S09E01 The path integral

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Gell-Mann and Low theorem

We have, using $\rightsquigarrow$ and $\rightarrow$ diagrams introduced in S03E05,

$$\begin{aligned}\phi_3\phi_2\phi_1 &= U_{03}\phi_{3I}U_{30}U_{02}\phi_{2I}U_{20}U_{01}\phi_{1I}U_{10}\\ &= U_{03}\phi_{3I}U_{32}\phi_{2I}U_{21}\phi_{1I}U_{10}\end{aligned}$$

where ϕ_j are time-dependent Heisenberg operators. We also have

$$\begin{aligned}U_{0\,-\infty}|\Omega_0\rangle &= \lim_{t_-\nwarrow -\infty} e^{iEt_-}e^{-iE_0t_-}|\Omega\rangle\langle\Omega|\Omega_0\rangle\\ \langle\Omega_0|U_{+\infty\,0} &= \lim_{t_+\searrow +\infty} e^{iE_0t_+}e^{-iEt_+}\langle\Omega_0|\Omega\rangle\langle\Omega|\end{aligned}$$

Notice that $U_{0\,-\infty}$ and $U_{+\infty\,0}$ are not unitary! since t_- and t_+ are not real.

$$\langle\Omega|\phi_3\phi_2\phi_1|\Omega\rangle = \frac{\langle\Omega_0|U_{+\infty 3}\phi_{3I}U_{32}\phi_{2I}U_{21}\phi_{1I}U_{1\,-\infty}|\Omega_0\rangle}{\langle\Omega_0|U_{+\infty\,-\infty}|\Omega_0\rangle}$$

The path-integral

$$\langle\Omega|\phi_3\phi_2\phi_1|\Omega\rangle = \frac{\langle\Omega_0|U_{+\infty 0}\phi_3\phi_2\phi_1 U_{0-\infty}|\Omega_0\rangle}{\langle\Omega_0|U_{+\infty 0}U_{0-\infty}|\Omega_0\rangle}$$

After canceling some factors,

$$\langle\Omega|\phi_3\phi_2\phi_1|\Omega\rangle = \lim_{t_-\nwarrow-\infty}\lim_{t_+\searrow+\infty}\frac{\langle\Omega_0|e^{-iHt_+}\phi_3\phi_2\phi_1 e^{iHt_-}|\Omega_0\rangle}{\langle\Omega_0|e^{-iHt_+}e^{iHt_-}|\Omega_0\rangle}$$

In fact, we have even more:

$$\langle\Omega|\phi_3\phi_2\phi_1|\Omega\rangle = \lim_{t_-\nwarrow-\infty}\lim_{t_+\searrow+\infty}\frac{\langle\Omega'|e^{-iHt_+}\phi_3\phi_2\phi_1 e^{iHt_-}|\Omega''\rangle}{\langle\Omega'|e^{-iHt_+}e^{iHt_-}|\Omega''\rangle}$$

for any $|\Omega'\rangle$ and $|\Omega''\rangle$. This is because

$$\lim_{t\searrow+\infty} e^{-iHt} = \lim_{t\searrow+\infty} e^{-iEt}|\Omega\rangle\langle\Omega|$$