FLORIDA ATLANTIC UNIVERSITY

DOCTORAL THESIS

Thesis Title

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Chapter 1

Background Information

1.1 Introduction

According to [1] Game theory is a mathematical tool and conceptual framework that can be used to study complex, self-interested interactions among rational players. A simpler explanation is that Game theory is the science of decision making. It originated in the minds of great economists like Von Nuemann and Morgeston as early as the 1940's as a way to explain the behaviour economics. It was then further developed to be used in financial markets. The theory really thrived in the 1950's when John Nash published several works around non-cooperative Games that where applied more broadly. Unfortunately in later years the discovery that a well-defined value system must be first defined before expecting the solutions dervied through Game theory to actually be accurate accoring to [1]. The varying nature of human value systems prevented significant in-roads in areas where human players where considered, but it did not stop the application to various fields with defined constraints such as control-loops, war-fare, ecology, etc..[1]

There are many definitions for what a *game* is in Game theory, but lets consider the definition proposed by [2] where they state that a game is any situation involving more than one individual, each of which can make more than one action, such that the outcome to each individual, called the *payoff*, is influenced by their own action, and the choice of action of at least one other individual. In other words a situation where rational agents take actions according to a strategy to maximize their own utility. Many situations can be arranged in this way to solve complex systems involving many decisions. [2]

A way to think of what is meant by *strategy* in Game theory lets consider the analogy proposed by [19] by imagining you where playing a game of chess. The caveot is that you are playing via a representative, so in order to play the game you must create a set of instructions for every possible circumstance that may arise during the game. That means for the first move they make you must design outline the response for every reaction of the opponents move. The entire set of these decisions all the way until the end-game is a strategy for the player. A deterministic game like chess can be mapped out this way (even if it would be a very tedious excercise). These types of games are called finite because tehre are a limited number of moves. They are also considered a zero-sum game because one players gain equals the other players loss. Also, this game can be considered a compettitive game or a *Non-Cooperative* game because both players are competing to increase their own utility at the expense of the other.

Game theory can be categorised into three main branches: *Non-Cooperative* game theory, as mentioned above, *Cooperative* game theory (coalitional game theory) and relatively recent branch called *Evolutionary* game theory (EGT). The contributions to game theory can be found in Figure 1. as presented in [3], it shows cooperative game theory was founded by Neumann and Morgenstern [10], and then non-cooperative game was represented by Nash's work [11],[12],[13],[14].

John Nash proved the existence of non-cooperative game solution, that is, the existence of Nash equilibrium (further explained later), thus laying the theoretical foundation of the modern non-cooperative game. Non-cooperative game theory essentially studies the settings where multiple payoff-maximising players, who have partially or totally contradictory interests over the system and/or personal outcomes, interact with each other.

Cooperative games are based on the Shapley value, a value that calculates the *fair* payoff of a player in a coalition of players. In a cooperative game all players are aware of the strategies chosen by other players. The goal in a cooperative game is to maximize the reward of all players involved. An example would be a group of homes with energy storage capacity and roof-top solar power generation working together to maximize overall revenue by selling energy at peak demand and meeting dynamic loads between them.

As for the evolutionary game, it is generally recognized that it was officially founded by Maynard Smith and Price [15]. This theory can be regarded as an organic combination of general game theory and dynamic evolution process [16]. Among these, the former focuses on the game problem within the framework of *bounded rationality* rather than *complete rationality*, while the latter draws on the biological evolution theory in biology field. In short, the decision-making stakeholders (i.e., playersor participants) in an evolutionary game constantly adjust their own strategies according to environmental changes and the strategies of other decision-making stakeholders in order to adapt to the game environment under the conditions of limited knowledge, information and reasoning ability [9], [17].

In conclusion Game theory has evolved from combinatorial analysis of decision making in economics, to an encompassing theory that can apply to acting on complex strategies in various disciplines. A game consists of rational agents called players, a set of decisions called strategies, and payoff's that maximize the players utility in some way. The major branches of Game theory are cooperative, non-cooperative and evolutionary. Each branch addresses various kinds of games found in nature and in many different disciplines. In the following section we explain the element sof a game in greater detail. The chapter will then continue to review the details of each branch of game theory and provide examples to illustrate the approaches.

1.2 Elements of Game Theory

1.2.1 Pay-Off Matrix

In an n-player game, the goal of the participants is usually to minimize payments or maximize incomes. Based on this, a typical n-player game [9] can be expressed as

$$G = [N; S_1, S_2, ..., S_n; u_1, u_2, ..., u_n]$$

(1)

Where the game is G, and N represents a set of players in the n-player game. The S series represents the strategies of each n-player. Finally the u represents the payment for each n-player.

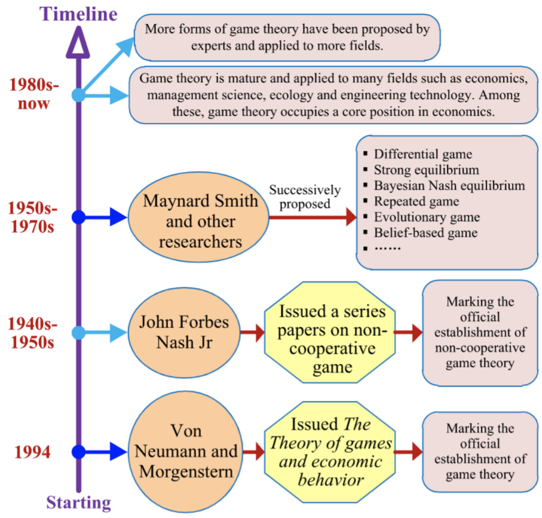

FIGURE 1.1: Evolution of Game Theory

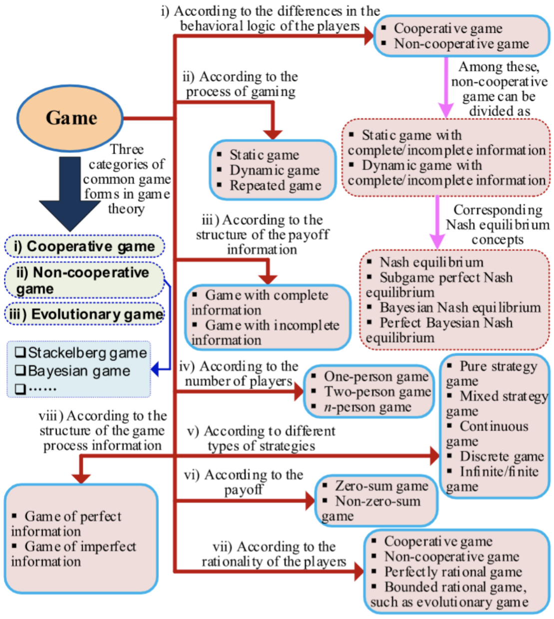

FIGURE 1.2: Classification of Game Theory

1.2.2 Pay-Off Matrix

The normal-form representation for the standard game model is the pay-off matrix as described by [2]. Each column and row represents the associated payoffs for each strategy as a combination of the two players. Given a strategy 'A' and a strategy 'B' a matrix of the game can be drawn to represent the responses of Player1's (P1) decision of strategy A and B while Player 2 (P2) chooses either A or B. The value at each cell then represents the reward/loss of the decision for each player.

1.2.3 Pay-Off Matrix Example

A good example to illustrate the pay-off matrix is to consider the idea of stop lights in a busy intersection. Both drivers can be considered as players in this game and the decisions they can take (stop,go) can be expressed in a combinatory manner using the normal-form representation.

The rewards for the game can be arbitrarily estimated given the desire of either players. If both players decide to "Go" at the same time then they will end up in a collision. This does not benefit either player so we place the reward for each accordingly (-5,-5). If either of the drivers have to stop to let the other go then the player who is stopped is inconvenience (reward of 0), while the driver who was able to go is now on its way and happy (reward of 1). The final strategy is if both players are stopped and niether move forward wasting time but not in a collision so not as bad but still a loss (reward of -1).

		Player 1	
		Stop	Go
Player 2	Stop	-1,-1	0,1
	Go	1,0	-5,-5

FIGURE 1.3: Pay-off matrix for 2 players.

The possible decisions of the players in the game are mapped out in the game matrix. The strategy for each player can now be assesed. The goal would be to find the best strategies for player 1 and for player 2 respectively. For player 1, the strategy where he is able to go is when player 2 must stop (1,0). The best strategy for player 2 is the opposite (0,1). Since there are no better moves for each player to take these are both known as Nash Equilibriums. The equilibrium of a non-cooperative game is when either player has no other decisions available that would better thier individual positions. There can be more than one nash equilibrium in a game .

1.2.4 Components of a Game

1) Players/Participants : The participants in the game who can decide their own strategies. An important assumption is that the players are rational, which means

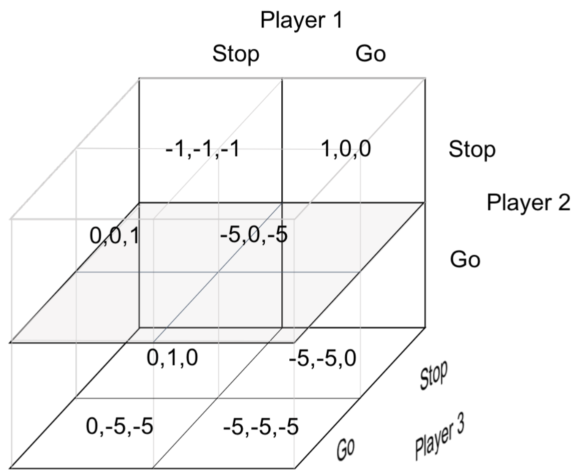

FIGURE 1.4: Pay-off matrix for 3 players.

that they don't leave things to chance and don't take advantage of others' mistakes. If we assume rational agents that have all rationality and all relevant information then the agent can model the game as an optimization problem. Unfortunately people do not always choose the most rational choice in real life. This then reduces the accuracy of classical game theory. Optimization is not just a single value system but many that also optmizes accross social and cultural systems.[1]

2) Strategy : A collection of all the strategies available for all players. According to whether a player's strategies are finite, we can divide games into three groups: *finite games* and *infinite games*. Sometimes a strategy set can be infinite since it can by almost any option, or only 3 for example in the game of rock, papers, scissors. There are some strategies that do not involve randomness (finite strategy) such as to tic-tac-toe.finite games are also known as *matrix games*. An *infinite game* is said to be a *continuous-kernel* game if the action sets of the players are continua, and the players' objective functions are continuous with respect to action variables of all players.[5]

3) Pay-offs/Payment: After each player in the game chooses a strategy from their own strategies set, a relevant result (a group of data) is provided to show each player's gain or loss. A good payoff is the fundamental goal of a player, and is the main basis of a player's judgment and behavior. Pay-offs are also known as the players preferences described as 'utility' and must be understood clearly for the model to be effective. The payoffs are very subjective. If there is a ranking to the payoff it is called *ordinal* , if it is just a subjective quantity value then it is a *cardinal* payoff.[1]

Honorable mention, *Order* , technically an overlooked aspect of games is the order in which players make a decision.

4) Orders: When different players are about to decide, there is a need to decide the orders. Sometimes players make their decisions at the same time to make sure that the game is fair; sometimes players make decisions one after another; most times in reality players may choose their strategies more than once. Different orders result in quite different situations.

1.3 Cooperative Games

1.3.1 Shapley

Write your subsection text here.

1.4 Non-Cooperative Games

1.4.1 Non-Cooperative Game Models

The different types of models as defined in [4] consists of the following:

1) Ordinary Non-Cooperative game: Analyze the problem and abstract the players, strategies and the pay-offs for the model. The properties of the model should result in equilibrium solutions that could solve the problem.

2) Generalized Nash equilibrium game: Each player's decision affects not only other players' payoffs, but also their feasible strategies set.

3) Cournot game: Derived from prisoner's dilemma model. Models the pay off capable in a game when both players will need to implement the strategy at the same time. Used in behavioral economics to determine pay-off of individual decisions.

4) Stackelberg game: Similar to the Cournot game, but assumes that there is a player that is a leader (goes first) and a player that goes after (follower). The model has the leader anticipate the followers response and include that in its own response. This technique helps the leader to optimize its utility in a market shared among two players.

5) Bounded rationality game: Models that attempt to remove the assumption that players are completely rational. Considering that players can take actions that do not immediately benefit them can possibly model the uncertainty of initial value assumptions of the utility modeled for each player.

6) Repeated game: A series of basic games one after the other. This type of model shows the varied results capable given the strategies implemented in each instance of a game.

1.4.2 Stackelberg Model Simulation

A Stackelberg Game is a type of noncooperative game that deals with a multi-level decision making process of a number of independent decision makers or players(followers) in response to the decisiont taken by a leading player (the leader).

The game has the following components:

a) Followers in the game that respond to a price set by the leader

b) The strategy of each of the followers in terms of satisfying some constraint.

c) The utility function of each follower that captures the benefit of consuming the demand.

d) The utility function of the leader which captures the total profit.

e) The price per unit quantity

The utility function of the follower represents the level of satisfaction of the follower. It is a function of the profit it recieves. The utility function is non-decreasing because each follower is interested in maximizing its profit. The marginal benefit of the follower is also considered a non-increasing function. The marginal benefit gets saturated the closer it gets to the maximum profit. The utility of a follower decreases as the price of a unit or cost of meeting a demand increases .

If the constraint is the same for all players then this gives rise to a noncooperative resource sharing game between the followers. A game l ike this represents a jointly convex generalized Nash equilibrium problem (GNDP) due to the same shared constraint. In game theory the coupling of the constraints has solution called the Gneralized Nash Equilibirum(GNE).[3]

The stackleberg duoply is a non-cooperative game between two players where one is a leader and the other is a follower. They are the only two players able to supply the needs of the market by creating some quantity of goods. Each player has a utility function that will be satisfied if they can maximize the profit of supply a quantity of goods to the demand curve of the market considering the cost to provide a unit of goods. [5],[7]

Define a pricing demand as a linear function and call it a "Market Demand Curve"

"' P = a - b * Q "'

Where Q is the market quantity demanded and P is the market price in dollars The firms create the quantity.

The quantity created for the market comes from multiple firms. The firms in a Stackleberg game provide some quantity after the leader firm goes first. The leader firm assumes the moves of the other firms and tries to maximize its profit by incurring the costs to meet the demand that it deems appropriate given its own costs. The demand is met through a total quantity that can be represented by the combination of all of the players quantity.

"' Q = q1 + q2 + ... qN "'

The cost to meet a single unit of demand per unit qunatity created. This is a predefined metric or can be a dynamic value. Knowing the marginal costs is critical to determining the other players moves. The Leader needs to know what it would cost other players to take action.

Begin backward induction to determine what the reaction would be of the other firms. Lets assume two firms A, and B. The procedure to determine the Leaders (firm A) move would be as follows:

1. Calculate Firm B's reaction 2. Calculate Firm A's response to B's reaction 3. Implement Firm A's response 4. Calculate Firm B's response given A's response 5. End Game

$$N_{firms} = 2$$

(2)

$$MC_a = 10$$

(3)

$$MC_b = 12$$

(4)

$$P_T = -Q_d m + b$$

(5)

$$P_T = -Q_d * 0.5 + 120$$

(6)

By inspecting the demand curve we can see that all the quantity generated by the payers (x-axis) will result in a total price for all the quantity (y-axis) at different levels of demand met. The pricing demand curve can now be used as a total demand and total quantity output that will be presented to the players. At some break even point the price demand for a unit is no longer advantageous considering the cost to the player to meet that demand.

The break even point would be the profit maximizing point for the player. In the stackleberg game the leader tries to maximize its output by looking at the break even point of the secondary player. If the marginal costs are lower for the follower they can generate more quantity and outsell the leader. This means the leader should make enough to break even and just enough to reduce the gains of the follower.

Again, by only looking at the demand curve the leader can only determine a break even point based on its own cost. As soon as it costs more to meet the demand than the price of the demand then the leader stops and no longer produces. If the demand must be met, then the rest of the demand is left for the second player to take on. If the second player looks at the remaining demand and only supplies what it can break even then both players are left supplying demand with diminishing returns.

To avoid having to supply diminishing returns the leader can take into account the maximizing move for the second player and then include that in determining its stake in the market based on its own costs and break even point.

Total Market Quantity Demand:

$$Q_d = q_a + q_b$$

(7)

Market Demand Curve:

$$-0.5 * Q_d + 120 = -0.5 * q_a + -0.5 * q_b + 120$$

(8)

$$TR_b = -0.5 * q_a * q_b - 0.5 * q_b^2 + 120 * q_b$$

(9)

Marginal revenue can be derived from the derivative of the total revenue equation (9), with respect to the firm.

$$MR_b = \frac{\partial d}{\partial q_b}(-0.5 * q_a * q_b - 0.5 * q_b^2 + 120 * q_b)$$

(10)

$$MR_b = -0.5 * q_a - q_b + 120$$

(11)

The reaction of the follower can be estimated by the leader by solving when the marginal cost of the follower will equal the marginal revenue of the follower. Setting them equal and solving for the quantity in terms of the quantity provided by the leader firm A we get the following reactionary quantity for firm B.

$$q_b^* = -0.5 * q_a + 108$$

(12)

The leader takes into account all the reactions and creates a leader response to the reactions. The approach is to use back induction to take the forecasted reaction of the follower when the marginal cost is equal to the marginal revenue (profit maximizing). That means that the follower will stop at some break even point that maximizes its profits. Given that information the leader can take that reaction and assume it is what the follower will do. The assumption is used in the price demand formula. By substituting equation (12) into equation (8) the demand for the leader is now shown below in (13). Knowing the demand the total revenue and marginal revenue can also be inferred.

$$P_{leader} = -0.25 * q_a + 66$$

(13)

$$TR_a = -0.25 * q_a^2 + 66 * q_a$$

(14)

$$MR_a = \frac{\partial d}{\partial q_a}(-0.25 * q_a^2 + 66 * q_a)$$

(15)

$$MR_a = -0.5 * q_a + 66$$

(16)

The last step in the backwards induction for the leader is to try to maximize his profit by only providing a quantity where the followers quantity has been considered in the game. The final quantity to be implemented by the leader is done just as before by setting marginal revenue equal to marginal cost and finding quantity.

$$q_a^F inal = 112$$

(17)

If we implement the leaders quantity output and then recalculate the resulting quantity for the follower, the follower will have a different output than what was orginally measured by the leader since the leader has now taken its maximum demand.

$$q_b^F inal = 52$$

(18)

Appendix A

Source Code

A.1 Stackelberg Example with Python

```
init_printing(use_latex='mathjax')
from IPython.display import display
import string
alpha = list(map(chr, range(97, 123)))

firms = 2

N = symbols('N_{Firms}')
display(Eq(N, firms))

# Marginal Cost
MC = [symbols('MC_%s'% i) for i in alpha]
for i in range(firms):
    cost = 10 + i*2
    display(Eq(MC[i],cost))
    MC[i]=cost

# General Market Demand Curve
b,m = symbols('b,m')
P_d = symbols('P_{T}')
Q_d = symbols('Q_{D}')
display(Eq(P_d,b-m*Q_d))
b = 120
m = 0.5
display(Eq(P_d,b-m*Q_d))
P_d=b-m*Q_d

# Total Market Quantity Demand
q = [symbols('q_%s'% i) for i in alpha]
Q = sum(q[i] for i in range(firms))
display(Eq(Q_d,Q))

# Market Demand Curve
P = b - m * Q
display(Eq(P_d,P))

# Total Revenue
```

```
TR = [symbols('TR_%s'% i) for i in alpha]
for i in range(firms-1):
    display(Eq(TR[i+1],expand(P * q[i+1])))
    TR[i+1]= expand(P * q[i+1])

# Marginal Revenue
MR = [symbols('MR_%s'% i) for i in alpha]
for i in range(firms-1):
    display(Eq(MR[i+1],Derivative(TR[i+1],q[i+1])))
    display(Eq(MR[i+1],Derivative(TR[i+1],q[i+1]).doit()))
    MR[i+1]= Derivative(TR[i+1],q[i+1]).doit()

# Reaction Functions :
qq = [symbols('q^{*}_%s'% i) for i in alpha]
for i in range(firms-1):
    display(Eq(qq[i+1],solve(MR[i+1] - MC[i+1],q[i+1])[0]))
    qq[i+1]=solve(MR[i+1] - MC[i+1],q[i+1])[0]

# Leaders market demand in terms of leader quantity
P_0 = P
P_a = symbols('P_Leader')
for i in range(firms - 1) :
    P_0 = P_0.subs(q[i+1],qq[i+1])

display(Eq(P_d,P))

display(Eq(P_a,P_0))

TR_0 = expand(P_0 * q[0])
TR_a = symbols('TR_a')
display(Eq(TR_a,TR_0))

MR_0 = Derivative(TR_0,q[0]).doit()
MR_a = symbols('MR_a')
display(Eq(MR_a,Derivative(TR_0,q[0])))
display(Eq(MR_a,MR_0))

# Most profit maximizing quantity for leader is q_0
q_0 = solve(MR_0-MC[0],q[0])[0]
q_a = symbols('q^Final_a')
display(Eq(q_a,q_0))

# Reactions Taken :
qqq = [symbols('q_%s__Final'% i) for i in alpha]
for i in range(firms-1):
    display(Eq(qqq[i+1],qq[i+1].subs(q[0],q_0)))
    qqq[i]=qq[i+1].subs(q[0],q_0)
P_d=b-m*Q_d
display(Eq(Q_d,Q))
q[0] = q_0
for i in range(firms-1):
```

```
        q[i+1] = qqq[i]
```