Integration on Manifolds

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1 Definitions

1.1 Tensors

Definition 1. A $k-$tensor is a multilinear map from $V^k \to \mathbb{R}$ where V is a vector space. The set of all k-tensors is denoted $\mathfrak{T}^k(V)$. If $S \in \mathfrak{T}^k(V)$ and $T \in \mathfrak{T}^l(V)$ define a tensor-product:

$$S \otimes T(v_1, ..., v_k, v_{k+1}, ..., v_{k+l}) = S(v_1, ..., v_k) \cdot T(v_{k+1}, ..., v_{k+l}).$$

This product is left and right distributive and associative if addition is defined the obvious way.
Theorem 1. *Let $v_1, ..., v_n$ be a basis for V, and let $\varphi_1, ..., \varphi_n$ be the dual basis, $\varphi_i(v_j) = \delta_{ij}$. Then the set of all*

$$\varphi_{i_1} \otimes ... \otimes \varphi_{i_k}, \quad 1 \leq i_1 \leq ... \leq i_k \leq n$$

is a basis for $\mathfrak{T}^k(V)$, which therefore has dimension n^k.
Definition 2. A k-tensor is alternating if

$$a(v_1, ..., v_i, v_j, ..., v_k) = -a(v_1, ..., v_j, v_i, ..., v_k)$$

i.e. two elements are swapped and the rest left alone.
Definition 3. If T is some k-tensor, then you can have an alternating tensor for free:

$$Alt(T) = \frac{1}{k!} \sum_{\sigma \in S_k} sign(\sigma) T(v_{\sigma 1}, ..., v_{\sigma k})$$

where S_k is the group of permutations on k letters.

In general, the tensor product of two alternating tensors is not likely to be alternating. Enter the wedge:
Definition 4. Let $\omega \in \Omega^k(V)$ and $\nu \in \Omega^l(V)$. Then

$$\omega \wedge \nu = \frac{(k+l)!}{k!l!} Alt(\omega \otimes \nu)$$

is in $\Omega^{k+l}(V)$.

1.2 Fields and Forms

Definition 5. For $p \in \mathbb{R}^n$ the tangent space of $\mathbb{R}^n$ at p is the set of all $(p, v), v \in \mathbb{R}^n$. Write v_p and say 'the vector v at p.'

Definition 6. The tangent space inherits the inner product from $\mathbb{R}^n$, by defining

$$\langle v_p, w_p \rangle_p = \langle v, w \rangle.$$

Definition 7. For the standard orientation in $\mathbb{R}^n_p$, the sensible choice is

$$[(e_1)_p, ..., (e_n)_p].$$

Definition 8. A vector field F is a function such that

$$F(p) \in \mathbb{R}^n_p$$

for every $p \in \mathbb{R}^n$. The component functions of F are the functions $F_i : \mathbb{R}^n \to \mathbb{R}$ such that

$$F(p) = F_1(p) \cdot (e_1)_p + ... + F_n(p) \cdot (e_n)_p.$$

Definition 9. A $k-$form or differential form is a function on $\mathbb{R}^n$ such that

$$\omega(p) \in \Omega^k(\mathbb{R}^n_p).$$

A $0-$form f takes a vector p and returns a tensor of rank 0, that is a number. Hence a f is considered a function from $\mathbb{R}^n$ to $\mathbb{R}$. Forms inherit the operations

$$\begin{aligned}(\omega + \mu)(p) &= \omega(p) + \mu(p) \\ (f \cdot \omega)(p)(v_1, ...v_k) &= f(p) \cdot \omega(p)(v_1, ..., v_k) \\ (\omega \wedge \mu)(p)(v_1, ..., v_k) &= \omega(p)(v_1, ...v(k)) \wedge \mu(p)(v_1, ..., v_k).\end{aligned}$$

If $\varphi_1(p), ..., \varphi_n(p)$ is the dual basis to $(e_1)_p, ..., (e_n)_p$ then any k-form can be written as

$$\omega(p) = \sum_{i_1 < ... < i_k} \omega_{i_1,...,i_k}(p) \cdot [\varphi_{i_1}(p) \wedge ... \wedge \varphi_{i_k}(p)]$$

for numbers $\omega i_1, ..., i_k$. The form ω is said to be continuous if each of these functions is.

Definition 10. If f is a $0-$form (a function $\mathbb{R}^n \to \mathbb{R}$) define a $1-$form df, the 'differential of f at p' by

$$df(p)(v_p) = Df(p)(v).$$

For example, writing x^i for the projection map π^i we have

$$dx^i(p)(v_p) = Dx^i(p)(v) = v^i$$

and so $dx^1(p), ..., dx^n(p)$ is the dual basis to $(e_1)_p, ..., (e_n)_p$. In particular, df in this basis is

$$df(p)(v_p) = D_1 f(p) \cdot dx^1(p)(v_p) + ... + D_n f(p) \cdot dx^n(p)(v_p)$$

or without arguments

$$df = D_1 f \cdot dx^1 + ... + D_n f \cdot dx^n.$$

Definition 11. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable define the transformation induced by f as

$$f_*(v_p) = (Df(p)(v))_{f(p)}.$$

Definition 12. For f as above we have $f_* : \mathbb{R}^n_p \to \mathbb{R}^m_{f(p)}$. Write f^* for the dual map[1] , i.e. if $T \in \Omega^k(\mathbb{R}^n_f(p))$ then

$$f^* : \Omega^k(\mathbb{R}^m_{f(p)}) \to \Omega^k(\mathbb{R}^n_p)$$
$$f^*(T)(v_1, ..., v_k) = T(f_*(v_1), ..., f_*(v_k)).$$

If ω is a k-form on $\mathbb{R}^m$ define the k-form $f^*\omega$ on $\mathbb{R}^n$ by $f^*\omega(p) = f^*(\omega(f(p)))$. To be explicit,

$$f^*(\omega(p))(v_1, ..., v_k) = \omega(f(p))(f_*(v_1), ..., f_*(v_k)).$$

Definition 13. Let ω be a k-form:

$$\omega = \sum_{i_1 < ... < i_k} \omega_{i_1,...,i_k} dx^{i_1} \wedge ... \wedge dx^{i_k}.$$

Define the 'differential of ω' by

$$d\omega = \sum_{i_1 < ... < i_k} d\omega_{i_1,...,i_k} dx^{i_1} \wedge ... \wedge dx^{i_k} = \sum_{i_1 < ... < i_k} \sum_{\alpha=1}^{n} D_\alpha \omega_{i_1,...,i_k} \cdot dx^\alpha \wedge dx^{i_1} \wedge ... \wedge dx^{i_k}.$$

Definition 14. A form ω is called closed if $d\omega = 0$ and exact if $\omega = d\mu$ for some μ.

1.3 Chains

Definition 15. A singular n-cube in $A \subset \mathbb{R}^m$ is a continuous function from $[0, 1]^n$ to A. Let $\mathbb{R}^0$ and $[0, 1]^0$ denote $\{0\}$. Then a singular 0-cube is a function from $\{0\}$ to A, that is a point in A. A singular 1-cube is often called a curve.

The book defines a n-cubes as a function from $[0, 1]^n$ to a subset of Euclidean space of the same dimension. This seems to be an error, for the following reason. The definition of a singular 0-cube is given as a function from $\{0\}$ to A, which must therefore be a subset of $\mathbb{R}^0$, i.e. the only singular 0-cube is the function $c : \{0\} \to \{0\}$, definitely at odds with the language of *a* singular 0-cube. Similarly the definition of a curve as a singular 1-cube is unnatural if the codomain is only one dimensional.

Definition 16. The standard n-cube I^n is the identity restricted to $[0, 1]^n$.

Definition 17. Let S be the set of all n-cubes in A. An n-chain in A is a function $f : S \to \mathbb{Z}$ such that all but finitely many cubes are mapped to zero. Define $f + g$ by $(f + g)(c) = f(c) + g(c)$ and $nf(c) = n \cdot f(c)$.

Definition 18. Starting with the standard n-cube define, for each $1 \leq i \leq n$, the $(i, 0)$ and $(i, 1)$ faces as the following functions on $[0, 1]^{n-1}$

$$I^n_{(i,0)}(x) = I^n(x^1, ..., x^{i-1}, 0, x^i, ..., x^{n-1})$$
$$I^n_{(i,1)}(x) = I^n(x^1, ..., x^{i-1}, 1, x^i, ..., x^{n-1})$$

[1]How can this be made precise?

Define the boundary of the standard n-cube as

$$\partial I^n = \sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha} I^n_{(i,\alpha)}.$$

For a general singular n-cube $c : [0,1]^n \to A$ define the (i,α) faces as $c_{(i,\alpha)} = c \circ I^n_{(i,\alpha)}$ and the boundary of c as

$$\partial c = \sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha} c_{(i,\alpha)}.$$

Finally define the boundary of a sum (i.e. a chain) as the sum of the boundaries:

$$\partial \sum a_i c_i = \sum a_i \partial c_i.$$

2 Issues

These are incomplete/problematic:

- 3.10 Would like a proof using the notions developed here rather than topological facts.
- 3.11, 3.12, 3.19, 3.25, 3.28, 3.29, 3.31, 3.40, 3.41 Needs proof.
- Still not sure I understand what 3.21 is getting at. Is it asking for a partition which doesn't contain any rectangles in $A - C$? If so its easy to draw JM sets with two connected components which seem to break this idea.
- 3.33 (b) straight up makes no sense. f is defined on $[a,b] \times [c,d]$ and $G(x)$ is defined, nonsensically, as $\int_a^{g(x)} f(t,x)dt$.
- Be more clear on the hypothesis of Fubinis for switching order in 3.35.
- 3-37 (b) is reportedly broken. Still needs doing with an improved hypothesis.
- 4.4 might need a better comment on orientation.
- Almost all the problems on tensors...
- Fix the brackets in 4.13.

It would be good to check whether an argument about absolute convergence should ever be made explicitly.

In other proofs Lemma 3.1 is used generously, in the sense that if (P') is a family of partitions subject to some condition which does not restrict the 'fine-ness' of P' then each P' is a refinement of some generic partition and so any result involving an infinumum/suprememum over all partitions holds equally well over the restricted set of partitions. Would be good to clear this up.

Personally I think the original statement 3.21 is pretty confusing. It could state explicitly that the partition contains no subrectangles contained in $A - C$.

3 Problems

Problem. 3.1 Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 0 & \text{if } 0 \leq x < \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Then f is integrable and the value of the integral is $\frac{1}{2}$.

Proof. For $\varepsilon > 0$ let P_ε be the partition with subrectangles

$$\begin{aligned} S_1 &= [0, \frac{1}{2} - \frac{\varepsilon}{4}] \times [0,1] \\ S_2 &= [\frac{1}{2} - \frac{\varepsilon}{4}, \frac{1}{2} + \frac{\varepsilon}{4}] \times [0,1] \\ S_3 &= [\frac{1}{2} + \frac{\varepsilon}{4}, 1] \times [0,1] \end{aligned}$$

Then $U(f, P_\varepsilon) - L(f, P_\varepsilon) = \dfrac{\varepsilon}{2} < \varepsilon$ i.e. f is integrable and $\int f = \inf U(f, P_\varepsilon) = \frac{1}{2}$. □

Problem. 3.2 Let $f : A \to \mathbb{R}$ be integrable and $g = f$ except at finitely many points. Then g is integrable and $\int f = \int g$.

Proof. Suppose f and g differ except at a single point x. Let P be some partition fine enough that x lies in a single subrectangle S. Then

$$(U - L)(g, P) = (U - L)(f, P) + (M_S - m_S)(g - f) \cdot v(S)$$

can be made arbitrarily small, as the term $(U - L)(f, P)$ vanishes by integrability of f and the other term can be made as small as we like by insiting that the rectangle S is sufficiently small. Furthermore if $g(x) > f(x)$ then

$$\int g = \inf[U(g, P)] = \inf[U(f, P) + (M_S(g - f) \cdot v(S))] = \inf[U(f, P)] = \int f$$

with a similar argument for when $g(x) < f(x)$. Repetition of the argument proves the result for a finite collection of points. □

Problem. 3.3 Linearity of the Integral. Let $f, g : A \to \mathbb{R}$ be integrable. Then $\int f + g = \int f + \int g$ and $\int cf = c \int f$.

Proof. Since

$$m_S(f) + m_S(g) \leq f(x) + g(x) \quad \forall x \in S$$

it follows that

$$m_S(f) + m_S(g) \leq \inf_S (f + g) = m_S(f + g).$$

Then $L(f, P) + L(g, P) \leq L(f + g, P)$ follows easily and by a similar argument $U(f + g, P) \leq U(f, P) + U(g, P)$. Hence the differece $(U - L)(f + g, P)$ is bounded from above by $(U - L)(f, P) + (U - L)(g, P)$ which vanishes by integrability of f and g.

To compute the value of the integral,

$$\int f + \int g = \sup L(f,P) + \sup L(g,P) = \sup[L(f,P) + L(g,P)] \leq \sup L(f+g,P) = \int f + g$$
$$\int f + \int g = \inf U(f,P) + \inf U(g,P) = \inf[U(f,P) + U(f,P)] \geq \inf U(f+g,P) = \int f + g$$

and so $\inf f + \int g = \int f + g$.

Finally for constants c,

$$\inf U(cf,P) = c \inf U(f,P) = c \sup L(f,P) = \sup L(cf,P)$$

i.e cf is integrable with the value $\int cf = \inf U(cf,P) = c \inf U(f,P) = c \int f$. $\square$

Problem. 3.4 Let $f : A \to \mathbb{R}$ and let P be a partition of A. Then f is integrable if and only if the restriction of f to each subrectangle is integrable, and in this case $\int_A f = \sum_P \int_S f|_S$.

Proof. Let the n subrectangles of P be S_i and suppose that each $f|_{S_i}$ is integrable. Then for any $\varepsilon > 0$ and each i there is a partitions P_i of S_i such that $(U-L)(f|_{S_i}, P_i) < \varepsilon/n$. Then

$$\sum_{S_i \in P} \sum_{S_{ij} \in P_i} [(M_{S_{ij}} - m_{S_{ij}})(f|S_i) \cdot v(S_{ij})] < \varepsilon \Longrightarrow \sum_{S_i \in P} \sum_{S_{ij} \in P_i} [(M_{S_{ij}} - m_{S_{ij}})(f) \cdot v(S_{ij})] < \varepsilon$$

where the latter double sum is $(U-L)f$ over the partition formed by taking the union of all the P_i. Hence f is integrable.

On the other hand for any rectangle $S \in P$ we have

$$(U-L)(f|_S, S) = (U-L)(f,S) \leq (U-L)(f,P)$$

since $U - L$ is non-negative and S is containined in P. If f is integrable then the RHS can be made arbitrarily small by refining P, in which case S may need to be replaced by the part of that refinement which covers it, $S = \cup P'$. Then $(U-L)(f|_S, P')$ vanishes and so $f|_S$ is integrable.

To compute the integral we have, for partitions P_S of each S

$$\sum_P \int_S f|_S = \sum_P \sup[L(f|_S, P_S)] = \sup\left[\sum_P L(f|_S, P_s)\right] = \sup\left[\sum_P \sum_{P_i} m_{S_ij}(f) \cdot v(S_{ij})\right].$$

The last term is the supremum over partitions P' which are generic partitions of A but for the refinement that each rectangle must be contained in one of the original $S \in P$. Hence the supremum over such partitions coincides with the supremum over generic ones and so $\sum \int f|_S = \int f$. $\square$

Problem. 3.5 Let $g, f : A \to \mathbb{R}$ be integrable and suppose $f \leq g$. Then $\int f \leq \int g$.

Proof.

$$\int f \quad = \quad \inf \sum M_S(f) \cdot v(S) \quad \leq \quad \inf \sum M_S(g) \cdot v(S) \quad = \quad \int g.$$

$\square$

Problem. If $f : A \to \mathbb{R}$ is integrable then $|f|$ is integrable and $\left|\int f\right| \leq \int |f|$.

Proof.

$$(U - L)(|f|, P) = \sum (M_S(|f|) - m_S(|f|)) \cdot v(S) \leq \sum (M_S(f) - m_S(f)) \cdot v(S)$$

which is as small as we like by integrability of f. The required comparison follows from taking infinums of

$$\left|\sum M_s(f) \cdot (S)\right| \leq \sum |M_s(f) \cdot (S)| = \sum M_s(|f|) \cdot (S).$$

□

Problem. 3.7 Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 0 & \text{if } x \text{ is irrational or y is irrational} \\ 1/q & \text{if } x \text{ is rational and } y = p/q \text{ in lowest terms.} \end{cases}$$

Then f is integrable and the integral is 0.

Proof. ? □

Problem. 3.8 The rectangle $A = \Pi[a_i, b_i] \subset \mathbb{R}^m$ does not have content zero if each $a_i < b_i$.

Proof. Let $U_1, ..., U_n$ be a finite cover of closed rectangles with each U_i contained in A and let $P_d = (a_d = t_{d0} < ... < t_{dk(d)} = b_d)$ be the partition containing all the endpoints of U_i in the dth dimension, $d = 1, .., m$. Then each $v(U_i) = \Pi_{d=1}^m \sum (t_{d,j} - t_{d,j-1})$ for a 'certain number' of summands. Moreover each $\Pi_{d=1}^m (t_{d,j} - t_{d,j-1})$ lies in at least one U_i, so

$$\sum v(U_i) \geq \prod_{d=1}^{m} \sum_{j=1}^{k(d)} (t_{d,j} - t_{d,j-1}) = \prod (b_d - a_d) = v(A)$$

which is positive so long as each $a_i < b_i$. □

Problem. 3.9 An unbounded set cannot have content zero, but can have measure zero.

Proof. If A has content zero then there is a finite cover of A by closed rectangles. Then each point of A lies in some closed rectangle, i.e. A is bounded.

The set $\{1, 2, 3, ...\}$ is clearly unbounded, but the cover $n \in [n - \frac{\varepsilon}{2^{n+1}}, n + \frac{\varepsilon}{2^{n+1}}]$ has volume ε, which is as small as we like. □

Problem. 3.10 A bounded set of content zero has boundary of content zero, but a bounded set of measure zero may have boundary with positive measure.

Proof. Using results from topology. Let C be a bounded set, $\overline{C}$ the closure of C and U_i a finite cover of C by closed rectangles. Then $U = \cup U_i$ is a finite union of closed sets, hence closed. The closure of C is by definition the smallest closed set containing C. Furthermore it can be shown that it also contains the bounday of C, whence $\partial C \subset \overline{C} \subset U$ and so ∂C has content zero.

On the other hand the set $\mathbb{Q} \cap [0, 1]$ is bounded and has zero measure but its boundary is that whole interval. □

Would be better to give a proof using only the theorems introduced in the book so far.

Problem. 3.11 Let $A \subset [0,1]$ be the union over countably many intervals (a_i, b_i) such that each rational number in $(0,1)$ is contianed in some interval. Problem $1-18$ states that $\partial A = [0,1] - A$. Show that if each $b_i > a_i$ then A does not have zero measure.

Proof. Follows from the subaditivity of a measure, but since that tool has not been introduced need a different proof. □

Problem. 3.12 Let $f : [a,b] \to \mathbb{R}$ be increasing. Then the set $X = \{x : f \textit{ discontinuous as } f\}$ has measure zero.

Proof. □

Problem. 3.13 Any open cover admits a countable subcover.

Proof. The first result is that the collection of 'rational rectangles' - closed rectangles with all rational endpoints - is countable. This is a consequence of theorem which says a finite cartesian product of countable sets is countable, after identifying the rectangle in $\mathbb{Q}^n$ with a point in $\mathbb{Q}^{2n}$.

For the main result, let C_i be any open cover. For any x in our set, there is some C_i containing x. Then there is a closed rectangle B with $x \in B \subset C_i$. By a generalisation of the argument that any interval contains infinitley many rational numbers, there is a rational rectangle A_x with $x \in A_x \subset B \subset C_i$. Then the union of all the A_x is obviously a cover, which cannot be larger than $\cup_{\mathbb{N}} A_j$ since there are only countably many rational rectangles A. Then each x is in some $A_j \subset C_j$, whence C_j is a cover. □

Problem. 3.14 If f and g are integrable then so is $f \cdot g$.

Proof. Let D_f, D_g be the sets on which f and g are not continuous. Then (Theorem 3.8) both are measure zero and so (Theorem 3.4 / subaditivity of measure) $D_{f \cdot g} \subset D_f \cup D_g$ is measure zero. By Theorem 3.8 again $f \cdot g$ is integrable. □

Problem. 3.15 If C has content zero then C is Jordan Measurable and then volume of C is zero.

Proof. Since C is content zero it can be covered by finitely many closed rectangles. Let a_i, b_i be the smallest and largest of the endpoints in each dimension, then define $A = \prod [a_i, b_i]$. Then A contains C which is therefore bounded. By problem 3.10 the boundary of C has content zero, hence measure zero and so the C is Jordan Measurable.

To compute the integral take $\varepsilon > 0$ and let $U_1, ..., U_n$ be some cover of C which has volume smaller than ε. Let P be some partition of A consisting of each U together with other rectangles V. Then

$$L(\chi_C, P) = \sum 1 \cdot v(U) + \sum 0 \cdot v(V) = \sum v(U) < \varepsilon$$

and so $\int_C 1 = 0$.

□

Problem. 3.16 The same does not hold for bounded sets of measure zero.

Proof. Consider $\mathbb{Q} \cap [0,1]$. The set is bounded and has zero measure but (becasue the boundary is large) not integrable. If P is any partition, then each subrectangle S contains both rational and irrational numbers, so $M_s(\chi_C) = 1$ and $m_s(\chi_C) = 0$. Clearly the upper and lower sums over P can never agree. □

Problem. 3.17 If C is a bounded set of measure zero and $\int_C 1$ exists then it must be zero.

Proof. Let A be some rectangle containing C so that $\int_C 1 = \int_A \chi_C$ and let P be a partition of A. Then

$$L(\chi_C, P) = \sum m(\chi_C, S) \cdot v(S) + \sum m(\chi_C, Q) \cdot v(Q) = \sum v(S)$$

where S are the subrectangles contained in C and Q are the rest. The inclusion $S \subset C$ implies S is also zero measure, and so by problem 3.8 is a degenerate rectangle. This is not allowed in our definition of a partition and so all subrectangles are of the type Q, so $L(\chi_C, P) = 0$ for any partition P. We have assumed that the integral exists, and so its value must be zero. □

Problem. 3.18 If $f : A \to \mathbb{R}$ is non-negative and $\int f = 0$ then the support of f has measure zero.

Proof. For any integer n and $\varepsilon > 0$ let P be a partition with $\sum M_S(f) \cdot v(S) < \varepsilon/n$. Then either $M_S(f) > 1/n$ or $M_S(f) \leq 1/n$ and so

$$\sum_{M \leq 1/n} M_S(f) \cdot v(S) + \sum_{M > 1/n} \frac{1}{n} v(S) < \varepsilon/n$$

from which (since both sums are finite)

$$\sum_{i=1,\ldots k} v(S_i) < \varepsilon.$$

If D_n is defined as $\{x : f(x) > 1/n\}$ then it is surely covered by these S, which the argument above shows to be a cover of content zero. The support of $supp(f) = D_1 \cup D_2 \cup \ldots$ is a countable union of measure zero sets, hence measure zero. □

Problem. 3.19 Let U be the open set in problem 3.11. If $f = \chi_U$ except on a set of measure zero then f is not integrable.

Proof. ? □

Problem. 3.20 An increasing function from $[a, b]$ to $\mathbb{R}$ is integrable.

Proof. By problem 3.12 the subset of the domain where the function is discontinuous has measure zero. The function is therefore integrable by theorem 3.8. □

Problem. 3.21 Let A be a closed rectangle and $C \subset A$ and let P be some partition in which every subrectangle is either contained in C, or intersects C (without being contained.) That is, there are no subrectangles in $A - C$.

Proof. ? □

Problem. 3.22 If A is a JM set and $\varepsilon > 0$ there is a compact JM set $C \subset A$ such that $\int_{A-C} 1 < \varepsilon$.

Proof. Baffling statement. $\square$

Problem. 3.23 Let $C = A \times B$ be a set of content zero. Let $A' \subset A$ be the set of $x \in A$ such that $\{y \in B : (x, y) \in C\}$ is not content zero. Then A' is measure zero.

Proof. By problem 3.15 C is Jordan Measurable and has volume 0. $\square$

Problem. 3.24 The result of problem 3.23 does not hold in general for sets of measure zero.

Proof. Let $C \subset [0, 1] \times [0, 1]$ be the union of all $\{p/q\} \times [0, 1]$ where p/q is a rational number in $[0, 1]$ with p/q in lowest terms. It's a werid comb kind of thing. The C has content zero as the bounded countable union of measure zero sets, but for any p/q the set $\{y \in [0, 1] : (p/q, y) \in C\} = [0, p/q]$ is not content zero for any p/q, i.e. A' is all rationals in $[0, 1]$, which is not content zero. $\square$

Problem. 3.25 Non-degenerate rectangles have positive measure. (Alternative proof to problem 3.8. Useful, because that proof sucks.)

Proof. Seems to be more going on than I thought. $\square$

Problem. 3.26 Let $f : [a, b] \to \mathbb{R}$ be integrable and non-negative and let A_f be the region between the graph of $f(x)$ and the x-axis. Then A_f is Jordan Measurable and has area $\int_a^b f$.

Proof. The boundary of A consists of three bounded line segments and the graph of f. Since f is integrable, it is continuous almost everywhere and so the the graph is a countable union[2] of continuous plane curves. Together with the obvious fact that A_f is bounded establishes that A_f is Jordan measurable.

Let M be some upper bound for f. Then

$$\mathcal{L}(x) = \int_{[0,M]} \chi_{A_f}(x, y) dy = f(x)$$

and so by Fubini the volume of A_f is

$$\int_{A_f} 1 = \int_{[a,b]\times[0,M]} \chi_{A_f} \int_{[a,b]} \mathcal{L}(x) = \int_a^b f.$$

$\square$

Problem. 3.27 If $f : [a, b] \times [a, b] \to \mathbb{R}$ is continuous then

$$\int_a^b \int_a^y f(x, y) dx dy = \int_a^b \int_x^b f(x, y) dy dx.$$

[2] A measure zero subset of an interval must be a countable union of singletons.

Proof. Let C be the region above the diagonal, that is

$$\{(x,y) : y \in [a,b] \text{ and } x \in [a,y]\} = \{(x,y) : x \in [a,b] \text{ and } y \in [x,b]\}.$$

Then

$$\int_C f = \int_{[a,b],[a,b]} \chi_C \cdot f = \int_{[a,b]} \left(\int_{[a,b]} (f \cdot \chi_C)(x,y) dy \right) dx = \int_a^b \left(\int_x^b f dy \right) dx$$

with the other side following from a similar calculation. □

Problem. 3.28 $D_{1,2}f = D_{2,1}f$ so long as these are continuous.

Proof. ? □

Problem. 3.29 Find the volume of the solid obtained by rotating a JM set in the yz plane about the z-axis.

Proof. ? □

Problem. 3.30 Let C be the set in problem 1.17 (the evil dense subset of the unit square containing at most one point on any horizontal or vertical line.) Then both $\int_0^1 \int_0^1 \chi_C dydx$ and $\int_0^1 \int_0^1 \chi_C dxdy$ are zero but the integral over C does not exist.

In other words, the converse of Fubini's theorem does not hold in general.

Proof. For any fixed x, the integral $\int_0^1 \chi_C(x,y)dy$ is clearly zero since C contains at most a single point on the line $\{x\} \times [0,1]$ from which it follows that the first double integral is 0. The argument for the other one is exactly the same.

Since C is dense, any rectangle contains both a point inside C and a point outside C and so $M_S = 1, m_S = 0$ for any rectangle S. Clearly the upper and lower integrals cannot coincide. □

Problem. 3.31 For $A = [a_1, b_1] \times ... \times [a_n, b_n]$ and $f : A \to \mathbb{R}$ define $F : A : \mathbb{R}$ by

$$F(x) = \int_{[a_1,b_1]\times...\times[a_n,b_n]} f.$$

What is $D_i f(x)$ for x in the interior of A?

Proof. ? □

Problem. 3.32: Leibnitz' Rule Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be continuous and suppose $D_2 f$ is continuous. Define $F(y) = \int_a^b f(x,y)dx$. Then $F'(y) = \int_a^b D_2 f(x,y)dx$.

Proof. Rewrite $F(y)$ as

$$\int_a^b \int_c^y D_2 f(x,t) dt dx + \int_a^b f(x,c) dx.$$

By continuity of $D_2 f$ the order of integration may be reversed. The result follows from the fundamental theorem of calculus applied to the first integral and the y-independence of the second. □

Problem. 3.33 If $f : [a, b] \times [c, d] \to \mathbb{R}$ is continuous, define $F(x, y) = \int_a^x f(t, y)dt$. Find D_1F and D_2F.

Proof. By the FTC, $D_1F(x, y) = f(x, y)$ and by Leibnitz' rule $D_2F(x, y) = \int_a^b D_2f(t, y)dt$. □

Problem. 3.34 Let $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable and suppose $D_1g_2 = D_2g_1$. Define

$$f(x, y) = \int_0^x g_1(t, 0)dt + \int_0^y g_2(x, t)dt.$$

Then $D_1f(x, y) = g_1(x, y)$.

Proof. By FTC and Leibnitz' rule

$$\begin{aligned} D_1f(x, y) &= g_1(x, 0) + \int_0^y D_1g_2(x, t)dt \\ &= g_1(x, 0) + \int_0^y D_2g_1(x, t)dt \\ &= g_1(x, 0) + g_1(x, y) - g_1(x, 0) \\ &= g_1(x, y). \end{aligned}$$

□

Problem. 3.35 If $G : \mathbb{R}^n \to \mathbb{R}^n$ is a invertible linear transformation and U a rectangle, then the volume of $G(U)$ is $|\det G| \cdot v(U)$.

Proof. Let $U = [a_1, b_1] \times ... \times [a_n, b_n]$ and consider first of all three kinds of elementary transformations.

1. G takes one of the basis vectors e^i to ce^i and fixes the rest. If c is positive then

$$\begin{aligned} v(G(U)) &= v([a_1, b_1] \times ... \times [ca_i, c_bi] \times ... \times [a_n, b_n]) \\ &= (b_1 - a_1) \cdot ... \cdot (cb_i - ca_i) \cdot ... \cdot (b_n - a_n) \\ &= c \cdot v(U). \end{aligned}$$

   The matrix representation of G is the identity with the i-th entry replaced by c, which has determinant c. If c is negative then the absolute value signs are needed but the volume is unchanged, although $[a, b]$ must become $[cb, ca]$.

2. G swaps two dimensions and fixes the rest, i.e. $g(e_i) = e_j$ and $g(e_j) = e_i$. As a matrix G is the identity with zeros in the i and j-th positions, and ones in the $G_{i,j}$ and $G_{j,i}$ positions. In $\mathbb{R}^3$ it might look like

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

   and in any dimension larger than one has determinant -1. Clearly this operation has no effect on the volume of the rectangle

3. G adds basis vector to another and fixes the rest, i.e.

$$G(x^1, ..., x^i, ..., x^j, ..., x^n) = (x^1, ..., x^i + x^j, ..., x^j, ..., x^n).$$

   As a matrix G is the identity with a single extra one somewhere, and so it is easy to see that $\det G = 1$.

   Linear operators on $\mathbb{R}^n$ are bounded, so the volume of $G(U)$ can be calculated as

$$\int_{[-M_n,M_n]\times...\times[-M_1,M_1]} \chi_{G(U)} dx^1...dx^n$$

   for some upper bound M of both U and $G(U)$. Since Fubini's theorem will allow us to interchange the order of integration, we can assume that G adds e_2 to e_1 and leaves the rest. Then $v(G(U)) =$

$$\int_{[-M_n,M_n]\times...\times[-M_3,M_3]} \left(\int_{[-M_2,M_2]\times[-M_1,M_1]} \chi_{G(U)}(x^1, x^2, ...)dx^1 dx^2 \right) dx^3...dx^n.$$

   The value of the inner integral is the area of a parallelogram, which coincides with the area of the rectangle which fell over. Hence this inner integral would not be changed by replacing $\chi_{G(U)}$ with χ_U. Furthermore, for fixed a^1 and a^2 and varying coordinates in the remaining positions, $\chi_U(a_1, a_2, x_3, ..., x_n)$ and $\chi_{G(U)}(a_1, a_2, x_3, ..., x_n)$ conincide exactly. So for the purposes of evaulating this integral, χ_U will do just as well as $\chi_{U(G)}$, or in other words their volumes are equal.

Finally if G is the composition of elementary operations as described above then G alters the volume of U by a factor of

$$|\det G_1| \cdot .. \cdot |\det G_m| = |\det G_1 \cdot ... \cdot \det G_m| = |\det(G_1 \cdot ... \cdot G_m)| = |\det G|.$$

We take as a fact that these elemetary operations generate the whole group of invertible matrices and so this scaling factor is good for any invertible linear map. □

Problem. 3.36 Cavalieri's Principle. Let A and B be JM subsets of $\mathbb{R}^3$. Let $A_c = \{(x, y) : (x, y, c) \in A\}$ and define B_c similarly. If each A_c and B_c are JM and have the same area then A and B have the same volume.

Proof. Let $[-M, M]^3$ be a rectangle bounding A and B. Then

$$\begin{aligned}
\int_{[-M,M]^3} \chi_A &= \int_{-M}^{M} \left(\int_{[-M,M]^2} \chi_A \right) dz \\
&= \int_{-M}^{M} \left(\int_{A_z} 1 \right) dz = \int_{-M}^{M} \left(\int_{B_z} 1 \right) dz \\
&= \int_{-M}^{M} \left(\int_{[-M,M]^2} \chi_B \right) dz = \int_{[-M,M]^3} \chi_B.
\end{aligned}$$

□

Problem. 3.37 Suppose that $f : (0, 1) \to \mathbb{R}$ is non-negative and continuous. Then $\int_{(0,1)} f$ exists if and only if $\lim_{\varepsilon\to 0} \int_{\varepsilon}^{1-\varepsilon} f$ exists.

Proof. Let o be the cover of $(0, 1)$ consisting of the intervals $(1/n, 1-1/n)$ and Φ a C^∞ partition of unity for $(0, 1)$ subordinate to o. Then each φ_i in Φ has support contained in some $(1/m_i, 1-1/m_i)$ in o. Choose an ordering of Φ so that if $i < j$ then $m_i \leq m_j$, that way the support of φ_i is in $(1/m_j, 1 - 1/m_j)$ for all $j > i$.

Then the partial sums can be written as

$$\sum_{k=1}^{n} \int_{(0,1)} \varphi_k \cdot f = \sum_{k=1}^{n} \int_{1/m_k}^{1-1/m_k} \varphi_k \cdot f = \sum_{k=1}^{n} \int_{1/m_n}^{1-1/m_n} \varphi_k \cdot f = \int_{1/m_n}^{1-1/m_n} \sum_{k=1}^{n} \varphi_k \cdot f = \int_{1/m_n}^{1-1/m_n} f.$$

Since $\varphi \cdot f$ is non-negative, convergence and absolute conergence are the same thing. Thus the extended integral exists if and only if the given limit exists. $\square$

Problem. 3.38 Let A_n be a closed set in $(n, n + 1)$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $\int_{A_n} f = (-1)^n/n$ and $f = 0$ outside A_n. Then f is not integrable.

Proof. Let Φ_n be a partition of unity for A_n and extend each φ to a function on $\mathbb{R}$ by setting $\varphi = 0$ outside A_n. Then their union Φ is a partion of unity for all of $\mathbb{R}$ and

$$\sum_{\Phi} \int_{\mathbb{R}} \varphi \cdot |f| = \sum_{\mathbb{N}} \left(\sum_{\Phi_n} \int_{\mathbb{R}} \varphi \cdot |f| \right) = \sum_{\mathbb{N}} \left(\sum_{\Phi_n} \int_{A_n} \varphi \cdot |f| \right) = \sum_{\mathbb{N}} \int_{A_n} f = -\log 2.$$

Following the example in the wiki for conditional convergence, define the sets

$$B_n = A_{2n-1} \cup A_{2(2n-1)} \cup A_{4n}$$

and let Φ_n be a partition of unity for B_n, taking the value 0 outside B_n. Then as above

$$\sum_{\Phi} \int_{\mathbb{R}} \varphi \cdot |f| = \sum_{\mathbb{N}} \left(\sum_{\Phi_n} \int_{\mathbb{R}} \varphi \cdot |f| \right) = \sum_{\mathbb{N}} \left(\sum_{\Phi_n} \int_{B_n} \varphi \cdot |f| \right) = \sum_{\mathbb{N}} \int_{B_n} f.$$

Each of the integrals in the last sum has the value

$$\begin{aligned} \int_{B_n} f &= \int_{A_{2n-1}} f + \int_{A_{2(2n-1)}} f + \int_{A_{4n}} f \\ &= \frac{(-1)^{2n-1}}{2n-1} + \frac{(-1)^{2(2n-1)}}{2(2n-1)} + \frac{(-1)^{4n}}{4n} \\ &= \frac{1}{2(2n-1)} + \frac{1}{4n} - \frac{1}{2n-1} \\ &= \frac{1}{2}\left(\frac{-1}{2n-1} + \frac{1}{2n} \right) \end{aligned}$$

and so the sum converges to $-1/2 \log 2$. $\square$

Problem. 3.39 The condition that $\det g$ in the formula for change of variables is superfluous, due to Sard's Theorem.

Proof. ? $\square$

Problem. 3-40 If $g : \mathbb{R}^n \to \mathbb{R}^n$ and $\det g'(x) \neq 0$ then there is an open set containing x in which $g = T \circ g_n \circ ... \circ g_1$ where each g_i is of the form $g_i(x) = (x^i, ..., f_i(x^i), ..., x^n)$ and T is a linear transformation. Furthermore g can be written as $g = g_n \circ ... \circ g_1$ if and only if g' is diagonal.

Proof. ? $\square$

Problem.

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

Proof.

1. Define $f : \{r : r > 0\} \times (0, 2\pi) \to \mathbb{R}^2$ by $f(r, \vartheta) = r(\cos\vartheta, \sin\vartheta)$. If $r(\cos\vartheta, \sin\vartheta) = r'(\cos\vartheta', \sin\vartheta')$ then the Pythagorean identity easily gives $r = r'$, and so

$$\cos\vartheta = \cos\vartheta', \quad \sin\vartheta = \sin\vartheta'.$$

$\square$

4 Integration on Chains

Problem. 4.1 Let $e_1, ..., e_n$ be the usual basis of $\mathbb{R}^n$ and let φ_i be the dual basis. Then (a) $\varphi_{i_1} \wedge ... \wedge \varphi_{i_k}(e_{i_1}, e_{i_k}) = 1$ and (b) $\varphi_{i_1} \wedge ... \wedge \varphi_{i_k}(v_1, ..., v_k)$ is the determinant of the $k \times k$ minor of

$$\begin{pmatrix} v_1 \\ ... \\ v_k \end{pmatrix}$$

obtained by selecting $i_1, ..., i_k$.

Proof. By induction on k. If $k = 1$ then $\varphi_i(e_i) = 1$. Assume the result for $k-1$. Then

$$(\varphi_{i_1} \wedge ... \wedge \varphi_{i_{k-1}}) \wedge \varphi_{i_k}(e_{i_1}, ..., e_{i_{k-1}}, e_{i_k}) = \frac{1}{(k-1)!} \sum_{S_k} sign(\sigma)(\varphi_{i_1} \wedge ... \wedge \varphi_{i_{k-1}})(e_{\sigma(i_1)}, ..., e_{\sigma(i_{k-1})}) \cdot \varphi_{i_k}(e_{\sigma(i_k)}).$$

Since $\varphi_{i_k}(e_{\sigma(i_k)})$ is one if σ fixes i_k and zero otherwise and $\varphi_{i_1} \wedge ... \wedge \varphi_{i_{k-1}}$ is alternating this reduces to

$$\frac{1}{(k-1)!} \sum_{S_{k-1}} sign(\sigma)(\varphi_{i_1} \wedge ... \wedge \varphi_{i_{k-1}})(e_{\sigma(i_1)}, ..., e_{\sigma(i_{k-1})}) =$$
$$\frac{1}{(k-1)!} \sum_{S_{k-1}} sign(\sigma)^2(\varphi_{i_1} \wedge ... \wedge \varphi_{i_{k-1}})(e_{i_1}, ..., e_{i_{k-1}}) =$$
$$\frac{|S_{k-1}|}{(k-1)!} \cdot \varphi_{i_1} \wedge ... \wedge \varphi_{i_{k-1}}(e_{i_1}, ..., e_{i_{k-1}}) = 1$$

by the inductive hypothesis.

Part (b) ? □

Problem. 4.2 Let $f : V \to V$ be linear and $\dim V = n$. Then $f^* : \Omega^n(V) \to \Omega^n(V)$ is multiplication by $\det f$.

Proof. Let $v_1, ..., v_n$ be a basis and let $w_1, ..., w_n$ be n vectors. Let A be the matrix such that $w_i = Av_i$ and let F be the matrix of f with respect to the basis v_i. Then, taking huge liberties with notation, $\omega(f(w_1), f(w_n)) =$

$$\omega(F(w_1, w_n)) = \omega(FA(v_1, v_n)) = \det(FA) \cdot \omega(v_1, v_n) = \det F \cdot \omega(A(v_1, v_n)) = \det F \cdot \omega(w_1, w_n).$$

□

Problem. 4.3 If ω is the volume element determined by T and μ and $w_1, ..., w_n$ are vectors in V then

$$|\omega(w_1, ..., w_n)| = \sqrt{det(g_{ij})}.$$

Note that the hint for the problem has the indicies around the wrong way.

Proof. Let $v_1, ...v_n$ be an orthonormal basis with resepect to T and $A = a_{ih}$ numbers such that $w_i = \sum_{h=1}^n a_{ih} v_h$. Then

$$g_{ij} := T(w_i, w_j) = \sum_{h,g=1}^n a_{ih} a_{jg} T(v_i, v_j) = \sum_{k=1}^n a_{ik} a_{jk},$$

that is $(g_{ij}) = A \cdot A^T$ and

$$\sqrt{\det G} = \det A.$$

By theorem 4.6 we have $\omega(w_1, ..., w_n) = \det A \cdot \omega(v_1, ...v_n) = \pm \det(A)$ (since ω is the volume element with respect to the orthonormal basis v_i) which proves the result. □

Problem. 4.4 If ω is the volume element of V determined by T and μ and $f : \mathbb{R}^n \to V$ is an isomorphism such that $f^*T = \langle , \rangle$ and such that $[f(e_1), ..., f(e_n)] = \mu$ then $f * \omega = \det$.

Proof. If $(f(e_i))$ were an orthonormal basis with positive orientation then

$$1 = \omega(f(e_1), ..., f(e_n)) = f * \omega(e_1, ..., e_n)$$

i.e. $f^*\omega$ is the volume element in $\mathbb{R}^n$. Since this is unique it equals det.

Orthonormailty of $(f(e_i))$ follows from the easy calculation

$$T(f(e_i), f(e_j)) = f^*T(e_i, e_j) = \langle e_i, e_j \rangle = \delta ij.$$

□

Problem. 4.5 If $c : [0, 1] \to (\mathbb{R}^n)^n$ is continuous and each $(c^1(t), ..., c^n(t))$ is a basis, then any two have the same orientation.

Proof. Since $c(t)$ is a basis for any t it must always be non-singular. Hence $\det \circ c \neq 0$ on $[0, 1]$, which means it is always positive, or always negative. Let C_1 and C_2 be the matrices of two bases in this family, and A the transformation between them. Then

$$\det A = \det(C_1^{-1}) \cdot \det C_2 > 0$$

as the argument above ensures the determinants on the right have the same sign. □

Problem. 4.6 If $v_1, ..., v_{n-1} \in \mathbb{R}^n$ are linearly independent then $[v_1, .., v_{n-1}, v_1 \times ... \times v_{n-1}]$ is the usual orientation for $\mathbb{R}^n$.

Proof. If $n = 2$ then it follows from the definition that

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \times = \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix}.$$

It is clear that $\times v$ is just v rotated 90 degrees anticlockwise, which is the usual orientation. To be extra sure, the matrix taking $(1, 0)$ to (v_1, v_2) and $(0, 1)$ to $(-v_2, v_1)$ is

$$\begin{pmatrix} v_1 & -v_2 \\ v_2 & v_1 \end{pmatrix}$$

which has positive determinant so long as at least one of v is non-zero.
Rest of it??? □

Problem. 4.8 If $\omega \in \Omega^n(V)$ is a volume element define a 'cross product' in terms of ω.
Problem. 4.11 Let f be a self-adjoint linear map on V with respect to the inner product T. If v_1, v_n is an orthonormal basis and $A = (a_{ij})$ is the matrix of f with respect to v_i then A is symmetric.

Proof. Going straight into jumbo calculations, $T(x, f(y)) =$

$$T\left(\sum_i x^i v_i, f\left(\sum_j y^j v_j\right)\right) = \sum_{i,j=1}^n x^i y^j T(v_i, f(v_j)) = \sum_{i,j=1}^n x^i y^j \sum_k a_{jk} T(v_i, v_k) = \sum_{i,j=1}^n x^i y^j a_{ji}.$$

By a similar calculation $T(f(x), y) = \sum_{i,j=1}^n x^i y^j a_{ij}$. Since f is self-adjoint we have

$$\sum_{i,j=1}^n x^i y^j a_{ji} = \sum_{i,j=1}^n x^i y^j a_{ij}$$

for all x, y, and so a_{ij} must equal a_{ji}. □

Problem. 4.13(a) If $f : \mathbb{R}^{n \to m}$ and $g : \mathbb{R}^{m \to p}$ then $(g \circ f)_* = g_* \circ f_*$.

Proof.

$$\begin{aligned}(g \circ f)_*(v_q) = [D(g \circ f)(q)](v) = [Dg(f(q)) \circ Df(q)](v) = \\ Dg(f(q))(Df(q)(v)) = Dg(f(q))(f_*(v_q)) = g_*(f_*(v_q))\end{aligned}$$

□

Problem. 4.13(b) Let $f, g : \mathbb{R}^n \to \mathbb{R}$. Then $d(f \cdot g) = f \cdot dg + g \cdot df$.

Proof.

$$d(f \cdot g) = \sum_{j=1}^n D_j(f \cdot g) \cdot dx^j = \sum_{j=1}^n [f \cdot D_j g + g \cdot D_j f] \cdot dx^j.$$

□

Problem. 4.14 Let c be a differentiable curve in $\mathbb{R}^n$, that is, a differentiable function $c : [0, 1] \to \mathbb{R}^n$. Define the tangent vector of c at t as $v = c_*((e_1)_t) = ((c^1)'(t), ..., (c^n)'(t))_{c(t)}$. If $f : \mathbb{R}^n \to \mathbb{R}^m$, show that the tangent vector to $f \circ c$ at t is $f_*(v)$.

Proof. By the previous problem the tangent to $f \circ c$ at t is

$$(f \circ c)_*((e_1)_t) = f_* \circ c_*((e_1)_t) = f_*(c_*((e_1)_t)) = f_*(v). \tag{1}$$

□

Problem. 4.15 Let $f : \mathbb{R} \to \mathbb{R}$ and define the planar curve c by $c(t) = (t, f(t))$. Show that the end point of the tangent vector of c at t lies on the tangent line to the graph of f at $(t, f(t))$.

Proof. The tangent line to the graph of f at $(t, f(t))$ is

$$L: \quad y - f(t) = f'(t)(x - t).$$

The tangent vector to the curve c at t is

$$c_*((e_1)_t) = (1, f'(t))_{c(t)}$$

which starts at $(t, f(t))$ and ends at

$$H: \quad (t + 1, f(t) + f'(t)).$$

It is easy to see that the point H lies on the line L. $\square$

Problem. Let $c : [0, 1] \to \mathbb{R}^n$ be a curve such that $|c(t)| = 1$ for all t. Then $c(t)_{c(t)}$ and the tangent vector to c at t are perpendicular.

Proof. Computing directly

$$c(t)_{c(t)} \cdot c_*((e_1)_t) = \sum c^i(t) \cdot (c^i)'(t).$$

Differentiating the hypothesis $|c(t)|^2 = 1$,

$$\sum 2 \cdot c^i(t) \cdot (c^i)'(t) = 0.$$

$\square$

Problem. If $f : \mathbb{R}^n \to \mathbb{R}^n$, define a vector field $\boldsymbol{f}$ by $\boldsymbol{f}(p) = f(p)_p$. Every vector field F on $\mathbb{R}^n$ is of the form $\boldsymbol{f}$ for some f and $div\boldsymbol{f} = trace f'$

Proof. If F is some vector field, define f by $F(p) = v_p \iff f(p) = v$. Then[3]

$$div\boldsymbol{f}(p) = \sum D_i \boldsymbol{f}^i(p) = \sum D_i f^i(p) = trace f'.$$

$\square$

Problem. 4.18 For $f : \mathbb{R}^n \to \mathbb{R}$ define the vector field

$$\nabla f(p) = D_1 f(p) \cdot (e_1)_p + ...D_n f(p) \cdot (e_n)_p.$$

If $\nabla f(p) = w_p$ then prove that $D_v f(p) = \langle v, w \rangle$ and conclude that $\nabla f(p)$ is the direction in which f is changing fastest at p.

Proof. If $w_p = \nabla f(p) = \sum D_i f(p) \cdot (e_i)_p$ then $w = \sum D_i f(p) \cdot e_i$. If $v = \sum v^i \cdot e_i$ then by properties of the directional derivative

$$\langle v, w \rangle = \sum v^i D_i f(p) = D_v f(p). \tag{2}$$

$\square$

[3] This feels sketchy.

Problem. 4.19 If F is a vector field on $\mathbb{R}^3$ define the forms

$$\omega_F^1 = F^1 dx + F^2 dy + F^3 dz$$
$$\omega_F^2 = F^1 dy \wedge dz + F^2 dz \wedge dx + F^3 dx \wedge dy.$$

(a) Prove that

$$df = \omega^1_{\nabla f}$$
$$d(\omega^1_F) = \omega^2_{curl\ F}$$
$$d(\omega^2_F) = (divF) dx \wedge dy \wedge dz.$$

(b) Use part (a) to prove

$$curl\ grad\ f = 0$$
$$div\ curl\ F = 0.$$

(c) If F is a vector field on a star-shaped set A and $curl\ F = 0$, show that $F = grad\ f$ for some function $f : A \to \mathbb{R}$. Similarly, if $\div F = 0$, show that $F = curl\ G$ for some vector field G on A.

Proof. (a) Write $\nabla f(p) = (D_1 f^1(p), D_2 f^2(p), D_3 f^3(p))_p$. Then

$$\omega^1_{\nabla f}(p) = D_1 f^1(p) dx + D_2 f^2(p) dy + D_3 f^3(p) dz = df(p).$$

The equation $d(\omega^1_F) = \omega^2_{curl\ F}$ follows from the definitions and the facts $dx^i \wedge dx^i = 0$ and $dx \wedge dy = -dy \wedge dx$. For the third equation

$$d(\omega^2_F) = [d_1 F^1 dx \wedge dy \wedge dz] + [d_2 F^2 dy \wedge dz \wedge dx] + [d_3 F^3 dz \wedge dx \wedge dy]$$

which equals $(div f) dx \wedge dy \wedge dz$ after an even number of swaps of dx^i in the latter two summands.

(b) Although these are easy to prove without part (a) I guess you should do it that way to set up part (c).

□

Problem. 4.20 Let $f : U \to \mathbb{R}^n$ be a differentiable function with differentiable inverse $f^{-1} : f(U) \to \mathbb{R}^n$. If every closed form on U is exact, show that the same is true for $f(U)$.

Proof. Let ω be a form on $f(U)$ so that $f^*\omega$ is a form on U. Then if ω is closed

$$d(f^*\omega) = f^*(d\omega) = f^*(0) = 0$$

and so $f^*\omega$ is closed, therefore exact. Let μ be such that $d(f^*\omega) = \mu$. Then $(f^{-1})^*\mu$ is a form on $f(U)$ and it would be nice if $d((f^{-1})^*\mu) = \omega$, so that ω is exact. We have

$$d((f^{-1})^*\mu) = (f^{-1})^* d\mu = (f^{-1})^* f^* \omega$$

and so we need to show that $(f^{-1})^* f^*$ is the identity on forms. Since

$$(f^{-1})^* f^* \omega(f(p))(v) = \omega(f(p))(f_* f_*^{-1}(v))$$

we need to show that $f_* f_*^{-1}$ is the identity on vectors. This follows from the inverse function theorem as follows:

$$\begin{aligned}
f_* f_*^{-1} v_{f(p)} &= f_*[Df^{-1}(f(p))(v)]_{f^{-1}f(p)} \\
&= [Df(p) \circ Df^{-1}(f(p))(v)]_{f(p)} \\
&= [f'(p) \cdot f^{-1\prime}(f(p)) \cdot v]_{f(p)} \\
&= [f'(p) \cdot [f'(f^{-1}(f(p)))]^{-1} \cdot v]_{f(p)} = v_{f(p)}.
\end{aligned}$$

□

Problem. 4.22 Let S be the set of all n-cubes. Define a singular n-chain as a function f from S to $\mathbb{Z}$ such that all but finitely many $f(c)$ are zero.

(a) Define $f+g$ and nf by $(f+g)(c) = f(c) + g(c)$ and $nf(c) = n \cdot f(c)$. Then $f+g$ and nf are n-chains.

(b) For an n-cube c let c also denote the n-chain which is 1 at c and 0 at every other cube. Then every n-chain can be written as $a_1 c_1 + ... + a_k c_k$ for integers a and singular n-cubes n.

Proof. (a) These are obvious.

(b) Let the finite collection of cubes at which f is non-zero be $c_1, ..., c_k$. Then

$$f(c) = \sum_S f(c)c(c) = \sum_{i=1}^k f(c_i)c_i(c_i) = \left(\sum_{i=1}^k a_i c_i\right)(c)$$

□

Problem. 2.23 For $R > 0$ and n an integer, define the singular 1-cube $C_{R,n} : [0,1] \to \mathbb{R}^2 - 0$ by

$$c_{R,n}(t) = (R\cos 2\pi nt, R\sin 2\pi nt).$$

Show that there is a singular 2-cube $c : [0,1]^2 \to \mathbb{R}^2 - 0$ such that $\partial c = c_{R_1,n} - c_{R_2,n}$.

Proof. Let $c : [0,1] \times [0,1] \to \mathbb{R}^2 - 0$ be defined by

$$c(x,y) = [(R_1 - R_2)x + R_2](\cos 2\pi ny, \sin 2\pi ny).$$

Then

$$\begin{aligned}
c_{(1,0)}(x,y) &= c \circ I_{(1,0)}(x,y) = c(0,y) = (R_2 \cos 2\pi nt, R_2 \sin 2\pi nt) \\
c_{(1,1)}(x,y) &= c \circ I_{(1,1)}(x,y) = c(1,y) = (R_1 \cos 2\pi nt, R_1 \sin 2\pi nt).
\end{aligned}$$

Since $c_{(2,0)} = c_{(2,1)}$ they cancel in the expanded sum of ∂c leaving

$$\partial c = c_{(1,1)} - c_{(1,0)} = c_{R_1,n} - c_{R_2,n}.$$

□