No. 1 Local truncation error for $f(y,t)$:

$$LTE_f(t) \equiv \frac{y(t+\Delta t) - y(t)}{\Delta t} - \tilde{f}(t,y)$$

- Second order Runge-Kutta method: $\tilde{f}(t,y) = \frac{1}{2}(k_1+k_2)$

$$\Rightarrow \tilde{f}(t,y) = \frac{1}{2}\cdot\left(f(y_n,t_n) + f(y_n + k_1\Delta t, t_n+\Delta t)\right) = \frac{1}{2}\cdot\left(f(y_n,t_n) + f(y_n + f(y_n,t_n)\Delta t, t_n+\Delta t)\right)$$

$\hookrightarrow$ Taylor-expansion of f: $f(y_n + f(y_n,t_n)\Delta t, t_n + \Delta t) = f(y_n,t_n)$
$+ f_t(y_n,t_n)\Delta t + f_y f(y_n,t_n)\Delta t$
$+ \mathcal{O}(\Delta t^2)$

$$\Rightarrow \tilde{f} = \frac{1}{2}\cdot\left(f + f + f_t\Delta t + f_y\cdot f\cdot\Delta t\right) + \mathcal{O}(\Delta t^2)$$

$\hookrightarrow$ Taylor-expansion of first term: $\frac{y(t+\Delta t)-y(t)}{\Delta t} = \frac{y + y'\Delta t + \frac{1}{2}y''\Delta t^2 - y}{\Delta t} + \mathcal{O}(\Delta t^3)$

$$\Leftrightarrow \frac{y(t+\Delta t)-y(t)}{\Delta t} = y' + \frac{1}{2}y''\Delta t + \mathcal{O}(\Delta t^2)$$

$$\Rightarrow LTE_f(t) = y' + \frac{1}{2}y''\Delta t - f - \frac{1}{2}f_t\Delta t - \frac{1}{2}f_y f\Delta t + \mathcal{O}(\Delta t^2)$$

with: $y' = \frac{dy}{dt} = f$ and: $y'' = \frac{df}{dt} = f_t + f_y f$

$$\Leftrightarrow LTE_f(t) = \cancel{f} + \cancel{\frac{1}{2}f_t\Delta t} + \cancel{\frac{1}{2}f_y f\Delta t} - \cancel{f} - \cancel{\frac{1}{2}f_t\Delta t} - \cancel{\frac{1}{2}f_y f\Delta t} + \mathcal{O}(\Delta t^2)$$

$$\Leftrightarrow LTE_f(t) = \mathcal{O}(\Delta t^2)$$

$\Rightarrow$ Local truncation error for trajectory y: $LTE_y(t) = LTE_f(t)\cdot\Delta t$

$$\Leftrightarrow LTE_y(t) = \mathcal{O}(\Delta t^3) \quad ■$$

Nr. 2. Second Lagrange equation: $\frac{d}{dt}\frac{\partial L}{\partial \dot\phi} - \frac{\partial L}{\partial \phi} = 0$

a.)

$\Leftrightarrow d_t q = \partial_\phi L$ with: $q \mathrel{\hat=}$ conjugate momenta

Lagrangian of the system:

$$\mathcal{L} = \frac{m_1}{2}\cdot(l_1\dot\phi_1)^2 + \frac{m_2}{2}\cdot\left[(l_1\dot\phi_1)^2+(l_2\dot\phi_2)^2+2l_1l_2\dot\phi_1\dot\phi_2\cos(\phi_1-\phi_2)\right]$$
$$- m_1 g l_1\cdot(1-\cos\phi_1) - m_2 g\cdot\left[l_1(1-\cos\phi_1)+l_2(1-\cos\phi_2)\right]$$

for ϕ_1: $\dot q_1 = -m_2\cdot l_1 l_2 \dot\phi_1\dot\phi_2\cdot \sin(\phi_1-\phi_2) - m_1 g l_1\cdot \sin\phi_1 - m_2 g l_1 \sin\phi_1$ (I)

$\Leftrightarrow \dot q_1 = -g l_1 \sin\phi_1\cdot(m_1+m_2) - m_2 l_1 l_2 \dot\phi_1\dot\phi_2\cdot \sin(\phi_1-\phi_2)$

for ϕ_2: $\dot q_2 = m_2 l_1 l_2 \dot\phi_1\dot\phi_2\cdot \sin(\phi_1-\phi_2) - m_2 g l_2\cdot \sin\phi_2$ (II)

b.) Define state vector: $\vec y \equiv (\phi_1, \phi_2, q_1, q_2)^T$ with: $d_t\vec y = \vec F \in \mathbb{R}^4$

$$q_1 = \frac{\partial L}{\partial\dot\phi_1} = m_1\cdot l_1^2\cdot\dot\phi_1 + m_2\cdot l_2^2\cdot\dot\phi_1 + m_2\cdot l_1 l_2\cdot\dot\phi_2\cdot\cos(\phi_1-\phi_2) \quad \text{(III)}$$

$$q_2 = \frac{\partial L}{\partial\dot\phi_2} = m_2\cdot l_2^2\cdot\dot\phi_2 + m_2\cdot l_1 l_2\cdot\dot\phi_1\cdot\cos(\phi_1-\phi_2) \quad \text{(IV)}$$

(III) & (IV) $\Rightarrow q_1 - \frac{l_1\cdot\cos(\phi_1-\phi_2)}{l_2} q_2 = (m_1\cdot l_1^2 + m_2 l_2^2)\cdot\dot\phi_1 - m_2\cdot l_1^2\cdot\dot\phi_1\cdot\cos^2(\phi_1-\phi_2)$

$\Leftrightarrow q_1 - \frac{l_1}{l_2}\cos(\phi_1-\phi_2)\, q_2 = \dot\phi_1\cdot\left[m_1\cdot l_1^2 + m_2\cdot\left(l_2^2 - l_1^2\cdot\cos^2(\phi_1-\phi_2)\right)\right]$

$$\Leftrightarrow \dot\phi_1 = \frac{q_1 - \frac{l_1}{l_2}\cos(\phi_1-\phi_2)\, q_2}{m_1 l_1^2 + m_2\cdot(l_2^2 - l_1^2\cdot\cos^2(\phi_1-\phi_2))} \equiv F_1$$

(III) & (IV) $\Rightarrow q_2 - \frac{m_2\cdot l_1\cdot l_2\cdot\cos(\phi_1-\phi_2)}{m_1 l_1^2 + m_2 l_2^2}\cdot q_1 = m_2 l_2^2\dot\phi_2 - \frac{m_2^2\cdot l_1^2\cdot l_2^2\cdot\cos^2(\phi_1-\phi_2)}{m_1 l_1^2 + m_2 l_2^2}\cdot\dot\phi_2$

$\Leftrightarrow (m_1 l_1^2 + m_2 l_2^2)\, q_2 - m_2 l_1 l_2 \cos(\phi_1-\phi_2)\, q_1 = \dot\phi_2\cdot\left[m_2 l_2^2\cdot\left(m_1 l_1^2 + m_2\cdot(l_2^2 - l_1^2\cos^2(\phi_1-\phi_2))\right)\right]$

$$\Leftrightarrow \dot\phi_2 = \frac{(m_1 l_1^2 + m_2 l_2^2)\, q_2 - m_2 l_1 l_2\cos(\phi_1-\phi_2)\, q_1}{m_2 l_2^2\cdot\left(m_1 l_1^2 + m_2\cdot(l_2^2 - l_1^2\cdot\cos^2(\phi_1-\phi_2))\right)} \equiv F_2$$

b.)

$$\dot{q}_1 = -m_2 \cdot l_1 \cdot l_2 \cdot f_1 \cdot f_2 \cdot \sin(\phi_1 - \phi_2) - g \cdot l_1 \cdot (m_1 + m_2) \cdot \sin\phi_1 \equiv f_3$$

$$\dot{q}_2 = m_2 \cdot l_1 \cdot l_2 \cdot f_1 \cdot f_2 \cdot \sin(\phi_1 - \phi_2) - m_2 \cdot g \cdot l_2 \cdot \sin\phi_2 \equiv f_4$$