MATH 131BH (Real Analysis)

April 4, 2022

1 3.28 Monday Week 1: Intro to the course. Review of material covered in 131AH: foundations (definition and constructions of naturals and reals), metric space convergence, continuity.

2 3.30 Wednesday Week 1: Limit of a function: definition and alternative formulations via images of balls and sequential characterization. Limit on a set, left and right limits for functions on $\mathbb{R}$. Discontinuities of first and second kind. Monotone functions have no discontinuities of second kind.

<u>**Limits of functions**</u>

Recall: $f: X \to Y$ is said to be **continuous at $x \in X$** if

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall z \in x : \rho_X(x, z) < \delta \Rightarrow \rho(f(z), f(x)) < \varepsilon.$$

Alternatives:

- $f(B_X(x, \delta)) \subseteq B_Y(f(x), \varepsilon)$;
- $\forall \{x_n\}_{n \in \mathbb{N}} \in X^{\mathbb{N}} : x_n \to x \Rightarrow f(x_n) \to f(x)$.

A function $f: X \to Y$ is **continuous** if

$$\forall x \in X : f \text{ is continuous at } x,$$

or, alternatively,

$$\forall O \subseteq Y \text{ open} : f^{-1}(O) \text{ open}.$$

Definition 2.1

A function $f: X \to Y$ **has limit $y \in Y$ at $x \in X$**, notation $\lim_{z \to x} f(z) = y$, if

$$\forall \varepsilon \, \exists \delta > 0 \, \forall z \in X : 0 < \rho_X(x, z) < \delta \Rightarrow \rho_Y(f(z), y) < \varepsilon.$$

Alternatives:

- $f(B_X(x, \delta) \setminus \{x\}) \subseteq B_Y(y, \varepsilon))$;
- $\forall \{x_n\}_{n \in \mathbb{N}} \in X^{\mathbb{N}} : (\forall n \in \mathbb{N} : x_n \neq x) \wedge x_n \to x \Rightarrow f(x_n) \to y$;
- $g(z) := \begin{cases} f(z) & z \neq x \\ y & z = x \end{cases}$ is continuous at x.

Definition 2.2

f has a **removable discontinuity** at x if $\lim_{z\to x} f(z)$ exists but $\neq f(x)$.

Definition 2.3

Let $A \subseteq X$ be nonempty, $x \in \overline{A}$ be not an isolated point. Then $\lim_{z\to x} f(z) = \lim_{z\to x} f_A(z)$ where f_A is the restriction of f to A.

Definition 2.4

For $f : \mathbb{R} \to \mathbb{R}$, let $x \in \overline{\mathrm{Dom}(f)}$ be such that $\mathrm{Dom}(f) \cap (x, \infty) \neq \varnothing$ and $\mathrm{Dom}(f) \cap (-\infty, x) \neq \varnothing$. Then $\lim_{z\to x^+} f(z) := \lim_{z\to x, z\in\mathrm{Dom}(f)\cap(x,\infty)} f(z) \wedge \lim_{z\to x^-} f(z) := \lim_{z\to x, z\in\mathrm{Dom}(f)\cap(-\infty,x)} f(z)$ are the **right / left limits of f at x.**

Alternative notation: $f(x^+), f(x^-)$.

Example 2.5.

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases} \tag{2.1}$$

has no right or left limits.

Example 2.6.

$$f(x) = \begin{cases} \frac{1}{n+1} & x = q_n \text{ where } \{q_n\}_{n\in\mathbb{N}} \text{ enumerates } \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases} \tag{2.2}$$

Then $\forall\, x \notin \mathbb{Q} : \lim_{z\to x} f(z) = 0$ so f is continuous on $\mathbb{R} \setminus \mathbb{Q}$, and $\forall\, x \in \mathbb{Q} : \lim_{z\to x} f(z) = 0$ but f is not continuous at x.

Lemma 2.7

$$\forall\, r > 0\, \forall\, \varepsilon > 0 : \{x \in \mathbb{R} : |x| < r \wedge |f(x)| > \varepsilon\} \text{ finite} \Rightarrow \forall\, x \in \mathbb{R} : \lim_{z\to x} f(z) = 0.$$

Definition 2.8

A function $f : \mathbb{R} \to \mathbb{R}$ has a **discontinuity of**

- **first kind** at x if $f(x^+)$ and $f(x^-)$ exist but are not both equal to $f(x)$;
- **second kind** at x if one or both of $f(x^+)$ and $f(x^-)$ don't exist.

Example 2.9.

$$f(x) := \begin{cases} (-1)^n & x = \frac{1}{n+1}, n \in \mathbb{N} \\ \text{linear} & x \in (0, \infty) \setminus \left\{\frac{1}{n+1} : n \in \mathbb{N}\right\} \\ 0 & x \leq 0. \end{cases} \tag{2.3}$$

This function has a discontinuity of second kind at 0.

Lemma 2.10

Let $f : \mathbb{R} \to \mathbb{R}$ ($\mathrm{Dom}(f) = \mathbb{R}$) be monotone. Then $\forall\, x \in \mathbb{R} : f(x^+), f(x^-)$ exist and so f has no discontinuities of second kind.

Proof. Let $x \in \mathbb{R}$ and assume f is nondecreasing. We claim that $\lim_{z \to x^+} f(z) = \inf\left\{f(z) : z > x\right\} =: L$. Indeed, $\forall\, z > x : f(z) \geq f(x)$, so $L \geq f(x)$ and so $L \in \mathbb{R}$. Then $(\forall\, z > x : L \leq f(z)) \wedge (\forall\, \varepsilon > 0\, \exists\, z_\varepsilon > x : f(z_\varepsilon) < L + \varepsilon)$. Let $\delta := z_\varepsilon - x$. Then $\forall\, z \in (x, x + \delta) : f(z) \leq f(z_\varepsilon) < L + \varepsilon$. Then $\forall\, z \in (x, x + \delta) : L \leq f(z) < L + \varepsilon$ and therefore $\left|f(z) - L\right| < \varepsilon$. Then $\lim_{z \to x^+} f(z) = L$. $\square$

3 3.31 Thursday Week 1: Monotone functions have only countably many discontinuities. Functions of bounded variation. Jordan decomposition theorem. Comments on uniqueness. Rectifiability of curves. Limsup and liminf of a function.

<u>**Limits of functions**</u>

Last time we showed that monotone functions have no discontinuities of second time.

Lemma 3.1

Let $f : \mathbb{R} \to \mathbb{R}$ be monotone. Then $\left\{x \in \mathbb{R} : f(x^+) \neq f(x^-)\right\}$ is countable.

Proof. Pick $k, m \in \mathbb{N}$ and let $A_{m,k} := \left\{x \in [-m, m] : \left|f(x^+) - f(x^-)\right| > \frac{1}{k+1}\right\}$. We claim that $A_{m,k}$ is finite. Let $x_0 < x_1 < \cdots < x_n$ be such that $\forall\, i \leq n : x_i \in A_{k,m}$. Assume (without loss of generality) that f is non-decreasing. Then

$$\begin{aligned} f(m+1) \geq f(x_n^+) &= f(x_0^+) + \sum_{i=1}^{n} \left(f(x_i^+) - f(x_{i-1}^+)\right) \\ &\geq f(m-1) + \sum_{i=1}^{n} \left(f(x_i^+) - f(x_i^-)\right) \\ &\geq f(-m+1) + \frac{n}{k+1}. \end{aligned} \tag{3.4}$$

Then $n \leq (k+1)$. Since $\left\{x \in \mathbb{R} : f(x^+) \neq f(x^-)\right\} = \bigcup_{k \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} A_{k,m}$, we are done. $\square$

Q: Can these be generalized to other functions?

Definition 3.2

A **partition** Π of an interval $[a,b]$ is a sequence $\{t_i\}_{i=0}^n$ such that

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b.$$

Definition 3.3

Given $f\colon [a,b] \to \mathbb{R}$, its **total variation** on $[a,b]$

$$V(f,[a,b]) := \sup_{\Pi=\{t_i\}_{i=0}^n} \sum_{i=1}^n \left|f(t_i) - f(t_{i-1})\right|$$

where the supremum if over the partitions of $[a,b]$.

Definition 3.4

f is said to be of **bounded variation** on $[a,b]$ if $V(f,[a,b]) < \infty$.

Lemma 3.5

If $f\colon \mathbb{R} \to \mathbb{R}$ is of bounded variation on $[-m,m]$ for all $m \in \mathbb{N}$, then f has only discontinuities of first kind and the set $\left\{x \in \mathbb{R} : f(x^+) \neq f(x^-)\right\}$ is countable.

Theorem 3.6: Jordan decomposition (1881)

Let $f\colon [a,b] \to \mathbb{R}$ obey $V(f,[a,b]) < \infty$. Then $\exists\, h, g\colon [a,b] \to \mathbb{R}$ nondecreasing such that $\forall\, t \in [a,b]$: $f(t) = h(t) - g(t)$.

Proof. Define $h(t) := V(f,[a,t])$ and $g(t) := V(f,[a,t]) - f(t)$. Note that $h(t) - g(t) = f(t)$.

We need to show that h and g are nondecreasing.

Let $a \le t < t' \le b$. Then for any partition Π of $[a,t]$. $\Pi' = \Pi \cup \{t'\}$ is a partition of $[a,t']$. Then

$$V(f,[0,t']) \ge \sum_{i=1}^m \left|f(t_i) - f(t_{i-1})\right| + \left|f(t') - f(t)\right|. \tag{3.5}$$

Taking supremum over Π gives

$$V(f,[a,t']) \ge V(f,[a,t]) + \left|f(t') - f(t)\right|. \tag{3.6}$$

Note that $\left|f(t') - f(t)\right| \ge 0$ and $\left|f(t') - f(t)\right| \ge f(t') - f(t)$. Then $h(t') \ge h(t)$ and $g(t') \ge g(t)$. $\square$

The representation of $f = h - g$ is called a Jordan decomposition. This is not unique because a nondecreasing function can be added to both h and g.

However, there is a minimal decomposition $f = h_0 - g_0$ such that $g_0(a) = 0$ such that for any other Jordan decomposition $f = h - g$ we have $h - h_0$, $g - g_0$ nondecreasing. This is then *the* Jordan decomposition.

Rectifiability of curves

Definition 3.7

Let (X, ρ) be a metric space. A curve C is $\mathrm{Ran}(f)$ for an $f : \mathbb{R} \to X$ continuous such that $\mathrm{Dom}(f)$ is nonempty and connected. This f is called a **parametrization** of C.

Definition 3.8

Assuming $\mathrm{Dom}(f) = [a, b]$, the **length of C** is

$$\ell(C) := \sup_{\Pi = \{t_i\}_{i=1}^n} \sum_{i=1}^n \rho(f(t_{i-1}), f(t_i)).$$

Definition 3.9

A curve is **rectifiable** if $\ell(C) < \infty$.

Definition 3.10

Let (X, ρ) be a metric space and $f : X \to \mathbb{R}$. Then

$$\limsup_{z \to x} f(z) := \inf_{\delta > 0} \sup_{z \in B(x,\delta) \setminus \{x\}} f(z)$$

and

$$\liminf_{z \to x} f(z) := \sup_{\delta > 0} \inf_{z \in B(x,\delta) \setminus \{x\}} f(z).$$

Lemma 3.11

$$\lim_{z \to x} f(z) \text{ exists in } \mathbb{R} \Leftrightarrow \limsup_{z \to x} f(z) = \liminf_{z \to x} f(z) \in \mathbb{R}.$$

4 4.1 Friday Week 1: Discussion

Definition 4.1

Let (X, ρ_X), (Y, ρ_Y) be metric spaces, $E \subseteq X$, $f : E \to Y$, and $x \in \overline{E}$. Then $\lim_{t \to x} f(t) = \alpha$ is defined by

$$\forall\, \varepsilon > 0\, \exists\, \delta > 0 : \forall\, x \in E \wedge 0 < \rho_X(t, x) < \delta \Rightarrow \rho_Y(f(t), \alpha) < \varepsilon.$$

Equivalently,

$$\forall\, \{t_n\}_{n \in \mathbb{N}} \in (E \setminus \{x\})^{\mathbb{N}} : t_n \to x \Rightarrow f(t_n) \to \alpha.$$

Note. f need not be defined at x.

Remark.

$$\limsup_{t \to x} f(t) := \inf_{\delta > 0} \sup_{t \in B(x,\delta) \setminus \{x\}} f(t) = \lim_{\delta \to 0} \sup_{t \in B(x,\delta) \setminus \{x\}} f(t).$$

lim inf is similarly defined.

Remark.

$$\limsup = \liminf \Rightarrow \lim \text{ exists.}$$

<u>**Discontiuities**</u>

Definition 4.2

Let $f : (a, b) \to \mathbb{R}$ be not continuous at x. Then f has a **discontinuity of first kind** at x if $f(x+)$ and $f(x-)$ both exist. Otherwise it is of **second kind**.

Remark. Discontinuities of the first kind are also known as **simple discontinuities**. The cases include

- $f(x+) = f(x-) \neq f(x)$: **removable discontinuity**, and
- $f(x+) \neq f(x-)$: **jump discontinuity**.

Example 4.3.

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases} \tag{4.7}$$

has a discontinuity of second kind at 0.

Example 4.4.

$$f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \tag{4.8}$$

is continuous on $\mathbb{R} \setminus \mathbb{Q}$ and has discontinuities of first kind (removable) at every point in $\mathbb{Q}$.

Recall: A monotone function has no discontinuity of second kind and has at most countably many discontinuities of first kind. One can deduce this from the fact that the real line is a union of countably many open intervals (indexed by rationals).

Definition 4.5

A function $f : (a, b) \to \mathbb{R}$ is convex if

$$\forall\, x, y \in (a, b) : x \le y \Rightarrow (\forall\, \lambda \in [0, 1] : f(\lambda x + (1 - \lambda)y)) \le \lambda f(x) + (1 - \lambda)f(y).$$

In words, this means that for any interval, the secant line is above the graph.

5 4.4 Monday Week 2

__Last time:__ $\lim_{z\to x} f(z)$, $\limsup_{z\to x} f(z) = \inf_{\delta>0} \sup_{z\in B(x,\delta)\setminus\{x\}x} f(z)$

Lemma 5.1

$$\lim_{z\to x} f(z) \text{ exists (in } \mathbb{R}) \Leftrightarrow \limsup_{z\to x} f(z) = \liminf_{z\to x} f(z) \in \mathbb{R}.$$

Proof. Both are equivalent:

$$\forall\, \varepsilon > 0\, \exists\, \delta > 0 : 0 \le \sup_{z\in B(x,\delta) f(z)\setminus\{x\}} - \inf_{z\in B(x,\delta)\setminus\{x\}} f(z) \le 2\varepsilon.$$

□

Definition 5.2

$$\lim_{z\to x} f(z) = \begin{cases} +\infty & \limsup_{z\to x} f(z) = \liminf_{z\to x} f(z) = +\infty \\ -\infty & \limsup_{z\to x} f(z) = \liminf_{z\to x} f(z) = -\infty. \end{cases}$$

Note. This characterization works even outside $\mathbb{R}$-valued functions:

$$\lim_{z\to x} f(z) \text{ exists} \Leftrightarrow \lim_{\delta\to 0^+} \sup \underbrace{\{\rho(f(z), f(u)) : z, u \in B(x, \delta) \setminus \{x\}\}}_{=\mathrm{diam}(f(B(x,\delta)\setminus\{x\}))} = 0.$$

__**The derivative**__

Definition 5.3

Let $f : \mathbb{R} \to \mathbb{R}$, $x \in \mathrm{int}(\mathrm{Dom}(f))$. We say that f has **derivative** or **is differentiable at x** if

$$f'(x) := \lim_{z\to x} \frac{f(z) - f(x)}{z - f} \text{ exists in } \mathbb{R}.$$

We call $f'(x)$ (Lagrange notation) the **derivative at x**, alternative notation $\frac{df}{dx}$ (Leibnitz notation).

Lemma 5.4

$$f'(x) \text{ exists} \Rightarrow f \text{ continuous at } x.$$

Proof. The existence of $f'(x)$ implies that $\exists\, \delta_0 > 0\, \forall\, z \in \mathbb{R} : 0 < |z - x| < \delta_0 \Rightarrow \left|\frac{f(z)-f(x)}{z-x}\right| \leq 1 + \left|f'(x)\right|$. Then, choosing $\varepsilon > 0$ and letting $\delta := \frac{\varepsilon}{1+\left|f'(x)\right|}$, we get

$$\forall\, z \in \mathbb{R} : 0 < |z - x| < \sigma \Rightarrow \left|f(z) - f(x)\right| \leq (1 + \left|f'(x)\right|)\,|z - x| < (1 + \left|f'(x)\right|) = \varepsilon.$$

Since $f(z) - f(x) = 0$ for $z = x$, we are done (in fact, we have shown that f is lipschitz continuous). □

Another way to write existence of $f'(x)$:

$$f(z) - f(x) = (f'(x) + u_x(z))(z - x)$$

where $\lim_{z\to x} u_x(z) = 0$. (Just define: $u_x(z) := \frac{f(z)-f(x)}{z-x} - f'(x)$ for $z \neq x$)

Lemma 5.5: Linear approximation

$$f'(x) \text{ exists} \Leftrightarrow \exists\, L \in \mathbb{R} : \lim_{\delta\to 0^+} \sup_{|z-x|<\delta} \frac{1}{\delta}\left|f(z) - f(x) - L(z - x)\right| = 0.$$

Lemma 5.6: Sum & product rule

Let f, g be differentiable at x. Then so are $f + g$ and $f \cdot g$ and

$$(f + g)'(x) = f'(x) + g'(x)$$
$$(f \cdot g)'(x) = f'(x)g(x) + g'(x)f(x)\text{(Leibnizt rule)}.$$

Proof. [Product rule]

$$f(z)g(z) - f(x)g(x) = (f(z) - f(x))g(z) + (g(z) - g(x))f(z).$$

Then

$$\frac{f(z)g(z) - f(x)g(x)}{z - x} = \frac{f(z) - f(x)}{z - x}g(z) + \frac{g(z) - g(x)}{z - x}f(z).$$

Since $g(z) \to g(x)$ by continuity of g, formula follows by sum & product rule for limit. □

Lemma 5.7: Chain rule

Let f be differentiable at x and g at $f(x)$. Then $g \circ f$ is differentiable at x and

$$(g \circ f)'(x) = g'(f(x))f'(x) \quad \left(\frac{dg}{df}\frac{df}{dx}\right)$$

Proof. Define $v_{f(x)}$ such that $g(y) - g(f(x)) = (g'(f(x))) + v_{f(x)}(y))(y - f(x))$ and u_x such that $f(z) - f(x) = (f'(x) + u_x(z))(z - x)$.

$$\begin{aligned}(g \circ f)(z) - (g \circ f)(x) &= [g'(f(x)) + v_{f(x)(f(z))}](f(z) - f(x)) \\ &= [g'(f(x)) + v_{f(x)}(f(z))][f'(x) + u_x(z)](z - x)\end{aligned}$$

Dividing by $z - x \neq 0$, note that $f(z) \to f(x)$ implies $v_{f(x)}(f(z)) \to 0$ as $z \to x$, we are done. □

Lemma 5.8

Let $f : \mathbb{R} \to \mathbb{R}$ be injective on $\mathrm{Dom}(f)$ and differentiable at $x \in \mathrm{int}(\mathrm{Dom}(f))$. Assume $f'(x) \neq 0$ and $f(x) \in \mathrm{int}(\mathrm{Ran}(f))$. Then f^{-1} is differentiable at $f(x)$ and

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}.$$

In Leibnitz notation:

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

Lemma 5.9: First derivative test

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then if $x \in (a, b)$ is a local maximum of f (i.e. $\exists\, \delta > 0\, \forall\, z \in \mathbb{R} : |z - x| < \delta \Rightarrow f(x) \geq f(z)$) then $f'(x) = 0$.

Proof.

$$z > x \wedge |z - x| < \delta \Rightarrow \frac{f(z) - f(x)}{z - x} \leq 0 \Rightarrow f'(x) \leq 0$$

and

$$z < x \wedge |z - x| < \delta \Rightarrow \frac{f(z) - f(x)}{z - x} \geq 0 \Rightarrow f'(x) \geq 0.$$

□