MATH 131BH (Real Analysis)

May 6, 2022

1 3.28 Monday Week 1: Intro to the course. Review of material covered in 131AH: foundations (definition and constructions of naturals and reals), metric space convergence, continuity.

2 3.30 Wednesday Week 1: Limit of a function: definition and alternative formulations via images of balls and sequential characterization. Limit on a set, left and right limits for functions on $\mathbb{R}$. Discontinuities of first and second kind. Monotone functions have no discontinuities of second kind.

<u>**Limits of functions**</u>

Recall: $f\colon X \to Y$ is said to be **continuous at $x \in X$** if

$$\forall\, \varepsilon > 0\, \exists\, \delta > 0\, \forall\, z \in x : \rho_X(x,z) < \delta \Rightarrow \rho(f(z), f(x)) < \varepsilon.$$

Alternatives:

- $f(B_X(x,\delta)) \subseteq B_Y(f(x),\varepsilon)$;
- $\forall\, \{x_n\}_{n\in\mathbb{N}} \in X^{\mathbb{N}} : x_n \to x \Rightarrow f(x_n) \to f(x)$.

A function $f\colon X \to Y$ is **continuous** if

$$\forall\, x \in X : f \text{ is continuous at } x,$$

or, alternatively,

$$\forall\, O \subseteq Y \text{ open} : f^{-1}(O) \text{ open}.$$

Definition 2.1

A function $f\colon X \to Y$ **has limit $y \in Y$ at $x \in X$**, notation $\lim_{z\to x} f(z) = y$, if

$$\forall\, \varepsilon > 0\, \exists\, \delta > 0\, \forall\, z \in X : 0 < \rho_X(x,z) < \delta \Rightarrow \rho_Y(f(z), y) < \varepsilon.$$

Alternatives:

- $f(B_X(x,\delta) \setminus \{x\}) \subseteq B_Y(y,\varepsilon)$;
- $\forall\, \{x_n\}_{n\in\mathbb{N}} \in X^{\mathbb{N}} : (\forall\, n \in \mathbb{N} : x_n \neq x) \wedge x_n \to x \Rightarrow f(x_n) \to y$;
- $g(z) := \begin{cases} f(z) & z \neq x \\ y & z = x \end{cases}$ is continuous at x.

Definition 2.2

f has a **removable discontinuity** at x if $\lim_{z\to x} f(z)$ exists but $\neq f(x)$.

Definition 2.3

Let $A \subseteq X$ be nonempty, $x \in \overline{A}$ be not an isolated point. Then $\lim_{z\to x} f(z) = \lim_{z\to x} f_A(z)$ where f_A is the restriction of f to A.

Definition 2.4

For $f : \mathbb{R} \to \mathbb{R}$, let $x \in \overline{\mathrm{Dom}(f)}$ be such that $\mathrm{Dom}(f) \cap (x, \infty) \neq \varnothing$ and $\mathrm{Dom}(f) \cap (-\infty, x) \neq \varnothing$. Then $\lim_{z\to x^+} f(z) := \lim_{z\to x, z\in \mathrm{Dom}(f)\cap(x,\infty)} f(z)$ and $\lim_{z\to x^-} f(z) := \lim_{z\to x, z\in\mathrm{Dom}(f)\cap(-\infty,x)} f(z)$ are the **right / left limits of f at x**.

Alternative notation: $f(x^+)$, $f(x^-)$.

Example 2.5.

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

has no right or left limits.

Example 2.6.

$$f(x) = \begin{cases} \frac{1}{n+1} & x = q_n \text{ where } \{q_n\}_{n\in\mathbb{N}} \text{ enumerates } \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

Then $\forall\, x \notin \mathbb{Q} : \lim_{z\to x} f(z) = 0$ so f is continuous on $\mathbb{R} \setminus \mathbb{Q}$, and $\forall\, x \in \mathbb{Q} : \lim_{z\to x} f(z) = 0$ but f is not continuous at x.

Lemma 2.7

$$\forall\, r > 0 \,\forall\, \varepsilon > 0 : \left\{x \in \mathbb{R} : |x| < r \wedge \left|f(x)\right| > \varepsilon\right\} \text{ finite} \Rightarrow \forall\, x \in \mathbb{R} : \lim_{z\to x} f(z) = 0.$$

Definition 2.8

A function $f : \mathbb{R} \to \mathbb{R}$ has a **discontinuity of**

- **first kind** at x if $f(x^+)$ and $f(x^-)$ exist but are not both equal to $f(x)$;
- **second kind** at x if one or both of $f(x^+)$ and $f(x^-)$ don't exist.

Example 2.9.

$$f(x) := \begin{cases} (-1)^n & x = \frac{1}{n+1}, n \in \mathbb{N} \\ \text{linear} & x \in (0,\infty) \setminus \left\{\frac{1}{n+1} : n \in \mathbb{N}\right\} \\ 0 & x \le 0. \end{cases}$$

This function has a discontinuity of second kind at 0.

Lemma 2.10

Let $f : \mathbb{R} \to \mathbb{R}$ ($\mathrm{Dom}(f) = \mathbb{R}$) be monotone. Then $\forall\, x \in \mathbb{R} : f(x^+), f(x^-)$ exist and so f has no discontinuities of second kind.

Proof. Let $x \in \mathbb{R}$ and assume f is nondecreasing. We claim that $\lim_{z \to x^+} f(z) = \inf\left\{f(z) : z > x\right\} =: L$. Indeed, $\forall\, z > x : f(z) \ge f(x)$, so $L \ge f(x)$ and so $L \in \mathbb{R}$. Then $(\forall\, z > x : L \le f(z)) \wedge (\forall\, \varepsilon > 0\, \exists\, z_\varepsilon > x : f(z_\varepsilon) < L + \varepsilon)$. Let $\delta := z_\varepsilon - x$. Then $\forall\, z \in (x, x+\delta) : f(z) \le f(z_\varepsilon) < L + \varepsilon$. Then $\forall\, z \in (x, x+\delta) : L \le f(z) < L + \varepsilon$ and therefore $\left|f(z) - L\right| < \varepsilon$. Then $\lim_{z \to x^+} f(z) = L$. $\square$

3 3.31 Thursday Week 1: Monotone functions have only countably many discontinuities. Functions of bounded variation. Jordan decomposition theorem. Comments on uniqueness. Rectifiability of curves. Limsup and liminf of a function.

<u>**Limits of functions**</u>

Last time we showed that monotone functions have no discontinuities of second kind.

Lemma 3.1

Let $f : \mathbb{R} \to \mathbb{R}$ be monotone. Then $\left\{x \in \mathbb{R} : f(x^+) \ne f(x^-)\right\}$ is countable.

Proof. Pick $k, m \in \mathbb{N}$ and let $A_{m,k} := \left\{x \in [-m, m] : \left|f(x^+) - f(x^-)\right| > \frac{1}{k+1}\right\}$. We claim that $A_{m,k}$ is finite. Let $x_0 < x_1 < \cdots < x_n$ be such that $\forall\, i \le n : x_i \in A_{k,m}$. Assume (without loss of generality) that f is non-decreasing. Then

$$\begin{aligned} f(m+1) \ge f(x_n^+) &= f(x_0^+) + \sum_{i=1}^{n} \left(f(x_i^+) - f(x_{i-1}^+)\right) \\ &\ge f(m-1) + \sum_{i=1}^{n} \left(f(x_i^+) - f(x_i^-)\right) \\ &\ge f(-m+1) + \frac{n}{k+1}. \end{aligned} \tag{3.1}$$

Then $n \le (k+1)$. Since $\left\{x \in \mathbb{R} : f(x^+) \ne f(x^-)\right\} = \bigcup_{k \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} A_{k,m}$, we are done. $\square$

Q: Can these be generalized to other functions?

Definition 3.2

A **partition** Π of an interval $[a,b]$ is a sequence $\{t_i\}_{i=0}^n$ such that

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b.$$

Definition 3.3

Given $f\colon [a,b] \to \mathbb{R}$, its **total variation** on $[a,b]$

$$V(f,[a,b]) := \sup_{\Pi=\{t_i\}_{i=0}^n} \sum_{i=1}^n \left|f(t_i) - f(t_{i-1})\right|$$

where the supremum is over the partitions of $[a,b]$.

Definition 3.4

f is said to be of **bounded variation** on $[a,b]$ if $V(f,[a,b]) < \infty$.

Lemma 3.5

If $f\colon \mathbb{R} \to \mathbb{R}$ is of bounded variation on $[-m,m]$ for all $m \in \mathbb{N}$, then f has only discontinuities of first kind and the set $\left\{x \in \mathbb{R} : f(x^+) \neq f(x^-)\right\}$ is countable.

Theorem 3.6: Jordan decomposition (1881)

Let $f\colon [a,b] \to \mathbb{R}$ obey $V(f,[a,b]) < \infty$. Then $\exists\, h, g\colon [a,b] \to \mathbb{R}$ nondecreasing such that $\forall\, t \in [a,b]$: $f(t) = h(t) - g(t)$.

Proof. Define $h(t) := V(f,[a,t])$ and $g(t) := V(f,[a,t]) - f(t)$. Note that $h(t) - g(t) = f(t)$.

We need to show that h and g are nondecreasing.

Let $a \leq t < t' \leq b$. Then for any partition Π of $[a,t]$, $\Pi' = \Pi \cup \{t'\}$ is a partition of $[a,t']$. Then

$$V(f,[a,t']) \geq \sum_{i=1}^m \left|f(t_i) - f(t_{i-1})\right| + \left|f(t') - f(t)\right|.$$

Taking supremum over Π gives

$$V(f,[a,t']) \geq V(f,[a,t]) + \left|f(t') - f(t)\right|.$$

Note that $\left|f(t') - f(t)\right| \geq 0$ and $\left|f(t') - f(t)\right| \geq f(t') - f(t)$. Then $h(t') \geq h(t)$ and $g(t') \geq g(t)$. $\square$

The representation of $f = h - g$ is called a Jordan decomposition. This is not unique because a nondecreasing function can be added to both h and g.

However, there is a minimal decomposition $f = h_0 - g_0$ such that $g_0(a) = 0$ such that for any other Jordan decomposition $f = h - g$ we have $h - h_0$, $g - g_0$ nondecreasing. This is then *the* Jordan decomposition.

Rectifiability of curves

Definition 3.7

Let (X, ρ) be a metric space. A curve C is $\mathrm{Ran}(f)$ for an $f : \mathbb{R} \to X$ continuous such that $\mathrm{Dom}(f)$ is nonempty and connected. This f is called a **parametrization** of C.

Definition 3.8

Assuming $\mathrm{Dom}(f) = [a, b]$, the **length of C** is

$$\ell(C) := \sup_{\Pi = \{t_i\}_{i=0}^n} \sum_{i=1}^n \rho(f(t_{i-1}), f(t_i)).$$

Definition 3.9

A curve is **rectifiable** if $\ell(C) < \infty$.

Definition 3.10

Let (X, ρ) be a metric space and $f : X \to \mathbb{R}$. Then

$$\limsup_{z \to x} f(z) := \inf_{\delta > 0} \sup_{z \in B(x,\delta) \setminus \{x\}} f(z)$$

and

$$\liminf_{z \to x} f(z) := \sup_{\delta > 0} \inf_{z \in B(x,\delta) \setminus \{x\}} f(z).$$

Lemma 3.11

$$\lim_{z \to x} f(z) \text{ exists in } \mathbb{R} \Leftrightarrow \limsup_{z \to x} f(z) = \liminf_{z \to x} f(z) \in \mathbb{R}.$$

4 4.1 Friday Week 1: Discussion

Definition 4.1

Let (X, ρ_X), (Y, ρ_Y) be metric spaces, $E \subseteq X$, $f : E \to Y$, and $x \in \overline{E}$. Then $\lim_{t \to x} f(t) = \alpha$ is defined by

$$\forall\, \varepsilon > 0\, \exists\, \delta > 0 : \forall\, x \in E \wedge 0 < \rho_X(t, x) < \delta \Rightarrow \rho_Y(f(t), \alpha) < \varepsilon.$$

Equivalently,

$$\forall\, \{t_n\}_{n \in \mathbb{N}} \in (E \setminus \{x\})^{\mathbb{N}} : t_n \to x \Rightarrow f(t_n) \to \alpha.$$

Note. f need not be defined at x.

Remark.

$$\limsup_{t \to x} f(t) := \inf_{\delta > 0} \sup_{t \in B(x,\delta) \setminus \{x\}} f(t) = \lim_{\delta \to 0} \sup_{t \in B(x,\delta) \setminus \{x\}} f(t).$$

lim inf is similarly defined.

Remark.

$$\limsup = \liminf \Rightarrow \lim \text{ exists}.$$

Discontiuities

Definition 4.2

Let $f : (a, b) \to \mathbb{R}$ be not continuous at x. Then f has a **discontinuity of first kind** at x if $f(x+)$ and $f(x-)$ both exist. Otherwise it is of **second kind**.

Remark. Discontinuities of first kind are also known as **simple discontinuities**. The cases include

- $f(x+) = f(x-) \neq f(x)$: **removable discontinuity**, and
- $f(x+) \neq f(x-)$: **jump discontinuity**.

Example 4.3.

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

has a discontinuity of second kind at 0.

Example 4.4.

$$f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is continuous on $\mathbb{R} \setminus \mathbb{Q}$ and has discontinuities of first kind (removable) at every point in $\mathbb{Q}$.

Recall: A monotone function has no discontinuity of second kind and has at most countably many discontinuities of first kind. One can deduce this from the fact that the real line is a union of countably many open intervals (indexed by rationals).

Definition 4.5

A function $f : (a, b) \to \mathbb{R}$ is convex if

$$\forall\, x, y \in (a, b) : x \leq y \Rightarrow (\forall\, \lambda \in [0, 1] : f(\lambda x + (1 - \lambda)y)) \leq \lambda f(x) + (1 - \lambda)f(y).$$

In words, this means that for any interval, the secant line is above the graph.

5 4.4 Monday Week 2: Existence of limit is equivalent to equality and finiteness of limsup and liminf. Derivative of a real valued function of one real variable. Differentiability implies continuity. Connection with linear approximation. Sum and product rule, chain rule and inverse function rule. First-derivative test and discussion of important counterexamples.

Last time: $\lim_{z\to x} f(z)$, $\limsup_{z\to x} f(z) = \inf_{\delta>0} \sup_{z\in B(x,\delta)\setminus\{x\}x} f(z)$

Lemma 5.1

$$\lim_{z\to x} f(z) \text{ exists (in } \mathbb{R}) \Leftrightarrow \limsup_{z\to x} f(z) = \liminf_{z\to x} f(z) \in \mathbb{R}.$$

Proof. Both are equivalent:

$$\forall\, \varepsilon > 0\, \exists\, \delta > 0 : 0 \leq \sup_{z\in B(x,\delta)\setminus\{x\}} f(z) - \inf_{z\in B(x,\delta)\setminus\{x\}} f(z) \leq 2\varepsilon.$$

□

Definition 5.2

$$\lim_{z\to x} f(z) = \begin{cases} +\infty & \limsup_{z\to x} f(z) = \liminf_{z\to x} f(z) = +\infty \\ -\infty & \limsup_{z\to x} f(z) = \liminf_{z\to x} f(z) = -\infty. \end{cases}$$

Note. This characterization works even outside $\mathbb{R}$-valued functions:

$$\lim_{z\to x} f(z) \text{ exists} \Leftrightarrow \lim_{\delta\to 0^+} \sup \underbrace{\left\{\rho(f(z), f(u)) : z, u \in B(x, \delta) \setminus \{x\}\right\}}_{=\operatorname{diam}(f(B(x,\delta)\setminus\{x\}))} = 0.$$

The derivative

Definition 5.3

Let $f:\mathbb{R}\to\mathbb{R}$, $x\in\operatorname{int}(\operatorname{Dom}(f))$. We say that f has **derivative** or **is differentiable at** x if

$$f'(x):=\lim_{z\to x}\frac{f(z)-f(x)}{z-x}\text{ exists in }\mathbb{R}.$$

We call $f'(x)$ (Lagrange notation) the **derivative at** x, alternative notation $\frac{df}{dx}$ (Leibniz notation).

Lemma 5.4

$$f'(x)\text{ exists}\Rightarrow f\text{ continuous at }x.$$

Proof. The existence of $f'(x)$ implies that $\exists\,\delta_0>0\,\forall\,z\in\mathbb{R}:0<|z-x|<\delta_0\Rightarrow\left|\frac{f(z)-f(x)}{z-x}\right|\leq 1+\left|f'(x)\right|$. Then, choosing $\varepsilon>0$ and letting $\delta:=\frac{\varepsilon}{1+|f'(x)|}$, we get

$$\forall\,z\in\mathbb{R}:0<|z-x|<\delta\Rightarrow\left|f(z)-f(x)\right|\leq(1+\left|f'(x)\right|)\,|z-x|<(1+\left|f'(x)\right|)\frac{\epsilon}{1+\left|f'(x)\right|}=\varepsilon.$$

Since $f(z)-f(x)=0$ for $z=x$, we are done (in fact, we have shown that f is lipschitz continuous). □

Another way to write existence of $f'(x)$:

$$f(z)-f(x)=(f'(x)+u_x(z))(z-x)$$

where $\lim_{z\to x}u_x(z)=0$. (Just define: $u_x(z):=\frac{f(z)-f(x)}{z-x}-f'(x)$ for $z\neq x$)

Lemma 5.5: Linear approximation

$$f'(x)\text{ exists}\Leftrightarrow\exists\,L\in\mathbb{R}:\lim_{\delta\to0^+}\sup_{|z-x|<\delta}\frac{1}{\delta}\left|f(z)-f(x)-L(z-x)\right|=0.$$

Lemma 5.6: Sum & product rule

Let f,g be differentiable at x. Then so are $f+g$ and $f\cdot g$ and

$$\begin{aligned}(f+g)'(x)&=f'(x)+g'(x)\\(f\cdot g)'(x)&=f'(x)g(x)+g'(x)f(x)\quad\text{(Leibniz rule).}\end{aligned}$$

Proof. For product rule, note that

$$f(z)g(z)-f(x)g(x)=(f(z)-f(x))g(z)+(g(z)-g(x))f(z).$$

Then

$$\frac{f(z)g(z)-f(x)g(x)}{z-x}=\frac{f(z)-f(x)}{z-x}g(z)+\frac{g(z)-g(x)}{z-x}f(z).$$

Since $g(z)\to g(x)$ by continuity of g, formula follows by sum & product rule for limit. □

Lemma 5.7: Chain rule

Let f be differentiable at x and g at $f(x)$. Then $g \circ f$ is differentiable at x and

$$(g \circ f)'(x) = g'(f(x))f'(x) \quad \left(\frac{dg}{df}\frac{df}{dx}\right)$$

Proof. Define $v_{f(x)}$ such that $g(y) - g(f(x)) = (g'(f(x))) + v_{f(x)}(y))(y - f(x))$ and u_x such that $f(z) - f(x) = (f'(x) + u_x(z))(z - x)$.

$$\begin{aligned}(g \circ f)(z) - (g \circ f)(x) &= [g'(f(x)) + v_{f(x)}(f(z))](f(z) - f(x)) \\ &= [g'(f(x)) + v_{f(x)}(f(z))][f'(x) + u_x(z)](z - x)\end{aligned}$$

Dividing by $z - x \neq 0$, note that $f(z) \to f(x)$ implies $v_{f(x)}(f(z)) \to 0$ as $z \to x$, we are done. □

Lemma 5.8

Let $f \colon \mathbb{R} \to \mathbb{R}$ be injective on $\mathrm{Dom}(f)$ and differentiable at $x \in \mathrm{int}(\mathrm{Dom}(f))$. Assume $f'(x) \neq 0$ and $f(x) \in \mathrm{int}(\mathrm{Ran}(f))$. Then f^{-1} is differentiable at $f(x)$ and

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}.$$

In Leibniz notation:

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

Lemma 5.9: First derivative test

Let $f \colon [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then if $x \in (a, b)$ is a local maximum of f (i.e. $\exists\, \delta > 0\, \forall\, z \in \mathbb{R} : |z - x| < \delta \Rightarrow f(x) \geq f(z)$) then $f'(x) = 0$.

Proof.

$$z > x \wedge |z - x| < \delta \Rightarrow \frac{f(z) - f(x)}{z - x} \leq 0 \Rightarrow f'(x) \leq 0$$

and

$$z < x \wedge |z - x| < \delta \Rightarrow \frac{f(z) - f(x)}{z - x} \geq 0 \Rightarrow f'(x) \geq 0.$$

□

6 4.6 Wednesday Week 2: Discussion

Recall: For, $x \colon [a, b] \to \mathbb{R}$, the total variation

$$V(f, [a, b]) = \sup_{\Pi} \sum_{i=1}^{n} \left| f(x_i) - f(x_{i-1}) \right|$$

where $\Pi = \{a = x_0 < x_1 < \cdots < x_n = b\}$. We say $f \in BV([a, b])$ if $V(f, [a, b]) < \infty$.

Theorem 6.1: Jordan decomposition

$$\forall f \in BV([a,b]) \, \exists h, g\colon [a,b] \to \mathbb{R} \text{ nondecreasing} : f = h - g.$$

Corollary 6.2

$f \in BV([a,b])$ can only have discontinuities of first kind and countably many of them.

Example 6.3. $f(x) = \sin x \in BV([-1,1])$ since f is nondecreasing on $[-1,1]$ and hence $V(f,[a,b]) = f(b) - f(a)$.

Example 6.4. $f(x) = \sin x \in BV([-M,M])$ by additive property of V.

Q. Does $BV([a,b])$ imply bounded on $[a,b]$?

Yes. By triangle inequality,

$$\left|f(x)\right| \leq \left|f(a)\right| + \left|f(a) - f(x)\right| \leq \left|f(a)\right| + V(f,[a,b]) < \infty.$$

Q. Does being bounded on $[a,b]$ imply $BV([a,b])$.

No. A counterexample is

$$f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

on $[0,1]$.

Choose $x_n = 1/(n\pi/2)$ such that $\sin(1/x_n) = \sin(n\pi/2)$. Then $\sum_{i=1}^{2n} \left|f(x_i) - f(x_{i-1})\right| = \sum_{k=1}^{n} \left|f(x_{2k+1})\right| = n \to \infty$.

Example 6.5. Is

$$f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

on $[0,1]$ of bounded variation?

No. Choose the same x_n as above. Note that $f(x_n) = \frac{2}{n\pi}\sin(n\pi/2)$. Then $\sum_{i=1}^{2n} \left|f(x_i) - f(x_{i-1})\right| = \sum_{k=1}^{n} \frac{2}{(2k-1)\pi} \to \infty$.

Example 6.6. Is

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

on $[0,1]$ of bounded variation?

Yes. Note that

$$f'(0) = \lim_{t \to 0} \frac{t^2 \sin \frac{1}{t} - 0}{t} = \lim_{t \to 0} t \sin \frac{1}{t} = 0.$$

Note that for $x \neq 0$,

$$f'(x) = 2x \sin \frac{1}{x} + x^2 \left(-\frac{1}{x^2}\right) \cos \frac{1}{x} = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$$

is bounded: $\left|f'(x)\right| \leq 2\left|x\right| + 1 \leq 3$.

Note that by mean value theorem, we have

$$\sum \left|f(x_i) - f(x_{i-1})\right| \le \sum \left|f'(\xi)\right|(x_i - x_{i-1}) \le M(b-a) < \infty$$

where $\left|f'(\xi)\right| \le M$.

Then f is of bounded variation on $[0,1]$.

Theorem 6.7

If f' exists and is bounded on $[a,b]$ then f is of bounded variation.

Q. Does the existence f' on $[a,b]$ and f being of bounded variation on $[a,b]$ imply f' is bounded on $[a,b]$?

7 4.7 Thursay Week 2: Mean-Value Theorems of Rolle, Lagrange and Cauchy. Applications: Monotone differentiable functions have derivative of one sign. Derivative of a differentiable function has no discontinuities of first kind (but those of second kind can occur densely). L'Hospital's Rule and its proof from Cauchy's MVT.

Mean value theorems

Last time: $f'(x)$ = derivative is linked to the local maxima and minima (first derivative test).

Theorem 7.1: Mean value theorem

Let $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then

1. (Rolle's theorem, 1691) $f(a) = f(b) \Rightarrow \exists\, x \in (a,b) : f'(x) = 0$,
2. (Lagrange's mean value theorem) $\exists\, x \in (a,b) : f'(x) = \frac{f(b)-f(a)}{b-a}$, and
3. (Cauchy mean value theorem, 1823) if also $g\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), then

$$\forall\, x \in (a,b) : g'(x) \neq 0 \Rightarrow g(a) \neq g(b) \wedge \exists\, x \in (a,b) : \frac{f'(x)}{g'(x)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Proof.

1. $f(a) = f(b) \wedge$ continuous function on $[a,b]$ achieves one of maximum and minimum on (a,b) $\Rightarrow \exists\, x \in (a,b) : x$ is local maximum or local minimum of f. Then $f'(x) = 0$.
2. Let $h(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$. Then $h(a) = f(a)$, $h(b) = f(b) - \frac{f(b)-f(a)}{b-a}(b-a) = f(a)$. Then, by 1., $\exists\, x \in (a,b) : h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a} = 0$.
3. Let $h(x) = f(x) - \frac{f(b)-f(a)}{g(b)-g(a)}(g(x)-g(a))$. Note that this is well defined since by 1. we have $g(b) \neq g(x)$. Then $h(a) = f(a) = h(b)$ so by 1. we have $\exists\, x \in (a,b) : h'(x) = f'(x) - \frac{f(b)-f(a)}{g(b)-g(a)}g'(x) = 0$.

□

Applications

Lemma 7.2

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then

$$\forall x \in (a, b) : f'(x) \geq 0 \Leftrightarrow \forall x, y \in [a, b] : x \leq y \Rightarrow f(x) \leq f(y).$$

Proof. The $\Leftarrow$ direction is immediate from the definition of limit $\left(\frac{f(y)-f(x)}{y-x} \geq 0\right)$.

For the $\Rightarrow$ direction, if $\exists\, x \geq y : f(y) < f(x)$ then by the mean value theorem $\exists\, z \in (x, y) : f'(z) = \frac{f(y)-f(x)}{y-x} < 0$.

□

8 4.8 Friday Week 2: Taylor's theorem via Mean Value Theorem (Rolle suffices). Riemann integral: motivation, definitions of marked partition, mesh of partition and Riemann sum. Notion of a function being Riemann integrable. Linearity of integral.

Taylor's theorem

Definition 8.1: Higher order derivatites

Define $f^{(0)} := f$ and for all $n \in \mathbb{N}$ define $f^{(n+1)}(x) := \left(f^{(n)}\right)'(x)$ assuming the derivatives exist. We call $f^{(n)}$ the ***n*-th derivative of *f***.

Theorem 8.2: Taylor's theorem (Taylor 1715, Gregory 1671)

Let $n \in \mathbb{N}$ and $f : (a, b) \to \mathbb{R}$ an $(n + 1)$-times differentiable function. Then

$$\forall\, x_0 \in (a, b)\,\forall\, x \in (x_0, b)\,\exists\, \xi \in (x_0, x) : f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k}_{n\text{-th order Taylor polynomial at } x_0} + \frac{f^{(n+1)}(\xi)}{(n + 1)!}(x - x_0)^{n+1}.$$

Proof. Based on MVT.

Denote

$$P_n(z) := \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(z - x_0)^k.$$

Pick $x \in (x_0, b)$ and denote

$$A := \frac{f(x) - P_n(x)}{(x - x_0)^{n+1}}.$$

Set
$$h(z) := f(z) - P_n(z) - A(z - x_0)^{n+1}.$$
Note that
$$\forall\, k \in \mathbb{N} : k \le n \Rightarrow f^{(k)}(x_0) = 0.$$
We claim that
$$\forall\, k \in \mathbb{N} : 1 \le k \le n+1 \Rightarrow \exists\, \xi_k \in (x_0, x) : h^{(k)}(\xi_k) = 0.$$
For $k = 1$, the choice of A implies $h(x) = 0$ so since $h(x_0) = 0$, by Rolle's theorem
$$\exists\, \xi_1 \in (x_0, x) : h'(\xi) = 0.$$
Assume true for some $k \in \mathbb{N}$ such that $1 \le k \le n$. Then $h^{(k)}(x_0) = 0$ and $h^{(k)}(\xi_k) = 0$ for $\xi_k \in (x_0, x)$. Then by Rolle's theorem
$$\exists\, \xi_{k+1} \in (x_0, \xi_k) : h^{(n+1)}(\xi_{k+1}) = 0.$$
Now observe that $P_n^{(n+1)} = 0$. Then $0 = h^{(n+1)}(\xi_{n+1}) = f^{(n+1)}(\xi_{n+1}) - A(n+1)!$. Then
$$f(x) - P_n(x) = A(x - x_0)^{n+1} = \frac{f^{(n+1)}(\xi_{n+1})}{(n+1)!}(x - x_0)^{n+1}.$$
□

Riemann integral (Riemann 1854)

Goal: Given $f\colon [a,b] \to \mathbb{R}$, assign meaning to the area under the graph of f on $[a,b]$; namely to the set
$$\left\{(x,y) \in \mathbb{R}^2 : x \in [a,b] \wedge 0 \le y \le f(x)\right\} \quad \text{(for } f \ge 0\text{)}.$$
Idea: Approximate f with a piecewise constant function and use that the area of a rectangle is "known."

Definition 8.3

Given $[a,b]$, a **marked partition Π** of $[a,b]$ is two sequences $\{t_i\}_{i=0}^n$, $\left\{t_i^*\right\}_{i=1}^n$ such that

- $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ and
- $\forall\, i = 1, \ldots, n : t_i^* \in [t_{i-1}, t_i]$.

Definition 8.4

The **mesh of Π** is defined by $||\Pi|| := \max_{i=1,\ldots,n} |t_i - t_{i-1}|$.

Definition 8.5

Given $f\colon [a,b] \to \mathbb{R}$ and a marked partition Π, the associated **Riemann sum** is
$$R(f,\Pi) := \sum_{i=1}^{n} f(t_i^*)(t_i - t_{i-1}).$$

Definition 8.6

A function $f\colon [a,b] \to \mathbb{R}$ is said to be **Riemann integrable** (on $[a,b]$) if there exists $L \in \mathbb{R}$ such that

$$\forall\, \varepsilon > 0\, \exists\, \delta > 0\, \forall\, \Pi = \text{marked partition of } [a,b] : \|\Pi\| < \delta \Rightarrow \left|R(f,\Pi) - L\right| < \varepsilon.$$

We sometimes write this as $\lim_{\|\Pi\| \to 0} R(f,\Pi) = L$ (this L is unique). Notation for L is $\int_a^b f(x)\, dx$.

Lemma 8.7: Additivity and homogeneity of Reimann integral

Let $f,g\colon [a,b] \to \mathbb{R}$ be Riemann integrable on $[a,b]$. Let $\alpha, \beta \in \mathbb{R}$. Then $\alpha f + \beta g$ is Riemann integrable on $[a,b]$ and

$$\int_a^b (\alpha f(x) + \beta g(x))\, dx = \alpha \int_a^b f(x)\, dx + \beta \int_a^b g(x)\, dx.$$

Proof. Given $\varepsilon > 0$, pick $\delta > 0$ such that $\|\Pi\| < \delta$ implies

$$\left|R(f,\Pi) - \int_a^b f(x)\, dx\right| < \varepsilon \wedge \left|R(g,\Pi) - \int_a^b g(x)\, dx\right| < \varepsilon.$$

Since $R(\alpha f + \beta g, \Pi) = \alpha R(f,\Pi) + \beta R(g,\Pi)$,

$$\begin{aligned}
&\left|R(\alpha f + \beta g, \Pi) - \alpha \int_a^b f(x)\, dx - \beta \int_a^b g(x)\, dx \leq |\alpha|\right| \\
&\leq |\alpha| \left|R(f,\Pi) - \int_a^b f(x)\, dx\right| + \left|\beta\right| \left|R(g,\Pi) - \int_a^b g(x)\, dx\right| \\
&\leq (|\alpha| + \left|\beta\right|)\varepsilon.
\end{aligned}$$

□

Corollary 8.8

Let $f,g : [0,\infty) \to \mathbb{R}$ be continuous on $[0,\infty)$ and differentiable on $(0,\infty)$. Then

$$f(0) \leq g(0) \wedge \forall\, x \in (0,\infty) : f'(x) \leq g'(x) \Rightarrow \forall\, x \in [0,\infty] : f(x) \leq g(x).$$

Example 8.9. $\forall\, x \geq 0 : e^x \geq 1 + x$.

Lemma 8.10

Let $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then f' has the intermediate value property.

Proof. Without loss of generality assume f' exists on $[\tilde{a}, \tilde{b}]$ such that $\tilde{a} < a < b < \tilde{b}$. Without loss of generality assume $f'(a) < f'(b)$. Let $t \in (f'(a), f'(b))$. Let $h(x) := f(x) - tx$. Then

$$h'(a) < 0 \Rightarrow \exists\, x \in (a, b) : h(x) < h(a).$$

With the same reasoning, we have

$$h'(b) > 0 \Rightarrow \exists\, y \in (a, b) : h(y) < h(b).$$

Then

$$\exists\, z \in (a, b) \text{ local minimum} \Rightarrow h'(z) = f(z) - t = 0.$$

□

Corollary 8.11

The derivative of a differentiable function does not have discontinuities of first kind.

Example 8.12. Let

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Then $\forall\, x \neq 0 : f'(x) = x \sin(1/x) - \cos(1/x)$. $\lim_{x \to 0^\pm} f'(x)$ does not exist.

Also note that

$$\frac{f(x) - f(0)}{x - 0} = x \sin(1/x) \xrightarrow[x \to 0]{} 0$$

so $f'(0) = 0$.

Theorem 8.13: L'Hopital's rule, proved by Bernoulli 1694

Let $f, g \colon \mathbb{R} \to \mathbb{R}$ be continuous and differentiable on $(a - \delta, a + \delta)$ where $a \in \mathbb{R}$ and $\delta > 0$. Assume

$$f(a) = 0 = g(a) \wedge \forall\, x \in (a - \delta, a + \delta) \setminus \{a\} : g(x) \neq 0 \wedge g'(x) \neq 0.$$

Theni

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} \text{ exists} \Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} \text{ exists} \wedge \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Proof. Let $x \in (a - \delta, a + \delta) \setminus \{a\}$. Then for $x > a$ we have

$$\frac{f(x)}{g(x)} \xlongequal[f(a)=0, g(a)=0]{} \frac{f(x) - f(a)}{g(x) - g(a)} \xlongequal[\text{Cauchy MVT}]{\exists z_x \in (a,x)} \frac{f'(z_x)}{g'(z_x)}.$$

Since $x \to a$ implies $z_x \to a$, existence of $\lim_{z \to a} \frac{f'(z)}{g'(z)}$ gives

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{z \to a} \frac{f'(z)}{g'(z)}.$$

□

Example 8.14. $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$.

9 4.11 Monday Week 3

Last time: $f : [a,b] \to \mathbb{R}$ is Riemann integrable (RI) if

$$\exists\, L \in \mathbb{R}\, \forall\, \varepsilon > 0\, \exists\, \delta > 0\, \forall\, \Pi = \text{marked partition of } [a,b] : ||\Pi|| < \delta \Rightarrow \left|R(f,\Pi) - L\right| < \varepsilon.$$

Notation: $L = \int_a^b f(x)\, dx$.

We proved **linearity**:

$$\int_a^b (\alpha f(x) + \beta f(x))\, dx = \alpha \int_a^b f(x)\, dx + \beta \int_a^b g(x)\, dx.$$

Lemma 9.1

If f is RI on $[a,b]$ then f is bounded on $[a,b]$.

Proof. RI $\Rightarrow \exists\, \delta > 0\, \forall\, \Pi$ = marked partition : $R(f,\Pi) \leq L+1$. Then $\forall\, i = 1,\ldots,n\, \forall\, \tilde{t}_i : f(\tilde{t}_i)(t_i - t_{i-1}) + \sum_{j=1,\ldots,n,j\neq i} f(t_j^*)(t_j - t_{j-1}) \leq L+1$, which means $\sup_{\tilde{t}_i \in [t_{i-1},t_i]} f(\tilde{t}_i) < \infty$. Then $\sup_{x\in[a,b]} f(x) < \infty$. □

Lemma 9.2: Additivity

Let $a < c < b$ be reals. If f is RI on $[a,c]$ and on $[c,b]$, then it is RI on $[a,b]$ and

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

Proof. Let $\varepsilon > 0$ and let $\delta > 0$ be such that $\forall\, \Pi$ = marked partition of $[a,c]$ and $\forall\, \Pi'$ = marked partition of $[c,b]$ such that $||\Pi|| < \delta \wedge ||\Pi'|| < \delta$ we have

$$\left|R(f,\Pi) - \int_a^c f(x)\, dx\right| < \varepsilon \quad \wedge \quad \left|R(f,\Pi') - \int_c^b f(x)\, dx\right| < \varepsilon.$$

If $\tilde{\Pi}$ is a marked partition of $[a,b]$ with $||\tilde{\Pi}|| < \delta$ containing c then

$$\left|R(f,\Pi) - \int_a^c f(x)\, dx - \int_c^b f(x)\, dx\right| < 2\varepsilon.$$

Suppose $\tilde{\Pi}$ does not contain c. Then adding c to $\tilde{\Pi}$ changes $R(f,\tilde{\Pi})$ by at most $2 \cdot 3\delta \sup_{x\in[a,b]} \left|f(x)\right|$. □

Lemma 9.3

If f is RI on $[a,b]$ then

$$\left|\int_a^b f(x)\, dx\right| \leq (b-a) \underbrace{\sup_{x\in[a,b]} \left|f(x)\right|}_{<\infty}.$$

Proof. Note that

$$\left|R(f,\Pi)\right| = \left|\sum_{i=1}^{n} f(t_i^*)(t_i - t_{i-1})\right| \leq \sum_{i=1}^{n} \left|f(t_i^*)(t_i - t_{i-1})\right| = R(\left|f\right|,\Pi) \leq \sup_{x\in[a,b]} \left|f(x)\right| \underbrace{\sum_{i=1}^{n}(t_i - t_{i-1})}_{=b-a}$$

□

Note. If we knew that $\left|f\right|$ is RI, then this gives

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b \left|f(x)\right|\,dx.$$

Q: Sufficient conditions for RI?

A: We will answer this using Darboux's version of Riemann integral.

Definition 9.4

Let $f\colon [a,b] \to \mathbb{R}$ be bounded. Given an unmarked partition $\Pi = \{t_i\}_{i=1}^{n}$ of $[a,b]$, set

$$U(f,\Pi) := \sum_{i=1}^{n} \sup\left\{f(x) : x \in [t_{i-1},t_i]\right\}(t_i - t_{i-1})$$

and

$$L(f,\Pi) := \sum_{i=1}^{n} \inf\left\{f(x) : x \in [t_{i-1},t_i]\right\}(t_i - t_{i-1})$$

to be the **upper and lower Darboux sums**.

Note. $L(f,\Pi) \leq R(f,\Pi) \leq U(f,\Pi)$ for any marked partition Π.

Lemma 9.5

For all unmarked partitions Π and Π' of $[a,b]$ we have

$$L(f,\Pi) \leq U(f,\Pi').$$

Proof. Assume first Π is a subset of Π', meaning that all points of Π are included in Π'. We claim that $U(f,\Pi') \leq U(f,\Pi)$ and $L(f,\Pi') \geq L(f,\Pi)$.

Note that if $\Pi' = \Pi \cup \{t\}$, let $[t_{i-1},t_i]$ be the interval containing t. Then

$$\max\left\{\sup_{x\in[t_{i-1},t]} f(x),\ \sup_{x\in[t,t_i]} f(x)\right\} \sup_{x\in[t_{i-1},t_i]} f(x),$$

resulting in $U(f,\Pi') \leq U(f,\Pi)$.

Now let Π and Π' be arbitrary and $\Pi \cup \Pi'$ be their common refinement. Then

$$L(f,\Pi) \leq L(f,\Pi\cup\Pi') \leq U(f,\Pi\cup\Pi') \leq U(f,\Pi').$$

□

Definition 9.6

Set
$$\underline{\int_a^b} f(x)\,dx := \sup\left\{L(f,\Pi) : \Pi = \text{partition of } [a,b]\right\}$$
and
$$\overline{\int_a^b} f(x)\,dx := \inf\left\{U(f,\Pi) : \Pi = \text{partition of } [a,b]\right\}$$
to be the **lower and upper Darboux integrals**.

Note.
$$\underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx.$$

Definition 9.7

We say that a bounded f is **Darboux integrable on $[a,b]$** if
$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx.$$

10 4.13 Wednesday Week 3

Riemann integral continued

Last time: $U(f,\Pi)$ and $L(f,\Pi)$ are the upper and lower Darboux sums. Note that
$$\forall\,\Pi,\Pi' : L(f,\Pi) \leq U(f,\Pi').$$
Then
$$\overline{\int_a^b} f(x)\,dx = \inf\left\{U(f,\Pi) : \Pi \text{ partition}\right\}$$
and
$$\underline{\int_a^b} f(x)\,dx = \sup\left\{L(f,\Pi) : \Pi \text{ partition}\right\}$$
obey
$$\underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx.$$

Definition 10.1

$f : [a, b] \to \mathbb{R}$ bounded is **Darboux integrable** if

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx.$$

Lemma 10.2

For every $f : [a, b] \to \mathbb{R}$:

$$f \text{ Darboux integrable} \Leftrightarrow \forall\, \varepsilon > 0\, \exists\, \Pi \text{ partition} : U(f, \Pi) - L(f, \Pi) < \varepsilon.$$

Proof. By definition,

$$\forall\, \varepsilon > 0\, \exists\, \Pi, \tilde{\Pi} : U(f, \Pi) < \overline{\int_a^b} f(x)\,dx + \varepsilon \quad \wedge \quad L(f, \tilde{\Pi}) > \underline{\int_a^b} f(x)\,dx - \varepsilon.$$

Then

$$U(f, \Pi \cup \tilde{\Pi}) - L(f, \Pi \cup \tilde{\Pi}) \le U(f, \Pi) - L(f, \tilde{\Pi}) \le \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx + 2\varepsilon.$$

Then the equality of the Darboux integrals implies the left to right direction of the lemma.

For the converse,

$$0 \le \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \le U(f, \Pi) - L(f, \Pi) < \varepsilon.$$

□

Lemma 10.3

Let Π and Π' be unmarked partitions. Then

$$U(f, \Pi') \ge U(f, \Pi) - 2\,|\Pi'|\, ||\Pi|| ||f||$$

and

$$L(f, \Pi') \le L(f, \Pi) + 2\,|\Pi'|\, ||\Pi|| ||f||.$$

where $||f|| := \sup_{x \in [a,b]} \left|f(x)\right|$.

Proof. Note that

$$U(f, \Pi') \ge U(f, \Pi \cup \Pi')$$

and for $f \ge 0$, dropping intervals of Π that receive points in Π' from $U(f, \Pi)$ changes the result by at most $2\,|\Pi'|\, ||\Pi|| ||f||$. □

Theorem 10.4

For every $f\colon [a,b] \to \mathbb{R}$ bounded:

$$f \text{ Riemann integrable} \Leftrightarrow f \text{ Darboux integrable.}$$

If both are true then

$$\int_a^b f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx.$$

Proof. $\Rightarrow$: RI means that

$$\exists L \in \mathbb{R}\, \exists\, \delta > 0\, \forall\, \Pi \text{ partition with } ||\Pi|| < \delta : \left|R(f,\Pi) - L\right| < \varepsilon.$$

Pick $N \in \mathbb{N}$ such that $N > (b-a)/\delta$, define $\Pi = \{t_i\}_{i=1}^n$ such that $t_i - t_{i-1} = \frac{b-a}{N} < \delta$. Now pick $t_i^* \in [t_{i-1}, t_i]$ such that

$$f(t_i^*) \geq \sup\left\{f(x) : x \in [t_{i-1}, t_i]\right\} - \frac{\varepsilon}{b-a}$$

and $\tilde{t}_i^* \in [t_{i-1}, t_i]$ such that

$$f(\tilde{t}_i^*) \leq \inf\left\{f(x) : x \in [t_{i-1}, t_i]\right\} + \frac{\varepsilon}{b-a}.$$

Then let Π be the partition with marked points $\left\{t_i^*\right\}_{i=1}^N$ and $\tilde{\Pi}$ be the partition with marked points $\left\{\tilde{t}_i^*\right\}_{i=1}^N$. Then

$$U(f,\Pi) \leq \sum_{i=1}^N \left(f(t_i^*) + \frac{\varepsilon}{b-a}\right)(t_i - t_{i-1}) = R(f,\Pi) + \varepsilon$$

and

$$L(f,\tilde{\Pi}) \geq \sum_{i=1}^n \left(f(\tilde{t}_i^* - \frac{\varepsilon}{b-a})\right)(t_i - t_{i-1}) = R(f,\tilde{\Pi}) - \varepsilon.$$

Now

$$\begin{aligned} U(f,\Pi \cup \tilde{\Pi}) - L(f,\Pi \cup \tilde{\Pi}) &\leq U(f,\Pi) - L(f,\tilde{\Pi}) \\ &\leq R(f,\Pi) - R(f,\tilde{\Pi}) + 2\varepsilon \\ &\leq \left|R(f,\Pi) - L\right| + \left|R(f,\tilde{\Pi}) - L\right| + 2\varepsilon \\ &\leq 4\varepsilon. \end{aligned}$$

$\Leftarrow$: $\forall\, \varepsilon > 0 \exists\, \Pi'$ partition such that $U(f,\Pi') - L(f,\Pi') < \varepsilon$. Pick any Π and $\tilde{\Pi}$ marked partitions with $||\tilde{\Pi}||, ||\Pi|| < \delta := \varepsilon/(|\Pi'|\,||f||)$ $\quad (f \neq 0)$.

Then

$$R(f,\Pi) \leq U(f,\Pi) \overset{\text{by Lemma 10.3}}{\leq} U(f,\Pi') + 2\underbrace{|\Pi'|\,||\Pi||||f||}_{\leq \varepsilon}$$

and

$$R(f,\tilde{\Pi}) \geq L(f,\tilde{\Pi}) \overset{\text{by Lemma 10.3}}{\geq} L(f,\Pi') - 2\underbrace{|\Pi'|\,||\tilde{\Pi}||||f||}_{\leq \varepsilon}.$$

Then
$$\left|R(f,\tilde{\Pi}) - R(f,\tilde{\Pi})\right| \le U(f,\Pi') - L(f,\Pi') + 4\varepsilon \le 5\varepsilon.$$
Let $\{\Pi_n\}$ be an arbitrary sequence of marked partitions such that
$$||\Pi_n|| \to 0 \quad \wedge \quad L := \lim_{n\to\infty} R(f,\Pi_n) \text{ exists.}$$
This exists by Bolzano-Weierstrass theorem.

Then
$$\left|R(f,\Pi) - L\right| \le \left|R(f,\Pi_n) - L\right| + \left|R(f,\Pi) - R(f,\Pi_n)\right| \underset{\text{once } ||\Pi_n||<\delta}{\le} \left|R(f,\Pi_n) - L\right| + 5\varepsilon \underset{n\to\infty}{\longrightarrow} 5\varepsilon.$$
Then we showed that
$$\exists L \in \mathbb{R}\, \forall \varepsilon > 0\, \exists \delta > 0\, \forall \Pi \text{ marked partition} : ||\Pi|| < \delta \Rightarrow \left|R(f,\Pi) - L\right| \le 5\varepsilon.$$
□

Corollary 10.5

Let $f: [a,b] \to \mathbb{R}$ be bounded. Then
$$f \text{ is RI} \Leftrightarrow \forall \varepsilon > 0\, \exists \Pi = \{t_i\}_{i=1}^n \text{ unmarked partition} : \sum_{i=1}^N \operatorname{osc}(f,[t_{i-1},t_i])(t_i - t_{i-1}) < \varepsilon$$
where $\operatorname{osc}(f,A) = \sup_{x\in A} f(x) - \inf_{x\in A} f(x)$

Example 10.6. The dirichlet function
$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$
is not RI.

Example 10.7. The function
$$f(x) = \begin{cases} \frac{1}{n+1} & x = q_n \\ 0 & x \notin \mathbb{Q} \end{cases}$$
is RI.

11 4.15 Friday Week 3

Riemann Integrability - criteria and characterization

Last time: $\forall f: [a,b] \to \mathbb{R}$ bounded,
$$f \text{ RI} \Leftrightarrow \forall \varepsilon > 0\, \exists \Pi = \{t_i\}_{i=1}^n \text{ partition of } [a,b] : \sum_{i=1}^n \operatorname{osc}(f,[t_{i-1},t_i])\, |t_i - t_{i-1}| < \varepsilon$$
where
$$\begin{aligned}\operatorname{osc}(f,A) &:= \sup\left\{\left|f(y) - f(x)\right| : x, y \in A\right\} \\ &= \sup_{x\in A} f(x) - \inf_{x\in A} f(x) (A \neq \varnothing).\end{aligned}$$

Lemma 11.1

Let $f, g\colon [a,b] \to \mathbb{R}$. Then

1. f RI $\Rightarrow \left|f\right|$ RI and
2. f, g RI $\Rightarrow f \cdot g$ RI.

Proof. Note that

$$\left|\left|f\right|(x) - \left|f\right|(y)\right| = \left|\left|f(x)\right| - \left|f(y)\right|\right| \leq \left|f(x) - f(y)\right|.$$

Then

$$\operatorname{osc}(f, A) \leq \operatorname{osc}(\left|f\right|, A).$$

Then

$$f \text{ RI} \Rightarrow \left|f\right| \text{ RI}.$$

Note that a counterexample for the converse is Dirichlet's function. □

Theorem 11.2

For all $f\colon [a,b] \to \mathbb{R}$ we have

$$f \text{ continuous} \Rightarrow f \text{ RI}.$$

Proof. Note that $[a,b]$ compact and f continuous implies that f is uniformly continuous. Then for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $s, t \in [a,b]$ we have

$$0 < |s-t| < \delta \Rightarrow \operatorname{osc}(f, [s,t]) < \frac{\varepsilon}{b-a}.$$

Then for all

$$\forall \Pi : ||\Pi|| < \delta \Rightarrow \sum_{i=1}^{n} \operatorname{osc}(f, [t_{i-1}, t_i])(t_i - t_{i-1}) \leq \sum_{i=1}^{n} \frac{\varepsilon}{b-a}(t_i - t_{i-1}) \leq \varepsilon.$$

□

Lemma 11.3

Let $f\colon [a,b] \to \mathbb{R}$ be bounded and such that f has only finitely many discontinuities. Then f is RI.

Proof. Let $x_1, \ldots, x_m$ enumerate discontinuity points of f. Pick $\varepsilon > 0$. Suppose without loss of generality $||f|| \neq 0$. Let $\delta < \frac{\varepsilon}{m||f||}$. Then

$$\operatorname{osc}(f, [x_i - \delta, x_i + \delta] \cap [a,b]) \leq 2||f||.$$

Next, note that $[a,b] \setminus \bigcup_{i=1}^{m} (x_i - \delta, x_i + \delta)$ is closed and thus compact. Then f is uniformly continuous on this set. Then there exists $\delta' > 0$ such that for all $[s,t] \subseteq$ this set we have

$$0 < |s-t| \leq \delta' \Rightarrow \operatorname{osc}(f, [s,t]) \leq \frac{\varepsilon}{b-a}.$$

Now partition $[a,b] \setminus \bigcup_{i=1}^{n} (x_i - \delta, x_i + \delta)$ into intervals of length $\leq \delta'$. Combine them with intervals $[x_i - \delta, x_i + \delta]$. Now take Π = set of endpoints of these intervals. Then

$$\sum_{i=1}^{n} \operatorname{osc}(f, [t_{i-1}, t_i])(t_i - t_{i-1}) \leq m \cdot 2||f|| \cdot 2\delta + \frac{\varepsilon}{b-a}(b-a) \leq 5\varepsilon.$$

□

Lemma 11.4

Let $f\colon [a,b] \to \mathbb{R}$ be bounded.

$$f \text{ has no discontinuities of second kind} \Rightarrow f \text{ RI}.$$

Proof. Key idea:

$$\forall\, \eta > 0 : \left\{ x \in (a,b) : \operatorname{diam}\{ \lim_{z \to x^+} f(z), \lim_{z \to x^-} f(z), f(x) \} > \eta \right\} \text{ is finite.}$$

□

Example 11.5.

$$f(x) = \begin{cases} \frac{1}{n+1} & x = q_n \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

It gets worse: Let

$$C := \left\{ \sum_{i \in \mathbb{N}} \frac{2\sigma_i}{3^{i+1}} : \{\sigma_i\}_{i \in \mathbb{N}} \in \{0,1\}^{\mathbb{N}} \right\}$$

be Cantor's ternary set.

Then $C = \bigcap_{n \in \mathbb{N}} C_n$ where

$$C_n = \left\{ \sum_{i=1}^{n} \frac{\sigma_i}{3^{i+1}} + [0, 3^{-n-1}] : \sigma_1, \dots, \sigma_n \ \{0,1\} \right\}.$$

Lemma 11.6

The function

$$1_C(x) = \begin{cases} 1 & x \in C \\ 0 & x \notin C \end{cases}$$

is RI.

Proof. Let $I_1, \dots, I_{2^n}$ be intervals constituting C_n. Define

$$J_k = \left\{ x \in [0,1] : \operatorname{dist}(x, I_k) < \frac{1}{3^{n+1}} \right\}.$$

Then

$$\operatorname{length}(J_k) = \operatorname{length}(I_k) + 2 \cdot \frac{1}{3^{n+1}} \le \frac{1}{3^n}.$$

Take Π to be the endpoints of $\{J_k\}_{k=1}^{2^n}$. Then

$$\sum_{i=1}^{m} \operatorname{osc}(f, [t_{i-1}, t_i])\, |t_i - t_{i-1}| \le \sum_{k=1}^{2^n} \operatorname{length}(J_k) \le 2^n \cdot \frac{1}{3^n} \xrightarrow[n \to \infty]{} 0.$$

□

12 4.18 Monday Week 4

Characterizing Riemann integrability

Sufficient conditions for RI: continuity, finite number of discontinuities, no discontinuities of second kind.

Necessary condition for RI: boundedness.

Definition 12.1

A set $A \subseteq \mathbb{R}$ is of **zero length** if

$$\forall\, \varepsilon > 0\, \exists\, \{(a_i, b_i)\}_{i\in\mathbb{N}} \text{ intervals} : A \leq \bigcup_{i\in\mathbb{N}} (a_i, b_i) \quad \wedge \quad \sum_{i\in\mathbb{N}} (b_i - a_i) < \varepsilon.$$

Lemma 12.2

In the definition of zero length, closed intervals can be used.

Proof. If $A \subseteq \bigcup_{i\in\mathbb{N}}[a_i, b_i]$, let $\tilde{a}_i = a_i - \varepsilon/2^i$ and $\tilde{b_i} = b_i + \varepsilon/2^i$. Then

$$A \subseteq \bigcup_{i\in\mathbb{N}} (\tilde{a}_i, \tilde{b}_i)$$

and

$$\sum_{i\in\mathbb{N}} (\tilde{b_i} - \tilde{a}_i) = \sum_{i\in\mathbb{N}} (b_i - a_i) + \sum_{i\in\mathbb{N}} 2 \cdot \frac{\varepsilon}{2^i} = \sum_{i\in\mathbb{N}} (b_i - a_i) + 4\varepsilon.$$

□

Lemma 12.3

Let $f : \mathbb{R} \to \mathbb{R}$ be bounded. Set

$$M_f(x) = \inf_{\delta>0} \sup_{z:|z-x|<\delta} f(z)$$

and

$$m_f(x) = \sup_{\delta>0} \inf_{z:|z-x|<\delta} f(z).$$

Then

1. $\forall\, x \in \mathbb{R} : f$ continuous at $x \Leftrightarrow M_f(x) = m_f(x)$,
2. $\forall\, x \in \mathbb{R}\, \forall\, \delta : \max\left\{\operatorname{osc}(f, [x-\delta, x]), \operatorname{osc}(f, [x-\delta, x])\right\} \geq M_f(x) - m_f(x)$, and
3. $\forall\, x \in \mathbb{R} : \lim_{\delta\to 0} \operatorname{osc}(f, [x-\delta, x+\delta]) = M_f(x) - m_f(x)$.

Theorem 12.4: Lebesgue's characterization of Riemann integrability

Let $f : [a, b] \to \mathbb{R}$ be bounded. Then

$$f \text{ RI} \Leftrightarrow \left\{x \in [a, b] : f \text{ discontinuous at } x\right\} \text{ is zero length.}$$

Proof. $\Rightarrow$: Let $f : [a,b] \to \mathbb{R}$ be bounded and RI.

Pick $\varepsilon > 0$. Then RI implies

$$\forall\, n \in \mathbb{N}\, \exists\, \Pi = \left\{t_i^n\right\}_{i=1}^{m(n)} \text{ partition of } [a,b] : \sum_{i=1}^{m(n)} \operatorname{osc}(f,[t_{i-1}^n, t_i^n])(t_i^n - t_{i-1}^n) < \varepsilon 4^{-n}.$$

Set $I_n := \left\{i = 1, \dots, m(n) : \operatorname{osc}(f,[t_{i-1}^n, t_i^n]) > 2^{-n}\right\}$. Then

$$\sum_{i\in I_n}(t_i^n - t_{i-1}^n) \overset{\text{Markov's inequality}}{\leq} \sum_{i\in I_n} \frac{\operatorname{osc}(f,[t_{i-1}^n, t_i^n])}{2^{-n}}(t_i^n - t_{i-1}^n) \leq 2^n \sum_{i=1}^{m(n)} \operatorname{osc}(f,[t_{i-1}^n, t_i^n])(t_i^n - t_{i-1}^n) \leq 2^n \cdot 4^{-n} = \varepsilon 2^{-n}.$$

Now

$$\left\{x \in [a,b] : M_f(x) \neq m_f(x)\right\} \subseteq \bigcup_{n\geq 1}\bigcup_{i\in I_n}[t_{i-1}^n, t_i^n].$$

Then

$$\sum_{n\geq 1}\sum_{i\in I_n}(t_i^n - t_{i-1}^n) \leq \sum_{n\geq 1} \varepsilon 2^{-n} = \varepsilon.$$

Then f RI $\Rightarrow \left\{x \in [a,b] : M_f(x) \neq m_f(x)\right\}$ is zero length.

$\Leftarrow$: Let $\varepsilon > 0$ and let $\{J_i\}_{i\in\mathbb{N}}$ be open intervals such that

$$\left\{x \in [a,b] : M_f(x) \neq m_f(x)\right\} \subseteq \bigcup_{i\in\mathbb{N}} J_i \quad \wedge \quad \sum_{i\in\mathbb{N}} \operatorname{length}(J) < \frac{\varepsilon}{2\varepsilon||f||}(f \neq 0).$$

Since $M_f(x) = m_f(x) \Rightarrow x$ is continuous:

$$\forall\, x \in [a,b] : M_f(x) = m_f(x) \Rightarrow \exists\, \delta_x > 0 : \operatorname{osc}(f,(x - \delta x, x + \delta x)) < \frac{\varepsilon}{b-a}.$$

Then intervals $\{J_i\}_{i\in\mathbb{N}} \cup \left\{(x-\delta, x+\delta) : M_f(x) = m_f(x)\right\}$ cover $[a,b]$. Then by Heine-Borel theorem,

$$\exists\, m, n \in \mathbb{N}\, \exists\, x_0, \dots, x_m \in \left\{x \in [a,b] : M_f(x) = m_f(x))\right\} : [a,b] \subseteq \bigcup_{i=0} J_i \cup \bigcup_{j=0}^{m}(x_j - \delta_{x_j}, x_j + \delta_{x_j}).$$

Let $\Pi = \{t_i\}_{i=1}^N$ be a partition containing of all endpoints of the intervals $(x_j - \delta_{x_j}, x_j + \delta_{x_j})$. Let $k = \left\{i = 1, \dots, N : [t_{i-1}, t_i] \subseteq \bigcup_{j=1}^m (x_j - \delta_{x_j}, x_j + \delta_{x_j})\right\}$. Then

$$\forall\, i \in K : \operatorname{osc}(f,[t_{i-1}, t_i]) < \frac{\varepsilon}{b-a}$$

and

$$\sum_{i\notin K} \operatorname{osc}(f,[t_{i-1}, t_i]) \leq 2||f|| \cdot \sum_{i\notin K}(t_i - t_{i-1}) < 2||f|| \sum_{i\in\mathbb{N}} \operatorname{length}(J_i) < \varepsilon.$$

Then

$$\sum_{i=1}^{n} \operatorname{osc}(f,[t_{i-1}, t_i])(t_i - t_{i-1}) \leq \sum_{i\in K} \operatorname{osc}(f,[t_{i-1}, t_i])(t_i - t_{i-1}) + \sum_{i\notin K} \operatorname{osc}(f,[t_{i-1}, t_i])(t_i - t_{i-1}) \leq \frac{\varepsilon}{b-a}(b-a) + \varepsilon = 2\varepsilon.$$

$\square$

13 4.20 Wednesday Week 4

Derivative vs. integral, FTC, . . .

Last time: f RI $\Leftrightarrow \left\{x \in [a,b] : f \text{ discnotinuous at } x\right\}$ is of zero length.

Corollary 13.1

$$\text{f RI} \wedge \left\{x \in [a,b] : g(x) \neq f(x)\right\} \text{ is of zero length} \Rightarrow g \text{ RI} \wedge \int_a^b g(x)\,dx = \int_a^b f(x)\,dx.$$

Today: Newton / Leibniz FTC:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \quad \wedge \quad \int_a^b \frac{d}{dx} f(t)\,dt = f(b) - f(a).$$

Note. These are not true without conditions.

Lemma 13.2

Let $a < b$ be reals and $f\colon [a,b] \to \mathbb{R}$ be an RI function on $[a,b]$. Set $F(x) = \int_a^x f(t)\,dt$. Then F is Lipschitz continuous.

Proof. If $a \leq x < y \leq b$ then additivity implies

$$F(y) - F(x) = \int_0^y f(t)\,dt - \int_0^x f(t)\,dt = \int_x^y f(t)\,dt.$$

Note that f RI $\Rightarrow f$ bounded. Then

$$\left|F(y) - F(x)\right| = \left|\int_x^y f(t)\,dt\right| \leq ||f|| \cdot \left|y - x\right|.$$

□

Example 13.3.

$$|x| = \int_0^x t(1_{[0,\infty)} - 1_{(-\infty,0)})\,dt.$$

Q: Is every Lipschitz function a Riemann integral?

Lemma 13.4

Let f be RI on $[a,b]$. Set $F(x) = \int_a^b f(t)\,dt$. Then

$$\forall\, x \in (a,b) : f \text{ continuous at } x \Rightarrow F'(x) \text{ exists} \wedge F'(x) = f(x).$$

Proof. Let $y \in (x,b)$. Then

$$F(y) - F(x) - f(x)(y-x) = \int_x^y (f(t) - f(x))\,dt.$$

Then
$$\left|\frac{F(y)-F(x)}{y-x}-f(x)\right| \leq \sup_{t\in[x,y]}\left|f(y)-f(x)\right| \underset{y\to x^+}{\longrightarrow} 0 \text{ by right continuity of } f.$$

Then $F'^{+}(x)=f(x)$. Similarly, $F'^{-}(x)=f(x)$. □

Example 13.5.
$$f(x)=1_{1/(n+1),n\in\mathbb{N}}.$$

Note that $F(x)=0$ for all $x\in\mathbb{R}$.

Corollary 13.6: Fundamental theorem of calculus I

Let $f\colon[a,b]\to\mathbb{R}$ be continuous. Then
$$\forall\, x\in(a,b): \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Note.

- The integral is an antiderivative / primitive function. Notation $\int f(t)\,dt$;
- $\frac{d}{dt}\int_x^b f(t)\,dt=-f(x)$;
- $\frac{d}{dx}\int_{g(x)}^{h(x)}=f(h(x))h'(x)-f(g(x))g'(x)$.

Theorem 13.7

Let $f\colon[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). (Choose $f'(a),'(b))$ arbitrarily. Then
$$f \text{ RI} \Rightarrow \int_a^b f'(t)\,dt = f(b)-f(a).$$

Proof. Let $\varepsilon>0$. Then f' RI implies
$$\exists\,\delta>0\,\forall\,\Pi: ||\Pi||<\varepsilon \Rightarrow \left|R(f',\Pi)-\int_a^b f'(t)\,dt\right|<\varepsilon.$$

Pick $n\in\mathbb{N}$ such that $n\delta>(b-a)$. Set $t_i := a+\frac{i}{n}(b-a)$ where $i=0,\dots,n$.

Then, by the mean value theorem, for all $i=1,\dots,n$ we have
$$\exists\, t_i^*\in(t_{i-1},t_i): f(t_i)-f(t_{i-1})=f'(t_i^*)(t_i-t_{i-1}).$$

Let $\Pi := (\{t_i\},\{t_i^*\})$. Then
$$f(b)-f(a)=\sum_{i=1}^n(f(t_i)-f(t_{i-1}))=\sum_{i=1^n}f'(t_i^*)(t_i-t_{i-1})=R(f',\Pi).$$

Then
$$\left|f(b)-f(a)-\int_a^b f'(t)\,dt\right|<\varepsilon.$$

□

Volterra's example

$$\exists F\colon [0,1] \to \mathbb{R} \text{ continuous} : F'(x) \text{ exists for all } x \in [0,1] \wedge F' \text{ bounded} \wedge F' \text{ is not RI.}$$

This is a major deficiency in Riemann's theory that led Lebesgue to the formulation of the Lebesgue integral.

Corollary 13.8: Integration by parts

Let $f, g\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then

$$f'g \text{ RI} \wedge fg' \text{ RI} \Rightarrow \int_0^b f'(x)g(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\,dx.$$

Proof. Note that

$$f'g \text{ RI} \wedge g'f \text{ RI} \Rightarrow (fg)' \text{ RI}.$$

Then

$$f(b)g(b) - f(a)g(a) = \int_a^b (fg)'(x)\,dx = \int_a^b f'(x)g(x)\,dx + \int_a^b f(x)g'(x)\,dx.$$

□

Corollary 13.9

Let $f\colon [c,d] \to \mathbb{R}$ and $\varphi\colon [a,b] \to \mathbb{R}$ be functions. Assume

1. φ is continuous on $[a,b]$ and differentiable on (a,b),
2. f is continuous on $[c,d]$, and
3. $(f \circ \varphi)\varphi'$ is RI on $[a,b]$.

Then

$$\int_{\varphi(a)}^{\varphi(b)} f(x)\,dx = \int_a^b (f \circ \varphi)(t)\varphi'(t)\,dt.$$

Proof. Note that

$$F(x) = \int_a^x f(t)\,dt \overset{\text{FTC I}}{\Longrightarrow} F'(x) = f(x).$$

Then

$$\int_{\varphi(a)}^{\varphi(b)} f(t)\,dt \overset{\text{FTC II}}{=\!=\!=} F(\varphi(b)) - F(\varphi(a)) \overset{\text{FTC II}}{=\!=\!=} \int_a^b \frac{d}{dx}(F \circ \varphi)(x)\,dx = \int_a^b (f \circ \varphi)(x)\varphi'(x)\,dx.$$

□

14 4.25 Monday Week 5

Taylor's theorem

Last time: FTC I:

$$f \text{ continuous} \Rightarrow F(x) = \int_a^x f(t)\,dt \text{ differentiable} \wedge F'(x) = f(x).$$

FTC II:

$$F \text{ continuous on } [a,b] \wedge F' \text{ exists on } (a,b) \wedge F' \text{ RI} \Rightarrow F(b) - F(a) = \int_a^b F'(x)\,dx.$$

Cantor's function ("Devil's staircase"):

$$x \in \sum_{i=0}^{n} \frac{2\sigma_i}{3^{in}} + [0, 3^{-n-1}] \mapsto F(x) = \sum_{i=b}^{n} \frac{\sigma_i}{2^{i+1}}.$$

This is simply not an integral of a derivative (not Lipschitz but Holder continuous with a coefficient less than 1). F' exists at every point excluding the Cantor set, which is 0.

Consequences of the FTC:

- Substitution rule
- Integration by parts

Theorem 14.1: Taylor's theorem with remainder

Let $f\colon (a,b) \to \mathbb{R}$ be $(n+1)$-times differentiable with $f^{(n+1)}$ Riemann integrable. Then

$$\forall\, x, x_0 \in (a,b)\colon f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(z)(x-z)^n\,dz.$$

Proof. $n = 0$: FTC: f' exists and f' is RI by assumption.

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(z)\,dz.$$

$n \Rightarrow n+1$: Assume $f^{(n+2)}$ exists and is RI. Then $f^{(n+1)}$ is continuous and therefore RI. Then

$$\begin{aligned}
\frac{1}{n!}\int f^{(n+1)}(z)(x-z)^n\,dz &= \frac{1}{n!}\int f^{(n+1)}(z)\frac{d}{dz}\left(-\frac{(x-z)^{n+1}}{n+1}\right)dz \\
&\overset{IBP}{=\!=} \frac{1}{n!} f^{(n+1)}(z)\left(-\frac{(x-z)^{n+1}}{n+1}\right)\Bigg|_{x_0}^{x} - \int_{x_0}^{x} f^{(n+2)}(z)\left(-\frac{(x-z)^{n+1}}{n+1}\right)dz \\
&= \frac{f^{(n+1)}(x_0)}{(n+1)!}(x-x_0)^n + \int_{x_0}^{x}\frac{f^{(n+2)}(z)}{(n+1)!}(x-z)^{n+1}\,dz.
\end{aligned}$$

Then

$$f(x) - P_n(x) \overset{(n)}{=\!=} \text{LHS} = P_{n+1}(x) - P_n(x) + \int_{x_0}^{x} \frac{f^{(n+2)}(z)}{n+1}(x-z)^{n+1}.$$

□

Stieljes integral

Idea: Measure length of intervals using other functions than just $g(x) = x$.

Definition 14.2

Let $\Pi = (\{t_i\}_{i=0}^n, \{t_i^*\}_{i=1}^n)$ be a marked partition of $[a,b]$. For $f, g\colon [a,b] \to \mathbb{R}$,

$$S(f, dg, \Pi) := \sum_{i=1}^n f(t_i^*)[g(t_i) - g(t_{i-1})]$$

is the **Riemann-Stieljes sum** of f with respect to g.

Definition 14.3

Let $f, g\colon [a,b] \to \mathbb{R}$. We say that **"$f$ is Stieljes integrable with respect to g on $[a,b]$"** if

$$\exists\, L \in \mathbb{R}\, \forall\, \varepsilon > 0\, \exists\, \delta > 0\, \forall\, \Pi \text{ marked partition of } [a,b] : ||\Pi|| < \delta \Rightarrow \left|S(f, dg, \Pi) - L\right| < \varepsilon$$

or in short,

$$\lim_{||\Pi|| \to 0} S(f, dg, \Pi) \text{ exists.}$$

Note.

- Such an L is unique (if it exists) and so we denote it $\int_a^b f(x)\, dg(x) = \int_a^b f\, dg$.
- For $g(x) = x$, we get the Riemann integral.
- If "length" of $[t,s]$ is given by $g(s) - g(t)$, then $\int f\, dg$ corresponds to "area" with lengths in $\mathbb{R}$ measured using g.
- In probability: g = cumulative distribution function of a random variable x ($g(t) := P(x \leq t)$) then

  $$\int f(x)\, dg(x) = E(f(X)) = \text{expectation of } f(X).$$

- In economics: $f(t)$ = price of stock at time t, $g(t)$ = current holding of the stock then

  $$\int_a^b f\, dg = \text{total money earned in time interval } [a,b].$$

  This shows g may not be monotone.

15 4.27 Wednesday Week 5

Stieljes integral

Last time:

$$S(f, dg, \Pi) = \sum_{i=1}^{n} f(t_i^*)(g(t_i) - g(t_{i-1}))$$

$$\int_a^b f\, dg := \lim_{||\Pi|| \to 0} S(f, dg, \Pi) \text{ wherever it exists.}$$

We call this the Stieljes integral in the Riemann sense.

Notation: $RS(g, [a,b]) := \left\{ f : [a,b] \to \mathbb{R} : \int_a^b f\, dg \text{ exists} \right\}$

Lemma 15.1: Linearity

Let $h : [a,b] \to \mathbb{R}$ be given. Then

$$\forall f, g \in RS(h, [a,b])\, \forall \alpha, \beta \in \mathbb{R} : \alpha f + \beta g \in RS(h, [a,b]) \wedge \int_a^b (\alpha f + \beta g)\, dh = \alpha \int_a^b f\, dh + \beta \int_a^b g\, dh.$$

Lemma 15.2: Additivity

Let $g : [a,b] \to \mathbb{R}$ be given. Then

$$\forall f \in RS(g, [a,b])\, \forall c \in (a,b) : f \in RS(g, [a,b]) \wedge f \in RS(g, [c,b]) \wedge \int_a^b f\, dg = \int_a^c f\, dg + \int_c^b f\, dg.$$

Lemma 15.3

Let $f \in RS(g, [a,b])$. Then

$$\left\{ x \in [a,b] : f \text{ discontinuous at } x \right\} \cap \left\{ x \in [a,b] : g \text{ discontinuous at } x \right\} = \varnothing.$$

Note. $f \in RS(g, [a,b])$ need not be bounded on intervals where g is constant.

Definition 15.4

We say f is **generalized Stieljes integrable** with respect to g if

$$\exists L \in \mathbb{R}\, \forall \varepsilon > 0\, \exists \delta > 0\, \exists \Pi_\varepsilon \text{ unmarked partition}\, \forall \Pi \text{ marked partition :}$$
$$||\Pi|| < \delta \wedge \Pi_\varepsilon \subseteq \Pi \Rightarrow \left| S(f, dg, \Pi) - L \right| < \varepsilon.$$

Criteria for Stieljes integrability

Theorem 15.5: Reduction to Riemann integral

Let $f, g\colon [a,b] \to \mathbb{R}$ be such that

1. f is Riemann integrable and
2. g is continuous on $[a,b]$, differentiable on (a,b) with g' Riemann integrable.

Then

$$f \in RS(g,[a,b]) \quad \wedge \quad \int_a^b f\, dg = \int_a^b f(x)g'(x)\, dx.$$

Proof. Let $\Pi = (\{t_i\}_{i=0}^n, \{t_i^*\}_{i=1}^n)$ be a marked partition of $[a,b]$. For each $i = 1,\dots,n$, let $\tilde{t}_i$ be a point such that $g(t_i) - g(t_{i-1}) = g'(\tilde{t}_i)(t_i - t_{i-1})$ given by the mean value theorem. Let $\tilde{\Pi} = (\{t_i\}_{i=0}^n, \{\tilde{t}_i\}_{i=1}^n)$. Then

$$\begin{aligned} S(f,dg,\Pi) - R(fg',\tilde{\Pi}) &= \sum_{i=1}^n f(t_i^*)(g(t_i) - g(t_{i-1})) - \sum_{i=1}^n f(\tilde{t}_i)g(\tilde{t}_i)(t_i - t_{i-1}) \\ &= \sum_{i=1}^n [f(t_i^*) - f(\tilde{t}_i)]g'(\tilde{t}_i)(t_i - t_{i-1}). \end{aligned}$$

Note that

$$|\text{RHS}| \le ||g'|| \sum_{i=1}^n \operatorname{osc}(f,[t_{i-1},t_i])(t_i - t_{i-1}) \underset{||\Pi||\to 0}{\overset{fg' \text{ RI}}{\longrightarrow}} 0.$$

Hence

$$\lim_{||\Pi||\to 0} S(f,g,\Pi) = \lim_{||\Pi||\to 0} R(fg',\Pi) = \int_a^b fg'\, dx.$$

$\square$

Theorem 15.6: BV condition

Let $f, g \in [a,b] \to \mathbb{R}$ be such that

1. f is continuous and
2. g is of bounded variation ($V(g,[a,b]) < \infty$).

Then $f \in RS(g,[a,b])$ and

$$\left| \int_a^b f\, dg \right| \le ||f|| V(g,[a,b]).$$

Proof. Let $\Pi = \{t_i\}_{i=0}^n$, $\tilde{\Pi} = \{s_i\}_{j=0}^m$ be unmarked partitions of $[a,b]$. Assume $\Pi \subseteq \tilde{\Pi}$ and the set $J_i = \{j = 1, \dots, m : [s_{j-1}, s_j] \subseteq [t_{i-1}, t_i]\}$. Now choose any marked points $t_i^* \in [t_{i-1}, t_i]$ and $s_j^* \in [s_{j-1}, s_j]$. Then

$$\begin{aligned}
S(f, dg, \Pi) - S(f, dg, \tilde{\Pi}) &= \sum_{i=1}^n f(t_i^*)(g(t_i) - g(t_{i-1})) - \sum_{j=1}^m f(s_i^*)[g(s_j) - g(s_{j-1})] \\
&= \sum_{i=1}^n \sum_{j \in J_i} [f(t_i^*) - f(s_j^*)][g(s_j) - g(s_{j-1})] \\
&\leq \sum_{i=1}^n \sum_{j \in J_i} \operatorname{osc}(f, [t_{i-1}, t_i]) \left| g(s_j) - g(s_{j-1}) \right| \\
&\leq \sum_{i=1}^n \operatorname{osc}(f, [t_{i-1}, t_i]) V(g, [t_{i-1}, t_i]).
\end{aligned}$$

If f is continuous then f is uniformly continuous. Then

$$\forall\, \varepsilon > 0\, \exists\, \delta > 0 : ||\Pi|| < \delta \Rightarrow \operatorname{osc}(f, [t_{i-1}, t_i]) < \varepsilon.$$

Then $|\text{RHS}| \leq \varepsilon V(g, [a,b])$. Then for any marked partitions Π, Π' of $[a,b]$ we have

$$||\Pi||, ||\Pi'|| < \delta \Rightarrow \left| S(f, dg, \Pi) - S(f, dg, \Pi') \right| \leq 2\varepsilon V(g, [a,b]).$$

□

Theorem 15.7: Loéve-Young condition, 1936

Let $f, g \colon [a,b] \to \mathbb{R}$ be such that

$$\exists\, \alpha, \beta \in (a,b] : f \text{ is } \alpha\text{-Hölder} \wedge g \text{ is } \beta\text{-Hölder} \wedge \alpha + \beta > 1.$$

Then $f \in RS(g, [a,b])$.

Note. f is α-Hölder if

$$\exists\, C > 0\, \forall\, x, y \in [a,b] : \left| f(x) - f(y) \right| \leq C \left| x - y \right|^\alpha.$$

16 4.29 Friday Week 5

Wrapping up Stieljes integral

Remark.

- Stieljes integral includes sums:

$$F(x) = \sum_{i=1}^n 1_{[x_i, \infty)} \text{ where } a < x_1 < x_2 < \cdots < x_n \leq b.$$

  Then for g continuous:

$$\int_a^b g\, dF = \sum_{i=1}^n g(x_i).$$

We can combine these with the *continuous part*:

$$F(x) = \sum_{i=1}^{n} 1_{[x_i,\infty)}(x) + \int_a^x f(x)\,dt \Rightarrow \int_a^b g\,dF = \sum_{i=1}^{r} g(x_i) + \int_a^b g(t)f(t)\,dt.$$

- Standard facts apply:

Lemma 16.1: Integration by parts

If $f \in RS(g,[a,b])$ and $g \in RS(f,[a,b])$ then

$$\int_a^b f\,dg + \int_a^b g\,df = fg\Big|_a^b = f(b)g(b) - f(a)g(a).$$

Lemma 16.2: Substitution

If $g \in RS(h,[a,b])$ and $G(x) := \int_a^x g\,dh$ then

$$f \in RS(G,[a,b]) \Leftrightarrow fg \in RS(h,[a,b])$$

and if (both) true then

$$\int_a^b f\,dG = \int_a^b fg\,dh.$$

- The definition is unchanged if f and g are $\mathbb{C}$-valued. This allows us to define **curve integrals**

$$\int_\gamma f(x)\,dz := \int_0^1 f(\gamma(t))\,d\gamma(t)$$

where $\gamma\colon [0,1] \to \mathbb{C}$ continuous.

This is independent of the parametrization.

- We can even generalize this to one of f or g being vector-valued and the other being scalar-valued.
- The length of a curve $\gamma\colon [a,b] \to X$ where (X,ρ) is a metric space is given by

$$\text{length}(\gamma) = \sup_{n\geq 1} \sup_{0=t_0<\cdots<t_n=1} \sum_{i=1}^{n} \rho(\gamma(t_{i-1}),\gamma(t_i)).$$

A curve is **rectifiable** if the length is finite.

If $X = \mathbb{R}^n$ or some other normed space then

$$\rho(\gamma(t_{i-1}),\gamma(t_i)) = ||\gamma(t_i) - \gamma(t_{i-1})||.$$

This allows us to think of length(γ) as

$$\int_0^1 1\,d||\gamma||.$$

If γ is differentiable then

$$\gamma(t_i) - \gamma(t_{i-1}) \approx \gamma'(t_{i-1})(t_i - t_{i-1}).$$

Then

$$\text{length}(\gamma) = \int_0^1 ||\gamma'||(t)\, dt.$$

Extensions of Riemann-Stieljes theory

Lebesgue integral: The idea is that instead of partitioning the domain of a function, partition the range. This requires developing a theory of measure of rather complicated sets.

Note. f is Lebesgue integrable $\Rightarrow \left|f\right|$ is Lebesgue integrable.

This is because Lebesgue integral mimics Darboux's approach.

FTC II does not hold.

The fix is given by:

Definition 16.3

$f\colon [a,b] \to \mathbb{R}$ is said to be **Henstock-Kurzweil integrable** if

$$\exists\, L \in \mathbb{R}\, \forall\, \varepsilon > 0\, \exists\, \delta\colon [a,b] \to (0,\infty)\, \forall\, \Pi = (\{t_i\}_{i=0}^n, \{t_i^*\}_{i=1}^n) :$$
$$\forall\, i = 1,\ldots,n : |t_i - t_{i-1}| < \delta(t_i^*) \Rightarrow \left|R(f,\Pi) - L\right| < \varepsilon$$

where δ is called the **guage function**.

For bounded $f\colon [a,b] \to \mathbb{R}$,

$$f \text{ HK-integrable} \Leftrightarrow f \text{ measurable} \wedge f \text{ Lebesgue integrable.}$$

FTC holds: suppose $F\colon [a,b] \to \mathbb{R}$ is differentiable on (a,b). Then F' is HK-integrable and

$$F(b) - F(a) = \int_a^b F(x)\, dx.$$

However note that this is restricted to the real line since it uses a partition.

Uniform convergence

Q: Let $\{a_{n,m}\}_{n,m\in\mathbb{N}}$ be real such that

$$\forall\, m \in \mathbb{N} : b_m := \lim_{n\to\infty} a_{m,n} \text{ exists}$$

and

$$\forall\, n \in \mathbb{N} : c_n := \lim_{m\to\infty} a_{m,n} \text{ exists.}$$

When is $\lim_{n\to\infty} c_n = \lim_{m\to\infty} b_m$?

Lemma 16.4

Suppose
$$\forall\, m \in \mathbb{N}\, \exists\, b_m \in \mathbb{R} : \lim_{n\to\infty} \sup_{n\in\mathbb{N}} |a_{m,n} - b_m| = 0,$$
or that $\lim_{m\to\infty} a_{m,n}$ is **uniform** in n. Then
$$\forall\, n \in \mathbb{N} : c_n := \lim_{m\to\infty} a_{m,n} \text{ exists} \Rightarrow \lim_{m\to\infty} b_m \text{ and } \lim_{n\to\infty} c_n \text{ exist} \wedge \lim_{n\to\infty} c_n = \lim_{m\to\infty} b_m.$$
This means
$$\lim_{n\to\infty}\lim_{m\to\infty} a_{m,n} = \lim_{m\to\infty}\lim_{n\to\infty} a_{m,n}.$$

17 5.2 Monday Week 6

18 5.4 Wednesday Week 6

Uniform convergence

Last time: $f_n \to f$ uniformly on $A := \lim_{n\to\infty} \sup_{x\in A} \rho(f_n(x), f(x)) = 0$.

Definition 18.1

A sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ where $f_n : A \to X$ is **uniformly Cauchy** if
$$\lim_{N\to\infty} \sup_{n,m\geq N} \underbrace{\sup_{x\in A} \rho(f_n(x), f_m(x))}_{\substack{\text{metric on space of functions } A \to X,\\ \text{assuming supremum finite}}}.$$

Lemma 18.2

Let $f_n, f : A \to X$ where (X, ρ) is a metric space. Then

1. $f_n \to f$ uniformly $\Rightarrow \{f_n\}_{n\in\mathbb{N}}$ is uniformly Cauchy and
2. if (X, ρ) is complete then also
$$\{f_n\} \text{ uniformly Cauchy} \Rightarrow \exists\, f : A \to X : f_n \to f \text{ uniformly}.$$

Proof.

1. Note that
$$\rho(f_n(x), f_m(x)) \leq \rho(f_n(x), f(x)) + \rho(f_m(x), f(x)).$$
Then
$$\sup_{n,m\geq N} \sup_{x\in A} \rho(f_n(x), f_m(x)) \leq 2 \sup_{n\geq N} \sup_{x\in A} \rho(f_n(x), f(x)) \xrightarrow[n\to\infty]{} \lim_{n\to\infty} \sup \sup_{x\in A} \rho(f_n(x), f(x)) \xlongequal{f_n\to f \text{ uniformly}} 0.$$

2. Assume $\{f_n\}_{n\in\mathbb{N}}$ uniformly Cauchy. Then $\forall\, x \in X : \{f_n(x)\}_{n\in\mathbb{N}}$ is Cauchy in (X,ρ). Then

$$(X,\rho) \text{ complete} \Rightarrow f(x) := \lim_{n\to\infty} f_n(x) \text{ exists } \forall\, x \in X.$$

Then $f_n \to f$ pointwise.

Note that

$$\rho(f_n(x), f(x)) = \lim_{m\to\infty} \rho(f_n(x), f_m(x)) \le \sup_{m\ge n} \rho(f_n(x), f_m(x)).$$

Then

$$\sup_{x\in A} \rho(f_n(x), f(x)) \le \sup_{m\ge n} \sup_{x\in A} \rho(f_n(x), f_m(x)).$$

Then

$$\limsup_{n\to\infty} \sup_{x\in A} \rho(f_n(x), f(x)) \le \lim_{N\to\infty} \sup_{m,n\ge N} \sup_{x\in A} \rho(f_n(x), f_m(x)) \xlongequal{\{f_n\} \text{ uniformly Cauchy}} 0.$$

□

Theorem 18.3

Let $a < b$ be reals and $f_n : (a,b) \to \mathbb{R}$ where $n \in \mathbb{N}$ be differentiable functions. Assume

1. $\exists\, x_0 \in (a,b) : \lim_{n\to\infty} f_n(x_0)$ exists and
2. $\{f_n'\}_{n\in\mathbb{N}}$ is uniformly Cauchy.

Then there exists $f : (a,b) \to \mathbb{R}$ differentiable such that

$$f_n \to f \text{ uniformly} \quad \wedge \quad f_n' \to f' \text{ uniformly}.$$

Proof. For all $n \in \mathbb{N}$, let $\phi_n : (a,b) \times (a,b) \to \mathbb{R}$ be defined by

$$\phi_n(x,y) := \begin{cases} \frac{f_n(y)-f_n(x)}{y-x} & x \ne y \\ f_n'(x) & x = y. \end{cases}$$

Note that ϕ_n is continuous.

We then show that $\{\phi_n\}$ is uniformly Cauchy. Note that

$$\phi_n(x,y) - \phi_m(x,y) \xlongequal{x\ne y} \frac{(f_n - f_m(x))(y) - (f_n - f(m))(x)}{y-x} \xlongequal{\text{MVT}} (f_n' - f_m')(\xi).$$

Then

$$\sup_{x,y\in(a,b)} \left|\phi_n(x,y) - \phi_m(x,y)\right| \le \sup_{x\in(a,b)} \left|f_n'(x) - f_m'(x)\right|.$$

Since $\mathbb{R}$ is complete in $|\cdot|$-norm, Lemma 18.2 implies that there exists $\phi : (a,b)\times(a,b) \to \mathbb{R}$ such that $\phi_n \to \phi$ uniformly on $(a,b)\times(a,b)$.

Then

$$f_n(x) = f_n(x_0) + (x - x_0)\phi_n(x, x_0).$$

Then $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all $x \in (a,b)$ and obeys

$$f(x) = f(x_0) + (x - x_0)\phi(x, x_0).$$

The limit $f_n \to f$ is uniform because $\phi_n \to \phi$ is.

Finally,

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = \lim_{y \to x} \lim_{n \to \infty} \phi_n(x, y) \overset{\phi_n \to \phi \text{ uniformly}}{=\!=\!=\!=\!=} \lim_{n \to \infty} \lim_{y \to x} \phi_n(x, y) = \lim_{n \to \infty} f_n'(x).$$

Then $f'(x)$ exists and $f'(x) = \lim_{n \to \infty} f_n'(x) = \phi(x, x)$. Then, since $\phi_n \to \phi$ uniformly, $f_n' \to f'$ uniformly.

□

Applications

Lemma 18.4

Let $f(x) := \sum_{n=0}^{\infty} a_n (x - x_0)^n$ for $x \in (x_0 - R, x_0 + R)$ where $R := (\limsup_{n \to \infty} |a_n|^{1/n})^{-1}$ is the radius of convergence.

Then f is differentiable on $(x_0 - R, x_0 + R)$ and

$$\forall\, x \in (x_0 - R, x_0 + R) : f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}$$

where the series has radius of convergence R.

Proof. Note that

$$\limsup_{n \to \infty} |n a_n|^{1/(n-1)} = \limsup_{n \to \infty} |a_n|^{1/n}.$$

Then both series have the same radius of convergence. Hence

$$f_N(x) = \sum_{k=0}^{N} a_k (x - x_0)^k \quad \wedge \quad f_N'(x) = \sum_{k=1}^{N} k a_k (x - x_0)^{k-1}.$$

Then the family $\{f_n'\}$ is uniformly Cauchy on any closed subinterval of $(x_0 - R, x_0 + R)$.

Since $f_N(x_0) = a_0$, Theorem 18.3 tells us that

$$f_N(x) \to \sum_{k=0}^{\infty} a_k (x - x_0)^k =: f,$$

$$f_N'(x) \to \sum_{k=1}^{\infty} k a_k (x - x_0)^{k-1},$$

and

$$f' = \sum_{k=1}^{\infty} k a_k (x - x_0)^{k-1}.$$

□

19 5.6 Friday Week 6

Uniform convergence

Last time:

- The derivative commutes with uniform convergence (of derivatives).
- Power series are ∞-differentiable on interval of locally uniform convergence.

Lemma 19.1

The power series

$$e^x := \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$\sin x := \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$
$$\cos x := \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

have radius of convergence $R = \infty$ and so they converge on all of $\mathbb{R}$ and uniformly on compact subsets thereof.

Proof. Note that

$$n! \underset{\forall k<n}{\geq} k^{n-k}.$$

Then

$$\left(\frac{1}{n!}\right)^{1/n} \leq \left(\frac{1}{k}\right)^{(n-k)/n} \underset{k \leq n/2}{\leq} \left(\frac{1}{k}\right)^{1/2}.$$

Then

$$\limsup \left(\frac{1}{n!}\right)^{1/n} \underset{\forall k \geq 1}{\leq} \frac{1}{\sqrt{k}} \underset{k\to\infty}{\longrightarrow} 0.$$

Then

$$R = \left(\limsup_{n\to\infty} \left(\frac{1}{n!}\right)^{1/n}\right)^{-1} = \infty.$$

□

Lemma 19.2

- $\frac{d}{dx} e^x = e^x$,
- $\frac{d}{dx} \sin x = \cos x$, and
- $\frac{d}{dx} \cos x = -\sin x$.

Proof.

$$\begin{aligned} \frac{d}{dx}\sum_{n=0}^{\infty}\frac{x^n}{n!} &= \sum_{n=1}^{\infty}\frac{nx^{n-1}}{n!} \\ &= \sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!} = e^x. \end{aligned}$$

$$\begin{aligned} \frac{d}{dx}\cos(x) &= \sum_{n=1}^{\infty}(-1)^n\frac{2nx^{2n-1}}{(2n)!} \\ &= \sum_{n=1}^{\infty}(-1)^n\frac{x^{2n-1}}{(2n-1)!} \\ &\underset{m=n-1}{=} \sum_{m=0}^{\infty}(-1)^{m+1}\frac{x^{2m+1}}{(2m+1)!} \\ &= -\sin x. \end{aligned}$$

□

Why writing e^x?

Lemma 19.3

For all $x, y \in \mathbb{R}$ we have

$$e^{x+y} = e^x e^y.$$

Lemma 19.4

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and such that

$$\forall\, x, y \in \mathbb{R} : f(x+y) = f(x)f(y).$$

Then either

$$\forall\, x \in \mathbb{R} : f(x) = 0$$

or

$$\exists\, c \forall\, x \in \mathbb{R} : f(x) = e^{cx}.$$

Lemma 19.5

1. For all $x \in R$ we have
$$\sin^2 x + \cos^2 x = 1 \quad \wedge \sin x, \cos x \in [-1, 1].$$

2. For all $x, y \in \mathbb{R}$ we have
$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y.$$

3. Set $\pi := 2 \inf \{t \geq 0 : \cos(t) = 0\}$. Then for all $x \in \mathbb{R}$ we have
$$\sin x = -\cos\left(x + \frac{\pi}{2}\right) = -\sin(x + \pi)$$
and so
$$\sin(x + 2\pi) = \sin x \quad \wedge \quad \cos(x + 2\pi) = \cos(x).$$

Singular functions

Let
$$h(x) := \begin{cases} \frac{x^2}{1+x^2} \sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$
Note that
$$h'(x) = \begin{cases} -\frac{1}{1+x^2} \cos(1/x) - \frac{2x}{(1+x^2)^2} \sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$
Let $\{q_n\}_{n\in\mathbb{N}}$ enumerate $\mathbb{Q}$. Set
$$f(x) = \sum_{n=0}^{\infty} 2^{-n} h(x - q_n).$$
Then, since $||h||, ||h'|| < \infty$, we can differentiate term-by-term and so
$$f'(x) \sum_{n=0}^{\infty} 2^{-n} h'(x - q_n).$$
Then f' is discontinuous on $\mathbb{Q}$ and continuous on $\mathbb{R} \setminus \mathbb{Q}$.

The space $C(X)$, etc.

Definition 19.6

Let (X, ρ) be a metric space. Set
$$C(X) := \{\overbrace{f\colon X \to \mathbb{R}}^{f\in\mathbb{R}^X} : \text{continuous}\}$$
$$C_b(X) := \{f\colon X \to \mathbb{R} : \text{continuous} \wedge \text{bounded}\}$$
$$||f|| := \sup_{x\in X} |f(x)|.$$

Lemma 19.7

- $C(X)$ and $C_b(X)$ are linear vector spaces with respect to

$$(\lambda f)(x) = \lambda f(x) \quad \wedge \quad (f+g)(x) := f(x) + g(x).$$

- $||\cdot||$ is a norm on $C_b(X)$.

Lemma 19.8

$(C_b(X), ||\cdot||\text{-metric})$ is complete.

Proof. Note that $(\mathbb{R}, |\cdot|)$ is complete.
Also note that from last time we know that

$$\text{uniformly Cauchy on such spaces} \Rightarrow \text{uniformly convergent.}$$

$\square$