MATH 131BH (Real Analysis)

April 11, 2022

1 3.28 Monday Week 1: Intro to the course. Review of material covered in 131AH: foundations (definition and constructions of naturals and reals), metric space convergence, continuity.

2 3.30 Wednesday Week 1: Limit of a function: definition and alternative formulations via images of balls and sequential characterization. Limit on a set, left and right limits for functions on $\mathbb{R}$. Discontinuities of first and second kind. Monotone functions have no discontinuities of second kind.

Limits of functions

Recall: $f\colon X \to Y$ is said to be **continuous at $x \in X$** if

$$\forall\, \varepsilon > 0\, \exists\, \delta > 0\, \forall\, z \in x : \rho_X(x,z) < \delta \Rightarrow \rho(f(z), f(x)) < \varepsilon.$$

Alternatives:

- $f(B_X(x,\delta)) \subseteq B_Y(f(x),\varepsilon)$;
- $\forall\, \{x_n\}_{n\in\mathbb{N}} \in X^{\mathbb{N}} : x_n \to x \Rightarrow f(x_n) \to f(x)$.

A function $f\colon X \to Y$ is **continuous** if

$$\forall\, x \in X : f \text{ is continuous at } x,$$

or, alternatively,

$$\forall\, O \subseteq Y \text{ open} : f^{-1}(O) \text{ open}.$$

Definition 2.1

A function $f\colon X \to Y$ **has limit $y \in Y$ at $x \in X$**, notation $\lim_{z\to x} f(z) = y$, if

$$\forall\, \varepsilon > 0\, \exists\, \delta > 0\, \forall\, z \in X : 0 < \rho_X(x,z) < \delta \Rightarrow \rho_Y(f(z), y) < \varepsilon.$$

Alternatives:

- $f(B_X(x,\delta) \setminus \{x\}) \subseteq B_Y(y,\varepsilon)$;
- $\forall\, \{x_n\}_{n\in\mathbb{N}} \in X^{\mathbb{N}} : (\forall\, n \in \mathbb{N} : x_n \neq x) \wedge x_n \to x \Rightarrow f(x_n) \to y$;
- $g(z) := \begin{cases} f(z) & z \neq x \\ y & z = x \end{cases}$ is continuous at x.

Definition 2.2

f has a **removable discontinuity** at x if $\lim_{z\to x} f(z)$ exists but $\neq f(x)$.

Definition 2.3

Let $A \subseteq X$ be nonempty, $x \in \overline{A}$ be not an isolated point. Then $\lim_{z\to x} f(z) = \lim_{z\to x} f_A(z)$ where f_A is the restriction of f to A.

Definition 2.4

For $f : \mathbb{R} \to \mathbb{R}$, let $x \in \overline{\mathrm{Dom}(f)}$ be such that $\mathrm{Dom}(f) \cap (x, \infty) \neq \varnothing$ and $\mathrm{Dom}(f) \cap (-\infty, x) \neq \varnothing$. Then $\lim_{z\to x^+} f(z) := \lim_{z\to x, z\in \mathrm{Dom}(f)\cap(x,\infty)} f(z)$ and $\lim_{z\to x^-} f(z) := \lim_{z\to x, z\in \mathrm{Dom}(f)\cap(-\infty,x)} f(z)$ are the **right / left limits of f at x**.

Alternative notation: $f(x^+)$, $f(x^-)$.

Example 2.5.

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

has no right or left limits.

Example 2.6.

$$f(x) = \begin{cases} \frac{1}{n+1} & x = q_n \text{ where } \{q_n\}_{n\in\mathbb{N}} \text{ enumerates } \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

Then $\forall x \notin \mathbb{Q} : \lim_{z\to x} f(z) = 0$ so f is continuous on $\mathbb{R} \setminus \mathbb{Q}$, and $\forall x \in \mathbb{Q} : \lim_{z\to x} f(z) = 0$ but f is not continuous at x.

Lemma 2.7

$$\forall r > 0 \, \forall \varepsilon > 0 : \{x \in \mathbb{R} : |x| < r \wedge |f(x)| > \varepsilon\} \text{ finite} \Rightarrow \forall x \in \mathbb{R} : \lim_{z\to x} f(z) = 0.$$

Definition 2.8

A function $f : \mathbb{R} \to \mathbb{R}$ has a **discontinuity of**

- **first kind** at x if $f(x^+)$ and $f(x^-)$ exist but are not both equal to $f(x)$;
- **second kind** at x if one or both of $f(x^+)$ and $f(x^-)$ don't exist.

Example 2.9.

$$f(x) := \begin{cases} (-1)^n & x = \frac{1}{n+1}, n \in \mathbb{N} \\ \text{linear} & x \in (0,\infty) \setminus \left\{\frac{1}{n+1} : n \in \mathbb{N}\right\} \\ 0 & x \leq 0. \end{cases}$$

This function has a discontinuity of second kind at 0.

Lemma 2.10

Let $f : \mathbb{R} \to \mathbb{R}$ ($\mathrm{Dom}(f) = \mathbb{R}$) be monotone. Then $\forall\, x \in \mathbb{R} : f(x^+), f(x^-)$ exist and so f has no discontinuities of second kind.

Proof. Let $x \in \mathbb{R}$ and assume f is nondecreasing. We claim that $\lim_{z\to x^+} f(z) = \inf\left\{f(z) : z > x\right\} =: L$. Indeed, $\forall\, z > x : f(z) \geq f(x)$, so $L \geq f(x)$ and so $L \in \mathbb{R}$. Then $(\forall\, z > x : L \leq f(z)) \wedge (\forall\, \varepsilon > 0\, \exists\, z_\varepsilon > x : f(z_\varepsilon) < L + \varepsilon)$. Let $\delta := z_\varepsilon - x$. Then $\forall\, z \in (x, x+\delta) : f(z) \leq f(z_\varepsilon) < L + \varepsilon$. Then $\forall\, z \in (x, x+\delta) : L \leq f(z) < L + \varepsilon$ and therefore $\left|f(z) - L\right| < \varepsilon$. Then $\lim_{z\to x^+} f(z) = L$. □

3 3.31 Thursday Week 1: Monotone functions have only countably many discontinuities. Functions of bounded variation. Jordan decomposition theorem. Comments on uniqueness. Rectifiability of curves. Limsup and liminf of a function.

Limits of functions

Last time we showed that monotone functions have no discontinuities of second time.

Lemma 3.1

Let $f : \mathbb{R} \to \mathbb{R}$ be monotone. Then $\left\{x \in \mathbb{R} : f(x^+) \neq f(x^-)\right\}$ is countable.

Proof. Pick $k, m \in \mathbb{N}$ and let $A_{m,k} := \left\{x \in [-m, m] : \left|f(x^+) - f(x^-)\right| > \frac{1}{k+1}\right\}$. We claim that $A_{m,k}$ is finite. Let $x_0 < x_1 < \cdots < x_n$ be such that $\forall\, i \leq n : x_i \in A_{k,m}$. Assume (without loss of generality) that f is non-decreasing. Then

$$\begin{aligned} f(m+1) \geq f(x_n^+) &= f(x_0^+) + \sum_{i=1}^{n} \left(f(x_i^+) - f(x_{i-1}^+)\right) \\ &\geq f(m-1) + \sum_{i=1}^{n} \left(f(x_i^+) - f(x_i^-)\right) \\ &\geq f(-m+1) + \frac{n}{k+1}. \end{aligned} \tag{3.1}$$

Then $n \leq (k+1)$. Since $\left\{x \in \mathbb{R} : f(x^+) \neq f(x^-)\right\} = \bigcup_{k\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} A_{k,m}$, we are done. □

Q: Can these be generalized to other functions?

Definition 3.2

A **partition** Π of an interval $[a, b]$ is a sequence $\{t_i\}_{i=0}^n$ such that

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b.$$

Definition 3.3

Given $f\colon [a, b] \to \mathbb{R}$, its **total variation** on $[a, b]$

$$V(f, [a, b]) := \sup_{\Pi=\{t_i\}_{i=0}^n} \sum_{i=1}^{n} \left|f(t_i) - f(t_{i-1})\right|$$

where the supremum if over the partitions of $[a, b]$.

Definition 3.4

f is said to be of **bounded variation** on $[a, b]$ if $V(f, [a, b]) < \infty$.

Lemma 3.5

If $f\colon \mathbb{R} \to \mathbb{R}$ is of bounded variation on $[-m, m]$ for all $m \in \mathbb{N}$, then f has only discontinuities of first kind and the set $\left\{x \in \mathbb{R} : f(x^+) \neq f(x^-)\right\}$ is countable.

Theorem 3.6: Jordan decomposition (1881)

Let $f\colon [a, b] \to \mathbb{R}$ obey $V(f, [a, b]) < \infty$. Then $\exists\, h, g\colon [a, b] \to \mathbb{R}$ nondecreasing such that $\forall\, t \in [a, b]$: $f(t) = h(t) - g(t)$.

Proof. Define $h(t) := V(f, [a, t])$ and $g(t) := V(f, [a, t]) - f(t)$. Note that $h(t) - g(t) = f(t)$.

We need to show that h and g are nondecreasing.

Let $a \leq t < t' \leq b$. Then for any partition Π of $[a, t]$, $\Pi' = \Pi \cup \{t'\}$ is a partition of $[a, t']$. Then

$$V(f, [0, t']) \geq \sum_{i=1}^{m} \left|f(t_i) - f(t_{i-1})\right| + \left|f(t') - f(t)\right|.$$

Taking supremum over Π gives

$$V(f, [a, t']) \geq V(f, [a, t]) + \left|f(t') - f(t)\right|.$$

Note that $\left|f(t') - f(t)\right| \geq 0$ and $\left|f(t') - f(t)\right| \geq f(t') - f(t)$. Then $h(t') \geq h(t)$ and $g(t') \geq g(t)$. $\square$

The representation of $f = h - g$ is called a Jordan decomposition. This is not unique because a nondecreasing function can be added to both h and g.

However, there is a minimal decomposition $f = h_0 - g_0$ such that $g_0(a) = 0$ such that for any other Jordan decomposition $f = h - g$ we have $h - h_0$, $g - g_0$ nondecreasing. This is then *the* Jordan decomposition.

Rectifiability of curves

Definition 3.7

Let (X, ρ) be a metric space. A curve C is $\mathrm{Ran}(f)$ for an $f : \mathbb{R} \to X$ continuous such that $\mathrm{Dom}(f)$ is nonempty and connected. This f is called a **parametrization** of C.

Definition 3.8

Assuming $\mathrm{Dom}(f) = [a, b]$, the **length of C** is

$$\ell(C) := \sup_{\Pi = \{t_i\}_{i=0}^n} \sum_{i=1}^{n} \rho(f(t_{i-1}), f(t_i)).$$

Definition 3.9

A curve is **rectifiable** if $\ell(C) < \infty$.

Definition 3.10

Let (X, ρ) be a metric space and $f : X \to \mathbb{R}$. Then

$$\limsup_{z \to x} f(z) := \inf_{\delta > 0} \sup_{z \in B(x,\delta) \setminus \{x\}} f(z)$$

and

$$\liminf_{z \to x} f(z) := \sup_{\delta > 0} \inf_{z \in B(x,\delta) \setminus \{x\}} f(z).$$

Lemma 3.11

$$\lim_{z \to x} f(z) \text{ exists in } \mathbb{R} \Leftrightarrow \limsup_{z \to x} f(z) = \liminf_{z \to x} f(z) \in \mathbb{R}.$$

4 4.1 Friday Week 1: Discussion

Definition 4.1

Let (X, ρ_X), (Y, ρ_Y) be metric spaces, $E \subseteq X$, $f : E \to Y$, and $x \in \overline{E}$. Then $\lim_{t \to x} f(t) = \alpha$ is defined by

$$\forall\, \varepsilon > 0\, \exists\, \delta > 0 : \forall\, x \in E \wedge 0 < \rho_X(t, x) < \delta \Rightarrow \rho_Y(f(t), \alpha) < \varepsilon.$$

Equivalently,

$$\forall\, \{t_n\}_{n \in \mathbb{N}} \in (E \setminus \{x\})^{\mathbb{N}} : t_n \to x \Rightarrow f(t_n) \to \alpha.$$

Note. f need not be defined at x.

Remark.

$$\limsup_{t \to x} f(t) := \inf_{\delta > 0} \sup_{t \in B(x,\delta) \setminus \{x\}} f(t) = \lim_{\delta \to 0} \sup_{t \in B(x,\delta) \setminus \{x\}} f(t).$$

lim inf is similarly defined.

Remark.

$$\limsup = \liminf \Rightarrow \lim \text{ exists.}$$

Discontiuities

Definition 4.2

Let $f : (a, b) \to \mathbb{R}$ be not continuous at x. Then f has a **discontinuity of first kind** at x if $f(x+)$ and $f(x-)$ both exist. Otherwise it is of **second kind**.

Remark. Discontinuities of first kind are also known as **simple discontinuities**. The cases include

- $f(x+) = f(x-) \neq f(x)$: **removable discontinuity**, and
- $f(x+) \neq f(x-)$: **jump discontinuity**.

Example 4.3.

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

has a discontinuity of second kind at 0.

Example 4.4.

$$f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is continuous on $\mathbb{R} \setminus \mathbb{Q}$ and has discontinuities of first kind (removable) at every point in $\mathbb{Q}$.

Recall: A monotone function has no discontinuity of second kind and has at most countably many discontinuities of first kind. One can deduce this from the fact that the real line is a union of countably many open intervals (indexed by rationals).

Definition 4.5

A function $f : (a, b) \to \mathbb{R}$ is convex if

$$\forall\, x, y \in (a, b) : x \le y \Rightarrow (\forall\, \lambda \in [0, 1] : f(\lambda x + (1 - \lambda)y)) \le \lambda f(x) + (1 - \lambda) f(y).$$

In words, this means that for any interval, the secant line is above the graph.

5 4.4 Monday Week 2: Existence of limit is equivalent to equality and finiteness of limsup and liminf. Derivative of a real valued function of one real variable. Differentiability implies continuity. Connection with linear approximation. Sum and product rule, chain rule and inverse function rule. First-derivative test and discussion of important counterexamples.

Last time: $\lim_{z\to x} f(z)$, $\limsup_{z\to x} f(z) = \inf_{\delta>0} \sup_{z\in B(x,\delta)\setminus\{x\}x} f(z)$

Lemma 5.1

$$\lim_{z\to x} f(z) \text{ exists (in } \mathbb{R}) \Leftrightarrow \limsup_{z\to x} f(z) = \liminf_{z\to x} f(z) \in \mathbb{R}.$$

Proof. Both are equivalent:

$$\forall\, \varepsilon > 0\, \exists\, \delta > 0 : 0 \le \sup_{z\in B(x,\delta)\setminus\{x\}} f(z) - \inf_{z\in B(x,\delta)\setminus\{x\}} f(z) \le 2\varepsilon.$$

□

Definition 5.2

$$\lim_{z\to x} f(z) = \begin{cases} +\infty & \limsup_{z\to x} f(z) = \liminf_{z\to x} f(z) = +\infty \\ -\infty & \limsup_{z\to x} f(z) = \liminf_{z\to x} f(z) = -\infty. \end{cases}$$

Note. This characterization works even outside $\mathbb{R}$-valued functions:

$$\lim_{z\to x} f(z) \text{ exists} \Leftrightarrow \lim_{\delta\to 0^+} \sup \underbrace{\left\{\rho(f(z), f(u)) : z, u \in B(x, \delta) \setminus \{x\}\right\}}_{=\operatorname{diam}(f(B(x,\delta)\setminus\{x\}))} = 0.$$

The derivative

Definition 5.3

Let $f : \mathbb{R} \to \mathbb{R}$, $x \in \operatorname{int}(\operatorname{Dom}(f))$. We say that f has **derivative** or **is differentiable at x** if

$$f'(x) := \lim_{z\to x} \frac{f(z)-f(x)}{z-x} \text{ exists in } \mathbb{R}.$$

We call $f'(x)$ (Lagrange notation) the **derivative at x**, alternative notation $\frac{df}{dx}$ (Leibniz notation).

Lemma 5.4

$$f'(x) \text{ exists} \Rightarrow f \text{ continuous at } x.$$

Proof. The existence of $f'(x)$ implies that $\exists\, \delta_0 > 0\, \forall\, z \in \mathbb{R} : 0 < |z-x| < \delta_0 \Rightarrow \left|\frac{f(z)-f(x)}{z-x}\right| \le 1 + \left|f'(x)\right|$. Then, choosing $\varepsilon > 0$ and letting $\delta := \frac{\varepsilon}{1+\left|f'(x)\right|}$, we get

$$\forall\, z \in \mathbb{R} : 0 < |z-x| < \delta \Rightarrow \left|f(z)-f(x)\right| \le (1+\left|f'(x)\right|)\,|z-x| < (1+\left|f'(x)\right|)\frac{\epsilon}{1+\left|f'(x)\right|} = \varepsilon.$$

Since $f(z)-f(x) = 0$ for $z = x$, we are done (in fact, we have shown that f is lipschitz continuous). □

Another way to write existence of $f'(x)$:

$$f(z) - f(x) = (f'(x) + u_x(z))(z-x)$$

where $\lim_{z\to x} u_x(z) = 0$. (Just define: $u_x(z) := \frac{f(z)-f(x)}{z-x} - f'(x)$ for $z \neq x$)

Lemma 5.5: Linear approximation

$$f'(x) \text{ exists} \Leftrightarrow \exists\, L \in \mathbb{R} : \lim_{\delta\to 0^+} \sup_{|z-x|<\delta} \frac{1}{\delta}\left|f(z)-f(x)-L(z-x)\right| = 0.$$

Lemma 5.6: Sum & product rule

Let f, g be differentiable at x. Then so are $f+g$ and $f\cdot g$ and

$$\begin{aligned}(f+g)'(x) &= f'(x) + g'(x)\\ (f\cdot g)'(x) &= f'(x)g(x) + g'(x)f(x) \quad \text{(Leibniz rule).}\end{aligned}$$

Proof. For product rule, note that

$$f(z)g(z) - f(x)g(x) = (f(z)-f(x))g(z) + (g(z)-g(x))f(z).$$

Then

$$\frac{f(z)g(z)-f(x)g(x)}{z-x} = \frac{f(z)-f(x)}{z-x}g(z) + \frac{g(z)-g(x)}{z-x}f(z).$$

Since $g(z) \to g(x)$ by continuity of g, formula follows by sum & product rule for limit. □

Lemma 5.7: Chain rule

Let f be differentiable at x and g at $f(x)$. Then $g \circ f$ is differentiable at x and

$$(g \circ f)'(x) = g'(f(x))f'(x) \quad \left(\frac{dg}{df}\frac{df}{dx}\right)$$

Proof. Define $v_{f(x)}$ such that $g(y) - g(f(x)) = (g'(f(x))) + v_{f(x)}(y))(y - f(x))$ and u_x such that $f(z) - f(x) = (f'(x) + u_x(z))(z - x)$.

$$\begin{aligned}(g \circ f)(z) - (g \circ f)(x) &= [g'(f(x)) + v_{f(x)}(f(z))](f(z) - f(x)) \\ &= [g'(f(x)) + v_{f(x)}(f(z))][f'(x) + u_x(z)](z - x)\end{aligned}$$

Dividing by $z - x \neq 0$, note that $f(z) \to f(x)$ implies $v_{f(x)}(f(z)) \to 0$ as $z \to x$, we are done. □

Lemma 5.8

Let $f : \mathbb{R} \to \mathbb{R}$ be injective on Dom(f) and differentiable at $x \in$ int(Dom(f)). Assume $f'(x) \neq 0$ and $f(x) \in$ int(Ran(f)). Then f^{-1} is differentiable at $f(x)$ and

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}.$$

In Leibniz notation:

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

Lemma 5.9: First derivative test

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then if $x \in (a, b)$ is a local maximum of f (i.e. $\exists\, \delta > 0\, \forall\, z \in \mathbb{R} : |z - x| < \delta \Rightarrow f(x) \geq f(z)$) then $f'(x) = 0$.

Proof.

$$z > x \wedge |z - x| < \delta \Rightarrow \frac{f(z) - f(x)}{z - x} \leq 0 \Rightarrow f'(x) \leq 0$$

and

$$z < x \wedge |z - x| < \delta \Rightarrow \frac{f(z) - f(x)}{z - x} \geq 0 \Rightarrow f'(x) \geq 0.$$

□

6 4.6 Wednesday Week 2: Discussion

Recall: For, $x : [a, b] \to \mathbb{R}$, the total variation

$$V(f, [a, b]) = \sup_{\Pi} \sum_{i=1}^{n} \left| f(x_i) - f(x_{i-1}) \right|$$

where $\Pi = \{a = x_0 < x_1 < \cdots < x_n = b\}$. We say $f \in BV([a, b])$ if $V(f, [a, b]) < \infty$.

Theorem 6.1: Jordan decomposition

$$\forall f \in BV([a,b])\, \exists h, g\colon [a,b] \to \mathbb{R} \text{ nondecreasing} : f = h - g.$$

Corollary 6.2

$f \in BV([a,b])$ can only have discontinuities of first kind and countably many of them.

Example 6.3. $f(x) = \sin x \in BV([-1,1])$ since f is nondecreasing on $[-1,1]$ and hence $V(f,[a,b]) = f(b) - f(a)$.

Example 6.4. $f(x) = \sin x \in BV([-M,M])$ by additive property of V.

Q. Does $BV([a,b])$ imply bounded on $[a,b]$?

Yes. By triangle inequality,

$$\left|f(x)\right| \le \left|f(a)\right| + \left|f(a) - f(x)\right| \le \left|f(a)\right| + V(f,[a,b]) < \infty.$$

Q. Does being bounded on $[a,b]$ imply $BV([a,b])$.

No. A counterexample is

$$f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

on $[0,1]$.

Choose $x_n = 1/(n\pi/2)$ such that $\sin(1/x_n) = \sin(n\pi/2)$. Then $\sum_{i=1}^{2n} \left|f(x_i) - f(x_{i-1})\right| = \sum_{k=1}^{n} \left|f(x_{2k+1})\right| = n \to \infty$.

Example 6.5. Is

$$f(x) = \begin{cases} x\sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

on $[0,1]$ of bounded variation?

No. Choose the same x_n as above. Note that $f(x_n) = \frac{2}{n\pi}\sin(n\pi/2)$. Then $\sum_{i=1}^{2n} \left|f(x_i) - f(x_{i-1})\right| = \sum_{k=1}^{n} \frac{2}{(2k-1)\pi} \to \infty$.

Example 6.6. Is

$$f(x) = \begin{cases} x^2\sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

on $[0,1]$ of bounded variation?

Yes. Note that

$$f'(0) = \lim_{t\to 0} \frac{t^2 \sin\frac{1}{t} - 0}{t} = \lim_{t \to 0} t \sin \frac{1}{t} = 0.$$

Note that for $x \neq 0$,

$$f'(x) = 2x\sin\frac{1}{x} + x^2\left(-\frac{1}{x^2}\right)\cos\frac{1}{x} = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$$

is bounded: $\left|f'(x)\right| \le 2\left|x\right| + 1 \le 3$.

Note that by mean value theorem, we have

$$\sum \left|f(x_i) - f(x_{i-1})\right| \leq \sum \left|f'(\xi)\right|(x_i - x_{i-1}) \leq M(b-a) < \infty$$

where $\left|f'(\xi)\right| \leq M$.

Then f is of bounded variation on $[0,1]$.

Theorem 6.7

If f' exists and is bounded on $[a,b]$ then f is of bounded variation.

Q. Does the existence f' on $[a,b]$ and f being of bounded variation on $[a,b]$ imply f' is bounded on $[a,b]$?

7 4.7 Thursay Week 2: Mean-Value Theorems of Rolle, Lagrange and Cauchy. Applications: Monotone differentiable functions have derivative of one sign. Derivative of a differentiable function has no discontinuities of first kind (but those of second kind can occur densely). L'Hospital's Rule and its proof from Cauchy's MVT.

Mean value theorems

Last time: $f'(x)$ = derivative is linked to the local maxima and minima (first derivative test).

Theorem 7.1: Mean value theorem

Let $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then

1. (Rolle's theorem, 1691) $f(a) = f(b) \Rightarrow \exists\, x \in (a,b) : f'(x) = 0$,
2. (Lagrange's mean value theorem) $\exists\, x \in (a,b) : f'(x) = \frac{f(b)-f(a)}{b-a}$, and
3. (Cauchy mean value theorem, 1823) if also $g\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), then

$$\forall\, x \in (a,b) : g'(x) \neq 0 \Rightarrow g(a) \neq g(b) \wedge \exists\, x \in (a,b) : \frac{f'(x)}{g'(x)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Proof.

1. $f(a) = f(b) \wedge$ continuous function on $[a,b]$ achieves one of maximum and minimum on (a,b) $\Rightarrow \exists\, x \in (a,b) : x$ is local maximum or local minimum of f. Then $f'(x) = 0$.
2. Let $h(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$. Then $h(a) = f(a)$, $h(b) = f(b) - \frac{f(b)-f(a)}{b-a}(b-a) = f(a)$. Then, by 1., $\exists\, x \in (a,b) : h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a} = 0$.
3. Let $h(x) = f(x) - \frac{f(b)-f(a)}{g(b)-g(a)}(g(x)-g(a))$. Note that this is well defined since by 1. we have $g(b) \neq g(x)$. Then $h(a) = f(a) = h(b)$ so by 1. we have $\exists\, x \in (a,b) : h'(x) = f'(x) - \frac{f(b)-f(a)}{g(b)-g(a)}g'(x) = 0$.

$\square$

Applications

Lemma 7.2

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then

$$\forall\, x \in (a, b) : f'(x) \geq 0 \Leftrightarrow \forall\, x, y \in [a, b] : x \leq y \Rightarrow f(x) \leq f(y).$$

Proof. The $\Leftarrow$ direction is immediate from the definition of limit $\left(\frac{f(y)-f(x)}{y-x} \geq 0\right)$.

For the $\Rightarrow$ direction, if $\exists\, x \geq y : f(y) < f(x)$ then by the mean value theorem $\exists\, z \in (x, y) : f'(z) = \frac{f(y)-f(x)}{y-x} < 0$. $\square$

8 4.8 Friday Week 2: Taylor's theorem via Mean Value Theorem (Rolle suffices). Riemann integral: motivation, definitions of marked partition, mesh of partition and Riemann sum. Notion of a function being Riemann integrable. Linearity of integral.

Taylor's theorem

Definition 8.1: Higher order derivatites

Define $f^{(0)} := f$ and for all $n \in \mathbb{N}$ define $f^{(n+1)}(x) := \left(f^{(n)}\right)'(x)$ assuming the derivatives exist. We call $f^{(n)}$ the ***n*-th derivative of *f***.

Theorem 8.2: Taylor's theorem (Taylor 1715, Gregory 1671)

Let $n \in \mathbb{N}$ and $f : (a, b) \to \mathbb{R}$ an $(n + 1)$-times differentiable function. Then

$$\forall\, x_0 \in (a, b)\, \forall\, x \in (x_0, b)\, \exists\, \xi \in (x_0, x) : f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k}_{n\text{-th order Taylor polynomial at } x_0} + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}.$$

Proof. Based on MVT.

Denote

$$P_n(z) := \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(z - x_0)^k.$$

Pick $x \in (x_0, b)$ and denote

$$A := \frac{f(x) - P_n(x)}{(x - x_0)^{n+1}}.$$

Set

$$h(z) := f(z) - P_n(z) - A(z - x_0)^{n+1}.$$

Note that

$$\forall\, k \in \mathbb{N} : k \leq n \Rightarrow f^{(k)}(x_0) = 0.$$

We claim that

$$\forall\, k \in \mathbb{N} : 1 \leq k \leq n+1 \Rightarrow \exists\, \xi_k \in (x_0, x) : h^{(k)}(\xi_k) = 0.$$

For $k = 1$, the choice of A implies $h(x) = 0$ so since $h(x_0) = 0$, by Rolle's theorem

$$\exists\, \xi_1 \in (x_0, x) : h'(\xi) = 0.$$

Assume true for some $k \in \mathbb{N}$ such that $1 \leq k \leq n$. Then $h^{(k)}(x_0) = 0$ and $h^{(k)}(\xi_k) = 0$ for $\xi_k \in (x_0, x)$. Then by Rolle's theorem

$$\exists\, \xi_{k+1} \in (x_0, \xi_k) : h^{(n+1)}(\xi_{k+1}) = 0.$$

Now observe that $P_n^{(n+1)} = 0$. Then $0 = h^{(n+1)}(\xi_{n+1}) = f^{(n+1)}(\xi_{n+1}) - A(n+1)!$. Then

$$f(x) - P_n(x) = A(x - x_0)^{n+1} = \frac{f^{(n+1)}(\xi_{n+1})}{(n+1)!}(x - x_0)^{n+1}.$$

□

Riemann integral (Riemann 1854)

Goal: Given $f\colon [a,b] \to \mathbb{R}$, assign meaning to the area under the graph of f on $[a,b]$; namely to the set

$$\left\{(x,y) \in \mathbb{R}^2 : x \in [a,b] \wedge 0 \leq y \leq f(x)\right\} \quad (\text{for } f \geq 0).$$

Idea: Approximate f with a piecewise constant function and use that the area of a rectangle is "known."

Definition 8.3

Given $[a,b]$, a **marked partition Π** of $[a,b]$ is two sequences $\{t_i\}_{i=0}^n$, $\left\{t_i^*\right\}_{i=1}^n$ such that

- $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ and
- $\forall\, i = 1, \ldots, n : t_i^* \in [t_{i-1}, t_i]$.

Definition 8.4

The **mesh of Π** is defined by $||\Pi|| := \max_{i=1,\ldots,n} |t_i - t_{i-1}|$.

Definition 8.5

Given $f\colon [a,b] \to \mathbb{R}$ and a marked partition Π, the associated **Riemann sum** is

$$R(f,\Pi) := \sum_{i=1}^{n} f(t_i^*)(t_i - t_{i-1}).$$

Definition 8.6

A function $f\colon [a,b] \to \mathbb{R}$ is said to be **Riemann integrable** (on $[a,b]$) if there exists $L \in \mathbb{R}$ such that

$$\forall\, \varepsilon > 0\, \exists\, \delta > 0\, \forall\, \Pi = \text{marked partition of } [a,b] : ||\Pi|| < \delta \Rightarrow \left|R(f,\Pi) - L\right| < \varepsilon.$$

We sometimes write this as $\lim_{||\Pi|| \to 0} R(f,\Pi) = L$ (this L is unique). Notation for L is $\int_a^b f(x)\, dx$.

Lemma 8.7: Additivity and homogeneity of Reimann integral

Let $f, g\colon [a,b] \to \mathbb{R}$ be Riemann integrable on $[a,b]$. Let $\alpha, \beta \in \mathbb{R}$. Then $\alpha f + \beta g$ is Riemann integrable on $[a,b]$ and

$$\int_a^b (\alpha f(x) + \beta g(x))\, dx = \alpha \int_a^b f(x)\, dx + \beta \int_a^b g(x)\, dx.$$

Proof. Given $\varepsilon > 0$, pick $\delta > 0$ such that $||\Pi|| < \delta$ implies

$$\left|R(f,\Pi) - \int_a^b f(x)\, dx\right| < \varepsilon \wedge \left|R(g,\Pi) - \int_a^b g(x)\, dx\right|.$$

Since $R(\alpha f + \beta g, \Pi) = \alpha R(f,\Pi) + \beta R(g,\Pi)$,

$$\begin{aligned}
&\left|R(\alpha f + \beta g, \Pi) - \alpha \int_a^b f(x)\, dx - \beta \int_a^b g(x)\, dx \leq |\alpha|\right| \\
&\leq \alpha \left|R(f,\Pi) - \int_a^b f(x)\, dx\right| + \left|\beta\right| \left|R(g,\Pi) - \int_a^b g(x)\, dx\right| \\
&\leq (|\alpha| + \left|\beta\right|)\varepsilon.
\end{aligned}$$

□

Corollary 8.8

Let $f, g : [0,\infty) \to \mathbb{R}$ be continuous on $[0,\infty)$ and differentiable on $(0,\infty)$. Then

$$f(0) \leq g(0) \wedge \forall\, x \in (0,\infty) : f'(x) \leq g'(x) \Rightarrow \forall\, x \in [0,\infty] : f(x) \leq g(x).$$

Example 8.9. $\forall\, x \geq 0 : e^x \geq 1 + x$.

Lemma 8.10

Let $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then f' has the intermediate value property.

Proof. Without loss of generality assume f' exists on $[\tilde{a}, \tilde{b}]$ such that $\tilde{a} < a < b < \tilde{b}$. Without loss of generality assume $f'(a) < f'(b)$. Let $t \in (f'(a), f'(b))$. Let $h(x) := f(x) - tx$. Then

$$h'(a) < 0 \Rightarrow \exists\, x \in (a, b) : h(x) < h(a).$$

With the same reasoning, we have

$$h'(b) > 0 \Rightarrow \exists\, y \in (a, b) : h(y) < h(b).$$

Then

$$\exists\, z \in (a, b) \text{ local minimum} \Rightarrow h'(z) = f(z) - t = 0.$$

□

Corollary 8.11

The derivative of a differentiable function does not have discontinuities of first kind.

Example 8.12. Let

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Then $\forall\, x \neq 0 : f'(x) = x \sin(1/x) - \cos(1/x)$. $\lim_{x \to 0^\pm} f'(x)$ does not exist.

Also note that

$$\frac{f(x) - f(0)}{x - 0} = x \sin(1/x) \underset{x \to 0}{\longrightarrow} 0$$

so $f'(0) = 0$.

Theorem 8.13: L'Hopital's rule, proved by Bernoulli 1694

Let $f, g \colon \mathbb{R} \to \mathbb{R}$ be continuous and differentiable on $(a - \delta, a + \delta)$ where $a \in \mathbb{R}$ and $\delta > 0$. Assume

$$f(a) = 0 = g(a) \wedge \forall\, x \in (a - \delta, a + \delta) \setminus \{a\} : g(x) \neq 0 \wedge g'(x) \neq 0.$$

Then

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} \text{ exists} \Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} \text{ exists} \wedge \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Proof. Let $x \in (a - \delta, a + \delta) \setminus \{a\}$. Then for $x > a$ we have

$$\frac{f(x)}{g(x)} \underset{f(a)=0, g(a)=0}{=\!=\!=\!=} \frac{f(x) - f(a)}{g(x) - g(a)} \underset{\text{Cauchy MVT}}{\overset{\exists\, z_x \in (a,x)}{=\!=\!=\!=}} \frac{f'(z_x)}{g'(z_x)}.$$

Since $x \to a$ implies $z_x \to a$, existence of $\lim_{z \to a} \frac{f'(z)}{g'(z)}$ gives

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{z \to a} \frac{f'(z)}{g'(z)}.$$

□

Example 8.14. $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{cos x}{1} = 1$.

9 4.11 Monday Week 3

Last time: $f : [a,b] \to \mathbb{R}$ is Riemann integrable (RI) if

$$\exists\, L \in \mathbb{R}\, \forall\, \varepsilon > 0\, \exists\, \delta > 0\, \forall\, \Pi = \text{marked partition of } [a,b] : ||\Pi|| < \delta \Rightarrow \left|R(f,\Pi) - L\right| < \varepsilon.$$

Notation: $L = \int_a^b f(x)\,dx$.

We proved **linearity**:

$$\int_a^b (\alpha f(x) + \beta f(x))\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx.$$

Lemma 9.1

If f is RI on $[a,b]$ then f is bounded on $[a,b]$.

Proof. RI $\Rightarrow \exists\,\delta > 0\, \forall\, \Pi =$ marked partition : $R(f,\Pi) \le L + 1$. Then $\forall\, i = 1, \ldots, n\, \forall\, \tilde{t}_i : f(\tilde{t}_i)(t_i - t_{i-1}) + \sum_{j=1,\ldots,n, j\neq i} f(t_j^*)(t_j - t_{j-1}) \le L + 1$, which means $\sup_{\tilde{t}_i \in [t_{i-1}, t_i]} f(\tilde{t}_i) < \infty$. Then $\sup_{x\in[a,b]} f(x) < \infty$. $\square$

Lemma 9.2: Additivity

Let $a < c < b$ be reals. If f is RI on $[a,c]$ and on $[c,b]$, then it is RI on $[a,b]$ and

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Proof. Let $\varepsilon > 0$ and let $\delta > 0$ be such that $\forall\,\Pi =$ marked partition of $[a,c]$ and $\forall\,\Pi' =$ marked partition of $[c,b]$ such that $||\Pi|| < \delta \wedge ||\Pi'|| < \delta$ we have

$$\left|R(f,\Pi) - \int_a^c f(x)\,dx\right| < \varepsilon \quad \wedge \quad \left|R(f,\Pi') - \int_c^b f(x)\,dx\right| < \varepsilon.$$

If $\tilde{\Pi}$ is a marked partition of $[a,b]$ with $||\tilde{\Pi}|| < \delta$ containing c then

$$\left|R(f,\Pi) - \int_a^c f(x)\,dx - \int_c^b f(x)\,dx\right| < 2\varepsilon.$$

Suppose $\tilde{\Pi}$ does not contain c. Then adding c to $\tilde{\Pi}$ changes $R(f,\tilde{\Pi})$ by at most $2 \cdot 3\delta \sup_{x\in[a,b]} \left|f(x)\right|$. $\square$

Lemma 9.3

If f is RI on $[a,b]$ then

$$\left|\int_a^b f(x)\,dx\right| \le (b-a) \underbrace{\sup_{x\in[a,b]} \left|f(x)\right|}_{<\infty}.$$

Proof. Note that

$$\left|R(f,\Pi)\right| = \left|\sum_{i=1}^{n} f(t_i^*)(t_i - t_{i-1})\right| \leq \sum_{i=1}^{n} \left|f(t_i^*)(t_i - t_{i-1})\right| = R(\left|f\right|,\Pi) \leq \sup_{x\in[a,b]} \left|f(x)\right| \underbrace{\sum_{i=1}^{n}(t_i - t_{i-1})}_{=b-a}$$

□

Note. If we knew that $\left|f\right|$ is RI, then this gives

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b \left|f(x)\right|\,dx.$$

Q: Sufficient conditions for RI?

A: We will answer this using Darboux's version of Riemann integral.

Definition 9.4

Let $f\colon [a,b] \to \mathbb{R}$ be bounded. Given an unmarked partition $\Pi = \{t_i\}_{i=1}^{n}$ of $[a,b]$, set

$$U(f,\Pi) := \sum_{i=1}^{n} \sup\left\{f(x) : x \in [t_{i-1}, t_i]\right\}(t_i - t_{i-1})$$

and

$$L(f,\Pi) := \sum_{i=1}^{n} \inf\left\{f(x) : x \in [t_{i-1}, t_i]\right\}(t_i - t_{i-1})$$

to be the **upper and lower Darboux sums**.

Note. $L(f,\Pi) \leq R(f,\Pi) \leq U(f,\Pi)$ for any marked partition Π.

Lemma 9.5

For all unmarked partitions Π and Π' of $[a,b]$ we have

$$L(f,\Pi) \leq U(f,\Pi').$$

Proof. Assume first Π is a subset of Π', meaning that all points of Π are included in Π'. We claim that $U(f,\Pi') \leq U(f,\Pi)$ and $L(f,\Pi') \geq L(f,\Pi)$.

Note that if $\Pi' = \Pi \cup \{t\}$, let $[t_{i-1}, t_i]$ be the interval containing t. Then

$$\max\left\{\sup_{x\in[t_{i-1},t]} f(x), \sup_{x\in[t,t_i]} f(x)\right\} \sup_{x\in[t_{i-1},t_i]} f(x),$$

resulting in $U(f,\Pi') \leq U(f,\Pi)$.

Now let Π and Π' be arbitrary and $\Pi \cup \Pi'$ be their common refinement. Then

$$L(f,\Pi) \leq L(f,\Pi\cup\Pi') \leq U(f,\Pi\cup\Pi') \leq U(f,\Pi').$$

□