MATH 131AH (Real Analysis): Homework 6

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Problem 1.

An ultrametric on X is a metric ρ on X such that

$$\forall x, y, z \in X : \rho(x, y) \leq \max\left\{\rho(x, z), \rho(y, z)\right\}.$$

Prove that, in this metric, every open ball $B(x, r) := \left\{y \in X \ : \ \rho(x, y) < r\right\}$ is closed and every closed ball $B'(x, r) := \left\{y \in X \ : \ \rho(x, y) \leq r\right\}$ is open. Determine the topological boundary $\partial B(x, r)$ of $B(x, r)$.

To give an example of such a setting, let $X := \{0, 1\}^{\mathbb{N}}$. Prove that then $\rho : X \times X \to \mathbb{R}$ defined for $\sigma, \sigma' \in X$ with $\sigma \neq \sigma'$ by

$$\rho(\sigma, \sigma') := 2^{-\inf\{k \in \mathbb{N} \ : \ \sigma(k) \neq \sigma'(k)\}}$$

(and by $\rho(\sigma, \sigma) := 0$) is an ultrametric.

Solution.

Note that for all $y \in X \setminus B(x, r)$ there is $\rho(x, y) \geq r$. Then for all $z \in B(y, r)$ there is $\rho(x, y) \leq \max\left\{\rho(x, z), \rho(z, y)\right\}$. Since $\rho(z, y) < r \leq \rho(x, y)$, there must be $\rho(x, z) \geq \rho(x, y) \geq r$. Then $z \in X \setminus B(x, r)$. Then $X \setminus B(x, r)$ is open and $B(x, r)$ is closed.

Note that $\partial B(x, r) = \{b \in X \ : \ \forall r > 0 : B(x, r) \cap B(b, r) \neq \varnothing \wedge B(b, r) \setminus B(x, r) \neq \varnothing\}$.
Suppose $b \in \partial B(x, r)$. Then there exists $y \in X$ such that $\rho(x, y) < r$ and $\rho(b, y) < r$. There also exists $z \in X$ such that $\rho(z, b) < r$ and $\rho(z, x) \leq r$. However, there is $\rho(z, x) \leq \max\left\{\rho(z, b), \rho(b, x)\right\} = \max\left\{\rho(z, b), \rho(b, y), \rho(y, x)\right\} < r$, resulting in a contradiction. Then $\partial B(x, r) = \varnothing$.

Note that exponential functions are positive over $\mathbb{R}$. Then $\rho(\sigma, \sigma') \geq 0$ and equality holds only when $\sigma = \sigma'$.

Since inequality is symmetric ($\sigma(k) \neq \sigma'(k) \Leftrightarrow \sigma'(k) \neq \sigma(k)$), ρ is symmetric.

Suppose for the sake of contradiction that for some $\sigma, \sigma', \sigma'' \in X$, $\rho(\sigma, \sigma'') > \max\left\{\rho(\sigma, \sigma'), \rho(\sigma', \sigma'')\right\}$. Let $i = \inf\{k \in \mathbb{N} \ : \ \sigma(k) \neq \sigma''(k)\}$, $j = \inf\{k \in \mathbb{N} \ : \ \sigma(k) \neq \sigma'(k)\}$, and $l = \inf\{k \in \mathbb{N} \ : \ \sigma'(k) \neq \sigma''(k)\}$. Then $i < j$ and $i < k$. Then $\sigma(k) = \sigma'(k) = \sigma''(k)$ for all $k = 1, \ldots, \min\left\{j, l\right\}$ and i is not the greatest lower bound of $\{k \in \mathbb{N} \ : \ \sigma(k) \neq \sigma''(k)\}$, resulting in a contradiction. Then $\rho(\sigma, \sigma'') \leq \max\left\{\rho(\sigma, \sigma'), \rho(\sigma', \sigma'')\right\} \leq \rho(\sigma, \sigma') + \rho(\sigma', \sigma'')$.

Then ρ is an ultrametric on X.

Problem 2 (Baby Rudin p.43 exercise 9).

Let E° denote the set of all interior points of a set E.

2.1 Prove that E° is always open.

Solution.
E° is open by definition: for all $x \in E^\circ$ there exists $r > 0$ such that $B(x, r) \subseteq A$.

2.2 Prove that E is open if and only if $E^\circ = E$.

Solution.
If $E = E^\circ$ then E is clearly open.

If E is open then for all $x \in E$ there exists $r > 0$ such that $B(x, r) \subseteq E$. Then $x \in E^\circ$. Then $E \subseteq E^\circ$. For all $x \in E^\circ$, there exists $r > 0$ such that $B(x, r) \subseteq E$. Then $x \in E$. Then $E^\circ \subseteq E$. Then $E = E^\circ$.

2.3 If $G \subseteq E$ and G is open, prove that $G \subseteq E^\circ$.

Solution.
Suppose G is open and $G \subseteq E$. Then for all $x \in G$ there exists $r > 0$ such that $B(x, r) \subseteq G \subseteq E$. Then $x \in E^\circ$.

2.4 Prove that the complement of E° is the closure of the complement of E.

Solution.
Note that since for all open subsets $G \subseteq E$ there is $G \subseteq E^\circ$, $E^\circ = \bigcup \left\{G \subseteq E \;:\; G \text{ is open}\right\}$. Then, by De Morgan's law, there is $X \setminus E^\circ = \bigcap \left\{X \setminus G \;:\; G \subseteq E \wedge G \text{ is open}\right\} = \bigcap \{F \subseteq X \;:\; X \setminus E \subseteq F \wedge F \text{ is closed}\} = \overline{X \setminus E}$.

2.5 Do E and $\overline{E}$ always have the same interiors?

Solution.
No. Let $X = \mathbb{R}$ and $E = \mathbb{Q}$. Note that $E^\circ = \varnothing$ while $\overline{E} = \mathbb{R}$ and $\overline{E}^\circ = \mathbb{R}$.

2.6 Do E and E° always have the same closures?

Solution.
No. Let $X = \mathbb{R}$ and $E = \mathbb{Q}$. Note that $\overline{E} = \mathbb{R}$ while $E^\circ = \varnothing$ and $\overline{E^\circ} = \varnothing$.