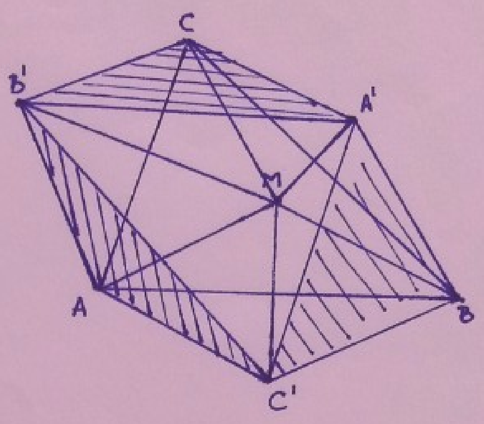
**4/6** Dat je trougao ABC i tačka M u trouglu. Neka su A', B' i C' redom tačke simetrične sa tačkom M u odnosu na stranice BC, CA i AB. Dokazati da su trouglovi C'AB', B'CA' i A'BC' jednakokraki.

Rješenje:




Primijetimo da je

$$S_{AB}(M) = C' . \qquad (1)$$

$$A \in AB \Rightarrow S_{AB}(A) = A . \qquad (2)$$

Iz (1) i (2) $\Rightarrow \boxed{AM = AC'}$ (3)

$$\left.\begin{array}{l} A \in AC \Rightarrow S_{AC}(A) = A \\ \qquad\qquad S_{AC}(M) = B' \end{array}\right\} \Rightarrow \boxed{AM = AB'} \qquad (4)$$

Iz (3) i (4) imamo:

$$AC' = AM = AB' \quad \Rightarrow \quad \triangle AC'B' \text{ je jednakokraki.}$$

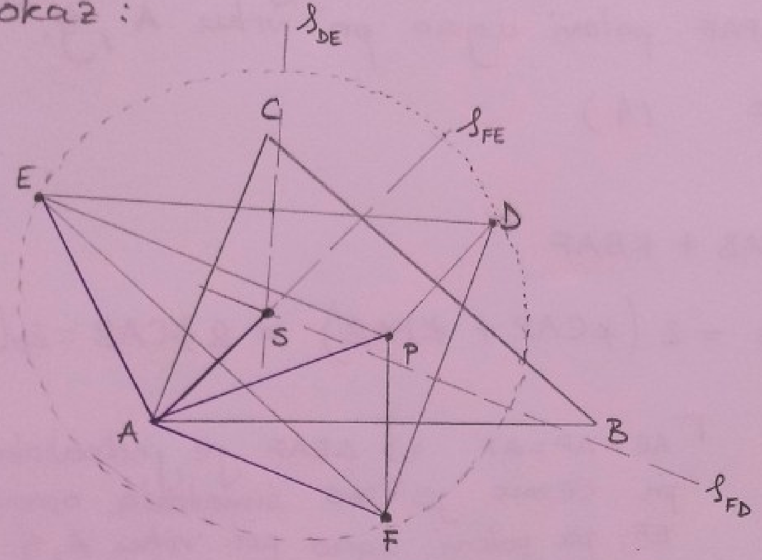
$$\left.\begin{array}{l} \left.\begin{array}{l} B \in BC \Rightarrow S_{BC}(B) = B \\ \qquad\qquad S_{BC}(M) = A' \end{array}\right\} \Rightarrow BM = BA' \\ \left.\begin{array}{l} B \in BA \Rightarrow S_{BA}(B) = B \\ \qquad\qquad S_{BA}(M) = C' \end{array}\right\} \Rightarrow BM = BC' \end{array}\right\} \Rightarrow BA' = BM = BC', \text{ tj. } \triangle A'BC' \text{ je jednakokraki.}$$

$$\left.\begin{array}{l} \left.\begin{array}{l} C \in BC \Rightarrow S_{BC}(C) = C \\ \qquad\qquad S_{BC}(M) = A' \end{array}\right\} \Rightarrow CM = CA' \\ \left.\begin{array}{l} C \in AC \Rightarrow S_{AC}(C) = C \\ \qquad\qquad S_{AC}(M) = B' \end{array}\right\} \Rightarrow CM = CB' \end{array}\right\} \Rightarrow CA' = CM = CB', \text{ tj. } \triangle B'CA' \text{ je jednakokraki.}$$

217. U ravni datog $\triangle ABC$ data je tačka $P$. Neka su $D, E$ i $F$ tačke simetrične tački $P$ u odnosu na prave $BC$, $CA$ i $AB$, a $S$ centar opisanog kruga oko $\triangle DEF$. Dokazati da je

a) $AS \perp EF$ b) $\measuredangle EAF = 2\alpha$ c) $\measuredangle PAB = \measuredangle SAC$.

Dokaz:




$S$-centar opisanog kruga oko $\triangle DEF \Rightarrow$

$s_{DE} \cap s_{FE} \cap s_{FD} = \{S\}$, tj. $S$ je presjek simetrala stranica $\triangle DEF$. (*)

a) $\left.\begin{array}{l} S_{AB}(P) = F \\ S_{AB}(A) = A \end{array}\right\} \Longrightarrow AP = AF \quad (1)$

$\left.\begin{array}{l} S_{AC}(P) = E \\ S_{AC}(A) = A \end{array}\right\} \Longrightarrow AP = AE \quad (2)$

Iz (1) i (2) $\Rightarrow AE = AF \Rightarrow \triangle EAF$ je jednakokraki. Otuda, simetrala osnovice $FE$ prolazi kroz vrh $A$, tj. $A \in s_{FE}$, pa zbog (*) (tj. $S \in s_{FE}$) slijedi da simetrala osnovice $FE$ je prava određena tačkama $A$ i $S$. Dakle,

$$s_{FE} = p(A,S) \Longrightarrow FE \perp AS.$$

b) $\left.\begin{array}{l} S_{AC}(P) = E \\ S_{AC}(A) = A \end{array}\right\} \Rightarrow AP = AE \Rightarrow \triangle PAE$ je jednakokraki $\triangle$.

$\Rightarrow$ simetrala osnovice $EP$ polovi ugao pri vrhu $A$ jednakokrakog $\triangle PAE$. Dakle,

$\measuredangle EAC = \measuredangle CAP = \frac{1}{2}\measuredangle EAP$ (3)

Dalje, vrijedi

$$\left.\begin{array}{l} S_{AB}(P) = F \\ S_{AB}(A) = A \end{array}\right\} \Rightarrow AP = AF \Rightarrow \Delta PAF \text{ je jednakokraki } \Delta$$

$\Rightarrow$ simetrala osnovice $PF$ jednakokrakog $\Delta PAF$ polovi ugao pri vrhu $A$, tj.

$\measuredangle PAB = \measuredangle BAF = \frac{1}{2}\measuredangle PAF$ (4)

$$\measuredangle EAF = \measuredangle EAC + \measuredangle CAP + \measuredangle PAB + \measuredangle BAF$$

$$\overset{(3),(4)}{=} 2\measuredangle CAP + 2\measuredangle PAB = 2(\measuredangle CAP + \measuredangle PAB) = 2\measuredangle CAB = 2\alpha$$

c) $\measuredangle SAC = \measuredangle SAE - \measuredangle EAC$

$\ulcorner$ $AE = AP = AF \Rightarrow \Delta EAF$ je jednakokraki pri čemu je $AS$ simetrala osnovice $EF$, pa polovi ugao pri vrhu $A$, tj. $\measuredangle EAF = 2\alpha \Rightarrow \measuredangle SAE = \alpha$. $\lrcorner$

(3)

$= \alpha - \measuredangle CAP$

$= \measuredangle CAB - \measuredangle CAP = \measuredangle PAB$.

**Z/8.** Dat je $\Delta ABC$. Neka je tačka $A_1$ presjek simetrale $\sphericalangle BAC$ sa stranicom $BC$, a $P$ i $Q$ tačke simetrične tački $A_1$ u odnosu na stranice $AB$ i $AC$. Neka je $p$ prava paralelna s $AC$ i $P\in p$, a $q$ prava paralelna s $AB$ i $Q\in q$. Dalje, neka je

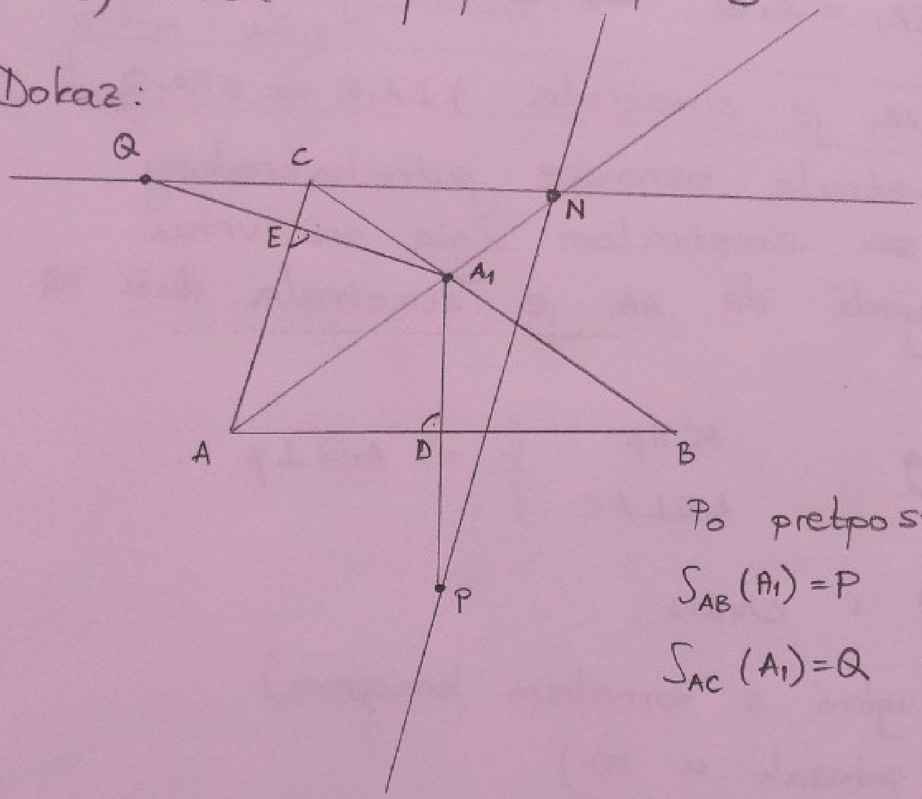
$$p\cap q=\{N\},\quad PA_1\cap AB=\{D\}\quad i\quad QA_1\cap AC=\{E\}.$$

Dokazati da

a) Prava $AA_1$ je simetrala duži $PQ$

b) Tačka $N$ pripada pravoj $AA_1$.

Dokaz:




Po pretpostavci zadatka je:

$S_{AB}(A_1)=P \implies AB\perp A_1P$ (1)

$S_{AC}(A_1)=Q \implies AC\perp A_1Q$ (2)

a) $PA_1\cap AB=\{D\} \xRightarrow{(1)} \Delta ADA_1$ pravougli $\Delta$

$QA_1\cap AC=\{E\} \xRightarrow{(2)} \Delta AEA_1$ pravougli $\Delta$

Posmatrajmo $\Delta ADA_1$ i $\Delta AEA_1$:

$AA_1=AA_1$ (zajednička stranica)

$\sphericalangle A_1AD=\frac{\alpha}{2}=\sphericalangle A_1AE$ ($A_1$ - presječna tačka simetrale $\sphericalangle BAC$ i str. $BC$)

$\sphericalangle A_1DA=90^\circ=\sphericalangle A_1EA$

$\implies \Delta ADA_1\cong\Delta AEA_1 \implies \underset{(3)}{\boxed{DA_1=EA_1}}$ i $\underset{(4)}{\boxed{\sphericalangle AA_1D=\sphericalangle AA_1E}}$

* $S_{AB}(A_1) = P \Rightarrow AB$ polovi duž $A_1P$, a kako je $PA_1 \cap AB = \{D\}$ to je $D$ središte duži $A_1P$, pa vrijedi

$$\boxed{A_1P = 2A_1D} \quad (5)$$

* $S_{AC}(A_1) = Q \Rightarrow AC$ polovi duž $A_1Q$, a kako je $QA_1 \cap AC = \{E\}$ to je $E$ središte duži $A_1Q$, pa vrijedi

$$\boxed{A_1Q = 2EA_1} \quad (6)$$

Iz (3), (5), (6) slijedi (7)

$$A_1P = 2A_1D = 2EA_1 = A_1Q \Rightarrow \boxed{\triangle PA_1Q \text{ je jednakokraki } \triangle}$$

* Iz (4) slijedi da $\boxed{AA_1 \text{ je simetrala } \sphericalangle DA_1E \overset{P-D-A_1 \text{ i } A_1-E-Q}{=} \sphericalangle PA_1Q.}$ (8)

Budući da se simetrala osnovice jednakokrakog trougla poklapa sa simetralom ugla pri vrhu iz (7) i (8) slijedi da $AA_1$ je simetrala duži $PQ$.

b)

$$\left.\begin{array}{l} AB \parallel q \\ A_1P \perp AB \end{array}\right\} \Rightarrow A_1P \perp q \qquad \left.\begin{array}{l} AC \parallel p \\ A_1Q \perp AC \end{array}\right\} \Rightarrow A_1Q \perp p.$$

Posmatrajmo $\triangle A_1PN$ i $\triangle A_1QN$:

$\sphericalangle A_1PN = \sphericalangle A_1QN$ (uglovi s normalnim kracima)

$A_1P = A_1Q$ (pokazali u a))

$A_1N = A_1N$ (zajednička stranica)

$\Rightarrow \triangle A_1PN \cong \triangle A_1QN$

$\Rightarrow \underbrace{PN = QN}$ i $\underbrace{\sphericalangle A_1NP = \sphericalangle A_1NQ}$

$\Rightarrow \triangle PNQ$ je jednakokraki

$\Rightarrow NA_1$ je simetrala $\sphericalangle PNQ$ jednakokrakog $\triangle PNQ$

$\Rightarrow NA_1$ se poklapa sa simetralom osnovice $PQ$, pa zbog dokaza a) slijedi da se prave $NA_1$ i $AA_1$ poklapaju, tj. tačke $A, A_1, N$ su kolinearne