STACKELBERG SOLUTION OF DYNAMIC GAME WITH CONSTRAINTS

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Abstract

The paper presents the solution of the dynamic closed-loop Stackelberg game, discusses the interpretation of the leader's strategy as the formalization of the intuitive notion of threat or incentive, and considers the limitations of the Stackelberg solution concept which within the dynamic context is applicable only to the situations where the realization of the leader's strategy in the course of the game is ensured by a binding contract. It is shown that the supremum S of the leader's payoff over his strategy space can be computed by solving a maximin problem and an optimization problem. In general the leader can achieve the payoff arbitrarily close to S by means of ε- Stackelberg strategies, but the Stackelberg strategy ensuring the payoff equal to S exists only in rather exceptional cases /e.g. the linear-quadratic game/.

Introduction

The recent activity in the area of dynamic closed-loop Stackelberg games has resulted in the formulation of several Stackelberg strategies for the linear-quadratic problem [1,2,3,4] and, very importantly, in the realization that by formulating his strategy the leader actually defines threats or incentives aimed to induce the follower to act in the way that benefits the leader [3,4,5]. The linear-quadratic game corresponds to a rather specific situation where the leader can decrease the follower's payoff by an arbitrary number, or in other words, the leader's ability to threaten the follower is unlimited. Thus it is not surprising that in such a game the leader can formulate a strategy enabling him to achieve the absolute maximum of his payoff, moreover the family of such strategies is very large and some of them have some additional desirable properties as regularity or insensitivity to unintentional errors made by the players in the course of the game.[3] If the game is not linear-quadratic then the leader's ability to formulate threats or incentives may be limited, for example by constraints on the set of admissible controls. In such a case the leader may not be able to enforce on the follower the solution maximizing his own payoff and he can be much more limited in his choice of the Stackelberg strategy.

The purpose of this paper is to present the solution of the general dynamic closed-loop Stackelberg game and to discuss limitations of the Stackelberg solution concept within the context of dynamic problems. It should be noted that the approach of this paper has some common points with the method proposed in reference [6] for the solution of the differential Stackelberg game. However, the formulation of ε- Stackelberg strategy given there is incorrect due mainly to the author's failure to recognize the closed-loop memory, and not pure feedback, character of the Stackelberg strategy within the dynamic context.

The Stackelberg Game

Consider a nonzero-sum game with two players L and F. Let Γ_L and Γ_F be the strategy sets for L and F respectively, with $\gamma_L \in \Gamma_L$, $\gamma_F \in \Gamma_F$. Let $J_L : \Gamma_L \times \Gamma_F \to R$ and $J_F : \Gamma_L \times \Gamma_F \to R$ be their corresponding payoffs /utility functions/. Assume that the following conditions are satisfied:

(S1) L declares his strategy $\gamma_L \in \Gamma_L$ before F.

(S2) L is compelled to act according to the declared strategy throughout the whole game /as the result of a binding contract or following his long-term interests/.

In the sequel such a game will be referred to as the Stackelberg game /SG/, with L being the leader and F being the follower. The equilibrium solution of SG is based on the following idea[7].
For any given strategy γ_L chosen by L, F tries to choose a reaction strategy $\gamma_F^o \in \Gamma_F$ which maximizes his own payoff J_F. Knowing that L seeks to announce a strategy $\gamma_L^s \in \Gamma_L$ such that with this strategy, and F's reaction to it, L's maximum payoff is achieved. Let

$$R(\gamma_L) = \{\gamma_F^o \in \Gamma_F \mid J_F(\gamma_L, \gamma_F^o) \geq J_F(\gamma_L, \gamma_F) \text{ for all } \gamma_F \in \Gamma_F\} \tag{1}$$

be the rational reaction set of F. Then γ_L^s is the Stackelberg strategy of L if

$$\inf_{\gamma_F \in R(\gamma_L^s)} J_L(\gamma_L^s, \gamma_F) \geq \inf_{\gamma_F \in R(\gamma_L)} J_L(\gamma_L, \gamma_F) \quad \text{for all } \gamma_L \in \Gamma_L \tag{2}$$

Moreover, any $\gamma_F^s \in R(\gamma_L^s)$ is said to be the Stackelberg strategy of F, and the pair $\gamma^s = (\gamma_L^s, \gamma_F^s)$ is called the Stackelberg solution of the game.

The classical definition of the Stackelberg solution is concerned with the game in the strategic form. It appears however /see references[3,4,5,8]/ that the actual meaning of this solution concept depends in a crucial way on more specific characteristics

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of the game problem, involving the definition of the game in the extensive form, and namely on an information structure of the game characterising the precise information gained or recalled by each player at every stage of the game. A game in the extensive form can be defined by specifying for each player his decision set, information set, payoff function and strategy set which consists of a class of mappings from the information set to the decision set. Let U_L, E_L and U_F, E_F be the decision and information sets for L and F respectively, with $u_L \in U_L$, $e_L \in E_L$, $u_F \in U_F$, $e_F \in E_F$. Let $J_L : U_L \times U_F \to R$ and $J_F : U_L \times U_F \to R$ be their corresponding payoffs. Assuming $E_L = \emptyset$, $E_F = U_L$, $\Gamma_L = U_L$, Γ_F to be the set of all mappings from U_L to U_F, and $J_L(\gamma_L, \gamma_F) = J_L(u_L, \gamma_F(u_F))$, $J_F(\gamma_L, \gamma_F) = J_F(u_L, \gamma_F(u_F))$ one obtains the classical Stackelberg problem. Quite a different problem is obtained however if the information structure of the game is of the form $E_L = U_F$, $E_F = \emptyset$, and consequently Γ_L is the set of all mappings from U_F to U_L, while $\Gamma_F = U_F$. Such a game, called in reference[5] the reversed Stackelberg problem, can be characterized as follows: F is required to select his decision first, L acts after F knowing the actual choice of F, so his strategy can be a function of F's decision. That is, while L announces his strategy first, F actually acts first. Not surprisingly the reversed information structure is very advantageous to L as it enables him to include in the formulation of his strategy incentives or threats aimed to induce F to act in the way that benefits L. At the same time it should be emphasized that in this case not only the standard condition (S1) but also (S2) is necessary for the applicability of the Stackelberg solution concept. (S2) ensures that L will always act according to his declared strategy, or in other words, that F can always trust L to keep any promise and execute any threat.

The solution to the reversed Stackelberg problem is provided by the following result[8]:

<u>Theorem 1</u>

Let

$$M = \sup_{u_F \in U_F} \inf_{u_L \in U_L} J_F(u_L, u_F), \tag{3}$$

$$D = \{(u_L, u_F) \in U_L \times U_F \mid J_F(u_L, u_F) \geq M\}, \tag{4}$$

$$D^0 = \{(u_L, u_F) \in U_L \times U_F \mid J_F(u_L, u_F) > M\}, \tag{5}$$

and assume that U_L, U_F are nonempty subsets of some topological spaces, J_L, J_F are defined continuous and bounded from above on $U_L \times U_F$, D^0 is nonempty and its enclosure is equal to D. Then

$$S \overset{\text{def.}}{=} \sup_{\gamma_L \in \Gamma_L} \inf_{\gamma_F \in R(\gamma_L)} J_L(\gamma_L, \gamma_F) = \sup_{(u_L, u_F) \in D} J_L(u_L, u_F) \overset{\text{def.}}{=} K. \tag{6}$$

Moreover, if the supremum of J_L is attained at a point $\tilde{u} = (\tilde{u}_L, \tilde{u}_F) \in D^0$, that is $J_L(\tilde{u}) = K$, then there exists a mapping $p: U_F \setminus \{\tilde{u}_F\} \to U_L$ such that the strategy

$$\gamma_L^s(u_F) = \begin{cases} \tilde{u}_L & \text{if } u_F = \tilde{u}_F \\ p(u_F) & \text{otherwise} \end{cases} \tag{7}$$

satisfies $R(\gamma_L^s) = \tilde{u}_F$ and $J_L(\gamma_L^s, R(\gamma_L^s)) = J_L(\tilde{u}) = S$, that is γ_L^s is L's Stackelberg strategy. If the supremum of J_L is not attained at any point of D^0, then for any $\varepsilon > 0$ there exists a strategy

$$\gamma_L^\varepsilon(u_F) = \begin{cases} u_L^\varepsilon & \text{if } u_F = u_F^\varepsilon \\ p^\varepsilon(u_F) & \text{otherwise} \end{cases} \tag{8}$$

such that $R(\gamma_L^\varepsilon) = u_F^\varepsilon$ and $J_L(\gamma_L^\varepsilon, R(\gamma_L^\varepsilon)) = J_L(u_L^\varepsilon, u_F^\varepsilon) \geq S - \varepsilon$.

Theorem 1 reduces the solution of the reversed Stackelberg problem to the solution of the maximin problem and the constrained maximization problem. Observe that M is the payoff that F can guarantee himself under any punitive action of L and D^0 is the set of points that L can enforce on F by threatening him with payoffs less than M. The Stackelberg strategy exists if the maximum of J_L on D is attained at a point belonging to D^0 /e.g. this is the case when L can enforce on F the absolute maximum of J_L/. In general S can be approximated by means of ε- Stackelberg strategies with arbitrarily small $\varepsilon > 0$. /Note however that when ε approaches zero then the incentive for F to make the equilibrium decision may become negligible/. As an illustration consider the following example:

<u>Example</u>. The static duopoly

Suppose that L and F are two firms selling the same product on the market and seeking to maximize their profits $J_L = qu_L$ and $J_F = qu_F$ respectively, where u_L, u_F are the quantities of the product each firm chooses to supply on the market, with $0 \leq u_L \leq A_L$, $0 \leq u_F \leq A_F$, and q is the market price of the product determined by the relation $q = a - u_L - u_F$ for some given $a > 0$. Assume also that $0 < A_L \leq a$ and $a/2 \leq A_F \leq a$. It is easily seen that

$$M = \max_{u_F} \min_{u_L} J_F(u_L, u_F) = J_F(A_L, (a - A_L)/2) = (a - A_L)^2/4,$$

$$K = \max_{(u_L, u_F) \in D} J_L(u_L, u_F) = J_L(A_L(2a - A_L)/2a, (a - A_L)^2/2a) = A_L(2a - A_L)/4,$$

D^0 is nonempty and encl. $D^0 = D$.
Moreover for any $\varepsilon > 0$ and
$u^\varepsilon = (u_L^\varepsilon, u_F^\varepsilon) = ((A_L(2a - A_L) - 4\varepsilon)/2a, ((a - A_L)^2 + 4\varepsilon)/2a)$
one has

$J_F(u^\varepsilon) = M+\varepsilon$ and $J_L(u^\varepsilon) = K-\varepsilon$, so for $\varepsilon > 0$ the strategy

$$\gamma_L^\varepsilon(u_F) = \begin{cases} u_L^\varepsilon & \text{if } u_F = u_F^\varepsilon \\ A_L & \text{otherwise} \end{cases}$$

satisfies the conditions

$R(\gamma_L^\varepsilon) = u_F^\varepsilon$ and $J_L(\gamma_L^\varepsilon, R(\gamma_L^\varepsilon)) = S - \varepsilon$.

The relation between ε and the incentive for F to select u_F^ε is obvious here. Note that after F's action u_F L can be tempted to pick up $u_L = m(u_F) = (a - u_F)/2$ maximizing his profit for any given u_F, instead of $u_L = \gamma_L^\varepsilon(u_F)$ as $m(u_F) \neq A_L$ and excluding the case $A_L = a$, when L is a virtual monopolist, $m(u_F^\varepsilon) \neq u_L^\varepsilon$. This illustrates the relevance of the condition /S2/.

In the next section we'll consider the dynamic game with closed-loop information available to the leader, which is more complex formally but conceptually is very similar to the reversed Stackelberg problem.

The Dynamic Game with the Closed-Loop Information Structure

Consider a deterministic dynamic system with two inputs

$$x_{t+1} = f_t(x_t, u_{Lt}, u_{Ft}), \quad t = 0,1,\ldots,T-1, \qquad x_0 \text{ given}, \tag{9}$$

where T is a finite time horizon, x_t /n-vector/ is the state variable, u_{Lt} /m-vector/ and u_{Ft} /r-vector/ are the control variables handled by players L and F respectively. The players are supposed to maximize their payoffs $J_L(0, x_0; u_L, u_F)$ and $J_F(0, x_0; u_L, u_F)$, where

$$J_L(s, x_s; u_L^s, u_F^s) = \sum_{t=s+1}^{T} g_{Lt}(x_t), \tag{10}$$

$$J_F(s, x_s; u_L^s, u_F^s) = \sum_{t=s+1}^{T} g_{Ft}(x_t), \tag{11}$$

with u_L^s, u_F^s denoting the control sequences $(u_{Ls}, u_{Ls+1}, \ldots, u_{LT-1})$, $(u_{Fs}, u_{Fs+1}, \ldots, u_{FT-1})$ for $s = 0,1,\ldots,T-1$, and $u_L = u_L^0$, $u_F = u_F^0$. It is assumed that $u_{Lt} \in U_{Lt}$, $u_{Ft} \in U_{Ft}$ with U_{Lt}, U_{Ft} being compact subsets of R^m and R^r respectively, and that f_t, g_{Lt}, g_{Ft} are continuous functions of their arguments. Let X_0 be a set of initial states, and for $t = 1,2,\ldots,T-1$ X_t be the set of states generated by $x_0 \in X_0$ and all admissible control sequences $(u_{L0}, \ldots, u_{Lt-1}), (u_{F0}, \ldots, u_{Ft-1})$. We consider the game with the so-called closed-loop information structure, implying that at time t the players gain information about x_t and recall $x_0, x_1, \ldots, x_{t-1}$. As a consequence strategy spaces Γ_L, Γ_F are defined as the sets of all mappings of the form $\gamma_L = (\gamma_{L0}, \ldots, \gamma_{LT-1})$, $\gamma_F = (\gamma_{F0}, \ldots, \gamma_{FT-1})$, where

$\gamma_{Lt} : X_0 \times \ldots \times X_t \to U_{Lt}$ and

$\gamma_{Ft} : X_0 \times \ldots \times X_t \to U_{Ft}$, $t = 0,1,\ldots,T-1$.

The solution of the dynamic Stackelberg game with the closed-loop information structure is based on the same idea as the solution of the static Stackelberg problem with the reversed information structure. At time t, $1 \leq t \leq T-1$, L recalls the whole history of the game, so he is in a position to punish F for an undesirable behaviour at stages $0,1,\ldots,t-1$, if only he can decrease F's payoff by means of a control sequence u_L^t. Let $M(t, x_t)$ be the payoff F can secure himself at point (t, x_t), $1 \leq t \leq T-1$, $x_t \in X_t$, against any threat announced by L. That is

$$M(t, x_t) = \inf_{u_L^t \in U_L^t} \sup_{u_F^t \in U_F^t} J_F(t, x_t; u_L^t, u_F^t) \tag{12}$$

where $U_L^t = U_{Lt} \times \ldots \times U_{LT-1}$, $U_F^t = U_{Ft} \times \ldots \times U_{FT-1}$ or using the dynamic programming approach

$$M(T-1, x_{T-1}) = \inf_{u_{LT-1} \in U_{LT-1}} \sup_{u_{FT-1} \in U_{FT-1}} g_{FT}(x_T) \tag{13}$$

and

$$M(t, x_t) = \inf_{u_{Lt} \in U_{Lt}} \sup_{u_{Ft} \in U_{Ft}} \left[g_{Ft+1}(x_{t+1}) + M(t+1, x_{t+1}) \right]$$

$$t = 1,2,\ldots,T-2. \tag{14}$$

$M(t, x_t)$ is the quantity L can use in the formulation of the threat influencing F's choice of $u_{F0}, u_{F1}, \ldots, u_{Ft-1}$ for $t = 1,2,\ldots,T-1$. The last decision of F, namely u_{FT-1}, cannot be influenced by any threat so it satisfies the condition

$$J_F(T-1, x_{T-1}; u_{LT-1}, u_{FT-1}) = \sup_{u'_{FT-1} \in U_{FT-1}} J_F(T-1, x_{T-1}; u_{LT-1}, u'_{FT-1}) \tag{15}$$

Thus the set of solutions L can enforce on F has the form

$$D^0 = \{(u_L, u_F) \in U_L \times U_F \mid J_F(t, x_t; u_L^t, u_F^t) > \sup_{u'_{Ft} \in U_{Ft}} [g_{t+1}(x'_{t+1}) + M(t+1, x'_{t+1})], \text{ where } x'_{t+1} = f_t(x_t, u_{Lt}, u'_{Ft}), \text{ for } t = 0,1,\ldots,T-2 \text{ and } u_{FT-1} \text{ satisfies (15)}\}. \tag{16}$$

Furthermore let D be the set defined as D^0 with the only difference that relation $\geq$ is substituted for $>$. The equilibrium solution of the dynamic closed-loop Stackelberg game is characterized by the following result:

Theorem 2.

If D^0 is nonempty and its enclosure contains D then

$$S \overset{\text{def.}}{=} \sup_{\gamma_L \in \Gamma_L} \inf_{\gamma_F \in R(\gamma_L)} J_L(\gamma_L, \gamma_F) =$$

$$= \sup_{(u_L, u_F) \in D} J_L(u_L, u_F) \overset{\text{def.}}{=} K. \tag{17}$$

Moreover, the ε- Stackelberg strategy of L is of the form

$$\gamma^{\varepsilon}_{L0} = u^{\varepsilon}_{L0}$$

$$\gamma^{\varepsilon}_{Lt}(x_0,\dots,x_t) = \begin{cases} u^{\varepsilon}_{Lt} & \text{if } x_s = x^{\varepsilon}_s \text{ for } s=1,\dots,t \\ p^{\varepsilon}_t(x_t) & \text{otherwise} \end{cases}$$

for $t=1,\dots,T-1$, (18)

where $(u^{\varepsilon}_L, u^{\varepsilon}_F) \in D^0$, trajectory x^{ε} is defined as $x^{\varepsilon}_0 = x_0$, $x^{\varepsilon}_{t+1} = f_t(x^{\varepsilon}_t, u^{\varepsilon}_{Lt}, u^{\varepsilon}_{Ft})$ for $t=0,1,\dots,T-1$, and functions $p^{\varepsilon}_t(x_t)$ satisfy conditions

$$J_F(t, x_t; u^{\varepsilon t}_L, u^{\varepsilon t}_F) \geq \sup_{u^t_F} J_F(t, x_t; u^{\varepsilon}_{Lt}, p^{\varepsilon}_{t+1}(x_{t+1}),$$

$$\dots, p^{\varepsilon}_{T-1}(x_{T-1}), u^t_F), \quad t=1,\dots,T-1 \tag{19}$$

Proof.

Suppose that $S > K$. This means that there exists $\gamma_L \in \Gamma_L$ such that for any $\tilde{u}_F$ maximizing

$$J_F(0, x_0; \gamma_L, u_F) \text{ subject to} \tag{20}$$

$$x_{t+1} = f_t(x_t, \gamma_{Lt}(x_0,\dots,x_t), u_{Ft}), t=0,1,\dots,T-1$$

one has

$$J_L(0, x_0; \gamma_L, \tilde{u}_F) = J_L(0, x_0; \tilde{u}_L, \tilde{u}_F) > K, \tag{21}$$

where $\tilde{u}_{Lt} = \gamma_{Lt}(\tilde{x}_0,\dots,\tilde{x}_t)$, with $\tilde{x}_0 = x_0$ and $\tilde{x}_{t+1} = f_t(\tilde{x}_t, \tilde{u}_{Lt}, \tilde{u}_{Ft})$. The fact that $\tilde{u}_F$ is the solution to the optimal control problem (20) implies that for $t=0,1,\dots,T-1$ and any $u^t_F \in U^t_F$

$$J_F(t, \tilde{x}_t; \tilde{u}^t_L, \tilde{u}^t_F) \geq J_F(t, \tilde{x}_t; \gamma^t_L, u^t_F) =$$

$$= \sum_{s=t+1}^{T} g_{Ls}(x_s), \tag{22}$$

with $x_{s+1} = f_s(x_s, \gamma_{Ls}(\tilde{x}_0,\dots,\tilde{x}_t, x_{t+1},\dots,x_s), u_{Fs})$

for $s = t, t+1,\dots,T-1$, and $x_t = \tilde{x}_t$.

This means however that $\tilde{u}_{FT-1}$ must satisfy (15), and that for $t=0,1,\dots,T-2$ and any $u^t_F \in U^t_F$ there exists u^{t+1}_L satisfying

$$J_F(t, \tilde{x}_t; \tilde{u}^t_L, \tilde{u}^t_F) \geq J_f(t, \tilde{x}_t; \tilde{u}_{Lt}, u^{t+1}_L, u^t_F), \tag{23}$$

or in other words

$$J_F(t, \tilde{x}_t; \tilde{u}^t_L, \tilde{u}^t_F) \geq$$

$$\inf_{u^{t+1}_L \in U^{t+1}_L} \sup_{u^t_F \in U^t_F} J_F(t, x_t; \tilde{u}_{Lt}, u^{t+1}_L, u^t_F) \tag{24}$$

Thus $(\tilde{u}_L, \tilde{u}_F) \in D$ and (21) contradicts the definition of K. So we have shown that $S \leq K$.

From the other hand by the definition of K for any $\varepsilon > 0$ there exists $(u^{\varepsilon/2}_L, u^{\varepsilon/2}_F) \in D$ satisfying

$$J_L(0, x_0; u^{\varepsilon/2}_L, u^{\varepsilon/2}_F) \geq K - \varepsilon/2 \tag{25}$$

Moreover, the continuity of J_L and the fact that in any neighbourhood of any point from D there are also points from D^0 imply the existence of $(u^{\varepsilon}_L, u^{\varepsilon}_F) \in D^0$ and the corresponding trajectory x^{ε} satisfying

$$J_L(0, x_0; u^{\varepsilon}_L, u^{\varepsilon}_F) = \sum_{t=1}^{T} g_t(x^{\varepsilon}_t) \geq K - \varepsilon \tag{26}$$

Now, the definition of D^0 implies the existence of functions $p^{\varepsilon}_t(x_t)$, $t=1,\dots,T-1$, satisfying (19), and this means that

$$R(\gamma^{\varepsilon}_L) = \{u_F \in U_F \mid x^{\varepsilon}_{t+1} = f_t(x^{\varepsilon}_t, u^{\varepsilon}_{Lt}, u_{Ft}), t=0,1,\dots,T-1\}$$

where γ^{ε}_L is defined by relation (18). So we have

$$\inf_{u_F \in R(\gamma^{\varepsilon}_L)} J_L(0, x_0; \gamma^{\varepsilon}_L, u_F) = J_L(0, x_0; u^{\varepsilon}_L, u^{\varepsilon}_F) \geq K - \varepsilon \tag{27}$$

(27) implies that $S \geq K$, and in consequence $S=K$.

q.e.d.

<u>Corollary 1</u>

If the supremum of J_L on D is attained at a point $\tilde{u} = (\tilde{u}_L, \tilde{u}_F) \in D^0$ then there exist mapping $p_t : X_t \to U_{Lt}$, $t=1,\dots,T-1$, such that the strategy

$$\gamma^s_{L0} = \tilde{u}_{L0},$$

$$\gamma^s_{Lt}(x_0,\dots,x_t) = \begin{cases} \tilde{u}_{Lt} & \text{if } x_s = \tilde{x}_s \text{ for } s=1,\dots,t \\ p_t(x_t) & \text{otherwise} \end{cases} \tag{28}$$

is the Stackelberg strategy of L.

In particular the Stackelberg solution exists if D^0 contains the absolute maximum of J_L on $U_L \times U_F$. For example this is the case of linear-quadratic games considered in [1,2,3], where L's power to threaten and punish F is unlimited. It is important to note that in general strategies (18) and (28) do not satisfy the principle of optimality, that is they are not ε-Stackelberg or Stackelberg strategies for the games starting at points (t, x^{ε}_t) or $(t, \tilde{x}_t)$ for $0 < t \leq T-1$.

This means that in the course of the game L can be tempted to change his strategy, so the condition /S2/ is crucial for the applicability of the Stackelberg solution concept. One notable exception is the game where L can enforce on F the absolute maximum of his payoff. In this case strategy (28) is optimal for L along the trajectory $\tilde{x}$, nevertheless /S2/ is needed to ensure that L will execute his threats if necessary.

Finally observe that F's strategy results from the solution of an optimal control problem, so within the deterministic context the state information gained or recalled by F is for him inessential.

Conclusion

The dynamic closed-loop game has the ε- Stackelberg solution under very mild assumptions, and the Stackelberg solution under the assumptions which are significantly stronger. Both solutions can be obtained by solving a maximin problem and a constrained maximization problem. Practically the applicability of the Stackelberg solution concept to dynamic games is limited by the condition implying the existence of the contract binding the leader to follow the strategy declared at the begining of the game, even if in the course of the game it ceases to be optimal. In other words, the leader is supposed to always keep his promise, not a very realistic assumption in many competitive situations. Theoretically the study of dynamic Stackelberg problems resulted in a valuable contribution, namely the formalization of the concept of incentives /threats/. It seems that this concept is important also for games without binding contracts and a priori given leaders. This is a very interesting topic which we plan to discuss in another paper.

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