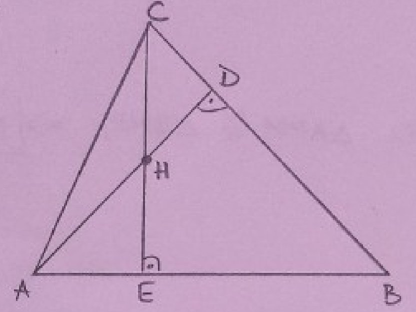
Z11. Visine CE i AD ΔABC sijeku se u tački H. Ako je $CH = AB$ izračunati $\measuredangle ACD$.

Rješenje:



ΔCDH - pravougli $\Delta \Rightarrow \measuredangle DHC = 90° - \measuredangle DCH$ (1)

ΔABD - pravougli $\Delta \Rightarrow \measuredangle ABD = 90° - \measuredangle BAD$ (2)

(3) $\measuredangle BAD = \measuredangle DCH$ (uglovi s normalnim kracima)

$AB = CH$

$\measuredangle ABD = \measuredangle CHD$ (zbog (1), (2), (3))

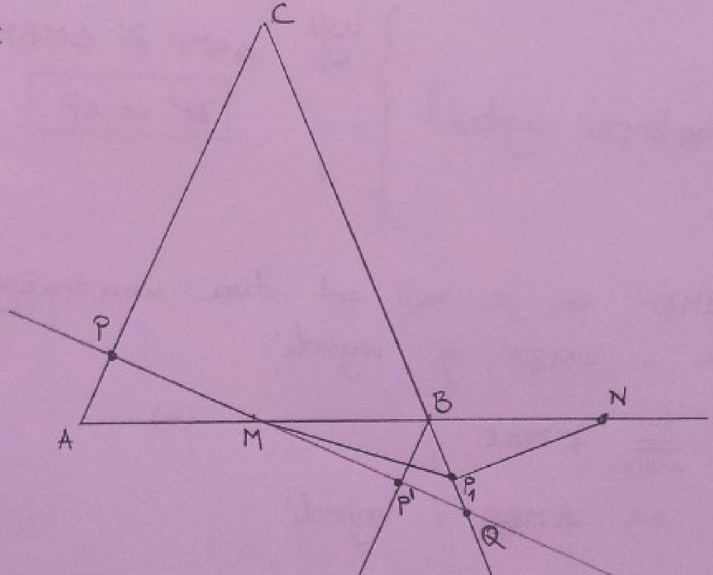
$\overset{USU}{\Rightarrow} \Delta ABD \cong \Delta CHD \Rightarrow \boxed{AD = CD}$

Kako je $AD = CD \Rightarrow \Delta ACD$ je jednakokraki pravougli Δ, pa je

$$\measuredangle ACD = \measuredangle DAC = 45°.$$

Z12. Kroz središte M osnovice AB jednakokrakog ΔABC prolazi prava koja siječe $p(A,C)$ i $p(B,C)$ u tačkama P i Q, redom, tako da je $P-M-Q$. Dokazati da je $PQ > AB$.

Rješenje:



* Bez gubitka opštosti pretpostavimo da prava, koja prolazi kroz središte M osnovice AB, siječe prave $p(A,C)$ i $p(B,C)$ u tačkama P i Q tako da vrijedi

$$A-P-C \quad i \quad C-B-Q.$$

* Na $pp[A,B)$ odredimo tačku N tako da je $A-B-N$ i

$$BN = BM, \qquad (1)$$

a na $pp[B,Q)$ tačku P_1 tako da

$$BP_1 = AP \qquad (2)$$

* Posmatrajmo $\triangle APM$ i $\triangle BP_1N$:

$$\left.\begin{aligned} AP &\overset{(2)}{=} BP_1 \\ \sphericalangle MAP &\underset{\triangle ABC \text{ j. kraki}}{=} \sphericalangle MBC \overset{\text{unakrsni}}{=} \sphericalangle NBP_1 \\ AM &= MB \overset{(1)}{=} BN \end{aligned}\right\} \xrightarrow{\text{sus}} \triangle APM \cong \triangle BNP_1 \Rightarrow \boxed{MP \cong NP_1} \qquad (3)$$

* $\sphericalangle ABQ$ je spoljašnji ugao jednakokrakog $\triangle ABC$ ($\sphericalangle ABC = \sphericalangle BAC$ - oštar), pa zaključujemo da $\sphericalangle ABQ$ – tupi i da

$$\boxed{\sphericalangle ABQ > \sphericalangle ABC} \qquad (4)$$

Otuda, u unutrašnjosti $\sphericalangle ABQ$ postoji prava koja ishodi iz tjemena B i koja siječe duž MQ u tački P' takva da

$$\boxed{\sphericalangle ABP' = \sphericalangle ABC.} \qquad (5)$$

* Posmatrajmo $\triangle AMP$ i $\triangle MBP'$:

$$\left.\begin{aligned} AM &= MB \\ \sphericalangle AMP &= \sphericalangle BMP' \text{ (unakrsni uglovi)} \\ \sphericalangle MAP &\overset{(5)}{=} \sphericalangle MBP' \end{aligned}\right\} \xrightarrow{\text{usu}} \triangle AMP \cong \triangle MBP' \quad \boxed{BP' = AP} \qquad (6)$$

* $\sphericalangle BP'Q$ je spoljašnji za $\triangle MBP'$, pa je veći od dva unutrašnja njemu nesusjedna ugla u $\triangle MBP'$, tj. vrijedi

$$\sphericalangle BP'Q > \sphericalangle MBP' \underset{A-M-B}{\overset{(5)}{=}} \sphericalangle MBC, \qquad (7)$$

a $\sphericalangle MBC$ je spoljašnji za $\triangle MBQ$ i vrijedi

$$\sphericalangle MBC > \sphericalangle MQB \qquad (8)$$

Iz (7) i (8) slijedi

$$\sphericalangle BP'Q > \sphericalangle MQB \overset{M-P'-Q}{=} \sphericalangle P'QB \qquad (9)$$

* Kako naspram većeg ugla u trouglu leži veća stranica, to iz (9) za $\triangle BP'Q$ vrijedi

$$BQ > BP' \overset{(6)}{=} AP \overset{(2)}{=} BP_1 . \qquad (10)$$

Kako za tačku $P_1 \in pp[B,Q)$ vrijedi (10) $\Longrightarrow$ $B-P_1-Q$.

* Primijetimo da $\sphericalangle MBP_1 = \sphericalangle ABQ$, tj zbog (4) je $\sphericalangle MBP_1$ tupi. Kako je $\sphericalangle MP_1Q$ spoljašnji ugao $\triangle MBP_1$ slijedi da

$$\sphericalangle MP_1Q > \sphericalangle MBP_1 - \text{tup} \implies \sphericalangle MP_1Q - \text{tupi} \qquad (11)$$

Stranica naspram tupog ugla u trouglu je najveća stranica u trouglu, pa zbog (11) je

$$MQ > MP_1 \qquad (12)$$

* Konačno,

$$PQ = PM + MQ \overset{(3)}{=} NP_1 + MQ \overset{(12)}{>} NP_1 + MP_1 > MN \qquad (*)$$

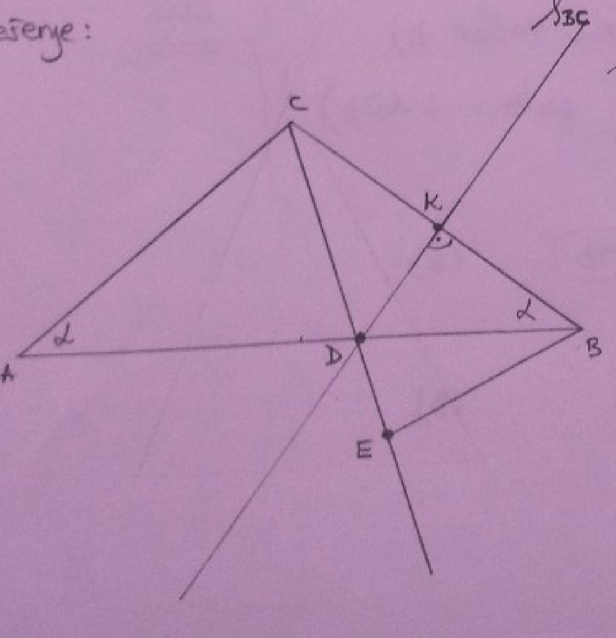
$$MN = MB + BN \overset{(1)}{=} MB + BM \underset{\substack{\uparrow \\ M \text{ je središte} \\ \text{osnovice } AB, \text{ tj. } MB = AM}}{=} AM + MB = AB \qquad (**)$$

Iz (*) i (**) vrijedi

$$PQ > AB.$$

213. U jednakokrakom $\triangle ABC$ simetrala kraka BC siječe osnovicu AB u tački D tako da je $A-D-B$. Neka je na pravoj $p(C,D)$ tačka E takva da je $CE \cong AD$ i $C-D-E$. Dokazati da je $\triangle DBE$ jednakokrak.

Rješenje:



s_{BC} - simetrala kraka BC

$s_{BC} \cap BC = \{K\} \implies BK = KC$

$\triangle ABC$ - j.kraki $\implies \sphericalangle ABC = \sphericalangle BAC = \alpha$

* Posmatrajmo $\triangle BKD$ i $\triangle CKD$

$$\left.\begin{array}{l} BK = KC \\ \sphericalangle BKD = 90^\circ = \sphericalangle CKD \\ DK = DK \end{array}\right\} \overset{SUS}{\Longrightarrow}$$

$$\triangle BKD \cong \triangle CKD \implies \boxed{\begin{array}{l} CD = DB \\ \sphericalangle KCD = \sphericalangle KBD = \alpha \end{array}} \quad (1)$$

Posmatrajmo $\triangle CBE$ i $\triangle ACD$

$$\left.\begin{array}{l} CE = AD \quad \text{(pp. zadatka)} \\ \sphericalangle BCE = \sphericalangle KCD \overset{(1)}{=} \alpha = \sphericalangle CAD \\ BC = AC \end{array}\right\} \begin{array}{l} \overset{SUS}{\Longrightarrow} \triangle CBE \cong \triangle ACD \\ \Longrightarrow \boxed{BE = CD} \quad (2) \end{array}$$

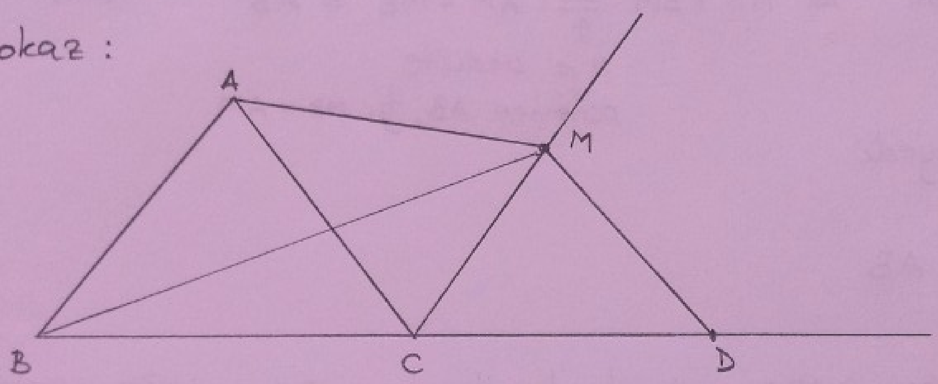
Iz (1) i (2)

$$BE \overset{(2)}{=} CD \overset{(1)}{=} DB \Longrightarrow \triangle DBE \text{ jednakokraki trougao.}$$

2/4. Neka je M proizvoljna tačka simetrale spoljašnjeg ugla kod tjemena C $\triangle ABC$. Dokazati da je

$$MA + MB > CA + CB.$$

Dokaz:



* Na pp$[B,C)$ odredimo tačku D tako da je $B-C-D$ i $AC \cong CD$. Tada je:

$$BD = BC + CD = BC + AC \qquad (1)$$

* Posmatrajmo $\triangle ACM$ i $\triangle DCM$:

$$\left.\begin{array}{ll} AC = CD & \text{(prema izboru tačke D)} \\ \sphericalangle ACM = \sphericalangle DCM & (\text{pp}[C,M) \text{ polovi } \sphericalangle ACD) \\ CM = CM & \end{array}\right\} \overset{SUS}{\Longrightarrow}$$

$$\triangle ACM \cong \triangle DCM \Longrightarrow \boxed{MA = MD} \qquad (2)$$

* U $\triangle BMD$ vrijedi

$$MD + MB > BD. \qquad (3)$$

* Iz (1), (2), (3) slijedi

$$MA + MB > BC + AC. \quad \blacksquare$$