

Miscellaneous

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1 The guiding center coordinates

are conserved, namely they commute with the Hamiltonian, and we have

$$[X, Y] = [x + l^2 \pi_y, y - l^2 \pi_x] = -il^2$$

Notice that they travel on equipotential lines:

$$\begin{aligned} i\dot{X}_1 &= [X_1, V] = -il^2 \partial_2 V \\ i\dot{X}_2 &= [X_2, V] = +il^2 \partial_1 V \end{aligned}$$

2 Spin-waves and magnons

The Schwinger-boson representation

$$\begin{aligned} S_+ &= a^\dagger b \\ S_- &= b^\dagger a \\ 2S_z &= a^\dagger a - b^\dagger b \end{aligned}$$

gives us the Holstein-Primakoff transformation if we "use the constraint"

$$a^\dagger a + b^\dagger b = 2S$$

The spin-wave is obtained by expanding around $S_z = \pm S$, for example setting

$$b = b^\dagger = \sqrt{2S - a^\dagger a} \approx \sqrt{2S}$$

3 The Bogoliubov transformation

In order to diagonalize a Hamiltonian of the form

$$\Psi_{i\dagger} A_{ij} \Psi_j$$

while keeping the commutation relations, we want

$$U^\dagger A U = \Lambda$$

where Λ is a diagonal matrix, as well as

$$C_{ij} = [\Psi_i, \Psi_{j\dagger}] = [U_{ik} \Phi_k, U_{jl\dagger} \Phi_{l\dagger}] = U_{ik} U_{jl\dagger} C_{kl} \Rightarrow C = U C U^\dagger$$

where C is, by assumption, a diagonal matrix. Therefore we have

$$C A U = U C U^\dagger A U = U C \Lambda$$

which means that U diagonalizes the matrix CA , with eigenvalues $C\Lambda$.

4 The three-plus-one decomposition

In **Minkowski spacetime**, we have

$$V^a = -V^b Z_b Z^a + W^a$$

such that W^a is a spatial vector

$$\begin{aligned} W^a Z_a &= V^a Z_a + V^b Z_b Z^a Z_a \\ &= V^a Z_a - V^b Z_b \\ &= 0 \end{aligned}$$

Notice also that $Z^b \nabla_b Z^a$ is a spatial vector, since $Z^b \nabla_b Z^a Z_a = 0$.

5 Incompressible quantum fluids

If the energy is an extensive quantity, then we have the **Gibbs-Duhem relation**:

$$SdT - Vdp + Nd\mu = 0$$

Therefore, if we stay at zero temperature,

$$-V \frac{\partial p}{\partial V} = -N \frac{\partial \mu}{\partial V} = n^2 \frac{\partial \mu}{\partial n}$$

all partial derivatives holding N fixed. We see that **gapped** quantum fluids are incompressible at zero temperature.