Contents

Chapter 1

Time-dependent perturbation theory

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The interaction picture

Suppose that the Hamiltonian $H = H_0 + H_1$, where H_0 is time-independent and H_1 is time-dependent. Suppose that we prepare or measure the system such that it's in an eigenstate of H_0, namely $H_0\psi_i = E_i\psi_i$. After some time t, due to the influence of H_1, the initial state ψ_i will generally evolve to a superposition of eigenstates of H_0, and the transition probability to the state ψ_j is

$$p_{i\to j} = |\langle\psi_j|e^{-iHt}|\psi_i\rangle|^2$$

and we have $\sum_j p_{i\to j} = 1$ due to the completeness of the eigenstates $|\psi_j\rangle$.

Notice that Hamiltonian at different times generally do not commute, and in that case the operator e^{-iHt} should really mean the time-ordered version of it. What we shall derive in the following should be true in the general case.

We would like to calculate the effect of H_1 on the transition probabilities $\{p_{i\to j}\}$ perturbatively – in terms of a series in H_1. If we have $H_1 = 0$, then $p_{i\to j} = \delta_{ij}$. A method called the "interaction picture" simplifies the calculation somewhat. In effect, it does the following:

$$p_{i\to j} = |\langle\psi_j|e^{-iHt}|\psi_i\rangle|^2 = |\langle\psi_j|e^{iH_0t}e^{-iHt}|\psi_i\rangle|^2$$

and the unitary operator $e^{iH_0t}e^{-iHt}$ satisfies the following differential equation

$$\begin{aligned} d_t e^{iH_0t}e^{-iHt} &= -ie^{iH_0t}H_1e^{-iHt} \\ &= -ie^{iH_0t}H_1e^{-iH_0t}e^{iH_0t}e^{-iHt} \\ &=: -iH_{1I}e^{iH_0t}e^{-iHt} \end{aligned}$$

with solution given by the Dyson series

$$e^{iH_0t}e^{-iHt} = 1 - i\int_0^t dt' H_{1I} - \int_0^t dt' \int_0^{t'} dt'' H_{1I}H_{1I} + \cdots$$

and thus to first order in H_1, the transition amplitude (up to a phase) is

$$\begin{aligned} \langle\psi_j|e^{iH_0t}e^{-iHt}|\psi_i\rangle &= \delta_{ij} - i\int_0^t dt' \langle\psi_j|H_{1I}|\psi_i\rangle \\ &= \delta_{ij} - i\int_0^t dt' e^{-i\omega_{ij}t'}\langle\psi_j|H_1|\psi_i\rangle \end{aligned}$$

The unitary operator $e^{iH_0t}e^{-iHt}$ can also be written as

$$e^{iH_0t}e^{-iHt} = \mathcal{T}\exp\left(-i\int_0^t dt' H_{1I}\right)$$

which may be generalized to

$$U_{21} := e^{iH_0t_2}e^{-iHt_{21}}e^{-iH_0t_1} = \mathcal{T}\exp\left(-i\int_{t_1}^{t_2} dt' H_{1I}\right)$$

Case I: Constant perturbation

Suppose that $H_1 = V$ is a constant operator, then the transition amplitude above evaluates to

$$\langle\psi_j|e^{iH_0t}e^{-iHt}|\psi_i\rangle = \delta_{ij} + V_{ji}\frac{e^{-i\omega_{ij}t} - 1}{\omega_{ij}}$$

and if $j \neq i$, the transition probability is

$$p_{i\to j} = |V_{ji}|^2\left(\frac{\sin\omega_{ij}t/2}{\omega_{ij}/2}\right)^2$$

which takes the form of the familiar "sinc" function. Interestingly, when t is very large, the transition probability $p_{i\to j}$ has the limiting behavior

$$\lim_{t\to\infty} p_{i\to j} = |V_{ji}|^2 2\pi t \delta^1(\omega_{ij})$$

This is a first sign of Fermi's golden rule.

However, the result above should only hold for sufficiently small t, since the expression for the transition probability does not always satisfy $p_{i\to j} \leq 1$. For example,

$$\omega_{ij} = 0 \Rightarrow p_{i\to j} = |V_{ji}|^2 t^2$$

and thus we require that $t \ll |V_{ji}|^{-1}$.

Case II: Monochromatic perturbation

Suppose that $H_1 = Ve^{-i\omega t} + V^\dagger e^{i\omega t}$, where V is a constant operator, then the transition amplitude is equal to

$$\langle\psi_j|e^{iH_0 t}e^{-iHt}|\psi_i\rangle = \delta_{ij} + V_{ji}\frac{e^{-i\omega_{ij}t}e^{-i\omega t}-1}{\omega_{ij}+\omega} + V_{ij*}\frac{e^{-i\omega_{ij}t}e^{i\omega t}-1}{\omega_{ij}-\omega}$$

which contains two terms if $j \neq i$, thus $p_{i\to j}$ will contain four terms if $j \neq i$.

In order to make some simplification, we may attempt to take $t \to \infty$. However, as we have remarked earlier, t cannot be taken too large, otherwise perturbation theory will break down.

We may focus on the case when ω is close to ω_{ij} or $-\omega_{ij}$. If for example ω is close to ω_{ij}, then the second term dominates, and the transition probability $p_{i\to j}$ when $j \neq i$ is approximately equal to

$$p_{i\to j} \approx |V_{ij}|^2 \left(\frac{\sin\omega_{ij-}t/2}{\omega_{ij-}/2}\right)^2$$

where $\omega_{ij-} := \omega_{ij} - \omega$.

Although we may not take $t\to\infty$, when the final state lies in a spectrum continuum, it can be justified (see Tong Chen's notes) that we may take t large, such that

$$\langle\psi_j|e^{iH_0 t}e^{-iHt}|\psi_i\rangle \approx \delta_{ij} - iV_{ji}2\pi\delta^1(\omega_{ij}+\omega) - iV_{ij*}2\pi\delta^1(\omega_{ij}-\omega)$$

which means that if $j \neq i$,

$$p_{i\to j} \approx |V_{ji}|^2 2\pi t \delta^1(\omega_{ij}+\omega) + |V_{ij}|^2 2\pi t \delta^1(\omega_{ij}-\omega)$$

since $\omega \neq 0$, the cross terms vanishes. Here we have used one of the delta functions to evaluate the other delta function integral, and obtained t.

Einstein's coefficients

$$\begin{aligned} d_t n_1 &= -B_{21}\rho_\omega n_1 + A_{12} n_2 + B_{12}\rho_\omega n_2 \\ d_t n_2 &= -d_t n_1 \end{aligned}$$

and Einstein: when $d_t n_1 = d_t n_2 = 0$,

$$\frac{n_2}{n_1} = \frac{B_{21}\rho_\omega}{A_{12} + B_{12}\rho_\omega} = e^{-\beta\omega}$$

The electromagnetic field is quantized

Let us suppose that

$$H_1 = Va + V^\dagger a^\dagger$$

which gives us the transition probability

$$\begin{aligned} p_{1,n\to 2,n-1} &= n_\omega\, |V_{21}|^2\, 2\pi t\, \delta^1(\omega_{21} - \omega) \\ p_{2,n\to 1,n+1} &= (n_\omega + 1)\, |V_{21*}|^2\, 2\pi t\, \delta^1(\omega_{21} - \omega) \end{aligned}$$

and thus

$$\begin{aligned} d_t n_1 &\propto -n_\omega\, |V_{21}|^2 n_1 + (n_\omega + 1)\, |V_{21}|^2 n_2 \\ d_t n_2 &= -d_t n_1 \end{aligned}$$

For phonons, the energy density is

$$\frac{E}{V} = 2\int \frac{d^3k}{(2\pi)^3} \frac{\omega}{e^{\beta\omega} - 1} = \int \frac{k^2 dk}{\pi^2} \frac{\omega}{e^{\beta\omega} - 1} = \int \frac{\omega^2 d\omega}{\pi^2} \frac{\omega}{e^{\beta\omega} - 1} = \int \frac{\omega^3 n_\omega d\omega}{\pi^2}$$

which gives us the spectral energy density

$$\rho_\omega = \frac{\omega^3 n_\omega}{\pi^2}$$

and we read

$$\begin{aligned} B_{21} &= B_{12} \\ A_{12} &= B_{12} \frac{\omega^3}{\pi^2} \end{aligned}$$

Chapter 2

The family of Green's functions

Linear response theory

We have

$$\begin{aligned}\mathcal{O} &= e^{iHt}\mathcal{O}_S e^{-iHt} \\ &= e^{iHt}e^{-iH_0t}e^{iH_0t}\mathcal{O}_S e^{-iH_0t}e^{iH_0t}e^{-iHt} \\ &= e^{iHt}e^{-iH_0t}\mathcal{O}_I e^{iH_0t}e^{-iHt}\end{aligned}$$

and we may expand $e^{iH_0t}e^{-iHt}$ to first order in H_1, and obtain

$$\mathcal{O} = \mathcal{O}_I - i\int_0^t dt'\mathcal{O}_I H_{1I} + i\int_0^t dt' H_{1I}\mathcal{O}_I$$

which means that the linear response is characterized by the commutator

$$\mathcal{O}_I H_{1I} - H_{1I}\mathcal{O}_I$$

at different times: $\mathcal{O}_I$ is at time t while H_{1I} is at time t', and we have $\int_0^t dt'$.

$$\begin{aligned}e^{iHt_2}\mathcal{O}_{2S}e^{-iHt_2} = e^{iH_0t_2}\mathcal{O}_{2S}e^{-iH_0t_2} - i\int_0^{t_2} dt_1 e^{iH_0t_2}\mathcal{O}_{2S}e^{-iH_0t_{21}}H_{1S}e^{-iH_0t_1} \\ + i\int_0^{t_2} dt_1 e^{iH_0t_1}H_{1S}e^{iH_0t_{21}}\mathcal{O}_{2S}e^{-iH_0t_2}\end{aligned}$$

which is easily proved by breaking the unitary evolution operator e^{-iHt} into pieces and expanding. This result is very intuitive.

An example: The simple harmonic oscillator

We have the response function at zero temperature

$$D_{\beta\to\infty} = \frac{i}{2\omega}\, e^{-i\omega t_{21}}\Theta_{21} - \frac{i}{2\omega}\, e^{-i\omega t_{12}}\Theta_{21}$$

and thus the linear response is

$$\int dt_1 \left(\frac{i}{2\omega}\, e^{-i\omega t_{21}}\Theta_{21} - \frac{i}{2\omega}\, e^{-i\omega t_{12}}\Theta_{21}\right) e^{0^+ t_1} = \frac{i}{2\omega}\left(\frac{1}{i\omega} - \frac{1}{-i\omega}\right) = \frac{1}{\omega^2}$$

which is correct, since the new equilibrium position is at $\frac{\lambda}{\omega^2}$ for $H_1 = -\lambda\phi$.

The Lehmann representation

1. The real-time Green's function.

$$\begin{aligned} iG_\beta &= \Theta_{21} p_i \langle\psi_i|\phi_2\phi_1|\psi_i\rangle + \eta\Theta_{12} p_i \langle\psi_i|\phi_1\phi_2|\psi_i\rangle \\ &= \Theta_{21} p_i \langle\psi_i|\phi_2|\psi_j\rangle\langle\psi_j|\phi_1|\psi_i\rangle + \eta\Theta_{12} p_i \langle\psi_i|\phi_1|\psi_j\rangle\langle\psi_j|\phi_2|\psi_i\rangle \\ &= \Theta_{21} p_i e^{iE_{ij}t_{21}} (\phi_{2S})_{ij}(\phi_{1S})_{ji} + \eta\Theta_{12} p_i e^{iE_{ij}t_{12}} (\phi_{1S})_{ij}(\phi_{2S})_{ji} \end{aligned}$$

and we have

$$\widetilde{G}_\beta = p_i \frac{(\phi_{2S})_{ij}(\phi_{1S})_{ji}}{\omega + E_{ij} + i0^+} - \eta p_i \frac{(\phi_{1S})_{ij}(\phi_{2S})_{ji}}{\omega - E_{ij} - i0^+}$$

2.a. The retarded Green's function.

$$\begin{aligned} iD_{\beta+} &= \Theta_{21} p_i \langle\psi_i|\phi_2\phi_1|\psi_i\rangle - \eta\Theta_{21} p_i \langle\psi_i|\phi_1\phi_2|\psi_i\rangle \\ &= \Theta_{21} p_i \langle\psi_i|\phi_2|\psi_j\rangle\langle\psi_j|\phi_1|\psi_i\rangle - \eta\Theta_{21} p_i \langle\psi_i|\phi_1|\psi_j\rangle\langle\psi_j|\phi_2|\psi_i\rangle \\ &= \Theta_{21} p_i e^{iE_{ij}t_{21}} (\phi_{2S})_{ij}(\phi_{1S})_{ji} - \eta\Theta_{21} p_i e^{iE_{ij}t_{12}} (\phi_{1S})_{ij}(\phi_{2S})_{ji} \end{aligned}$$

and we have

$$\widetilde{D}_{\beta+} = p_i \frac{(\phi_{2S})_{ij}(\phi_{1S})_{ji}}{\omega + E_{ij} + i0^+} - \eta p_i \frac{(\phi_{1S})_{ij}(\phi_{2S})_{ji}}{\omega - E_{ij} + i0^+}$$

2.b. The advanced Green's function.

$$\begin{aligned} -iD_{\beta-} &= \Theta_{12} p_i \langle\psi_i|\phi_2\phi_1|\psi_i\rangle - \eta\Theta_{12} p_i \langle\psi_i|\phi_1\phi_2|\psi_i\rangle \\ &= \Theta_{12} p_i \langle\psi_i|\phi_2|\psi_j\rangle\langle\psi_j|\phi_1|\psi_i\rangle - \eta\Theta_{12} p_i \langle\psi_i|\phi_1|\psi_j\rangle\langle\psi_j|\phi_2|\psi_i\rangle \\ &= \Theta_{12} p_i e^{iE_{ij}t_{21}} (\phi_{2S})_{ij}(\phi_{1S})_{ji} - \eta\Theta_{12} p_i e^{iE_{ij}t_{12}} (\phi_{1S})_{ij}(\phi_{2S})_{ji} \end{aligned}$$

and we have

$$\widetilde{D}_{\beta-} = p_i \frac{(\phi_{2S})_{ij}(\phi_{1S})_{ji}}{\omega + E_{ij} - i0^+} - \eta p_i \frac{(\phi_{1S})_{ij}(\phi_{2S})_{ji}}{\omega - E_{ij} - i0^+}$$

3. The imaginary-time Green's function.

$$\begin{aligned}
-\mathcal{G}_\beta &= \Theta_{21} p_i \langle \psi_i | \phi_2 \phi_1 | \psi_i \rangle + \eta \Theta_{12} p_i \langle \psi_i | \phi_1 \phi_2 | \psi_i \rangle \\
&= \Theta_{21} p_i \langle \psi_i | \phi_2 | \psi_j \rangle \langle \psi_j | \phi_1 | \psi_i \rangle + \eta \Theta_{12} p_i \langle \psi_i | \phi_1 | \psi_j \rangle \langle \psi_j | \phi_2 | \psi_i \rangle \\
&= \Theta_{21} p_i e^{E_{ij}\tau_{21}} (\phi_{2S})_{ij}(\phi_{1S})_{ji} + \eta \Theta_{12} p_i e^{E_{ij}\tau_{12}} (\phi_{1S})_{ij}(\phi_{2S})_{ji}
\end{aligned}$$

and we have, setting $t_1 = 0$ such that the Θ_{12} term does not contribute,

$$\widetilde{\mathcal{G}}_\beta = p_i \frac{(\phi_{2S})_{ij}(\phi_{1S})_{ji}}{i\omega_l + E_{ij}} - \eta p_i \frac{(\phi_{1S})_{ij}(\phi_{2S})_{ji}}{i\omega_l - E_{ij}}$$

The "master" Green's function

is defined as

$$\widetilde{\mathcal{M}}_\beta = p_i \frac{(\phi_{2S})_{ij}(\phi_{1S})_{ji}}{z + E_{ij}} - \eta p_i \frac{(\phi_{1S})_{ij}(\phi_{2S})_{ji}}{z - E_{ij}}$$

or equivalently,

$$\widetilde{\mathcal{M}}_\beta = \frac{(p_i - \eta p_j)(\phi_{2S})_{ij}(\phi_{1S})_{ji}}{z + E_{ij}}$$

For a non-interacting Hamiltonian, when $\phi_{1S} = a^\dagger$ and $\phi_{2S} = a$, the master Green's function takes a very simple form:

$$\widetilde{\mathcal{M}}_\beta = \frac{1}{z - \varepsilon}$$

which is most easily proved by taking $|\psi_i\rangle$ to be the occupation number basis.

The spectral function

is defined as

$$\widetilde{A}_\beta = i\widetilde{D}_{\beta+} - i\widetilde{D}_{\beta-}$$

which is equal to

$$p_i (\phi_{2S})_{ij}(\phi_{1S})_{ji} 2\pi\delta^1(\omega + E_{ij}) - \eta p_i (\phi_{1S})_{ij}(\phi_{2S})_{ji} 2\pi\delta^1(\omega - E_{ij})$$

or equivalently,

$$(p_i - \eta p_j)(\phi_{2S})_{ij}(\phi_{1S})_{ji}\, 2\pi\delta^1(\omega + E_{ij})$$

We have the following exact identity

$$\int \frac{d\omega}{2\pi}\, \widetilde{A}_\beta = 1$$

for a general Hamiltonian, when $\phi_{1S} = a^\dagger$ and $\phi_{2S} = a$.

The Kramers-Kronig relations

It can be easily verified that

$$\widetilde{\mathcal{M}}_\beta = \int \frac{d\omega}{2\pi} \frac{\widetilde{A}_\beta}{z-\omega}$$

thanks to the delta function. Now we have, if $\operatorname{im} z > 0$,

$$\begin{aligned}\widetilde{\mathcal{M}}_\beta &= \int \frac{d\omega}{2\pi} \frac{\widetilde{A}_\beta}{z-\omega} \\ &= i\int \frac{d\omega}{2\pi} \frac{\widetilde{D}_{\beta+} - \widetilde{D}_{\beta-}}{z-\omega} = i\int_{C-} \frac{d\omega}{2\pi} \frac{\widetilde{D}_{\beta+} - \widetilde{D}_{\beta-}}{z-\omega} = i\int_{C-} \frac{d\omega}{2\pi} \frac{\widetilde{D}_{\beta+}}{z-\omega} = i\int \frac{d\omega}{2\pi} \frac{\widetilde{D}_{\beta+}}{z-\omega}\end{aligned}$$

which gives us

$$\begin{aligned}\widetilde{D}_{\beta+} &= i\int \frac{d\omega'}{2\pi} \frac{\widetilde{D}_{\beta+}}{\omega-\omega'+i0^+} \\ \Rightarrow \widetilde{D}_{\beta+} &= i\mathcal{P}\int \frac{d\omega'}{\pi} \frac{\widetilde{D}_{\beta+}}{\omega-\omega'}\end{aligned}$$

or equivalently, the Kramers-Kronig relations,

$$\begin{aligned}\operatorname{re}\widetilde{D}_{\beta+} &= -\mathcal{P}\int \frac{d\omega'}{\pi} \frac{\operatorname{im}\widetilde{D}_{\beta+}}{\omega-\omega'} \\ \operatorname{im}\widetilde{D}_{\beta+} &= \mathcal{P}\int \frac{d\omega'}{\pi} \frac{\operatorname{re}\widetilde{D}_{\beta+}}{\omega-\omega'}\end{aligned}$$

Chapter 3

Scattering theory

A useful calculation

$$\begin{aligned}
-i\int_{-\infty}^{t} dt' \langle\psi_j|H_{1I}|\psi_i\rangle &= -i\int_{-\infty}^{t} dt' e^{-i\omega_{ij}t'}\langle\psi_j|H_1|\psi_i\rangle \\
&= -i\int_{-\infty}^{t} dt' e^{-i\omega_{ij}t'} e^{0^+ t}\langle\psi_j|H_1|\psi_i\rangle \\
&= \frac{1}{\omega_{ij}+i0^+} e^{-i\omega_{ij}t'} e^{0^+ t}\langle\psi_j|H_1|\psi_i\rangle \\
&= \frac{1}{\omega_{ij}+i0^+} e^{-i\omega_{ij}t'}\langle\psi_j|H_1|\psi_i\rangle
\end{aligned}$$

The scattering S-matrix

is defined by

$$S := U_{+\infty-\infty}$$

We have, using the calculation above,

$$S_{ji} = \langle\psi_j|U_{+\infty-\infty}|\psi_i\rangle = \delta_{ji} - i\,T_{ji}2\pi\delta^1(\omega_{ij})$$

The transition probability

$$p_{i\to j} = |T_{ji}|^2\, 2\pi\, t\,\delta^1(\omega_{ij})$$

where we have defined

$$T = V + V\frac{1}{E_i - H_0 + i0^+}V + V\frac{1}{E_i - H_0 + i0^+}V\frac{1}{E_i - H_0 + i0^+}V + \cdots$$

The Lippmann-Schwinger equation

We define

$$|\psi_{i+}\rangle := U_{0-\infty}|\psi_i\rangle$$
$$|\psi_{i-}\rangle := U_{0+\infty}|\psi_i\rangle$$

or equivalently,

$$\lim_{t\to-\infty} e^{-iHt}|\psi_{i+}\rangle = \lim_{t\to-\infty} e^{-iH_0t}|\psi_i\rangle$$
$$\lim_{t\to+\infty} e^{-iHt}|\psi_{i-}\rangle = \lim_{t\to+\infty} e^{-iH_0t}|\psi_i\rangle$$

and by definition,

$$\begin{aligned} S_{ji} &= \langle\psi_j|U_{+\infty-\infty}|\psi_i\rangle \\ &= \langle\psi_j|U_{+\infty 0}U_{0-\infty}|\psi_i\rangle \\ &= \langle\psi_{j-}|\psi_{i+}\rangle \end{aligned}$$

To obtain the expression of $|\psi_{i+}\rangle$, we compute using the "useful calculation",

$$\langle\psi_j|\psi_{i+}\rangle = \langle\psi_j|U_{0-\infty}|\psi_i\rangle = \langle\psi_j|\psi_i\rangle + \langle\psi_j|\frac{1}{E_i - H_0 + i0^+}T|\psi_i\rangle$$

from which we read the Lippmann-Schwinger equation

$$\begin{aligned} |\psi_{i+}\rangle &= |\psi_i\rangle + \frac{1}{E_i - H_0 + i0^+}T|\psi_i\rangle \\ &= |\psi_i\rangle + \frac{1}{E_i - H_0 + i0^+}V|\psi_{i+}\rangle \end{aligned}$$

Since by definition $H_0|\psi_i\rangle = E_i|\psi_i\rangle$, we have

$$H_0|\psi_{i+}\rangle + V|\psi_{i+}\rangle = E_i|\psi_{i+}\rangle$$

namely $|\psi_{i+}\rangle$ is an eigenstate of the total H, with the same eigenvalue E_i. Therefore, if $|\psi_i\rangle$ is a scattering eigenstate of H_0, then $|\psi_{i+}\rangle$ is a scattering eigenstate of H.

Scattering problem in three dimensions

The scattering problem in three dimensions is a good place to show the usefulness of the Lippmann-Schwinger equation. To set up the problem, notice that the scattering eigenstates of the free Hamiltonian H_0 are the plane waves, and thus we may take the state $|\psi_i\rangle$ to be the plane wave with wave vector k_i.

We then use the Lippmann-Schwinger equation to calculate the scattering eigenstates of the total Hamiltonian

$$|\psi_{i+}\rangle = |\psi_i\rangle + \frac{1}{E_i - H_0 + i0^+} T|\psi_i\rangle$$

Now let us compute the coordinate space wave function of the scattering eigenstates, and focus on its asymptotic behavior. Noticing the following calculation,

$$\langle x|\frac{1}{E_i - H_0 + i0^+}|y\rangle = 2m\int \frac{d^3 l}{(2\pi)^3}\frac{e^{il\cdot x}e^{-il\cdot y}}{k^2 - l^2 + i0^+} = -\frac{m}{2\pi}\frac{e^{ik_i\eta}}{\eta}$$

where η is the distance between x and y. In the limit $x \to \infty$, we have

$$\frac{e^{ik_i\eta}}{\eta} = \frac{e^{ik_i r}}{r}\, e^{-ik_i y\cdot\hat{x}} = \frac{e^{ik_i r}}{r}\,\langle\psi_j|y\rangle$$

where $|\psi_j\rangle$ is the plane wave with wave vector $k_j = k_i\hat{x}$. Thus we read

$$\langle x|\psi_{i+}\rangle = \langle x|\psi_i\rangle - \frac{m}{2\pi}\frac{e^{ik_i r}}{r}\,\langle\psi_j|T|\psi_i\rangle$$

which means that asymptotically, the coordinate space wave function of the scattering eigenstates of the total H is the superposition of the incoming plane wave $|\psi_i\rangle$ and an outgoing spherical wave, modulated by an angular distribution function $-\frac{m}{2\pi}T_{ji}$. Since we already know that the transition probability is proportional to $|T_{ji}|^2$, we now know that it is, equivalently, proportional to the square of the angular distribution function.

The Wigner-Eckart theorem

The statement of the Wigner-Eckart theorem is quite intuitive: if some operators T_{jm} transform like the $|jm\rangle$ states, where j is fixed, then we have

$$\langle\alpha, jm|T_{j_1m_1}|\alpha_2, j_2m_2\rangle = \langle jm|j_1m_1, j_2m_2\rangle\langle\alpha, j\|T_{j_1}\|\alpha_2, j_2\rangle$$

where $\langle\alpha, j\|T_{j_1}\|\alpha_2, j_2\rangle$ is a number that does not depend on m, m_1 or m_2.

Rotational symmetry

In the special case that the scattering potential V is rotationally invariant, the general non-perturbative analysis above can be carried out further. In this case, the total H is rotationally invariant, and thus the scattering S- and T-matrices are rotationally invariant. In other words, they are scalar operators. The Wigner-Eckart theorem:

$$\begin{aligned}\langle E, lm|S|E', l'm'\rangle &= \delta_{ll'}\delta_{mm'}\langle E, l\|S\|E', l'\rangle \\ &= \delta_{ll'}\delta_{mm'}\langle E, l\|S\|E', l\rangle\end{aligned}$$

Since we also know that the result should be proportional to $\delta_{EE'}$, we arrive at the conclusion that the scattering S-matrix is diagonal in the $|E, lm\rangle$ basis, with diagonal elements depending only on E and l,

$$\langle E, lm|S|E', l'm'\rangle = \delta_{ll'}\delta_{mm'}\delta_{EE'}S_{E,l}$$

and due to unitarity, the diagonal element $S_{E,l}$ can only be a phase.

The T-matrix is also a scalar operator, and we have, considering the case $E' = E$,

$$\begin{aligned}\langle E, lm|T|E, l'm'\rangle &= \delta_{ll'}\delta_{mm'}\langle E, l\|T\|E, l'\rangle \\ &= \delta_{ll'}\delta_{mm'}\langle E, l\|T\|E, l\rangle \\ &= \delta_{ll'}\delta_{mm'}T_{E,l}\end{aligned}$$

Thus we conclude that

$$S_{E,l} = 1 - 2\pi i T_{E,l}$$

Spherical harmonics

$$\mathcal{D}_{\phi\theta\psi}|jm\rangle = \mathcal{D}_{\phi\theta\psi,jm'jm}|jm'\rangle$$

We have

$$\begin{aligned}Y_{lm,\hat{n}*} &= \langle lm|\hat{n}\rangle \\ &= \langle lm|\mathcal{D}_{\phi\theta\psi}|\hat{z}\rangle \\ &= \langle lm|\mathcal{D}_{\phi\theta\psi}|l'm'\rangle\langle l'm'|\hat{z}\rangle \\ &= \langle lm|\mathcal{D}_{\phi\theta\psi}|lm'\rangle\langle lm'|\hat{z}\rangle \\ &= \langle lm|\mathcal{D}_{\phi\theta\psi}|l0\rangle\langle l0|\hat{z}\rangle \\ &= \mathcal{D}_{\phi\theta\psi,lml0}Y_{l0,\hat{z}*}\end{aligned}$$

where $Y_{l0,\hat{z}} = \sqrt{\frac{2l+1}{4\pi}}$. This calculation shows that the spherical harmonics are related to the matrix representation of the rotation operators in a simple way. Notice that

$$\mathcal{D}_{\phi\theta\psi,lml0} = \mathcal{D}_{\phi\theta 0,lml0}$$

We also have

$$\begin{aligned}
\langle E, lm|p\rangle &= \langle E, lm|\mathcal{D}_{\phi\theta\psi}|p\hat{z}\rangle \\
&= \langle E, lm|\mathcal{D}_{\phi\theta\psi}|E', l'm'\rangle\langle E', l'm'|p\hat{z}\rangle \\
&= \langle E, lm|\mathcal{D}_{\phi\theta\psi}|E, lm'\rangle\langle E, lm'|p\hat{z}\rangle \\
&= \langle E, lm|\mathcal{D}_{\phi\theta\psi}|E, l0\rangle\langle E, l0|p\hat{z}\rangle \\
&= \mathcal{D}_{\phi\theta\psi,lml0}\langle E, l0|p\hat{z}\rangle
\end{aligned}$$

Chapter 4

Quantum field theory

Gell-Mann and Low theorem

We have, using $\rightsquigarrow$ and $\rightarrow$ diagrams, for example,

$$\begin{aligned}\phi_3\phi_2\phi_1 &= U_{03}\phi_{3I}U_{30}U_{02}\phi_{2I}U_{20}U_{01}\phi_{1I}U_{10}\\ &= U_{03}\phi_{3I}U_{32}\phi_{2I}U_{21}\phi_{1I}U_{10}\end{aligned}$$

where ϕ_j are time-dependent Heisenberg operators. We also have

$$\begin{aligned}U_{0\,-\infty}|\Omega_0\rangle &= \lim_{t_-\nwarrow-\infty} e^{iEt_-}e^{-iE_0t_-}|\Omega\rangle\langle\Omega|\Omega_0\rangle\\ \langle\Omega_0|U_{+\infty\,0} &= \lim_{t_+\searrow+\infty} e^{iE_0t_+}e^{-iEt_+}\langle\Omega_0|\Omega\rangle\langle\Omega|\end{aligned}$$

Notice that $U_{0\,-\infty}$ and $U_{+\infty\,0}$ are not unitary! since t_- and t_+ are not real.

$$\langle\Omega|\phi_3\phi_2\phi_1|\Omega\rangle = \frac{\langle\Omega_0|U_{+\infty3}\phi_{3I}U_{32}\phi_{2I}U_{21}\phi_{1I}U_{1\,-\infty}|\Omega_0\rangle}{\langle\Omega_0|U_{+\infty\,-\infty}|\Omega_0\rangle}$$

The path-integral

We have

$$\langle\Omega|\phi_3\phi_2\phi_1|\Omega\rangle = \frac{\langle\Omega_0|U_{+\infty\,0}\phi_3\phi_2\phi_1U_{0\,-\infty}|\Omega_0\rangle}{\langle\Omega_0|U_{+\infty\,0}U_{0\,-\infty}|\Omega_0\rangle}$$

after canceling some factors,

$$\langle\Omega|\phi_3\phi_2\phi_1|\Omega\rangle = \lim_{t_-\nwarrow-\infty}\lim_{t_+\searrow+\infty}\frac{\langle\Omega_0|e^{-iHt_+}\phi_3\phi_2\phi_1e^{iHt_-}|\Omega_0\rangle}{\langle\Omega_0|e^{-iHt_+}e^{iHt_-}|\Omega_0\rangle}$$

In fact, we have

$$\langle\Omega|\phi_3\phi_2\phi_1|\Omega\rangle = \lim_{t_-\searrow-\infty}\lim_{t_+\searrow+\infty}\frac{\langle\Omega'|e^{-iHt_+}\phi_3\phi_2\phi_1 e^{iHt_-}|\Omega''\rangle}{\langle\Omega'|e^{-iHt_+}e^{iHt_-}|\Omega''\rangle}$$

for any $|\Omega'\rangle$ and $|\Omega''\rangle$. This is because

$$\lim_{t\searrow+\infty} e^{-iHt} = \lim_{t\searrow+\infty} e^{-iEt}|\Omega\rangle\langle\Omega|$$

Second quantization

$$\Psi = \Psi^0 + \Psi^1 + \Psi^2 + \Psi^3 + \cdots$$

or more precisely,

$$\mathcal{F} = \mathcal{H}^0 \oplus \mathcal{H}^1 \oplus \mathcal{H}^2 \oplus \mathcal{H}^3 \oplus \cdots$$

We have

$$\Psi = \Psi^0 + C_i|\psi_i\rangle + C_{ij}|\psi_i\rangle|\psi_j\rangle + C_{ijk}|\psi_i\rangle|\psi_j\rangle|\psi_k\rangle + \cdots$$

and

$$\begin{aligned}\mathcal{P}_{12}\Psi &= \Psi^0 + C_i|\psi_i\rangle + C_{ij}|\psi_j\rangle|\psi_i\rangle + C_{ijk}|\psi_j\rangle|\psi_i\rangle|\psi_k\rangle + \cdots \\ &= \Psi^0 + C_i|\psi_i\rangle + C_{ji}|\psi_i\rangle|\psi_j\rangle + C_{jik}|\psi_i\rangle|\psi_j\rangle|\psi_k\rangle + \cdots\end{aligned}$$

which means that

$$C_{ji\cdots} = \zeta C_{ij\cdots}$$

Therefore, we define

$$|\psi_i\psi_j\cdots\rangle = \frac{1}{\sqrt{n!}}\sum_{\mathcal{P}}\zeta_{\mathcal{P}}|\psi_{\mathcal{P}i}\rangle|\psi_{\mathcal{P}j}\rangle\cdots$$

which is a over-complete basis of the n-particle subspace $\mathcal{H}^n$. Notice that

$$\begin{aligned}\langle\phi_i\phi_j\cdots|\psi_k\psi_l\cdots\rangle &= \frac{1}{n!}\sum_{\mathcal{P}}\sum_{\mathcal{Q}}\zeta_{\mathcal{P}}\zeta_{\mathcal{Q}}\langle\phi_{\mathcal{P}i}|\psi_{\mathcal{Q}k}\rangle\langle\phi_{\mathcal{P}j}|\psi_{\mathcal{Q}l}\rangle\cdots \\ &= \sum_{\mathcal{P}}\zeta_{\mathcal{P}}\langle\phi_i|\psi_{\mathcal{P}k}\rangle\langle\phi_j|\psi_{\mathcal{P}l}\rangle\cdots \\ &= \det\begin{pmatrix}\langle\phi_i|\psi_k\rangle & \langle\phi_i|\psi_l\rangle & \cdots \\ \langle\phi_j|\psi_k\rangle & \langle\phi_j|\psi_l\rangle & \cdots \\ \vdots & \vdots & \ddots\end{pmatrix}\end{aligned}$$

which, in particular, means that

$$\langle \psi_i \psi_i \cdots \psi_i \psi_j \psi_j \cdots \psi_j \cdots | \psi_i \psi_i \cdots \psi_i \psi_j \psi_j \cdots \psi_j \cdots \rangle = n_i! n_j! \cdots$$

A natural complete orthonormal basis of the n-particle subspace $\mathcal{H}^n$ is the occupation number basis

$$|\cdots n_i \cdots n_j \cdots \rangle = \frac{|\cdots \psi_i \psi_i \cdots \psi_i \cdots \psi_j \psi_j \cdots \psi_j \cdots \rangle}{\sqrt{\cdots n_i! \cdots n_j! \cdots}}$$

Creation and annihilation operators

We define

$$(a_\psi)^\dagger |\psi_i \psi_j \cdots \rangle := |\psi \psi_i \psi_j \cdots \rangle$$

such that using

$$\begin{aligned}
\langle \phi_i \phi_j \cdots | a_\psi | \psi_k \psi_l \cdots \rangle &= \langle \psi_k \psi_l \cdots | (a_\psi)^\dagger | \phi_i \phi_j \cdots \rangle^* \\
&= \langle \psi_k \psi_l \cdots | \psi \phi_i \phi_j \cdots \rangle^* \\
&= \langle \psi \phi_i \phi_j \cdots | \psi_k \psi_l \cdots \rangle
\end{aligned}$$

one can prove that

$$(a_\phi)^\dagger (a_\psi)^\dagger - \zeta (a_\psi)^\dagger (a_\phi)^\dagger = 0$$

as well as

$$\boxed{a_\phi (a_\psi)^\dagger - \zeta (a_\psi)^\dagger a_\phi = \langle \phi | \psi \rangle}$$

Second quantized operators

In the Fock space, one-body operators take the form

$$\widetilde{\mathcal{O}}_1 = \widetilde{\mathcal{O}}_1|_{\mathcal{H}0} \oplus \widetilde{\mathcal{O}}_1|_{\mathcal{H}1} \oplus \widetilde{\mathcal{O}}_1|_{\mathcal{H}2} \oplus \widetilde{\mathcal{O}}_1|_{\mathcal{H}3} \oplus \cdots$$

where $\widetilde{\mathcal{O}}_1|_{\mathcal{H}n}$ is like the total angular momentum of n particles

$$\widetilde{\mathcal{O}}_1|_{\mathcal{H}n} = \left(\mathcal{O}_1 \otimes 1 \otimes \cdots \otimes 1\right) + \left(1 \otimes \mathcal{O}_1 \otimes \cdots \otimes 1\right) + \cdots + \left(1 \otimes 1 \otimes \cdots \otimes \mathcal{O}_1\right)$$

It is easy to verify that

$$\begin{aligned}
\widetilde{\mathcal{O}}_1 &= \mathcal{O}_{1i} (a_{\psi i})^\dagger a_{\psi i} \\
&= \mathcal{O}_{1i} \langle \phi_j | \psi_i \rangle \langle \psi_i | \phi_k \rangle (a_j)^\dagger a_k \\
&= \langle \phi_j | \mathcal{O}_1 | \phi_k \rangle (a_j)^\dagger a_k
\end{aligned}$$

For two-body operators

$$\widetilde{\mathcal{O}}_2 = \widetilde{\mathcal{O}}_2|_{\mathcal{H}0} \oplus \widetilde{\mathcal{O}}_2|_{\mathcal{H}1} \oplus \widetilde{\mathcal{O}}_2|_{\mathcal{H}2} \oplus \widetilde{\mathcal{O}}_2|_{\mathcal{H}3} \oplus \cdots$$

Notice that we may always write

$$\mathcal{O}_2 = \sum_{I\leq J} C_{IJ}\left[\left(\mathcal{O}_{1I}\otimes\mathcal{O}_{1J}\right)+\left(\mathcal{O}_{1J}\otimes\mathcal{O}_{1I}\right)\right]$$

since the two-body operator is always symmetric under permutation.

For concreteness, take $\mathcal{H}^3$ as an example, we have

$$\begin{aligned}\widetilde{\mathcal{O}}_2|_{\mathcal{H}3} = {} & C_{IJ}\left[\left(\mathcal{O}_{1I}\otimes\mathcal{O}_{1J}\otimes 1\right)+\left(\mathcal{O}_{1I}\otimes 1\otimes\mathcal{O}_{1J}\right)+\left(1\otimes\mathcal{O}_{1I}\otimes\mathcal{O}_{1J}\right)\right]\\ & + C_{IJ}\left[\left(\mathcal{O}_{1J}\otimes\mathcal{O}_{1I}\otimes 1\right)+\left(\mathcal{O}_{1J}\otimes 1\otimes\mathcal{O}_{1I}\right)+\left(1\otimes\mathcal{O}_{1J}\otimes\mathcal{O}_{1I}\right)\right]\end{aligned}$$

where we have suppressed the $\sum_{I\leq J}$ symbol. Now this can be written as

$$\widetilde{\mathcal{O}}_2|_{\mathcal{H}3} = C_{IJ}\,\widetilde{\mathcal{O}}_{1I}|_{\mathcal{H}3}\widetilde{\mathcal{O}}_{1J}|_{\mathcal{H}3} - C_{IJ}\,\widetilde{\mathcal{O}}_{1I}\widetilde{\mathcal{O}}_{1J}|_{\mathcal{H}3}$$

where

$$\widetilde{\mathcal{O}}_{1I}\widetilde{\mathcal{O}}_{1J}|_{\mathcal{H}3} = \left(\mathcal{O}_{1I}\mathcal{O}_{1J}\otimes 1\otimes 1\right)+\left(1\otimes\mathcal{O}_{1I}\mathcal{O}_{1J}\otimes 1\right)+\left(1\otimes 1\otimes\mathcal{O}_{1I}\mathcal{O}_{1J}\right)$$

The two-particle operator can now be written as

$$\begin{aligned}\widetilde{\mathcal{O}}_2 &= C_{IJ}\langle\phi_i|\mathcal{O}_{1I}|\phi_j\rangle\langle\phi_k|\mathcal{O}_{1J}|\phi_l\rangle(a_i)^\dagger a_j(a_k)^\dagger a_l - C_{IJ}\langle\phi_i|\mathcal{O}_{1I}\mathcal{O}_{1J}|\phi_l\rangle(a_i)^\dagger a_l\\ &= C_{IJ}\langle\phi_i|\mathcal{O}_{1I}|\phi_j\rangle\langle\phi_k|\mathcal{O}_{1J}|\phi_l\rangle(a_i)^\dagger a_j(a_k)^\dagger a_l - C_{IJ}\langle\phi_i|\mathcal{O}_{1I}|\phi_j\rangle\langle\phi_k|\mathcal{O}_{1J}|\phi_l\rangle(a_i)^\dagger a_l\delta_{jk}\\ &= C_{IJ}\langle\phi_i|\mathcal{O}_{1I}|\phi_j\rangle\langle\phi_k|\mathcal{O}_{1J}|\phi_l\rangle(a_i)^\dagger(a_k)^\dagger a_l a_j\\ &= C_{IJ}\langle\phi_k|\mathcal{O}_{1J}|\phi_l\rangle\langle\phi_i|\mathcal{O}_{1I}|\phi_j\rangle(a_k)^\dagger(a_i)^\dagger a_j a_l\end{aligned}$$

and we finally conclude that

$$\widetilde{\mathcal{O}}_2 = \frac{1}{2}\langle\phi_i|\langle\phi_k|\mathcal{O}_2|\phi_j\rangle|\phi_l\rangle(a_i)^\dagger(a_k)^\dagger a_l a_j$$

The number operator

is the identity operator $1:\mathcal{H}^1\to\mathcal{H}^1$ lifted to the Fock space: $\mathcal{N}=(a_j)^\dagger a_j$.

Quantum statistical mechanics

is governed by

$$\rho = Z^{-1}e^{-\beta\mathcal{H}+\beta\mu\mathcal{N}}$$

The Legendre transformation

We have

$$Z = e^{iW} = \int D\phi\, e^{iS+iJ\cdot\phi}$$

such that the expectation value

$$\phi_{\text{exp}} = \frac{1}{iZ}\frac{\partial Z}{\partial J} = \frac{\partial W}{\partial J}$$

The effective action Γ is defined by the following Legendre transformation

$$-\Gamma = J\cdot\phi - W$$

and should be viewed as a function of ϕ,

$$-\delta\Gamma = \delta J\cdot\phi + J\cdot\delta\phi - \frac{\partial W}{\partial J}\cdot\delta J = J\cdot\delta\phi$$

The effective action

We define a path-integral using the effective action in place of the action

$$Z_\Gamma = \int D\phi\, e^{i\Gamma+iJ\cdot\phi}$$

We may use the saddle point approximation, which requires that

$$\frac{\partial\Gamma}{\partial\phi} + J = 0$$

and we have

$$Z_\Gamma \approx e^{i\Gamma+iJ\cdot\phi} = e^{iW}$$

which means that the original Z and W can be computed by drawing tree diagrams of the effective action only! We thus conclude that the propagator and interaction vertices in the expression of the effective action are **exact**.

The quantum variational principle

Finally, notice that at $J = 0$, we have

$$\delta\Gamma = -J\cdot\delta\phi = 0$$

and the field is now equal to

$$\frac{\partial W}{\partial J}|_{J=0} = \phi_{\text{exp}}|_{J=0}$$

namely $\phi_{\text{exp}}|_{J=0}$ is the solution to the equation $\delta\Gamma = 0$. This is to be compared with the fact that ϕ_{cl} is the solution to the equation $\delta S = 0$.

Conformal transformations

A map $\phi : (M_1, g_1) \to (M_2, g_2)$ is called conformal if

$$\phi^* g_2 = \Omega^2 g_1$$

or equivalently in local coordinates,

$$g_{2\mu\nu}\frac{\partial x^{2\mu}\circ\phi}{\partial x^{1\rho}}\frac{\partial x^{2\nu}\circ\phi}{\partial x^{1\sigma}} = \Omega^2 g_{1\rho\sigma}$$

Noether's theorem

For the shift $\psi' = \psi + \delta\psi$, where $\delta\psi$ may depend on ψ,

$$\begin{aligned}
\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\delta\partial_\mu\psi \\
&= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\right)\delta\psi + \frac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\partial_\mu\delta\psi + \frac{\partial S}{\partial\psi}\delta\psi \\
&= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\delta\psi\right) + \frac{\partial S}{\partial\psi}\delta\psi
\end{aligned}$$

namely some current is conserved **on-shell**.

The Ward identity

For the shift $\phi' = \phi + \delta\phi$, where $\delta\phi$ may depend on ϕ,

$$\begin{aligned}
\int D\phi\, e^{-S} X &= \int D\phi'\, e^{-S'} X' \\
&= \int D\phi\, e^{-S} \left(1 - \int d^2x\, T_{\mu\nu}\partial_\mu\varepsilon_\nu\right)\left(X + \delta X\right)
\end{aligned}$$

$$\begin{aligned}
0 &= \int D\phi\, e^{-S} \left(\delta X - X \int d^2x\, T_{\mu\nu}\partial_\mu\varepsilon_\nu\right) \\
&= \int D\phi\, e^{-S} \left(\delta X + X \int d^2x\, \varepsilon_\nu\partial_\mu T_{\mu\nu}\right)
\end{aligned}$$

Integrating the Ward identity over a small pillbox shows that $\int dx\, T_{0\nu}$ is the generator of spacetime symmetries.

The conformal Ward identity

The right-hand-side of the equation above can actually be written as

$$\begin{aligned}
0 &= \int D\phi\, e^{-S} \left(\delta X + X \int_{\mathcal{B}} d^2x\, \varepsilon_\nu\partial_\mu T_{\mu\nu}\right) \\
&= \int D\phi\, e^{-S} \left(\delta X - X \int_{\mathcal{B}} d^2x\, T_{\mu\nu}\partial_\mu\varepsilon_\nu + X \int_{\partial\mathcal{B}} \varepsilon_{\mu\rho}\, dx_\rho\, \varepsilon_\nu T_{\mu\nu}\right)
\end{aligned}$$

$$0 = \int D\phi\, e^{-S} \left(\delta' X + X \int_{\partial\mathcal{B}} \varepsilon_{\mu\rho}\, dx_\rho\, \varepsilon_\nu T_{\mu\nu}\right)$$

Chapter 5

The Kondo problem

The effective Hamiltonian method

We have, since $\mathcal{P}_j\mathcal{P}_j = 1$,

$$\mathcal{P}_i H \mathcal{P}_j \mathcal{P}_j \psi = E \mathcal{P}_i \psi$$

and when $\mathcal{P}_1 + \mathcal{P}_2 = 1$,

$$\begin{aligned} H_{11}\psi_1 + H_{12}\psi_2 &= E\psi_1 \\ H_{21}\psi_1 + H_{22}\psi_2 &= E\psi_2 \end{aligned}$$

The second equation can be used to solve for ψ_2,

$$\psi_2 = \frac{1}{E - H_{22}} H_{21}\psi_1$$

which gives us a self-consistent equation

$$\left(H_{11} + H_{12} \frac{1}{E - H_{22}} H_{21} \right) \psi_1 = E\psi_1$$

Applications to degenerate perturbation theory

To second order in the perturbation,

$$\begin{aligned} H_{\text{eff}} &= E_1 1 + V_{11} + V_{12} \frac{1}{E - H_{0\,22} - V_{22}} V_{21} \\ &\approx E_1 1 + V_{11} + V_{12} \frac{1}{E_1 - H_{0\,22}} V_{21} \end{aligned}$$

AFM coupling in the Hubbard model

$$H = -t\sum_{i\neq j}(a_{is})^\dagger a_{js} + U\sum_i n_{i+}n_{i-}$$

The effective Hamiltonian is equal to

$$\begin{aligned}
H_{\text{eff}} &= \frac{t^2}{E-U}\sum_{i\neq j}\sum_{k\neq l}(a_{is})^\dagger a_{js}(a_{kr})^\dagger a_{lr} \\
&= \frac{t^2}{E-U}\sum_{i\neq j}(a_{is})^\dagger a_{js}(a_{jr})^\dagger a_{ir}
\end{aligned}$$

Notice that

$$\begin{aligned}
\sum_{i\neq j} S_i\cdot S_j &= \frac{1}{2}\sum_{i\neq j}(a_{is})^\dagger(a_{jr})^\dagger a_{js}a_{ir} - \frac{1}{4}\sum_{i\neq j}(a_{is})^\dagger(a_{jr})^\dagger a_{jr}a_{is} \\
&= \frac{1}{2}\sum_{i\neq j}(a_{is})^\dagger a_{is} - \frac{1}{2}\sum_{i\neq j}(a_{is})^\dagger a_{js}(a_{jr})^\dagger a_{ir} - \frac{1}{4}\sum_{i\neq j}(a_{is})^\dagger a_{is}(a_{jr})^\dagger a_{jr} \\
&= \frac{1}{2} - \frac{1}{2}\sum_{i\neq j}(a_{is})^\dagger a_{js}(a_{jr})^\dagger a_{ir}
\end{aligned}$$

which means that the effective Hamiltonian contains an interaction

$$H_{\text{eff}} \approx \frac{2t^2}{U}\sum_{i\neq j} S_i\cdot S_j + \frac{t^2}{-U}$$

The Anderson impurity model

$$H = \sum_{ks}\varepsilon_k n_{ks} + \sum_{ks} V_k(a_{ks})^\dagger A_s + \text{h.c.} + \sum_s \mathcal{E} N_s + U N_+ N_-$$

Notice that we have

$$H_{02} = H_{20} = 0$$

The low-energy effective Hamiltonian

In the low-energy subspace, we have

$$\begin{aligned}
H_{10}\frac{1}{E-H_{00}}H_{01} &= \sum_{ks}\sum_{lr} V_{k*}V_l (A_s)^\dagger a_{ks}\frac{1}{E-H_{00}}(a_{lr})^\dagger A_r \\
&= \sum_{ks}\sum_{lr} V_{k*}V_l (A_s)^\dagger a_{ks}(a_{lr})^\dagger A_r \frac{1}{E-H_{00}-\varepsilon_l} \\
&\approx \sum_{ks}\sum_{lr} V_{k*}V_l (A_s)^\dagger a_{ks}(a_{lr})^\dagger A_r \frac{1}{\mathcal{E}-\varepsilon_l}
\end{aligned}$$

as well as

$$\begin{aligned}
H_{12}\frac{1}{E-H_{22}}H_{21} &= \sum_{ks}\sum_{lr} V_k V_{l*} (a_{ks})^\dagger A_s\frac{1}{E-H_{22}}(A_r)^\dagger a_{lr} \\
&= \sum_{ks}\sum_{lr} V_k V_{l*} (a_{ks})^\dagger A_s (A_r)^\dagger a_{lr} \frac{1}{E-H_{22}+\varepsilon_l} \\
&\approx \sum_{ks}\sum_{lr} V_k V_{l*} (a_{ks})^\dagger A_s (A_r)^\dagger a_{lr} \frac{1}{-\mathcal{E}-U+\varepsilon_l}
\end{aligned}$$

and thus

$$H_{\text{l-e}} \approx H_{11} - \sum_{ks}\sum_{lr} V_{k*}V_l \left[\frac{(A_s)^\dagger a_{ks}(a_{lr})^\dagger A_r}{-\mathcal{E}+\varepsilon_l} + \frac{(a_{lr})^\dagger A_r (A_s)^\dagger a_{ks}}{\mathcal{E}+U-\varepsilon_k} \right]$$

The Kondo model

Notice that

$$\begin{aligned}
s_{lk}\cdot S &= \frac{1}{2}\sum_{sr}(a_{lr})^\dagger (A_s)^\dagger A_r a_{ks} - \frac{1}{4}\sum_{sr}(a_{lr})^\dagger (A_s)^\dagger A_s a_{kr} \\
&= \frac{1}{2}\sum_{sr}(a_{lr})^\dagger (A_s)^\dagger A_r a_{ks} - \frac{1}{4}\sum_{r}(a_{lr})^\dagger a_{kr}
\end{aligned}$$

we see that the low-energy effective Hamiltonian contains an interaction

$$2\sum_{kl} V_{k*}V_l \left[\frac{1}{-\mathcal{E}+\varepsilon_l} + \frac{1}{\mathcal{E}+U-\varepsilon_k} \right] s_{lk}\cdot S$$

which is anti-ferromagnetic near the Fermi surface.

Chapter 6

Constraints and the Hamiltonian formalism

The transverse vector field

The Lagrangian is

$$
\begin{aligned}
-\mathcal{L} &= \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \tfrac{1}{2} m^2 A_\mu A^\mu \\
&= \tfrac{1}{2} F_{0i} F^{0i} + \tfrac{1}{4} F_{ij} F^{ij} + \tfrac{1}{2} m^2 A_0 A^0 + \tfrac{1}{2} m^2 A_i A^i
\end{aligned}
$$

and thus

$$
\begin{aligned}
\Pi_0 &= \frac{\partial \mathcal{L}}{\partial \partial_0 A^0} = 0 \qquad \text{constraint} \\
\Pi_i &= \frac{\partial \mathcal{L}}{\partial \partial_0 A^i} = F_{0i}
\end{aligned}
$$

therefore the Hamiltonian is

$$
\begin{aligned}
\mathcal{H} &= \Pi_\mu \partial_0 A^\mu - \mathcal{L} \\
&= \Pi_0 \partial_0 A^0 + \Pi_i \partial_0 A^i - \mathcal{L} \\
&= \tfrac{1}{2} \Pi_i \Pi_i - \Pi_i \partial_i A^0 + \tfrac{1}{4} F_{ij} F^{ij} + \tfrac{1}{2} m^2 A_0 A^0 + \tfrac{1}{2} m^2 A_i A^i \\
&= \tfrac{1}{2} \Pi_i \Pi_i + A^0 \partial_i \Pi_i + \tfrac{1}{4} F_{ij} F^{ij} + \tfrac{1}{2} m^2 A_0 A^0 + \tfrac{1}{2} m^2 A_i A^i
\end{aligned}
$$

The equations of motion

$$\partial_0 A^\mu = \frac{\partial \mathcal{H}}{\partial \Pi_\mu} + \lambda \frac{\partial \phi}{\partial \Pi_\mu}$$
$$-\partial_0 \Pi_\mu = \frac{\partial \mathcal{H}}{\partial A^\mu} + \lambda \frac{\partial \phi}{\partial A^\mu}$$

where the constraint $\phi \equiv \Pi_0$. To be more specific,

$$\partial_0 A^0 = \lambda$$
$$\partial_0 A^i = \Pi_i - \partial_i A^0$$
$$-\partial_0 \Pi_0 = \partial_i \Pi_i - m^2 A^0$$
$$-\partial_0 \Pi_i = -\nabla^2 A^i + \partial_i \partial_j A^j + m^2 A^i$$

The massive case

The consistency condition $\partial_0 \Pi_0 = 0$ requires that

$$A^0 = m^{-2} \partial_i \Pi_i$$

and the consistency condition $\partial_0 \partial_0 \Pi_0 = 0$ requires that

$$0 = \partial_0 \partial_i \Pi_i - m^2 \partial_0 A^0 = \partial_0 \partial_i \Pi_i - \partial_0 \partial_i \Pi_i = 0 \qquad \text{automatically satisfied}$$

Therefore, no constraints other than the above two are needed. Notice that

$$\{\Pi_{0x}, \partial_0 \Pi_{0y}\} = -\{\Pi_{0x}, \partial_i \Pi_{iy} - m^2 A^{0y}\} = -m^2 \delta_{xy} \neq 0$$

and thus by definition, the two constraints are second class.

The massless case

The consistency condition $\partial_0 \Pi_0 = 0$ requires that

$$\partial_i \Pi_i = 0$$

and the consistency condition $\partial_0 \partial_0 \Pi_0 = 0$ requires that

$$0 = \partial_i \partial_0 \Pi_i = \nabla^2 \partial_i A^i - \nabla^2 \partial_j A^j = 0 \qquad \text{automatically satisfied}$$

Therefore, no constraints other than the above two are needed. Notice that

$$\begin{aligned}
\left\{\Pi_{0x}, \Pi_{0y}\right\} &= 0 \\
\left\{\partial_0 \Pi_{0x}, \partial_0 \Pi_{0y}\right\} &= \left\{\partial_i \Pi_{ix}, \partial_j \Pi_{jy}\right\} = 0 \\
\left\{\Pi_{0x}, \partial_0 \Pi_{0y}\right\} &= -\left\{\Pi_{0x}, \partial_i \Pi_{iy}\right\} = 0
\end{aligned}$$

and thus by definition, the two constraints are first class.

Chapter 7

Miscellaneous

The guiding center coordinates

are conserved, namely they commute with the Hamiltonian, and we have

$$[X, Y] = [x + l^2\pi_y, y - l^2\pi_x] = -il^2$$

Notice that they travel on equipotential lines

$$i\dot{X}_1 = [X_1, V] = -il^2\partial_2 V$$
$$i\dot{X}_2 = [X_2, V] = +il^2\partial_1 V$$

Spin-waves and magnons

The Schwinger-boson representation

$$S_+ = a^\dagger b$$
$$S_- = b^\dagger a$$
$$2S_z = a^\dagger a - b^\dagger b$$

gives us the Holstein-Primakoff transformation if we use the constraint

$$a^\dagger a + b^\dagger b = 2S$$

The spin-wave is obtained by, e.g., expanding around $S_z = -S$, namely setting

$$b = b^\dagger = \sqrt{2S - a^\dagger a} \approx \sqrt{2S}$$

The Bogoliubov transformation

In order to diagonalize a Hamiltonian of the form

$$\Psi_{i\dagger} A_{ij} \Psi_j$$

while keeping the commutation relations, we want

$$U^\dagger A U = \Lambda$$

where Λ is a diagonal matrix, as well as

$$C_{ij} = \left[\Psi_i, \Psi_{j\dagger}\right] = \left[U_{ik}\Phi_k, U_{jl*}\Phi_{l\dagger}\right] = U_{ik}U_{jl*}C_{kl} \Rightarrow C = UCU^\dagger$$

where C is, by assumption, a diagonal matrix. Therefore we have

$$CAU = UCU^\dagger AU = UC\Lambda$$

which means that U diagonalizes the matrix CA, with eigenvalues $C\Lambda$.

The Wilsonian renormalization group

1. The "effective action".

$$Z = \int D\phi_\Lambda \exp iS_\Lambda = \int D\phi_{\Lambda'} \exp iS_{\Lambda'} = \cdots$$

2. Effective field theory.

$$q_\Lambda = \frac{Q_\Lambda}{\Lambda^{\dim Q}} \text{ is meaningful}$$

3. Renormalization group fix points.

$$\{q_{\Lambda'}\} = \{q_\Lambda\} \text{ up to a field rescaling}$$