The Kondo problem

Liqin Huang

1 The effective Hamiltonian method

The stationary state Schrodinger equation

$$H\psi = E\psi$$

can be written as, for example,

$$\begin{aligned} H_{11}\psi_1 + H_{12}\psi_2 &= E\psi_1 \\ H_{21}\psi_1 + H_{22}\psi_2 &= E\psi_2 \end{aligned}$$

The second equation can be used to solve for ψ_2:

$$\psi_2 = \frac{1}{E - H_{22}} H_{21}\psi_1$$

which gives us a self-consistent equation

$$\left(H_{11} + H_{12}\frac{1}{E - H_{22}} H_{21}\right)\psi_1 = E\psi_1$$

2 Applications to degenerate perturbation theory

To second order in the perturbation,

$$\begin{aligned} H_{\text{eff}} &= E_1 1 + V_{11} + V_{12}\frac{1}{E - H_{0\,22} - V_{22}} V_{21} \\ &\approx E_1 1 + V_{11} + V_{12}\frac{1}{E_1 - H_{0\,22}} V_{21} \end{aligned}$$

3 AFM coupling in the Hubbard model

Consider the Hubbard model with only two sites

$$H = -t \sum_{i \neq j} (a_{is})^\dagger a_{js} + U \sum_i n_{i+} n_{i-}$$

The effective Hamiltonian is equal to

$$\begin{aligned} H_{\text{eff}} &= \frac{t^2}{E-U} \sum_{i \neq j} \sum_{k \neq l} (a_{is})^\dagger a_{js} (a_{kr})^\dagger a_{lr} \\ &= \frac{t^2}{E-U} \sum_{i \neq j} (a_{is})^\dagger a_{js} (a_{jr})^\dagger a_{ir} \end{aligned}$$

Notice that

$$\begin{aligned} \sum_{i \neq j} S_i \cdot S_j &= \frac{1}{2} \sum_{i \neq j} (a_{is})^\dagger a_{ir} (a_{jr})^\dagger a_{js} - \frac{1}{4} \sum_{i \neq j} (a_{is})^\dagger a_{is} (a_{jr})^\dagger a_{jr} \\ &= \frac{1}{2} \sum_{i \neq j} (a_{is})^\dagger a_{is} - \frac{1}{2} \sum_{i \neq j} (a_{is})^\dagger a_{ir} a_{js} (a_{jr})^\dagger \\ &= \frac{1}{2} - \frac{1}{2} \sum_{i \neq j} (a_{is})^\dagger a_{js} (a_{jr})^\dagger a_{ir} \end{aligned}$$

and thus the effective Hamiltonian contains an anti-ferromagnetic interaction

$$H_{\text{eff}} \approx \frac{2t^2}{U} \sum_{i \neq j} S_i \cdot S_j + \frac{t^2}{-U}$$

4 The Anderson impurity model

The Hamiltonian of the Anderson model is

$$H = \sum_{ks} \varepsilon_k n_{ks} + \sum_{ks} V_k (a_{ks})^\dagger A_s + \text{h.c.} + \sum_s \mathcal{E} N_s + U N_+ N_-$$

Notice that we have

$$H_{02} = H_{20} = 0$$

5 The low-energy effective Hamiltonian

In the low-energy subspace, we have

$$\begin{aligned}
H_{10}\,\frac{1}{E-H_{00}}\,H_{01} &= \sum_{ks}\sum_{lr} V_{k*}\,V_l\,(A_s)^\dagger a_{ks}\,\frac{1}{E-H_{00}}\,(a_{lr})^\dagger A_r \\
&= \sum_{ks}\sum_{lr} V_{k*}\,V_l\,(A_s)^\dagger a_{ks}\,(a_{lr})^\dagger A_r\,\frac{1}{E-H_{00}-\varepsilon_l} \\
&\approx \sum_{ks}\sum_{lr} V_{k*}\,V_l\,(A_s)^\dagger a_{ks}\,(a_{lr})^\dagger A_r\,\frac{1}{\mathcal{E}-\varepsilon_l}
\end{aligned}$$

as well as

$$\begin{aligned}
H_{12}\,\frac{1}{E-H_{22}}\,H_{21} &= \sum_{ks}\sum_{lr} V_k V_{l*}\,(a_{ks})^\dagger A_s\,\frac{1}{E-H_{22}}\,(A_r)^\dagger a_{lr} \\
&= \sum_{ks}\sum_{lr} V_k V_{l*}\,(a_{ks})^\dagger A_s\,(A_r)^\dagger a_{lr}\,\frac{1}{E-H_{22}+\varepsilon_l} \\
&\approx \sum_{ks}\sum_{lr} V_k V_{l*}\,(a_{ks})^\dagger A_s\,(A_r)^\dagger a_{lr}\,\frac{1}{-\mathcal{E}-U+\varepsilon_l}
\end{aligned}$$

The low-energy effective Hamiltonian

$$H_{\text{l-e}} \approx H_{11} - \sum_{ks}\sum_{lr} V_{k*}\,V_l\left[\frac{(A_s)^\dagger a_{ks}\,(a_{lr})^\dagger A_r}{-\mathcal{E}+\varepsilon_l} + \frac{(a_{lr})^\dagger A_r\,(A_s)^\dagger a_{ks}}{\mathcal{E}+U-\varepsilon_k}\right]$$

Notice that

$$\begin{aligned}
s_{lk}\cdot S &= \frac{1}{2}\sum_{sr}(a_{lr})^\dagger a_{ks}\,(A_s)^\dagger A_r - \frac{1}{4}\sum_{sr}(a_{lr})^\dagger a_{kr}\,(A_s)^\dagger A_s \\
&= \frac{1}{2}\sum_{sr}(a_{lr})^\dagger a_{ks}\,(A_s)^\dagger A_r - \frac{1}{4}\sum_{r}(a_{lr})^\dagger a_{kr}
\end{aligned}$$

We see that the low-energy effective Hamiltonian contains an interaction

$$2\sum_{kl} V_{k*}\,V_l\left[\frac{1}{-\mathcal{E}+\varepsilon_l} + \frac{1}{\mathcal{E}+U-\varepsilon_k}\right] s_{lk}\cdot S$$

which is **anti-ferromagnetic** at low energy, i.e., near the Fermi surface.

6 Anderson: poor man's scaling

The Hamiltonian of the Kondo model is

$$H = \sum_{ks} \varepsilon_k n_{ks} + \sum_{kl} \left[J_z S_z (s_{kl})_z + \tfrac{1}{2} J_+ S_+ (s_{kl})_- + \tfrac{1}{2} J_- S_- (s_{kl})_+ \right]$$

where for example,

$$(s_{kl})_+ = (a_{k+})^\dagger a_{l-}$$
$$(s_{kl})_- = (a_{k-})^\dagger a_{l+}$$