n \in [0, N-1] \\ 0 & n \in [N, N+M-1] \end{cases}$
- $G[k] = X(\Omega) \Big|_{\Omega = \frac{2\pi}{N+M} k}$

11.3 dft properties

- $x[n] \rightarrow X[k]$
- time shifting property
 - $x[n-m] \rightarrow e^{-j\omega km} X[k]$
 - wrong because sampling window doesn't change
 - solution: cyclic shift, periodic extension
 - $g[n] = x[(n-m) \bmod N]$
 - $x[(n-m) \bmod N] \rightarrow e^{-j\omega km} X[k]$
 - where $\omega = \frac{2\pi}{N}$
- linearity
 - $x_1[n] \rightarrow X_1[k]$
 - $x_2[n] \rightarrow X_2[k]$
 - $\alpha x_1[n] + \beta x_2[n] \rightarrow \alpha X_1[k] + \beta X_2[k]$

11.4 determine the dtft

- $x[n] = (0.7)^n \cos(0.2\pi n)u[n]$
 - $x_1[n] = (0.7)^n u[n] \rightarrow X_1(\Omega) = \frac{1}{1-0.7e^{-j\Omega}}$
 - $X(\Omega) = \text{DTFT}(\cos(0.2\pi n)x_1[n])$

$$= \left(\frac{1}{2}e^{j0.2\pi n} + \frac{1}{2}e^{-j0.2\pi n} \right) x_1[n]$$

$$= \frac{1}{2}e^{j0.2\pi n} x_1[n] + \frac{1}{2}e^{-j0.2\pi n} x_1[n]$$

$$= \frac{1}{2}X_1(\Omega - 0.2\pi) + \frac{1}{2}X_1(\Omega + 0.2\pi)$$

$$= \frac{1}{2} \cdot \frac{1}{1 - 0.7e^{-j(\Omega-0.2\pi)}} + \frac{1}{2} \cdot \frac{1}{1 - 0.7e^{-j(\Omega+0.2\pi)}}$$
- $x[n] = nx[n-1]$
 - time shift: $x[n-1] \rightarrow e^{-j\Omega \cdot 1}X(\Omega)$
 - differentiation in freq: $nx[n-1] \rightarrow j \frac{d}{d\Omega} [e^{-j\Omega}X(\Omega)]$
- $x_2[n] = e^{j\frac{n\Omega}{2}} (x[n] * x[n])$

12 11.14 8m

12.1 dft conv thm

- conv thm for dtft
 - $x[n] \rightarrow h[n] \rightarrow y[n] = x[n] * h[n]$
 - $\xleftrightarrow{\text{dtft}} X(\Omega) \rightarrow H(\Omega) \rightarrow Y(\Omega) = X(\Omega)H(\Omega), \Omega \in \mathbb{R}$
- in practice, we compute dft, so does the property hold?
 - $X[k] = \text{DFT}(x[n]), k = 0, 1, \dots, N_x - 1$
 - $H[k] = \text{DFT}(h[n]), k = 0, 1, \dots, N_h - 1$
 - $Y[k] = \text{DFT}(y[n]), k = 0, 1, \dots, N_y - 1$
- $X[k] \cdot H[k] \stackrel{?}{=} Y[k]$
 - $N_y = N_x + N_h - 1$
 - no
- try to derive an analog of conv thm
 - recall that we got dft from “periodic” extension of a finite signal
 - assumption: $x[n], h[n]$ both have length N
 - $x[n] \rightarrow \tilde{x}[n] \xrightarrow{\text{dtfs}} \tilde{c}_k \rightarrow c_k \mid X(k) := c_k \cdot N$
 - $h[n] \rightarrow \tilde{h}[n] \xrightarrow{\text{dtfs}} \tilde{d}_k \rightarrow d_k \mid H(k) := d_k \cdot N$
- periodic convolution
 - $\tilde{y}[n] = \tilde{x}[n] \otimes \tilde{h}[n] = \sum_{k=0}^{N-1} \tilde{x}[k] \tilde{h}[n-k]$

- $\tilde{y}[n] \xrightarrow{\text{dtfs}} \tilde{e}_k$
- recall $\tilde{e}_k = N\tilde{c}_k\tilde{d}_k$ proven in an earlier lecture
- taking N samples of $\tilde{y}[n]$ such that $y[n] = \tilde{y}[n], n = 0, 1, \dots, N-1$
- $y[n] = \tilde{y}[n] = \sum_{k=0}^{N-1} \tilde{x}[k]\tilde{h}[n-k], n = 0, 1, \dots, N-1$
- $y[n] = \sum_{k=0}^{N-1} x[k]\tilde{h}[n-k], n = 0, 1, \dots, N-1$
- $y[n] = \sum_{k=0}^{N-1} x[k]h[(n-k) \bmod N]$
- circular convolution: applies to two finite duration signals of length N
- $y[n] = x[n] \circ h[n]$
- example
 - $x[n] = \{1, 2, 3, -4, 6\}$
 - $h[n] = \{5, 4, 3, 2, 1\}$
 - $y[0] = \sum_{k=0}^{N-1} x[k]h[(-k) \bmod N] = 5 \cdot 1 + 2 \cdot 1 + 3 \cdot 2 + (-4) \cdot 3 + 6 \cdot 4 = 25$
 - $y[n] = \sum_{k=0}^{N-1} x[k]h[(n-k) \bmod N]$
 - $\xrightarrow{N\text{-point DFT}} Y[k] = X[k]H[k]$
 - $y[n] = \text{IDFT}(Y[k]) = \text{IDFT}(X[k] \cdot H[k])$
 - recall: $\tilde{e}_k = N\tilde{d}_k\tilde{c}_k$
 - $y[n] = \tilde{y}[n] \rightarrow \tilde{e}_k, n = 0, \dots, N-1$
 - $x[n] = \tilde{x}[n] \rightarrow \tilde{c}_k, n = 0, \dots, N-1$
 - $h[n] = \tilde{h}[n] \rightarrow \tilde{d}_k, n = 0, \dots, N-1$
 - $\tilde{e}_k = N\tilde{c}_k\tilde{d}_k \forall k$
 - dft relationship:
 - $X[k] = Nc_k$
 - $H[k] = Nd_k$
 - $Y[k] = Ne_k$
 - $c_k = \frac{X[k]}{N}$
 - $d_k = \frac{H[k]}{N}$
 - $e_k = \frac{Y[k]}{N}$
- generalize this insight to not same length sequences
 - $y_l[n] = x[n] * h[n]$
 - $x[n]$ length N_x
 - $h[n]$ length N_h
 - $y_l[n]$ length $N_y = N_x + N_h - 1$
 - trick: zero pad x and h to have the same length
 - $x_p[n] = \begin{cases} x[n] & n = 0, \dots, N_x - 1 \\ 0 & n = N_x, \dots, N_y - 1 \end{cases}$
 - $h_p[n] = \begin{cases} h[n] & n = 0, \dots, N_h - 1 \\ 0 & n = N_h, \dots, N_y - 1 \end{cases}$
 - $y^c[n] = x_p[n] \circ h_p[n] = \sum_{k=0}^{N_y-1} x_p[k]h_p[(n-k) \bmod N_y]$
 - $y_l[n] = \sum_{k=0}^{N_y-1} x[k] \cdot h[n-k]$
 - $X[k] = \text{DFT}(x_p[n])$
 - $H[k] = \text{DFT}(h_p[n])$

- $Y[k] = H[k] \cdot X[k] = \text{DFT}(x_p(n) \circ h_p(n)) = \text{DFT}(x[n] * h[n])$
- summary
 - $y[n] = x[n] * h[n] = \text{IDFT}(X[k] \cdot H[k])$ assuming $X[k]$ is DFT of zero padded x : $X(\Omega) \Big|_{\Omega=\frac{2\pi}{N}k}$

12.2 fast fourier theorem (fft) + algo analysis

- what is the runtime of a dft?
 - $x[n] \rightarrow N$ values
 - $X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}, k = 0, 1, \dots, N-1$
 - for 1 coefficient: N complex multiplications, $N-1$ complex additions
 - repeating for N coefficients
 - $N \cdot (N + (N-1))$ operations = $O(N^2)$
- radix-2 fft
 - preliminaries
 - N -dft has $\Theta(N^2)$ complexity
 - $\omega_N = e^{-j\frac{2\pi}{N}}, X[k] = \sum_{n=0}^{N-1} x[n] \omega_N^{kn}$
 - properties of ω_N :
 - $\omega_N^2 = \omega_{N/2}$
 - $e^{-j\frac{2\pi}{N} \cdot 2} = e^{-j\frac{2\pi}{N/2}}$
 - $\omega_N^{2+N/2} = -\omega_N^2$
 - $e^{-j\frac{2\pi}{N}(2+N/2)} = e^{-j\frac{2\pi}{N} \cdot 2} \cdot e^{-j\pi} = -\omega_N^2$
 - $\omega_N^{2(N/2)} = (-1)^2$
 - $e^{-j\frac{2\pi}{N} \cdot 2(N/2)} = (e^{-j\pi})^2 = (-1)^2$
 - assumption: $N = 2^p$
 - even samples = $\{x[0], x[2], x[4], \dots, x[N-2]\}$
 - odd samples = $\{x[1], x[3], x[5], \dots, x[N-1]\}$
 - $X[k] = \sum_{n=\text{even}} x[n] \omega_N^{kn} + \sum_{n=\text{odd}} x[n] \omega_N^{kn}$

$$= \sum_{m=0}^{N/2-1} x[2m] \omega_N^{k \cdot 2m} + \sum_{m=0}^{N/2-1} x[2m+1] \omega_N^{k(2m+1)}$$

$$= \sum_{m=0}^{N/2-1} x[2m] \omega_{N/2}^{km} + \omega_N^k \sum_{m=0}^{N/2-1} x[2m+1] \omega_{N/2}^{km}$$

$$= X_e[k] + \omega_N^k X_o[k]$$
 - $X_e[k]$ has length $N/2$
 - observe that $X_e[k] = X_e[k + N/2], X_o[k] = X_o[k + N/2]$
 - now:
 - $X[k] = X_e[k] + \omega_N^k X_o[k]$
 - $X[k + N/2] = X_e[k + N/2] + \omega_N^{k+N/2} X_o[k + N/2]$
 - $= X_e[k] - \omega_N^k X_o[k]$
 - complexity: $\Theta((N/2)^2) \rightarrow 2\Theta(N^2/4) + \Theta(N/2) + \Theta(N)$

13 11.16 8w

13.1 warm up problem

- suppose we have $\{y[0], y[1]\}$, compute the 2-point dft
 - $Y[k] = y[0]e^0 + y[1]e^{-j\frac{2\pi}{N}k}$
 - $Y[0] = y[0] + y[1]$
 - $Y[1] = y[0] - y[1]$

13.2 radix-2 fft

- $X[k] = \sum_{m=0}^{N/2-1} x[2m]\omega_N^{k \cdot 2m} + \sum_{m=0}^{N/2-1} x[2m+1]\omega_N^{k(2m+1)}$
- $X[k] = X_e[k] - \omega_N^k X_o[k]$
- $X_e[k] \rightarrow \Theta((N/2)^2) = \Theta(N^2/4)$
- $X_o[k] \rightarrow \Theta((N/2)^2) = \Theta(N^2/4)$
- $X[k + N/2] = X_e[k + N/2] - \omega_N^{k+N/2} X_o[k + N/2]$
- complexity: $2 \cdot N^2/4 + N/2$ multiplications + N additions = $\Theta(N^2 + N) = \Theta(N^2)$
- continue splitting
 - $X_e[k] = X_{ee}[k] + \omega_{N/2}^k X_{eo}[k]$
 - $X_e[k + N/4] = X_{ee}[k] - \omega_{N/2}^k X_{eo}[k]$
 - $X_o[k] = X_{oe}[k] + \omega_{N/2}^k X_{oo}[k]$
 - $X_o[k + N/4] = X_{oe}[k] - \omega_{N/2}^k X_{oo}[k]$
- can continue splitting p times
- $\Theta(N)$ complexity in each state
- total complexity: $\Theta(N \log N)$
- $N \log N \ll N^2$

13.3 application of fft (compression)

- 1 hour song at 44kHz
 - humans can only hear 20 - 20000 Hz
 - very few sounds are above 10000 Hz
 - song is usually on a major scale, few dominant notes
 - just look at fft and keep top 100 frequencies by magnitude
 - \rightarrow 100 floating points
- 2d fft to compress image
 - $x[n_1, n_2] \rightarrow X[k_1, k_2]$
 - bright disc in the center and everything else is dark
 - lossy compression

13.4 z-transform

- motivation
 - dtft of a step function $u[n]$
 - doesn't exist
 - recall that a signal must be absolutely summable to have a dtft
 - intuition (102)
 - fourier transform of an exponential function (e.g. e^3): kill / damp it by multiplying by $e^{-3} \rightarrow$ laplace transform
 - z-transform helps with analysis of exploding signals
 - z-transform helps solve diff eqs
- next phase of sig + sys
 - $y[n] = ax[n-1] + x[n-2] \rightarrow H(\Omega)$
 - reverse the idea
 - given what $H(\Omega)$ looks like
 - design the system engineering accordingly

14 11.21 9m

14.1 z-transform

- $x[n] \xrightarrow{\mathbb{Z}} X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}, z = re^{j\Omega} \in \mathbb{C}$
- $X(z) = \mathbb{Z}\{x[n]\}$
- observe: z-transform is a superset of dtft
- $X(z)|_{z=re^{j\Omega}} = \sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-j\Omega n}$
 - $r = 1 \Rightarrow X(\Omega)$ (dtft)
- dtft is obtained from z-transform for setting $z = e^{j\Omega}$ if $X(z)$ exists for $e^{j\Omega}$
- example
 - $x[n] = \{1, 5, 7, 9, 13\}$: finite duration, causal
 - $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$

$$= 1z^0 + 5z^{-1} + 7z^{-2} + 9z^{-3} + 13z^{-4}$$
 - $X(z)$ exists for $z \neq 0 + j \cdot 0$ or $|z| \neq 0$
 - concept: region of convergence: values of z for which $X(z)$ "exists"
 - "exists" means it is not infinity
- example
 - $x[n] = \{3, 1, 2, 4, 5\}$: non-causal, finite duration
 - $\mathbb{Z}\{x[n]\} = X(z) = 3z^2 + z + 2 + 4z^{-1} + 5z^{-2}$
 - region of convergence: $|z| \neq 0$ and $|z| \nrightarrow \infty$
- example

- try to compute $\mathbb{Z}\{\delta[n]\}$
- $\delta[n] \xrightarrow{\mathbb{Z}} \delta[0]z^{-0} = 1$
- roc: entire z -plane
- example
 - $\mathbb{Z}\{\delta[n-k]\} = z^{-k}$
 - roc: $|z| \neq 0$ if $k > 0$, $|z| \rightarrow \infty$ if $k < 0$, entire z -plane if $k = 0$
- example
 - $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$
 - $\mathbb{Z}\{u[n]\} = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$
 - roc: $|z| > 1$
 - summary: dtft, which is $X(z)|_{z=e^{j\Omega}}$ is not in the roc; this is consistent with our understanding that dtft of $u[n]$ does not exist
- example
 - compute the z -transform of $x[n] = a^n u[n]$: infinite duration, causal
 - $X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$
 - roc: $|az^{-1}| < 1$ or $|z| > |a|$
 - dtft exists if $|a| < 1$
- observation: z -transform are fractional polynomials
 - $X(z)$ is defined by z_1, z_2, \dots, z_M zeroes and p_1, p_2, \dots, p_N poles (roc does not contain poles)
- examples
 - $x[n] = \begin{cases} 1 & 0 \leq n \leq N-1 \\ 0 & n < 0 \vee n > N \end{cases}$
 - $X(z) = \sum_{n=0}^{N-1} z^{-n} = \frac{1-z^{-N}}{1-z^{-1}} = \frac{z^N-1}{z^{N-1}(z-1)}$
 - roc: $|z| \neq 0$
 - note: $z = 1$ works
- example
 - $x[n] = -a^n u[-n-1] = \begin{cases} -a^n & n < 0 \\ 0 & n \geq 0 \end{cases}$
 - $X(z) = \sum_{n=-\infty}^{\infty} -a^n u[-n-1] z^{-n} = \sum_{n=-\infty}^{-1} -a^n z^{-n} = -\sum_{m=1}^{\infty} a^{-m} z^m = -\frac{a^{-1}z}{1-a^{-1}z} = \frac{z}{z-a}$
 - roc: $|a^{-1}z| < 1$ or $|z| < |a|$
 - z -transform is uniquely defined by $X(z)$ and roc
- example
 - $x[n] = 3^{-n} u[n] - 2^{-n} u[-n-1]$
 - $x_1[n] = 3^{-n} u[n]$
 - $x_2[n] = 2^{-n} u[-n-1]$
 - $X(z) = X_1(z) - X_2(z) = \frac{z}{z-3^{-1}} - \frac{z}{z-2^{-1}}$
 - roc: $|z| \in (3^{-1}, \infty) \cap [0, 2^{-1}) = (3^{-1}, 2^{-1})$

- summary
 - roc is circularly shaped in z-transform
 - roc cannot have any poles
 - causal signals are outside the circle
 - anticausal signals are inside the circle
 - signals that decompose into causal and anticausal signals have a ring shaped roc

15 11.23 9w

15.1 z-transform properties

- recall: $x[n] \xrightarrow{\mathbb{Z}} X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$ where $z \in \mathbb{C}$
 - $z = e^{j\Omega} \rightarrow \text{dtft}$
- linearity
 - if $x_1[n] \xrightarrow{\mathbb{Z}} X_1(z)$ and $x_2[n] \xrightarrow{\mathbb{Z}} X_2(z)$
 - then $\alpha x_1[n] + \beta x_2[n] \xrightarrow{\mathbb{Z}} \alpha X_1(z) + \beta X_2(z)$
 - roc: recalculate poles
- time shifting
 - if $x[n] \xrightarrow{\mathbb{Z}} X(z)$
 - then $x[n-k] \xrightarrow{\mathbb{Z}} X(z)z^{-k}$
 - proof
 - $\mathbb{Z}\{x[n-k]\} = \sum_{n=-\infty}^{\infty} x[n-k]z^{-n} = \sum_{m=-\infty}^{\infty} x[m]z^{-(m+k)} = z^{-k} \sum_{m=-\infty}^{\infty} x[m]z^{-m} = z^{-k} X(z)$
 - new roc \neq roc
 - $k > 0$: roc excludes $|z| = 0$
 - $k < 0$: roc excludes $|z| \rightarrow \infty$
- convolution property
 - if $x_1[n] \xrightarrow{\mathbb{Z}} X_1(z)$ and $x_2[n] \xrightarrow{\mathbb{Z}} X_2(z)$
 - then $x_1[n] * x_2[n] \xrightarrow{\mathbb{Z}} X_1(z)X_2(z)$
 - proof
 - $x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]$
 - $\mathbb{Z}\left\{\sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]\right\} = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]\right)z^{-n}$
 - $= \sum_{k=-\infty}^{\infty} x_1[k] \sum_{n=-\infty}^{\infty} x_2[n-k]z^{-n}$
 - $= x_1[k] \sum_{k=-\infty}^{\infty} X_2(z)z^{-k}$
 - $= X_2(z) \sum_{k=-\infty}^{\infty} x_1[k]z^{-k}$
 - $= X_1(z)X_2(z)$

- combined roc depends on the poles (pay attention to zero-pole cancellations)

15.2 inverse z-transform

- $X(z) = \frac{B(z)}{(z-p_1)(z-p_2)\cdots(z-p_N)}$
- recall
 - $a^n u[n] \xrightarrow{\mathbb{Z}} \frac{z}{z-a}, |z| > |a|$
 - $-a^n u[-n-1] \xrightarrow{\mathbb{Z}} \frac{z}{z-a}, |z| < |a|$
- $X(z) = \frac{k_1 z}{z-p_1} + \frac{k_2 z}{z-p_2} + \cdots + \frac{k_N z}{z-p_N}$
- example
 - $X(z) = \frac{(z-1)(z+2)}{(z-1/2)(z-2)}$
 - $\frac{X(z)}{z} = \frac{(z-1)(z+2)}{z(z-1/2)(z-2)} = \frac{k_1}{z} + \frac{k_2}{z-1/2} + \frac{k_3}{z-2}$
 - $k_1 = \left(\frac{k_1}{z} + \frac{k_2}{z-1/2} + \frac{k_3}{z-2} \right) \cdot (z-0) \Big|_{z=0} = (z-0) \frac{X(z)}{z} = X(z) \Big|_{z=0} = \frac{(0-1)(0+2)}{(0-1/2)(0-2)} = -2$
 - $k_2 = \frac{(1/2-1)(1/2+2)}{(1/2)(1/2-2)} = \frac{5}{3}$
 - $k_3 = \frac{(2-1)(2+2)}{2(2-1/2)} = \frac{4}{3}$
 - $X(z) = -2 + \frac{(5/3)z}{z-1/2} + \frac{(4/3)z}{z-2}$
 - $X(z) \xrightarrow{\mathbb{Z}^{-1}} x[n] = -2\delta[n] + \frac{5}{3}x_1[n] + \frac{4}{3}x_2[n]$
 - roc $|z| > 2$: $x_1[n] = 2^{-n}u[n]$, $x_2[n] = 2^n u[n]$
 - roc $|z| < 2^{-1}$: $x_1[n] = -2^n u[-n-1]$, $x_2[n] = -2^{-n} u[-n-1]$
 - roc $2^{-1} < |z| < 2$: $x_1[n] = 2^{-n}u[n]$, $x_2[n] = -2^{-n}u[-n-1]$
- summary
 - find poles $\{p_i\}, i = 1, \dots, N$
 - find coefficients for partial fraction $k_i = X(z)(z-p_i) \Big|_{z=p_i}$
 - based on roc, figure out which causality of canonical z-transform pairs to use

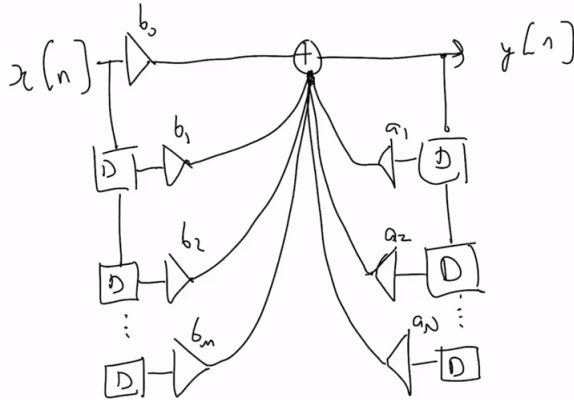
15.3 using z-transform to analyze / design dltti systems

- $H(z) = \mathbb{Z}\{h[n]\}$
- $H(z) = \frac{Y(z)}{X(z)}$: easier to obtain
- recall: dltti is defined by a constant coefficient difference equation w suitable initial conditions
- $\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$
 - z-tranform on both sides: $\sum_{k=0}^N a_k z^{-k} Y(z) = \sum_{k=0}^M b_k X(z) z^{-k}$
 - $\frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}} = H(z)$
- given $H(z)$, say smth abt the system
 - causal? $h[n]$ only defined for $n \geq 0$
 - check that $|z| \rightarrow \infty$ is defined on the z-plane
 - $\lim_{z \rightarrow \infty} H(z) = \lim_{z \rightarrow \infty} \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}} < \infty$
 - for causality: $N \geq M$
 - stable? $h[n]$ bounded or dtft bounded / exists
 - check that unit circle is in roc

15.4 design of dtlti

$$- H(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}}$$

$$- y[n] = b_0 x[n] + b_1 x[n-1] + \dots + b_M x[n-M] - a_1 y[n-1] - \dots - a_N y[n-N]$$



- steps
 - given $H(z)$ in pole / zero notation
 - given $H(z)$ in polynomial
 - constant coefficient diff eq
 - system circuit / design
- gap: how to go from high level idea, e.g. "filter out frequencies from 30 - 50Hz", to $H(z)$

16 11.28 10m

16.1 review

- recall: $H(z) = \frac{(z-z_1)\dots(z-z_N)}{(z-p_1)\dots(z-p_M)}$ (poles and zeroes)
- $H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}}$ (polynomial coefficients / constant coefficient difference equation \rightarrow block diagram)

16.2 design question

- low pass filter with passband of 20000 Hz
- stable system: dtft exists \rightarrow roc includes unit circle
- $H(z) = \frac{\prod_{i=0}^M (z-z_i)}{\prod_{i=0}^N (z-p_i)}$
- $H(e^{j\Omega}) = \frac{\prod_{i=0}^M (e^{j\Omega}-z_i)}{\prod_{i=0}^N (e^{j\Omega}-p_i)}$
- $|H(e^{j\Omega})| = \frac{\prod_{i=0}^M |e^{j\Omega}-z_i|}{\prod_{i=0}^N |e^{j\Omega}-p_i|}$
 - Ω can express in Hz if we want to
- key insight

- a frequency in Hz that is near a zero is attenuated
- a frequency near a pole is amplified

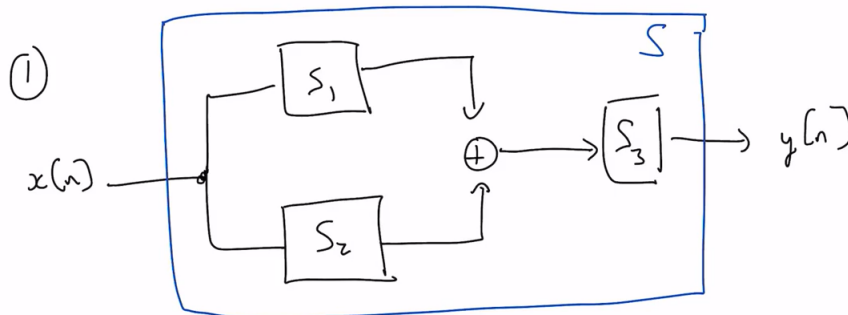
16.3 fir filter

- “finite impulse response”
- $H(e^{j\omega}) = \sum_{k=0}^M b_k z^{-k}$
- $y[n] = x[n] + b_1 x[n-1] + b_2 x[n-2]$: ~convolution
- easy to do optimization to find b_k 's
- find frequency / phase response
- with enough b_k 's, we can make pretty much any filter
- problem: requires large M

16.4 iir filter

- $H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}}$
- add feedback
- complex designs with fewer coefficients
- nonlinear effects (phase unpredictable)

16.5 practice



- “detective / sleuth” question
 - evaluate $H(z)$
 - S_1 : has a pole at 0.5, a zero at 0, and $H_1(1) = -3$
 - S_2 : $y[n] - 0.7y[n-1] = 2.5x[n] - 1.6x[n-1]$
 - S_3 : $h_3[n] = \frac{5}{4}(0.8)^n u[n-1]$
 - $H(z) = (H_1(z) + H_2(z)) \cdot H_3(z)$
 - $H_1(z) = A \cdot \frac{z}{z-0.5} = -\frac{3z}{2(z-0.5)}$
 - roc: $|z| > 0.5$ (check if H goes to infinity as $z \rightarrow \infty$)
 - $H_2(z) = \frac{2.5-1.6z^{-1}}{1-0.7z^{-1}} = \frac{2.5z-1.6}{z-0.7}$
 - roc: $|z| > 0.7$
 - $H_3(z) = \mathbb{Z}\{h_3[n]\} = \mathbb{Z}\{(0.8)^{n-1}u[n-1]\} = z^{-1} \left(\frac{z}{z-0.8}\right) = \frac{1}{z-0.8}$

- roc: $|z| > 0.8$
- $H(z) = \frac{(z-1)(z-0.8)}{(z-0.5)(z-0.7)(z-0.8)} = \frac{z-1}{(z-0.5)(z-0.7)}$
- roc: $|z| > 0.7$ because of zero-pole cancellation