

The Dirac equation

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1 The Dirac equation

The Dirac Hamiltonian in the presence of an electromagnetic field is

$$H = \alpha \cdot \pi + \beta m + q\phi$$

with the Dirac matrices in the Dirac representation

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad H = \begin{pmatrix} m + q\phi & \sigma \cdot \pi \\ \sigma \cdot \pi & -m + q\phi \end{pmatrix}$$

The effective Hamiltonian of the upper-left subspace is

$$\left(m + q\phi + \sigma \cdot \pi \frac{1}{E + m - q\phi} \sigma \cdot \pi \right) \psi = E\psi$$

and let us write $E = m + \varepsilon$, such that we obtain

$$\left(q\phi + \sigma \cdot \pi \frac{1}{2m + \varepsilon - q\phi} \sigma \cdot \pi \right) \psi = \varepsilon\psi$$

2 The Pauli Hamiltonian

In the case where ϕ vanishes, we have the leading-order correction

$$\begin{aligned} \frac{1}{2m} \sigma \cdot \pi \sigma \cdot \pi &= \frac{1}{2m} \pi^2 + \frac{1}{2m} i \sigma \cdot \pi \times \pi \\ &= \frac{1}{2m} \pi^2 - \frac{q}{2m} \sigma \cdot \nabla \times \mathbf{A} \quad \pi \equiv \mathbf{p} - q\mathbf{A} \\ &= \frac{1}{2m} \pi^2 - \frac{q}{2m} \sigma \cdot \mathbf{B} \end{aligned}$$

3 The spin-orbit coupling

In the case where ϕ does not vanish, we have the leading-order correction

$$q\phi + \frac{1}{2m} \sigma \cdot \pi \sigma \cdot \pi = q\phi + \frac{1}{2m} \pi^2 - \frac{q}{2m} \sigma \cdot \mathbf{B}$$

Now we want the next-to-leading-order correction:

$$\left(q\phi + \frac{1}{2m} \sigma \cdot \pi \sigma \cdot \pi - \sigma \cdot \pi \frac{\varepsilon - q\phi}{4m^2} \sigma \cdot \pi \right) \psi = \varepsilon \psi$$

$$\left(q\phi + \frac{1}{2m} \sigma \cdot \pi \sigma \cdot \pi + \sigma \cdot \pi \frac{q\phi}{4m^2} \sigma \cdot \pi \right) \psi = \varepsilon \left(1 + \frac{1}{4m^2} \sigma \cdot \pi \sigma \cdot \pi \right) \psi$$

such that

$$\left(1 - \frac{1}{8m^2} \sigma \cdot \pi \sigma \cdot \pi \right) \left(q\phi + \frac{1}{2m} \sigma \cdot \pi \sigma \cdot \pi + \sigma \cdot \pi \frac{q\phi}{4m^2} \sigma \cdot \pi \right) \left(1 - \frac{1}{8m^2} \sigma \cdot \pi \sigma \cdot \pi \right) \left(1 + \frac{1}{8m^2} \sigma \cdot \pi \sigma \cdot \pi \right) \psi = \varepsilon \left(1 + \frac{1}{8m^2} \sigma \cdot \pi \sigma \cdot \pi \right) \psi$$

where the effective Hamiltonian is the first line

$$\begin{aligned} H_{\text{eff}} &= q\phi + \frac{1}{2m} \sigma \cdot \pi \sigma \cdot \pi + \sigma \cdot \pi \frac{q\phi}{4m^2} \sigma \cdot \pi - \frac{1}{8m^2} \sigma \cdot \pi \sigma \cdot \pi q\phi - \frac{1}{8m^2} q\phi \sigma \cdot \pi \sigma \cdot \pi \\ &= q\phi + \frac{1}{2m} \sigma \cdot \pi \sigma \cdot \pi - \frac{1}{8m^2} \sigma \cdot \pi [\sigma \cdot \pi, q\phi] - \frac{1}{8m^2} [q\phi, \sigma \cdot \pi] \sigma \cdot \pi \end{aligned}$$

and the final result is

$$H_{\text{eff}} = q\phi + \frac{1}{2m} \pi^2 - \frac{q}{2m} \sigma \cdot \mathbf{B} + \frac{1}{8m^2} q \nabla^2 \phi + \frac{1}{4m^2} \sigma \cdot q \nabla \phi \times \pi$$

Remark We also have the relativistic correction

$$-\frac{1}{8m^3} \sigma \cdot \pi \sigma \cdot \pi \sigma \cdot \pi \sigma \cdot \pi \sim -\frac{1}{8m^3} p^4$$

4 Transformation properties

The following is what we want:

$$\begin{aligned} i\gamma^\mu \frac{\partial}{\partial x^\mu} \psi(x) &= m\psi(x) \\ \Rightarrow i\gamma^\mu \frac{\partial}{\partial x^\mu} D(\Lambda) \psi(\Lambda^{-1}x) &= mD(\Lambda) \psi(\Lambda^{-1}x) \end{aligned}$$

and we write to first order

$$D(\Lambda)\psi(\Lambda^{-1}x) = \left(1 + \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\left(1 + \frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\right)\psi(x) = \left(1 + \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right)\psi(x)$$