

# MATH 131BH (Real Analysis)

April 27, 2022

- 1 **3.28 Monday Week 1: Intro to the course. Review of material covered in 131AH: foundations (definition and constructions of naturals and reals), metric space convergence, continuity.**
- 2 **3.30 Wednesday Week 1: Limit of a function: definition and alternative formulations via images of balls and sequential characterization. Limit on a set, left and right limits for functions on  $\mathbb{R}$ . Discontinuities of first and second kind. Monotone functions have no discontinuities of second kind.**

### Limits of functions

**Recall:**  $f: X \rightarrow Y$  is said to be **continuous at  $x \in X$**  if

$$\forall \varepsilon > 0 \exists \delta > 0 \forall z \in X : \rho_X(x, z) < \delta \Rightarrow \rho_Y(f(z), f(x)) < \varepsilon.$$

Alternatives:

- $f(B_X(x, \delta)) \subseteq B_Y(f(x), \varepsilon)$ ;
- $\forall \{x_n\}_{n \in \mathbb{N}} \in X^{\mathbb{N}} : x_n \rightarrow x \Rightarrow f(x_n) \rightarrow f(x)$ .

A function  $f: X \rightarrow Y$  is **continuous** if

$$\forall x \in X : f \text{ is continuous at } x,$$

or, alternatively,

$$\forall O \subseteq Y \text{ open} : f^{-1}(O) \text{ open}.$$

### **Definition 2.1**

A function  $f: X \rightarrow Y$  **has limit  $y \in Y$  at  $x \in X$** , notation  $\lim_{z \rightarrow x} f(z) = y$ , if

$$\forall \varepsilon > 0 \exists \delta > 0 \forall z \in X : 0 < \rho_X(x, z) < \delta \Rightarrow \rho_Y(f(z), y) < \varepsilon.$$

Alternatives:

- $f(B_X(x, \delta) \setminus \{x\}) \subseteq B_Y(y, \varepsilon)$ ;
- $\forall \{x_n\}_{n \in \mathbb{N}} \in X^{\mathbb{N}} : (\forall n \in \mathbb{N} : x_n \neq x) \wedge x_n \rightarrow x \Rightarrow f(x_n) \rightarrow y$ ;
- $g(z) := \begin{cases} f(z) & z \neq x \\ y & z = x \end{cases}$  is continuous at  $x$ .

**Definition 2.2**

$f$  has a **removable discontinuity** at  $x$  if  $\lim_{z \rightarrow x} f(z)$  exists but  $\neq f(x)$ .

**Definition 2.3**

Let  $A \subseteq X$  be nonempty,  $x \in \overline{A}$  be not an isolated point. Then  $\lim_{z \rightarrow x} f(z) = \lim_{z \rightarrow x} f_A(z)$  where  $f_A$  is the restriction of  $f$  to  $A$ .

**Definition 2.4**

For  $f: \mathbb{R} \rightarrow \mathbb{R}$ , let  $x \in \overline{\text{Dom}(f)}$  be such that  $\text{Dom}(f) \cap (x, \infty) \neq \emptyset$  and  $\text{Dom}(f) \cap (-\infty, x) \neq \emptyset$ . Then  $\lim_{z \rightarrow x^+} f(z) := \lim_{z \rightarrow x, z \in \text{Dom}(f) \cap (x, \infty)} f(z)$  and  $\lim_{z \rightarrow x^-} f(z) := \lim_{z \rightarrow x, z \in \text{Dom}(f) \cap (-\infty, x)} f(z)$  are the **right / left limits of  $f$  at  $x$** .

Alternative notation:  $f(x^+)$ ,  $f(x^-)$ .

**Example 2.5.**

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

has no right or left limits.

**Example 2.6.**

$$f(x) = \begin{cases} \frac{1}{n+1} & x = q_n \text{ where } \{q_n\}_{n \in \mathbb{N}} \text{ enumerates } \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

Then  $\forall x \notin \mathbb{Q} : \lim_{z \rightarrow x} f(z) = 0$  so  $f$  is continuous on  $\mathbb{R} \setminus \mathbb{Q}$ , and  $\forall x \in \mathbb{Q} : \lim_{z \rightarrow x} f(z) = 0$  but  $f$  is not continuous at  $x$ .

**Lemma 2.7**

$$\forall r > 0 \forall \varepsilon > 0 : \{x \in \mathbb{R} : |x| < r \wedge |f(x)| > \varepsilon\} \text{ finite} \Rightarrow \forall x \in \mathbb{R} : \lim_{z \rightarrow x} f(z) = 0.$$

**Definition 2.8**

A function  $f: \mathbb{R} \rightarrow \mathbb{R}$  has a **discontinuity of**

- **first kind** at  $x$  if  $f(x^+)$  and  $f(x^-)$  exist but are not both equal to  $f(x)$ ;
- **second kind** at  $x$  if one or both of  $f(x^+)$  and  $f(x^-)$  don't exist.

**Example 2.9.**

$$f(x) := \begin{cases} (-1)^n & x = \frac{1}{n+1}, n \in \mathbb{N} \\ \text{linear} & x \in (0, \infty) \setminus \left\{ \frac{1}{n+1} : n \in \mathbb{N} \right\} \\ 0 & x \leq 0. \end{cases}$$

This function has a discontinuity of second kind at 0.

**Lemma 2.10**

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  ( $\text{Dom}(f) = \mathbb{R}$ ) be monotone. Then  $\forall x \in \mathbb{R} : f(x^+), f(x^-)$  exist and so  $f$  has no discontinuities of second kind.

*Proof.* Let  $x \in \mathbb{R}$  and assume  $f$  is nondecreasing. We claim that  $\lim_{z \rightarrow x^+} f(z) = \inf \{f(z) : z > x\} =: L$ .

Indeed,  $\forall z > x : f(z) \geq f(x)$ , so  $L \geq f(x)$  and so  $L \in \mathbb{R}$ . Then  $(\forall z > x : L \leq f(z)) \wedge (\forall \varepsilon > 0 \exists z_\varepsilon > x : f(z_\varepsilon) < L + \varepsilon)$ . Let  $\delta := z_\varepsilon - x$ . Then  $\forall z \in (x, x + \delta) : f(z) \leq f(z_\varepsilon) < L + \varepsilon$ . Then  $\forall z \in (x, x + \delta) : L \leq f(z) < L + \varepsilon$  and therefore  $|f(z) - L| < \varepsilon$ . Then  $\lim_{z \rightarrow x^+} f(z) = L$ .  $\square$

### 3 3.31 Thursday Week 1: Monotone functions have only countably many discontinuities. Functions of bounded variation. Jordan decomposition theorem. Comments on uniqueness. Rectifiability of curves. Limsup and liminf of a function.

#### Limits of functions

Last time we showed that monotone functions have no discontinuities of second kind.

**Lemma 3.1**

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be monotone. Then  $\{x \in \mathbb{R} : f(x^+) \neq f(x^-)\}$  is countable.

*Proof.* Pick  $k, m \in \mathbb{N}$  and let  $A_{m,k} := \{x \in [-m, m] : |f(x^+) - f(x^-)| > \frac{1}{k+1}\}$ . We claim that  $A_{m,k}$  is finite.

Let  $x_0 < x_1 < \dots < x_n$  be such that  $\forall i \leq n : x_i \in A_{m,k}$ . Assume (without loss of generality) that  $f$  is non-decreasing. Then

$$\begin{aligned} f(m+1) &\geq f(x_n^+) = f(x_0^+) + \sum_{i=1}^n (f(x_i^+) - f(x_{i-1}^+)) \\ &\geq f(m-1) + \sum_{i=1}^n (f(x_i^+) - f(x_i^-)) \\ &\geq f(-m+1) + \frac{n}{k+1}. \end{aligned} \tag{3.1}$$

Then  $n \leq (k+1)$ . Since  $\{x \in \mathbb{R} : f(x^+) \neq f(x^-)\} = \bigcup_{k \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} A_{m,k}$ , we are done.  $\square$

**Q:** Can these be generalized to other functions?

**Definition 3.2**

A **partition**  $\Pi$  of an interval  $[a, b]$  is a sequence  $\{t_i\}_{i=0}^n$  such that

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b.$$

**Definition 3.3**

Given  $f: [a, b] \rightarrow \mathbb{R}$ , its **total variation** on  $[a, b]$

$$V(f, [a, b]) := \sup_{\Pi = \{t_i\}_{i=0}^n} \sum_{i=1}^n |f(t_i) - f(t_{i-1})|$$

where the supremum is over the partitions of  $[a, b]$ .

**Definition 3.4**

$f$  is said to be of **bounded variation** on  $[a, b]$  if  $V(f, [a, b]) < \infty$ .

**Lemma 3.5**

If  $f: \mathbb{R} \rightarrow \mathbb{R}$  is of bounded variation on  $[-m, m]$  for all  $m \in \mathbb{N}$ , then  $f$  has only discontinuities of first kind and the set  $\{x \in \mathbb{R} : f(x^+) \neq f(x^-)\}$  is countable.

**Theorem 3.6: Jordan decomposition (1881)**

Let  $f: [a, b] \rightarrow \mathbb{R}$  obey  $V(f, [a, b]) < \infty$ . Then  $\exists h, g: [a, b] \rightarrow \mathbb{R}$  nondecreasing such that  $\forall t \in [a, b] : f(t) = h(t) - g(t)$ .

*Proof.* Define  $h(t) := V(f, [a, t])$  and  $g(t) := V(f, [a, t]) - f(t)$ . Note that  $h(t) - g(t) = f(t)$ .

We need to show that  $h$  and  $g$  are nondecreasing.

Let  $a \leq t < t' \leq b$ . Then for any partition  $\Pi$  of  $[a, t]$ ,  $\Pi' = \Pi \cup \{t'\}$  is a partition of  $[a, t']$ . Then

$$V(f, [a, t']) \geq \sum_{i=1}^m |f(t_i) - f(t_{i-1})| + |f(t') - f(t)|.$$

Taking supremum over  $\Pi$  gives

$$V(f, [a, t']) \geq V(f, [a, t]) + |f(t') - f(t)|.$$

Note that  $|f(t') - f(t)| \geq 0$  and  $|f(t') - f(t)| \geq f(t') - f(t)$ . Then  $h(t') \geq h(t)$  and  $g(t') \geq g(t)$ . □

The representation of  $f = h - g$  is called a Jordan decomposition. This is not unique because a nondecreasing function can be added to both  $h$  and  $g$ .

However, there is a minimal decomposition  $f = h_0 - g_0$  such that  $g_0(a) = 0$  such that for any other Jordan decomposition  $f = h - g$  we have  $h - h_0, g - g_0$  nondecreasing. This is then *the* Jordan decomposition.

## Rectifiability of curves

### Definition 3.7

Let  $(X, \rho)$  be a metric space. A curve  $C$  is  $\text{Ran}(f)$  for an  $f: \mathbb{R} \rightarrow X$  continuous such that  $\text{Dom}(f)$  is nonempty and connected. This  $f$  is called a **parametrization** of  $C$ .

### Definition 3.8

Assuming  $\text{Dom}(f) = [a, b]$ , the **length of  $C$**  is

$$\ell(C) := \sup_{\Pi = \{t_i\}_{i=0}^n} \sum_{i=1}^n \rho(f(t_{i-1}), f(t_i)).$$

### Definition 3.9

A curve is **rectifiable** if  $\ell(C) < \infty$ .

### Definition 3.10

Let  $(X, \rho)$  be a metric space and  $f: X \rightarrow \mathbb{R}$ . Then

$$\limsup_{z \rightarrow x} f(z) := \inf_{\delta > 0} \sup_{z \in B(x, \delta) \setminus \{x\}} f(z)$$

and

$$\liminf_{z \rightarrow x} f(z) := \sup_{\delta > 0} \inf_{z \in B(x, \delta) \setminus \{x\}} f(z).$$

### Lemma 3.11

$$\lim_{z \rightarrow x} f(z) \text{ exists in } \mathbb{R} \Leftrightarrow \limsup_{z \rightarrow x} f(z) = \liminf_{z \rightarrow x} f(z) \in \mathbb{R}.$$

## 4 4.1 Friday Week 1: Discussion

### Definition 4.1

Let  $(X, \rho_X), (Y, \rho_Y)$  be metric spaces,  $E \subseteq X$ ,  $f: E \rightarrow Y$ , and  $x \in \bar{E}$ . Then  $\lim_{t \rightarrow x} f(t) = \alpha$  is defined by

$$\forall \varepsilon > 0 \exists \delta > 0 : \forall x \in E \wedge 0 < \rho_X(t, x) < \delta \Rightarrow \rho_Y(f(t), \alpha) < \varepsilon.$$

Equivalently,

$$\forall \{t_n\}_{n \in \mathbb{N}} \in (E \setminus \{x\})^{\mathbb{N}} : t_n \rightarrow x \Rightarrow f(t_n) \rightarrow \alpha.$$

**Note.**  $f$  need not be defined at  $x$ .

**Remark.**

$$\limsup_{t \rightarrow x} f(t) := \inf_{\delta > 0} \sup_{t \in B(x, \delta) \setminus \{x\}} f(t) = \lim_{\delta \rightarrow 0} \sup_{t \in B(x, \delta) \setminus \{x\}} f(t).$$

$\liminf$  is similarly defined.

**Remark.**

$$\limsup = \liminf \Rightarrow \lim \text{ exists.}$$

### Discontinuities

### Definition 4.2

Let  $f: (a, b) \rightarrow \mathbb{R}$  be not continuous at  $x$ . Then  $f$  has a **discontinuity of first kind** at  $x$  if  $f(x+)$  and  $f(x-)$  both exist. Otherwise it is of **second kind**.

**Remark.** Discontinuities of first kind are also known as **simple discontinuities**. The cases include

- $f(x+) = f(x-) \neq f(x)$ : **removable discontinuity**, and
- $f(x+) \neq f(x-)$ : **jump discontinuity**.

**Example 4.3.**

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

has a discontinuity of second kind at 0.

**Example 4.4.**

$$f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is continuous on  $\mathbb{R} \setminus \mathbb{Q}$  and has discontinuities of first kind (removable) at every point in  $\mathbb{Q}$ .

**Recall:** A monotone function has no discontinuity of second kind and has at most countably many discontinuities of first kind. One can deduce this from the fact that the real line is a union of countably many open intervals (indexed by rationals).

**Definition 4.5**

A function  $f: (a, b) \rightarrow \mathbb{R}$  is convex if

$$\forall x, y \in (a, b) : x \leq y \Rightarrow (\forall \lambda \in [0, 1] : f(\lambda x + (1 - \lambda)y)) \leq \lambda f(x) + (1 - \lambda)f(y).$$

In words, this means that for any interval, the secant line is above the graph.

## 5 4.4 Monday Week 2: Existence of limit is equivalent to equality and finiteness of limsup and liminf. Derivative of a real valued function of one real variable. Differentiability implies continuity. Connection with linear approximation. Sum and product rule, chain rule and inverse function rule. First-derivative test and discussion of important counterexamples.

Last time:  $\lim_{z \rightarrow x} f(z)$ ,  $\limsup_{z \rightarrow x} f(z) = \inf_{\delta > 0} \sup_{z \in B(x, \delta) \setminus \{x\}} f(z)$

**Lemma 5.1**

$$\lim_{z \rightarrow x} f(z) \text{ exists (in } \mathbb{R}) \Leftrightarrow \limsup_{z \rightarrow x} f(z) = \liminf_{z \rightarrow x} f(z) \in \mathbb{R}.$$

*Proof.* Both are equivalent:

$$\forall \varepsilon > 0 \exists \delta > 0 : 0 \leq \sup_{z \in B(x, \delta) \setminus \{x\}} f(z) - \inf_{z \in B(x, \delta) \setminus \{x\}} f(z) \leq 2\varepsilon.$$

□

**Definition 5.2**

$$\lim_{z \rightarrow x} f(z) = \begin{cases} +\infty & \limsup_{z \rightarrow x} f(z) = \liminf_{z \rightarrow x} f(z) = +\infty \\ -\infty & \limsup_{z \rightarrow x} f(z) = \liminf_{z \rightarrow x} f(z) = -\infty. \end{cases}$$

**Note.** This characterization works even outside  $\mathbb{R}$ -valued functions:

$$\lim_{z \rightarrow x} f(z) \text{ exists} \Leftrightarrow \lim_{\delta \rightarrow 0^+} \underbrace{\sup \{ \rho(f(z), f(u)) : z, u \in B(x, \delta) \setminus \{x\} \}}_{= \text{diam}(f(B(x, \delta) \setminus \{x\}))} = 0.$$



## The derivative

### Definition 5.3

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $x \in \text{int}(\text{Dom}(f))$ . We say that  $f$  has **derivative** or is **differentiable at  $x$**  if

$$f'(x) := \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x} \text{ exists in } \mathbb{R}.$$

We call  $f'(x)$  (Lagrange notation) the **derivative at  $x$** , alternative notation  $\frac{df}{dx}$  (Leibniz notation).

### Lemma 5.4

$$f'(x) \text{ exists} \Rightarrow f \text{ continuous at } x.$$

*Proof.* The existence of  $f'(x)$  implies that  $\exists \delta_0 > 0 \forall z \in \mathbb{R} : 0 < |z - x| < \delta_0 \Rightarrow \left| \frac{f(z) - f(x)}{z - x} \right| \leq 1 + |f'(x)|$ . Then, choosing  $\varepsilon > 0$  and letting  $\delta := \frac{\varepsilon}{1 + |f'(x)|}$ , we get

$$\forall z \in \mathbb{R} : 0 < |z - x| < \delta \Rightarrow |f(z) - f(x)| \leq (1 + |f'(x)|) |z - x| < (1 + |f'(x)|) \frac{\varepsilon}{1 + |f'(x)|} = \varepsilon.$$

Since  $f(z) - f(x) = 0$  for  $z = x$ , we are done (in fact, we have shown that  $f$  is Lipschitz continuous).  $\square$

Another way to write existence of  $f'(x)$ :

$$f(z) - f(x) = (f'(x) + u_x(z))(z - x)$$

where  $\lim_{z \rightarrow x} u_x(z) = 0$ . (Just define:  $u_x(z) := \frac{f(z) - f(x)}{z - x} - f'(x)$  for  $z \neq x$ )

### Lemma 5.5: Linear approximation

$$f'(x) \text{ exists} \Leftrightarrow \exists L \in \mathbb{R} : \lim_{\delta \rightarrow 0^+} \sup_{|z - x| < \delta} \frac{1}{\delta} |f(z) - f(x) - L(z - x)| = 0.$$

### Lemma 5.6: Sum & product rule

Let  $f, g$  be differentiable at  $x$ . Then so are  $f + g$  and  $f \cdot g$  and

$$\begin{aligned} (f + g)'(x) &= f'(x) + g'(x) \\ (f \cdot g)'(x) &= f'(x)g(x) + g'(x)f(x) \quad (\text{Leibniz rule}). \end{aligned}$$

*Proof.* For product rule, note that

$$f(z)g(z) - f(x)g(x) = (f(z) - f(x))g(z) + (g(z) - g(x))f(x).$$

Then

$$\frac{f(z)g(z) - f(x)g(x)}{z - x} = \frac{f(z) - f(x)}{z - x} g(z) + \frac{g(z) - g(x)}{z - x} f(x).$$

Since  $g(z) \rightarrow g(x)$  by continuity of  $g$ , formula follows by sum & product rule for limit.  $\square$

**Lemma 5.7: Chain rule**

Let  $f$  be differentiable at  $x$  and  $g$  at  $f(x)$ . Then  $g \circ f$  is differentiable at  $x$  and

$$(g \circ f)'(x) = g'(f(x))f'(x) \quad \left( \frac{dg}{df} \frac{df}{dx} \right)$$

*Proof.* Define  $v_{f(x)}$  such that  $g(y) - g(f(x)) = (g'(f(x)) + v_{f(x)}(y))(y - f(x))$  and  $u_x$  such that  $f(z) - f(x) = (f'(x) + u_x(z))(z - x)$ .

$$\begin{aligned} (g \circ f)(z) - (g \circ f)(x) &= [g'(f(x)) + v_{f(x)}(f(z))](f(z) - f(x)) \\ &= [g'(f(x)) + v_{f(x)}(f(z))][f'(x) + u_x(z)](z - x) \end{aligned}$$

Dividing by  $z - x \neq 0$ , note that  $f(z) \rightarrow f(x)$  implies  $v_{f(x)}(f(z)) \rightarrow 0$  as  $z \rightarrow x$ , we are done.  $\square$

**Lemma 5.8**

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be injective on  $\text{Dom}(f)$  and differentiable at  $x \in \text{int}(\text{Dom}(f))$ . Assume  $f'(x) \neq 0$  and  $f(x) \in \text{int}(\text{Ran}(f))$ . Then  $f^{-1}$  is differentiable at  $f(x)$  and

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}.$$

In Leibniz notation:

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

**Lemma 5.9: First derivative test**

Let  $f: [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Then if  $x \in (a, b)$  is a local maximum of  $f$  (i.e.  $\exists \delta > 0 \forall z \in \mathbb{R} : |z - x| < \delta \Rightarrow f(x) \geq f(z)$ ) then  $f'(x) = 0$ .

*Proof.*

$$z > x \wedge |z - x| < \delta \Rightarrow \frac{f(z) - f(x)}{z - x} \leq 0 \Rightarrow f'(x) \leq 0$$

and

$$z < x \wedge |z - x| < \delta \Rightarrow \frac{f(z) - f(x)}{z - x} \geq 0 \Rightarrow f'(x) \geq 0.$$

$\square$

## 6 4.6 Wednesday Week 2: Discussion

**Recall:** For,  $x: [a, b] \rightarrow \mathbb{R}$ , the total variation

$$V(f, [a, b]) = \sup_{\Pi} \sum_{i=1}^n |f(x_i) - f(x_{i-1})|$$

where  $\Pi = \{a = x_0 < x_1 < \dots < x_n = b\}$ . We say  $f \in BV([a, b])$  if  $V(f, [a, b]) < \infty$ .

**Theorem 6.1: Jordan decomposition**

$$\forall f \in BV([a, b]) \exists h, g : [a, b] \rightarrow \mathbb{R} \text{ nondecreasing} : f = h - g.$$

**Corollary 6.2**

$f \in BV([a, b])$  can only have discontinuities of first kind and countably many of them.

**Example 6.3.**  $f(x) = \sin x \in BV([-1, 1])$  since  $f$  is nondecreasing on  $[-1, 1]$  and hence  $V(f, [a, b]) = f(b) - f(a)$ .

**Example 6.4.**  $f(x) = \sin x \in BV([-M, M])$  by additive property of  $V$ .

**Q.** Does  $BV([a, b])$  imply bounded on  $[a, b]$ ?

Yes. By triangle inequality,

$$|f(x)| \leq |f(a)| + |f(a) - f(x)| \leq |f(a)| + V(f, [a, b]) < \infty.$$

**Q.** Does being bounded on  $[a, b]$  imply  $BV([a, b])$ .

No. A counterexample is

$$f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

on  $[0, 1]$ .

Choose  $x_n = 1/(n\pi/2)$  such that  $\sin(1/x_n) = \sin(n\pi/2)$ . Then  $\sum_{i=1}^{2n} |f(x_i) - f(x_{i-1})| = \sum_{k=1}^n |f(x_{2k+1})| = n \rightarrow \infty$ .

**Example 6.5.** Is

$$f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

on  $[0, 1]$  of bounded variation?

No. Choose the same  $x_n$  as above. Note that  $f(x_n) = \frac{2}{n\pi} \sin(n\pi/2)$ . Then  $\sum_{i=1}^{2n} |f(x_i) - f(x_{i-1})| = \sum_{k=1}^n \frac{2}{(2k-1)\pi} \rightarrow \infty$ .

**Example 6.6.** Is

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

on  $[0, 1]$  of bounded variation?

Yes. Note that

$$f'(0) = \lim_{t \rightarrow 0} \frac{t^2 \sin \frac{1}{t} - 0}{t} = \lim_{t \rightarrow 0} t \sin \frac{1}{t} = 0.$$

Note that for  $x \neq 0$ ,

$$f'(x) = 2x \sin \frac{1}{x} + x^2 \left( -\frac{1}{x^2} \right) \cos \frac{1}{x} = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$$

is bounded:  $|f'(x)| \leq 2|x| + 1 \leq 3$ .

Note that by mean value theorem, we have

$$\sum |f(x_i) - f(x_{i-1})| \leq \sum |f'(\xi)| (x_i - x_{i-1}) \leq M(b-a) < \infty$$

where  $|f'(\xi)| \leq M$ .

Then  $f$  is of bounded variation on  $[0, 1]$ .

### Theorem 6.7

If  $f'$  exists and is bounded on  $[a, b]$  then  $f$  is of bounded variation.

Q. Does the existence  $f'$  on  $[a, b]$  and  $f$  being of bounded variation on  $[a, b]$  imply  $f'$  is bounded on  $[a, b]$ ?

## 7 4.7 Thursay Week 2: Mean-Value Theorems of Rolle, Lagrange and Cauchy. Applications: Monotone differentiable functions have derivative of one sign. Derivative of a differentiable function has no discontinuities of first kind (but those of second kind can occur densely). L'Hospital's Rule and its proof from Cauchy's MVT.

### Mean value theorems

Last time:  $f'(x)$  = derivative is linked to the local maxima and minima (first derivative test).

### Theorem 7.1: Mean value theorem

Let  $f: [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Then

1. (Rolle's theorem, 1691)  $f(a) = f(b) \Rightarrow \exists x \in (a, b) : f'(x) = 0$ ,
2. (Lagrange's mean value theorem)  $\exists x \in (a, b) : f'(x) = \frac{f(b)-f(a)}{b-a}$ , and
3. (Cauchy mean value theorem, 1823) if also  $g: [a, b] \rightarrow \mathbb{R}$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , then

$$\forall x \in (a, b) : g'(x) \neq 0 \Rightarrow g(a) \neq g(b) \wedge \exists x \in (a, b) : \frac{f'(x)}{g'(x)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof.

1.  $f(a) = f(b) \wedge$  continuous function on  $[a, b]$  achieves one of maximum and minimum on  $(a, b) \Rightarrow \exists x \in (a, b) : x$  is local maximum or local minimum of  $f$ . Then  $f'(x) = 0$ .
2. Let  $h(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$ . Then  $h(a) = f(a)$ ,  $h(b) = f(b) - \frac{f(b)-f(a)}{b-a}(b-a) = f(a)$ . Then, by 1.,  $\exists x \in (a, b) : h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a} = 0$ .
3. Let  $h(x) = f(x) - \frac{f(b)-f(a)}{g(b)-g(a)}(g(x) - g(a))$ . Note that this is well defined since by 1. we have  $g(b) \neq g(a)$ . Then  $h(a) = f(a) = h(b)$  so by 1. we have  $\exists x \in (a, b) : h'(x) = f'(x) - \frac{f(b)-f(a)}{g(b)-g(a)}g'(x) = 0$ .

□

## Applications

### Lemma 7.2

Let  $f: [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Then

$$\forall x \in (a, b) : f'(x) \geq 0 \Leftrightarrow \forall x, y \in [a, b] : x \leq y \Rightarrow f(x) \leq f(y).$$

*Proof.* The  $\Leftarrow$  direction is immediate from the definition of limit  $\left(\frac{f(y)-f(x)}{y-x} \geq 0\right)$ .

For the  $\Rightarrow$  direction, if  $\exists x \geq y : f(y) < f(x)$  then by the mean value theorem  $\exists z \in (x, y) : f'(z) = \frac{f(y)-f(x)}{y-x} < 0$ . □

## 8 4.8 Friday Week 2: Taylor's theorem via Mean Value Theorem (Rolle suffices). Riemann integral: motivation, definitions of marked partition, mesh of partition and Riemann sum. Notion of a function being Riemann integrable. Linearity of integral.

### Taylor's theorem

#### Definition 8.1: Higher order derivatives

Define  $f^{(0)} := f$  and for all  $n \in \mathbb{N}$  define  $f^{(n+1)}(x) := (f^{(n)})'(x)$  assuming the derivatives exist. We call  $f^{(n)}$  the  $n$ -th derivative of  $f$ .

#### Theorem 8.2: Taylor's theorem (Taylor 1715, Gregory 1671)

Let  $n \in \mathbb{N}$  and  $f: (a, b) \rightarrow \mathbb{R}$  an  $(n+1)$ -times differentiable function. Then

$$\forall x_0 \in (a, b) \forall x \in (x_0, b) \exists \xi \in (x_0, x) : f(x) = \underbrace{\sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k}_{n\text{-th order Taylor polynomial at } x_0} + \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1}.$$

*Proof.* Based on MVT.

Denote

$$P_n(z) := \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (z - x_0)^k.$$

Pick  $x \in (x_0, b)$  and denote

$$A := \frac{f(x) - P_n(x)}{(x - x_0)^{n+1}}.$$

Set

$$h(z) := f(z) - P_n(z) - A(z - x_0)^{n+1}.$$

Note that

$$\forall k \in \mathbb{N} : k \leq n \Rightarrow f^{(k)}(x_0) = 0.$$

We claim that

$$\forall k \in \mathbb{N} : 1 \leq k \leq n+1 \Rightarrow \exists \xi_k \in (x_0, x) : h^{(k)}(\xi_k) = 0.$$

For  $k = 1$ , the choice of  $A$  implies  $h(x) = 0$  so since  $h(x_0) = 0$ , by Rolle's theorem

$$\exists \xi_1 \in (x_0, x) : h'(\xi_1) = 0.$$

Assume true for some  $k \in \mathbb{N}$  such that  $1 \leq k \leq n$ . Then  $h^{(k)}(x_0) = 0$  and  $h^{(k)}(\xi_k) = 0$  for  $\xi_k \in (x_0, x)$ . Then by Rolle's theorem

$$\exists \xi_{k+1} \in (x_0, \xi_k) : h^{(k+1)}(\xi_{k+1}) = 0.$$

Now observe that  $P_n^{(n+1)} = 0$ . Then  $0 = h^{(n+1)}(\xi_{n+1}) = f^{(n+1)}(\xi_{n+1}) - A(n+1)!$ . Then

$$f(x) - P_n(x) = A(x - x_0)^{n+1} = \frac{f^{(n+1)}(\xi_{n+1})}{(n+1)!}(x - x_0)^{n+1}.$$

□

### Riemann integral (Riemann 1854)

**Goal:** Given  $f : [a, b] \rightarrow \mathbb{R}$ , assign meaning to the area under the graph of  $f$  on  $[a, b]$ ; namely to the set

$$\{(x, y) \in \mathbb{R}^2 : x \in [a, b] \wedge 0 \leq y \leq f(x)\} \quad (\text{for } f \geq 0).$$

**Idea:** Approximate  $f$  with a piecewise constant function and use that the area of a rectangle is “known.”

#### Definition 8.3

Given  $[a, b]$ , a **marked partition**  $\Pi$  of  $[a, b]$  is two sequences  $\{t_i\}_{i=0}^n, \{t_i^*\}_{i=1}^n$  such that

- $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$  and
- $\forall i = 1, \dots, n : t_i^* \in [t_{i-1}, t_i]$ .

#### Definition 8.4

The **mesh of  $\Pi$**  is defined by  $||\Pi|| := \max_{i=1, \dots, n} |t_i - t_{i-1}|$ .

#### Definition 8.5

Given  $f : [a, b] \rightarrow \mathbb{R}$  and a marked partition  $\Pi$ , the associated **Riemann sum** is

$$R(f, \Pi) := \sum_{i=1}^n f(t_i^*)(t_i - t_{i-1}).$$

**Definition 8.6**

A function  $f: [a, b] \rightarrow \mathbb{R}$  is said to be **Riemann integrable** (on  $[a, b]$ ) if there exists  $L \in \mathbb{R}$  such that

$$\forall \varepsilon > 0 \exists \delta > 0 \forall \Pi = \text{marked partition of } [a, b] : \|\Pi\| < \delta \Rightarrow |R(f, \Pi) - L| < \varepsilon.$$

We sometimes write this as  $\lim_{\|\Pi\| \rightarrow 0} R(f, \Pi) = L$  (this  $L$  is unique). Notation for  $L$  is  $\int_a^b f(x) dx$ .

**Lemma 8.7: Additivity and homogeneity of Riemann integral**

Let  $f, g: [a, b] \rightarrow \mathbb{R}$  be Riemann integrable on  $[a, b]$ . Let  $\alpha, \beta \in \mathbb{R}$ . Then  $\alpha f + \beta g$  is Riemann integrable on  $[a, b]$  and

$$\int_a^b (\alpha f(x) + \beta g(x)) dx = \alpha \int_a^b f(x) dx + \beta \int_a^b g(x) dx.$$

*Proof.* Given  $\varepsilon > 0$ , pick  $\delta > 0$  such that  $\|\Pi\| < \delta$  implies

$$\left| R(f, \Pi) - \int_a^b f(x) dx \right| < \varepsilon \wedge \left| R(g, \Pi) - \int_a^b g(x) dx \right| < \varepsilon.$$

Since  $R(\alpha f + \beta g, \Pi) = \alpha R(f, \Pi) + \beta R(g, \Pi)$ ,

$$\begin{aligned} & \left| R(\alpha f + \beta g, \Pi) - \alpha \int_a^b f(x) dx - \beta \int_a^b g(x) dx \right| \\ & \leq |\alpha| \left| R(f, \Pi) - \int_a^b f(x) dx \right| + |\beta| \left| R(g, \Pi) - \int_a^b g(x) dx \right| \\ & \leq (|\alpha| + |\beta|) \varepsilon. \end{aligned}$$

□

**Corollary 8.8**

Let  $f, g: [0, \infty) \rightarrow \mathbb{R}$  be continuous on  $[0, \infty)$  and differentiable on  $(0, \infty)$ . Then

$$f(0) \leq g(0) \wedge \forall x \in (0, \infty) : f'(x) \leq g'(x) \Rightarrow \forall x \in [0, \infty] : f(x) \leq g(x).$$

**Example 8.9.**  $\forall x \geq 0 : e^x \geq 1 + x$ .

**Lemma 8.10**

Let  $f: [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Then  $f'$  has the intermediate value property.

*Proof.* Without loss of generality assume  $f'$  exists on  $[\tilde{a}, \tilde{b}]$  such that  $\tilde{a} < a < b < \tilde{b}$ . Without loss of generality assume  $f'(a) < f'(b)$ . Let  $t \in (f'(a), f'(b))$ . Let  $h(x) := f(x) - tx$ . Then

$$h'(a) < 0 \Rightarrow \exists x \in (a, b) : h(x) < h(a).$$

With the same reasoning, we have

$$h'(b) > 0 \Rightarrow \exists y \in (a, b) : h(y) < h(b).$$

Then

$$\exists z \in (a, b) \text{ local minimum} \Rightarrow h'(z) = f'(z) - t = 0.$$

□

### Corollary 8.11

The derivative of a differentiable function does not have discontinuities of first kind.

**Example 8.12.** Let

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Then  $\forall x \neq 0 : f'(x) = x \sin(1/x) - \cos(1/x)$ .  $\lim_{x \rightarrow 0^\pm} f'(x)$  does not exist.

Also note that

$$\frac{f(x) - f(0)}{x - 0} = x \sin(1/x) \xrightarrow{x \rightarrow 0} 0$$

so  $f'(0) = 0$ .

### Theorem 8.13: L'Hopital's rule, proved by Bernoulli 1694

Let  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  be continuous and differentiable on  $(a - \delta, a + \delta)$  where  $a \in \mathbb{R}$  and  $\delta > 0$ . Assume

$$f(a) = 0 = g(a) \wedge \forall x \in (a - \delta, a + \delta) \setminus \{a\} : g(x) \neq 0 \wedge g'(x) \neq 0.$$

Then

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} \text{ exists} \Rightarrow \lim_{x \rightarrow a} \frac{f(x)}{g(x)} \text{ exists} \wedge \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}.$$

*Proof.* Let  $x \in (a - \delta, a + \delta) \setminus \{a\}$ . Then for  $x > a$  we have

$$\frac{f(x)}{g(x)} \xrightarrow{f(a)=0, g(a)=0} \frac{f(x) - f(a)}{g(x) - g(a)} \xrightarrow[\text{Cauchy MVT}]{\exists z_x \in (a, x)} \frac{f'(z_x)}{g'(z_x)}.$$

Since  $x \rightarrow a$  implies  $z_x \rightarrow a$ , existence of  $\lim_{z \rightarrow a} \frac{f'(z)}{g'(z)}$  gives

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{z \rightarrow a} \frac{f'(z)}{g'(z)}.$$

□

**Example 8.14.**  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{\cos x}{1} = 1$ .



## 9 4.11 Monday Week 3

**Last time:**  $f: [a, b] \rightarrow \mathbb{R}$  is Riemann integrable (RI) if

$$\exists L \in \mathbb{R} \forall \varepsilon > 0 \exists \delta > 0 \forall \Pi = \text{marked partition of } [a, b] : \|\Pi\| < \delta \Rightarrow |R(f, \Pi) - L| < \varepsilon.$$

Notation:  $L = \int_a^b f(x) dx$ .

We proved **linearity**:

$$\int_a^b (\alpha f(x) + \beta g(x)) dx = \alpha \int_a^b f(x) dx + \beta \int_a^b g(x) dx.$$

### Lemma 9.1

If  $f$  is RI on  $[a, b]$  then  $f$  is bounded on  $[a, b]$ .

*Proof.* RI  $\Rightarrow \exists \delta > 0 \forall \Pi = \text{marked partition} : R(f, \Pi) \leq L + 1$ . Then  $\forall i = 1, \dots, n \forall \tilde{t}_i : f(\tilde{t}_i)(t_i - t_{i-1}) + \sum_{j=1, \dots, n, j \neq i} f(\tilde{t}_j^*)(t_j - t_{j-1}) \leq L + 1$ , which means  $\sup_{\tilde{t}_i \in [t_{i-1}, t_i]} f(\tilde{t}_i) < \infty$ . Then  $\sup_{x \in [a, b]} f(x) < \infty$ .  $\square$

### Lemma 9.2: Additivity

Let  $a < c < b$  be reals. If  $f$  is RI on  $[a, c]$  and on  $[c, b]$ , then it is RI on  $[a, b]$  and

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

*Proof.* Let  $\varepsilon > 0$  and let  $\delta > 0$  be such that  $\forall \Pi = \text{marked partition of } [a, c]$  and  $\forall \Pi' = \text{marked partition of } [c, b]$  such that  $\|\Pi\| < \delta \wedge \|\Pi'\| < \delta$  we have

$$\left| R(f, \Pi) - \int_a^c f(x) dx \right| < \varepsilon \quad \wedge \quad \left| R(f, \Pi') - \int_c^b f(x) dx \right| < \varepsilon.$$

If  $\tilde{\Pi}$  is a marked partition of  $[a, b]$  with  $\|\tilde{\Pi}\| < \delta$  containing  $c$  then

$$\left| R(f, \tilde{\Pi}) - \int_a^c f(x) dx - \int_c^b f(x) dx \right| < 2\varepsilon.$$

Suppose  $\tilde{\Pi}$  does not contain  $c$ . Then adding  $c$  to  $\tilde{\Pi}$  changes  $R(f, \tilde{\Pi})$  by at most  $2 \cdot 3\delta \sup_{x \in [a, b]} |f(x)|$ .  $\square$

### Lemma 9.3

If  $f$  is RI on  $[a, b]$  then

$$\left| \int_a^b f(x) dx \right| \leq (b - a) \underbrace{\sup_{x \in [a, b]} |f(x)|}_{< \infty}.$$

*Proof.* Note that

$$|R(f, \Pi)| = \left| \sum_{i=1}^n f(t_i^*)(t_i - t_{i-1}) \right| \leq \sum_{i=1}^n |f(t_i^*)(t_i - t_{i-1})| = R(|f|, \Pi) \leq \sup_{x \in [a, b]} |f(x)| \underbrace{\sum_{i=1}^n (t_i - t_{i-1})}_{=b-a}$$

□

**Note.** If we knew that  $|f|$  is RI, then this gives

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$$

**Q:** Sufficient conditions for RI?

**A:** We will answer this using Darboux's version of Riemann integral.

#### Definition 9.4

Let  $f: [a, b] \rightarrow \mathbb{R}$  be bounded. Given an unmarked partition  $\Pi = \{t_i\}_{i=1}^n$  of  $[a, b]$ , set

$$U(f, \Pi) := \sum_{i=1}^n \sup \{f(x) : x \in [t_{i-1}, t_i]\} (t_i - t_{i-1})$$

and

$$L(f, \Pi) := \sum_{i=1}^n \inf \{f(x) : x \in [t_{i-1}, t_i]\} (t_i - t_{i-1})$$

to be the **upper and lower Darboux sums**.

**Note.**  $L(f, \Pi) \leq R(f, \Pi) \leq U(f, \Pi)$  for any marked partition  $\Pi$ .

#### Lemma 9.5

For all unmarked partitions  $\Pi$  and  $\Pi'$  of  $[a, b]$  we have

$$L(f, \Pi) \leq U(f, \Pi').$$

*Proof.* Assume first  $\Pi$  is a subset of  $\Pi'$ , meaning that all points of  $\Pi$  are included in  $\Pi'$ . We claim that  $U(f, \Pi') \leq U(f, \Pi)$  and  $L(f, \Pi') \geq L(f, \Pi)$ .

Note that if  $\Pi' = \Pi \cup \{t\}$ , let  $[t_{i-1}, t_i]$  be the interval containing  $t$ . Then

$$\max \left\{ \sup_{x \in [t_{i-1}, t]} f(x), \sup_{x \in [t, t_i]} f(x) \right\} \sup_{x \in [t_{i-1}, t_i]} f(x),$$

resulting in  $U(f, \Pi') \leq U(f, \Pi)$ .

Now let  $\Pi$  and  $\Pi'$  be arbitrary and  $\Pi \cup \Pi'$  be their common refinement. Then

$$L(f, \Pi) \leq L(f, \Pi \cup \Pi') \leq U(f, \Pi \cup \Pi') \leq U(f, \Pi').$$

□

**Definition 9.6**

Set

$$\int_a^b f(x) dx := \sup \{L(f, \Pi) : \Pi = \text{partition of } [a, b]\}$$

and

$$\overline{\int_a^b f(x) dx} := \inf \{U(f, \Pi) : \Pi = \text{partition of } [a, b]\}$$

to be the **lower and upper Darboux integrals**.

**Note.**

$$\int_a^b f(x) dx \leq \overline{\int_a^b f(x) dx}.$$

**Definition 9.7**

We say that a bounded  $f$  is **Darboux integrable on  $[a, b]$**  if

$$\int_a^b f(x) dx = \overline{\int_a^b f(x) dx}.$$

## 10 4.13 Wednesday Week 3

### Riemann integral continued

**Last time:**  $U(f, \Pi)$  and  $L(f, \Pi)$  are the upper and lower Darboux sums. Note that

$$\forall \Pi, \Pi' : L(f, \Pi) \leq U(f, \Pi').$$

Then

$$\overline{\int_a^b f(x) dx} = \inf \{U(f, \Pi) : \Pi \text{ partition}\}$$

and

$$\int_a^b f(x) dx = \sup \{L(f, \Pi) : \Pi \text{ partition}\}$$

obey

$$\int_a^b f(x) dx \leq \overline{\int_a^b f(x) dx}.$$

**Definition 10.1**

$f: [a, b] \rightarrow \mathbb{R}$  bounded is **Darboux integrable** if

$$\int_a^b f(x) dx = \overline{\int_a^b f(x) dx}.$$

**Lemma 10.2**

For every  $f: [a, b] \rightarrow \mathbb{R}$ :

$$f \text{ Darboux integrable} \Leftrightarrow \forall \varepsilon > 0 \exists \Pi \text{ partition : } U(f, \Pi) - L(f, \Pi) < \varepsilon.$$

*Proof.* By definition,

$$\forall \varepsilon > 0 \exists \Pi, \tilde{\Pi} : U(f, \Pi) < \overline{\int_a^b f(x) dx} + \varepsilon \quad \wedge \quad L(f, \tilde{\Pi}) > \underline{\int_a^b f(x) dx} - \varepsilon.$$

Then

$$U(f, \Pi \cup \tilde{\Pi}) - L(f, \Pi \cup \tilde{\Pi}) \leq U(f, \Pi) - L(f, \tilde{\Pi}) \leq \overline{\int_a^b f(x) dx} - \underline{\int_a^b f(x) dx} + 2\varepsilon.$$

Then the equality of the Darboux integrals implies the left to right direction of the lemma.

For the converse,

$$0 \leq \overline{\int_a^b f(x) dx} - \underline{\int_a^b f(x) dx} \leq U(f, \Pi) - L(f, \Pi) < \varepsilon.$$

□

**Lemma 10.3**

Let  $\Pi$  and  $\Pi'$  be unmarked partitions. Then

$$U(f, \Pi') \geq U(f, \Pi) - 2 |\Pi'| \|\Pi\| \|f\|$$

and

$$L(f, \Pi') \leq L(f, \Pi) + 2 |\Pi'| \|\Pi\| \|f\|.$$

where  $\|f\| := \sup_{x \in [a, b]} |f(x)|$ .

*Proof.* Note that

$$U(f, \Pi') \geq U(f, \Pi \cup \Pi')$$

and for  $f \geq 0$ , dropping intervals of  $\Pi$  that receive points in  $\Pi'$  from  $U(f, \Pi)$  changes the result by at most  $2 |\Pi'| \|\Pi\| \|f\|$ . □

**Theorem 10.4**

For every  $f: [a, b] \rightarrow \mathbb{R}$  bounded:

$$f \text{ Riemann integrable} \Leftrightarrow f \text{ Darboux integrable.}$$

If both are true then

$$\int_a^b f(x) dx = \int_a^b f(x) dx = \overline{\int_a^b f(x) dx}.$$

*Proof.*  $\Rightarrow$ : RI means that

$$\exists L \in \mathbb{R} \exists \delta > 0 \forall \Pi \text{ partition with } \|\Pi\| < \delta : |R(f, \Pi) - L| < \varepsilon.$$

Pick  $N \in \mathbb{N}$  such that  $N > (b - a)/\delta$ , define  $\Pi = \{t_i\}_{i=1}^N$  such that  $t_i - t_{i-1} = \frac{b-a}{N} < \delta$ . Now pick  $t_i^* \in [t_{i-1}, t_i]$  such that

$$f(t_i^*) \geq \sup \{f(x) : x \in [t_{i-1}, t_i]\} - \frac{\varepsilon}{b-a}$$

and  $\tilde{t}_i^* \in [t_{i-1}, t_i]$  such that

$$f(\tilde{t}_i^*) \leq \inf \{f(x) : x \in [t_{i-1}, t_i]\} + \frac{\varepsilon}{b-a}.$$

Then let  $\Pi$  be the partition with marked points  $\{t_i^*\}_{i=1}^N$  and  $\tilde{\Pi}$  be the partition with marked points  $\{\tilde{t}_i^*\}_{i=1}^N$ .

Then

$$U(f, \Pi) \leq \sum_{i=1}^N \left( f(t_i^*) + \frac{\varepsilon}{b-a} \right) (t_i - t_{i-1}) = R(f, \Pi) + \varepsilon$$

and

$$L(f, \tilde{\Pi}) \geq \sum_{i=1}^n \left( f(\tilde{t}_i^*) - \frac{\varepsilon}{b-a} \right) (t_i - t_{i-1}) = R(f, \tilde{\Pi}) - \varepsilon.$$

Now

$$\begin{aligned} U(f, \Pi \cup \tilde{\Pi}) - L(f, \Pi \cup \tilde{\Pi}) &\leq U(f, \Pi) - L(f, \tilde{\Pi}) \\ &\leq R(f, \Pi) - R(f, \tilde{\Pi}) + 2\varepsilon \\ &\leq |R(f, \Pi) - L| + |R(f, \tilde{\Pi}) - L| + 2\varepsilon \\ &\leq 4\varepsilon. \end{aligned}$$

$\Leftarrow$ :  $\forall \varepsilon > 0 \exists \Pi'$  partition such that  $U(f, \Pi') - L(f, \Pi') < \varepsilon$ . Pick any  $\Pi$  and  $\tilde{\Pi}$  marked partitions with  $\|\tilde{\Pi}\|, \|\Pi\| < \delta := \varepsilon / (\|\Pi'\| \|f\|)$  ( $f \neq 0$ ).

Then

$$R(f, \Pi) \leq U(f, \Pi) \stackrel{\text{by Lemma 10.3}}{\leq} U(f, \Pi') + \underbrace{2 \|\Pi'\| \|\Pi\| \|f\|}_{\leq \varepsilon}$$

and

$$R(f, \tilde{\Pi}) \geq L(f, \tilde{\Pi}) \stackrel{\text{by Lemma 10.3}}{\geq} L(f, \Pi') - \underbrace{2 \|\Pi'\| \|\tilde{\Pi}\| \|f\|}_{\leq \varepsilon}.$$

Then

$$|R(f, \tilde{\Pi}) - R(f, \Pi)| \leq U(f, \Pi') - L(f, \Pi') + 4\varepsilon \leq 5\varepsilon.$$

Let  $\{\Pi_n\}$  be an arbitrary sequence of marked partitions such that

$$\|\Pi_n\| \rightarrow 0 \quad \wedge \quad L := \lim_{n \rightarrow \infty} R(f, \Pi_n) \text{ exists.}$$

This exists by Bolzano-Weierstrass theorem.

Then

$$|R(f, \Pi) - L| \leq |R(f, \Pi_n) - L| + |R(f, \Pi) - R(f, \Pi_n)| \underset{\text{once } \|\Pi_n\| < \delta}{\leq} |R(f, \Pi_n) - L| + 5\varepsilon \xrightarrow{n \rightarrow \infty} 5\varepsilon.$$

Then we showed that

$$\exists L \in \mathbb{R} \forall \varepsilon > 0 \exists \delta > 0 \forall \Pi \text{ marked partition : } \|\Pi\| < \delta \Rightarrow |R(f, \Pi) - L| \leq 5\varepsilon.$$

□

### Corollary 10.5

Let  $f: [a, b] \rightarrow \mathbb{R}$  be bounded. Then

$$f \text{ is RI} \Leftrightarrow \forall \varepsilon > 0 \exists \Pi = \{t_i\}_{i=1}^n \text{ unmarked partition : } \sum_{i=1}^N \text{osc}(f, [t_{i-1}, t_i])(t_i - t_{i-1}) < \varepsilon$$

where  $\text{osc}(f, A) = \sup_{x \in A} f(x) - \inf_{x \in A} f(x)$

**Example 10.6.** The dirichlet function

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

is not RI.

**Example 10.7.** The function

$$f(x) = \begin{cases} \frac{1}{n+1} & x = q_n \\ 0 & x \notin \mathbb{Q} \end{cases}$$

is RI.

## 11 4.15 Friday Week 3

### Riemann Integrability - criteria and characterization

**Last time:**  $\forall f: [a, b] \rightarrow \mathbb{R}$  bounded,

$$f \text{ RI} \Leftrightarrow \forall \varepsilon > 0 \exists \Pi = \{t_i\}_{i=1}^n \text{ partition of } [a, b] : \sum_{i=1}^n \text{osc}(f, [t_{i-1}, t_i]) |t_i - t_{i-1}| < \varepsilon$$

where

$$\begin{aligned} \text{osc}(f, A) &:= \sup \{|f(y) - f(x)| : x, y \in A\} \\ &= \sup_{x \in A} f(x) - \inf_{x \in A} f(x) (A \neq \emptyset). \end{aligned}$$

**Lemma 11.1**

Let  $f, g: [a, b] \rightarrow \mathbb{R}$ . Then

1.  $f$  RI  $\Rightarrow |f|$  RI and
2.  $f, g$  RI  $\Rightarrow f \cdot g$  RI.

*Proof.* Note that

$$||f|(x) - |f|(y)| = ||f(x) - f(y)|| \leq |f(x) - f(y)|.$$

Then

$$\text{osc}(f, A) \leq \text{osc}(|f|, A).$$

Then

$$f \text{ RI} \Rightarrow |f| \text{ RI}.$$

Note that a counterexample for the converse is Dirichlet's function. □

**Theorem 11.2**

For all  $f: [a, b] \rightarrow \mathbb{R}$  we have

$$f \text{ continuous} \Rightarrow f \text{ RI}.$$

*Proof.* Note that  $[a, b]$  compact and  $f$  continuous implies that  $f$  is uniformly continuous. Then for all  $\varepsilon > 0$  there exists  $\delta > 0$  such that for all  $s, t \in [a, b]$  we have

$$0 < |s - t| < \delta \Rightarrow \text{osc}(f, [s, t]) < \frac{\varepsilon}{b - a}.$$

Then for all

$$\forall \Pi : ||\Pi|| < \delta \Rightarrow \sum_{i=1}^n \text{osc}(f, [t_{i-1}, t_i])(t_i - t_{i-1}) \leq \sum_{i=1}^n \frac{\varepsilon}{b - a} (t_i - t_{i-1}) \leq \varepsilon.$$

□

**Lemma 11.3**

Let  $f: [a, b] \rightarrow \mathbb{R}$  be bounded and such that  $f$  has only finitely many discontinuities. Then  $f$  is RI.

*Proof.* Let  $x_1, \dots, x_m$  enumerate discontinuity points of  $f$ . Pick  $\varepsilon > 0$ . Suppose without loss of generality  $||f|| \neq 0$ . Let  $\delta < \frac{\varepsilon}{m||f||}$ . Then

$$\text{osc}(f, [x_i - \delta, x_i + \delta] \cap [a, b]) \leq 2||f||.$$

Next, note that  $[a, b] \setminus \bigcup_{i=1}^m (x_i - \delta, x_i + \delta)$  is closed and thus compact. Then  $f$  is uniformly continuous on this set. Then there exists  $\delta' > 0$  such that for all  $[s, t] \subseteq$  this set we have

$$0 < |s - t| \leq \delta' \Rightarrow \text{osc}(f, [s, t]) \leq \frac{\varepsilon}{b - a}.$$

Now partition  $[a, b] \setminus \bigcup_{i=1}^m (x_i - \delta, x_i + \delta)$  into intervals of length  $\leq \delta'$ . Combine them with intervals  $[x_i - \delta, x_i + \delta]$ . Now take  $\Pi =$  set of endpoints of these intervals. Then

$$\sum_{i=1}^n \text{osc}(f, [t_{i-1}, t_i])(t_i - t_{i-1}) \leq m \cdot 2||f|| \cdot 2\delta + \frac{\varepsilon}{b - a} (b - a) \leq 5\varepsilon.$$

□

**Lemma 11.4**

Let  $f : [a, b] \rightarrow \mathbb{R}$  be bounded.

$f$  has no discontinuities of second kind  $\Rightarrow f$  RI.

*Proof.* Key idea:

$$\forall \eta > 0 : \left\{ x \in (a, b) : \text{diam} \left\{ \lim_{z \rightarrow x^+} f(z), \lim_{z \rightarrow x^-} f(z), f(x) \right\} > \eta \right\} \text{ is finite.}$$

□

**Example 11.5.**

$$f(x) = \begin{cases} \frac{1}{n+1} & x = q_n \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

**It gets worse:** Let

$$C := \left\{ \sum_{i \in \mathbb{N}} \frac{2\sigma_i}{3^{i+1}} : \{\sigma_i\}_{i \in \mathbb{N}} \in \{0, 1\}^{\mathbb{N}} \right\}$$

be Cantor's ternary set.

Then  $C = \bigcap_{n \in \mathbb{N}} C_n$  where

$$C_n = \left\{ \sum_{i=1}^n \frac{\sigma_i}{3^{i+1}} + [0, 3^{-n-1}] : \sigma_1, \dots, \sigma_n \in \{0, 1\} \right\}.$$

**Lemma 11.6**

The function

$$1_C(x) = \begin{cases} 1 & x \in C \\ 0 & x \notin C \end{cases}$$

is RI.

*Proof.* Let  $I_1, \dots, I_{2^n}$  be intervals constituting  $C_n$ . Define

$$J_k = \left\{ x \in [0, 1] : \text{dist}(x, I_k) < \frac{1}{3^{n+1}} \right\}.$$

Then

$$\text{length}(J_k) = \text{length}(I_k) + 2 \cdot \frac{1}{3^{n+1}} \leq \frac{1}{3^n}.$$

Take  $\Pi$  to be the endpoints of  $\{J_k\}_{k=1}^{2^n}$ . Then

$$\sum_{i=1}^m \text{osc}(f, [t_{i-1}, t_i]) |t_i - t_{i-1}| \leq \sum_{k=1}^{2^n} \text{length}(J_k) \leq 2^n \cdot \frac{1}{3^n} \xrightarrow{n \rightarrow \infty} 0.$$

□



## 12 4.18 Monday Week 4

### Characterizing Riemann integrability

Sufficient conditions for RI: continuity, finite number of discontinuities, no discontinuities of second kind.

Necessary condition for RI: boundedness.

#### Definition 12.1

A set  $A \subseteq \mathbb{R}$  is of **zero length** if

$$\forall \varepsilon > 0 \exists \{(a_i, b_i)\}_{i \in \mathbb{N}} \text{ intervals} : A \subseteq \bigcup_{i \in \mathbb{N}} (a_i, b_i) \quad \wedge \quad \sum_{i \in \mathbb{N}} (b_i - a_i) < \varepsilon.$$

#### Lemma 12.2

In the definition of zero length, closed intervals can be used.

*Proof.* If  $A \subseteq \bigcup_{i \in \mathbb{N}} [a_i, b_i]$ , let  $\tilde{a}_i = a_i - \varepsilon/2^i$  and  $\tilde{b}_i = b_i + \varepsilon/2^i$ . Then

$$A \subseteq \bigcup_{i \in \mathbb{N}} (\tilde{a}_i, \tilde{b}_i)$$

and

$$\sum_{i \in \mathbb{N}} (\tilde{b}_i - \tilde{a}_i) = \sum_{i \in \mathbb{N}} (b_i - a_i) + \sum_{i \in \mathbb{N}} 2 \cdot \frac{\varepsilon}{2^i} = \sum_{i \in \mathbb{N}} (b_i - a_i) + 4\varepsilon.$$

□

#### Lemma 12.3

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be bounded. Set

$$M_f(x) = \inf_{\delta > 0} \sup_{z: |z-x| < \delta} f(z)$$

and

$$m_f(x) = \sup_{\delta > 0} \inf_{z: |z-x| < \delta} f(z).$$

Then

1.  $\forall x \in \mathbb{R} : f \text{ continuous at } x \Leftrightarrow M_f(x) = m_f(x),$
2.  $\forall x \in \mathbb{R} \forall \delta : \max \{ \text{osc}(f, [x - \delta, x]), \text{osc}(f, [x - \delta, x]) \} \geq M_f(x) - m_f(x),$  and
3.  $\forall x \in \mathbb{R} : \lim_{\delta \rightarrow 0} \text{osc}(f, [x - \delta, x + \delta]) = M_f(x) - m_f(x).$

### Theorem 12.4: Lebesgue's characterization of Riemann integrability

Let  $f: [a, b] \rightarrow \mathbb{R}$  be bounded. Then

$$f \text{ RI} \Leftrightarrow \{x \in [a, b] : f \text{ discontinuous at } x\} \text{ is zero length.}$$

*Proof.*  $\Rightarrow$ : Let  $f : [a, b] \rightarrow \mathbb{R}$  be bounded and RI.

Pick  $\varepsilon > 0$ . Then RI implies

$$\forall n \in \mathbb{N} \exists \Pi = \{t_i^n\}_{i=1}^{m(n)} \text{ partition of } [a, b] : \sum_{i=1}^{m(n)} \text{osc}(f, [t_{i-1}^n, t_i^n])(t_i^n - t_{i-1}^n) < \varepsilon 4^{-n}.$$

Set  $I_n := \{i = 1, \dots, m(n) : \text{osc}(f, [t_{i-1}^n, t_i^n]) > 2^{-n}\}$ . Then

$$\sum_{i \in I_n} (t_i^n - t_{i-1}^n) \stackrel{\text{Markov's inequality}}{\leq} \sum_{i \in I_n} \frac{\text{osc}(f, [t_{i-1}^n, t_i^n])}{2^{-n}} (t_i^n - t_{i-1}^n) \leq 2^n \sum_{i=1}^{m(n)} \text{osc}(f, [t_{i-1}^n, t_i^n])(t_i^n - t_{i-1}^n) \leq 2^n \cdot \varepsilon 4^{-n} = \varepsilon 2^{-n}.$$

Now

$$\{x \in [a, b] : M_f(x) \neq m_f(x)\} \subseteq \bigcup_{n \geq 1} \bigcup_{i \in I_n} [t_{i-1}^n, t_i^n].$$

Then

$$\sum_{n \geq 1} \sum_{i \in I_n} (t_i^n - t_{i-1}^n) \leq \sum_{n \geq 1} \varepsilon 2^{-n} = \varepsilon.$$

Then  $f$  RI  $\Rightarrow \{x \in [a, b] : M_f(x) \neq m_f(x)\}$  is zero length.

$\Leftarrow$ : Let  $\varepsilon > 0$  and let  $\{J_i\}_{i \in \mathbb{N}}$  be open intervals such that

$$\{x \in [a, b] : M_f(x) \neq m_f(x)\} \subseteq \bigcup_{i \in \mathbb{N}} J_i \quad \wedge \quad \sum_{i \in \mathbb{N}} \text{length}(J_i) < \frac{\varepsilon}{2\varepsilon \|f\|} (f \neq 0).$$

Since  $M_f(x) = m_f(x) \Rightarrow x$  is continuous:

$$\forall x \in [a, b] : M_f(x) = m_f(x) \Rightarrow \exists \delta_x > 0 : \text{osc}(f, (x - \delta_x, x + \delta_x)) < \frac{\varepsilon}{b - a}.$$

Then intervals  $\{J_i\}_{i \in \mathbb{N}} \cup \{(x - \delta, x + \delta) : M_f(x) = m_f(x)\}$  cover  $[a, b]$ . Then by Heine-Borel theorem,

$$\exists m, n \in \mathbb{N} \exists x_0, \dots, x_m \in \{x \in [a, b] : M_f(x) = m_f(x)\} : [a, b] \subseteq \bigcup_{i=0}^m J_i \cup \bigcup_{j=0}^m (x_j - \delta_{x_j}, x_j + \delta_{x_j}).$$

Let  $\Pi = \{t_i\}_{i=1}^N$  be a partition containing of all endpoints of the intervals  $(x_j - \delta_{x_j}, x_j + \delta_{x_j})$ . Let  $k = \{i = 1, \dots, N : [t_{i-1}, t_i] \subseteq \bigcup_{j=1}^m (x_j - \delta_{x_j}, x_j + \delta_{x_j})\}$ . Then

$$\forall i \in K : \text{osc}(f, [t_{i-1}, t_i]) < \frac{\varepsilon}{b - a}$$

and

$$\sum_{i \notin K} \text{osc}(f, [t_{i-1}, t_i]) \leq 2\|f\| \cdot \sum_{i \notin K} (t_i - t_{i-1}) < 2\|f\| \sum_{i \in \mathbb{N}} \text{length}(J_i) < \varepsilon.$$

Then

$$\sum_{i=1}^n \text{osc}(f, [t_{i-1}, t_i])(t_i - t_{i-1}) \leq \sum_{i \in K} \text{osc}(f, [t_{i-1}, t_i])(t_i - t_{i-1}) + \sum_{i \notin K} \text{osc}(f, [t_{i-1}, t_i])(t_i - t_{i-1}) \leq \frac{\varepsilon}{b - a} (b - a) + \varepsilon = 2\varepsilon.$$

□

# 13 4.20 Wednesday Week 4

## Derivative vs. integral, FTC, ...

**Last time:**  $f$  RI  $\Leftrightarrow \{x \in [a, b] : f \text{ discontinuous at } x\}$  is of zero length.

### Corollary 13.1

$$f \text{ RI} \wedge \{x \in [a, b] : g(x) \neq f(x)\} \text{ is of zero length} \Rightarrow g \text{ RI} \wedge \int_a^b g(x) dx = \int_a^b f(x) dx.$$

**Today:** Newton / Leibniz FTC:

$$\frac{d}{dx} \int_a^x f(t) dt = f(x) \quad \wedge \quad \int_a^b \frac{d}{dx} f(t) dt = f(b) - f(a).$$

**Note.** These are not true without conditions.

### Lemma 13.2

Let  $a < b$  be reals and  $f : [a, b] \rightarrow \mathbb{R}$  be an RI function on  $[a, b]$ . Set  $F(x) = \int_a^x f(t) dt$ . Then  $F$  is Lipschitz continuous.

*Proof.* If  $a \leq x < y \leq b$  then additivity implies

$$F(y) - F(x) = \int_0^y f(t) dt - \int_0^x f(t) dt = \int_x^y f(t) dt.$$

Note that  $f$  RI  $\Rightarrow f$  bounded. Then

$$|F(y) - F(x)| = \left| \int_x^y f(t) dt \right| \leq \|f\| \cdot |y - x|.$$

□

**Example 13.3.**

$$|x| = \int_0^x t(1_{[0, \infty)} - 1_{(-\infty, 0)}) dt.$$

**Q:** Is every Lipschitz function a Riemann integral?

### Lemma 13.4

Let  $f$  be RI on  $[a, b]$ . Set  $F(x) = \int_a^b f(t) dt$ . Then

$$\forall x \in (a, b) : f \text{ continuous at } x \Rightarrow F'(x) \text{ exists} \wedge F'(x) = f(x).$$

*Proof.* Let  $y \in (x, b)$ . Then

$$F(y) - F(x) - f(x)(y - x) = \int_x^y (f(t) - f(x)) dt.$$

Then

$$\left| \frac{F(y) - F(x)}{y - x} - f(x) \right| \leq \sup_{t \in [x, y]} |f(y) - f(x)| \xrightarrow{y \rightarrow x^+} 0 \text{ by right continuity of } f.$$

Then  $F'^+(x) = f(x)$ . Similarly,  $F'^-(x) = f(x)$ . □

**Example 13.5.**

$$f(x) = 1_{1/(n+1), n \in \mathbb{N}}.$$

Note that  $F(x) = 0$  for all  $x \in \mathbb{R}$ .

### Corollary 13.6: Fundamental theorem of calculus I

Let  $f: [a, b] \rightarrow \mathbb{R}$  be continuous. Then

$$\forall x \in (a, b) : \frac{d}{dx} \int_a^x f(t) dt = f(x).$$

**Note.**

- The integral is an antiderivative / primitive function. Notation  $\int f(t) dt$ ;
- $\frac{d}{dt} \int_x^b f(t) dt = -f(x)$ ;
- $\frac{d}{dx} \int_{g(x)}^{h(x)} f(t) dt = f(h(x))h'(x) - f(g(x))g'(x)$ .

### Theorem 13.7

Let  $f: [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . (Choose  $f'(a), f'(b)$ ) arbitrarily. Then

$$f \text{ RI} \Rightarrow \int_a^b f'(t) dt = f(b) - f(a).$$

*Proof.* Let  $\varepsilon > 0$ . Then  $f'$  RI implies

$$\exists \delta > 0 \forall \Pi : \|\Pi\| < \varepsilon \Rightarrow \left| R(f', \Pi) - \int_a^b f'(t) dt \right| < \varepsilon.$$

Pick  $n \in \mathbb{N}$  such that  $n\delta > (b - a)$ . Set  $t_i := a + \frac{i}{n}(b - a)$  where  $i = 0, \dots, n$ .

Then, by the mean value theorem, for all  $i = 1, \dots, n$  we have

$$\exists t_i^* \in (t_{i-1}, t_i) : f(t_i) - f(t_{i-1}) = f'(t_i^*)(t_i - t_{i-1}).$$

Let  $\Pi := (\{t_i\}, \{t_i^*\})$ . Then

$$f(b) - f(a) = \sum_{i=1}^n (f(t_i) - f(t_{i-1})) = \sum_{i=1}^n f'(t_i^*)(t_i - t_{i-1}) = R(f', \Pi).$$

Then

$$\left| f(b) - f(a) - \int_a^b f'(t) dt \right| < \varepsilon.$$

□

### Volterra's example

$\exists F: [0, 1] \rightarrow \mathbb{R}$  continuous :  $F'(x)$  exists for all  $x \in [0, 1] \wedge F'$  bounded  $\wedge F'$  is not RI.

This is a major deficiency in Riemann's theory that led Lebesgue to the formulation of the Lebesgue integral.

#### Corollary 13.8: Integration by parts

Let  $f, g: [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Then

$$f'g \text{ RI} \wedge fg' \text{ RI} \Rightarrow \int_a^b f'(x)g(x) dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x) dx.$$

*Proof.* Note that

$$f'g \text{ RI} \wedge g'f \text{ RI} \Rightarrow (fg)' \text{ RI}.$$

Then

$$f(b)g(b) - f(a)g(a) = \int_a^b (fg)'(x) dx = \int_a^b f'(x)g(x) dx + \int_a^b f(x)g'(x) dx.$$

□

#### Corollary 13.9

Let  $f: [c, d] \rightarrow \mathbb{R}$  and  $\varphi: [a, b] \rightarrow \mathbb{R}$  be functions. Assume

1.  $\varphi$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ ,
2.  $f$  is continuous on  $[c, d]$ , and
3.  $(f \circ \varphi)\varphi'$  is RI on  $[a, b]$ .

Then

$$\int_{\varphi(a)}^{\varphi(b)} f(x) dx = \int_a^b (f \circ \varphi)(t)\varphi'(t) dt.$$

*Proof.* Note that

$$F(x) = \int_a^x f(t) dt \xrightarrow{\text{FTC I}} F'(x) = f(x).$$

Then

$$\int_{\varphi(a)}^{\varphi(b)} f(t) dt \xrightarrow{\text{FTC II}} F(\varphi(b)) - F(\varphi(a)) \xrightarrow{\text{FTC II}} \int_a^b \frac{d}{dx}(F \circ \varphi)(x) dx = \int_a^b (f \circ \varphi)(x)\varphi'(x) dx.$$

□

## 14 4.25 Monday Week 5

### Taylor's theorem

Last time: FTC I:

$$f \text{ continuous} \Rightarrow F(x) = \int_a^x f(t) dt \text{ differentiable} \wedge F'(x) = f(x).$$

FTC II:

$$F \text{ continuous on } [a, b] \wedge F' \text{ exists on } (a, b) \wedge F' \text{ RI} \Rightarrow F(b) - F(a) = \int_a^b F'(x) dx.$$

Cantor's function ("Devil's staircase"):

$$x \in \sum_{i=0}^n \frac{2\sigma_i}{3^{in}} + [0, 3^{-n-1}] \mapsto F(x) = \sum_{i=0}^n \frac{\sigma_i}{2^{i+1}}.$$

This is simply not an integral of a derivative (not Lipschitz but Holder continuous with a coefficient less than 1).  $F'$  exists at every point excluding the Cantor set, which is 0.

Consequences of the FTC:

- Substitution rule
- Integration by parts

### Theorem 14.1: Taylor's theorem with remainder

Let  $f: (a, b) \rightarrow \mathbb{R}$  be  $(n+1)$ -times differentiable with  $f^{(n+1)}$  Riemann integrable. Then

$$\forall x, x_0 \in (a, b): f(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{1}{n!} \int_{x_0}^x f^{(n+1)}(z) (x - z)^n dz.$$

*Proof.*  $n = 0$ : FTC:  $f'$  exists and  $f'$  is RI by assumption.

$$f(x) = f(x_0) + \int_{x_0}^x f'(z) dz.$$

$n \Rightarrow n+1$ : Assume  $f^{(n+2)}$  exists and is RI. Then  $f^{(n+1)}$  is continuous and therefore RI. Then

$$\begin{aligned} \frac{1}{n!} \int f^{(n+1)}(z) (x - z)^n dz &= \frac{1}{n!} \int f^{(n+1)}(z) \frac{d}{dz} \left( -\frac{(x - z)^{n+1}}{n+1} \right) dz \\ &\stackrel{IBP}{=} \frac{1}{n!} f^{(n+1)}(z) \left( -\frac{(x - z)^{n+1}}{n+1} \right) \Big|_{x_0}^x - \int_{x_0}^x f^{(n+2)}(z) \left( -\frac{(x - z)^{n+1}}{n+1} \right) dz \\ &= \frac{f^{(n+1)}(x_0)}{(n+1)!} (x - x_0)^n + \int_{x_0}^x \frac{f^{(n+2)}(z)}{(n+1)!} (x - z)^{n+1} dz. \end{aligned}$$

Then

$$f(x) - P_n(x) \stackrel{(n)}{=} \text{LHS} = P_{n+1}(x) - P_n(x) + \int_{x_0}^x \frac{f^{(n+2)}(z)}{n+1} (x - z)^{n+1} dz.$$

□

## Stieljes integral

**Idea:** Measure length of intervals using other functions than just  $g(x) = x$ .

### Definition 14.2

Let  $\Pi = (\{t_i\}_{i=0}^n, \{t_i^*\}_{i=1}^n)$  be a marked partition of  $[a, b]$ . For  $f, g: [a, b] \rightarrow \mathbb{R}$ ,

$$S(f, dg, \Pi) := \sum_{i=1}^n f(t_i^*)[g(t_i) - g(t_{i-1})]$$

is the **Riemann-Stieljes sum** of  $f$  with respect to  $g$ .

### Definition 14.3

Let  $f, g: [a, b] \rightarrow \mathbb{R}$ . We say that “ $f$  is Stieljes integrable with respect to  $g$  on  $[a, b]$ ” if

$$\exists L \in \mathbb{R} \forall \varepsilon > 0 \exists \delta > 0 \forall \Pi \text{ marked partition of } [a, b] : \|\Pi\| < \delta \Rightarrow |S(f, dg, \Pi) - L| < \varepsilon$$

or in short,

$$\lim_{\|\Pi\| \rightarrow 0} S(f, dg, \Pi) \text{ exists.}$$

**Note.**

- Such an  $L$  is unique (if it exists) and so we denote it  $\int_a^b f(x) dg(x) = \int_a^b f dg$ .
- For  $g(x) = x$ , we get the Riemann integral.
- If “length” of  $[t, s]$  is given by  $g(s) - g(t)$ , then  $\int f dg$  corresponds to “area” with lengths in  $\mathbb{R}$  measured using  $g$ .
- In probability:  $g$  = cumulative distribution function of a random variable  $x$  ( $g(t) := P(x \leq t)$ ) then

$$\int f(x) dg(x) = E(f(X)) = \text{expectation of } f(X).$$

- In economics:  $f(t)$  = price of stock at time  $t$ ,  $g(t)$  = current holding of the stock then

$$\int_a^b f dg = \text{total money earned in time interval } [a, b].$$

This shows  $g$  may not be monotone.

## 15 4.27 Wednesday Week 5

### Stieljes integral

Last time:

$$S(f, dg, \Pi) = \sum_{i=1}^n f(t_i^*)(g(t_i) - g(t_{i-1}))$$

$$\int_a^b f dg := \lim_{\|\Pi\| \rightarrow 0} S(f, dg, \Pi) \text{ wherever it exists.}$$

We call this the Stieljes integral in the Riemann sense.

Notation:  $RS(g, [a, b]) := \left\{ f : [a, b] \rightarrow \mathbb{R} : \int_a^b f dg \text{ exists} \right\}$

#### Lemma 15.1: Linearity

Let  $h : [a, b] \rightarrow \mathbb{R}$  be given. Then

$$\forall f, g \in RS(h, [a, b]) \forall \alpha, \beta \in \mathbb{R} : \alpha f + \beta g \in RS(h, [a, b]) \wedge \int_a^b (\alpha f + \beta g) dh = \alpha \int_a^b f dh + \beta \int_a^b g dh.$$

#### Lemma 15.2: Additivity

Let  $g : [a, b] \rightarrow \mathbb{R}$  be given. Then

$$\forall f \in RS(g, [a, b]) \forall c \in (a, b) : f \in RS(g, [a, b]) \wedge f \in RS(g, [c, b]) \wedge \int_a^b f dg = \int_a^c f dg + \int_c^b f dg.$$

#### Lemma 15.3

Let  $f \in RS(g, [a, b])$ . Then

$$\{x \in [a, b] : f \text{ discontinuous at } x\} \cap \{x \in [a, b] : g \text{ discontinuous at } x\} = \emptyset.$$

**Note.**  $f \in RS(g, [a, b])$  need not be bounded on intervals where  $g$  is constant.

#### Definition 15.4

We say  $f$  is **generalized Stieljes integrable** with respect to  $g$  if

$$\exists L \in \mathbb{R} \forall \varepsilon > 0 \exists \delta > 0 \exists \Pi_\varepsilon \text{ unmarked partition } \forall \Pi \text{ marked partition :}$$

$$\|\Pi\| < \delta \wedge \Pi_\varepsilon \subseteq \Pi \Rightarrow |S(f, dg, \Pi) - L| < \varepsilon.$$

### Criteria for Stieljes integrability



**Theorem 15.5: Reduction to Riemann integral**

Let  $f, g: [a, b] \rightarrow \mathbb{R}$  be such that

1.  $f$  is Riemann integrable and
2.  $g$  is continuous on  $[a, b]$ , differentiable on  $(a, b)$  with  $g'$  Riemann integrable.

Then

$$f \in RS(g, [a, b]) \quad \wedge \quad \int_a^b f \, dg = \int_a^b f(x)g'(x) \, dx.$$

*Proof.* Let  $\Pi = (\{t_i\}_{i=0}^n, \{t_i^*\}_{i=1}^n)$  be a marked partition of  $[a, b]$ . For each  $i = 1, \dots, n$ , let  $\tilde{t}_i$  be a point such that  $g(t_i) - g(t_{i-1}) = g'(\tilde{t}_i)(t_i - t_{i-1})$  given by the mean value theorem. Let  $\tilde{\Pi} = (\{t_i\}_{i=0}^n, \{\tilde{t}_i\}_{i=1}^n)$ . Then

$$\begin{aligned} S(f, dg, \Pi) - R(fg', \Pi') &= \sum_{i=1}^n f(t_i^*)(g(t_i) - g(t_{i-1})) - \sum_{i=1}^n f(\tilde{t}_i)g'(\tilde{t}_i)(t_i - t_{i-1}) \\ &= \sum_{i=1}^n [f(t_i^*) - f(\tilde{t}_i)]g'(\tilde{t}_i)(t_i - t_{i-1}). \end{aligned}$$

Note that

$$|\text{RHS}| \leq \|g'\| \sum_{i=1}^n \text{osc}(f, [t_{i-1}, t_i])(t_i - t_{i-1}) \xrightarrow[\|\Pi\| \rightarrow 0]{f g' \text{ RI}} 0.$$

Hence

$$\lim_{\|\Pi\| \rightarrow 0} S(f, g, \Pi) = \lim_{\|\Pi\| \rightarrow 0} R(fg', \Pi) = \int_a^b f g' \, dx.$$

□

**Theorem 15.6: BV condition**

Let  $f, g \in [a, b] \rightarrow \mathbb{R}$  be such that

1.  $f$  is continuous and
2.  $g$  is of bounded variation ( $V(g, [a, b]) < \infty$ ).

Then  $f \in RS(g, [a, b])$  and

$$\left| \int_a^b f \, dg \right| \leq \|f\| V(g, [a, b]).$$

*Proof.* Let  $\Pi = \{t_i\}_{i=0}^n$ ,  $\tilde{\Pi} = \{s_i\}_{i=0}^m$  be unarked partitions of  $[a, b]$ . Assume  $\Pi \subseteq \tilde{\Pi}$  and the set  $J_i = \{j = 1, \dots, m : [s_{j-1}, s_j] \subseteq [t_{i-1}, t_i]\}$ . Now choose any marked points  $t_i^* \in [t_{i-1}, t_i]$  and  $s_j^* \in [s_{j-1}, s_j]$ . Then

$$\begin{aligned} S(f, dg, \Pi) - S(f, dg, \tilde{\Pi}) &= \sum_{i=1}^n f(t_i^*)(g(t_i) - g(t_{i-1})) - \sum_{j=1}^m f(s_j^*)(g(s_j) - g(s_{j-1})) \\ &= \sum_{i=1}^n \sum_{j \in J_i} [f(t_i^*) - f(s_j^*)][g(s_j) - g(s_{j-1})] \\ &\leq \sum_{i=1}^n \sum_{j \in J_i} \text{osc}(f, [t_{i-1}, t_i]) |g(s_j) - g(s_{j-1})| \\ &\leq \sum_{i=1}^n \text{osc}(f, [t_{i-1}, t_i]) V(g, [t_{i-1}, t_i]). \end{aligned}$$

If  $f$  is continuous then  $f$  is uniformly continuous. Then

$$\forall \varepsilon > 0 \exists \delta > 0 : ||\Pi|| < \delta \Rightarrow \text{osc}(f, [t_{i-1}, t_i]) < \varepsilon.$$

Then  $|RHS| \leq \varepsilon V(g, [a, b])$ . Then for any marked partitions  $\Pi, \Pi'$  of  $[a, b]$  we have

$$||\Pi||, ||\Pi'|| < \delta \Rightarrow |S(f, dg, \Pi) - S(f, dg, \Pi')| \leq 2\varepsilon V(g, [a, b]).$$

□

#### Theorem 15.7: Loéve-Young condition, 1936

Let  $f, g : [a, b] \rightarrow \mathbb{R}$  be such that

$$\exists \alpha, \beta \in (0, 1] : f \text{ is } \alpha\text{-Hölder} \wedge g \text{ is } \beta\text{-Hölder} \wedge \alpha + \beta > 1.$$

Then  $f \in RS(g, [a, b])$ .

**Note.**  $f$  is  $\alpha$ -Hölder if

$$\exists C > 0 \forall x, y \in [a, b] : |f(x) - f(y)| \leq C |x - y|^\alpha.$$