The Dirac equation

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1 The Dirac equation

The Dirac Hamiltonian in the presence of an electromagnetic field is

$$H = \alpha \cdot \pi + \beta m + q\phi$$

with the Dirac matrices in the Dirac representation

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad H = \begin{pmatrix} m + q\phi & \sigma \cdot \pi \\ \sigma \cdot \pi & -m + q\phi \end{pmatrix}$$

The effective Hamiltonian of the upper-left subspace is

$$\left(m + q\phi + \sigma \cdot \pi \frac{1}{E + m - q\phi} \sigma \cdot \pi \right) \psi = E\psi$$

and let us write $E = m + \varepsilon$, such that we obtain

$$\left(q\phi + \sigma \cdot \pi \frac{1}{2m + \varepsilon - q\phi} \sigma \cdot \pi \right) \psi = \varepsilon\psi$$

2 The Pauli Hamiltonian

In the case where ϕ vanishes, we have the leading-order correction

$$\begin{aligned} \frac{1}{2m} \sigma \cdot \pi \, \sigma \cdot \pi &= \frac{1}{2m} \pi^2 + \frac{1}{2m} i\sigma \cdot \pi \times \pi \\ &= \frac{1}{2m} \pi^2 - \frac{q}{2m} \sigma \cdot \nabla \times \mathbf{A} \qquad \pi \equiv \mathbf{p} - q\mathbf{A} \\ &= \frac{1}{2m} \pi^2 - \frac{q}{2m} \sigma \cdot \mathbf{B} \end{aligned}$$

from which we read the electron magnetic moment

$$\mathbf{m} = \frac{q}{2m} \sigma$$

3 The spin-orbit coupling

In the case where ϕ does not vanish, we have the leading-order correction

$$q\phi + \frac{1}{2m}\sigma\cdot\pi\,\sigma\cdot\pi = q\phi + \frac{1}{2m}\pi^2 - \frac{q}{2m}\sigma\cdot\mathbf{B}$$

Now we want the next-to-leading-order correction:

$$\left(q\phi + \frac{1}{2m}\sigma\cdot\pi\,\sigma\cdot\pi - \sigma\cdot\pi\,\frac{\varepsilon - q\phi}{4m^2}\,\sigma\cdot\pi\right)\psi = \varepsilon\psi$$

$$\left(q\phi + \frac{1}{2m}\sigma\cdot\pi\,\sigma\cdot\pi + \sigma\cdot\pi\,\frac{q\phi}{4m^2}\,\sigma\cdot\pi\right)\psi = \varepsilon\left(1 + \frac{1}{4m^2}\sigma\cdot\pi\,\sigma\cdot\pi\right)\psi$$

such that, approximately,

$$\begin{gathered}\left(1 - \frac{1}{8m^2}\sigma\cdot\pi\,\sigma\cdot\pi\right)\left(q\phi + \frac{1}{2m}\sigma\cdot\pi\,\sigma\cdot\pi + \sigma\cdot\pi\,\frac{q\phi}{4m^2}\,\sigma\cdot\pi\right)\left(1 - \frac{1}{8m^2}\sigma\cdot\pi\,\sigma\cdot\pi\right)\\ \left(1 + \frac{1}{8m^2}\sigma\cdot\pi\,\sigma\cdot\pi\right)\psi = \varepsilon\left(1 + \frac{1}{8m^2}\sigma\cdot\pi\,\sigma\cdot\pi\right)\psi\end{gathered}$$

The effective Hamiltonian is the first line

$$\begin{aligned}H_{\text{eff}} &= q\phi + \frac{1}{2m}\sigma\cdot\pi\,\sigma\cdot\pi + \sigma\cdot\pi\,\frac{q\phi}{4m^2}\,\sigma\cdot\pi - \frac{1}{8m^2}\sigma\cdot\pi\,\sigma\cdot\pi\,q\phi - \frac{1}{8m^2}q\phi\,\sigma\cdot\pi\,\sigma\cdot\pi\\ &= q\phi + \frac{1}{2m}\sigma\cdot\pi\,\sigma\cdot\pi - \frac{1}{8m^2}\sigma\cdot\pi\left[\sigma\cdot\pi, q\phi\right] - \frac{1}{8m^2}\left[q\phi, \sigma\cdot\pi\right]\sigma\cdot\pi\end{aligned}$$

and the final result is

$$H_{\text{eff}} = q\phi + \frac{1}{2m}\pi^2 - \frac{q}{2m}\sigma\cdot\mathbf{B} + \frac{1}{8m^2}q\nabla^2\phi + \frac{1}{4m^2}\sigma\cdot q\nabla\phi\times\pi$$

In the case where ϕ is a central potential, the last term is a spin-orbit coupling term:

$$\frac{1}{4m^2}\sigma\cdot q\nabla\phi\times\pi = \frac{q}{2m^2 r}\frac{d\phi}{dr}\mathbf{S}\cdot\mathbf{L}$$

Remark We also have the relativistic correction

$$-\frac{1}{8m^3}\sigma\cdot\pi\,\sigma\cdot\pi\,\sigma\cdot\pi\,\sigma\cdot\pi \sim -\frac{1}{8m^3}p^4$$

4 Transformation properties

The following is what we want:

$$i\gamma^{\mu}\frac{\partial}{\partial x^{\mu}}\psi\left(x\right)=m\psi\left(x\right)\qquad\text{would imply that}$$
$$i\gamma^{\mu}\frac{\partial}{\partial x^{\mu}}D\left(\Lambda\right)\psi\left(\Lambda^{-1}x\right)=mD\left(\Lambda\right)\psi\left(\Lambda^{-1}x\right)$$

which determines the desired transformation property of the gamma matrices.

5 Quantum Lorentz transformations

An infinitesimal quantum Lorentz transformation of the Dirac field is

$$\begin{aligned}D\left(\Lambda\right)\psi\left(\Lambda^{-1}x\right)&=\left(1+\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\left(1+\frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\right)\psi\left(x\right)\\&\equiv\left(1+\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right)\psi\left(x\right)\end{aligned}$$

and generally we have

$$U\left(\Lambda,a\right)=\left(1+\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}-ia_{\rho}P^{\rho}\right)$$