S16E01 The quantum effective action

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1 The Legendre transformation

The partition function in the presence of an external field is

$$Z = e^{iW} = \int D\phi \, e^{iS+iJ\cdot\phi}$$

The effective action is defined by the Legendre transformation

$$\begin{aligned} -\Gamma &\equiv J\cdot\phi - W \\ -\delta\Gamma &= \delta J\cdot\phi + J\cdot\delta\phi - \frac{\partial W}{\partial J}\cdot\delta J = J\cdot\delta\phi \qquad \phi_{\text{exp}} = \frac{\partial W}{\partial J} \end{aligned}$$

2 Exact propagator and vertices

We define a path integral using the effective action in place of the action

$$Z_\Gamma = \int D\phi \, e^{i\Gamma+iJ\cdot\phi}$$

If we use the saddle point approximation, which requires that

$$\frac{\partial\Gamma}{\partial\phi} + J = 0$$

then we have

$$Z_\Gamma \approx e^{i\Gamma+iJ\cdot\phi} = e^{iW}$$

Notice that the saddle point approximation is also the tree approximation! This means that the propagator and interaction vertices of the effective action are **exact**.

3 The quantum variational principle

Notice that when $J=0$, we have

$$\delta\Gamma = -J\cdot\delta\phi = 0$$

and the corresponding field configuration is equal to

$$\frac{\partial W}{\partial J}|_{J=0} = \phi_{\mathrm{exp}}|_{J=0}$$

namely the vacuum expectation value is the solution to the equation $\delta\Gamma=0$. This is to be compared with the fact that ϕ_{cl} is the solution to the equation $\delta S=0$.

4 The Slavnov-Taylor identity

For the shift $\phi' = \phi+\delta\phi$, where $\delta\phi$ may depend on ϕ,

$$\begin{aligned}\int D\phi\, e^{iS+iJ\cdot\phi} &= \int D\phi'\, e^{iS'+iJ\cdot\phi'} \\ &= \int D\phi\, e^{iS+iJ\cdot\phi'} \qquad \text{an assumption!} \\ &= \int D\phi\, e^{iS+iJ\cdot\phi+iJ\cdot\delta\phi}\end{aligned}$$

from which we read the Slavnov-Taylor identity

$$\begin{aligned}0 &= iJ^*\cdot\int D\phi\, e^{iS+iJ^*\cdot\phi}\delta\phi \qquad \text{for all } J^* \\ &= -i\,\frac{\partial\Gamma}{\partial\phi^*}\cdot\int D\phi\, e^{iS+iJ^*\cdot\phi}\delta\phi \qquad \text{for all } J^*\leftrightarrow\phi^*\end{aligned}$$

Notice that by the definition of J^* we have

$$\int D\phi\, e^{iS+iJ^*\cdot\phi}\phi = \phi^*$$

and thus

$$0 = \frac{\partial\Gamma}{\partial\phi^*}\cdot\int D\phi\, e^{iS+iJ^*\cdot\phi}\delta\phi = \frac{\partial\Gamma}{\partial\phi^*}\cdot\delta\phi^* = \delta\Gamma$$

for $\delta\phi = A\cdot\phi+b$.

5 A shifting property of the effective action

For the shift $\phi' = \phi + \delta\phi$,

$$\begin{aligned} e^{iW'} &= \int D\phi \, e^{iS' + iJ\cdot\phi} \\ &= \int D\phi' \, e^{iS' + iJ\cdot\phi} \qquad \text{an assumption} \\ &= \int D\phi' \, e^{iS' + iJ\cdot\phi' - iJ\cdot\delta\phi} \end{aligned}$$

and for $\delta\phi$ independent of ϕ,

$$\begin{aligned} e^{iW'} &= e^{-iJ\cdot\delta\phi} \int D\phi' \, e^{iS' + iJ\cdot\phi'} \\ &= e^{-iJ\cdot\delta\phi} \int D\phi \, e^{iS + iJ\cdot\phi} \qquad \text{trivially} \\ &= e^{-iJ\cdot\delta\phi} e^{iW} \end{aligned}$$

Therefore,

$$\frac{\partial W'}{\partial J} = \frac{\partial W}{\partial J} - \delta\phi$$

which means that

$$J'^{*}\left(\phi^{*}\right) = J^{*}\left(\phi^{*} + \delta\phi^{*}\right)$$

and the effective action is

$$\begin{aligned} -\Gamma'\left(\phi^{*}\right) &= J'^{*}\left(\phi^{*}\right) \cdot \phi^{*} - W'\left(J'^{*}\left(\phi^{*}\right)\right) \\ &= J'^{*}\left(\phi^{*}\right) \cdot \left(\phi^{*} + \delta\phi^{*}\right) - W\left(J'^{*}\left(\phi^{*}\right)\right) \\ &= J^{*}\left(\phi^{*} + \delta\phi^{*}\right) \cdot \left(\phi^{*} + \delta\phi^{*}\right) - W\left(J^{*}\left(\phi^{*} + \delta\phi^{*}\right)\right) \\ &= -\Gamma\left(\phi^{*} + \delta\phi^{*}\right) \end{aligned}$$

6 An interesting formula: Weinberg eqn. 16.1.17

We have, setting $\phi^{*} = 0$ in the shifting property above,

$$\Gamma\left(\delta\phi^{*}\right) = W'\left(J'^{*}(0)\right)$$

where

$$e^{iW'} = \int D\phi \, e^{iS' + iJ'^{*}\cdot\phi}$$

Now, if we really think about it, the above path integral is equal to

$$e^{i\Gamma} = e^{iW'} = \int D\phi\, e^{iS' + iJ'^{*}\cdot\phi} = \int_{\text{NO TADPOLES}} D\phi\, e^{iS'} = \int_{1\text{PI}} D\phi\, e^{iS'}$$

or, taking the logarithm,

$$i\Gamma\left(\delta\phi^{*}\right) = \int_{1\text{PI CONNECTED}} D\phi\, e^{iS'} \qquad 16.1.17$$

Amazing!