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On the variational equilibrium as a refinement of the generalized Nash equilibrium[☆]

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ABSTRACT

We are concerned with a class of games in which the players' strategy sets are coupled by a shared constraint. A widely employed solution concept for such *generalized* Nash games is the generalized Nash equilibrium (GNE). The variational equilibrium (VE) (Facchinei & Kanzow, 2007) is a specific kind of GNE characterized by the solution of the variational inequality formed from the common constraint and the mapping of the gradients of player objectives. Our contribution is a theory that provides sufficient conditions for ensuring that the existence of a GNE implies the existence of a VE; in such an instance, the VE is said to be a *refinement* of the GNE. For certain games, these conditions are shown to be necessary. This theory rests on a result showing the equality of the Brouwer degree of two suitably defined functions, whose zeros are the GNE and VE, respectively. This theory has a natural extension to the primal–dual space of strategies and Lagrange multipliers corresponding to the shared constraint. Our results unify some known results pertaining to such equilibria and provide mathematical substantiation for ideas that were known to be appealing to economic intuition.

1. Introduction

This paper concerns noncooperative N-player *generalized Nash games* (Harker, 1991) (or *coupled constrained games* (Rosen, 1965)) where players are assumed to have continuous strategy sets that are *dependent* on the strategies of their adversaries. Such games represent generalizations of classical noncooperative games that have traditionally allowed for strategic interactions between players to be expressed only through their objective functions. In a frequently encountered class of generalized Nash games, player strategies are required to satisfy a *common* coupling constraint. These games are called generalized Nash games with *shared constraints* (Facchinei & Kanzow, 2007; Rosen, 1965) and they form the focus of this paper.

Let $\mathcal{N} = \{1, 2, \ldots, N\}$ be a set of players, $m_1, \ldots, m_N$ be positive integers and $m = \sum_{i=1}^{N} m_i$. For each $i \in \mathcal{N}$, let $U_i \subseteq \mathbb{R}^{m_i}$ represent player i's strategy set, $x_i \in U_i$ be his strategy and $\varphi_i : \mathbb{R}^m \to \mathbb{R}$ be his objective function. We use the following notation: by x we denote the tuple $(x_1, x_2, \ldots, x_N)$, x^{-i} denotes the tuple $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$ and (y_i, x^{-i}) the tuple $(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_N)$. A *shared constraint* is a requirement that the tuple x be constrained to lie in a set $\mathbb{C} \subseteq \mathbb{R}^m$. In the generalized Nash game with shared constraint $\mathbb{C}$, player i is assumed to solve the parameterized optimization problem,

$$\boxed{\begin{array}{lll} A_i(x^{-i}) & \underset{x_i}{\text{minimize}} & \varphi_i(x_i; x^{-i}) \\ & \text{subject to} & x_i \in K_i(x^{-i}). \end{array}}$$

For each $i \in \mathcal{N}$, the set-valued maps $K_i : \prod_{j \neq i} \mathbb{R}^{m_j} \to 2^{\mathbb{R}^{m_i}}$ and the map $K : \mathbb{R}^m \to 2^{\mathbb{R}^m}$, are defined as

$$K_i(x^{-i}) := \{y_i \in \mathbb{R}^{m_i} \mid (y_i, x^{-i}) \in \mathbb{C}\}, \quad \forall i \in \mathcal{N}$$
$$\text{and} \quad K(x) := \prod_{i \in \mathcal{N}} K_i(x^{-i}) \quad \forall\, x \in \mathbb{R}^m \tag{1}$$

where the notation 2^X stands for the set of all subsets of a set X. For simplicity, we have dropped the sets U_i in the above optimization problems and have assumed that $\mathbb{C}$ is contained in $\prod_{i \in \mathcal{N}} U_i$.

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Throughout this paper, we denote the game resulting from the above optimization problems by $\mathscr{G}$. The solution concept applied to analyze such games is called the *generalized Nash equilibrium* (GNE).

Definition 1.1 (*Generalized Nash Equilibrium*)**.** A strategy tuple $x \equiv (x_1, x_2, \ldots, x_N)$ is a generalized Nash equilibrium of $\mathscr{G}$ if $x_i \in SOL(A_i(x^{-i}))$ for all $i \in \mathscr{N}$.

Here SOL(P) refers to the solution set of an optimization problem P. The GNE is a special case of the *social equilibrium* proposed by Debreu (1952) for the case of general coupling constraints; see also Rosen (1965), Arrow and Debreu (1954) and the recent survey (Facchinei & Kanzow, 2007) for more on this. We now introduce another solution concept: the variational equilibrium (VE) which is a specific kind of GNE defined in Facchinei and Kanzow (2007), Facchinei and Pang (2009):

Definition 1.2 (*Variational equilibrium (VE)*)**.** A strategy tuple x is said to be a variational equilibrium of $\mathscr{G}$ if x is a solution of (VI($\mathbb{C}$, F)).

The notation (VI($\mathbb{C}$, F)) denotes a *variational inequality* with mapping F and a set $\mathbb{C}$ (see Section 1.1), where $F:\mathbb{R}^m \to \mathbb{R}^m$ is the function given by

$$F(x) = (\nabla_{x_1}\varphi_1(x), \ldots, \nabla_{x_N\varphi_N(x)}) \quad \forall\, x \in \mathbb{R}^m, \tag{2}$$

where ∇_{x_i} (henceforth abbreviated as ∇_i) denotes the partial derivative with respect to x_i.

The goal of this paper is to provide a theory that gives sufficient conditions for the VE to be a *refinement* of the GNE. A refinement of the set of equilibria of a game is (a) a *subset* satisfying a certain *rule*, where this rule has the property that (b) any game with a nonempty set of equilibria also possesses an equilibrium satisfying this rule. Both the refined equilibria and the generating rule are collectively referred to as the refinement. From an economic standpoint, the notion of the refinement of an equilibrium is rooted in the belief that the concept of this equilibrium may be far too weak to serve as a solution concept. For instance, if the weakness of the original concept is on the count that certain equilibria have less economic justification, then a refinement should formalize this by excluding such equilibria. Refinements of equilibria have been previously sought in both static and dynamic games: trembling hand perfect (Selten, 1975) and proper (Myerson, 1978) equilibria are refinements of mixed Nash equilibria in static finite strategy games (Başar & Olsder, 1999; Myerson, 1997; Weibull, 1997); the subgame-perfect Nash equilibrium is a refinement of the Nash equilibrium of a dynamic game (see Nisan, Roughgarden, Tardos, and Vazirani (2007, ch. 3.8)). It is known from Facchinei, Fischer, and Piccialli (2007) that every VE is a GNE. Thus this paper focuses on showing that, under suitable conditions, the existence of a GNE implies the existence of a VE.

GNEs of games such as $\mathscr{G}$ have properties that, we believe, warrant a refinement. These games are known to admit a large number, and in some cases, a manifold of GNEs (see Facchinei and Kanzow (2007); also Theorem 16 in Appendix A.1). In fact, in the following example, *every* strategy tuple in $\mathbb{C}$ is a GNE.

Example 1 (*Game Where Every Strategy Tuple is a GNE*)**.** Consider a game where player i has real valued strategies and solves

$$\boxed{\begin{array}{lll} A_i(x^{-i}) & \underset{x_i}{\text{minimize}} & x_i\ell(X) \\ & \text{subject to} & X = \alpha : \lambda_i, \end{array}}$$

where $X = \sum_{i\in\mathscr{N}} x_i$ for $x_i \in \mathbb{R}$ for each $i \in \mathscr{N}$ and λ_i is the Lagrange multiplier for the constraint $X = \alpha$ for player i. Such games arise commonly in network routing problems. The Karush–Kuhn–Tucker (KKT) conditions characterizing the GNE, x^*, of this game are given by

$$(x_i^*\ell(X^*))' = \lambda_i, \quad \forall i \in \mathscr{N} \text{ and } X^* = \alpha.$$

Clearly, every point in the set $\mathbb{C} = \{x \mid X = \alpha\}$ is a GNE of this game. Does a subset of these characterize economically justifiable strategic behavior? □

Another shortcoming of the GNE is that there are settings in which not every GNE is meaningful from a real-world standpoint. This shortcoming provides the *first* motivation for our study which is to present a refinement of the GNE that will retain a set of GNEs that is smaller, yet economically meaningful, even under these settings. It may be argued that the VE does indeed possess such a property. Consider a game similar to that in the above example where the Lagrange multipliers corresponding to the shared constraints can be interpreted as *prices* charged on the players by an administrator for whom the players are anonymous. The VE is also known to be the GNE with equal Lagrange multipliers corresponding to the shared constraint (Facchinei et al., 2007). Thus for this game the VE has the additional property of being an equilibrium with *uniform prices* whereas the GNE corresponds to one with discriminatory prices. Since players are anonymous and hence indistinguishable from each other, it is unreasonable to assume that the administrator can charge discriminatory prices and the only equilibria that make sense are those where the same price is charged to all players, i.e. the VE.

Our *second* motivation arises from the need to characterize and compute GNEs. In general, obtaining a GNE requires a solution of an ill-posed system which leads to a *quasi-variational inequality* in the primal-space and a non-square complementarity problem in the primal–dual space. The VE, on the other hand, requires the solution of either a variational inequality (primal space) or a square complementarity problem (primal–dual space) both of which are far more tractable objects. To demonstrate this, consider the game $\mathscr{G}$ in which the set $\mathbb{C} = \{x \mid c(x) \geq 0, x \geq 0\}$ for a continuously differentiable concave function $c:\mathbb{R}^m \to \mathbb{R}^n$. Assuming that an appropriate constraint qualification holds (Facchinei & Pang, 2003), a vector x is a GNE of $\mathscr{G}$ if the KKT conditions for optimality of x_i for problem $A_i(x^{-i})$ hold for each player $i \in \mathscr{N}$, i.e., for each $i \in \mathscr{N}$, x_i satisfies

$$\begin{aligned} &0 \leq x_i \perp \nabla_i\varphi_i(x) - \nabla_i c(x)^T\lambda_i \geq 0 \\ &0 \leq \lambda_i \perp c(x) \geq 0, \end{aligned} \tag{KKT$_i$}$$

for some Lagrange multipliers $\lambda_i \in \mathbb{R}^n$ corresponding to the constraint $c(\cdot) \geq 0$. Note that λ_i is a vector in $\mathbb{R}^n$ and the index i corresponds to player i. For $u, v \in \mathbb{R}^n$, the notation $0 \leq u \perp v \geq 0$ means $u, v \geq 0$ and $u_j v_j = 0$ for $j = 1, \ldots, n$. In the system $\{\mathrm{KKT}_1, \ldots, \mathrm{KKT}_N\}$, each vector λ_i, $i \in \mathscr{N}$ is orthogonal to the same requirement "$c(x) \geq 0$", suggesting that the system is ill-posed. On the other hand, a VE is a strategy tuple x which satisfies the above system of equations for some $\lambda := \lambda_1 = \cdots = \lambda_N$, thereby resulting a square well-posed complementarity problem. Consequently, it has been common practice (Facchinei & Kanzow, 2007; Leyffer & Munson, 2005; Pang & Fukushima, 2005) to compute the VE instead of the GNE.

Let $\mathscr{S}$ be the set of games $\mathscr{G}$ for which the VE is a refinement of the GNE and suppose the subset of games for which the VE exists is denoted by $\mathscr{S}_2$. Traditional sufficiency conditions for the solvability of variational inequalities (such as those in Facchinei and Pang (2003)) do not exploit the existence of a GNE to show a solution to (VI($\mathbb{C}$, F)), thereby limiting computation to only those cases where the existence of a VE can be claimed *independently* of knowledge of the existence of a GNE. We refer to this class as $\mathscr{S}_2'$ and it is a subset of $\mathscr{S}_2$.

Our theory gives sufficient conditions for membership in $\mathcal{S}$ – for certain games these conditions are also shown to be necessary – and it shows that $\mathcal{S} \setminus \mathcal{S}_2'$ is a nonempty class. Games in $\mathcal{S}_2'$ emerge as a special case of the theory. Ours is perhaps the first work on the refinement of equilibria in the context of generalized Nash games. Our sufficient conditions are expressed in terms of the *Brouwer degree*, which is seen to relate the GNE and the VE in a profound manner. Specifically, in both, primal and primal–dual space, we find functions v and g whose zeros characterize VEs and GNEs respectively, such that the Brouwer degrees of these functions with respect to zero are equal.

The paper is organized as follows. Sections 2 and 3 deal with the treatment of the refinement question in primal and primal–dual spaces, respectively. We conclude in Section 4. An appendix is included in the end with some supplementary results, proofs of lemmas and some examples of shared constraint games with unusual properties (Appendix A.5). Before proceeding, we outline our assumptions and provide some technical background.

1.1. Background

We make the following assumptions throughout the paper.

Assumption 1. For each $i \in \mathcal{N}$, the objective function $\varphi_i \in C^2$ and $\varphi_i(x_i; x^{-i})$ is convex in x_i for all x^{-i}. Unless otherwise mentioned, $\mathbb{C}$ is closed, convex and has a nonempty interior.

A brief background on variational inequalities and Brouwer degree theory follows.

Recall problems A_i from Section 1. Under Assumption 1, x_i is optimal for $A_i(x^{-i})$ if and only if $\nabla_i\varphi_i(x)^T(y_i - x_i) \geq 0$, for all $y_i \in K_i(x^{-i})$, where K_i is as defined in (1). Thus $x = (x_1, \ldots, x_N)$ is a GNE of $\mathcal{G}$ if and only if it solves the *quasi-variational inequality* (QVI) (Facchinei & Pang, 2003) below.

$$\text{Find } x \in K(x) \text{ such that } F(x)^T(y-x) \geq 0 \quad \forall y \in K(x), \qquad \text{(QVI(K, F))}$$

where F is the function defined in (2). The variational inequality (VI($\mathbb{C}$, F)) is the following problem, a solution of which was defined to be the VE in Definition 1.2.

$$\text{Find } x \in \mathbb{C} \text{ such that } F(x)^T(y-x) \geq 0 \quad \forall y \in \mathbb{C}. \qquad \text{(VI}(\mathbb{C}, \text{F}))$$

The *natural map* of (VI($\mathbb{C}$, F)), $\mathbf{F}^{\text{nat}}{}_{\mathbb{C}}: \mathbb{R}^m \to \mathbb{R}^m$, defined as $\mathbf{F}^{\text{nat}}_{\mathbb{C}}(v) = v - \Pi_{\mathbb{C}}(v - F(v))$ where $\Pi_{\mathbb{C}}: \mathbb{R}^m \to \mathbb{C}$ is the Euclidean projection on $\mathbb{C}$, provides an equation reformulation of the VI. Let $\text{dom}(K) := \{v \mid K(v) \neq \emptyset\}$ and $\widetilde{\mathbf{F}}^{\text{nat}}_K: \text{dom}(K) \to \mathbb{R}^m$ denote a similar natural map for (QVI(K,F)) defined as $\widetilde{\mathbf{F}}^{\text{nat}}_K(v) := v - \Pi_{K(v)}(v - F(v))$, for $v \in \text{dom}(K)$. A vector v solves (VI($\mathbb{C}$, F)) if and only if $\mathbf{F}^{\text{nat}}_{\mathbb{C}}(v) = 0$ and v solves (QVI(K,F)) if and only if $\widetilde{\mathbf{F}}^{\text{nat}}_K(v) = 0$ (Facchinei & Pang, 2003). We frequently use the following result on projections on closed convex sets.

Lemma 1 (*Facchinei and Pang (2003)*). *Let $V \subseteq \mathbb{R}^m$ be a closed convex set and x be a point in $\mathbb{R}^m$. Then the projection of x on V, $\Pi_V(x)$, satisfies $(y - \Pi_V(x))^T(\Pi_V(x) - x) \geq 0$ for each y in V.*

$\mathbf{F}^{\text{nat}}_V$ is a continuous function when V is closed and convex but the continuity of $\widetilde{\mathbf{F}}^{\text{nat}}_K$ relies on the continuity of the set-valued map K (see Aubin and Frankowska (1990) for concepts of continuity in set-valued analysis).

Lemma 2 (*Lemma 2.8.2 (Facchinei & Pang, 2003)*). *Let $x \in \text{dom}(K)$ and y be any point in $\mathbb{R}^m$. Then $\phi(x, y) := \mathbf{F}^{\text{nat}}_{K(x)}(y)$ is continuous at (x, y) for all $y \in \mathbb{R}^m$ if and only if $K(\cdot)$ is continuous at x.*

Proposition 4.7.1 in (Facchinei & Pang, 2003, page 401) provides sufficient conditions for K to be continuous when $\mathbb{C}$ is given explicitly by a continuously differentiable constraint.

The Brouwer degree (Fonseca & Gangbo, 1995; Kesavan, 2004; O'Regan, Cho, & Chen, 2006) of a function is a topological concept that allows us to claim the existence of zeros of the function in a specified open bounded set. Let $\Omega \subset \mathbb{R}^m$ be an open bounded set, $f: \overline{\Omega} \to \mathbb{R}^m$ be continuous and $p \in \mathbb{R}^m \setminus f(\partial\Omega)$. For $p \notin f(\partial\Omega)$, the Brouwer degree of f with respect to p on Ω is considered well-defined and is an integer denoted by $\deg(f, \Omega, p)$. Let $\mathbf{1}: \mathbb{R}^m \to \mathbb{R}^m$ denote the identity map. $\deg(f, \Omega, p)$ has the following properties. Of these, the properties of normalization, homotopy invariance and additivity uniquely determine $\deg(\cdot)$ to be an integer-valued function on the space of such tuples (f, Ω, p) (Lloyd, 1978).

(1) (*Normalization*) $\deg(\mathbf{1}, \Omega, p) = 1$ if and only if $p \in \Omega$.
(2) (*Solvability*) If $\deg(f, \Omega, p) \neq 0$, then $f(x) = p$ for some $x \in \Omega$.
(3) (*Homotopy invariance*) $\deg(H(\cdot, t), \Omega, p)$ is independent of $t \in [0, 1]$ for any continuous function $H: \overline{\Omega} \times [0, 1] \to \mathbb{R}^m$ and $p \in \mathbb{R}^m$ such that $p \notin \cup_{t\in[0,1]} H(\partial\Omega, t)$. Such a function H is called a *homotopy*.
(4) (*Translation invariance*) $\deg(f - p, \Omega, 0) = \deg(f, \Omega, p)$.
(5) (*Degree of injective maps*) Let f be continuous and injective and $f(x) = p$ for some $x \in \Omega$. Then $\deg(f, \Omega, p) = \pm 1$.
(6) (*Additivity*) $\deg(f, \Omega, p) = \deg(f, \Omega_1, p) + \deg(f, \Omega_2, p)$ if Ω_1 and Ω_2 are disjoint open subsets of Ω and $p \notin f(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2))$.

Note that the converse of property 2 is *not* true in general, i.e., the existence of a point $x \in \Omega$ such that $f(x) = p$ for some $p \notin f(\partial\Omega)$ does not imply that $\deg(f, \Omega, p) \neq 0$. But if f is continuous and injective, such a claim can be made, cf. property 5. Degree theory has been previously applied to the study of variational inequalities (Facchinei & Pang, 2003; Gowda, 1993; Gowda & Pang, 1994).

2. Primal generalized Nash equilibria and variational equilibria

In this section, we begin the development of our theory of the refinement of the GNE. Our analysis is restricted to the primal space and does not require $\mathbb{C}$ to be specified by inequality or equality constraints, by relying mainly on convex analytic arguments. A *primal–dual* analysis that assumes $\mathbb{C}$ to be representable by finitely many continuously differentiable constraints is included in Section 3. The discussion pertaining specifically to the refinement of the GNE is encompassed in Section 2.2 while Section 2.1 establishes some preliminary results. We begin by recalling that every VE is a GNE, a result shown by Facchinei et al. (2007, Theorem 2.1).

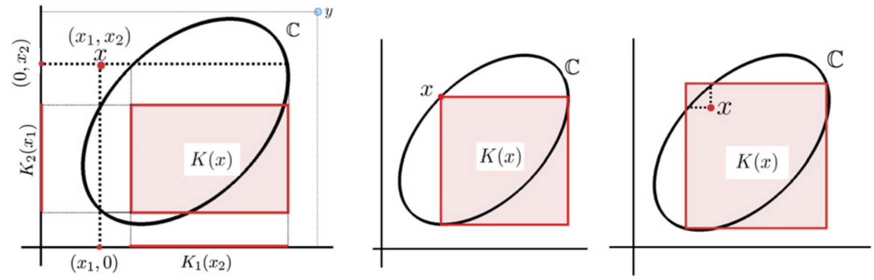
Theorem 3. *For any continuous function $\Gamma: \mathbb{R}^m \to \mathbb{R}^m$, if x is a solution of* VI($\mathbb{C}$, Γ) *then x is a solution of* QVI(K, Γ).

The next set of results help develop a deeper understanding of the set-valued map K.

2.1. The properties of K

Fig. 1(a) shows a convex set $\mathbb{C}$ and $K(x)$ for an $x \in \mathbb{R}^2$, assuming $m_1 = m_2 = 1$ and $N = 2$. Note that $K(x)$ is formed as a product, namely $K_1(x_2) \times K_2(x_1)$. In general, $\text{dom}(K)$ is not $\mathbb{R}^m$ and there may be points outside $\mathbb{C}$ whose image under at least one of the K_i's is empty. For instance, in Fig. 1(a), note the point $y = (y_1, y_2)$ for which both $K_1(y_2)$ and $K_2(y_1)$ are empty. The following Lemma mentions some more relationships between K and $\mathbb{C}$. See Appendix A.2 for proof.

Lemma 4. *Let $\mathbb{C}$ be a closed set in $\mathbb{R}^m$ and K be as given in (1). Then the following hold:*



(a) x and $K(x)$ for the set $\mathbb{C}$ and y such that $K(y) = \emptyset$.

(b) x on the boundary of $\mathbb{C}$ and in the interior of $\mathbb{C}$.

Fig. 1. K for a set $\mathbb{C}$ in $\mathbb{R}^2$.

(1) *If $\mathbb{C} = \prod_{i \in \mathcal{N}} \mathbb{C}_i$, where $\mathbb{C}_i \subseteq \mathbb{R}^{m_i}$ for every $i \in \mathcal{N}$, are nonempty, not necessarily convex sets, then $K(x) = \mathbb{C}$ for every x in $\mathbb{C}$ and is empty otherwise.*
(2) *For any $\mathbb{C}$, not necessarily convex, x is a fixed point of K if and only if $x \in \mathbb{C}$.*
(3) *If $\mathbb{C}$ is closed and convex, $K(x)$ is closed and convex for any $x \in$ dom(K).*
(4) *Let $\mathbb{C}$ be closed and convex and $x \in \mathbb{C}$. For this x, let $K(x)_\infty$ and $\mathbb{C}_\infty$ denote the recession cone (see Appendix A.4) of the sets $K(x)$ and $\mathbb{C}$, respectively. Then we have $K(x)_\infty \subseteq \mathbb{C}_\infty$. Consequently, if $\mathbb{C}$ is bounded, $K(x)$ is bounded for every x in $\mathbb{C}$.*

Lemma 4 (2) can be strengthened further: fixed points of K are in the interior of $\mathbb{C}$ if and only if they are in the interior of their image under K. This is illustrated below in Fig. 1(b) and proved in the following result. See Appendix A.3 for proof. The notation int($\bullet$) and $\partial\bullet$ stand for the interior and the boundary of '$\bullet$' respectively.

Lemma 5. *A point x belongs to the interior of $K(x)$ if and only if x is in the interior of $\mathbb{C}$.*

This concludes the preliminaries for this section. The following section addresses the issue of the refinement of the GNE.

2.2. Refinement of the GNE

Let $\mathcal{S}_1$ be the class of generalized Nash games that admit a GNE and recall that $\mathcal{S}_2$ is the class of games that admit a VE. If $\widetilde{\mathcal{S}}_1$ is the class of games for which a GNE does not exist, the class of games for which the VE is a refinement of the GNE is given by $\mathcal{S} = \mathcal{S}_2 \cup \widetilde{\mathcal{S}}_1$. By Theorem 3, we know that $\mathcal{S}_2 \subseteq \mathcal{S}_1$. To confirm the VE as a refinement for games in $\mathcal{S}$, we need to show that when QVI(K,F) is solvable (a GNE exists), VI($\mathbb{C}$, F) is solvable (a VE exists) or

$$\text{SOL}(\text{QVI}(K, F)) \neq \emptyset \Longrightarrow \text{SOL}(\text{VI}(\mathbb{C}, F)) \neq \emptyset. \tag{3}$$

A natural question one may ask is whether a VE exists whenever a GNE exists or in effect, is $\mathcal{S}$ the class of all games? This is answered in the negative by the following counter-example of a game with a (unique) GNE but no VE.

Example 2 (*Game with Unique GNE and no VE*)**.** Let $\mathbb{C} = \{(x_1, x_2) \mid x_2 \geq e^{-x_1},\ x_1 \geq 0\}$, so that $K(x) = \{(y_1, y_2) | y_2 \geq e^{-x_1}, y_1 \geq 0, e^{-y_1} \leq x_2\}$ if $x_1 \geq 0$ and empty if $x_1 < 0$. Note that the feasibility of a point (x_1, y_2) with respect to $\mathbb{C}$ requires that x_1 be nonnegative. Let $F(x) = (1 + x_1 - \frac{1}{x_2}, 1)$. It is easily verified that $x = (0, 1)$ satisfies

$$\left(1 + x_1 - \frac{1}{x_2}\right)(y_1 - x_1) + (y_2 - x_2) \geq 0, \tag{4}$$

$\forall y \in K(x)$ and thus $(0, 1)$ is a GNE. To show the uniqueness of this GNE, we assume that a GNE, denoted by x, exists in $\mathbb{C}$ where $x \neq (0, 1)$ and arrive at a contradiction. Since $x \in \mathbb{C}$, we must have $x_2 \geq e^{-x_1}$, but for x to be a GNE, we note that it must satisfy $x_2 = e^{-x_1}$. This follows by observing that the points $\{(x_1, y_2) \mid y_2 \in [e^{-x_1}, \infty)\}$ lie in $K(x)$, so if $x_2 > e^{-x_1}$, then the point $y = (x_1, e^{-x_1}) \in K(x)$ will not satisfy (4) and x cannot solve the QVI. Now, since $x_2 = e^{-x_1}$, the point $(y_1, x_2) = (y_1, e^{-x_1})$ lies in $K(x)$ for all $y_1 \in [x_1, \infty)$. If x is a solution of the QVI, for such points we require $\left(1 + x_1 - \frac{1}{x_2}\right)(y_1 - x_1) \geq 0$ for every $y_1 \in [x_1, \infty)$. The term in the first bracket is strictly negative since $x_2 = e^{-x_1}$ and $x \neq (0, 1)$, while the term in the second bracket can be made positive for $y_1 > x_1$. Thus x cannot be a solution and $(0, 1)$ is the only solution. Since every VE is a GNE, this game can have at most one VE, i.e. $(0, 1)$. But for $(0, 1)$ to be a VE, we require $(0, 1)^T[(y_1, y_2) - (0, 1)] \geq 0\ \forall\, y \in \mathbb{C}$. It is easy to check that $y = (2, e^{-2}) \in \mathbb{C}$ and does not satisfy this. Thus this game has no VE but a unique GNE. $\square$

In effect, $\mathcal{S}$ is smaller than the class of *all* generalized Nash games, and therefore our efforts in this paper are focused on identifying subclasses of $\mathcal{S}$. We will also be interested in whether (a) there is any class larger than $\mathcal{S}_2'$ included in $\mathcal{S}_2$ and (b) whether there is any unifying criterion that may be articulated in terms of F and $\mathbb{C}$ that determines $\mathcal{S}$. Recall that $\mathcal{S}_2'$ consists of games for which a VE can be shown to exist *without* using the hypothesis that a GNE exists (e.g. where $\mathbb{C}$ is compact or where F is coercive). (a) is answered in the affirmative in Section 2.3, whereas for (b), we see that the Brouwer degree holds promise in a way made precise below.

We begin by noting a simple consequence of Lemma 5—in the interior of $\mathbb{C}$ the GNE and VE are equivalent. Thus, the VE is a refinement for every game $\mathcal{G}$ that has a GNE in the interior of $\mathbb{C}$ and this GNE is also a VE. This is established in the theorem below.

Theorem 6. *Let $x \in$ int$(K(x))$. Then x is a GNE of $\mathcal{G}$ if and only if x is a VE.*

Proof. Due to Theorem 3, it suffices to prove the "only if" part of the claim. Suppose $x \in$ int$(K(x))$ is a GNE. By Lemma 5, $x \in$ int$(\mathbb{C})$. It follows that one can construct a ball, $B(x, r)$, centered at x with sufficiently small radius r, such that $B(x, r)$ is contained in $K(x) \cap \mathbb{C}$. Since x is a GNE, it follows that

$$F(x)^T(y - x) \geq 0, \quad \forall y \in B(x, r). \tag{5}$$

Putting $y = x + \frac{1}{2}re$ and $y = x - \frac{1}{2}re$ for an arbitrary unit vector e gives $F(x)^T e = 0$. Since this holds for each unit vector e, we must have $F(x) = 0$. As a consequence, x solves VI($\mathbb{C}$, F). $\square$

The result that our theory is built on is Theorem 7. We show that the Brouwer degrees of $\widetilde{\mathbf{F}}_K^{\text{nat}}$ and $\mathbf{F}_{\mathbb{C}}^{\text{nat}}$ (see Section 1.1) with respect to zero, whenever well-defined, are equal. Recall from Section 1.1 that if K is continuous, $\widetilde{\mathbf{F}}_K^{\text{nat}}$ is continuous.

Theorem 7. *Let Ω be an open bounded set such that $\overline{\Omega} \subseteq \mathrm{dom}(K)$ and suppose K is continuous on $\overline{\Omega}$. If $0 \notin \widetilde{\mathbf{F}}_K^{\mathrm{nat}}(\partial\Omega)$, then $\deg(\widetilde{\mathbf{F}}_K^{\mathrm{nat}}, \Omega, 0) = \deg(\mathbf{F}_{\mathbb{C}}^{\mathrm{nat}}, \Omega, 0)$.*

Proof. First, observe that, because every VE is a GNE, the assumption that $0 \notin \widetilde{\mathbf{F}}_K^{\mathrm{nat}}(\partial\Omega)$ implies that $\mathbf{F}_{\mathbb{C}}^{\mathrm{nat}}$ is not zero on $\partial\Omega$. Thus $\deg(\widetilde{\mathbf{F}}_K^{\mathrm{nat}}, \Omega, 0)$ and $\deg(\mathbf{F}_{\mathbb{C}}^{\mathrm{nat}}, \Omega, 0)$ are both well defined.

We will use the invariance of the Brouwer degree under homotopy (property 3 of from Section 1.1) to prove the claim. Define $H: [0,1] \times \overline{\Omega} :\to \mathbb{R}^m$ as

$$H(\bar{t}, v) = \bar{t}\mathbf{F}_{\mathbb{C}}^{\mathrm{nat}}(v) + (1-\bar{t})\widetilde{\mathbf{F}}_K^{\mathrm{nat}}(v) \quad \forall\, \bar{t} \in [0,1],\, v \in \overline{\Omega}.$$

By continuity of K, H is a homotopy between $\widetilde{\mathbf{F}}_K^{\mathrm{nat}}$ and $\mathbf{F}_{\mathbb{C}}^{\mathrm{nat}}$. By property 3 of the Brouwer degree, if $0 \notin \bigcup_{t\in[0,1]} H(t, \partial\Omega)$, we would have $\deg(H(1,\cdot), \Omega, 0) = \deg(H(0,\cdot), \Omega, 0)$, by which the required result would follow.

We have already seen $0 \notin H(1, \partial\Omega) \cup H(0, \partial\Omega)$. So it suffices that $0 \notin H(\bar{t}, \partial\Omega)$ for all $\bar{t} \in (0,1)$ for the result to follow. Assume that this is not so, i.e. assume that for some $t \in (0,1)$ and $z \in \partial\Omega$, $H(t,z) = 0$. Then $z = tx^c + (1-t)x^k$, where $x^k = \Pi_{K(z)}(z - F(z))$ and $x^c = \Pi_{\mathbb{C}}(z - F(z))$. Since $x^k \in K(z)$, $(x_i^k, z^{-i}) \in \mathbb{C}$ for every $i \in \mathcal{N}$, implying that the point x^a belongs to $\mathbb{C}$, where

$$x^a := \frac{1}{N}\sum_{i\in\mathcal{N}} (x_i^k, z^{-i}) = \frac{(N-1)}{N} z + \frac{1}{N} x^k.$$

Indeed, one may verify that

$$z = \frac{N(1-t)}{N(1-t)+t} x^a + \frac{t}{N(1-t)+t} x^c,$$

implying that z is also in $\mathbb{C}$, or equivalently in $K(z)$. Now using the property of projection in Lemma 1, we get

$$(y - x^c)^T(x^c - z + F(z)) \geq 0 \quad \forall y \in \mathbb{C}$$
$$\text{and} \quad (y - x^k)^T(x^k - z + F(z)) \geq 0, \quad \forall y \in K(z).$$

Since $z \in K(z) \cap \mathbb{C}$, we may put $y = z$ in both of the above inequalities to get

$$F(z)^T(z - x^c) \geq \|z - x^c\|^2 \geq 0$$
$$\text{and} \quad F(z)^T(z - x^k) \geq \|z - x^k\|^2 \geq 0. \tag{6}$$

On the other hand, since $z - x^c = -\frac{1-t}{t}(z - x^k)$, we have $-\frac{1-t}{t}F(z)^T(z - x^k) \geq 0$, which from (6) gives $F(z)^T(z - x^k) = 0$ and $z = x^k$. But this means that $\widetilde{\mathbf{F}}_K^{\mathrm{nat}}(z) = 0$, a contradiction to the hypothesis that $0 \notin \widetilde{\mathbf{F}}_K^{\mathrm{nat}}(\partial\Omega)$. Hence $\deg(H(t,\cdot), \Omega, 0)$ is well defined for all $t \in [0,1]$. By property 3 of the Brouwer degree, its value is independent of t, whence the result follows. □

The above result is of a deeper flavor than Theorem 3 of Facchinei et al., for it shows a symmetric relationship (equality, rather than a one-way inclusion) between the GNE and the VE. Moreover, the only assumptions the theorem makes are those necessary for these degrees to be well defined and the result may thereby be thought of as being germane to such games.

Combining Theorem 7 and the solvability property of the Brouwer degree, we can conclude the validity of the implication in (3). Specifically, if a game has the property that

$$\mathrm{SOL}(\mathrm{QVI}(K,F)) \neq \emptyset \Longrightarrow \deg(\widetilde{\mathbf{F}}_K^{\mathrm{nat}}, \Omega, 0) \neq 0 \tag{7}$$

holds for some Ω, i.e. the *converse* of property 2 of the Brouwer degree holds, then nonemptiness of $\mathrm{SOL}(\mathrm{QVI}(K,F))$ also implies the nonemptiness of $\mathrm{SOL}(\mathrm{VI}(\mathbb{C},F))$ and the game admits the VE as a refinement of the GNE. The next theorem shows that for a subclass of games, this argument is not just sufficient but also necessary for (3) to hold.

Theorem 8. *Let K be continuous, $\mathrm{SOL}(\mathrm{QVI}(K,F))$ be bounded and Ω be an open bounded set with $\overline{\Omega} \subseteq \mathrm{dom}(K)$ containing $\mathrm{SOL}(\mathrm{QVI}(K,F))$. If F is pseudo-monotone,[2] then the implication in (3) holds if and only if the one in (7) holds.*

Proof. We first recall Theorem 2.3.17 in Facchinei and Pang (2003) that since the solution set of $(\mathrm{VI}(\mathbb{C}, F))$ (if nonempty) is bounded and F is pseudo-monotone,

$$\mathrm{SOL}(\mathrm{VI}(\mathbb{C},F)) \neq \emptyset \iff \deg(\mathbf{F}_{\mathbb{C}}^{\mathrm{nat}}, \Omega, 0) \neq 0, \tag{8}$$

for the given Ω. Suppose (3) holds. Then combining (3), (8) and Theorem 7, we get (7). Conversely, if (7) holds, then using Theorem 7 we get

$$\mathrm{SOL}(\mathrm{QVI}(K,F)) \neq \emptyset \Longrightarrow \deg(\mathbf{F}_{\mathbb{C}}^{\mathrm{nat}}, \Omega, 0) \neq 0,$$

which by the solvability property of the Brouwer degree leads to (3). □

Note that (7) does not hold in general, since in general, the converse of solvability property 2 of the Brouwer degree is not true. A sufficient condition for (7) to hold is that $\widetilde{\mathbf{F}}_K^{\mathrm{nat}}$ be continuous and one-to-one. But since generalized Nash games are known to have manifolds of equilibria, we expect $\widetilde{\mathbf{F}}_K^{\mathrm{nat}}$ to have manifolds of zeros in fairly general cases (see Theorem 16). Consequently, it is unlikely that $\widetilde{\mathbf{F}}_K^{\mathrm{nat}}$ can be shown to be one-to-one in general and we do not attempt that line of research. Instead we observe that in (7), we may ask that $\deg(\mathbf{F}_{\mathbb{C}}^{\mathrm{nat}}, \Omega, 0)$ be nonzero, thanks to Theorem 7, and use this to identify subclasses of $\mathcal{S}$. We begin by noting that certain classes in $\mathcal{S}_2'$ are identified by this approach. In particular, if $\mathbb{C}$ is compact, the VE and GNE both exist and hence the VE is a refinement of the GNE.

Lemma 9 (*Brouwer's Fixed Point Theorem*)**.** *Let $\mathbb{C}$ be compact and suppose Ω is an open bounded set large enough to strictly contain $\mathbb{C}$. Then $\deg(\mathbf{F}_{\mathbb{C}}^{\mathrm{nat}}, \Omega, 0) = 1$ and the game admits a VE and a GNE.*

Furthermore, games where F is a coercive mapping also have this property.

Lemma 10. *Suppose there exists an $x^{\mathrm{ref}} \in \mathbb{C}$ such that $\liminf_{x\in\mathbb{C}, \|x\|\to\infty} F(x)^T(x - x^{\mathrm{ref}}) > 0$. There exists an open bounded set Ω such that $\deg(\mathbf{F}_{\mathbb{C}}^{\mathrm{nat}}, \Omega, 0) = 1$ and this game has a VE and GNE.*

The proofs of these lemmas can be easily adapted from Corollary 2.2.5 and Proposition 2.2.3 in Facchinei and Pang (2003) respectively. Notice that, since in Lemmas 9 and 10 we are essentially proving $\deg(\mathbf{F}_{\mathbb{C}}^{\mathrm{nat}}, \Omega, 0)$ to be nonzero, the requirement of the continuity of K imposed in Theorem 7 has been relaxed. The existence of the GNE is claimed using Theorem 3 after proving the existence of a VE from the nonzeroness of $\deg(\mathbf{F}_{\mathbb{C}}^{\mathrm{nat}}, \Omega, 0)$.

Remark. Corollary 2.8.4 in Facchinei and Pang (2003) provides a sufficient condition for $\deg(\widetilde{\mathbf{F}}_K^{\mathrm{nat}}, \Omega, 0)$ to be nonzero for an open bounded set Ω. A key requirement in this result is $\{x \in K(x) : (x - x^{\mathrm{ref}})^T F(x) < 0\} \cap \partial\Omega = \emptyset$. For a shared constraint this is equivalent to requiring that $\{x \in \mathbb{C} : (x - x^{\mathrm{ref}})^T F(x) < 0\} \cap \partial\Omega = \emptyset$. This in turn is known to be a sufficient condition for the nonzeroness of $\deg(\mathbf{F}_{\mathbb{C}}^{\mathrm{nat}}, \Omega, 0)$ via Proposition 2.2.3 in Facchinei and Pang (2003). Theorem 7 proves the underlying connection between Corollary 2.8.4 (Facchinei & Pang, 2003) and Proposition 2.2.3 (Facchinei & Pang, 2003), which are the main existence results for VIs and QVIs respectively, and unifies these results. □

[2] A mapping $F : \mathbb{R}^m \to \mathbb{R}^m$ is said to be pseudo-monotone if for all $x, y \in \mathbb{R}^m$, $F(y)^T(x-y) \geq 0 \Longrightarrow F(x)^T(x-y) \geq 0$.

2.3. Identification of subclasses of $\mathscr{S} \setminus \mathscr{S}_2'$

In this section, we concentrate on the identification of classes of games in $\mathscr{S}$ that do not lie in $\mathscr{S}_2'$. We show that the class of games for which F is a pseudo-monotone mapping have the VE as a refinement under certain other assumptions of the recession cone of $\mathbb{C}$. Recall that in the proof of Theorem 8, we had seen that if F is pseudo-monotone and $\mathrm{SOL}(\mathrm{VI}(\mathbb{C}, F))$ is nonempty and bounded, then $\deg(\mathbf{F}^{\mathrm{nat}}_{\mathbb{C}}, \Omega, 0)$ is well defined and nonzero over any neighborhood Ω containing $\mathrm{SOL}(\mathrm{VI}(\mathbb{C}, F))$. Our next result is an extension of this fact to $\mathrm{QVI}(K, F)$. Definitions of the normal cone ($\mathscr{N}$), tangent cone ($\mathscr{T}$), recession cone ($\mathbb{C}_\infty$) and dual cone (denoted by S^* for a cone S) used below can be found in Appendix A.4.

Theorem 11. *Suppose F is pseudo-monotone and x^{ref} is a GNE of $\mathscr{G}$. Consider the following conditions:*

(1) *Either $F(x^{\mathrm{ref}}) = 0$ or $-F(x^{\mathrm{ref}}) \in \mathrm{int}(\mathscr{N}(x^{\mathrm{ref}}; K(x^{\mathrm{ref}})))$,*
(1') *$\mathscr{T}(x^{\mathrm{ref}}; K(x^{\mathrm{ref}}))^* \setminus \{0\} \subseteq \mathrm{int}(\mathbb{C}^*_\infty)$,*
(2) *$\mathbb{C}_\infty \subseteq \mathscr{T}(x^{\mathrm{ref}}; K(x^{\mathrm{ref}}))$.*

If $\mathscr{G}$ has the property that condition (2) holds and either (1) or (1') holds then $\mathscr{G}$ admits a VE.

Proof. Our result is proved through the following set of steps.

Step 1: If $\mathbb{C}$ is bounded, the result follows from Lemma 9. Therefore, we assume that $\mathbb{C}$ is unbounded and prove the result by contradiction. Suppose $\mathscr{G}$ has no VE. Therefore for any open bounded set Ω containing x^{ref}, $0 \notin \mathbf{F}^{\mathrm{nat}}_{\mathbb{C}}(\partial\Omega)$. Fix such an Ω and consider the homotopy $H: [0, 1] \times \overline{\Omega} \to \mathbb{R}^m$ given by

$$H(t, x) = x - \Pi_{\mathbb{C}}(t(x - F(x)) + (1 - t)x^{\mathrm{ref}}), \tag{9}$$

$\forall x \in \overline{\Omega},\ t \in [0, 1]$. Since $x^{\mathrm{ref}} \in \Omega$, we have $0 \notin H(0, \partial\Omega) \cup H(1, \partial\Omega)$. If we also have $0 \notin \bigcup_{t\in(0,1)} H(t, \partial\Omega)$, then we would get $\deg(\mathbf{F}^{\mathrm{nat}}_{\mathbb{C}}, \Omega, 0) = \deg(\mathbf{1} - x^{\mathrm{ref}}, \Omega, 0) = 1$, implying that $\mathscr{G}$ has a VE in Ω. This contradicts our assumption. So we must have $H(t, x) = 0$ for some $x \in \partial\Omega \cap \mathbb{C}$ and $t \in (0, 1)$. i.e. for such x, t we must have $x = \Pi_{\mathbb{C}}(t(x-F(x))+(1-t)x^{\mathrm{ref}})$. Therefore by Lemma 1, $(y - x)^T(x - t(x - F(x)) - (1 - t)x^{\mathrm{ref}}) \geq 0$ for all $y \in \mathbb{C}$. Putting $y = x^{\mathrm{ref}}$ in this gives that x satisfies

$$F(x)^T(x - x^{\mathrm{ref}}) < -\frac{1 - t}{t}\|x - x^{\mathrm{ref}}\|^2 < 0.$$

Now since F is pseudo-monotone, it follows that x also satisfies $F(x^{\mathrm{ref}})^T(x - x^{\mathrm{ref}}) \leq 0$. Since Ω was arbitrary, we conclude that for each open bounded set Ω containing x^{ref}, there exists an $x \in \partial\Omega \cap \mathbb{C}$ such that $F(x^{\mathrm{ref}})^T(x - x^{\mathrm{ref}}) \leq 0$.

Step 2: Let $\{\Omega_k\}$ be a sequence of increasing open balls, each containing x^{ref}, such that $\bigcup_{k\in\mathbf{N}} \Omega_k = \mathbb{R}^m$. Let $x_k \in \partial\Omega_k \cap \mathbb{C}$ be such that $F(x^{\mathrm{ref}})^T(x_k - x^{\mathrm{ref}}) \leq 0$. Assume, without loss of generality, that the sequence $\left\{\frac{x_k - x^{\mathrm{ref}}}{\|x_k - x^{\mathrm{ref}}\|}\right\}$ is convergent and let its limit be d'. Clearly, $d' \neq 0$ and d' satisfies the inequality

$$F(x^{\mathrm{ref}})^T d' \leq 0. \tag{10}$$

Step 3: Next, we prove that $d' \in \mathbb{C}_\infty$. Let $\tau \geq 0$ be arbitrary. Since $\|x_k\| \to \infty$, for sufficiently large k the point u_k belongs to $\mathbb{C}$, where $u_k := x^{\mathrm{ref}} + \tau\frac{x_k - x^{\mathrm{ref}}}{\|x_k - x^{\mathrm{ref}}\|}$. By closedness of $\mathbb{C}$, $\lim_{k\to\infty} u_k = x^{\mathrm{ref}} + \tau d' \in \mathbb{C}$. Since τ is arbitrary and $\mathbb{C}$ is convex, d' is a recession direction of $\mathbb{C}$.

Step 4: To finish the proof, recall that the normal cone and the tangent cone of a convex set are related in the following way (Facchinei & Pang, 2003; Rockafellar, 1997): $-\mathscr{N}(x^{\mathrm{ref}}; K(x^{\mathrm{ref}})) = \mathscr{T}(x^{\mathrm{ref}}; K(x^{\mathrm{ref}}))^*$. Furthermore, since x^{ref} is a GNE of $\mathscr{G}$, $F(x^{\mathrm{ref}}) \in \mathscr{T}(x^{\mathrm{ref}}; K(x^{\mathrm{ref}}))^*$. Suppose condition (1) holds. If $F(x^{\mathrm{ref}}) = 0$ then x^{ref} itself is a VE and there is nothing to prove. If $-F(x^{\mathrm{ref}}) \in \mathrm{int}(\mathscr{N}(x^{\mathrm{ref}}; K(x^{\mathrm{ref}})))$ then for all nonzero vectors d in $\mathscr{T}(x^{\mathrm{ref}}; K(x^{\mathrm{ref}}))$ we must have

$$F(x^{\mathrm{ref}})^T d > 0. \tag{11}$$

But because condition (2) holds, (11) must also hold for all nonzero d in $\mathbb{C}_\infty$. Putting $d = d'$ in (11) contradicts (10). Now suppose condition (1') holds. Then $F(x^{\mathrm{ref}}) \in \mathrm{int}(\mathbb{C}^*_\infty)$ and so (11) is satisfied by $d = d'$; a contradiction to (10) is reached. Thus our initial assumption is incorrect; $\mathscr{G}$ must admit a VE. □

Note that the pseudo-monotonicity of F and the properties of $\mathbb{C}$ mentioned in Theorem 11(1'), (2) are by themselves insufficient for the existence of a VE of $\mathscr{G}$. But given that a GNE x^{ref} exists, the above theorem provides sufficient conditions for $\mathscr{G}$ to have a VE. The above theorem is thus seen to identify a class of games lying in $\mathscr{S} \setminus \mathscr{S}_2'$. It is not hard to see that (2) is satisfied in two cases in which we have already seen the VE to be a refinement: the case where $\mathbb{C}$ is compact (in this case $\mathbb{C}_\infty = \{0\}$ and is included in *any* cone; herein (1') from Theorem 11 also holds) and the case where $x^{\mathrm{ref}} \in \mathrm{int}(K(x^{\mathrm{ref}}))$ (here $\mathscr{T}(x^{\mathrm{ref}}; K(x^{\mathrm{ref}})) = \mathbb{R}^m$ and (1) from Theorem 11 holds, since $F(x^{\mathrm{ref}}) = 0$). Furthermore condition (1) is necessary, since the existence of a VE necessitates the existence of a GNE satisfying (1) (take $x^{\mathrm{ref}} = \mathrm{VE}$). Next, we discuss some instances where the above sufficiency condition may be applied.

Example 3 (*Generalized Nash Game with Affine Shared Constraints*). Consider a game $\mathscr{G}$ where F is pseudo-monotone and $\mathbb{C} = \{x \mid Ax \geq b, x \geq 0\}$ for some nonnegative $b \in \mathbb{R}^n$ and $n \times m$ matrix A with nonnegative elements. Let x^{ref} be a GNE such that $Ax^{\mathrm{ref}} = b$. Suppose $A = [a_1, \ldots, a_N]$, where $a_i \in \mathbb{R}^{n\times m_i}$. We have $\mathbb{C}_\infty = \{d \mid Ad \geq 0, d \geq 0\}$ implying that $\mathbb{C}^*_\infty = \{A^T\lambda \mid \lambda \geq 0\}$. Then $K(x^{\mathrm{ref}})$ and $\mathscr{T}(x^{\mathrm{ref}}; K(x^{\mathrm{ref}}))$ are given as

$$K(x^{\mathrm{ref}}) = \prod_{i\in\mathscr{N}}\left\{y_i \mid a_i y_i + \sum_{j\neq i} a_j x^{\mathrm{ref}}_j \geq b, y_i \geq 0\right\},$$

$$\mathscr{T}(x^{\mathrm{ref}}; K(x^{\mathrm{ref}})) = \prod_{i\in\mathscr{N}}\{d_i \mid a_i d_i \geq 0, d_i \geq 0\}.$$

Clearly, $\mathscr{T}(x^{\mathrm{ref}}; K(x^{\mathrm{ref}})) \subseteq \mathbb{C}_\infty$. But by noting that A has nonnegative entries, we have that $\mathbb{C}_\infty = \mathscr{T}(x^{\mathrm{ref}}; K(x^{\mathrm{ref}}))$. Therefore if (1) from Theorem 11 holds, we conclude from Theorem 11 that the game $\mathscr{G}$ also admits a VE. □

Example 4 (*Generalized Nash Game with Non-Affine Shared Constraints*). Evidently, Theorem 11 can apply to numerous other games with non-affine constraints since requirement (1') from Theorem 11 is not very restrictive but the lack of a general expression for $\mathbb{C}_\infty$ makes it harder to provide examples. To illustrate games where (1') from Theorem 11 may hold, we present the following example in $\mathbb{R}^2$ shown in Fig. 2. Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ be pseudo-monotone, $\mathbb{C} = \{(x_1, x_2) \mid x_1^2 - 2x_1x_2 + x_2^2 - x_1\sqrt{2} - x_2\sqrt{2} \leq 0\}$, be the epigraph of a tilted parabola. Thus $\mathbb{C}_\infty = \{(x_1, x_2) \mid x_1 = x_2 \geq 0\}$, whereas for x^{ref} as shown in Fig. 2, $\mathscr{T}(x^{\mathrm{ref}}; K(x^{\mathrm{ref}})) = [0, \infty) \times [0, \infty)$. Thus Theorem 11(1') holds (so Theorem 11 (2) also holds). Hence if x^{ref} is a GNE, $\mathscr{G}$ admits a VE. □

Following are some concluding remarks about the above result. Theorem 11 can be claimed via Theorem 2.3.5, in Facchinei and Pang (2003, page 158). We have avoided that path in order to demonstrate the reach of the degree-theoretic approach. The 'int' in conditions (1) and (1') presupposes that cones $\mathscr{N}(x^{\mathrm{ref}}; K(x^{\mathrm{ref}}))$ and $\mathbb{C}^*_\infty$ have an interior. The 'int' may be relaxed to 'relative interior' if the cones satisfy some regularity; see Facchinei and Pang

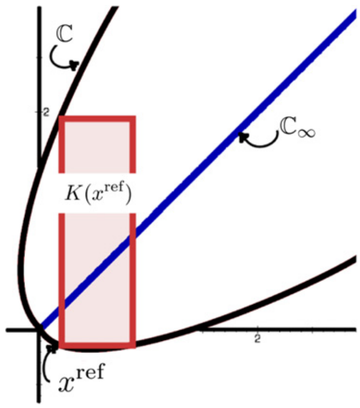


Fig. 2. Example where Theorem 11(1') holds.

(2003, ch. 2.4.1) for details. It is easy to show (see, e.g., Rockafellar and Wets (2009)) that

$$\mathcal{N}(x^{\mathrm{ref}}; K(x^{\mathrm{ref}})) = \prod_{i\in\mathcal{N}} \mathcal{N}(x_i^{\mathrm{ref}}; K_i(x^{\mathrm{ref},-i})).$$

When $\mathbb{C}$ is given as $\{v \mid c(v) \geq 0\}$ for a continuously differentiable function $c : \mathbb{R}^m \to \mathbb{R}^n$ and x^{ref} is on the boundary of $\mathbb{C}$, $\mathcal{N}(x_i^{\mathrm{ref}}; K_i(x^{\mathrm{ref},-i})) = \{\alpha^T \nabla_i c(x^{\mathrm{ref}}) \mid \alpha \geq 0\}$. Hence $\mathcal{N}(x^{\mathrm{ref}}; K(x^{\mathrm{ref}}))$ is at most nN dimensional. In this setting, for $\mathcal{N}(x^{\mathrm{ref}}; K(x^{\mathrm{ref}}))$ to have a nonempty interior, it is necessary that $m \leq nN$. When we do have $\mathrm{int}(\mathcal{N}(x^{\mathrm{ref}}; K(x^{\mathrm{ref}}))) \neq \emptyset$, Theorem 11 (1) says that either $F(x^{\mathrm{ref}}) = 0$ or the Lagrange multipliers corresponding to the constraint c are all strictly positive at x^{ref}. Interestingly, condition (1) in Theorem 11 and the requirement that $m \leq nN$ also appear in the sufficient condition for the existence of a manifold of GNEs (Theorem 16 in Appendix A.1). The connections between Theorems 11 and 16 are being studied further as part of ongoing research.

Review of sufficiency conditions: We now summarize the contributions of this section. The natural maps corresponding to the GNE and the VE were shown to have equal Brouwer degree (cf. Theorem 7). Therefore a sufficient condition for a game to have the VE as a refinement of the GNE is that the existence of a GNE implies the nonzeroness of the degree of these natural maps. This condition is also necessary for a certain class of games (cf. Theorem 8). Lemmas 9 and 10 showed that classes in $\mathcal{S}_2'$ are identified as special cases of our approach. Subclasses of $\mathcal{S} \setminus \mathcal{S}_2'$ were identified in Theorem 11 as games with a pseudo-monotone mapping F and with recession cones $\mathbb{C}_\infty$ admitting certain properties.

3. Primal–dual generalized Nash equilibria and variational equilibria

We now pursue the degree theoretic approach in addressing the question of the refinement of the GNE in the primal–dual space. Throughout, we assume that the set $\mathbb{C}$ is given by inequality constraints and that an appropriate constraint qualification holds. We construct nonlinear equations whose zeros are the GNE and the VE (in the primal–dual space) and show that the degree-theoretic approach of the previous section has a natural extension to the primal–dual space. While this extension may be intuitively expected, it must be emphasized that it is not a corollary of the primal approach. The maps used in the primal setting ($\mathbf{F}_{\mathbb{C}}^{\mathrm{nat}}$ and $\widetilde{\mathbf{F}}_K^{\mathrm{nat}}$) and the ones we use in the primal–dual setting ($\mathbf{G}^{\mathrm{nat}}$ and $\mathbf{J}^{\mathrm{nat}}$) are very different (in particular, defined on different spaces) and there is no obvious degree-theoretic connection that one can draw between them. The primal–dual characterization also offers some analytical advantages over the primal characterization. The assumptions are less abstract and easier to verify – specifically, the assumption of continuity of the set-valued map K is no longer required – and the analysis is somewhat simplified since we escape the QVI setting and now work with complementarity problems (CP). Moreover, the set over which these CPs defined are Cartesian products of sets (in the primal–dual space), for which numerous results are known that are simpler than those for general VIs.

Unless mentioned otherwise, we assume that $\mathbb{C} = \{x \mid x \geq 0,\ c(x) \geq 0\}$ where $c : \mathbb{R}^m \to \mathbb{R}$ is a concave continuously differentiable function. Note that c is assumed to be $\mathbb{R}$-valued as opposed to $\mathbb{R}^n$-valued, only to ease the exposition; in most cases, no generality is lost. We will make the generality or the absence thereof clear wherever necessary. Recall optimization problems A_i from Section 1 the system of KKT conditions, $\{\mathrm{KKT}_1, \ldots, \mathrm{KKT}_N\}$. We begin by recalling the characterization of the VE in terms of the KKT systems observed by Facchinei et al. (2007).

Theorem 12 (*Theorem 3.1, Facchinei et al. (2007)*). *Let x be a GNE for which the system $\{KKT_i\}_{i\in\mathcal{N}}$ is satisfied with $\lambda_1 = \lambda_2 = \cdots = \lambda_N$. Then x solves* $\mathrm{VI}(\mathbb{C}, F)$. *Conversely if x solves* $\mathrm{VI}(\mathbb{C}, F)$ *then there exist $\lambda \in \mathbb{R}$ such that $\{KKT_i\}_{i\in\mathcal{N}}$ hold with $\lambda_i = \lambda$ for all $i \in \mathcal{N}$.*

If there exist $(x, \lambda_1, \ldots, \lambda_N) \in \mathbb{R}^{m+N}$ that satisfied $\{\mathrm{KKT}_i\}_{i\in\mathcal{N}}$, x has been historically referred to as a GNE of $\mathcal{G}$ with *non-shared multipliers*. If for some $x \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}$, $\{\mathrm{KKT}_i\}_{i\in\mathcal{N}}$ were met with $\lambda_1 = \lambda_2 = \cdots = \lambda_N = \lambda$, then x was called a GNE of $\mathcal{G}$ with *shared multipliers*. Rosen (1965) was the first to spot the possibility of redundancy of multipliers when the constraint was shared – he termed the shared multiplier GNE as a *normalized equilibrium*. The fact that the GNE with shared multipliers was actually a solution of a VI in the primal space was a new insight in the context of such games.

3.1. Refinement of the primal–dual GNE

To derive a result similar to Theorem 7 in the primal–dual space, denote by Λ the tuple $(\lambda_1, \ldots, \lambda_N)$ and define the maps $\mathbf{G}^{\mathrm{nat}}$ and $\mathbf{J}^{\mathrm{nat}} : \mathbb{R}^{m+N} \to \mathbb{R}^{m+N}$ as follows.

$$\mathbf{G}^{\mathrm{nat}}(x, \Lambda) := \begin{pmatrix} x_1 - \Pi_+\left(x_1 - \nabla_1\varphi_1(x) + \lambda_1\nabla_1 c(x)\right) \\ \vdots \\ x_N - \Pi_+\left(x_N - \nabla_N\varphi_N(x) + \lambda_N\nabla_N c(x)\right) \\ \lambda_1 - \Pi_+(\lambda_1 - c(x)) \\ \vdots \\ \lambda_N - \Pi_+(\lambda_N - c(x)) \end{pmatrix},$$

$$\mathbf{J}^{\mathrm{nat}}(x, \Lambda) := \begin{pmatrix} x_1 - \Pi_+\left(x_1 - \nabla_1\varphi_1(x) + \lambda_1\nabla_1 c(x)\right) \\ \vdots \\ x_N - \Pi_+\left(x_N - \nabla_N\varphi_N(x) + \lambda_N\nabla_N c(x)\right) \\ \lambda_1 - \Pi_+(\lambda_1 - c(x)) \\ \lambda_2 - \lambda_1 \\ \vdots \\ \lambda_N - \lambda_1 \end{pmatrix},$$

$\forall x \in \mathbb{R}^m$, $\Lambda \in \mathbb{R}^N$, where $\Pi_+(\cdot)$ is the Euclidean projection on the nonnegative orthant of appropriate dimension. Recall that the relation $0 \leq u \perp v \geq 0$ is equivalent to $u = \Pi_+(u - v)$ Facchinei and Pang (2003). It follows from a comparison with $\{\mathrm{KKT}_i\}_{i\in\mathcal{N}}$ that x is a GNE if and only if there exists $\Lambda \in \mathbb{R}^N$ such that $\mathbf{G}^{\mathrm{nat}}(x, \Lambda) = 0$. Likewise, by Theorem 12, x is a VE if and only if there exists $\Lambda \in \mathbb{R}^N$ such that $\mathbf{J}^{\mathrm{nat}}(x, \Lambda) = 0$. Notice the structure of $\mathbf{J}^{\mathrm{nat}}$: contrary to popular approaches that express the "shared" multiplier as a point in $\mathbb{R}$, we treat the shared multiplier as a vector in $\mathbb{R}^N$ with identical coordinates. This creates equation reformulations of the GNE and VE with the same domain ($\mathbb{R}^{m+N}$); it is analytically convenient to have them so for the result below.

Theorem 13. *Let Ω be an open bounded set in $\mathbb{R}^{m+N}$ such that $0 \notin \mathbf{G}^{\text{nat}}(\partial\Omega)$. Then*

$$\deg(\mathbf{G}^{\text{nat}}, \Omega, 0) = \deg(\mathbf{J}^{\text{nat}}, \Omega, 0).$$

Proof. Observe that since $0 \notin \mathbf{G}^{\text{nat}}(\partial\Omega)$, $\mathbf{J}^{\text{nat}}$ is not zero on $\partial\Omega$ and $\deg(\mathbf{G}^{\text{nat}}, \Omega, 0)$ and $\deg(\mathbf{J}^{\text{nat}}, \Omega, 0)$ are well defined. We again will invoke the homotopy invariance of the Brouwer degree. Define $H : [0, 1] \times \overline{\Omega} \to \mathbb{R}^{m+N}$ as

$$H(\bar{t}, \bar{x}, \bar{\Lambda}) = \bar{t}\mathbf{G}^{\text{nat}}(\bar{x}, \bar{\Lambda}) + (1 - \bar{t})\mathbf{J}^{\text{nat}}(\bar{x}, \bar{\Lambda}),$$

$\forall \bar{t} \in [0, 1], (\bar{x}, \bar{\Lambda}) \in \overline{\Omega}$. We will show that $\deg(H(\bar{t}, \cdot), \Omega, 0)$ is well defined for each $\bar{t} \in [0, 1]$ and then invoke its invariance with respect to $\bar{t}$ to conclude the claim. We know that this degree is well defined for $t \in \{0, 1\}$. Assume that it is not so for some $t \in (0, 1)$, i.e. assume that for some $t \in (0, 1)$ and $(x, \Lambda) \in \partial\Omega$, $H(t, x, \Lambda) = 0$. Therefore

$$x_i - \Pi_+ (x_i - \nabla_i\varphi_i(x) + \nabla_i c(x)\lambda_i) = 0 \quad \forall\, i \in \mathcal{N},$$
$$\lambda_1 - \Pi_+(\lambda_1 - c(x)) = 0 \tag{12}$$
$$t\Pi_+(\lambda_i - c(x)) + (1 - t)\lambda_1 = \lambda_i. \quad \forall\, i \in \mathcal{N} \setminus \{1\}. \tag{13}$$

It follows that $(x, \Lambda) \in \mathbb{R}^{m+N}_+$. From (12) it is clear that $0 \le \lambda_1 \perp c(x) \ge 0$. Pick an arbitrary $i \ne 1$. We will show that $0 \le \lambda_i \perp c(x) \ge 0$ holds for this i. From (13) it follows that $\lambda_i \ge 0$. Since λ_i is a real number, two cases arise:

(a) If $\lambda_i \ge c(x)$, then $\Pi_+(\lambda_i - c(x)) = \lambda_i - c(x)$. So we get from (13) $(1 - t)(\lambda_i - \lambda_1) = -tc(x) \Longrightarrow (1 - t)\lambda_i c(x) = -tc(x)^2 \Longrightarrow \lambda_i c(x) = 0$, where the last implication is deduced by noting that $c(x) \ge 0$, $\lambda_i \ge 0$ and $t > 0$.
(b) On the other hand if $\lambda_i < c(x)$, $\lambda_i = (1 - t)\lambda_1 \Longrightarrow \lambda_i c(x) = 0$.

Hence in either case, $0 \le \lambda_i \perp c(x) \ge 0$ or $\lambda_i = \Pi_+(\lambda_i - c(x))$ and therefore, from (13), $\lambda_i = \lambda_1$. As i was arbitrary, we have $\lambda_1 = \lambda_2 = \cdots = \lambda_N$. But this means $\mathbf{J}^{\text{nat}}(x, \Lambda) = 0$, a contradiction to our hypothesis.

So $0 \notin H(\bar{t}, \partial\Omega)$ for all $\bar{t}$ in $[0, 1]$ and $\deg(H(\bar{t}, \cdot), \Omega, 0)$ is independent of $\bar{t}$. The claim follows. □

Remark. The above theorem can be generalized for games where $c : \mathbb{R}^m \to \mathbb{R}^n$, $n > 1$. This would merely require steps (a) and (b) above to be repeated for each of the n components of $c(x)$. □

Theorem 13 plays the same role in the primal–dual space as Theorem 7 did in the primal space, insofar as studying the VE as a refinement of the GNE. The VE is a refinement for the class of games for which the existence of a GNE implies that we can find an Ω as in Theorem 13 so that $\deg(\mathbf{G}^{\text{nat}}, \Omega, 0) \ne 0$. In the remainder of this section, we will develop a condition that ensures this. Let $\psi : \mathbb{R}^{m+N} \to \mathbb{R}^{m+N}$ be defined as follows

$$\psi(x, \Lambda) := \begin{pmatrix} \nabla_1\varphi_1(x) - \lambda_1\nabla_1 c(x) \\ \vdots \\ \nabla_N\varphi_N(x) - \lambda_N\nabla_N c(x) \\ c(x) \\ \vdots \qquad (N \text{ times}) \\ c(x) \end{pmatrix}.$$

Observe that $\mathbf{G}^{\text{nat}}$ is the natural map of $\text{VI}(\mathbb{R}^{m+N}_+, \psi)$: $\mathbf{G}^{\text{nat}}(z) = z - \Pi_+(z - \psi(z))$. Moreover, note that $\text{VI}(\mathbb{R}^{m+N}, \psi)$ is a *Cartesian* VI in the sense of Facchinei and Pang (2003) over the following set

$$\mathbb{R}^{m+N}_+ = \prod_{k=1}^{2N} \mathbb{R}^{\nu_k}_+, \quad \nu_k = \begin{cases} m_k & 1 \le k \le N \\ 1 & N + 1 \le k \le 2N. \end{cases} \tag{14}$$

Recall that the function ψ is said to be $\mathbf{P}_0$ on $\mathbb{R}^{m+N}_+$ partitioned as in (14) if for all distinct vectors $x, y \in \mathbb{R}^{m+N}$, there exists an index $i \in \{1, \ldots, 2N\}$ such that $x_i \ne y_i$ and $(\psi_i(x) - \psi_i(y))^T(x_i - y_i) \ge 0$. Suppose ψ is a $\mathbf{P}_0$ function and $\text{VI}(\mathbb{R}^{m+N}_+, \psi)$ has a bounded solution set. It can be shown that $\deg(\mathbf{G}^{\text{nat}}, \Omega, 0) = \pm 1$ for any open bounded set Ω that contains $\text{SOL}(\text{VI}(\mathbb{R}^{m+N}_+, \psi))$ (Facchinei & Pang, 2003, section 3.6.1). Combining with Theorem 13, we see that such a game also admits a VE. This is articulated in the following theorem which is the analogue of Theorem 8.

Theorem 14. *Suppose ψ is a $\mathbf{P}_0$ mapping and $(\mathbf{G}^{\text{nat}})^{-1}(0)$ (if nonempty) is bounded, then the implication*

$$(\mathbf{G}^{\text{nat}})^{-1}(0) \ne \emptyset \Longrightarrow (\mathbf{J}^{\text{nat}})^{-1}(0) \ne \emptyset,$$

holds and this game admits the VE as a refinement of the GNE.

Note that ψ being $\mathbf{P}_0$ is not sufficient for $\text{VI}(\mathbb{R}^{m+N}_+, \psi)$ to have a solution and $\mathcal{G}$ to have a GNE. Consequently, the game in the above result belongs to $\mathcal{S} \setminus \mathcal{S}'_2$. A sufficient condition for the boundedness of $\text{SOL}(\text{VI}(\mathbb{R}^{m+N}_+, \psi))$ can be seen in Theorem 5.5.15 in Facchinei and Pang (2003).

We conclude this section by giving a sufficient condition for ψ to be a $\mathbf{P}_0$ function and showing thereby that the class games which ψ is $\mathbf{P}_0$ is not vacuous. Recall that ψ is $\mathbf{P}_0$ if its Jacobian, $\nabla\psi(z)$, is a $\mathbf{P}_0$ matrix for any z. Assume that c is twice continuously differentiable and let the Jacobian of ψ be defined as

$$\Psi := \nabla\psi = \begin{pmatrix} \mathbf{H} & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{pmatrix},$$

where $\mathbf{0}$ is a $N \times N$ matrix of zeros, the submatrices $\mathbf{H}, \mathbf{B}, \mathbf{C}$ are given by

$$\mathbf{H} = \begin{bmatrix} \nabla_{11}\varphi_1(x) - \nabla_{11}c(x)\lambda_1 & \cdots & \nabla_{1N}\varphi_1(x) - \nabla_{1N}c(x)\lambda_1 \\ & \ddots & \\ \nabla_{N1}\varphi_N(x) - \nabla_{1N}c(x)\lambda_N & \cdots & \nabla_{NN}\varphi_N(x) - \nabla_{NN}c(x)\lambda_N \end{bmatrix},$$

$$\mathbf{C} = \mathbf{e}\nabla c(x)^T,$$

$$\mathbf{e} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^N,$$

$$\mathbf{B} = \begin{bmatrix} -\nabla_1 c(x) & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & -\nabla_N c(x) \end{bmatrix}.$$

$\nabla_{ij}f(x)$ is the second order partial derivative (matrix) with respect x_i and x_j of a $\mathbb{R}$-valued function f, i.e. $\nabla_{ij}f(x) = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}$.

Lemma 15. *Let $c : \mathbb{R}^m \to \mathbb{R}$ be a concave function in C^2. Assume that for all $x, \Lambda \ge 0$ $\mathbf{H}(x, \Lambda)$ is a block diagonal positive definite matrix with blocks $\mathbf{H}_{1,1}, \ldots, \mathbf{H}_{N,N}$ where for each $i \in \mathcal{N}$, the submatrix $\mathbf{H}_{i,i}$ is a positive definite matrix in $\mathbb{R}^{m_i \times m_i}$. Then $\Psi(x, \Lambda)$ is a $\mathbf{P}_0$ matrix.*

Proof. The proof follows by showing that every principal submatrix of $\Psi(x, \Lambda)$ has a nonnegative determinant. Consider an arbitrary submatrix D where D is given by $D = \Psi(x, \Lambda)_{\alpha,\alpha}$, the submatrix of $\Psi(x, \Lambda)$ with rows and columns drawn from a set of indices $\alpha, \alpha \subseteq \{1, \ldots, m + N\}$. Let $\alpha = \beta \cup \gamma$, where $\beta \subseteq \{1, \ldots, m\}$ and $\gamma \subseteq \{m + 1, \ldots, m + N\}$. Then $\Psi(x, \Lambda)_{\alpha,\alpha}$ is given by

$$\begin{pmatrix} \mathbf{H}(x, \Lambda)_{\beta,\beta} & \mathbf{B}(x)_{\beta,\gamma} \\ \mathbf{C}(x)_{\gamma,\beta} & \mathbf{0}_{\gamma,\gamma} \end{pmatrix},$$

We drop arguments (x) and (x, Λ) for brevity. Consider some $\beta \subseteq \{1, \ldots, m\}$. If $[\mathbf{C}_{\gamma,\beta}, \mathbf{0}_{\gamma,\gamma}]$, has at least 2 identical rows, it follows that $\det(D) = 0$. Since c is $\mathbb{R}$-valued, it suffices to consider the case where $|\gamma| \leq 1$ and $[\mathbf{C}_{\gamma,\beta}, \mathbf{0}_{\gamma,\gamma}]$ does not contain all zeros. For any $i \in \beta$, let $b_i := \min\{k \mid i \leq \sum_{j=1}^{k} m_j\}$ and let $\kappa_i = m + b_i$. For a row $i \in \beta$, κ_i is the column of $\mathbf{B}$ that contains the vector $-\nabla_j c$ through which row i passes. i.e. if $\mathbf{B}[i, j]$ denotes the element in the ith row and $j-m$th column of $\mathbf{B}$, we have, $j \neq \kappa_i \implies \mathbf{B}[i, j] = 0$. Recall that γ has at most one element. Based on the choice of γ, three cases arise: (1) $\gamma = \emptyset$: In this case, D is a principal submatrix of $\mathbf{H}$. Since $\mathbf{H} \succ 0$, it follows that $\det(D) \geq 0$. (2) $\gamma \neq \emptyset, \gamma \cap \bigcup_{i\in\beta} \kappa_i = \emptyset$: Let $\gamma = \{j\}$. This assumption ensures that for all $i \in \beta, j \neq \kappa_i$. Hence there is a column of $\mathbf{B}_{\beta,\gamma}$ that has all zeros. Consequently, there is a zero column of D. Hence $\det(D) = 0$. (3) $\gamma \neq \emptyset, \gamma \cap \bigcup_{i\in\beta} \kappa_i = \gamma$: let $\gamma = \{j\}$. This means that there is an $i \in \beta$ such that $j = \kappa_i$. Assume that $j = m+1$ and $\beta = \{1, \ldots, m\}$. We shall see that there is no loss of generality in this assumption. Then, recalling that $\mathbf{H}$ is block diagonal, D may be written as

$$D = \begin{pmatrix} \mathbf{H}_{1,1} & \ldots & \mathbf{0}_{1,N} & -\nabla_1 c \\ & \ddots & & 0 \\ \mathbf{0}_{N,1} & \ldots & \mathbf{H}_{N,N} & 0 \\ \nabla_1 c^T & \ldots & \nabla_N c^T & 0 \end{pmatrix}$$

where $\mathbf{H}_{i,i} \in \mathbb{R}^{m_i \times m_i}$. Using the Schur complement (Horn & Johnson, 1990), we may write the determinant of D as

$$\begin{aligned}\det(D) &= \det(\mathbf{H})\det(-\mathbf{C}_{\gamma,\beta}\mathbf{H}^{-1}\mathbf{B}_{\beta,\gamma}) \\ &= \det(\mathbf{H})\det(\nabla_1 c^T \mathbf{H}_{1,1}{}^{-1}\nabla_1 c) \geq 0.\end{aligned}$$

Since c is $\mathbb{R}$-valued, $\nabla_1 c$ is a vector. The nonnegativity of $\det(D)$ follows from the positive definiteness of $\mathbf{H}$ and of the inverse of $\mathbf{H}_{1,1}$.

It is easy to see that the above arguments would hold if we picked γ to comprise some other element ($\neq m+1$) and $\mathbf{H}_{\beta,\beta}$ was any other principal submatrix of $\mathbf{H}$. □

Remark. The above result has assumed that c is $\mathbb{R}$-valued. Extending this result to settings where c is $\mathbb{R}^n$-valued (with $n > 1$) may not be immediately possible without making stronger assumptions on the Jacobians of c. □

An example of a game where the hypotheses of Lemma 15 are satisfied is the network routing game with affine coupling constraints considered in Johari and Tsitsiklis (2004). We present a modification of this game below.

Example 5 (*A Network Routing Game*)**.** Assume that each player has real valued strategies and solves an optimization problem given by

$$\boxed{\begin{array}{lll} A_i(x^{-i}) & \underset{x_i}{\text{minimize}} & \varphi_i(x_i; x^{-i}) = \mathcal{U}_i(x_i) \\ & \text{subject to} & a^T x \geq b, \\ & & x \geq 0, \end{array}}$$

where $a \in \mathbb{R}^m, b \in \mathbb{R}$ and $\mathcal{U}_i$ is a convex continuously differentiable function. This is clearly a shared constraint game with $m_i = 1$ for all $i \in \mathcal{N}$ and $\mathbb{C} = \{x \,|x \geq 0, a^T x \geq b\}$. If $\mathcal{U}_i$ are strictly convex for each $i \in \mathcal{N}$, F is a strictly monotone function and $\mathbf{H}$ is a positive definite diagonal matrix. Ordinarily, this would not be sufficient to claim that $\mathrm{VI}(\mathbb{C}, F)$ has a solution and that this game has a VE primal–dual space), one may use Theorem 13 and thus this game belongs to the class $\mathcal{S} \setminus \mathcal{S}_2'$. Furthermore, if we are independently given the boundedness of the (possibly empty) set of GNEs in the primal–dual space, we may say that the game either admits a VE or admits no GNE at all. This is also compatible with economic intuition that for such games, the equilibrium with uniform prices is the more appealing solution concept and is the one applied in Johari and Tsitsiklis (2004). This is an example where our results provide a mathematical substantiation of ideas that were either previously known or had an intuitive appeal. □

4. Conclusions

In this paper, we presented a theory of the VE as a refinement of the GNE. We argued that this question is important for pure, applied and computational game theory. The theory was based on a topological relation between the GNE and the VE. The GNE and the VE were shown to be related via a Brouwer degree-theoretic equivalence in the primal and primal–dual space. This equivalence led to sufficiency conditions for a shared constraint game to have the VE as a refinement of the GNE. Furthermore, for certain games, these conditions were seen as necessary. Our results unified some known results and provided mathematical substantiation for ideas that were known to be appealing to economic intuition.

We also note that the refinement theory has several implications that extend beyond theoretical interest. For instance, VEs often have economic merit and this result paves the way for claiming when the presence of a GNE suffices for the existence of an economically desirable equilibrium, namely a VE. Another utility of such a result is that it immediately allows us to focus on the better posed computational problem of solving for a VE. We believe that the degree-theoretic result has the potential for furthering the understanding GNEs and VEs, beyond the application to the study of the refinement. Finally, we believe that our work will serve in reviving interest in the more general question of equilibrium refinement in the theory of games.

A.1. On the existence of a manifold of GNEs

A remark made in Facchinei et al. (2007) claimed that generalized Nash games often have a manifold of equilibria. Theorem 16 below shows that manifolds may exist under some conditions on the dimensions of the QVI. Here we assume $c : \mathbb{R}^m \to \mathbb{R}^n$ to be a C^1 concave function and that an appropriate constraint qualification holds. A similar conclusion has been drawn in Proposition 4 in Facchinei, Fischer, and Piccialli (2007). Our result also provides indications about the role that dimensions m, n and the number of players N play.

Theorem 16. *Consider a game in which $m = nN$. Let $(x^*, \Lambda^*) > 0$ be a GNE of such a game such that the square matrix $\mathbf{B}(x^*)$ is nonsingular. Then there exists a neighborhood $B(x^*, r) \subseteq \mathbb{R}^m$ of x^* of radius r such that for every $x \in B(x^*, r) \cap \{v \,|c(v) = 0\}$, there exists $\Lambda \geq 0$ so that (x, Λ) is a GNE.*

Proof. Since $(x^*, \Lambda^*) > 0$, it is easy to see from the KKT conditions and nonsingularity of $\mathbf{B}(x^*)$ that

$$\Lambda^* = -\mathbf{B}(x^*)^{-1}F(x^*).$$

$\det(\mathbf{B}(\cdot))$ is a $\mathbb{R}^m \to \mathbb{R}$ continuous function. By continuity, there is a neighborhood $B(x^*, r_1) \subseteq \mathbb{R}^m$ of x^* such that $\operatorname{sgn}\det(\mathbf{B}(x)) = \operatorname{sgn}\det(\mathbf{B}(x^*))$ for all $x \in B(x^*, r_1)$. Thus $\mathbf{B}(x)$ is nonsingular on $B(x^*, r_1)$. Furthermore, since $-\mathbf{B}(x^*)^{-1}F(x^*) > 0$ and $x^* > 0$ there is another neighborhood $B(x^*, r_2)$ of x^* such that for all $x \in B(x^*, r_2)$, $-\mathbf{B}(x)^{-1}F(x) \geq 0$ and $x > 0$. Finally, let $r = \min\{r_1, r_2\}$ and pick an arbitrary $x \in B(x^*, r) \cap \{v \mid c(v) = 0\}$. Since $r \leq r_2$, $x > 0$. Using this it is easy to see that for this x, the pair (x, Λ), where $\Lambda = -\mathbf{B}(x)^{-1}F(x)$, is a GNE. □

Theorem 16 can be extended to the case where $m < nN$ by replacing the hypothesis of nonsingularity of $\mathbf{B}(x^*)$ with one of full row-rank.

A.2. Proof of Lemma 4

Proof. (1) Take any $i \in \mathcal{N}$ and consider an $x \in \mathbb{R}^m$. Note from (1) and the Cartesian nature assumed on $\mathbb{C}$ that $K_i(x^{-i}) = \{y_i \in \mathbb{R}^{m_i} \mid (y_i, x^{-i}) \in \mathbb{C}\} = \{y_i \in \mathbb{R}^{m_i} \mid y_i \in \mathbb{C}_i, x_j \in \mathbb{C}_j, j \neq i\}$, which is nonempty if $x_j \in \mathbb{C}_j, \forall j \neq i$. Thus $K(x) = \prod K_i(x^{-i}) \neq \emptyset$ if and only if $x \in \mathbb{C}$. Similarly, for $x \in \mathbb{C}$, we have $y \in K(x)$ if and only if $y \in \mathbb{C}$. Therefore $K(x) = \mathbb{C}$ if and only if $x \in \mathbb{C}$.

(2) Let $x \in K(x)$ implying that $x_i \in K_i(x^{-i}),\ \forall i \in \mathcal{N}$, and therefore $(x_i, x^{-i}) \in \mathbb{C},\ \forall i \in \mathcal{N}$ and $x \in \mathbb{C}$. The converse follows by noting that $x \in \mathbb{C}$ is equivalent to $(x_i, x^{-i}) \in \mathbb{C}\ \forall i$, i.e. $x_i \in K_i(x^{-i}),\ \forall i$ and therefore $x \in K(x)$.

(3) Let $x \in \mathrm{dom}(K)$ and $y, z \in K(x)$, i.e. for each $i \in \mathcal{N}$, (y_i, x^{-i}) and $(z_i, x^{-i}) \in \mathbb{C}$. The convexity of $K(x)$ follows by noting that since $\mathbb{C}$ is convex, $((\alpha y_i + (1-\alpha) z_i), x^{-i}) \in \mathbb{C}$ for each i and $\alpha \in [0, 1]$. To show closedness, consider a sequence $\{y^k\} \subseteq K(x)$ with limit point $\bar{y}$. By closedness of $\mathbb{C}$, for each i, the sequence $\{(y_i^k, x^{-i})\} \subseteq \mathbb{C}$ and $\lim(y_i^k, x^{-i}) = (\bar{y}_i, x^{-i}) \in \mathbb{C}$. Thus $K(x)$ is closed.

(4) Suppose x is a point in $\mathbb{C}$, $d \in K(x)_\infty$ is an arbitrary recession direction and τ is an arbitrary nonnegative number. By convexity of $\mathbb{C}$, it suffices to show that $x + \tau d \in \mathbb{C}$. Since $d \in K(x)_\infty$, and $x \in K(x)$, the point $x + N\tau d$ belongs to $K(x)$. Therefore the points $z^i := (x_i + N\tau d_i, x^{-i}), i \in \mathcal{N}$, belong to $\mathbb{C}$. By convexity of $\mathbb{C}$, the average of these points

$$\frac{1}{N}\sum_{i\in\mathcal{N}} z^i = \frac{N-1}{N}x + \frac{1}{N}(x + N\tau d) = x + \tau d,$$

also belongs to $\mathbb{C}$, as required. If $\mathbb{C}$ is bounded, $\mathbb{C}_\infty = \{0\}$ and we get $K(x)_\infty \subseteq \{0\}$. Therefore $K(x)_\infty = \{0\}$, implying that $K(x)$ is bounded. □

A.3. Proof of Lemma 5

Proof. Suppose $x \in \mathrm{int}(\mathbb{C})$. Then there exist open sets $\mathcal{O}_i \subseteq \mathbb{R}^{m_i}$ containing x_i such that $x \in \mathcal{O} := \prod_{i\in\mathcal{N}} \mathcal{O}_i \subseteq \mathbb{C}$. Then $(\mathcal{O}_i, x^{-i}) := \cup_{y_i \in \mathcal{O}_i}(y_i, x^{-i}) \subseteq \mathbb{C}$, so that $\mathcal{O}_i \subseteq K_i(x^{-i})$, for each $i \in \mathcal{N}$. It follows that $\mathcal{O} \subseteq K(x)$. For the converse, let $\mathrm{int}(K(x))$ be nonempty and $x \in \mathrm{int}(K(x))$. Then for each $i \in \mathcal{N}$, x_i belongs to the interior of $K_i(x^{-i})$ (where $K_i(x^{-i})$ is considered a set in $\mathbb{R}^{m_i}$). Thus there exist open sets $\mathbb{R}^{m_i} \supseteq \mathcal{O}_i \subseteq K_i(x^{-i})$ containing x_i for all i. It follows that $(\mathcal{O}_i, x^{-i}) \subseteq \mathbb{C}$ for all $i \in \mathcal{N}$. Now since $\mathbb{C}$ is convex, the average of these sets is contained in $\mathbb{C}$, i.e.

$$\mathcal{A} := \sum_{i\in\mathcal{N}} \frac{(\mathcal{O}_i, x^{-i})}{N} = \frac{1}{N}\prod_{i\in\mathcal{N}} \mathcal{O}_i + \frac{N-1}{N}x \subseteq \mathbb{C}.$$

Since $x_i \in \mathcal{O}_i$, $\mathcal{A}$ contains x. Furthermore, $\mathcal{A}$ is open, implying that $x \in \mathrm{int}(\mathbb{C})$. □

A.4. Cones

For any closed convex set S, by S_∞ we denote its recession cone (Facchinei & Pang, 2003; Rockafellar, 1997, page 158): $S_\infty = \{d \mid S + \tau d \subseteq S,\ \forall \tau \geq 0\}$. For any set T and a point $z \in T$, let $\mathcal{N}(z;T)$, $\mathcal{T}(z;T)$ denote the *normal cone* and the *tangent cone* of T at z respectively.

$$\mathcal{N}(z;T) = \{d \mid d^T(y - z) \leq 0 \ \forall y \in T\}$$

$$\mathcal{T}(z;T) = \Big\{ d \mid \exists\, \{\tau_k\} \subseteq (0,\infty) \text{ and } \{y_k\} \subseteq T, \text{ s.t. } \tau_k \to 0, y_k \to z, \text{ and } d = \lim_{k\to\infty} \frac{y_k - x}{\tau_k} \Big\}.$$

For any cone T, we use T^* to denote its dual cone as $T^* = \{d \mid d^T x \geq 0 \ \forall x \in T\}$.

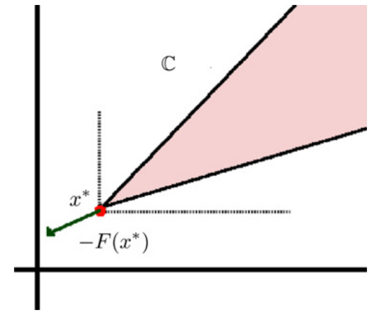


Fig. A.1. An example where $K(x^*) = \{x^*\}$. Independently of F, x^* solves $\mathrm{QVI}(K, F)$.

A.5. Some more examples of shared constraint games

The following examples illustrate the unusual properties that generalized Nash games with shared constraints can exhibit. We observe that these properties are a consequence of the structure of the set-valued map K and often occur for points on the boundary of the domain of K. Throughout we assume $N = 2$ and $m_1 = m_2 = 1$.

Example 6 (*Game with GNE Independent of F*)**.** Fig. A.1 shows $\mathbb{C}$ and a point $x^* \in \partial\mathbb{C}$ with the property that for *any* F, x^* solves $\mathrm{QVI}(K, F)$. This is because the image of x^* under K is a singleton, namely x^* itself. In Fig. A.1, dotted lines depict axes with their origin shifted to x^*. If $y \in K(x^*)$, the points (y_1, x_2^*) and (x_1^*, y_2) lie in on these 'axes'. Since these 'axes' intersect $\mathbb{C}$ at only one point, x^*, we have $K(x^*) = \{x^*\}$. Therefore trivially, x^* solves $\mathrm{SOL}(\mathrm{QVI}(K, F))$. Observe that x^* lies in $\partial\mathrm{dom}(K)$.

Example 7 (*Game with Unique GNE and VE*)**.** Suppose x^* in Fig. A.1 is the origin of the coordinate system, i.e. let $x^* = (0, 0)$. Assume that $\mathbb{C} = \{(x_1, x_2) \mid x \geq 0,\ x_1 \in [x_2, 2x_2]\}$, so that $K(x) = \{(y_1, y_2) \mid y \geq 0,\ y_1 \in [x_2, 2x_2],\ y_2 \leq x_1 \leq 2y_2,\ x \geq 0\}$. Let $F(x) = (x_1, 1)$ (arising from, say, $\varphi_1(x) = \frac{1}{2}x_1^2 + x_2$, $\varphi_2(x) = x_2 - x_1$). This game has a unique GNE, x^*. Indeed one may verify that $K(x^*) = \{x^*\}$, and conclude from the previous example that x^* is a solution for any F. To see that this is the only solution, suppose $x \in \mathbb{C} \setminus \{x^*\}$ solves $\mathrm{QVI}(K, F)$. i.e. $(x_1, 1)^T[(y_1, y_2) - (x_1, x_2)] \geq 0\ \forall y \in K(x)$. Observe that $y = (x_1, \frac{1}{2}x_1) \in K(x)$ since $x_1 \in [x_2, 2x_2]$. Substituting above gives $x_1 \geq 2x_2$, which implies that the solution must satisfy $x_1 = 2x_2$. Now observe that $y = (x_2, \frac{1}{2}x_1) = (\frac{1}{2}x_1, \frac{1}{2}x_1)$ also belongs to $K(x)$. Substituting this y above gives $\frac{1}{2}x_1^2 \leq 0$. It follows that $x = (0, 0) = x^*$, a contradiction. That makes x^* the only solution. Note that Theorem 16 does not apply here because $(x^*, \Lambda^*) \not> 0$. It is easy to check that x^* also solves $\mathrm{VI}(\mathbb{C}, F)$, i.e. x^* is a VE, and is the unique VE. □

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