

s16E01 The quantum effective action

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1 The Legendre transformation

The partition function in the presence of an external field is

$$Z = e^{iW} = \int D\phi e^{iS+iJ\cdot\phi}$$

The effective action is defined by the Legendre transformation

$$\begin{aligned} -\Gamma &\equiv J \cdot \phi - W \\ -\delta\Gamma &= \delta J \cdot \phi + J \cdot \delta\phi - \frac{\partial W}{\partial J} \cdot \delta J = J \cdot \delta\phi \quad \phi_{\text{exp}} = \frac{\partial W}{\partial J} \end{aligned}$$

2 Exact propagator and vertices

We define a path integral using the effective action in place of the action

$$Z_{\Gamma} = \int D\phi e^{i\Gamma+iJ\cdot\phi}$$

If we use the saddle point approximation, which requires that

$$\frac{\partial\Gamma}{\partial\phi} + J = 0$$

then we have

$$Z_{\Gamma} \approx e^{i\Gamma+iJ\cdot\phi} = e^{iW}$$

Notice that the saddle point approximation is also the tree approximation! This means that the propagator and interaction vertices of the effective action are **exact**.

3 The quantum variational principle

Notice that when $J = 0$, we have

$$\delta\Gamma = -J \cdot \delta\phi = 0$$

and the corresponding field configuration is equal to

$$\frac{\partial W}{\partial J}|_{J=0} = \phi_{\text{exp}}|_{J=0}$$

namely the **vacuum expectation value** is the solution to the equation $\delta\Gamma = 0$. This is to be compared with the fact that ϕ_{cl} is the solution to the equation $\delta S = 0$.

4 The Slavnov-Taylor identity

For the shift $\phi' = \phi + \delta\phi$, where $\delta\phi$ may depend on ϕ ,

$$\begin{aligned} \int D\phi e^{iS+iJ\cdot\phi} &= \int D\phi' e^{iS'+iJ\cdot\phi'} \\ &= \int D\phi e^{iS+iJ\cdot\phi'} \quad \text{an assumption!} \\ &= \int D\phi e^{iS+iJ\cdot\phi+iJ\cdot\delta\phi} \end{aligned}$$

from which we read the Slavnov-Taylor identity

$$\begin{aligned} 0 &= iJ^* \cdot \int D\phi e^{iS+iJ^*\cdot\phi} \delta\phi \quad \text{for all } J^* \\ &= -i \frac{\partial\Gamma}{\partial\phi^*} \cdot \int D\phi e^{iS+iJ^*\cdot\phi} \delta\phi \quad \text{for all } J^* \leftrightarrow \phi^* \end{aligned}$$

Notice that by the definition of J^* we have

$$\int D\phi e^{iS+iJ^*\cdot\phi} \phi = \phi^*$$

and thus

$$0 = \frac{\partial\Gamma}{\partial\phi^*} \cdot \int D\phi e^{iS+iJ^*\cdot\phi} \delta\phi = \frac{\partial\Gamma}{\partial\phi^*} \cdot \delta\phi^* = \delta\Gamma$$

for $\delta\phi = A \cdot \phi + b$.

5 A shifting property of the effective action

For the shift $\phi' = \phi + \delta\phi$, where $\delta\phi$ may depend on ϕ ,

$$\begin{aligned} e^{iW'} &= \int D\phi e^{iS' + iJ \cdot \phi} \\ &= \int D\phi' e^{iS' + iJ \cdot \phi} \quad \text{an assumption} \\ &= \int D\phi' e^{iS' + iJ \cdot \phi' - iJ \cdot \delta\phi} \end{aligned}$$

and for $\delta\phi$ independent of ϕ ,

$$\begin{aligned} e^{iW'} &= e^{-iJ \cdot \delta\phi} \int D\phi' e^{iS' + iJ \cdot \phi'} \\ &= e^{-iJ \cdot \delta\phi} \int D\phi e^{iS + iJ \cdot \phi} \quad \text{trivially} \\ &= e^{-iJ \cdot \delta\phi} e^{iW} \end{aligned}$$

Therefore,

$$\frac{\partial W'}{\partial J} = \frac{\partial W}{\partial J} - \delta\phi$$

which means that

$$J'^*(\phi^*) = J^*(\phi^* + \delta\phi^*)$$

and the effective action is

$$\begin{aligned} -\Gamma'(\phi^*) &= J'^*(\phi^*) \cdot \phi^* - W'(J'^*(\phi^*)) \\ &= J'^*(\phi^*) \cdot (\phi^* + \delta\phi^*) - W(J'^*(\phi^*)) \\ &= J^*(\phi^* + \delta\phi^*) \cdot (\phi^* + \delta\phi^*) - W(J^*(\phi^* + \delta\phi^*)) \\ &= -\Gamma(\phi^* + \delta\phi^*) \end{aligned}$$

6 An interesting formula: Weinberg eqn. 16.1.17

We have, setting $\phi^* = 0$ in the shifting property above,

$$\Gamma(\delta\phi^*) = W'(J^*(0))$$

where

$$e^{iW'} = \int D\phi e^{iS' + iJ'^* \cdot \phi}$$

Now, if we really think about it, the above path integral is equal to

$$e^{i\Gamma} = e^{iW'} = \int D\phi e^{iS' + iJ'^* \cdot \phi} = \int_{\text{NO TADPOLES}} D\phi e^{iS'} = \int_{\text{1PI}} D\phi e^{iS'}$$

or, taking the logarithm,

$$i\Gamma(\delta\phi^*) = \int_{\text{1PI CONNECTED}} D\phi e^{iS'} \quad 16.1.17$$

Amazing!