S07E03 Symmetries

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Noether's theorem

For the shift $\psi' = \psi + \delta\psi$, where $\delta\psi$ may depend on ψ,

$$\begin{aligned}
\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\psi}\,\delta\psi + \frac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\,\delta\partial_\mu\psi \\
&= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\right)\delta\psi + \frac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\,\partial_\mu\delta\psi + \frac{\partial S}{\partial\psi}\,\delta\psi \\
&= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\,\delta\psi\right) + \frac{\partial S}{\partial\psi}\,\delta\psi
\end{aligned}$$

namely some current is conserved **on-shell**.

The Ward identity

For the shift $\phi' = \phi + \delta\phi$, where $\delta\phi$ may depend on ϕ,

$$\begin{aligned}
\int D\phi\, e^{-S} X = \int D\phi'\, e^{-S'} X' \\
&= \int D\phi\, e^{-S}\left(1 - \int d^2x\, T_{\mu\nu}\partial_\mu\varepsilon_\nu\right)(X + \delta X)
\end{aligned}$$

$$\begin{aligned}
0 &= \int D\phi\, e^{-S}\left(\delta X - X\int d^2x\, T_{\mu\nu}\partial_\mu\varepsilon_\nu\right) \\
&= \int D\phi\, e^{-S}\left(\delta X + X\int d^2x\, \varepsilon_\nu\partial_\mu T_{\mu\nu}\right)
\end{aligned}$$

Integrating the Ward identity over a small pillbox shows that $\int dx\, T_{0\nu}$ is the generator of spacetime symmetries.

Conformal transformations

A map $\phi : (M_1, g_1) \to (M_2, g_2)$ is called conformal if

$$\phi^* g_2 = \Omega^2 g_1$$

or equivalently in local coordinates,

$$g_{2\mu\nu} \frac{\partial x^{2\mu} \circ \phi}{\partial x^{1\rho}} \frac{\partial x^{2\nu} \circ \phi}{\partial x^{1\sigma}} = \Omega^2 g_{1\rho\sigma}$$

The conformal Ward identity

The right-hand-side of the Ward identity above can actually be written as

$$\begin{aligned} 0 &= \int D\phi\, e^{-S} \left(\delta X + X \int_{\mathcal{B}} d^2 x\, \varepsilon_\nu \partial_\mu T_{\mu\nu} \right) \\ &= \int D\phi\, e^{-S} \left(\delta X - X \int_{\mathcal{B}} d^2 x\, T_{\mu\nu} \partial_\mu \varepsilon_\nu + X \int_{\partial\mathcal{B}} \varepsilon_{\mu\rho}\, d x_\rho\, \varepsilon_\nu T_{\mu\nu} \right) \end{aligned}$$

$$0 = \int D\phi\, e^{-S} \left(\delta' X + X \int_{\partial\mathcal{B}} \varepsilon_{\mu\rho}\, d x_\rho\, \varepsilon_\nu T_{\mu\nu} \right)$$