MATH 131BH (Real Analysis)

March 30, 2022

1 3.28 Monday Week 1

2 3.30 Wednesday Week 1

Recall: $f: X \to Y$ is said to be **continuous at** $x \in X$ if $\forall\, \varepsilon > 0\, \exists\, \delta > 0\, \forall\, z \in x : \rho_X(x, z) < \delta \Rightarrow \rho(f(z), f(x)) < \varepsilon$.

Alternatives:

- $f(B_X(x, \delta)) \subseteq B_Y(f(x), \varepsilon)$

Definition 2.1

A function $f: X \to Y$ **has limit** $y \in Y$ **at** $x \in X$, notation $\lim_{z \to x} f(z) = y$, if

$$\forall\, \varepsilon\, \exists\, \delta > 0\, \forall\, z \in X : 0 < \rho_X(x, z) < \delta \Rightarrow \rho_Y(f(z), y) < \varepsilon.$$

Alternatives:

- $f(B_X(x, \delta) \setminus \{x\}) \subseteq B_Y(y, \varepsilon))$;
- $\forall\, \{x_n\}_{n \in \mathbb{N}} \in X^{\mathbb{N}} : (\forall\, n \in \mathbb{N} : x_n \neq x) \wedge x_n \to x \Rightarrow f(x_n) \to y$;
- $g(z) := \begin{cases} f(z) & z \neq x \\ y & z = x \end{cases}$ is continuous at x.

Definition 2.2

f has a **removable discontinuity** at x if $\lim_{z \to x} f(z)$ exists but $\neq f(x)$.

Definition 2.3

Let $A \subseteq X$ be nonempty, $x \in \overline{A}$ be not an isolated point. Then $\lim_{z \to x} f(z) = \lim_{z \to x} f_A(z)$ where f_A is the restriction of f to A.

Definition 2.4

For $f: \mathbb{R} \to \mathbb{R}$, let $x \in \overline{\mathrm{Dom}(f)}$ be such that $\mathrm{Dom}(f) \cap (x, \infty) \neq \varnothing$ and $\mathrm{Dom}(f) \cap (-\infty, x) \neq \varnothing$. Then $\lim_{z \to x^+} f(z) := \lim_{z \to x, z \in \mathrm{Dom}(f) \cap (x, \infty)} f(z) \wedge \lim_{z \to x^-} f(z) := \lim_{z \to x, z \in \mathrm{Dom}(f) \cap (-\infty, x)} f(z)$ are the **right / left limits of** f **at** x.

Alternate notation: $f(x^+)$, $f(x^-)$.

Example 2.5.

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases} \tag{2.1}$$

has no right or left limits.

Example 2.6.

$$f(x) = \begin{cases} \frac{1}{n+1} & x = q_n \text{ where } \{q_n\}_{n \in \mathbb{N}} \text{ enumerates } \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases} \tag{2.2}$$

Then $\forall\, x \notin \mathbb{Q} : \lim_{z \to x} f(z) = 0$ so f is continuous on $\mathbb{R} \setminus \mathbb{Q}$, and $\forall\, x \in \mathbb{Q} : \lim_{z \to x} f(z) = 0$ but f is not continuous at x.

Lemma 2.7

$$\forall\, r > 0\, \forall\, \varepsilon > 0 : \left\{x \in \mathbb{R} : |x| < r \wedge \left|f(x)\right| > \varepsilon\right\} \text{ finite} \Rightarrow \forall\, x \in \mathbb{R} : \lim_{z \to x} f(z) = 0.$$

Definition 2.8

A function $f : \mathbb{R} \to \mathbb{R}$ has a **discontinuity of**

- **first kind** at x if $f(x^+)$ and $f(x^-)$ exist but are not both equal to $f(x)$;
- **second kind** at x if one or both of $f(x^+)$ and $f(x^-)$ don't exist.

Example 2.9.

$$f(x) := \begin{cases} (-1)^n & x = \frac{1}{n+1}, n \in \mathbb{N} \\ \text{linear} & x \in (0, \infty) \setminus \left\{\frac{1}{n+1} : n \in \mathbb{N}\right\} \\ 0 & x \leq 0. \end{cases} \tag{2.3}$$

This function has a discontinuity of second kind at 0.

Lemma 2.10

Let $f : \mathbb{R} \to \mathbb{R}$ ($\mathrm{Dom}(f) = \mathbb{R}$) be monotone. Then $\forall\, x \in \mathbb{R} : f(x^+), f(x^-)$ exist and so f has no discontinuities of second kind.

Proof. Let $x \in \mathbb{R}$ and assume f is nondecreasing. We claim that $\lim_{z \to x^+} f(z) = \inf\left\{f(z) : z > x\right\} =: L$.

Indeed, $\forall\, z > x : f(z) \geq f(x)$, so $L \geq f(x)$ and so $L \in \mathbb{R}$. Then $(\forall\, z > x : L \leq f(z)) \wedge (\forall\, \varepsilon > 0\, \exists\, z_\varepsilon > x : f(z_\varepsilon) < L + \varepsilon)$. Let $\delta := z_\varepsilon - x$. Then $\forall\, z \in (x, x + \delta) : f(z) \leq f(z_\varepsilon) < L + \varepsilon$. Then $\forall\, z \in (x, x + \delta) : L \leq f(z) < L + \varepsilon$ and therefore $\left|f(z) - L\right| < \varepsilon$. $\square$