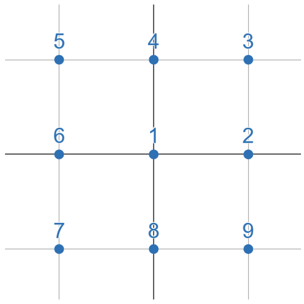
MA3K7 Assignment 5 Rubric

Huw Llewellyn (2103163)

The natural numbers can be arranged over $\mathbb{Z}^2$ in an outward anticlockwise spiral, starting at $(0, 0)$. Here are the first nine natural numbers arranged this way:



What natural number lies at (25,-32)?

Entry

INTRODUCE
I KNOW
I WANT

Let $\phi : \mathbb{N} \to \mathbb{Z}^2$ map out this spiral. It's clear that ϕ is injective, and so since Z^2 is countable, we get that ϕ must be a surjection and hence a bijection. We aim to find the inverse of this map (which we know exists, since ϕ is a bijection). The first step to finding ϕ^{-1} is to find ϕ.

AHA

At first glance, it looks like it might be possible to construct a recursive formula for ϕ: apart from $\phi(1)$ and $\phi(2)$, the positioning of $\phi(n+1)$ depends on whether we're able to place it immediately left of $\phi(n)$, with direction defined with respect to the vector joining $\phi(n-1)$ and $\phi(n)$. This doesn't look like it will help us find an explicit formula directly, but it may come in handy for writing a script to visualise the spiral for bigger set of naturals.

AHA

Another thing to notice is that the squares of the odd natural numbers seem to lie on the same ray. At least up to $n = 5$, numbers from the set $\{n^2 \mid n \text{ is odd}\}$ lie on the ray through the origin of gradient -1. We formally conjecture this:

Conjecture 1. *Let $n \in \mathbb{N}$ be odd. Then under ϕ, $n^2 \mapsto (\frac{n-1}{2}, \frac{1-n}{2})$.*

It's a good time to move to the attack phase and start looking for an explicit formula for ϕ (and in turn, ϕ^{-1}).

Attack

The first thing to do is to write a script that uses our recursive formula to continue the spiral. Given $N \in \mathbb{N}$, it should compute and visualise the spiral for all values of $n \leq N$.

Our algorithm will use a 2D array (square matrix) as it's main data structure. If we assume that our conjecture is true, we can exploit it to pre-determine the required dimension of this matrix (we round $\sqrt{n}$ up to the next odd integer, call it m, and we will require an $m \times m$ matrix).
We use it to generate the spiral up to $n = 99$. Any zero entries should be ignored, as they are an artefact of the construction algorithm (they correspond to points in $\mathbb{Z}^2$ that the spiral has not yet 'reached'). Note that for $n > 99$, the algorithm continues to work but the visualiser starts to fail. This is because we need to allocate more space on the image plane for numbers

with more digits. The issue could be solved by constructing a more sophisticated, adaptive coordinate transform between the spiral array and the image plane, but $n = 99$ is sufficient for now:

0	0	99	98	97	96	95	94	93	92	91
0	65	64	63	62	61	60	59	58	57	90
0	66	37	36	35	34	33	32	31	56	89
0	67	38	17	16	15	14	13	30	55	88
0	68	39	18	5	4	3	12	29	54	87
0	69	40	19	6	1	2	11	28	53	86
0	70	41	20	7	8	9	10	27	52	85
0	71	42	21	22	23	24	25	26	51	84
0	72	43	44	45	46	47	48	49	50	83
0	73	74	75	76	77	78	79	80	81	82
0	0	0	0	0	0	0	0	0	0	0

Figure 1: Spiral for n=99

AHA

So, we have verified Conjecture 1 up to the ninth square number (which is the fifth element of the set it describes). Extending the spiral also sheds light on the distribution of the squares of even numbers, notice how they also all seem to sit on a ray? We formalise this with another conjecture:

Conjecture 2. *Let $n \in \mathbb{N}$ be even. Then under ϕ, $n^2 \mapsto (-\frac{1}{2}n + 1, \frac{n}{2})$*

INTRODUCE

BUT WHY?

Now, a natural number's position under ϕ looks like it can be determined based on it's distance from the nearest odd square less than it. Each odd square defines a 'section' of the spiral, with a section consisting of two sets of vertically collinear points and two sets of horizontally collinear points (they correspond to rows and columns in our matrix). The cardinality of these sets (and hence the coordinates of their members) are determined by which section they are in, i.e. which odd square they round down to.

TRY

Look at the section of the spiral 'generated' by $n = 1$. It contains the numbers 2,3,4,5,6,7 and 8. Numbers 2 and 3 are vertically collinear, 4 and 5 are horizontally collinear, 6 and 7 are vertically collinear, and 8 is sort of on it's own.

TRY

Look at the spiral generated by $n = 9$. We get a similar scenario, whereby $\{10, 11, 12, 13\}$ and $\{18, 19, 20, 21\}$ are each vertically collinear, and $\{14, 15, 16, 17\}$ and $\{22, 23, 24\}$ are horizontally collinear.

The number of elements in these collinear sets is related to n. It is one more than the square root of the odd square that generates that section, apart from the final horizontally collinear set, which has one less. The set that a number lies in depends on it's magnitude compared to n.

With this in mind, we propose a piecewise formula for ϕ:

$$\phi(n) = \begin{cases} (\frac{m-1}{2}, \frac{1-m}{2}) & : \ n = m^2 \\ (\frac{m-1}{2} + 1, (\frac{1-m}{2} - 1) + (n - m^2)) & : \ m^2 < n \leq m^2 + (m+1) \\ ((\frac{m-1}{2} + 1) - (n - m^2) + (m+1), \frac{1-m}{2} + m) & : \ m^2 + (m+1) < n \leq m^2 + 2(m+1) \\ ((\frac{m-1}{2} - m), (\frac{1-m}{2} + m) - (n - m^2) + 2(m+1)) & : \ m^2 + 2(m+1) < n \leq m^2 + 3(m+1) \\ ((\frac{m-1}{2} - m) + (n - m^2) - 3(m+1), \frac{1-m}{2} - 1) & : \ m^2 + 3(m+1) < n \leq m^2 + 3(m+1) + m \end{cases}$$

where $m = 2\lfloor \frac{\sqrt{n}+1}{2} \rfloor - 1$, i.e. the result of rounding $\sqrt{n}$ down to the nearest odd integer (making m^2 the generator of the spiral section containing n).

JUSTIFY

This proves conjecture one, since if n is an odd square, then $m^2 = n$.

Now, we could look at finding an inverse for this function, as we set out to do. However during the construction of our function, it became obvious that we can find the pre-image of $(x, y) \in \mathbb{Z}^2$ by calculating it's distance (with respect to some spiral metric that I won't formalise) from the coordinates of the nearest odd square. We demonstrate this by finding the coordinates of (25,-32), as required by the question:

According to conjecture one (which we know is true), the natural number at coordinates (32,-32) is the square of the thirty third odd number. The thirty third odd number is 65, and it's square is 4225.

We know from the construction of ϕ that (25,-32) is going to lie in the second (smaller) horizontally collinear set within the section of the spiral generated by $63^2 = 3969$, because the cardinality of this set of numbers will be 63, which is far greater than $|25 - 32| = 7$. Hence, we can just subtract seven from 4225 to get the number at $(25, -32)$.

$$\phi^{-1}(25, -32) = 4218$$

Review

CHECK

Setting $n = 4218$ in my python code (and commenting out the earlier iterative approach, since the image it generates will be illegible) gives the coordinates (25,-32). I can hence be satisfied that the solution I have presented is the correct one.

REFLECT

I could've provided a bit more justification as to why ϕ was injective. I could've also proved that $\mathbb{Z}^2$ is countable, however I recall that the steps to do so involve winding the natural number in a spiral pattern, similarly to that described by the question. I thought it best to assume this result, so not to confuse things.

My python script worked very well. Unlike the previous assignment, my algorithms for computing $\phi(n)$ were very efficient (the iterative one having linear time complexity, the direct one having constant time complexity). The name of the variable `dim` was a bit misleading. This was because I didn't factor-in range checking until the interpreter raised an error, and at that point I just re-purposed it slightly. Another product of not thinking ahead was the visualiser, which didn't display arrows to clarify the spiral shape. The data structure I used didn't keep track of the order in which elements are added, and it would've been too much work to implement this as an afterthought (the whole algorithm would need re-designing).

I found it a bit difficult to explain the process of deriving the explicit formula for ϕ. I introduced some notation (spiral sections, collinear sets) in an attempt to alleviate the verbosity, but perhaps that section could've also benefited from a diagram to illustrate what those terms actually mean.

EXTEND

I set out to find an explicit formula for ϕ^{-1}. As it turned out, this wasn't necessary, but a nice extension would be to invert the formula I constructed for ϕ.

I made a conjecture regarding the distribution of even squares under ϕ. Although I proved my first conjecture (regarding the distribution of odd squares) using the formula for ϕ, it wasn't necessary to prove the second conjecture in order to solve the problem, so I left it unsolved. It doesn't look at straightforward as the trivial proof for the first conjecture, but perhaps there's something to be explored in starting the spiral at some other $(x, y) \in \mathbb{Z}^2$ so that the even squares lie on some ray through the origin, and then looking for a coordinate transform.

Another extension would be to look at the distribution of primes. I recall seeing a video published by 3Blue1Brown, in which he explores exactly this. The mathematics underlying his exploration however (if you were to seek proofs for any results uncovered) is perhaps slightly beyond what could be expected of a first year undergraduate.

Supplementary material

The code for this assignment can be found on my GitHub page:
`https://github.com/hllewellyn1/MA3K7.git`.