Integration Basics

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1 Comments

There's a theme of 'economy' in the presentation of the theorems. For example, Fubini's Theorem is given in pretty intense generality, but several more specific results follow trivially from it, such as Leibnitz' Rule, equality of continuous mixed derivatives, volumes of solids of revolution, Cavalieri's Principle and most importantly the volume of the image of a rectangle under a linear transformation.

2 Issues

These are incomplete/problematic:

- 3.10 Would like a proof using the notions developed here rather than topological facts.
- 3.11, 3.12, 3.19, 3.25, 3.28, 3.29, 3.31, 3.40, 3.41 Needs proof.
- Still not sure I understand what 3.21 is getting at. Is it asking for a partition which doesn't contain any rectangles in $A-C$? If so its easy to draw JM sets with two connected components which seem to break this idea.
- 3.33 (b) straight up makes no sense. f is defined on $[a,b]\times[c,d]$ and $G(x)$ is defined, nonsensically, as $\int_a^{g(x)} f(t,x)dt$.
- Be more clear on the hypothesis of Fubinis for switching order in 3.35.
- 3-37 (b) is reportedly broken. Still needs doing with an improved hypothesis.

It would be good to check whether an argument about absolute convergence should ever be made explicitly.

In other proofs Lemma 3.1 is used generously, in the sense that if (P') is a family of partitions subject to some condition which does not restrict the 'fine-ness' of P' then each P' is a refinement of some generic partition and so any result involving an infinumum/suprememum over all partitions holds equally well over the restricted set of partitions. Would be good to clear this up.

Personally I think the original statement 3.21 is pretty confusing. I could state explicitly that the partition contains no subrectangles contained in $A - C$.

3 Problems

Problem 1. 3.1 Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 0 & \text{if } 0 \leq x < \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Then f is integrable and the value of the integral is $\frac{1}{2}$.

Proof. For $\varepsilon > 0$ let P_ε be the partition with subrectangles

$$\begin{aligned} S_1 &= [0, \frac{1}{2} - \frac{\varepsilon}{4}] \times [0,1] \\ S_2 &= [\frac{1}{2} - \frac{\varepsilon}{4}, \frac{1}{2} + \frac{\varepsilon}{4}] \times [0,1] \\ S_3 &= [\frac{1}{2} + \frac{\varepsilon}{4}, 1] \times [0,1] \end{aligned}$$

Then $U(f, P_\varepsilon) - L(f, P_\varepsilon) = \dfrac{\varepsilon}{2} < \varepsilon$ i.e. f is integrable and $\int f = \inf U(f, P_\varepsilon) = \frac{1}{2}$. $\square$

Problem 2. 3.2 Let $f : A \to \mathbb{R}$ be integrable and $g = f$ except at finitely many points. Then g is integrable and $\int f = \int g$.

Proof. Suppose f and g differ except at a single point x. Let P be some partition fine enough that x lies in a single subrectangle S. Then

$$(U - L)(g, P) = (U - L)(f, P) + (M_S - m_S)(g - f) \cdot v(S)$$

can be made arbitrarily small, as the term $(U - L)(f, P)$ vanishes by integrability of f and the other term can be made as small as we like by insiting that the rectangle S is sufficiently small. Furthermore if $g(x) > f(x)$ then

$$\int g = \inf[U(g, P)] = \inf[U(f, P) + (M_S(g - f) \cdot v(S))] = \inf[U(f, P)] = \int f$$

with a similar argument for when $g(x) < f(x)$. Repetition of the argument proves the result for a finite collection of points. $\square$

Problem 3. 3.3 Linearity of the Integral. Let $f, g : A \to \mathbb{R}$ be integrable. Then $\int f + g = \int f + \int g$ and $\int cf = c \int f$.

Proof. Since
$$m_S(f) + m_S(g) \le f(x) + g(x) \quad \forall x \in S$$
it follows that
$$m_S(f) + m_S(g) \le \inf_S (f+g) = m_S(f+g).$$
Then $L(f,P) + L(g,P) \le L(f+g,P)$ follows easily and by a similar argument $U(f+g,P) \le U(f,P) + U(g,P)$. Hence the differece $(U-L)(f+g,P)$ is bounded from above by $(U-L)(f,P) + (U-L)(g,P)$ which vanishes by integrability of f and g.

To compute the value of the integral,
$$\int f + \int g = \sup L(f,P) + \sup L(g,P) = \sup[L(f,P) + L(g,P)] \le \sup L(f+g,P) = \int f + g$$
$$\int f + \int g = \inf U(f,P) + \inf U(g,P) = \inf[U(f,P) + U(f,P)] \ge \inf U(f+g,P) = \int f + g$$
and so $\inf f + \int g = \int f + g$.

Finally for constants c,
$$\inf U(cf,P) = c \inf U(f,P) = c \sup L(f,P) = \sup L(cf,P)$$
i.e cf is integrable with the value $\int cf = \inf U(cf,P) = c \inf U(f,P) = c \int f$. □

Problem 4. 3.4 Let $f : A \to \mathbb{R}$ and let P be a partition of A. Then f is integrable if and only if the restriction of f to each subrectangle is integrable, and in this case $\int_A f = \sum_P \int_S f|_S$.

Proof. Let the n subrectangles of P be S_i and suppose that each $f|_{S_i}$ is integrable. Then for any $\varepsilon > 0$ and each i there is a partitions P_i of S_i such that $(U-L)(f|_{S_i}, P_i) < \varepsilon/n$. Then
$$\sum_{S_i \in P} \sum_{S_{ij} \in P_i} [(M_{S_{ij}} - m_{S_{ij}})(f|S_i) \cdot v(S_{ij})] < \varepsilon \Longrightarrow \sum_{S_i \in P} \sum_{S_{ij} \in P_i} [(M_{S_{ij}} - m_{S_{ij}})(f) \cdot v(S_{ij})] < \varepsilon$$
where the latter double sum is $(U-L)f$ over the partition formed by taking the union of all the P_i. Hence f is integrable.

On the other hand for any rectangle $S \in P$ we have
$$(U-L)(f|_S, S) = (U-L)(f,S) \le (U-L)(f,P)$$
since $U - L$ is non-negative and S is containined in P. If f is integrable then the RHS can be made arbitrarily small by refining P, in which case S may need to be replaced by the part of that refinement which covers it, $S = \cup P'$. Then $(U-L)(f|_S, P')$ vanishes and so $f|_S$ is integrable.

To compute the integral we have, for partitions P_S of each S
$$\sum_P \int_S f|_S = \sum_P \sup[L(f|_S, P_S)] = \sup\left[\sum_P L(f|_S, P_s)\right] = \sup\left[\sum_P \sum_{P_i} m_{S_ij}(f) \cdot v(S_{ij})\right].$$
The last term is the supremum over partitions P' which are generic partitions of A but for the refinement that each rectangle must be contained in one of the original $S \in P$. Hence the supremum over such partitions coincides with the supremum over generic ones and so $\sum \int f|_S = \int f$. □

Problem 5. 3.5 Let $g, f : A \to \mathbb{R}$ be integrable and suppose $f \le g$. Then $\int f \le \int g$.

Proof.

$$\int f \;=\; \inf \sum M_S(f) \cdot v(S) \;\le\; \inf \sum M_S(g) \cdot v(S) \;=\; \int g.$$

□

Problem 6. If $f : A \to \mathbb{R}$ is integrable then $|f|$ is integrable and $\left|\int f\right| \le \int |f|$.

Proof.

$$(U - L)(|f|, P) = \sum (M_S(|f|) - m_S(|f|)) \cdot v(S) \le \sum (M_S(f) - m_S(f)) \cdot v(S)$$

which is as small as we like by integrability of f. The required comparison follows from taking infinums of

$$\left|\sum M_s(f) \cdot (S)\right| \le \sum |M_s(f) \cdot (S)| = \sum M_s(|f|) \cdot (S).$$

□

Problem 7. 3.7 Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 0 & \text{if } x \text{ is irrational or y is irrational} \\ 1/q & \text{if } x \text{ is rational and } y = p/q \text{ in lowest terms.} \end{cases}$$

Then f is integrable and the integral is 0.

Proof. ? □

Problem 8. 3.8 The rectangle $A = \Pi[a_i, b_i] \subset \mathbb{R}^m$ does not have content zero if each $a_i < b_i$.

Proof. Let $U_1, ..., U_n$ be a finite cover of closed rectangles with each U_i contained in A and let $P_d = (a_d = t_{d0} < ... < t_{dk(d)} = b_d)$ be the partition containing all the endpoints of U_i in the dth dimension, $d = 1, .., m$. Then each $v(U_i) = \Pi_{d=1}^m \sum (t_{d,j} - t_{d,j-1})$ for a 'certain number' of summands. Moreover each $\Pi_{d=1}^m (t_{d,j} - t_{d,j-1})$ lies in at least one U_i, so

$$\sum v(U_i) \ge \prod_{d=1}^{m} \sum_{j=1}^{k(d)} (t_{d,j} - t_{d,j-1}) = \prod (b_d - a_d) = v(A)$$

which is positive so long as each $a_i < b_i$. □

Problem 9. 3.9 An unbounded set cannot have content zero, but can have measure zero.

Proof. If A has content zero then there is a finite cover of A by closed rectangles. Then each point of A lies in some closed rectangle, i.e. A is bounded.

The set $\{1, 2, 3, ...\}$ is clearly unbounded, but the cover $n \in [n - \frac{\varepsilon}{2^{n+1}}, n + \frac{\varepsilon}{2^{n+1}}]$ has volume ε, which is as small as we like. □

Problem 10. 3.10 A bounded set of content zero has boundary of content zero, but a bounded set of measure zero may have boundary with positive measure.

Proof. Using results from topology. Let C be a bounded set, $\overline{C}$ the closure of C and U_i a finite cover of C by closed rectangles. Then $U = U_i$ is a finite union of closed sets, hence closed. The closure of a set is by definition the smallest closed set and yet also contains that set's boundary, whence $\partial C \subset \overline{C} \subset U$ and so ∂C has content zero.

On the other hand the set $\mathbb{Q} \cap [0, 1]$ is bounded and has zero measure but its boundary is that whole interval. $\square$

Would be better to give a proof using only the theorems introduced in the book so far.

Problem 11. 3.11 Let $A \subset [0, 1]$ be the union over countably many intervals (a_i, b_i) such that each rational number in $(0, 1)$ is contianed in some interval. Problem $1 - 18$ states that $\partial A = [0, 1] - A$. Show that if each $b_i > a_i$ then A does not have zero measure.

Proof. Follows from the subaditivity of a measure, but since that tool has not been introduced need a different proof. $\square$

Problem 12. 3.12 Let $f : [a, b] \to \mathbb{R}$ be increasing. Then the set $X = \{x : f \text{ } \textit{discontinuous as } f\}$ has measure zero.

Proof. $\square$

Problem 13. 3.13 Any open cover admits a countable subcover.

Proof. The first result is that the collection of 'rational rectangles' - closed rectangles with all rational endpoints - is countable. This is a consequence of theorem which says a finite cartesian product of countable sets is countable, after identifying the rectangle in $\mathbb{Q}^n$ with a point in $\mathbb{Q}^{2n}$.

For the main result, let C_i be any open cover. For any x in our set, there is some C_i containing x. Then there is a closed rectangle B with $x \in B \subset C_i$. By a generalisation of the argument that any interval contains infinitley many rational numbers, there is a rational rectangle A_x with $x \in A_x \subset B \subset C_i$. Then the union of all the A_x is obviously a cover, which cannot be larger than $\cup_{\mathbb{N}} A_j$ since there are only countably many rational rectangles A. Then each x is in some $A_j \subset C_j$, whence C_j is a cover. $\square$

Problem 14. 3.14 If f and g are integrable then so is $f \cdot g$.

Proof. Let D_f, D_g be the sets on which f and g are not continuous. Then (Theorem 3.8) both are measure zero and so (Theorem 3.4 / subaditivity of measure) $D_{f \cdot g} \subset D_f \cup D_g$ is measure zero. By Theorem 3.8 again $f \cdot g$ is integrable. $\square$

Problem 15. 3.15 If C has content zero then C is Jordan Measurable and then volume of C is zero.

Proof. Since C is content zero it can be covered by finitely many closed rectangles. Let a_i, b_i be the smallest and largest of the endpoints in each dimension, then define $A = \prod[a_i, b_i]$. Then A contains C which is therefore bounded. By problem 3.10 the boundary of C has content zero, hence measure zero and so the C is Jordan Measurable.

To compute the integral take $\varepsilon > 0$ and let $U_1, ..., U_n$ be some cover of C which has volume smaller than ε. Let P be some partition of A consisting of each U together with other rectangles V. Then

$$L(\chi_C, P) = \sum 1 \cdot v(U) + \sum 0 \cdot v(V) = \sum v(U) < \varepsilon$$

and so $\int_C 1 = 0$.

$\square$

Problem 16. 3.16 The same does not hold for bounded sets of measure zero.

Proof. Consider $\mathbb{Q} \cap [0, 1]$. The set is bounded and has zero measure but (becasue the boundary is large) not integrable. If P is any partition, then each subrectangle S contains both rational and irrational numbers, so $M_s(\chi_C) = 1$ and $m_s(\chi_C) = 0$. Clearly the upper and lower sums over P can never agree. $\square$

Problem 17. 3.17 If C is a bounded set of measure zero and $\int_C 1$ exists then it must be zero.

Proof. Let A be some rectangle containing C so that $\int_C 1 = \int_A \chi_C$ and let P be a partition of A. Then

$$L(\chi_C, P) = \sum m(\chi_C, S) \cdot v(S) + \sum m(\chi_C, Q) \cdot v(Q) = \sum v(S)$$

where S are the subrectangles contained in C and Q are the rest. The inclusion $S \subset C$ implies S is also zero measure, and so by problem 3.8 is a degenerate rectangle. This is not allowed in our definition of a partition and so all subrectangles are of the type Q, so $L(\chi_C, P) = 0$ for any partition P. We have assumed that the integral exists, and so its value must be zero. $\square$

Problem 18. 3.18 If $f : A \to \mathbb{R}$ is non-negative and $\int f = 0$ then the support of f has measure zero.

Proof. For any integer n and $\varepsilon > 0$ let P be a partition with $\sum M_S(f) \cdot v(S) < \varepsilon/n$. Then either $M_S(f) > 1/n$ or $M_S(f) \leq 1/n$ and so

$$\sum_{M \leq 1/n} M_S(f) \cdot v(S) + \sum_{M > 1/n} \frac{1}{n} v(S) < \varepsilon/n$$

from which (since both sums are finite)

$$\sum_{i=1,...k} v(S_i) < \varepsilon.$$

If D_n is defined as $\{x : f(x) > 1/n\}$ then it is surely covered by these S, which the argument above shows to be a cover of content zero. The support of $supp(f) = D_1 \cup D_2 \cup ...$ is a countable union of measure zero sets, hence measure zero. $\square$

Problem 19. 3.19 Let U be the open set in problem 3.11. If $f = \chi_U$ except on a set of measure zero then f is not integrable.

Proof. ? $\square$

Problem 20. 3.20 An increasing function from $[a, b]$ to $\mathbb{R}$ is integrable.

Proof. By problem 3.12 the subset of the domain where the function is discontinuous has measure zero. The function is therefore integrable by theorem 3.8. $\square$

Problem 21. 3.21 Let A be a closed rectangle and $C \subset A$ and let P be some partition in which every subrectangle is either contained in C, or intersects C (without being contained.) That is, there are no subrectangles in $A - C$.

Proof. ? $\square$

Problem 22. 3.22 If A is a JM set and $\varepsilon > 0$ there is a compact JM set $C \subset A$ such that $\int_{A-C} 1 < \varepsilon$.

Proof. Baffling statement. $\square$

Problem 23. 3.23 Let $C = A \times B$ be a set of content zero. Let $A' \subset A$ be the set of $x \in A$ such that $\{y \in B : (x, y) \in C\}$ is not content zero. Then A' is measure zero.

Proof. By problem 3.15 C is Jordan Measurable and has volume 0. $\square$

Problem 24. 3.24 The result of problem 3.23 does not hold in general for sets of measure zero.

Proof. Let $C \subset [0, 1] \times [0, 1]$ be the union of all $\{p/q\} \times [0, 1]$ where p/q is a rational number in $[0, 1]$ with p/q in lowest terms. It's a werid comb kind of thing. The C has content zero as the bounded countable union of measure zero sets, but for any p/q the set $\{y \in [0, 1] : (p/q, y) \in C\} = [0, p/q]$ is not content zero for any p/q, i.e. A' is all rationals in $[0, 1]$, which is not content zero. $\square$

Problem 25. 3.25 Non-degenerate rectangles have positive measure. (Alternative proof to problem 3.8. Useful, because that proof sucks.)

Proof. Seems to be more going on than I thought. $\square$

Problem 26. 3.26 Let $f : [a, b] \to \mathbb{R}$ be integrable and non-negative and let A_f be the region between the graph of $f(x)$ and the x-axis. Then A_f is Jordan Measurable and has area $\int_a^b f$.

Proof. The boundary of A consists of three bounded line segments and the graph of f. Since f is integrable, it is continuous almost everywhere and so the the graph is a countable union[1] of continuous plane curves. Together with the obvious fact that A_f is bounded establishes that A_f is Jordan measurable.

[1] A measure zero subset of an interval must be a countable union of singletons.

Let M be some upper bound for f. Then

$$\mathcal{L}(x) = \int_{[0,M]} \chi_{A_f}(x,y)dy = f(x)$$

and so by Fubini the volume of A_f is

$$\int_{A_f} 1 = \int_{[a,b]\times[0,M]} \chi_{A_f} \int_{[a,b]} \mathcal{L}(x) = \int_a^b f.$$

□

Problem 27. 3.27 If $f : [a,b] \times [a,b] \to \mathbb{R}$ is continuous then

$$\int_a^b \int_a^y f(x,y)dxdy = \int_a^b \int_x^b f(x,y)dydx.$$

Proof. Let C be the region above the diagonal, that is

$$\{(x,y) : y \in [a,b] \textit{ and } x \in [a,y]\} = \{(x,y) : x \in [a,b] \textit{ and } y \in [x,b]\}.$$

Then

$$\int_C f = \int_{[a,b],[a,b]} \chi_C \cdot f = \int_{[a,b]} \left(\int_{[a,b]} (f \cdot \chi_C)(x,y)dy \right) dx = \int_a^b \left(\int_x^b f dy \right) dx$$

with the other side following from a similar calculation. □

Problem 28. 3.28 $D_{1,2}f = D_{2,1}f$ so long as these are continuous.

Proof. ? □

Problem 29. 3.29 Find the volume of the solid obtained by rotating a JM set in the yz plane about the z-axis.

Proof. ? □

Problem 30. 3.30 Let C be the set in problem 1.17 (the evil dense subset of the unit square containing at most one point on any horizontal or vertical line.) Then both $\int_0^1 \int_0^1 \chi_C dydx$ and $\int_0^1 \int_0^1 \chi_C dxdy$ are zero but the integral over C does not exist.

In other words, the converse of Fubini's theorem does not hold in general.

Proof. For any fixed x, the integral $\int_0^1 \chi_C(x,y)dy$ is clearly zero since C contains at most a single point on the line $\{x\} \times [0,1]$ from which it follows that the first double integral is 0. The argument for the other one is exactly the same.

Since C is dense, any rectangle contains both a point inside C and a point outside C and so $M_S = 1, m_S = 0$ for any rectangle S. Clearly the upper and lower integrals cannot coincide. □

Problem 31. 3.31 For $A = [a_1, b_1] \times ... \times [a_n, b_n]$ and $f : A \to \mathbb{R}$ define $F : A : \mathbb{R}$ by

$$F(x) = \int_{[a_1,b_1]\times...\times[a_n,b_n]} f.$$

What is $D_i f(x)$ for x in the interior of A?

Proof. ? $\square$

Problem 32. 3.32: Leibnitz' Rule Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be continuous and suppose $D_2 f$ is continuous. Define $F(y) = \int_a^b f(x, y)dx$. Then $F'(y) = \int_a^b D_2 f(x, y)dx$.

Proof. Rewrite $F(y)$ as

$$\int_a^b \int_c^y D_2 f(x, t)dtdx + \int_a^b f(x, c)dx.$$

By continuity of $D_2 f$ the order of integration may be reversed. The result follows from the fundamental theorem of calculus applied to the first integral and the y-independence of the second. $\square$

Problem 33. 3.33 If $f : [a, b] \times [c, d] \to \mathbb{R}$ is continuous, define $F(x, y) = \int_a^x f(t, y)dt$. Find $D_1 F$ and $D_2 F$.

Proof. By the FTC, $D_1 F(x, y) = f(x, y)$ and by Leibnitz' rule $D_2 F(x, y) = \int_a^b D_2 f(t, y)dt$. $\square$

Problem 34. 3.34 Let $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable and suppose $D_1 g_2 = D_2 g_1$. Define

$$f(x, y) = \int_0^x g_1(t, 0)dt + \int_0^y g_2(x, t)dt.$$

Then $D_1 f(x, y) = g_1(x, y)$.

Proof. By FTC and Leibnitz' rule

$$\begin{aligned} D_1 f(x, y) &= g_1(x, 0) + \int_0^y D_1 g_2(x, t)dt \\ &= g_1(x, 0) + \int_0^y D_2 g_1(x, t)dt \\ &= g_1(x, 0) + g_1(x, y) - g_1(x, 0) \\ &= g_1(x, y). \end{aligned}$$

$\square$

Problem 35. 3.35 If $G : \mathbb{R}^n \to \mathbb{R}^n$ is a invertible linear transformation and U a rectangle, then the volume of $G(U)$ is $|\det G| \cdot v(U)$.

Proof. Let $U = [a_1, b_1] \times ... \times [a_n, b_n]$ and consider first of all three kinds of elementary transformations.

1. G takes one of the basis vectors e^i to ce^i and fixes the rest. If c is positive then

$$\begin{aligned} v(G(U)) &= v([a_1, b_1] \times ... \times [ca_i, c_b i] \times ... \times [a_n, b_n]) \\ &= (b_1 - a_1) \cdot ... \cdot (cb_i - ca_i) \cdot ... \cdot (b_n - a_n) \\ &= c \cdot v(U). \end{aligned}$$

   The matrix representation of G is the identity with the i-th entry replaced by c, which has determinant c. If c is negative then the absolute value signs are needed but the volume is unchanged, although $[a, b]$ must become $[cb, ca]$.

2. G swaps two dimensions and fixes the rest, i.e. $g(e_i) = e_j$ and $g(e_j) = e_i$. As a matrix G is the identity with zeros in the i and j-th positions, and ones in the $G_{i,j}$ and $G_{j,i}$ positions. In $\mathbb{R}^3$ it might look like

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

   and in any dimension larger than one has determinant -1. Clearly this operation has no effect on the volume of the rectangle

3. G adds basis vector to another and fixes the rest, i.e.

$$G(x^1, ..., x^i, ..., x^j, ..., x^n) = (x^1, ..., x^i + x^j, ..., x^j, ..., x^n).$$

   As a matrix G is the identity with a single extra one somewhere, and so it is easy to see that $\det G = 1$.

   Linear operators on $\mathbb{R}^n$ are bounded, so the volume of $G(U)$ can be calculated as

$$\int_{[-M_n, M_n] \times ... \times [-M_1, M_1]} \chi_{G(U)} dx^1 ... dx^n$$

   for some upper bound M of both U and $G(U)$. Since Fubini's theorem will allow us to interchange the order of integration, we can assume that G adds e_2 to e_1 and leaves the rest. Then $v(G(U)) =$

$$\int_{[-M_n, M_n] \times ... \times [-M_3, M_3]} \left(\int_{[-M_2, M_2] \times [-M_1, M_1]} \chi_{G(U)}(x^1, x^2, ...) dx^1 dx^2 \right) dx^3 ... dx^n.$$

   The value of the inner integral is the area of a parallelogram, which coincides with the area of the rectangle which fell over. Hence this inner integral would not be changed by replacing $\chi_{G(U)}$ with χ_U. Furthermore, for fixed a^1 and a^2 and varying coordinates in the remaining positions, $\chi_U(a_1, a_2, x_3, ..., x_n)$ and $\chi_{G(U)}(a_1, a_2, x_3, ..., x_n)$ conincide exactly. So for the purposes of evaulating this integral, χ_U will do just as well as $\chi_{U(G)}$, or in other words their volumes are equal.

Finally if G is the composition of elementary operations as described above then G alters the volume of U by a factor of

$$|\det G_1| \cdot .. \cdot |\det G_m| = |\det G_1 \cdot ... \cdot \det G_m| = |\det(G_1 \cdot ... \cdot G_m)| = |\det G|.$$

We take as a fact that these elemetary operations generate the whole group of invertible matrices and so this scaling factor is good for any invertible linear map. $\square$

Problem 36. 3.36 Cavalieri's Principle. Let A and B be JM subsets of $\mathbb{R}^3$. Let $A_c = \{(x,y) : (x,y,c) \in A\}$ and define B_c similarly. If each A_c and B_c are JM and have the same area then A and B have the same volume.

Proof. Let $[-M,M]^3$ be a rectangle bounding A and B. Then

$$\begin{aligned}\int_{[-M,M]^3} \chi_A &= \int_{-M}^{M} \left(\int_{[-M,M]^2} \chi_A \right) dz && (1)\\ &= \int_{-M}^{M} \left(\int_{A_z} 1 \right) dz = \int_{-M}^{M} \left(\int_{B_z} 1 \right) dz && (2)\\ &= \int_{-M}^{M} \left(\int_{[-M,M]^2} \chi_B \right) dz = \int_{[-M,M]^3} \chi_B. && (3)\end{aligned}$$

□

Problem 37. 3.37 Suppose that $f : (0,1) \to \mathbb{R}$ is non-negative and continuous. Then $\int_{(0,1)} f$ exists if and only if $\lim_{\varepsilon \to 0} \int_{\varepsilon}^{1-\varepsilon} f$ exists.

Proof. Let o be the cover of $(0,1)$ consisting of the intervals $(1/n, 1-1/n)$ and Φ a C^∞ partition of unity for $(0,1)$ subordinate to o. Then each ϕ_i in Φ has support contained in some $(1/m_i, 1-1/m_i)$ in o. Choose an ordering of Φ so that if $i < j$ then $m_i \leq m_j$, that way the support of ϕ_i is in $(1/m_j, 1-1/m_j)$ for all $j > i$.

Then the partial sums can be written as

$$\sum_{k=1}^{n} \int_{(0,1)} \phi_k \cdot f = \sum_{k=1}^{n} \int_{1/m_k}^{1-1/m_k} \phi_k \cdot f = \sum_{k=1}^{n} \int_{1/m_n}^{1-1/m_n} \phi_k \cdot f = \int_{1/m_n}^{1-1/m_n} \sum_{k=1}^{n} \phi_k \cdot f = \int_{1/m_n}^{1-1/m_n} f.$$

Since $\phi \cdot f$ is non-negative, convergence and absolute conergence are the same thing. Thus the extended integral exists if and only if the given limit exists. □

Problem 38. 3.38 Let A_n be a closed set in $(n, n+1)$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $\int_{A_n} f = (-1)^n/n$ and $f = 0$ outside A_n. Then f is not integrable.

Proof. Let Φ_n be a partition of unity for A_n and extend each ϕ to a function on $\mathbb{R}$ by setting $\phi = 0$ outside A_n. Then their union Φ is a partion of unity for all of $\mathbb{R}$ and

$$\sum_{\Phi} \int_{\mathbb{R}} \phi \cdot |f| = \sum_{\mathbb{N}} \left(\sum_{\Phi_n} \int_{\mathbb{R}} \phi \cdot |f| \right) = \sum_{\mathbb{N}} \left(\sum_{\Phi_n} \int_{A_n} \phi \cdot |f| \right) = \sum_{\mathbb{N}} \int_{A_n} f = -\log 2. \quad (4)$$

Following the example in the wiki for conditional convergence, define the sets

$$B_n = A_{2n-1} \cup A_{2(2n-1)} \cup A_{4n} \quad (5)$$

and let Φ_n be a partition of unity for B_n, taking the value 0 outside B_n. Then as above

$$\sum_{\Phi} \int_{\mathbb{R}} \phi \cdot |f| = \sum_{\mathbb{N}} \left(\sum_{\Phi_n} \int_{\mathbb{R}} \phi \cdot |f| \right) = \sum_{\mathbb{N}} \left(\sum_{\Phi_n} \int_{B_n} \phi \cdot |f| \right) = \sum_{\mathbb{N}} \int_{B_n} f. \quad (6)$$

Each of the integrals in the last sum has the value

$$\int_{B_n} f = \int_{A_{2n-1}} f + \int_{A_{2(2n-1)}} f + \int_{A_{4n}} f \tag{7}$$
$$= \frac{(-1)^{2n-1}}{2n-1} + \frac{(-1)^{2(2n-1)}}{2(2n-1)} + \frac{(-1)^{4n}}{4n} \tag{8}$$
$$= \frac{1}{2(2n-1)} + \frac{1}{4n} - \frac{1}{2n-1} \tag{9}$$
$$= \frac{1}{2}\left(\frac{-1}{2n-1} + \frac{1}{2n}\right) \tag{10}$$

and so the sum converges to $-1/2 \log 2$. □

Problem 39. 3.39 The condition that $\det g$ in the formula for change of variables is superfluous, due to Sard's Theorem.

Proof. ? □

Problem 40. 3-40 If $g : \mathbb{R}^n \to \mathbb{R}^n$ and $\det g'(x) \neq 0$ then there is an open set containing x in which $g = T \circ g_n \circ ... \circ g_1$ where each g_i is of the form $g_i(x) = (x^i, ..., f_i(x^i), ..., x^n)$ and T is a linear transformation. Furthermore g can be written as $g = g_n \circ ... \circ g_1$ if and only if g' is diagonal.

Proof. ? □

Problem 41.

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}. \tag{11}$$

Proof.

1. Define $f : \{r : r > 0\} \times (0, 2\pi) \to \mathbb{R}^2$ by $f(r, \vartheta) = r(\cos\vartheta, \sin\vartheta)$. If $r(\cos\vartheta, \sin\vartheta) = r'(\cos\vartheta', \sin\vartheta')$ then the Pythagorean identity easily gives $r = r'$, and so

$$\cos\vartheta = \cos\vartheta', \quad \sin\vartheta = \sin\vartheta'.$$

□

4 Definitions

Theorem 1. *Let $A \subset \mathbb{R}^n$ and o an open cover of A. Then there is countable collection Φ of C^∞ functions ϕ defined on A with the following properties:*

1. *For any $x \in A$, $0 \leq \phi(x) \leq 1$.*
2. *For any $x \in A$ there is an open set V containing x such that all but finitely many ϕ are zero on V.*
3. *For each $x \in A$, we have $\sum_\Phi \phi(x) = 1$ and by (2) this sum is finite in some open set around x.*

4. For each $\phi \in \Phi$ there is an open set $U \in o$ such that $\phi = 0$ outisde some closed set contained in U (the support of each ϕ is compact and contained in some $U \in o$.)

A colletion satisfying (1) - (3) is called a C^∞ partition of unity for A. If it also satisfies (4) it is said to be subordinate to the cover o.

Definition 1. Let o be an admissible open cover of an open set $A \subset \mathbb{R}^n$ (each $U \in o$ is contained in A) and Φ a partition of unity subordinate to o. If $f : A \to \mathbb{R}$ is bounded in some open set around each point in A and the subset of A where f is discontinuous is measure zero then, each $\int_A \phi \cdot f$ exists. If the series $\sum_\Phi \int_A \phi \cdot |f|$ converges then we say f is integrable write $\int_A f = \sum_\Phi \int_A \phi \cdot f$.

Definition 2. A $k - tensor$ is a multilinear map function from $V^k \to \mathbb{R}$ where V is some vector space. The set of all k-tensors is denoted $\mathfrak{T}^k(V)$ and becomes a is a vector space over $\mathbb{R}$. If $S \in \mathfrak{T}^k(V)$ and $T \in \mathfrak{T}^l(V)$ define a tensor-product:

$$S \otimes T(v_1, ..., v_k, v_{k+1}, ..., v_{k+l}) = S(v_1, ..., v_k) \cdot T(v_{k+1}, ..., v_{k+l}). \tag{12}$$

This product is left and right distributive and associative.

Theorem 2. *Let $v_1, ..., v_n$ be a basis for V, and let $\varphi_1, ..., \varphi_n$ be the dual basis, $\varphi_i(v_j) = \delta_{ij}$. Then the set of all k-fold tensor products of φ_i is a basis for $\mathfrak{T}^k(V)$, which therefore has dimension n^k.*
Definition 3. A k-tensor is alternating if

$$a(v_1, ..., v_i, v_j, ..., v_k) = -a(v_1, ..., v_j, v_i, ..., v_k) \tag{13}$$

i.e. two elements are swapped and the rest left alone.

Definition 4. If T is some k-tensor, then you can have an alternating tensor for free:

$$Alt(T) = \frac{1}{k!} \sum_{\sigma \in S_k} sign(\sigma) T(v_{\sigma 1}, ..., v_{\sigma k}) \tag{14}$$

where S_k is the group of permutations on k letters.

In general, the tensor product of two alternating tensors is not likely to be alternating. Enter the wedge:
Definition 5. Let $\omega \in \Omega^k(V)$ and $\nu \in \Omega^l(V)$. Then

$$\omega \wedge \nu = \frac{(k+l)!}{k!l!} Alt(\omega \times \nu) \tag{15}$$

is in $\Omega^{k+l}(V)$.