MATH 131BH (Real Analysis)

April 18, 2022

1 3.28 Monday Week 1: Intro to the course. Review of material covered in 131AH: foundations (definition and constructions of naturals and reals), metric space convergence, continuity.

2 3.30 Wednesday Week 1: Limit of a function: definition and alternative formulations via images of balls and sequential characterization. Limit on a set, left and right limits for functions on $\mathbb{R}$. Discontinuities of first and second kind. Monotone functions have no discontinuities of second kind.

Limits of functions

Recall: $f : X \to Y$ is said to be **continuous at** $x \in X$ if

$$\forall \varepsilon > 0 \exists \delta > 0 \forall z \in x : \rho_X(x, z) < \delta \Rightarrow \rho(f(z), f(x)) < \varepsilon.$$

Alternatives:

- $f(B_X(x, \delta)) \subseteq B_Y(f(x), \varepsilon)$;

- $\forall \{x_n\}_{n \in \mathbb{N}} \in X^{\mathbb{N}} : x_n \to x \Rightarrow f(x_n) \to f(x)$.

A function $f : X \to Y$ is **continuous** if

$$\forall x \in X : f \text{ is continuous at } x,$$

or, alternatively,

$$\forall O \subseteq Y \text{ open} : f^{-1}(O) \text{ open}.$$

Definition 2.1

A function $f : X \to Y$ **has limit** $y \in Y$ **at** $x \in X$, notation $\lim_{z \to x} f(z) = y$, if

$$\forall \varepsilon > 0 \exists \delta > 0 \forall z \in X : 0 < \rho_X(x, z) < \delta \Rightarrow \rho_Y(f(z), y) < \varepsilon.$$

Alternatives:

- $f(B_X(x, \delta) \setminus \{x\}) \subseteq B_Y(y, \varepsilon)$;

- $\forall \{x_n\}_{n \in \mathbb{N}} \in X^{\mathbb{N}} : (\forall n \in \mathbb{N} : x_n \neq x) \wedge x_n \to x \Rightarrow f(x_n) \to y$;

- $g(z) := \begin{cases} f(z) & z \neq x \\ y & z = x \end{cases}$ is continuous at x.

Definition 2.2

f has a **removable discontinuity** at x if $\lim_{z \to x} f(z)$ exists but $\neq f(x)$.

Definition 2.3

Let $A \subseteq X$ be nonempty, $x \in \overline{A}$ be not an isolated point. Then $\lim_{z \to x} f(z) = \lim_{z \to x} f_A(z)$ where f_A is the restriction of f to A.

Definition 2.4

For $f : \mathbb{R} \to \mathbb{R}$, let $x \in \overline{\text{Dom}(f)}$ be such that $\text{Dom}(f) \cap (x, \infty) \neq \varnothing$ and $\text{Dom}(f) \cap (-\infty, x) \neq \varnothing$. Then $\lim_{z \to x^+} f(z) := \lim_{z \to x, z \in \text{Dom}(f) \cap (x, \infty)} f(z)$ and $\lim_{z \to x^-} f(z) := \lim_{z \to x, z \in \text{Dom}(f) \cap (-\infty, x)} f(z)$ are the **right / left limits of f at x**.

Alternative notation: $f(x^+)$, $f(x^-)$.

Example 2.5.

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

has no right or left limits.

Example 2.6.

$$f(x) = \begin{cases} \frac{1}{n+1} & x = q_n \text{ where } \{q_n\}_{n \in \mathbb{N}} \text{ enumerates } \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

Then $\forall x \notin \mathbb{Q} : \lim_{z \to x} f(z) = 0$ so f is continuous on $\mathbb{R} \setminus \mathbb{Q}$, and $\forall x \in \mathbb{Q} : \lim_{z \to x} f(z) = 0$ but f is not continuous at x.

Lemma 2.7

$$\forall r > 0 \, \forall \varepsilon > 0 : \left\{ x \in \mathbb{R} : |x| < r \wedge |f(x)| > \varepsilon \right\} \text{ finite} \implies \forall x \in \mathbb{R} : \lim_{z \to x} f(z) = 0.$$

Definition 2.8

A function $f : \mathbb{R} \to \mathbb{R}$ has a **discontinuity of**

- **first kind** at x if $f(x^+)$ and $f(x^-)$ exist but are not both equal to $f(x)$;

- **second kind** at x if one or both of $f(x^+)$ and $f(x^-)$ don't exist.

Example 2.9.

$$f(x) := \begin{cases} (-1)^n & x = \frac{1}{n+1}, n \in \mathbb{N} \\ \text{linear} & x \in (0, \infty) \setminus \left\{ \frac{1}{n+1} : n \in \mathbb{N} \right\} \\ 0 & x \leq 0. \end{cases}$$

This function has a discontinuity of second kind at 0.

> **Lemma 2.10**
>
> Let $f : \mathbb{R} \to \mathbb{R}$ ($\mathrm{Dom}(f) = \mathbb{R}$) be monotone. Then $\forall x \in \mathbb{R} : f(x^+), f(x^-)$ exist and so f has no discontinuities of second kind.

Proof. Let $x \in \mathbb{R}$ and assume f is nondecreasing. We claim that $\lim_{z \to x^+} f(z) = \inf \left\{ f(z) : z > x \right\} =: L$.

Indeed, $\forall z > x : f(z) \geq f(x)$, so $L \geq f(x)$ and so $L \in \mathbb{R}$. Then $(\forall z > x : L \leq f(z)) \wedge (\forall \varepsilon > 0 \, \exists z_\varepsilon > x : f(z_\varepsilon) < L + \varepsilon)$. Let $\delta := z_\varepsilon - x$. Then $\forall z \in (x, x + \delta) : f(z) \leq f(z_\varepsilon) < L + \varepsilon$. Then $\forall z \in (x, x + \delta) : L \leq f(z) < L + \varepsilon$ and therefore $|f(z) - L| < \varepsilon$. Then $\lim_{z \to x^+} f(z) = L$. $\qquad\square$

3 3.31 Thursday Week 1: Monotone functions have only countably many discontinuities. Functions of bounded variation. Jordan decomposition theorem. Comments on uniqueness. Rectifiability of curves. Limsup and liminf of a function.

Limits of functions

Last time we showed that monotone functions have no discontinuities of second kind.

> **Lemma 3.1**
>
> Let $f : \mathbb{R} \to \mathbb{R}$ be monotone. Then $\left\{ x \in \mathbb{R} : f(x^+) \neq f(x^-) \right\}$ is countable.

Proof. Pick $k, m \in \mathbb{N}$ and let $A_{m,k} := \left\{ x \in [-m, m] : |f(x^+) - f(x^-)| > \frac{1}{k+1} \right\}$. We claim that $A_{m,k}$ is finite.

Let $x_0 < x_1 < \cdots < x_n$ be such that $\forall i \leq n : x_i \in A_{k,m}$. Assume (without loss of generality) that f is non-decreasing. Then

$$f(m+1) \geq f(x_n^+) = f(x_0^+) + \sum_{i=1}^{n} \left(f(x_i^+) - f(x_{i-1}^+) \right)$$

$$\geq f(m-1) + \sum_{i=1}^{n} \left(f(x_i^+) - f(x_i^-) \right)$$

$$\geq f(-m+1) + \frac{n}{k+1}. \tag{3.1}$$

Then $n \leq (k+1)$. Since $\left\{ x \in \mathbb{R} : f(x^+) \neq f(x^-) \right\} = \bigcup_{k \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} A_{k,m}$, we are done. $\qquad\square$

Q: Can these be generalized to other functions?

> **Definition 3.2**
>
> A **partition** Π of an interval $[a, b]$ is a sequence $\{t_i\}_{i=0}^n$ such that
>
> $$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b.$$

> **Definition 3.3**
>
> Given $f : [a, b] \to \mathbb{R}$, its **total variation** on $[a, b]$
>
> $$V(f, [a, b]) := \sup_{\Pi = \{t_i\}_{i=0}^n} \sum_{i=1}^n \left| f(t_i) - f(t_{i-1}) \right|$$
>
> where the supremum if over the partitions of $[a, b]$.

> **Definition 3.4**
>
> f is said to be of **bounded variation** on $[a, b]$ if $V(f, [a, b]) < \infty$.

> **Lemma 3.5**
>
> If $f : \mathbb{R} \to \mathbb{R}$ is of bounded variation on $[-m, m]$ for all $m \in \mathbb{N}$, then f has only discontinuities of first kind and the set $\left\{ x \in \mathbb{R} : f(x^+) \neq f(x^-) \right\}$ is countable.

> **Theorem 3.6: Jordan decomposition (1881)**
>
> Let $f : [a, b] \to \mathbb{R}$ obey $V(f, [a, b]) < \infty$. Then $\exists\, h, g : [a, b] \to \mathbb{R}$ nondecreasing such that $\forall t \in [a, b] :$
> $f(t) = h(t) - g(t)$.

Proof. Define $h(t) := V(f, [a, t])$ and $g(t) := V(f, [a, t]) - f(t)$. Note that $h(t) - g(t) = f(t)$.

We need to show that h and g are nondecreasing.

Let $a \leq t < t' \leq b$. Then for any partition Π of $[a, t]$, $\Pi' = \Pi \cup \{t'\}$ is a partition of $[a, t']$. Then

$$V(f, [0, t']) \geq \sum_{i=1}^m \left| f(t_i) - f(t_{i-1}) \right| + \left| f(t') - f(t) \right|.$$

Taking supremum over Π gives

$$V(f, [a, t']) \geq V(f, [a, t]) + \left| f(t') - f(t) \right|.$$

Note that $\left| f(t') - f(t) \right| \geq 0$ and $\left| f(t') - f(t) \right| \geq f(t') - f(t)$. Then $h(t') \geq h(t)$ and $g(t') \geq g(t)$. $\qquad \square$

The representation of $f = h - g$ is called a Jordan decomposition. This is not unique because a nondecreasing function can be added to both h and g.

However, there is a minimal decomposition $f = h_0 - g_0$ such that $g_0(a) = 0$ such that for any other Jordan decomposition $f = h - g$ we have $h - h_0$, $g - g_0$ nondecreasing. This is then *the* Jordan decomposition.

Rectifiability of curves

> **Definition 3.7**
>
> Let (X, ρ) be a metric space. A curve C is $\mathrm{Ran}(f)$ for an $f : \mathbb{R} \to X$ continuous such that $\mathrm{Dom}(f)$ is nonempty and connected. This f is called a **parametrization** of C.

> **Definition 3.8**
>
> Assuming $\mathrm{Dom}(f) = [a, b]$, the **length of C** is
>
> $$\ell(C) := \sup_{\Pi = \{t_i\}_{i=0}^n} \sum_{i=1}^n \rho(f(t_{i-1}), f(t_i)).$$

> **Definition 3.9**
>
> A curve is **rectifiable** if $\ell(C) < \infty$.

> **Definition 3.10**
>
> Let (X, ρ) be a metric space and $f : X \to \mathbb{R}$. Then
>
> $$\limsup_{z \to x} f(z) := \inf_{\delta > 0} \sup_{z \in B(x, \delta) \setminus \{x\}} f(z)$$
>
> and
>
> $$\liminf_{z \to x} f(z) := \sup_{\delta > 0} \inf_{z \in B(x, \delta) \setminus \{x\}} f(z).$$

> **Lemma 3.11**
>
> $$\lim_{z \to x} f(z) \text{ exists in } \mathbb{R} \Leftrightarrow \limsup_{z \to x} f(z) = \liminf_{z \to x} f(z) \in \mathbb{R}.$$

4 4.1 Friday Week 1: Discussion

> **Definition 4.1**
>
> Let $(X, \rho_X), (Y, \rho_Y)$ be metric spaces, $E \subseteq X$, $f : E \to Y$, and $x \in \overline{E}$. Then $\lim_{t \to x} f(t) = \alpha$ is defined by
>
> $$\forall \, \varepsilon > 0 \, \exists \, \delta > 0 : \forall \, x \in E \wedge 0 < \rho_X(t, x) < \delta \Rightarrow \rho_Y(f(t), \alpha) < \varepsilon.$$
>
> Equivalently,
>
> $$\forall \, \{t_n\}_{n \in \mathbb{N}} \in (E \setminus \{x\})^{\mathbb{N}} : t_n \to x \Rightarrow f(t_n) \to \alpha.$$
>
> **Note.** f need not be defined at x.

Remark.

$$\limsup_{t \to x} f(t) := \inf_{\delta > 0} \sup_{t \in B(x, \delta) \setminus \{x\}} f(t) = \lim_{\delta \to 0} \sup_{t \in B(x, \delta) \setminus \{x\}} f(t).$$

$\liminf$ is similarly defined.

Remark.

$$\limsup = \liminf \Rightarrow \lim \text{ exists}.$$

Discontiuities

> **Definition 4.2**
>
> Let $f : (a, b) \to \mathbb{R}$ be not continuous at x. Then f has a **discontinuity of first kind** at x if $f(x+)$ and $f(x-)$ both exist. Otherwise it is of **second kind**.

Remark. Discontinuities of first kind are also known as **simple discontinuities**. The cases include

- $f(x+) = f(x-) \neq f(x)$: **removable discontinuity**, and

- $f(x+) \neq f(x-)$: **jump discontinuity**.

Example 4.3.

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

has a discontinuity of second kind at 0.

Example 4.4.

$$f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is continuous on $\mathbb{R} \setminus \mathbb{Q}$ and has discontinuities of first kind (removable) at every point in $\mathbb{Q}$.

Recall: A monotone function has no discontinuity of second kind and has at most countably many discontinuities of first kind. One can deduce this from the fact that the real line is a union of countably many open intervals (indexed by rationals).

Definition 4.5

A function $f : (a, b) \to \mathbb{R}$ is convex if

$$\forall x, y \in (a, b) : x \le y \Rightarrow (\forall \lambda \in [0, 1] : f(\lambda x + (1 - \lambda)y)) \le \lambda f(x) + (1 - \lambda)f(y).$$

In words, this means that for any interval, the secant line is above the graph.

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5   4.4 Monday Week 2: Existence of limit is equivalent to equality and finiteness of limsup and liminf. Derivative of a real valued function of one real variable. Differentiability implies continuity. Connection with linear approximation. Sum and product rule, chain rule and inverse function rule. First-derivative test and discussion of important counterexamples.

Last time: $\lim_{z \to x} f(z)$, $\limsup_{z \to x} f(z) = \inf_{\delta > 0} \sup_{z \in B(x,\delta) \setminus \{x\} x} f(z)$

Lemma 5.1

$$\lim_{z \to x} f(z) \text{ exists (in } \mathbb{R}) \Leftrightarrow \limsup_{z \to x} f(z) = \liminf_{z \to x} f(z) \in \mathbb{R}.$$

Proof.   Both are equivalent:

$$\forall \varepsilon > 0 \, \exists \delta > 0 : 0 \le \sup_{z \in B(x,\delta) \setminus \{x\}} f(z) - \inf_{z \in B(x,\delta) \setminus \{x\}} f(z) \le 2\varepsilon.$$

$\square$

Definition 5.2

$$\lim_{z \to x} f(z) = \begin{cases} +\infty & \limsup_{z \to x} f(z) = \liminf_{z \to x} f(z) = +\infty \\ -\infty & \limsup_{z \to x} f(z) = \liminf_{z \to x} f(z) = -\infty. \end{cases}$$

Note.   This characterization works even outside $\mathbb{R}$-valued functions:

$$\lim_{z \to x} f(z) \text{ exists} \Leftrightarrow \lim_{\delta \to 0^+} \sup \underbrace{\left\{ \rho(f(z), f(u)) : z, u \in B(x, \delta) \setminus \{x\} \right\}}_{=\text{diam}(f(B(x,\delta)\setminus\{x\}))} = 0.$$

The derivative

Definition 5.3

Let $f : \mathbb{R} \to \mathbb{R}$, $x \in \text{int}(\text{Dom}(f))$. We say that f has **derivative** or **is differentiable at x** if

$$f'(x) := \lim_{z \to x} \frac{f(z) - f(x)}{z - x} \text{ exists in } \mathbb{R}.$$

We call $f'(x)$ (Lagrange notation) the **derivative at x**, alternative notation $\frac{df}{dx}$ (Leibniz notation).

Lemma 5.4

$$f'(x) \text{ exists} \Rightarrow f \text{ continuous at } x.$$

Proof. The existence of $f'(x)$ implies that $\exists \delta_0 > 0 \, \forall z \in \mathbb{R} : 0 < |z - x| < \delta_0 \Rightarrow \left| \frac{f(z)-f(x)}{z-x} \right| \leq 1 + |f'(x)|$. Then, choosing $\varepsilon > 0$ and letting $\delta := \frac{\varepsilon}{1+|f'(x)|}$, we get

$$\forall z \in \mathbb{R} : 0 < |z - x| < \delta \Rightarrow |f(z) - f(x)| \leq (1 + |f'(x)|) |z - x| < (1 + |f'(x)|) \frac{\epsilon}{1 + |f'(x)|} = \varepsilon.$$

Since $f(z) - f(x) = 0$ for $z = x$, we are done (in fact, we have shown that f is lipschitz continuous). $\square$

Another way to write existence of $f'(x)$:

$$f(z) - f(x) = (f'(x) + u_x(z))(z - x)$$

where $\lim_{z \to x} u_x(z) = 0$. (Just define: $u_x(z) := \frac{f(z)-f(x)}{z-x} - f'(x)$ for $z \neq x$)

Lemma 5.5: Linear approximation

$$f'(x) \text{ exists} \Leftrightarrow \exists L \in \mathbb{R} : \lim_{\delta \to 0^+} \sup_{|z-x|<\delta} \frac{1}{\delta} \left| f(z) - f(x) - L(z - x) \right| = 0.$$

Lemma 5.6: Sum & product rule

Let f, g be differentiable at x. Then so are $f + g$ and $f \cdot g$ and

$$(f + g)'(x) = f'(x) + g'(x)$$
$$(f \cdot g)'(x) = f'(x)g(x) + g'(x)f(x) \quad \text{(Leibniz rule)}.$$

Proof. For product rule, note that

$$f(z)g(z) - f(x)g(x) = (f(z) - f(x))g(z) + (g(z) - g(x))f(z).$$

Then

$$\frac{f(z)g(z) - f(x)g(x)}{z - x} = \frac{f(z) - f(x)}{z - x}g(z) + \frac{g(z) - g(x)}{z - x}f(z).$$

Since $g(z) \to g(x)$ by continuity of g, formula follows by sum & product rule for limit. $\square$

Lemma 5.7: Chain rule

Let f be differentiable at x and g at $f(x)$. Then $g \circ f$ is differentiable at x and

$$(g \circ f)'(x) = g'(f(x))f'(x) \quad \left(\frac{dg}{df}\frac{df}{dx}\right)$$

Proof. Define $v_{f(x)}$ such that $g(y) - g(f(x)) = (g'(f(x))) + v_{f(x)}(y))(y - f(x))$ and u_x such that $f(z) - f(x) = (f'(x) + u_x(z))(z - x)$.

$$(g \circ f)(z) - (g \circ f)(x) = [g'(f(x)) + v_{f(x)}(f(z))](f(z) - f(x))$$
$$= [g'(f(x)) + v_{f(x)}(f(z))][f'(x) + u_x(z)](z - x)$$

Dividing by $z - x \neq 0$, note that $f(z) \to f(x)$ implies $v_{f(x)}(f(z)) \to 0$ as $z \to x$, we are done. $\qquad\square$

Lemma 5.8

Let $f \colon \mathbb{R} \to \mathbb{R}$ be injective on $\mathrm{Dom}(f)$ and differentiable at $x \in \mathrm{int}(\mathrm{Dom}(f))$. Assume $f'(x) \neq 0$ and $f(x) \in \mathrm{int}(\mathrm{Ran}(f))$. Then f^{-1} is differentiable at $f(x)$ and

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}.$$

In Leibniz notation:

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

Lemma 5.9: First derivative test

Let $f \colon [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then if $x \in (a, b)$ is a local maximum of f (i.e. $\exists \delta > 0 \, \forall z \in \mathbb{R} : |z - x| < \delta \Rightarrow f(x) \geq f(z)$) then $f'(x) = 0$.

Proof.

$$z > x \wedge |z - x| < \delta \Rightarrow \frac{f(z) - f(x)}{z - x} \leq 0 \Rightarrow f'(x) \leq 0$$

and

$$z < x \wedge |z - x| < \delta \Rightarrow \frac{f(z) - f(x)}{z - x} \geq 0 \Rightarrow f'(x) \geq 0.$$

$\qquad\square$

6   4.6 Wednesday Week 2: Discussion

Recall: For, $x \colon [a, b] \to \mathbb{R}$, the total variation

$$V(f, [a, b]) = \sup_{\Pi} \sum_{i=1}^{n} \left| f(x_i) - f(x_{i-1}) \right|$$

where $\Pi = \{a = x_0 < x_1 < \cdots < x_n = b\}$. We say $f \in BV([a, b])$ if $V(f, [a, b]) < \infty$.

> **Theorem 6.1: Jordan decomposition**
>
> $$\forall f \in BV([a,b]) \, \exists h, g \colon [a,b] \to \mathbb{R} \text{ nondecreasing} : f = h - g.$$

> **Corollary 6.2**
>
> $f \in BV([a,b])$ can only have discontinuities of first kind and countably many of them.

Example 6.3. $f(x) = \sin x \in BV([-1,1])$ since f is nondecreasing on $[-1,1]$ and hence $V(f,[a,b]) = f(b) - f(a)$.

Example 6.4. $f(x) = \sin x \in BV([-M,M])$ by additive property of V.

Q. Does $BV([a,b])$ imply bounded on $[a,b]$?

Yes. By triangle inequality,

$$\left| f(x) \right| \le \left| f(a) \right| + \left| f(a) - f(x) \right| \le \left| f(a) \right| + V(f,[a,b]) < \infty.$$

Q. Does being bounded on $[a,b]$ imply $BV([a,b])$.

No. A counterexample is

$$f(x) = \begin{cases} \sin \frac{1}{x} & x \ne 0 \\ 0 & x = 0 \end{cases}$$

on $[0,1]$.

Choose $x_n = 1/(n\pi/2)$ such that $\sin(1/x_n) = \sin(n\pi/2)$. Then $\sum_{i=1}^{2n} \left| f(x_i) - f(x_{i-1}) \right| = \sum_{k=1}^{n} \left| f(x_{2k+1}) \right| = n \to \infty$.

Example 6.5. Is

$$f(x) = \begin{cases} x \sin \frac{1}{x} & x \ne 0 \\ 0 & x = 0 \end{cases}$$

on $[0,1]$ of bounded variation?

No. Choose the same x_n as above. Note that $f(x_n) = \frac{2}{n\pi} \sin(n\pi/2)$. Then $\sum_{i=1}^{2n} \left| f(x_i) - f(x_{i-1}) \right| = \sum_{k=1}^{n} \frac{2}{(2k-1)\pi} \to \infty$.

Example 6.6. Is

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \ne 0 \\ 0 & x = 0 \end{cases}$$

on $[0,1]$ of bounded variation?

Yes. Note that

$$f'(0) = \lim_{t \to 0} \frac{t^2 \sin \frac{1}{t} - 0}{t} = \lim_{t \to 0} t \sin \frac{1}{t} = 0.$$

Note that for $x \ne 0$,

$$f'(x) = 2x \sin \frac{1}{x} + x^2 \left(-\frac{1}{x^2} \right) \cos \frac{1}{x} = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$$

is bounded: $\left| f'(x) \right| \le 2 \left| x \right| + 1 \le 3$.

Note that by mean value theorem, we have

$$\sum \left| f(x_i) - f(x_{i-1}) \right| \le \sum \left| f'(\xi) \right| (x_i - x_{i-1}) \le M(b-a) < \infty$$

where $\left| f'(\xi) \right| \le M$.

Then f is of bounded variation on $[0,1]$.

> **Theorem 6.7**
>
> If f' exists and is bounded on $[a,b]$ then f is of bounded variation.

Q. Does the existence f' on $[a,b]$ and f being of bounded variation on $[a,b]$ imply f' is bounded on $[a,b]$?

7   4.7 Thursay Week 2: Mean-Value Theorems of Rolle, Lagrange and Cauchy. Applications: Monotone differentiable functions have derivative of one sign. Derivative of a differentiable function has no discontinuities of first kind (but those of second kind can occur densely). L'Hospital's Rule and its proof from Cauchy's MVT.

Mean value theorems

Last time: $f'(x)$ = derivative is linked to the local maxima and minima (first derivative test).

> **Theorem 7.1: Mean value theorem**
>
> Let $f: [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then
>
> 1. (Rolle's theorem, 1691) $f(a) = f(b) \Rightarrow \exists x \in (a,b) : f'(x) = 0$,
>
> 2. (Lagrange's mean value theorem) $\exists x \in (a,b) : f'(x) = \frac{f(b)-f(a)}{b-a}$, and
>
> 3. (Cauchy mean value theorem, 1823) if also $g: [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), then
>
> $$\forall x \in (a,b) : g'(x) \ne 0 \Rightarrow g(a) \ne g(b) \wedge \exists x \in (a,b) : \frac{f'(x)}{g'(x)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Proof.

1. $f(a) = f(b) \wedge$ continuous function on $[a,b]$ achieves one of maximum and minimum on (a,b) $\Rightarrow \exists x \in (a,b) : x$ is local maximum or local minimum of f. Then $f'(x) = 0$.

2. Let $h(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$. Then $h(a) = f(a)$, $h(b) = f(b) - \frac{f(b)-f(a)}{b-a}(b-a) = f(a)$. Then, by 1., $\exists x \in (a,b) : h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a} = 0$.

3. Let $h(x) = f(x) - \frac{f(b)-f(a)}{g(b)-g(a)}(g(x) - g(a))$. Note that this is well defined since by 1. we have $g(b) \ne g(x)$. Then $h(a) = f(a) = h(b)$ so by 1. we have $\exists x \in (a,b) : h'(x) = f'(x) - \frac{f(b)-f(a)}{g(b)-g(a)}g'(x) = 0$.

□

Applications

> **Lemma 7.2**
>
> Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then
>
> $$\forall x \in (a, b) : f'(x) \geq 0 \Leftrightarrow \forall x, y \in [a, b] : x \leq y \Rightarrow f(x) \leq f(y).$$

Proof. The $\Leftarrow$ direction is immediate from the definition of limit $\left(\frac{f(y) - f(x)}{y - x} \geq 0 \right)$.

For the $\Rightarrow$ direction, if $\exists x \geq y : f(y) < f(x)$ then by the mean value theorem $\exists z \in (x, y) : f'(z) = \frac{f(y) - f(x)}{y - x} < 0$.

□

8   4.8 Friday Week 2: Taylor's theorem via Mean Value Theorem (Rolle suffices). Riemann integral: motivation, definitions of marked partition, mesh of partition and Riemann sum. Notion of a function being Riemann integrable. Linearity of integral.

Taylor's theorem

> **Definition 8.1: Higher order derivatites**
>
> Define $f^{(0)} := f$ and for all $n \in \mathbb{N}$ define $f^{(n+1)}(x) := \left(f^{(n)} \right)'(x)$ assuming the derivatives exist. We call $f^{(n)}$ the **n-th derivative of f**.

> **Theorem 8.2: Taylor's theorem (Taylor 1715, Gregory 1671)**
>
> Let $n \in \mathbb{N}$ and $f : (a, b) \to \mathbb{R}$ an $(n + 1)$-times differentiable function. Then
>
> $$\forall x_0 \in (a, b) \, \forall x \in (x_0, b) \, \exists \xi \in (x_0, x) : f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k}_{n\text{-th order Taylor polynomial at } x_0} + \frac{f^{(n+1)}(\xi)}{(n + 1)!} (x - x_0)^{n+1}.$$

Proof. Based on MVT.

Denote

$$P_n(z) := \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (z - x_0)^k.$$

Pick $x \in (x_0, b)$ and denote

$$A := \frac{f(x) - P_n(x)}{(x - x_0)^{n+1}}.$$

Set

$$h(z) := f(z) - P_n(z) - A(z - x_0)^{n+1}.$$

Note that

$$\forall k \in \mathbb{N} : k \le n \Rightarrow f^{(k)}(x_0) = 0.$$

We claim that

$$\forall k \in \mathbb{N} : 1 \le k \le n + 1 \Rightarrow \exists \xi_k \in (x_0, x) : h^{(k)}(\xi_k) = 0.$$

For $k = 1$, the choice of A implies $h(x) = 0$ so since $h(x_0) = 0$, by Rolle's theorem

$$\exists \xi_1 \in (x_0, x) : h'(\xi) = 0.$$

Assume true for some $k \in \mathbb{N}$ such that $1 \le k \le n$. Then $h^{(k)}(x_0) = 0$ and $h^{(k)}(\xi_k) = 0$ for $\xi_k \in (x_0, x)$. Then by Rolle's theorem

$$\exists \xi_{k+1} \in (x_0, \xi_k) : h^{(n+1)}(\xi_{k+1}) = 0.$$

Now observe that $P_n^{(n+1)} = 0$. Then $0 = h^{(n+1)}(\xi_{n+1}) = f^{(n+1)}(\xi_{n+1}) - A(n+1)!$. Then

$$f(x) - P_n(x) = A(x - x_0)^{n+1} = \frac{f^{(n+1)}(\xi_{n+1})}{(n+1)!}(x - x_0)^{n+1}.$$

$\square$

Riemann integral (Riemann 1854)

Goal: Given $f : [a, b] \to \mathbb{R}$, assign meaning to the area under the graph of f on $[a, b]$; namely to the set

$$\big\{(x, y) \in \mathbb{R}^2 : x \in [a, b] \wedge 0 \le y \le f(x)\big\} \quad (\text{for } f \ge 0).$$

Idea: Approximate f with a piecewise constant function and use that the area of a rectangle is "known."

Definition 8.3

Given $[a, b]$, a **marked partition** Π of $[a, b]$ is two sequences $\{t_i\}_{i=0}^n$, $\{t_i^*\}_{i=1}^n$ such that

- $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ and

- $\forall i = 1, \ldots, n : t_i^* \in [t_{i-1}, t_i]$.

Definition 8.4

The **mesh of Π** is defined by $||\Pi|| := \max_{i=1,\ldots,n} |t_i - t_{i-1}|$.

Definition 8.5

Given $f : [a, b] \to \mathbb{R}$ and a marked partition Π, the associated **Riemann sum** is

$$R(f, \Pi) := \sum_{i=1}^n f(t_i^*)(t_i - t_{i-1}).$$

Definition 8.6

A function $f : [a, b] \to \mathbb{R}$ is said to be **Riemann integrable** (on $[a, b]$) if there exists $L \in \mathbb{R}$ such that

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall \Pi = \text{marked partition of } [a, b] : ||\Pi|| < \delta \Rightarrow \left| R(f, \Pi) - L \right| < \varepsilon.$$

We sometimes write this as $\lim_{||\Pi|| \to 0} R(f, \Pi) = L$ (this L is unique). Notation for L is $\int_a^b f(x) \, dx$.

Lemma 8.7: Additivity and homogeneity of Reimann integral

Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$. Let $\alpha, \beta \in \mathbb{R}$. Then $\alpha f + \beta g$ is Riemann integrable on $[a, b]$ and

$$\int_a^b (\alpha f(x) + \beta g(x)) \, dx = \alpha \int_a^b f(x) \, dx + \beta \int_a^b g(x) \, dx.$$

Proof. Given $\varepsilon > 0$, pick $\delta > 0$ such that $||\Pi|| < \delta$ implies

$$\left| R(f, \Pi) - \int_a^b f(x) \, dx \right| < \varepsilon \wedge \left| R(g, \Pi) - \int_a^b g(x) \, dx \right|.$$

Since $R(\alpha f + \beta g, \Pi) = \alpha R(f, \Pi) + \beta R(g, \Pi)$,

$$\left| R(\alpha f + \beta g, \Pi) - \alpha \int_a^b f(x) \, dx - \beta \int_a^b g(x) \, dx \leq |\alpha| \right|$$
$$\leq \alpha \left| R(f, \Pi) - \int_a^b f(x) \, dx \right| + |\beta| \left| R(g, \Pi) - \int_a^b g(x) \, dx \right|$$
$$\leq (|\alpha| + |\beta|) \varepsilon.$$

$\square$

Corollary 8.8

Let $f, g : [0, \infty) \to \mathbb{R}$ be continuous on $[0, \infty)$ and differentiable on $(0, \infty)$. Then

$$f(0) \leq g(0) \wedge \forall x \in (0, \infty) : f'(x) \leq g'(x) \Rightarrow \forall x \in [0, \infty] : f(x) \leq g(x).$$

Example 8.9. $\forall x \geq 0 : e^x \geq 1 + x$.

Lemma 8.10

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then f' has the intermediate value property.

Proof. Without loss of generality assume f' exists on $[\tilde{a}, \tilde{b}]$ such that $\tilde{a} < a < b < \tilde{b}$. Without loss of generality assume $f'(a) < f'(b)$. Let $t \in (f'(a), f'(b))$. Let $h(x) := f(x) - tx$. Then

$$h'(a) < 0 \Rightarrow \exists\, x \in (a,b) : h(x) < h(a).$$

With the same reasoning, we have

$$h'(b) > 0 \Rightarrow \exists\, y \in (a,b) : h(y) < h(b).$$

Then

$$\exists\, z \in (a,b) \text{ local minimum} \Rightarrow h'(z) = f(z) - t = 0.$$

$\square$

Corollary 8.11

The derivative of a differentiable function does not have discontinuities of first kind.

Example 8.12. Let

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Then $\forall\, x \neq 0 : f'(x) = x \sin(1/x) - \cos(1/x)$. $\lim_{x \to 0^{\pm}} f'(x)$ does not exist.

Also note that

$$\frac{f(x) - f(0)}{x - 0} = x \sin(1/x) \xrightarrow[x \to 0]{} 0$$

so $f'(0) = 0$.

Theorem 8.13: L'Hopital's rule, proved by Bernoulli 1694

Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous and differentiable on $(a - \delta, a + \delta)$ where $a \in \mathbb{R}$ and $\delta > 0$. Assume

$$f(a) = 0 = g(a) \wedge \forall\, x \in (a - \delta, a + \delta) \setminus \{a\} : g(x) \neq 0 \wedge g'(x) \neq 0.$$

Theni

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} \text{ exists} \Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} \text{ exists} \wedge \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Proof. Let $x \in (a - \delta, a + \delta) \setminus \{a\}$. Then for $x > a$ we have

$$\frac{f(x)}{g(x)} \underset{f(a)=0, g(a)=0}{=\!=\!=\!=\!=} \frac{f(x) - f(a)}{g(x) - g(a)} \underset{\text{Cauchy MVT}}{\overset{\exists\, z_x \in (a,x)}{=\!=\!=\!=\!=}} \frac{f'(z_x)}{g'(z_x)}.$$

Since $x \to a$ implies $z_x \to a$, existence of $\lim_{z \to a} \frac{f'(z)}{g'(z)}$ gives

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{z \to a} \frac{f'(z)}{g'(z)}.$$

$\square$

Example 8.14. $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1.$

9   4.11 Monday Week 3

Last time: $f : [a, b] \to \mathbb{R}$ is Riemann integrable (RI) if

$$\exists L \in \mathbb{R} \, \forall \, \varepsilon > 0 \, \exists \, \delta > 0 \, \forall \, \Pi = \text{marked partition of } [a, b] : ||\Pi|| < \delta \Rightarrow \left| R(f, \Pi) - L \right| < \varepsilon.$$

Notation: $L = \int_a^b f(x) \, dx$.

We proved **linearity**:

$$\int_a^b (\alpha f(x) + \beta f(x)) \, dx = \alpha \int_a^b f(x) \, dx + \beta \int_a^b g(x) \, dx.$$

Lemma 9.1

If f is RI on $[a, b]$ then f is bounded on $[a, b]$.

Proof.   RI $\Rightarrow \exists \, \delta > 0 \, \forall \, \Pi = \text{marked partition} : R(f, \Pi) \le L + 1$. Then $\forall \, i = 1, \dots, n \, \forall \, \tilde{t}_i : f(\tilde{t}_i)(t_i - t_{i-1}) + \sum_{j=1,\dots,n, j \ne i} f(t_j^*)(t_j - t_{j-1}) \le L + 1$, which means $\sup_{\tilde{t}_i \in [t_{i-1}, t_i]} f(\tilde{t}_i) < \infty$. Then $\sup_{x \in [a,b]} f(x) < \infty$. $\square$

Lemma 9.2: Additivity

Let $a < c < b$ be reals. If f is RI on $[a, c]$ and on $[c, b]$, then it is RI on $[a, b]$ and

$$\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.$$

Proof.   Let $\varepsilon > 0$ and let $\delta > 0$ be such that $\forall \, \Pi = \text{marked partition of } [a, c]$ and $\forall \, \Pi' = \text{marked partition of } [c, b]$ such that $||\Pi|| < \delta \wedge ||\Pi'|| < \delta$ we have

$$\left| R(f, \Pi) - \int_a^c f(x) \, dx \right| < \varepsilon \quad \wedge \quad \left| R(f, \Pi') - \int_c^b f(x) \, dx \right| < \varepsilon.$$

If $\tilde{\Pi}$ is a marked partition of $[a, b]$ with $||\tilde{\Pi}|| < \delta$ containing c then

$$\left| R(f, \Pi) - \int_a^c f(x) \, dx - \int_c^b f(x) \, dx \right| < 2\varepsilon.$$

Suppose $\tilde{\Pi}$ does not contain c. Then adding c to $\tilde{\Pi}$ changes $R(f, \tilde{\Pi})$ by at most $2 \cdot 3\delta \sup_{x \in [a,b]} \left| f(x) \right|$. $\square$

Lemma 9.3

If f is RI on $[a, b]$ then

$$\left| \int_a^b f(x) \, dx \right| \le (b - a) \underbrace{\sup_{x \in [a,b]} \left| f(x) \right|}_{< \infty}.$$

Proof. Note that

$$\left| R(f, \Pi) \right| = \left| \sum_{i=1}^{n} f(t_i^*)(t_i - t_{i-1}) \right| \leq \sum_{i=1}^{n} \left| f(t_i^*)(t_i - t_{i-1}) \right| = R(|f|, \Pi) \leq \sup_{x \in [a,b]} |f(x)| \underbrace{\sum_{i=1}^{n} (t_i - t_{i-1})}_{=b-a}$$

$\square$

Note. If we knew that $|f|$ is RI, then this gives

$$\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx.$$

Q: Sufficient conditions for RI?

A: We will answer this using Darboux's version of Riemann integral.

Definition 9.4

Let $f : [a, b] \to \mathbb{R}$ be bounded. Given an unmarked partition $\Pi = \{t_i\}_{i=1}^{n}$ of $[a, b]$, set

$$U(f, \Pi) := \sum_{i=1}^{n} \sup \{ f(x) : x \in [t_{i-1}, t_i] \} (t_i - t_{i-1})$$

and

$$L(f, \Pi) := \sum_{i=1}^{n} \inf \{ f(x) : x \in [t_{i-1}, t_i] \} (t_i - t_{i-1})$$

to be the **upper and lower Darboux sums**.

Note. $L(f, \Pi) \leq R(f, \Pi) \leq U(f, \Pi)$ for any marked partition Π.

Lemma 9.5

For all unmarked partitions Π and Π' of $[a, b]$ we have

$$L(f, \Pi) \leq U(f, \Pi').$$

Proof. Assume first Π is a subset of Π', meaning that all points of Π are included in Π'. We claim that $U(f, \Pi') \leq U(f, \Pi)$ and $L(f, \Pi') \geq L(f, \Pi)$.

Note that if $\Pi' = \Pi \cup \{t\}$, let $[t_{i-1}, t_i]$ be the interval containing t. Then

$$\max \left\{ \sup_{x \in [t_{i-1}, t]} f(x), \sup_{x \in [t, t_i]} f(x) \right\} \sup_{x \in [t_{i-1}, t_i]} f(x),$$

resulting in $U(f, \Pi') \leq U(f, \Pi)$.

Now let Π and Π' be arbitrary and $\Pi \cup \Pi'$ be their common refinement. Then

$$L(f, \Pi) \leq L(f, \Pi \cup \Pi') \leq U(f, \Pi \cup \Pi') \leq U(f, \Pi').$$

$\square$

Definition 9.6

Set

$$\underline{\int_a^b} f(x)\, dx := \sup \left\{ L(f, \Pi) : \Pi = \text{partition of } [a,b] \right\}$$

and

$$\overline{\int_a^b} f(x)\, dx := \inf \left\{ U(f, \Pi) : \Pi = \text{partition of } [a,b] \right\}$$

to be the **lower and upper Darboux integrals**.

Note.

$$\underline{\int_a^b} f(x)\, dx \le \overline{\int_a^b} f(x)\, dx.$$

Definition 9.7

We say that a bounded f is **Darboux integrable on $[a, b]$** if

$$\underline{\int_a^b} f(x)\, dx = \overline{\int_a^b} f(x)\, dx.$$

10 4.13 Wednesday Week 3

Riemann integral continued

Last time: $U(f, \Pi)$ and $L(f, \Pi)$ are the upper and lower Darboux sums. Note that

$$\forall\, \Pi, \Pi' : L(f, \Pi) \le U(f, \Pi').$$

Then

$$\overline{\int_a^b} f(x)\, dx = \inf \left\{ U(f, \Pi) : \Pi \text{ partition} \right\}$$

and

$$\underline{\int_a^b} f(x)\, dx = \sup \left\{ L(f, \Pi) : \Pi \text{ partition} \right\}$$

obey

$$\underline{\int_a^b} f(x)\, dx \le \overline{\int_a^b} f(x)\, dx.$$

Definition 10.1

$f : [a, b] \to \mathbb{R}$ bounded is **Darboux integrable** if

$$\underline{\int_a^b} f(x)\, dx = \overline{\int_a^b} f(x)\, dx.$$

Lemma 10.2

For every $f : [a, b] \to \mathbb{R}$:

$$f \text{ Darboux integrable} \Leftrightarrow \forall\, \varepsilon > 0\, \exists\, \Pi \text{ partition} : U(f, \Pi) - L(f, \Pi) < \varepsilon.$$

Proof. By definition,

$$\forall\, \varepsilon > 0\, \exists\, \Pi, \tilde{\Pi} : U(f, \Pi) < \overline{\int_a^b} f(x)\, dx + \varepsilon \quad \wedge \quad L(f, \tilde{\Pi}) > \underline{\int_a^b} f(x)\, dx - \varepsilon.$$

Then

$$U(f, \Pi \cup \tilde{\Pi}) - L(f, \Pi \cup \tilde{\Pi}) \leq U(f, \Pi) - L(f, \tilde{\Pi}) \leq \overline{\int_a^b} f(x)\, dx - \underline{\int_a^b} f(x)\, dx + 2\varepsilon.$$

Then the equality of the Darboux integrals implies the left to right direction of the lemma.

For the converse,

$$0 \leq \overline{\int_a^b} f(x)\, dx - \underline{\int_a^b} f(x)\, dx \leq U(f, \Pi) - L(f, \Pi) < \varepsilon.$$

$\square$

Lemma 10.3

Let Π and Π' be unmarked partitions. Then

$$U(f, \Pi') \geq U(f, \Pi) - 2\, |\Pi'|\, ||\Pi||||f||$$

and

$$L(f, \Pi') \leq L(f, \Pi) + 2\, |\Pi'|\, ||\Pi||||f||.$$

where $||f|| := \sup_{x \in [a,b]} |f(x)|$.

Proof. Note that

$$U(f, \Pi') \geq U(f, \Pi \cup \Pi')$$

and for $f \geq 0$, dropping intervals of Π that receive points in Π' from $U(f, \Pi)$ changes the result by at most $2\, |\Pi'|\, ||\Pi||||f||$. $\square$

> ### Theorem 10.4
>
> For every $f : [a,b] \to \mathbb{R}$ bounded:
>
> $$f \text{ Riemann integrable} \Leftrightarrow f \text{ Darboux integrable.}$$
>
> If both are true then
>
> $$\int_a^b f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx.$$

Proof. $\Rightarrow$: RI means that

$$\exists L \in \mathbb{R} \, \exists \delta > 0 \, \forall \Pi \text{ partition with } ||\Pi|| < \delta : \left| R(f, \Pi) - L \right| < \varepsilon.$$

Pick $N \in \mathbb{N}$ such that $N > (b-a)/\delta$, define $\Pi = \{t_i\}_{i=1}^n$ such that $t_i - t_{i-1} = \frac{b-a}{N} < \delta$. Now pick $t_i^* \in [t_{i-1}, t_i]$ such that

$$f(t_i^*) \geq \sup \left\{ f(x) : x \in [t_{i-1}, t_i] \right\} - \frac{\varepsilon}{b-a}$$

and $\tilde{t}_i^* \in [t_{i-1}, t_i]$ such that

$$f(\tilde{t}_i^*) \leq \inf \left\{ f(x) : x \in [t_{i-1}, t_i] \right\} + \frac{\varepsilon}{b-a}.$$

Then let Π be the partition with marked points $\{t_i^*\}_{i=1}^N$ and $\tilde{\Pi}$ be the partition with marked points $\{\tilde{t}_i^*\}_{i=1}^N$.
Then

$$U(f, \Pi) \leq \sum_{i=1}^N \left(f(t_i^*) + \frac{\varepsilon}{b-a} \right)(t_i - t_{i-1}) = R(f, \Pi) + \varepsilon$$

and

$$L(f, \tilde{\Pi}) \geq \sum_{i=1}^n \left(f(\tilde{t}_i^*) - \frac{\varepsilon}{b-a} \right)(t_i - t_{i-1}) = R(f, \tilde{\Pi}) - \varepsilon.$$

Now

$$\begin{aligned}
U(f, \Pi \cup \tilde{\Pi}) - L(f, \Pi \cup \tilde{\Pi}) &\leq U(f, \Pi) - L(f, \tilde{\Pi}) \\
&\leq R(f, \Pi) - R(f, \tilde{\Pi}) + 2\varepsilon \\
&\leq \left| R(f, \Pi) - L \right| + \left| R(f, \tilde{\Pi}) - L \right| + 2\varepsilon \\
&\leq 4\varepsilon.
\end{aligned}$$

$\Leftarrow$: $\forall \varepsilon > 0 \, \exists \Pi'$ partition such that $U(f, \Pi') - L(f, \Pi') < \varepsilon$. Pick any Π and $\tilde{\Pi}$ marked partitions with $||\tilde{\Pi}||, ||\Pi|| < \delta := \varepsilon / (|\Pi'| \, ||f||) \quad (f \neq 0)$.
Then

$$R(f, \Pi) \leq U(f, \Pi) \overset{\text{by Lemma 10.3}}{\leq} U(f, \Pi') + 2 \underbrace{|\Pi'| \, ||\Pi|| ||f||}_{\leq \varepsilon}$$

and

$$R(f, \tilde{\Pi}) \geq L(f, \tilde{\Pi}) \overset{\text{by Lemma 10.3}}{\geq} L(f, \Pi') - 2 \underbrace{|\Pi'| \, ||\tilde{\Pi}|| ||f||}_{\leq \varepsilon}.$$

Then

$$\left|R(f, \tilde{\Pi}) - R(f, \tilde{\Pi})\right| \le U(f, \Pi') - L(f, \Pi') + 4\varepsilon \le 5\varepsilon.$$

Let $\{\Pi_n\}$ be an arbitrary sequence of marked partitions such that

$$||\Pi_n|| \to 0 \quad \wedge \quad L := \lim_{n \to \infty} R(f, \Pi_n) \text{ exists.}$$

This exists by Bolzano-Weierstrass theorem.

Then

$$\left|R(f, \Pi) - L\right| \le \left|R(f, \Pi_n) - L\right| + \left|R(f, \Pi) - R(f, \Pi_n)\right| \underset{\text{once } ||\Pi_n|| < \delta}{\le} \left|R(f, \Pi_n) - L\right| + 5\varepsilon \xrightarrow[n \to \infty]{} 5\varepsilon.$$

Then we showed that

$$\exists L \in \mathbb{R} \, \forall \, \varepsilon > 0 \, \exists \delta > 0 \, \forall \Pi \text{ marked partition} : ||\Pi|| < \delta \Rightarrow \left|R(f, \Pi) - L\right| \le 5\varepsilon.$$

$\square$

Corollary 10.5

Let $f : [a, b] \to \mathbb{R}$ be bounded. Then

$$f \text{ is RI} \Leftrightarrow \forall \, \varepsilon > 0 \, \exists \Pi = \{t_i\}_{i=1}^n \text{ unmarked partition} : \sum_{i=1}^N \text{osc}(f, [t_{i-1}, t_i])(t_i - t_{i-1}) < \varepsilon$$

where $\text{osc}(f, A) = \sup_{x \in A} f(x) - \inf_{x \in A} f(x)$

Example 10.6. The dirichlet function

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

is not RI.

Example 10.7. The function

$$f(x) = \begin{cases} \frac{1}{n+1} & x = q_n \\ 0 & x \notin \mathbb{Q} \end{cases}$$

is not RI.

11   4.15 Friday Week 3

Riemann Integrability - criteria and characterization

Last time: $\forall \, f : [a, b] \to \mathbb{R}$ bounded,

$$f \text{ RI} \Leftrightarrow \forall \, \varepsilon > 0 \, \exists \Pi = \{t_i\}_{i=1}^n \text{ partition of } [a, b] : \sum_{i=1}^n \text{osc}(f, [t_{i-1}, t_i]) \, |t_i - t_{i-1}| < \varepsilon$$

where

$$\text{osc}(f, A) := \sup \left\{ \left|f(y) - f(x)\right| : x, y \in A \right\}$$
$$= \sup_{x \in A} f(x) - \inf_{x \in A} f(x) (A \neq \varnothing).$$

> **Lemma 11.1**
>
> Let $f, g: [a, b] \to \mathbb{R}$. Then
>
> 1. f RI $\Rightarrow |f|$ RI and
>
> 2. f, g RI $\Rightarrow f \cdot g$ RI.

Proof. Note that
$$\left| |f|(x) - |f|(y) \right| = \left| |f(x)| - |f(y)| \right| \leq |f(x) - f(y)|.$$

Then
$$\text{osc}(f, A) \leq \text{osc}(|f|, A).$$

Then
$$f \text{ RI} \Rightarrow |f| \text{ RI}.$$

Note that a counterexample for the converse is Dirichlet's function. $\qquad \square$

> **Theorem 11.2**
>
> For all $f: [a, b] \to \mathbb{R}$ we have
> $$f \text{ continuous} \Rightarrow f \text{ RI}.$$

Proof. Note that $[a, b]$ compact and f continuous implies that f is uniformly continuous. Then for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $s, t \in [a, b]$ we have
$$0 < |s - t| < \delta \Rightarrow \text{osc}(f, [s, t]) < \frac{\varepsilon}{b - a}.$$

Then for all
$$\forall \Pi : ||\Pi|| < \delta \Rightarrow \sum_{i=1}^{n} \text{osc}(f, [t_{i-1}, t_i])(t_i - t_{i-1}) \leq \sum_{i=1}^{n} \frac{\varepsilon}{b - a}(t_i - t_{i-1}) \leq \varepsilon.$$

$\qquad \square$

> **Lemma 11.3**
>
> Let $f: [a, b] \to \mathbb{R}$ be bounded and such that f has only finitely many discontinuities. Then f is RI.

Proof. Let $x_1, \ldots, x_m$ enumerate discontinuity points of f. Pick $\varepsilon > 0$. Suppose without loss of generality $||f|| \neq 0$. Let $\delta < \frac{\varepsilon}{m||f||}$. Then
$$\text{osc}(f, [x_i - \delta, x_i + \delta] \cap [a, b]) \leq 2||f||.$$

Next, note that $[a, b] \setminus \bigcup_{i=1}^{m}(x_i - \delta, x_i + \delta)$ is closed and thus compact. Then f is uniformly continuous on this set. Then there exists $\delta' > 0$ such that for all $[s, t] \subseteq$ this set we have
$$0 < |s - t| \leq \delta' \Rightarrow \text{osc}(f, [s, t]) \leq \frac{\varepsilon}{b - a}.$$

Now partition $[a, b] \setminus \bigcup_{i=1}^{n}(x_i - \delta, x_i + \delta)$ into intervals of length $\leq \delta'$. Combine them with intervals $[x_i - \delta, x_i + \delta]$. Now take $\Pi = $ set of endpoints of these intervals. Then
$$\sum_{i=1}^{n} \text{osc}(f, [t_{i-1}, t_i])(t_i - t_{i-1}) \leq m \cdot 2||f|| \cdot 2\delta + \frac{\varepsilon}{b - a}(b - a) \leq 5\varepsilon.$$

$\qquad \square$

Lemma 11.4

Let $f : [a, b] \to \mathbb{R}$ be bounded.

$$f \text{ has no discontinuities of second kind} \Rightarrow f \text{ RI.}$$

Proof. Key idea:

$$\forall \eta > 0 : \left\{ x \in (a, b) : \mathrm{diam}\{ \lim_{z \to x^+} f(z), \lim_{z \to x^-} f(z), f(x) \} > \eta \right\} \text{ is finite.}$$

$\square$

Example 11.5.

$$f(x) = \begin{cases} \frac{1}{n+1} & x = q_n \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

It gets worse: Let

$$C := \left\{ \sum_{i \in \mathbb{N}} \frac{2\sigma_i}{3^{i+1}} : \{\sigma_i\}_{i \in \mathbb{N}} \in \{0, 1\}^{\mathbb{N}} \right\}$$

be Cantor's ternary set.

Then $C = \bigcap_{n \in \mathbb{N}} C_n$ where

$$C_n = \left\{ \sum_{i=1}^{n} \frac{\sigma_i}{3^{i+1}} + [0, 3^{-n-1}] : \sigma_1, \dots, \sigma_n \{0, 1\} \right\}.$$

Lemma 11.6

The function

$$1_C(x) = \begin{cases} 1 & x \in C \\ 0 & x \notin C \end{cases}$$

is RI.

Proof. Let $I_1, \dots, I_{2n}$ be intervals constituting C_n. Define

$$J_k = \left\{ x \in [0, 1] : \mathrm{dist}(x, I_k) < \frac{1}{3^{n+1}} \right\}.$$

Then

$$\mathrm{length}(J_k) = \mathrm{length}(I_k) + 2 \cdot \frac{1}{3^{n+3}} \leq \frac{1}{3^n}.$$

Take Π to be the endpoints of $\{J_k\}_{k=1}^{2n}$. Then

$$\sum_{i=1}^{m} \mathrm{osc}(f, [t_{i-1}, t_i]) \, |t_i - t_{i-1}| \leq \sum_{k=1}^{2^n} \mathrm{length}(J_k) \leq 2^n \cdot \frac{1}{3^n} \xrightarrow[n \to \infty]{} 0.$$

$\square$

12   4.18 Monday Week 4

Characterizing Riemann integrability

Sufficient conditions for RI: continuity, finite number of discontinuities, no discontinuities of second kind.

Necessary condition for RI: boundedness.

Definition 12.1

A set $A \subseteq \mathbb{R}$ is of **zero length** if

$$\forall\, \varepsilon > 0 \,\exists\, \{(a_i, b_i)\}_{i\in\mathbb{N}} \text{ intervals} : A \leq \bigcup_{i\in\mathbb{N}}(a_i, b_i) \quad \wedge \quad \sum_{i\in\mathbb{N}}(b_i - a_i) < \varepsilon.$$

Lemma 12.2

In the definition of zero length, closed intervals can be used.

Proof. If $A \subseteq \bigcup_{i\in\mathbb{N}}[a_i, b_i]$, let $\tilde{a}_i = a_i - \varepsilon/2^i$ and $\tilde{b}_i = b_i + \varepsilon/2^i$. Then

$$A \subseteq \bigcup_{i\in\mathbb{N}}(\tilde{a}_i, \tilde{b}_i)$$

and

$$\sum_{i\in\mathbb{N}}(\tilde{b}_i - \tilde{a}_i) = \sum_{i\in\mathbb{N}}(b_i - a_i) + \sum_{i\in\mathbb{N}} 2\cdot\frac{\varepsilon}{2^i} = \sum_{i\in\mathbb{N}}(b_i - a_i) + 4\varepsilon.$$

$\square$

Lemma 12.3

Let $f : \mathbb{R} \to \mathbb{R}$ be bounded. Set

$$M_f(x) = \inf_{\delta>0} \sup_{z:|z-x|<\delta} f(z)$$

and

$$m_f(x) = \sup_{\delta>0} \inf_{z:|z-x|<\delta} f(z).$$

Then

1. $\forall\, x \in \mathbb{R} : f$ continuous at $x \Leftrightarrow M_f(x) = m_f(x)$,

2. $\forall\, x \in \mathbb{R}\,\forall\, \delta : \max\left\{\operatorname{osc}(f, [x-\delta, x]), \operatorname{osc}(f, [x-\delta, x])\right\} \geq M_f(x) - m_f(x)$, and

3. $\forall\, x \in \mathbb{R} : \lim_{\delta\to 0} \operatorname{osc}(f, [x-\delta, x+\delta]) = M_f(x) - m_f(x)$.

Theorem 12.4: Lebesgue's characterization of Riemann integrability

Let $f : [a,b] \to \mathbb{R}$ be bounded. Then

$$f \text{ RI} \Leftrightarrow \left\{x \in [a,b] : f \text{ discontinuous at } x\right\} \text{ is zero length.}$$

Proof. $\Rightarrow$: Let $f : [a,b] \to \mathbb{R}$ be bounded and RI.

Pick $\varepsilon > 0$. Then RI implies

$$\forall n \in \mathbb{N} \, \exists \Pi = \{t_i^n\}_{i=1}^{m(n)} \text{ partition of } [a,b] : \sum_{i=1}^{m(n)} \mathrm{osc}(f, [t_{i-1}^n, t_i^n])(t_i^n - t_{i-1}^n) < \varepsilon 4^{-n}.$$

Set $I_n := \{i = 1, \ldots, m(n) : \mathrm{osc}(f, [t_{i-1}^n, t_i^n]) > 2^{-n}\}$. Then

$$\sum_{i \in I_n}(t_i^n - t_{i-1}^n) \overset{\text{Markov's inequality}}{\leq} \sum_{i \in I_n} \frac{\mathrm{osc}(f, [t_{i-1}^n, t_i^n])}{2^{-n}}(t_i^n - t_{i-1}^n) \leq 2^n \sum_{i=1}^{m(n)} \mathrm{osc}(f, [t_{i-1}^n, t_i^n])(t_i^n - t_{i-1}^n) \leq 2^n \cdot 4^{-n} = \varepsilon 2^{-n}.$$

Now

$$\{x \in [a,b] : M_f(x) \neq m_f(x)\} \subseteq \bigcup_{n \geq 1} \bigcup_{i \in I_n} [t_{i-1}^n, t_i^n].$$

Then

$$\sum_{n \geq 1} \sum_{i \in I_n}(t_i^n - t_{i-1}^n) \leq \sum_{n \geq 1} \varepsilon 2^{-n} = \varepsilon.$$

Then f RI $\Rightarrow \{x \in [a,b] : M_f(x) \neq m_f(x)\}$ is zero length.

$\Leftarrow$: Let $\varepsilon > 0$ and let $\{J_i\}_{i \in \mathbb{N}}$ be open intervals such that

$$\{x \in [a,b] : M_f(x) \neq m_f(x)\} \subseteq \bigcup_{i \in \mathbb{N}} J_i \quad \wedge \quad \sum_{i \in \mathbb{N}} \mathrm{length}(J) < \frac{\varepsilon}{2\varepsilon ||f||} (f \neq 0).$$

Since $M_f(x) = m_f(x) \Rightarrow x$ is continuous:

$$\forall x \in [a,b] : M_f(x) = m_f(x) \Rightarrow \exists \delta_x > 0 : \mathrm{osc}(f, (x\text{-}\delta x, x + \delta x) < \frac{\varepsilon}{b-a}.$$

Then intervals $\{J_i\}_{i \in \mathbb{N}} \cup \{(x - \delta, x + \delta) : M_f(x) = m_f(x)\}$ cover $[a,b]$. Then by Heme-Bord theorem,

$$\exists m, n \in \mathbb{N} \, \exists x_0, \ldots, x_m \in \{x \in [a,b] : M_f(x) = m_f(x))\} : [a,b] \subseteq \bigcup_{i=0} J_i \cup \bigcup_{j=0}^{m}(x_j - \delta_{x_j}, x_j + \delta_{x_j}).$$

Let $\Pi = \{t_i\}_{i=1}^{N}$ be a partition containing of all endpoints of the intervals $(x_j - \delta_{x_j}, x_j + \delta_{x_j})$. Let $k = \{i = 1, \ldots, N : [t_{i-1}, t_i] \subseteq \bigcup_{j=1}^{m}(x_j - \delta_{x_j}, x_j + \delta_{x_j})\}$. Then

$$\forall i \in K : \mathrm{osc}(f, [t_{i-1}, t_i]) < \frac{\varepsilon}{b-a}$$

and

$$\sum_{i \notin K} \mathrm{osc}(f, [t_{i-1}, t_i]) \leq 2||f|| \cdot \sum_{i \notin K}(t_i - t_{i-1}) < 2||f|| \sum_{i \in \mathbb{N}} \mathrm{length}(J_i) < \varepsilon.$$

Then

$$\sum_{i=1}^{n} \mathrm{osc}(f, [t_{i-1}, t_i])(t_i - t_{i-1}) \leq \sum_{i \in K} \mathrm{osc}(f, [t_{i-1}, t_i])(t_i - t_{i-1}) + \sum_{i \notin K} \mathrm{osc}(f, [t_{i-1}, t_i])(t_i - t_{i-1}) \leq \frac{\varepsilon}{b-a}(b-a) + \varepsilon = 2\varepsilon.$$

$\square$