No. 1.1. (1) $-D\,\partial_x^2 T(x) = \varepsilon$ with Dirichlet boundary conditions

- Numerical form of second derivative:

$$\partial_x^2 T = \frac{1}{h}\cdot\left(\frac{T_{i+1}-T_i}{h} - \frac{T_i - T_{i-1}}{h}\right) = \frac{1}{h^2}\cdot\left(T_{i+1} - 2T_i + T_{i-1}\right)$$

$$\Rightarrow \quad (T_{i+1} - 2T_i + T_{i-1}) = -\frac{\varepsilon h^2}{D}$$

with: $h \mathrel{\hat=}$ step size

$h = \frac{2L}{N}$

(2) Consider fixed boundary conditions: $T_0 \overset{!}{=} T_N$

$$\Rightarrow \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ 1 & -2 & 1 & & \vdots \\ 0 & 1 & -2 & 1 & 0 \\ \vdots & & & \ddots & 1 \\ & \cdots & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} T_0 \\ T_1 \\ \vdots \\ T_{N-1} \\ T_0 \end{pmatrix} = -\frac{\varepsilon h^2}{D} \cdot \begin{pmatrix} -\frac{DT_0}{\varepsilon h^2} \\ 1 \\ 1 \\ \vdots \\ -\frac{DT_0}{\varepsilon h^2} \end{pmatrix} \in \mathbb{R}^{N+1}$$

with: $\varepsilon = D = T_0 = L = 1$: inhomogeneity $\vec{b} = \begin{pmatrix} 1 \\ -4/N^2 \\ \vdots \\ 1 \end{pmatrix}$

No. 1.2. $R^{(3)}\vec{u} = \vec{v}$ with: $\vec{u} \in \mathbb{R}^9 \mathrel{\hat=}$ fine mesh state vector

(1) $\vec{v} \in \mathbb{R}^5 \mathrel{\hat=}$ coarse grid state vector

$\Rightarrow R^{(3)} \in \mathbb{R}^{5\times 9} \mathrel{\hat=}$ restriction matrix for inverse interpolation

↳ $R^{(3)}$ takes the weighted average over five elements. The weights need to be chosen according to a binomial distribution with $B(k|\frac{1}{2}, 4)$. $\longrightarrow$ see Pascal's triangle!

$$\Rightarrow R^{(3)} = \frac{1}{16}\cdot\begin{bmatrix} 1&4&6&4&1&0&0&0&0 \\ 0&1&4&6&4&1&0&0&0 \\ 0&0&1&4&6&4&1&0&0 \\ 0&0&0&1&4&6&4&1&0 \\ 0&0&0&0&1&4&6&4&1 \end{bmatrix}$$

↳ Similar we can construct $R^{(2)}$, with: $\vec{u} \in \mathbb{R}^5$, $\vec{v} \in \mathbb{R}^3$. In this case, the weights are determined by the binomial distribution with the parameters $B(k|\frac{1}{2}, 2)$.

$$\Rightarrow R^{(2)} = \frac{1}{4}\cdot\begin{bmatrix} 1&2&1&0&0 \\ 0&1&2&1&0 \\ 0&0&1&2&1 \end{bmatrix} \mathrel{\hat=} \text{restriction matrix from } \mathbb{R}^5 \text{ to } \mathbb{R}^3.$$

No. 1.2. ↳ Following the same procedure we can construct
(1) $R^{(1)}$, with: $\vec{u} \in \mathbb{R}^3$, $\vec{v} \in \mathbb{R}^2$. In this case, the weights are determined by the binomial distribution with the paramters $B(k|\frac{1}{2},1)$.

$$\Rightarrow R^{(1)} = \frac{1}{2} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$ $\hat{=}$ restriction matrixe from $\mathbb{R}^3$ to $\mathbb{R}^2$.

(2) $P^{(2)} \vec{v} = \vec{u}$ with: $\vec{v} \in \mathbb{R}^2 \hat{=}$ coarse grid state vector
$\vec{u} \in \mathbb{R}^3 \hat{=}$ fine mesh state vector

$\Rightarrow P^{(2)} \in \mathbb{R}^{3 \times 2}$ $\hat{=}$ prolongation matrixe for interpolation

↳ $P^{(2)}$ performs an interpolation between nearest neighbours.

$$\Rightarrow P^{(2)} = \frac{1}{2} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{3 \times 2}$$, Contraint: zero boundary conditions.

↳ $P^{(1)}$ performs an interpolation between nearest neighbours under zero boundary conditions, with: $\vec{v} \in \mathbb{R}^3$ and: $\vec{u} \in \mathbb{R}^5$.

$$\Rightarrow P^{(1)} = \frac{1}{4} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{5 \times 3}$$

↳ $P^{(0)}$ performs an interpolation between the two nearest neighbours and zero boundary conditions, with: $\vec{v} \in \mathbb{R}^5$, $\vec{u} \in \mathbb{R}^9$.

$$\Rightarrow P^{(0)} = \frac{1}{16} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \\ 6 & 4 & 1 & 0 & 0 \\ 4 & 6 & 4 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \\ 0 & 1 & 4 & 6 & 4 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{9 \times 5}$$

$\Rightarrow R = P^T$, with this approach we can propagate as much information as possible between the different levels.