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# Random and frozen states in complex triangulations

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Triangulations of complex surfaces with different genera are studied within a statistical mechanics framework where an energy is associated to deviations from an ideal, ordered ground-state. We observe that the complexity of the embedding surface strongly affects the properties of the triangulations. At high temperatures the “random states” have degree distributions that broaden with the surface-genus. At low temperatures the “frozen states” can reach higher degree of order with the increasing genus. The dynamics between disordered to ordered states is also affected by the surface genus. High genus triangulations start from more disordered states at high temperatures but they quench faster into more ordered states than the low genus counterparts. However, the ground-state is never reached because at low temperatures the relaxation dynamics slows down into a glassy kind of behaviour. Topological frustration can also play a very important role when the surface-genus forces the average degree to a fractional number.

**Keywords:** Complex Networks, Triangulations, Topological Froths, Surface genus, Hyperbolic networks.

## 1. Introduction

In recent years there has been a strong and growing interest in the study of complex networks [1–4]. Indeed, it has been observed that in many natural systems, where a large variety of constituents are involved, the network of interactions between the constituents plays a fundamental role that can determine several properties of the system and can be responsible for its emerging global behaviour. There is, therefore, a great interest in the study and development of statistical physics approaches that can describe the formation and evolution of complex networks [5–7].

Networks are ubiquitous in nature where they can have a broad range of features and different degrees of complexity. One of the simplest and most common network is the soap froth that can form fascinating patterns and, when squeezed between two plates, reveals a very simple graph-structure with regular minimal degree equal to three. Despite such simplicity, these networks have highly intriguing structural properties which pose fundamental questions about the delicate balance between degrees of freedoms and constraints that govern the properties of disordered systems and that are at the basis of the emergence of universal rules [8]. In the literature, these three-connected networks, called *topological froths*, have been extensively studied in order to get insights into the properties of disordered struc-

tures [9–13]. *Topological froths* have been also studied as the simplest models for cellular structures which are of great relevance for the understanding of the organization and evolution of biological tissues [14]. The dual structure of a topological froth is a *triangulation* where to each vertex of the froth is associated a triangular face and -conversely- to each froth-cell is associated a vertex of the triangulation. These triangulations are in general disordered and the degree distribution can have a wide range of possibilities from exponential to power-law kinds [12, 15, 16].

Triangulations, and their dual froths, are networks that naturally develop on a surface. However, surfaces are not limited to the two-dimensional Euclidean plane. Indeed, surfaces can bend, they can stretch, they can close on themselves and they can form handles and holes, yielding to complex interfolding structures that can span spaces of arbitrary dimensions and, eventually, become hyperbolic [16]. The complexity of the surface is associated with a single parameter: its genus  $g$ , which is a non-negative integer number counting the handles of the surface [17]. For instance, a spherical surface has no handles and it has  $g = 0$ , a torus has one handle and it has  $g = 1$ , instead  $g = 2$  corresponds to the double torus, etc.. By increasing the genus, the surface becomes more complex and consequently also the triangulation that can be embedded onto it increases in complexity acquiring a larger number of edges ( $number\ of\ edges = 3n - 6 + 6g$ ) and increasing its interwovenness. Eventually, in the limit of very large genus ( $O(n^2)$ ), the complete graph  $K_n$  (a network with  $n$  vertices where every vertex is connected to all the other  $n - 1$  vertices) can be embedded on the surface [18] and the triangulation becomes fully interwoven. The genus is therefore a global measure for the network complexity which is associated to its degree of interwovenness. Let us stress that any network can be made from the pruning of a triangulation embedded on a surface with large enough genus. Therefore, the present studies have a broad applicability.

In a paper with David Sherrington [9] we investigated the relaxation dynamics of triangulations embedded on low genus surfaces ( $g = 1$ ) when they are driven towards the ordered, regular, triangular lattice by means of a greedy algorithm that uses local ordering rules. We discovered that the dynamics of these systems becomes extremely slow, the ground state is never reached in a finite time and the system rests trapped in frozen states characterized by a small fraction of topological defects that cannot be annihilated easily. The appealing aspect of such a *topological glass* is that the typical features of structural glasses are observed in a simple triangulation of a surface [9]. It has been shown that this model has some of the typical properties of a *strong glass* with aging behaviour, but without breakdown of the fluctuation-dissipation relation [9, 19–22]. In this paper, we extend the topological glass model, originally developed in [9], to the general case of triangulations embedded on complex surfaces with different genus.

The paper is organized as follows: in section 2 we introduce a statistical physics framework to study complex triangulations embedded on surfaces with arbitrary genus. In section 3 we present the results from extended numerical simulations, discussing the properties of the embedded triangulations at high and low temperatures and investigating the cooling dynamics between disordered and ordered structures. Conclusions and perspectives are reported in section 4.

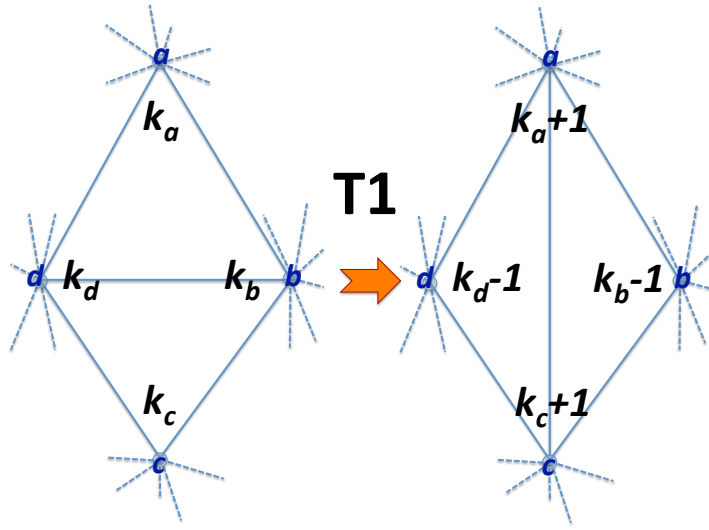


Figure 1. Exemplification of a T1 elementary move, which consists in the switching of one edge between four vertices ( $a, b, c, d$ ).

## 2. Statistical physics of embedded triangulations

In analogy with [9], we adopt a statistical physics framework where the triangulation has an energy

$$E = \sum_{i=1}^n (k_i - \langle k \rangle)^2, \quad (1)$$

where  $k_i$  is the degree of vertex ' $i$ ',  $n$  is the number of vertices in the network and  $\langle k \rangle = 1/n \sum_i k_i$  is the average degree. The ground state has  $E = 0$  and it is reached when all vertices have the same degree  $k = \langle k \rangle$  (if  $\langle k \rangle$  is an integer number). The energy accounts for a quadratic 'cost' associated to local deviations from the average degree. The triangulation can evolve through elementary moves T1 [23], which consist in the switching of an edge in a local configuration of four vertices in which two second neighbouring vertices become directly connected and, vice-versa, two first neighbour vertices become second neighbours, as shown in Fig.1. After the application of a T1 move two vertices (i.e.  $k_a$  and  $k_c$  in Fig.1) acquire a new edge and other two vertices (i.e.  $k_b$  and  $k_d$  in Fig.1) loose one edge leaving the overall  $\langle k \rangle$  unchanged. As a consequence of such a move the energy change is:

$$\Delta E = 2(k_a + k_c - k_b - k_d) + 4. \quad (2)$$

which does not depend on  $\langle k \rangle$  (i.e. it is independent on the embedding).

A 'temperature'  $\beta^{-1}$  can be introduced and a statistical mechanics description can be implemented by using a Glauber–Kawasaki type of dynamics where a T1 move is performed accordingly to the probability

$$\Pi(k_a, k_b, k_c, k_d) = \frac{1}{1 + \exp(\beta \Delta E)} (1 - \delta_{k_b, 3})(1 - \delta_{k_d, 3})(1 - \delta_{a, c}), \quad (3)$$

where the Kronecker deltas prevent the vertex-degrees to become smaller than 3 and avoid the formation of loops. (These two conditions are necessary to maintain

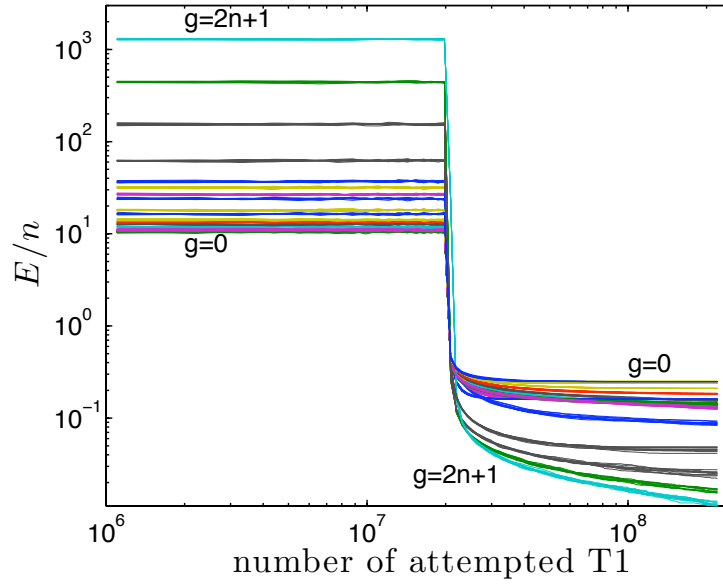


Figure 2. Evolution of the energy per vertex as function of the number of attempted T1 moves.

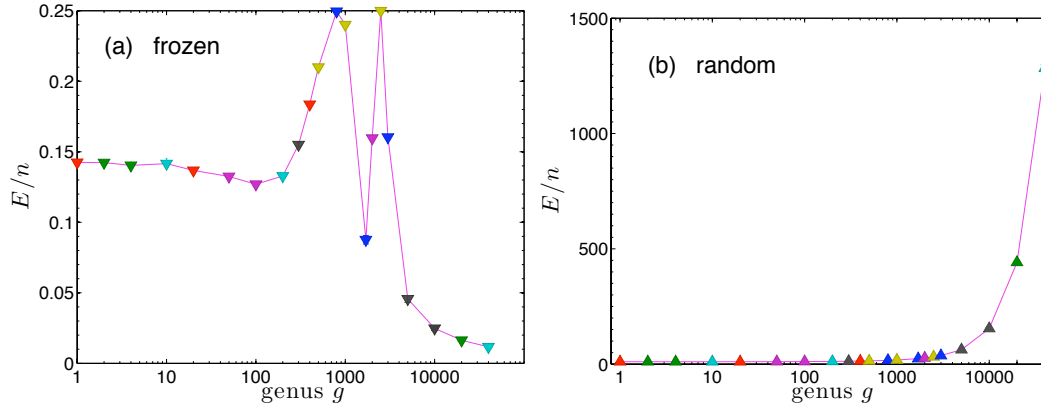


Figure 3. Average energy of the frozen states (a) and of the random states (b) vs. the surface genus  $g$ .

a proper triangulation, however they are not essential from a network perspective and they can be removed).

At high ‘temperatures’ (when  $\beta \rightarrow 0$ ) the probability becomes independent from the degrees of the four nodes (except for the forbidden moves) and the resulting equilibrium configuration, after a large number of moves, is the *random state*. When the temperature is lowered, the system becomes gradually less disordered because the T1 moves, mediated by the probability in Eq.3, favour the redistribution of the degrees locally. At very low temperatures ( $\beta \rightarrow \infty$ ), the probability allows only moves that lower the energy or leave it unchanged, driving therefore the system towards the ground state where all vertices have degree equal to  $\langle k \rangle$  (if it is an integer number). However, it was observed that, for low genus embeddings, a ‘freezing’ transition takes place and the system cannot reach its ground state in any finite time [9, 21, 22]. The general cases with arbitrary genus  $g$  have not been studied yet, and this is indeed the topic of the present paper.

### 3. Numerical simulations

We studied numerically-generated triangulations embedded on surfaces with different genera from  $g = 0$  to  $g = 2n + 1$ . We analyzed samples of various sizes from  $n = 50$  vertices up to  $n = 20,000$ . We followed the evolution of these triangulations from the random state at infinite temperatures to the frozen state at very low temperatures.

Fig.2 shows the changes in the system energy when, after  $n \times 10^3$  T1 moves at  $\beta = 0$  (infinite temperature), a quenching to  $\beta = 10^{+5}$  (very low temperature) is performed. The double-log scale in the plot reveals that the energy drops initially very fast but then it slows down into a glassy-like dynamics. Let us note that the energy is still residually decreasing after the attempted application of  $n \times 10^4 = 2 \times 10^8$  T1 moves. It is clear from the figure that the surface genus has a strong effect on both the random and frozen states. Indeed, we observe that triangulations with large genera have random states with larger energy and conversely they have frozen states with lower energy with respect to the triangulations with small  $g$ . This is shown in Fig.3 where the average energies per vertex ( $E/n$ ) of the frozen and random states are reported. Intriguingly, the dependence of the frozen state energy on the genus is rather complex with an overall decreasing behaviour which is regularly disrupted by peaks at larger energy  $E/n \sim 0.25$ . The decreasing overall trend of the minimal frozen energies with respect to the genus is rather counter-intuitive because it is uncommon that the dynamics becomes more efficient while the system becomes more complex. The origin of the peaks is instead due to a new phenomena associated to *topological frustration*. Indeed, let us note that the average degree of an embedded triangulation depends on the surface-genus

$$\langle k \rangle = 6 + 12 \frac{g-1}{n} , \quad (4)$$

consequently, the present simulations, we are investigating triangulations with a range of degrees between  $\langle k \rangle \simeq 6$  ( $g = 0$ ) and  $\langle k \rangle = 30$  ( $g = 2n + 1$ ). One can verify that the peaks in Fig.3 are associated with  $\langle k \rangle = 6.5$  and  $\langle k \rangle = 7.5$ . Other peaks (not computed) are occurring at  $\langle k \rangle = 8.5, 9.5, 10.5 \dots$ . Indeed, one can note that the energy per node  $E/n = 1/n \sum_{i=1}^n (k_i - \langle k \rangle)^2 = \sigma^2$  coincides with the variance of the distribution. When  $\langle k \rangle = k^*$  is an integer, the minimum variance can be zero. On the other hand, when the average degree is in between two integers  $\langle k \rangle = (k_1^* + k_2^*)/2$ , then the minimum variance is reached when half of the vertices have degree  $k_1^*$  and the other half have degree  $k_2^*$  yielding to  $\sigma^2 = 1/4$ . This is a topological frustration mechanism that forbids the system to reach an ordered ground state.

The frozen states have a simple degree distribution which is peaked around  $\langle k \rangle$  (Fig.4(a)). When  $\langle k \rangle = k^*$  is an integer (or close to an integer), the degree distribution  $p(k)$  is mostly populated by vertices with degree equal to  $k^*$  with only two other small fractions of vertices respectively with degree  $k^* - 1$  and degree  $k^* + 1$ . Similarly, when  $\langle k \rangle$  is in between two integers  $k_1^*$  and  $k_2^*$ ,  $p(k)$  is mostly populated by vertices with degrees equal to the two integers with eventually the marginal presence of some vertices with degrees respectively equal to the next or previous integers. We can observe from the distributions in Fig.4(a) that the large genus triangulations are less disordered with a smaller variance with respect to the triangulations with  $g = 0$ .

In the random state, the degree distribution is strongly dependent on the genus. For  $g = 0$  the functional form of the degree distribution is known analytically:  $p(k) = 16(3/16)^k (k-2)(2k-2)!/[k!(k-1)!]$  [24, 25], which is a function which

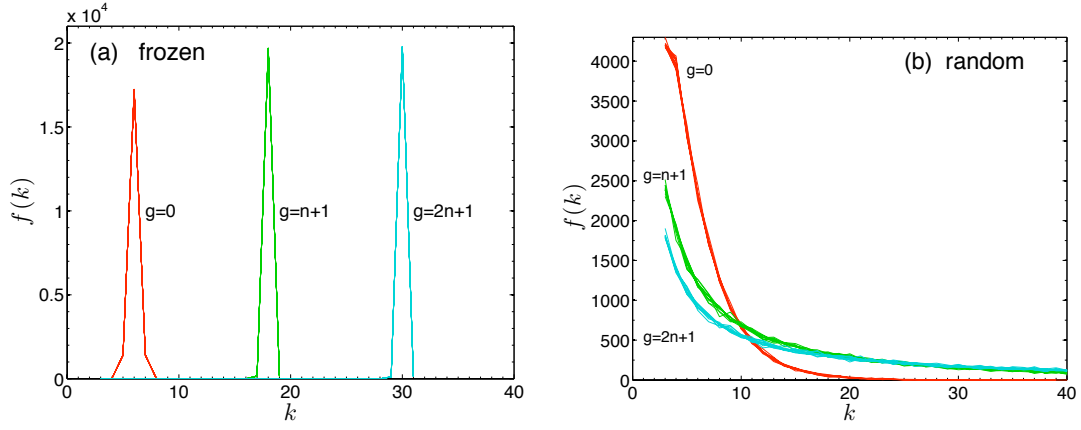


Figure 4. Frequency distributions ( $f(k) = np(k)$ ) of the degrees for (a) frozen and (b) random states at  $g = 0$ ,  $g = n + 1$  and  $g = 2n + 1$ .

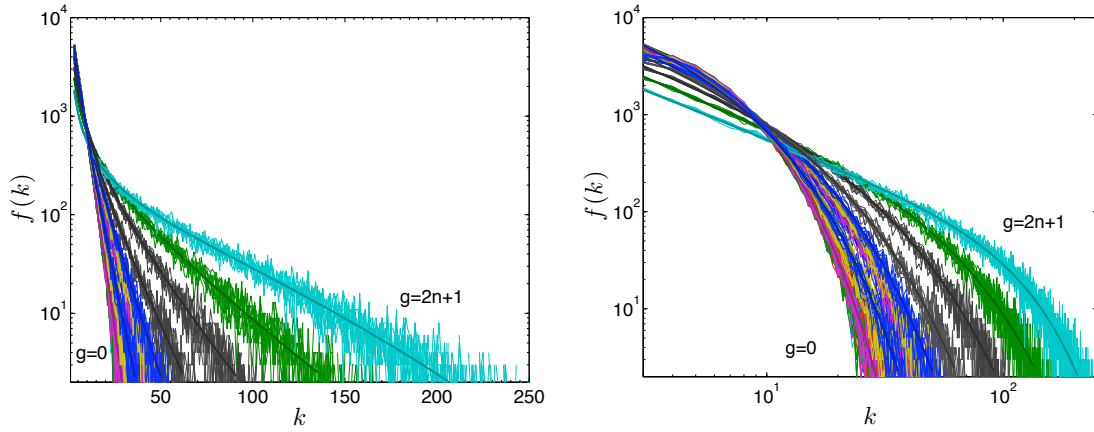


Figure 5. Frequency distribution of the degree for random states at different genera.

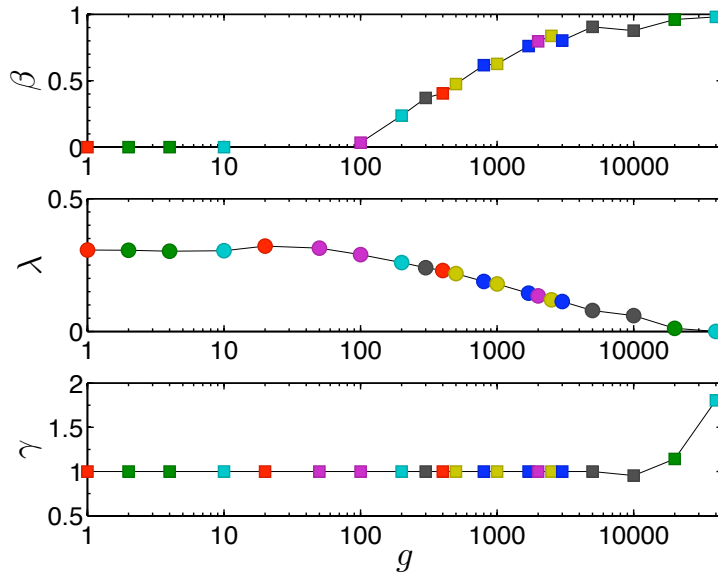


Figure 6. Variation of the best-fit parameters  $\beta$  and  $k_1$ , describing the functional approximant  $p(k) = cx^\beta \exp(-\lambda k^\gamma)$  for the degree distribution, as function of the surface-genus.

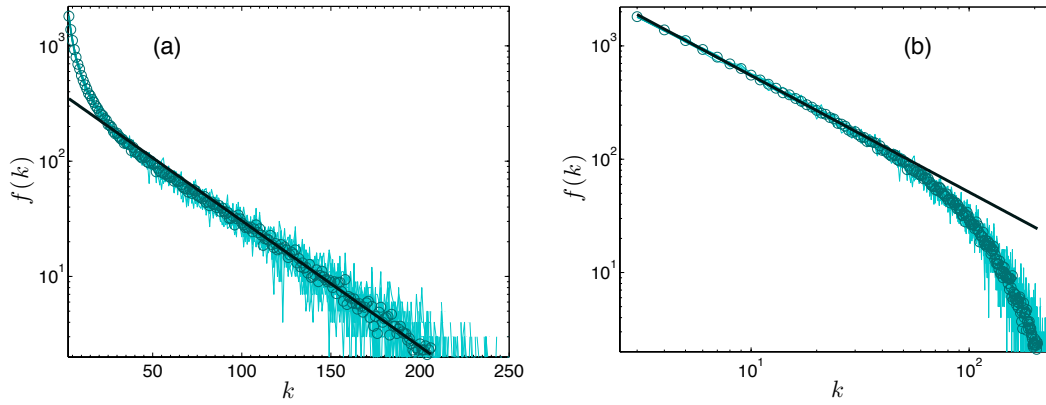


Figure 7. Frequency distribution of the degree for random states for the case  $g = 2n + 1$ . The tick straight lines are respectively: (a) (*log-normal scale*) the fit with an exponential law  $p(k) \sim \exp(-\lambda k)$  in the region  $k > 4\langle k \rangle$  ( $\lambda = 0.025$ ); (b) (*log-log scale*) the fit with power law  $p(k) \sim k^{-\beta}$  in the region  $k < \langle k \rangle$  ( $\beta = 1.03$ ).

decreases exponentially in the tail region of large  $k$  but it decreases slower than exponential for small  $k$ . On the other hand, we do not have an analytical expression for arbitrary genera. However, let us point out that the observed degree distributions can be described by using the functional form:

$$p(k) \sim ck^\beta \exp(-\lambda k^\gamma) . \quad (5)$$

The tick lines in Fig.6 show that the best-fitting plots from Eq.5 are in a rather well agreement with the behaviour of the observed frequency distributions ( $f(k) = np(k)$ ). The best-fitting values of the parameters at different genera are shown in Fig.6. We can observe that initially, at low genus, the curves are well fitted with large values of  $\lambda$ , small values of  $\beta$  and with  $\gamma \sim 1$ , revealing a good exponential behaviour. Conversely, at large genus, the value of  $\lambda$  decreases and  $\beta$  approaches  $-1$  revealing in this case an initial power law behaviour followed by a cut-off for  $k > 1/\lambda$ . Figure 7 shows in details this double behaviour for the case  $g = 2n + 1$ , in this case we also fitted the region of small  $k$  with a power law  $p(k) \sim \exp(-\lambda k)$  and the region of large  $k$  with an exponential law  $p(k) \sim \exp(-\lambda k)$  obtaining  $\beta = 1.03$ ,  $\lambda = 0.025$ . We can see that the exponential fit is satisfactory although the curve appears to be slightly bending downwards. Indeed, we can observe that in the fits with Eq.5 the parameter  $\gamma$  increases significantly above 1 for large genus (see Fig.6) indicating that the cut-off is faster than exponential.

#### 4. Discussion and conclusions

We have studied triangulations embedded on complex surfaces with different genera. Despite the fact that this is a very special class of graphs, we must stress that any graph can be embedded on a surface with large enough genus and triangulations are the embedded networks that maximize the number of edges for a given surface. Therefore, any network can be represented as a pruned triangulation of a surface. The genus is a global topological quantity that characterizes the network property and it is associated to the level of global interwovenness. We have shown that embedded triangulations can be conveniently described by means of a statistical mechanics model where the network can evolve within the embedding surface by local elementary moves T1. Numerical results, obtained from triangulations with broad ranges of sizes and different genera, show that the network properties are affected by the value of the genus. At high temperatures the degree distribution

becomes broader with increasing genus with an initial power-law kind of trend followed by a faster than exponential cut-off. The relaxation dynamics towards ideal, ordered states at low temperatures reveals a glass transition which is affected by the genus. We observe an overall trend where it turns out that high genus triangulation can order faster and better than the low genus counterparts. However, we also observe that the genus introduces a new element of topological frustration when the average connectivity is a fractional number forcing the freezing into irregular high-energy states. Future development would concern the application of these triangulated surfaces as network tools for information filtering extending to complex surfaces the PMFG technique introduced recently [26–29].

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