No. 1.1. • 1D advection problem: $\frac{\partial u}{\partial t} + v \cdot \frac{\partial u}{\partial x} = 0$, $u = u(x,t)$

• Consider: $u(x,t) = q(x - vt)$

$$\frac{\partial q}{\partial t} + v \cdot \frac{\partial q}{\partial x} = 0 \iff \frac{\partial q}{\partial x} \cdot \frac{\partial x}{\partial t} + v \cdot \frac{\partial q}{\partial x} = 0$$

$$\iff \frac{\partial q}{\partial x} \cdot (-v) + v \cdot \frac{\partial q}{\partial x} = 0$$

$$\iff 0 = 0 \implies q(x - vt) \text{ is a solution of the advection problem } \blacksquare$$

No. 1.2. The Euler equations are given by the conservation trio of:

• mass: $\partial_t \rho + \vec{\nabla}(\rho \vec{v}) = 0$, $\rho\vec{v} \mathrel{\hat{=}}$ mass flux

• momentum: $\partial_t(\rho \vec{v}) + \vec{\nabla}(\rho \vec{v}\vec{v}^T + P \cdot \mathbb{1}) = 0$, $\rho\vec{v}\vec{v}^T + P\mathbb{1} \mathrel{\hat{=}}$ stress tensor

• energy: $\partial_t(\rho e) + \vec{\nabla}((\rho e + P)\vec{v}) = 0$, $e \mathrel{\hat{=}}$ energy density

↳ Consider: Isothermal equation of state: $P = c_s^2 \cdot \rho$

↳ Leading order perturbation around the equilibrium solution:

$$\left.\begin{aligned} \rho(\vec{x},t) &= \rho_0 + \delta\rho(\vec{x},t) \equiv \rho_0 + \rho_1(\vec{x},t), \quad \text{with: } \rho_0 >> \rho_1 \\ \vec{v}(\vec{x},t) &= \vec{v}_0 + \delta\vec{v}(\vec{x},t) \equiv \vec{v}_0 + \vec{v}_1(\vec{x},t) \end{aligned}\right\} \begin{aligned} &\rho_0, \vec{v}_0 \text{ solve} \\ &\text{the Euler equations} \\ &\implies \rho_0, \vec{v}_0 \neq f(\vec{x},t) \end{aligned}$$

• PDE for mass and momentum of medium:

(I) $\partial_t \rho + \vec{\nabla}(\rho \vec{v}) = 0$

(II) $\partial_t(\rho \vec{v}) + \vec{\nabla}(\rho \vec{v}\vec{v}^T + c_s^2 \cdot \rho \cdot \mathbb{1}) = 0$

• Rewrite (I) and (II) in terms of leading order perturbations:

(III) $\partial_t \rho_1 + \vec{v}_0^T \cdot \vec{\nabla}\rho_1 + \rho_0 \cdot \vec{\nabla}\vec{v}_1 + \cancel{\rho_1 \vec{\nabla}\vec{v}_1} + \cancel{\vec{v}_1^T \vec{\nabla}\rho_1} = 0$

(IV) $\vec{v}_0 \cdot \partial_t \rho_1 + \rho_0 \cdot \partial_t \vec{v}_1 + c_s^2 \cdot \vec{\nabla}\rho_1 + \rho_0 \vec{\nabla}(\vec{v}_1 \vec{v}_0^T) + \rho_0 \cdot \vec{\nabla}(\vec{v}_0 \vec{v}_1^T) + \rho_0 \cancel{\vec{\nabla}(\vec{v}_1 \vec{v}_1^T)} + \ldots$

$\ldots \cancel{\rho_1 \partial_t \vec{v}_1} + \cancel{\vec{v}_1 \partial_t \rho_1} + \vec{\nabla}(\rho_1 \vec{v}_0 \vec{v}_0^T) + \vec{\nabla}(\cancel{\rho_1 \vec{v}_1 \vec{v}_0^T}) + \vec{\nabla}(\cancel{\rho_1 \vec{v}_0 \vec{v}_1^T}) + \vec{\nabla}(\cancel{\rho_1 \vec{v}_1 \vec{v}_1^T}) = 0$

↳ Higher order terms shall be neglected!

• Use the identity: $\vec{\nabla}(\vec{a}\vec{b}^T) = \vec{a}\vec{\nabla}\vec{b} + \vec{b}^T\vec{\nabla}\vec{a}^T$, to rewrite (IV):

(V) $\cancel{\vec{v}_0 \partial_t \rho_1} + \rho_0 \cdot \partial_t \vec{v}_1 + c_s^2 \vec{\nabla}\rho_1 + \cancel{\vec{v}_0 \vec{v}_0^T \cdot \vec{\nabla}\rho_1} + \cancel{\rho_0 \cdot \vec{v}_0 \vec{\nabla}\vec{v}_1} + \rho_0 \cdot \vec{v}_0^T \vec{\nabla}\vec{v}_1^T = 0$

↳ The marked terms add up to zero according to equation (III)!

No. 1.2. • Perturbed - PDE's:

$$(\mathrm{III}) \quad \partial_t \rho_1 + \vec{v}_0^{\,T} \cdot \vec{\nabla} \rho_1 + \rho_0 \vec{\nabla} \vec{v}_1 = 0$$

$$(\mathrm{V}) \quad \rho_0 \partial_t \vec{v}_1 + c_s^2 \vec{\nabla} \rho_1 + \rho_0 \cdot \vec{v}_0^{\,T} \cdot \vec{\nabla} \vec{v}_1^{\,T} = 0$$

↳ Case 1.: Static background density: $\vec{v}_0 \overset{!}{=} \vec{0}$

$$\Rightarrow \left\{ \begin{array}{l} \partial_t \rho_1 + \rho_0 \vec{\nabla} \vec{v}_1 = 0 \\ \rho_0 \partial_t \vec{v}_1 + c_s^2 \vec{\nabla} \rho_1 = 0 \end{array} \right\} \Rightarrow \left. \begin{array}{l} \partial_t^2 \rho_1 - c_s^2 \Delta \rho_1 = 0 \\ \Leftrightarrow \Box \rho_1 = 0 \quad \blacksquare \end{array} \right\} \text{wave equation}$$

↳ with: $\Box \hat{=}$ D'Alembert operator

↳ Case 2.: $\vec{v}_0 \overset{!}{\neq} \vec{0}$:

- Consider first exercise: $\partial_t u + v \partial_x u = 0$

$$\rightsquigarrow \partial_t u + \vec{v}^{\,T} \vec{\nabla} u = 0 \quad (\mathrm{VI})$$

- Solution for equation (VI) is given by: $u(\vec{x}, t) = q(\vec{x} - \vec{v} t)$
- Define comoving derivative: $D_t u \equiv \partial_t u + \vec{v}^{\,T} \vec{\nabla} u \quad (\mathrm{VII})$
- Rewrite (III) and (V) in terms of comoving derivative:

$$\left. \begin{array}{ll} (\mathrm{VIII}) & D_t \rho_1 + \rho_0 \vec{\nabla} \vec{v}_1 = 0 \\ (\mathrm{IX}) & \rho_0 \cdot D_t \vec{v}_1 + c_s^2 \cdot \vec{\nabla} \rho_1 = 0 \end{array} \right\} \Rightarrow D_t^2 \rho_1 - c_s^2 \cdot \Delta \rho_1 = 0$$

$\hat{=}$ wave equation in the comoving frame of the background medium. $\blacksquare$

- Conclusion: wave propagates with velocities: $\vec{v}_{+/-} = \vec{v}_0 \pm c_s = \frac{\omega}{k}$

No. 6.3. a.)

$R \cong 6371\,km$

circumference: $U = 2\pi R$

velocity of longitudinal p-wave: $V_p \approx 8000 \frac{m}{s}$

- time to travel around the earth: $t_p = \frac{U}{2V_p} = \frac{\pi R}{V_p} \approx 2.5e^3 s \mathrel{\hat{=}} 0.69h$

↳ We are neglecting the fact that the head-wave will be the first trigger of the earthquake since the earth crust is only ~35Km thick. In comparison to the overall travel distance of roughly 20000km this effect has no significant influence on the result for the travel time.

b.) • velocity of longitudinal sound-wave: $V_s \approx 300 \frac{m}{s}$

=> time to travel around the earth: $t_s = \frac{U}{2V_s} = \frac{\pi R}{V_s}$

=> $t_s \approx 66.7e^3 s$

$\mathrel{\hat{=}} 18.5h$

↳ This is not observable since the sound wave is exponentially decaying.