ADAPTIVE BEAMFORMING

Assignment 3: EEL 6935 - SPRING.2018 Hector Lopez 3-2-2018

Problem 1

Use the Matrix Inversion Lemma (Woodbury's identity) to derive a recursion for the inverse of the estimated input autocorrelation matrix $[\hat{R}(k)]^{-1}$ based on the sample average recursion

$$(1) \quad \hat{R}(k) = \frac{(k-1) * \hat{R}(k-1) + r_k r_k^H}{k}$$

where $r_k, k = 1,2, \ldots$, are the array input vectors.

Let's recall the Woodbury Identity.
Given: $A_{MxM}, B_{MxM}, C_{MxL}, D_{LxL}$ assume that A,B,D, are invertible.

If $A = B^{-1} + CD^{-1}C^H$;
Then $A^{-1} = B - BC(D + C^H BC)^{-1} + C^H B$

By direct multiplication the Woodbury Lemma shows:

$(AA^{-1} = \cdots = I)$ and $(A^{-1}A = \cdots = I)$

Lets first re-write our equation (1) so that it is in the form of the Woodbury formula. We will have to move the scalar (1/k) and remember to multiply our inverted matrix by it at the end.

$$(2.1) \quad \hat{R}(k) = \frac{(k-1)}{k} * \hat{R}(k-1) + \frac{1}{k} * \vec{r}_k \vec{r}_k^H$$

$$(2.2) \quad k * \hat{R}(k) = (k-1) * \hat{R}(k-1) + \vec{r}_k \vec{r}_k^H$$

We can now define the matrices that we will use in the inversion formula.

$$(3) \quad A = \hat{R}(k)\,, \;\; B^{-1} = (k-1) * \hat{R}(k-1), \;\; C = \vec{r}_k, \;\; D^{-1} = 1$$

These matrices can be used in the lemma to define our inverse A matrix.

$$(4) \quad A^{-1} = B - BC(D + C^H BC)^{-1} + C^H B$$

$$(5.1) \quad \hat{R}(k)^{-1} = inv\big[(k-1) * \hat{R}(k-1)\big] - inv[(k-1) * \hat{R}(k-1)]\vec{r}_k\big(1 + \vec{r}_k{}^H inv[(k-1) * \hat{R}(k-1)]\vec{r}_k\big)^{-1} + \vec{r}_k{}^H inv[(k-1) * \hat{R}(k-1)]$$

$$(5.2) \quad \hat{R}(k)^{-1} = \frac{1}{(k-1)} * \hat{R}(k-1)^{-1} - \frac{1}{(k-1)} * \hat{R}(k-1)^{-1} * \vec{r}_k \left(1 + \vec{r}_k{}^H * \frac{1}{(k-1)} * \hat{R}(k-1)^{-1} * \vec{r}_k\right)^{-1} + \vec{r}_k{}^H * \frac{1}{(k-1)} * \hat{R}(k-1)^{-1}$$

$$(5.3) \qquad \hat{R}(k)^{-1} = \frac{1}{(k-1)} * \left[\hat{R}(k-1)^{-1} - \frac{\hat{R}(k-1)^{-1} * \vec{r}_k {\vec{r}_k}^H * \hat{R}(k-1)^{-1}}{(k-1) + {\vec{r}_k}^H * \hat{R}(k-1)^{-1} * \vec{r}_k} \right]$$

We need to add the scalar we had before the Woodbury lemma was applied.

$$(6.1) \qquad k * \hat{R}(k)^{-1} = \frac{1}{(k-1)} * \left[\hat{R}(k-1)^{-1} - \frac{\hat{R}(k-1)^{-1} * \vec{r}_k {\vec{r}_k}^H * \hat{R}(k-1)^{-1}}{(k-1) + {\vec{r}_k}^H * \hat{R}(k-1)^{-1} * \vec{r}_k} \right]$$

$$(6.2) \qquad \hat{R}(k)^{-1} = \frac{1}{k * (k-1)} * \left[\hat{R}(k-1)^{-1} - \frac{\hat{R}(k-1)^{-1} * \vec{r}_k {\vec{r}_k}^H * \hat{R}(k-1)^{-1}}{(k-1) + {\vec{r}_k}^H * \hat{R}(k-1)^{-1} * \vec{r}_k} \right]$$

Finally, the equation (6.2) shows the final inversion as a recursive function in terms of the k , the input vector instance and the previous inversion value to get the new inversion value. This recursive formula needs to be initialized. We can initialize it with $\hat{R}(0)^{-1} = c * I$ Where I is the identity matrix and for some c << 1.

Problem 2

Show that,

$$(7) \;\; \alpha_1 \triangleq \frac{\alpha_1'}{SINR_{opt} * (1 - \alpha_1') + 1}$$

where $SINR_{opt}$ is the maximum attainable SINR by the antenna array and α_1, α_1' are as defined in the lectures.

We are examining the signal present adaptive beamformer versus the signal absent adaptive beamformer by analyzing the SINR of each, versus the optimal SINR of a beamformer given some linear antenna array.

$$(1) \;\; \alpha_1 = \frac{SINR_1}{SINR_{opt}} = \frac{\left({\vec{S}_\theta}^H * \hat{R}^{-1} * \vec{S}_\theta \right)^2}{\left({\vec{S}_\theta}^H * R_{I+N} * \vec{S}_\theta \right) \left({\vec{S}_\theta}^H * \hat{R}^{-1} * R_{I+N} * \hat{R}^{-1} * \vec{S}_\theta \right)}$$

$$(2) \;\; \alpha_2 = \frac{SINR_2}{SINR_{opt}} = \frac{\left({\vec{S}_\theta}^H * {\hat{R}_{I+N}}^{-1} * \vec{S}_\theta \right)^2}{\left({\vec{S}_\theta}^H * {R_{I+N}}^{-1} * \vec{S}_\theta \right) \left({\vec{S}_\theta}^H * {\hat{R}_{I+N}}^{-1} * {R_{I+N}}^{-1} * {\hat{R}_{I+N}}^{-1} * \vec{S}_\theta \right)}$$

We notice that there is no direct way to solve for the PDF of α_1 because there are two types of matrices in the denominator, $\hat{R}^{-1}$, ${\hat{R}_{I+N}}^{-1}$ for the eq.1 . Let's define another version of α_1. by replacing the inversed sampled disturbance matrix, ${\hat{R}_{I+N}}^{-1}$ with $\hat{R}^{-1}$ in eq.1. This creates a new formula of the ratio α_1'. The resulting equation is similar to α_2. The two formulas are almost identical in form with the difference being the swap between the disturbance matrix and the input signal autocorrelation matrices. Since the two matrices, $\hat{R}^{-1}$, ${\hat{R}_{I+N}}^{-1}$, have the same distribution, the ratios of ${\alpha_1}'$ and α_2 also have the same distribution.

$$\alpha_1' \sim \alpha_2 \; (identically\ distributed)$$

$$(8) \quad E\{\alpha_1'\} = E\{\alpha_2\}$$

The denominator of the right hand side can be evaluated to always be greater than zero. We can prove this by first observing that the random variable α_1' is a ratio of the SINR's that ranges from zero to one.

$$0 \leq \alpha_1' \leq 1$$

The next observation is that the $SINR_{opt}$ is also a ratio of the signal and the disturbances, noise and interference, so it will be greater than 0.

$$SINR_{opt} > 0$$

We can prove by solving for the limits of the variables that the expression will remain as a positive value.

$$SINR_{opt} * (1 - \alpha_1') + 1 > 0$$

$$SINR_{opt} * (1 - \alpha_1') + 1 > SINR_{opt} + 1 \because \alpha_1' > 0$$

$$\cancel{0 * (1 - \alpha_1')} + 1 = 1 \because SINR_{opt} = 0$$

$$\cancel{SINR_{opt} * (0)} + 1 = 1 \because \alpha_1' = 1$$

The optimal SINR should be much larger than 0, so if the $SINR_{opt}$ value gets larger. We can also observe that as we reduce the value of α_1' the largest value α_1 can achieve is also reduced proportionally.

$$(9) \;\; \alpha_1 \triangleq \frac{\alpha_1'}{SINR_{opt} * (1 - \alpha_1') + 1} \ll \alpha_1'$$

In conclusion given the statement (eq.9) the random variable of α_1 will always be less than α_1' and similarly will always be less than α_2 since the mean of $E\{\alpha_1'\} = E\{\alpha_2\}$.

Problem 3

3. Design a simulation to estimate $E\{\alpha_1\}, E\{\alpha_2\}, E\{\alpha_3\}$:

Consider BPSK transmissions of one user of interest and three interferers.
Assume M = 10 and arbitrary in (-90,90) but fixed angles of arrival.

$$BPSK = \vec{r} = \sqrt{E_1} * b_1 * \vec{S}_{\theta_1} + \sum_{k=2}^{4} \sqrt{E_k} * b_k * \vec{S}_{\theta_k} + \vec{n}$$

$$where\ b = \pm 1$$

How do we measure SNR? The SNR would be measured as the power of a signal's meaningful information over the power of the background noise. If the variance of the signal and noise are known, and the signal and noise are both zero mean, SNR can be:

$$SNR = \frac{P_{signal}}{P_{noise}} = \frac{\sigma^2_{signal}}{\sigma^2_{noise}}$$

(a) For some fixed SNR values for the user signals, plot your estimated means of α1, α2, and α3 as a function of N from N = 10 to 200.

Our user signals can be given fixed SNR's if we assume a distribution for the noise component and if we have a distribution for the signal component.

The signal component would just be the $\sqrt{E_i} * b_i$, for $i = 10,11, \ldots 200$ since we will be creating N=200 samples.

$$user\ signals = \sqrt{E_i} * b_i$$

We can determine that the SNR with signal absent and signal present:

$$SNR1 = \frac{E\{|\sqrt{E_1} * b_1|^2\} + E\{|\sum_{k=2}^{4} \sqrt{E_k} * b_k|^2\}}{\sigma^2_{noise}} = E + 3E$$

$$SNR2 = \frac{E\{|\sum_{k=2}^{4} \sqrt{E_k} * b_k|^2\}}{\sigma^2_{noise}} = 3E$$

For this BPSK arrangement the average signal power will always equal to E .
The noise component will be an AWGN signal and have a variance of 1.

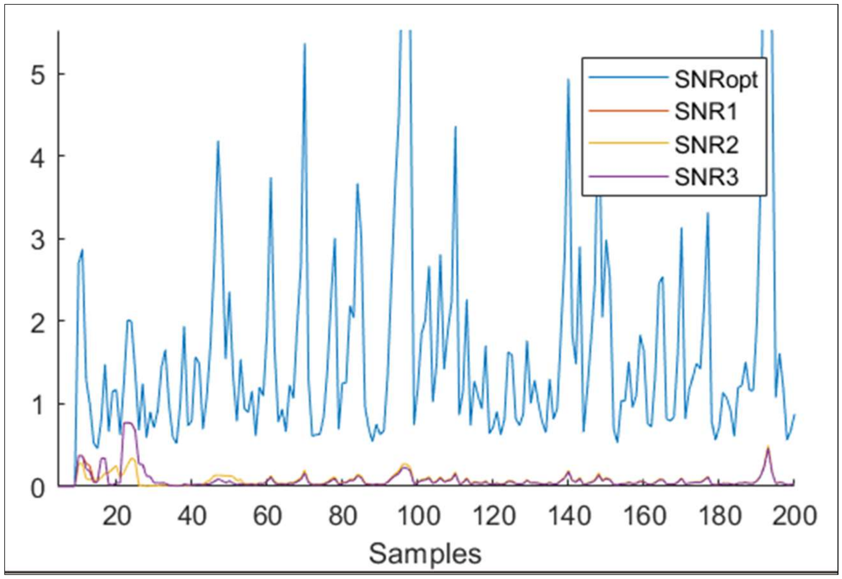
Our SNR optimal would come from the closed form solution for the optimal SNR beamformer applied to the input signals and then the power of that output would be used to calculate SNR again over an AWGN noise . Finally by taking the ratios we can calculate the alpha's.

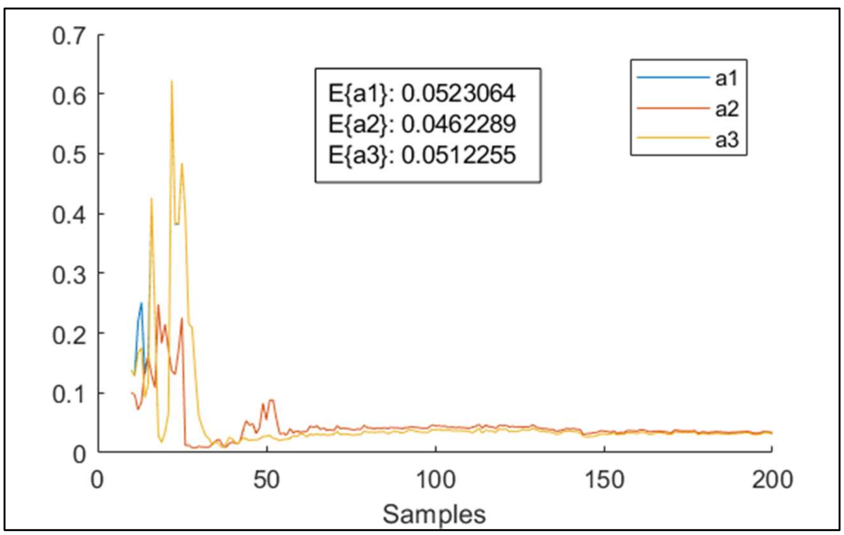
We have a beamformer that can give us the optimum SNR in the form of :

$$\hat{\vec{w}}_{maxS} = c * \vec{S}_\theta$$

We can multiply our received signal by this weight vector and be able to create an output value for each sample N=10 to 200. The output series can then be used to get the average power and divide it by the average AWGN noise power we can get the optimum SNR.

Since we are using a derived series of samples for the signal and for the noise lets use that same series and apply the beamformers for the signal present and signal absent SINR maximization. The outputs of these beamformers should on average equal to E.





(a) Keep the SNRs of the interferences as in Part (a) and plot the estimated means of α1, α2, and α3 as a function of the SNR of the user of interest for N = 10,50,100,200.

We can look at the SNR's over the optimum as alpha's 1,2, and 3. The mean of the signals shows us the performance of the simulation. The optimal curves between a1 and a2 can be found algebraically since we know the M and the N.

$$E\{\alpha_2\} = \frac{N + 2 - M}{N + 1} \text{ , } when\ (N > M - 3)$$

$$E\{\alpha_2\} = \frac{(200) + 2 - (10)}{(200) + 1} = \frac{208}{201} = 1.03$$