SI 10/24

Extra midterm practice problems

(1) Let $u = \langle 2t, -3, 5t \rangle$

$v = \langle 5, -t^2, -2 \rangle$

Find ALL values of t, if they exist, such that u and v are orthogonal

Let $u = \langle 2t, -3, 9 \rangle$

$v = \langle 0, -t^2, -3 \rangle$

Find ALL values of t, if they exist, such that u and v are orthogonal

$u \cdot v = 0$. $2t \cdot 0 + 3t^2 - 9 \cdot 3 = 0$

$$3t^2 - 27 = 0$$

$$3(t^2 - 9) = 0$$

$$t^2 - 9 = 0$$

$$(t+3)(t-3) = 0$$

$$t = -3, t = 3$$

2) $v = \langle 2, 3, -2 \rangle$

$u = i - 2j + k$

a) (i) Find $v \times u$

(ii) Find $u \times v$ (can we do this easily?)

b) What is the area of the parallelogram formed by u and v?

c) What is the volume of the parallelepiped formed by u, v, and the vector $w = \langle 2, 0, 2 \rangle$?

d) Are the three vectors coplanar?

a) $v = \langle 2, 3, -2 \rangle$

$u = i - 2j + k = \langle 1, -2, 1 \rangle$

$v \times u = \langle 3-4, -2-2, -4-3 \rangle$

$= \langle -1, -4, -7 \rangle$

$$\begin{matrix} 2 & 3 & -2 \\ 1 & -2 & 1 \end{matrix}$$

(ii) $u \times v = -(v \times u) = \langle 1, 4, 7 \rangle$

b) $\|v \times u\| = \sqrt{1^2 + 4^2 + 7^2}$

$= \sqrt{1 + 16 + 49}$

$= \sqrt{56}$

c) Triple scalar product

$u \cdot (v \times w) = w \cdot (u \times v)$

$= \langle 2, 0, 2 \rangle \cdot \langle 1, 4, 7 \rangle$

$= 2 \cdot 1 + 0 + 2 \cdot 7 = 2 + 14 = 16$

$|16| = 16$

d) Are they coplanar?

No!

Because $|u \cdot (v \times w)| \neq 0$ so the 3D shape has some volume, they cannot all lie in a plane.

3) a) Is the line $r(t) = \langle 0, 1, 5 \rangle + \langle 4, -2, 0 \rangle t$ parallel to the plane $x + 2y - z + 3 = 0$?

b) If so, does it lie on the plane?

3) a) Is the line $r(t) = \langle 0, 1, 5 \rangle + \langle 4, -2, 0 \rangle t$ parallel to the plane $x + 2y - z + 3 = 0$?

b) If so, does it lie on the plane?

a) Check if direction vector matches "slope" of the plane. How can we do this?

✲ must be orthogonal to plane's normal vector

$$\langle 4, -2, 0 \rangle \cdot \langle 1, 2, -1 \rangle = 4 - 4 + 0 = 0 \quad ✓$$

$\boxed{\text{Yes.}}$

b) Check if any point on the line lies on the plane. Since the line travels parallel to the plane, this suffices to show it lies on the plane (the line cannot leave the plane)

Pick $r(0) = \langle 0, 1, 5 \rangle + \vec{0}$

test to see if it works w/ plane eq.

$$0 + 2(1) - 1(5) + 3 = 0$$
$$2 - 5 + 3 = 0$$
$$0 = 0 \quad ✓$$

$\boxed{\text{Yes}}$

4) A UFO has been spotted in the Revelle College area! We don't know what they want, but it's important that we track its strange motions in 3 dimensions. According to our sensors, its position can be calculated as a function of time: $r(t) = 7t\,i + \frac{1}{2}\sin(2t)\,j + \frac{1}{2}\cos(2t)\,k$.

We need you to help figure out:

a) The UFO's velocity as a function of t

b) The UFO's speed

c) The UFO's acceleration

d) Cosine of the angle between v(t) and a(t)

$$r(t) = 7t\,i + \frac{1}{2}\sin(2t)\,j + \frac{1}{2}\cos(2t)\,k.$$

$$= \langle 7t, \tfrac{1}{2}\sin(2t), \tfrac{1}{2}\cos(2t) \rangle$$

a) $v(t) = r'(t) = \langle 7, \cos(2t), -\sin(2t) \rangle$

b) speed $= \|v(t)\| = \sqrt{7^2 + \cos^2(2t) + \sin^2 2t}$

$$= \sqrt{49 + 1} = \sqrt{50}$$

c) $a(t) = v'(t) = \langle 0, -2\sin(2t), -2\cos(2t) \rangle$

d) $u \cdot v = \|u\| \|v\| \cos\theta$

$$\cos\theta = \frac{v\,a}{\|v\|\|a\|}$$

$$\cos\theta = \frac{7\cdot 0 + (-2\sin(2t)\cos(2t)) + (-2\cos(2t)(-\sin(2t)))}{\|u\|\,\|v\|}$$

$$\cos\theta = \frac{-2\sin(2t)\cos(2t) + 2\sin(2t)\cos(2t)}{\|u\|\,\|v\|}$$

$$= \frac{0}{\|u\|\,\|v\|} = 0$$

$$\cos\theta = 0 \Rightarrow \boxed{\theta = \frac{\pi}{2}}$$

5) Find the vector projection of $\langle 3, 2, -1 \rangle$ onto $\langle 1, -1, -2 \rangle$

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$$\left(\frac{u \cdot v}{u \cdot u}\right) u = \frac{3-2+2}{1+1+4} u = \frac{3}{6} u = \frac{1}{2} u$$

$$= \boxed{\left\langle \frac{1}{2}, -\frac{1}{2}, -1 \right\rangle}$$

6) Let v be the vector from $(4,2,-3)$ to $(2,0,2)$. Let w be the vector from $(6,5,365)$ to $(7,-1,366)$.

(i) Find $v+w$

(ii) Find $2v-w$

(iii) Find $\text{proj}_w(v)$

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(iii) Find $\text{proj}_w(v)$

$v = (2,0,2)-(4,2,-3) = \langle -2,-2,5 \rangle$

$w = (7,-1,366)-(6,5,365) = \langle 1,-6,1 \rangle$

(i) $v+w = \langle 1-2, -6-2, 5+1 \rangle = \langle -1,-8,6 \rangle$

(ii) $2v-w = \langle -4,-4,10 \rangle - \langle 1,-6,1 \rangle = \langle -5,2,9 \rangle$

(iii) $\text{proj}_w v = \left(\frac{v \cdot w}{w \cdot w}\right) w$

$$= \frac{-2+12+6}{1+36+1} w = \frac{16}{38} w = \frac{8}{19} w$$

$$= \boxed{\frac{8}{19} \langle 1,-6,1 \rangle}$$