6. Suppose we model this disease by an SIR model in the form:

$$\dot{S} = -\beta I \tfrac{S}{N}$$
$$\dot{I} = \beta I \tfrac{S}{N} - \alpha I$$
$$\dot{R} = \alpha I$$

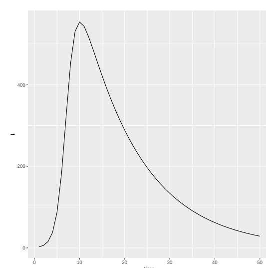
a. Estimate the value of α

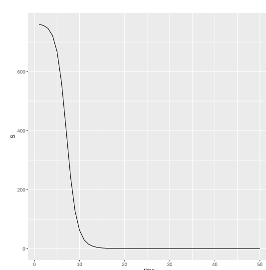
$$x = \frac{S}{N} \qquad y = \frac{I}{N} \qquad \tau = \alpha t$$
$$\frac{dx}{d\tau} = -\gamma x y \qquad \frac{dy}{d\tau} = \gamma x y - y \qquad \frac{d\bar{x}}{d\tau} = F(\bar{x})$$
$$\Rightarrow \qquad \gamma = \frac{\beta}{\alpha}$$

Along the line $x + y = 1$, the inward normal is

$$\bar{n} =< -1, -1 >$$
$$\bar{n} \cdot F(x, y) = y$$

If $\gamma > 1$, an epidemic occurs; i.e. if $S(0) = S_0 \approx 1$, then the number of infections increases before decreasing.





$$\frac{dI}{dt} = 0 \Rightarrow \qquad \gamma x y - y = 0$$
$$x = \frac{1}{\gamma}$$

$$\frac{dy}{dx} = \frac{\gamma x - 1}{-\gamma x} = \frac{1}{\gamma x} - 1$$
$$\int_{y_0}^{y} dy = \int_{x_0}^{x} (\frac{1}{\gamma x} - 1) dx$$
$$y = y_0 + \frac{1}{\gamma} \ln{(\frac{x}{x_0})} - x + x_0$$

The maximum number of infected at time t is:

$$y = y_0 + \frac{1}{\gamma}\ln(\frac{\gamma}{x_0}) - x + x_0$$

Assuming $x_0 \approx 1$ and $y_0 \approx 0$, we have:

$$\begin{aligned} y_{max} &= 1 - \gamma + \frac{1}{\gamma}\ln(\gamma) \\ y(x) &= 1 - \gamma + \frac{1}{\gamma}\ln(x) - x \end{aligned}$$

Computing the number that were infected during the epidemic, we let $z = \frac{R}{N}$

$$\begin{aligned} \frac{dz}{d\tau} &= y \\ \frac{dz}{dx} &= -\frac{1}{\gamma x} \\ \int_0^z &= dz = \int_1^x -\frac{1}{\gamma x}dx \\ z &= -\frac{1}{\gamma}\ln(x) \\ x &= e^{-\gamma z} \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{dz}{d\tau} &= (1 - x - z) \\ &= (1 - e^{-\gamma z} - z) \end{aligned}$$

The total number of infections is at the fixed point z^* when $\mathbf{z^* = 1 - e^{-\gamma z^*}}$.

b. Using the relationship between S and I derived in class, estimate the value of β. transmission rate, β

c. Plot S and I alongside the data. Do they fit?