

MATH 131BH (Real Analysis)

April 1, 2022

- 1 **3.28 Monday Week 1: Intro to the course. Review of material covered in 131AH: foundations (definition and constructions of naturals and reals), metric space convergence, continuity.**
- 2 **3.30 Wednesday Week 1: Limit of a function: definition and alternative formulations via images of balls and sequential characterization. Limit on a set, left and right limits for functions on \mathbb{R} . Discontinuities of first and second kind. Monotone functions have no discontinuities of second kind.**

Limits of functions

Recall: $f: X \rightarrow Y$ is said to be **continuous at $x \in X$** if

$$\forall \varepsilon > 0 \exists \delta > 0 \forall z \in X : \rho_X(x, z) < \delta \Rightarrow \rho_Y(f(z), f(x)) < \varepsilon.$$

Alternatives:

- $f(B_X(x, \delta)) \subseteq B_Y(f(x), \varepsilon)$;
- $\forall \{x_n\}_{n \in \mathbb{N}} \in X^{\mathbb{N}} : x_n \rightarrow x \Rightarrow f(x_n) \rightarrow f(x)$.

A function $f: X \rightarrow Y$ is **continuous** if

$$\forall x \in X : f \text{ is continuous at } x,$$

or, alternatively,

$$\forall O \subseteq Y \text{ open} : f^{-1}(O) \text{ open}.$$

Definition 2.1

A function $f: X \rightarrow Y$ **has limit $y \in Y$ at $x \in X$** , notation $\lim_{z \rightarrow x} f(z) = y$, if

$$\forall \varepsilon \exists \delta > 0 \forall z \in X : 0 < \rho_X(x, z) < \delta \Rightarrow \rho_Y(f(z), y) < \varepsilon.$$

Alternatives:

- $f(B_X(x, \delta) \setminus \{x\}) \subseteq B_Y(y, \varepsilon)$;
- $\forall \{x_n\}_{n \in \mathbb{N}} \in X^{\mathbb{N}} : (\forall n \in \mathbb{N} : x_n \neq x) \wedge x_n \rightarrow x \Rightarrow f(x_n) \rightarrow y$;
- $g(z) := \begin{cases} f(z) & z \neq x \\ y & z = x \end{cases}$ is continuous at x .

Definition 2.2

f has a **removable discontinuity** at x if $\lim_{z \rightarrow x} f(z)$ exists but $\neq f(x)$.

Definition 2.3

Let $A \subseteq X$ be nonempty, $x \in \overline{A}$ be not an isolated point. Then $\lim_{z \rightarrow x} f(z) = \lim_{z \rightarrow x} f_A(z)$ where f_A is the restriction of f to A .

Definition 2.4

For $f: \mathbb{R} \rightarrow \mathbb{R}$, let $x \in \overline{\text{Dom}(f)}$ be such that $\text{Dom}(f) \cap (x, \infty) \neq \emptyset$ and $\text{Dom}(f) \cap (-\infty, x) \neq \emptyset$. Then $\lim_{z \rightarrow x^+} f(z) := \lim_{z \rightarrow x, z \in \text{Dom}(f) \cap (x, \infty)} f(z) \wedge \lim_{z \rightarrow x^-} f(z) := \lim_{z \rightarrow x, z \in \text{Dom}(f) \cap (-\infty, x)} f(z)$ are the **right / left limits of f at x** .

Alternative notation: $f(x^+)$, $f(x^-)$.

Example 2.5.

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases} \quad (2.1)$$

has no right or left limits.

Example 2.6.

$$f(x) = \begin{cases} \frac{1}{n+1} & x = q_n \text{ where } \{q_n\}_{n \in \mathbb{N}} \text{ enumerates } \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases} \quad (2.2)$$

Then $\forall x \notin \mathbb{Q} : \lim_{z \rightarrow x} f(z) = 0$ so f is continuous on $\mathbb{R} \setminus \mathbb{Q}$, and $\forall x \in \mathbb{Q} : \lim_{z \rightarrow x} f(z) = 0$ but f is not continuous at x .

Lemma 2.7

$$\forall r > 0 \forall \varepsilon > 0 : \{x \in \mathbb{R} : |x| < r \wedge |f(x)| > \varepsilon\} \text{ finite} \Rightarrow \forall x \in \mathbb{R} : \lim_{z \rightarrow x} f(z) = 0.$$

Definition 2.8

A function $f: \mathbb{R} \rightarrow \mathbb{R}$ has a **discontinuity of**

- **first kind** at x if $f(x^+)$ and $f(x^-)$ exist but are not both equal to $f(x)$;
- **second kind** at x if one or both of $f(x^+)$ and $f(x^-)$ don't exist.

Example 2.9.

$$f(x) := \begin{cases} (-1)^n & x = \frac{1}{n+1}, n \in \mathbb{N} \\ \text{linear} & x \in (0, \infty) \setminus \left\{ \frac{1}{n+1} : n \in \mathbb{N} \right\} \\ 0 & x \leq 0. \end{cases} \quad (2.3)$$

This function has a discontinuity of second kind at 0.

Lemma 2.10

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ ($\text{Dom}(f) = \mathbb{R}$) be monotone. Then $\forall x \in \mathbb{R} : f(x^+), f(x^-)$ exist and so f has no discontinuities of second kind.

Proof. Let $x \in \mathbb{R}$ and assume f is nondecreasing. We claim that $\lim_{z \rightarrow x^+} f(z) = \inf \{f(z) : z > x\} =: L$.

Indeed, $\forall z > x : f(z) \geq f(x)$, so $L \geq f(x)$ and so $L \in \mathbb{R}$. Then $(\forall z > x : L \leq f(z)) \wedge (\forall \varepsilon > 0 \exists z_\varepsilon > x : f(z_\varepsilon) < L + \varepsilon)$. Let $\delta := z_\varepsilon - x$. Then $\forall z \in (x, x + \delta) : f(z) \leq f(z_\varepsilon) < L + \varepsilon$. Then $\forall z \in (x, x + \delta) : L \leq f(z) < L + \varepsilon$ and therefore $|f(z) - L| < \varepsilon$. Then $\lim_{z \rightarrow x^+} f(z) = L$. \square

3 3.31 Thursday Week 1: Monotone functions have only countably many discontinuities. Functions of bounded variation. Jordan decomposition theorem. Comments on uniqueness. Rectifiability of curves. Limsup and liminf of a function.

Limits of functions

Last time we showed that monotone functions have no discontinuities of second time.

Lemma 3.1

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be monotone. Then $\{x \in \mathbb{R} : f(x^+) \neq f(x^-)\}$ is countable.

Proof. Pick $k, - \in \mathbb{N}$ and let $A_{m,k} := \{x \in [-m, m] : |f(x^+) - f(x^-)| > \frac{1}{k+1}\}$. We claim that $A_{m,k}$ is finite.

Let $x_0 < x_1 < \dots < x_n$ be such that $\forall i \leq n : x_i \in A_{m,k}$. Assume (without loss of generality) that f is non-decreasing. Then

$$\begin{aligned} f(m+1) &\geq f(x_n^+) = f(x_0^+) + \sum_{i=1}^n (f(x_i^+) - f(x_{i-1}^+)) \\ &\geq f(m-1) + \sum_{i=1}^n (f(x_i^+) - f(x_i^-)) \\ &\geq f(-m+1) + \frac{n}{k+1}. \end{aligned} \quad (3.4)$$

Then $n \leq (k+1)$. Since $\{x \in \mathbb{R} : f(x^+) \neq f(x^-)\} = \bigcup_{k \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} A_{m,k}$, we are done. \square

Q: Can these be generalized to other functions?

Definition 3.2

A **partition** Π of an interval $[a, b]$ is a sequence $\{t_i\}_{i=0}^n$ such that

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b.$$

Definition 3.3

Given $f: [a, b] \rightarrow \mathbb{R}$, its **total variation** on $[a, b]$

$$V(f, [a, b]) := \sup_{\Pi = \{t_i\}_{i=0}^n} \sum_{i=1}^n |f(t_i) - f(t_{i-1})|$$

where the supremum is over the partitions of $[a, b]$.

Definition 3.4

f is said to be of **bounded variation** on $[a, b]$ if $V(f, [a, b]) < \infty$.

Lemma 3.5

If $f: \mathbb{R} \rightarrow \mathbb{R}$ is of bounded variation on $[-m, m]$ for all $m \in \mathbb{N}$, then f has only discontinuities of first kind and the set $\{x \in \mathbb{R} : f(x^+) \neq f(x^-)\}$ is countable.

Theorem 3.6: Jordan decomposition (1881)

Let $f: [a, b] \rightarrow \mathbb{R}$ obey $V(f, [a, b]) < \infty$. Then $\exists h, g: [a, b] \rightarrow \mathbb{R}$ nondecreasing such that $\forall t \in [a, b] : f(t) = h(t) - g(t)$.

Proof. Define $h(t) := V(f, [a, t])$ and $g(t) := V(f, [a, t]) - f(t)$. Note that $h(t) - g(t) = f(t)$.

We need to show that h and g are nondecreasing.

Let $a \leq t < t' \leq b$. Then for any partition Π of $[a, t]$, $\Pi' = \Pi \cup \{t'\}$ is a partition of $[a, t']$. Then

$$V(f, [0, t']) \geq \sum_{i=1}^m |f(t_i) - f(t_{i-1})| + |f(t') - f(t)|. \quad (3.5)$$

Taking supremum over Π gives

$$V(f, [a, t']) \geq V(f, [a, t]) + |f(t') - f(t)|. \quad (3.6)$$

Note that $|f(t') - f(t)| \geq 0$ and $|f(t') - f(t)| \geq f(t') - f(t)$. Then $h(t') \geq h(t)$ and $g(t') \geq g(t)$. \square

The representation of $f = h - g$ is called a Jordan decomposition. This is not unique because a nondecreasing function can be added to both h and g .

However, there is a minimal decomposition $f = h_0 - g_0$ such that $g_0(a) = 0$ such that for any other Jordan decomposition $f = h - g$ we have $h - h_0, g - g_0$ nondecreasing. This is then *the* Jordan decomposition.

Rectifiability of curves

Definition 3.7

Let (X, ρ) be a metric space. A curve C is $\text{Ran}(f)$ for an $f: \mathbb{R} \rightarrow X$ continuous such that $\text{Dom}(f)$ is nonempty and connected. This f is called a **parametrization** of C .

Definition 3.8

Assuming $\text{Dom}(f) = [a, b]$, the **length of C** is

$$\ell(C) := \sup_{\Pi = \{t_i\}_{i=1}^n} \sum_{i=1}^n \rho(f(t_{i-1}), f(t_i)).$$

Definition 3.9

A curve is **rectifiable** if $\ell(C) < \infty$.

Definition 3.10

Let (X, ρ) be a metric space and $f: X \rightarrow \mathbb{R}$. Then

$$\limsup_{z \rightarrow x} f(z) := \inf_{\delta > 0} \sup_{z \in B(x, \delta) \setminus \{x\}} f(z)$$

and

$$\liminf_{z \rightarrow x} f(z) := \sup_{\delta > 0} \inf_{z \in B(x, \delta) \setminus \{x\}} f(z).$$

Lemma 3.11

$$\lim_{z \rightarrow x} f(z) \text{ exists in } \mathbb{R} \Leftrightarrow \limsup_{z \rightarrow x} f(z) = \liminf_{z \rightarrow x} f(z) \in \mathbb{R}.$$

4 4.1 Friday Week 1: Discussion

Definition 4.1

Let $(X, \rho_X), (Y, \rho_Y)$ be metric spaces, $E \subseteq X$, $f: E \rightarrow Y$, and $x \in \bar{E}$. Then $\lim_{t \rightarrow x} f(t) = \alpha$ is defined by

$$\forall \varepsilon > 0 \exists \delta > 0 : \forall x \in E \wedge 0 < \rho_X(t, x) < \delta \Rightarrow \rho_Y(f(t), \alpha) < \varepsilon.$$

Equivalently,

$$\forall \{t_n\}_{n \in \mathbb{N}} \in (E \setminus \{x\})^{\mathbb{N}} : t_n \rightarrow x \Rightarrow f(t_n) \rightarrow \alpha.$$

Note. f need not be defined at x .

Remark.

$$\limsup_{t \rightarrow x} f(t) := \inf_{\delta > 0} \sup_{t \in B(x, \delta) \setminus \{x\}} f(t) = \lim_{\delta \rightarrow 0} \sup_{t \in B(x, \delta) \setminus \{x\}} f(t).$$

\liminf is similarly defined.

Remark.

$$\limsup = \liminf \Rightarrow \lim \text{ exists.}$$

Discontinuities

Definition 4.2

Let $f: (a, b) \rightarrow \mathbb{R}$ be not continuous at x . Then f has a **discontinuity of first kind** at x if $f(x+)$ and $f(x-)$ both exist. Otherwise it is of **second kind**.

Remark. Discontinuities of the first kind are also known as **simple discontinuities**. The cases include

- $f(x+) = f(x-) \neq f(x)$: **removable discontinuity**, and
- $f(x+) \neq f(x-)$: **jump discontinuity**.

Example 4.3.

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases} \quad (4.7)$$

has a discontinuity of second kind at 0.

Example 4.4.

$$f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \quad (4.8)$$

is continuous on $\mathbb{R} \setminus \mathbb{Q}$ and has discontinuities of first kind (removable) at every point in \mathbb{Q} .

Recall: A monotone function has no discontinuity of second kind and has at most countably many discontinuities of first kind. One can deduce this from the fact that the real line is a union of countably many open intervals (indexed by rationals).

Definition 4.5

A function $f: (a, b) \rightarrow \mathbb{R}$ is convex if

$$\forall x, y \in (a, b) : x \leq y \Rightarrow (\forall \lambda \in [0, 1] : f(\lambda x + (1 - \lambda)y)) \leq \lambda f(x) + (1 - \lambda)f(y).$$

In words, this means that for any interval, the secant line is above the graph.