S07E06 Constraints and the Hamiltonian formalism

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The transverse vector field

The Lagrangian is

$$\begin{aligned}-\mathcal{L} &= \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}m^2 A_\mu A^\mu \\ &= \tfrac{1}{2}F_{0i}F^{0i} + \tfrac{1}{4}F_{ij}F^{ij} + \tfrac{1}{2}m^2 A_0 A^0 + \tfrac{1}{2}m^2 A_i A^i\end{aligned}$$

and thus

$$\begin{aligned}\Pi_0 &= \frac{\partial\mathcal{L}}{\partial\partial_0 A^0} = 0 \qquad \text{constraint} \\ \Pi_i &= \frac{\partial\mathcal{L}}{\partial\partial_0 A^i} = F_{0i}\end{aligned}$$

therefore the Hamiltonian is

$$\begin{aligned}\mathcal{H} &= \Pi_\mu \partial_0 A^\mu - \mathcal{L} \\ &= \Pi_0\partial_0 A^0 + \Pi_i \partial_0 A^i - \mathcal{L} \\ &= \tfrac{1}{2}\Pi_i\Pi_i - \Pi_i\partial_i A^0 + \tfrac{1}{4}F_{ij}F^{ij} + \tfrac{1}{2}m^2 A_0 A^0 + \tfrac{1}{2}m^2 A_i A^i \\ &= \tfrac{1}{2}\Pi_i\Pi_i + A^0\partial_i\Pi_i + \tfrac{1}{4}F_{ij}F^{ij} + \tfrac{1}{2}m^2 A_0 A^0 + \tfrac{1}{2}m^2 A_i A^i\end{aligned}$$

The equations of motion

are

$$\begin{aligned}\partial_0 A^\mu &= \frac{\partial\mathcal{H}}{\partial\Pi_\mu} + \lambda\frac{\partial\phi}{\partial\Pi_\mu} \qquad \phi \equiv \Pi_0 \\ -\partial_0\Pi_\mu &= \frac{\partial\mathcal{H}}{\partial A^\mu} + \lambda\frac{\partial\phi}{\partial A^\mu}\end{aligned}$$

or more explicitly,

$$\begin{aligned} \partial_0 A^0 &= \lambda \\ -\partial_0 \Pi_0 &= \partial_i \Pi_i - m^2 A^0 \\ \partial_0 A^i &= \Pi_i - \partial_i A^0 \\ -\partial_0 \Pi_i &= -\partial_j \partial_j A^i + \partial_j \partial_i A^j + m^2 A^i \end{aligned}$$

The massive case

The consistency condition $\partial_0 \Pi_0 = 0$ requires that

$$A^0 = m^{-2} \partial_i \Pi_i \qquad \text{also determines } \lambda$$

and the condition $\partial_0 \partial_0 \Pi_0 = 0$ requires that

$$0 = \partial_0 \partial_i \Pi_i - m^2 \partial_0 A^0 = \partial_0 \partial_i \Pi_i - \partial_0 \partial_i \Pi_i = 0 \qquad \text{automatically satisfied}$$

Therefore, no constraints other than the above two are needed. Notice that

$$\begin{aligned} \left\{\Pi_{0\mathbf{x}}, \Pi_{0\mathbf{y}}\right\} &= 0 \\ \left\{\partial_0 \Pi_{0\mathbf{x}}, \partial_0 \Pi_{0\mathbf{y}}\right\} &= 0 \qquad -\partial_0 \Pi_0 = \partial_i \Pi_i - m^2 A^0 \\ \left\{\Pi_{0\mathbf{x}}, \partial_0 \Pi_{0\mathbf{y}}\right\} &\neq 0 \end{aligned}$$

and thus the two constraints are **second class**.

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Therefore, no constraints other than the above two are needed. Notice that

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and thus the two constraints are **first class**.