Online Bipartite Matching and Adwords

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Abstract

The simplicity of the recently obtained proof of the optimal algorithm for online bipartite matching (OBM), called RANKING [KVV90], naturally raises the possibility of extending this algorithm to the adwords problem, or its special case called SMALL, in which bids are small compared to budgets; the latter has been of considerable practical significance in ad auctions [MSVV07].

The attractive feature of our approach, in contrast to [MSVV07], was that it would yield a *budget-oblivious algorithm*, i.e., during its execution, the algorithm would not need to know what fraction of budget each bidder has spent — only whether there is budget left-over. This would immediately render the algorithm useful for autobidding platforms.

Our attempt met several hurdles, which are described in detail in the paper. In particular, a substantial probabilistic development led us to obtain an optimal, online, budget-oblivious algorithm for SINGLE-VALUED, which is intermediate between OMB and the adwords problem; this algorithm is a natural generalization of RANKING. For SMALL, we managed to overcome all but one hurdle, namely failure of a property, called the no-surpassing property. Interestingly enough, this property plays a minor role in the proofs of RANKING as well as our algorithm for SINGLE-VALUED.

In this paper, we have stated our result for SMALL, namely an optimal, online, budget-oblivious algorithm, as a Conditional Theorem, after assuming the no-surpassing property. Considering the importance of budget-obliviousness, it is important to remove the assumption — we leave this as an open problem. Another viewpoint is that to make the no-surpassing property fail, the instance has to be intricately "doctored up". Therefore, we believe that in typical instances, this property fails on very few queries, if any; this needs to be experimentally verified. If so, our Conditional Algorithm may be useful as such in practice.

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1 Introduction

The online bipartite matching problem (OBM) occupies a central place not only in online algorithms but also in matching-based market design, see details in Sections 1.2 and 1.1. For formal statements of problems studied in this paper, see Section 2. For OBM, a simple, optimal, randomized online algorithm, called RANKING, was given in [KVV90]. Its competitive ratio is $(1-\frac{1}{e})$ and [KVV90] showed that no randomized online algorithm can achieve a better ratio than $(1-\frac{1}{e})+o(1)$.

The analysis of RANKING given in [KVV90] was considered "extremely difficult". Over the years, several researchers contributed valuable ideas to simplifying its proof; indeed, our work would not have been possible without these ideas, see details in Section 1.2. These proofs involve methodology from two domains, probability theory and combinatorics, with the former playing a dominant role. The simplicity of the final proof [EFFS21] naturally raises the possibility of extending RANKING to generalizations of OBM. Our attempt at this had the effect of applying a stress test to the proof, as a result of which, the combinatorial part was found to be incomplete in a small though essential way, see details in Sections 1.2 and 1.3.

We have named the missing piece the *No-Surpassing Property*, see Property 10 and have given a simple combinatorial fact which helps prove this property. Our paper starts by presenting this analysis¹. Our goal was to extend RANKING all the way to the *adwords problem*, called GENERAL in this paper, or even its special case, called SMALL. Informally, GENERAL involves matching keyword queries, as they arrive online, to advertisers having budget limits, and SMALL is the special case in which bids are small compared to budgets.

The latter problem captures a key computational issue that arises in the context of ad auctions, for instance in Google's AdWords marketplace. An optimal algorithm achieving a competitive ratio of $\left(1-\frac{1}{e}\right)$ was first given in [MSVV07]; for the impact of this result in the marketplace, see 1.1. We note that previous attempts at extending RANKING to SMALL did not meet with success; the alternative approach followed by [MSVV07] is described in Section 1.2. In contrast, GENERAL is a notoriously difficult problem, one of the reasons being inherent structural difficulties, see Section 5.1, and has remained largely unresolved; see below for marginal progress made recently.

Whereas the approach of [MSVV07] attacks SMALL directly, without having to deal with GEN-ERAL, our approach, of building directly on RANKING, needs to first solve GENERAL. Our way of bypassing the structural difficulties of GENERAL was to use the idea of *fake money*, see Section 1.3, and then resort to the fact that in SMALL, the fake money used is negligible in the limit.

Given the complications in GENERAL and the already challenging probabilistic argument in the proof of RANKING, extending RANKING directly to GENERAL appeared quite hopeless. Fortunately, a clean setting was offered by an intermediate problem, which we call SINGLE-VALUED. Under this problem, each bidder can make bids of one value only, although the value may be different for different bidders. It turns out that SINGLE-VALUED had already been solved by a reduction to online vertex weighted matching, see Section 1.2.

However, for our purposes, SINGLE-VALUED needed to be solved from first principles, for

¹The proof presented in this paper is meant to be a "textbook quality" exposition and will appear in the chapter [EIV22] of an upcoming edited book on matching markets.

reasons explained in Section 1.3. This turned out to be valuable in three respects: our algorithm for SINGLE-VALUED, which can be seen as a natural generalization of RANKING, uses fewer random bits (see Section 1.3), the algorithm is budget-oblivious² (see explanation below), and the probabilistic part of the proof of RANKING got extended in a substantial enough manner that we could then attack GENERAL with the idea of fake money. In this last step, whereas the probabilistic argument panned out, the combinatorial part of the proof, in particular, the no-surpassing property, failed to hold. This came as a surprise, since in the proofs of RANKING and SINGLE-VALUED, this property was a small aspect of the combinatorial part, which was a small part of the entire proof!

Since the ideas underlying the algorithm and the rest of the proof of correctness are quite substantial, it was not appropriate to simply abandon them, especially considering the potential gains if the hurdle can be overcome; these gains are specified below. For these reasons, in Section 5.2, we have assumed that the property of no-surpassing and stated our result for SMALL as Conditional Theorem 41. We leave the open problem of filling the gap. Another viewpoint is that in order to make the no-surpassing property fail, the instance has to be intricately "doctored up". For this reason, we believe that in typical instances, this property fails on very few queries, if at all, and the competitiveness of the algorithm may be comparable to that of [MSVV07]; this needs to be experimentally verified. If so, our Conditional Algorithm may be useful as such in practice, especially because of its budget-obliviousness.

We note that our approach to SMALL is more basic than the one used in [MSVV07], since it builds directly on OBM and RANKING, as a result, our conditional algorithm for SMALL is more elementary than the one in [MSVV07]. The effective bid of each bidder j for a query is simply its bid multiplied by its price $p_j = e^{w_j-1}$, where w_j , called the rank of j, is picked at random from [0,1]. On the other hand, the effective bid in [MSVV07] is the bid multiplied by $(1-e^{L_j/B_j})$, where B_j and L_j are the total budget and the leftover budget of bidder j, respectively. As a result, whereas the algorithm of [MSVV07] needs to know the total budget of each bidder, our algorithm does not. During its run, our algorithm only needs to know whether the budget of a bidder has been exhausted. Yet, its revenue is compared to the optimal revenue generated by an offline algorithm with full knowledge of the budget. This budget-obliviousness gives our approach a distinct advantage, since it can be used in autobidding platforms [ABM19, DM22], which dynamically adjust the bids and budgets of advertisers over multiple search engines to improve performance.

For GENERAL, the greedy algorithm, which matches each query to the highest bidder, achieves a competitive ratio of 1/2. Until recently, that was the best possible. In [HZZ20] a marginally improved algorithm, with a ratio of 0.5016, was given. It is important to point out that this 60-page paper was a tour-de-force, drawing on a diverse collection of ideas — a testament to the difficulty of this problem.

Remark 1. The objective of all adwords problems studied in this paper is to maximize the total revenue accrued by the online algorithm. In economics, such a solution is referred to as *efficient*, since the amount bid by an advertiser is indicative of how useful the query is to it, and hence to

²The question of budget-obliviousness for the previous approach [AGKM11] does not arise since in the reduced instance, each bidder only makes one bid.

the economy.

1.1 Significance and Practical Impact

Google's AdWords marketplace generates multi-billion dollar revenues annually and the current annual worldwide spending on digital advertising is almost half a trillion dollars. These revenues of Google and other Internet services companies enable them to offer crucial services, such as search, email, videos, news, apps, maps etc. for free – services that have virtually transformed our lives.

We note that SMALL is the most relevant case of adwords for the search ads marketplace e.g., see [DM22]. A remarkable feature of Google, and other search engines, is the speed with which they are able to show search results, often in milliseconds. In order to show ads at the same speed, together with search results, the solution for SMALL needed to be minimalistic in its use of computing power, memory and communication.

The online algorithm of [MSVV07] satisfied these criteria and therefore had a substantial impact in this marketplace. Furthermore, the idea underlying their algorithm was extracted into a simple heuristic, called *bid scaling*, which uses even less computation and is widely used by search engine companies today. As mentioned above, our Conditional Algorithm for SMALL is even more elementary and is budget-oblivious.

It will be useful to view the AdWords marketplace in the context of a bigger revolution, namely the advent of the Internet and mobile computing, and the consequent resurgence of the area of matching-based market design. The birth of this area goes back to the seminal 1962 paper of Gale and Shapley on stable matching [GS62]. Over the decades, this area became known for its highly successful applications, having economic as well as sociological impact. These included matching medical interns to hospitals, students to schools in large cities, and kidney exchange.

The resurgence led to a host of highly innovative and impactful applications. Besides the Ad-Words marketplace, which matches queries to advertisers, these include Uber, matching drivers to riders; Upwork, matching employers to workers; and Tinder, matching people to each other, see [Ins19] for more details.

A successful launch of such markets calls for economic and game-theoretic insights, together with algorithmic ideas. The Gale-Shapley deferred acceptance algorithm and its follow-up works provided the algorithmic backbone for the "first life" of matching-based market design. The algorithm RANKING has become the paradigm-setting algorithmic idea in the "second life" of this area. Interestingly enough, this result was obtained in the pre-Internet days, over thirty years ago.

1.2 Related Works

We start by describing simplifications to the proof of RANKING for OBM. The first simplifications, in [GM08, BM08], got the ball rolling, setting the stage for the substantial simplification given in [DJK13], using a randomized primal-dual approach. [DJK13] introduced the idea of splitting the contribution of each matched edge into primal and dual contributions and lower-

bounding each part separately. Their method for defining prices p_j of goods, using randomization, was used by subsequent papers, including this one³.

Interestingly enough, the next simplification involved removing the scaffolding of LP-duality and casting the proof in purely probabilistic terms⁴, using notions from economics to split the contribution of each matched edge into the contributions of the buyer and the seller. This elegant analysis was given by [EFFS21]. We note that when we move to generalizations of OBM, even this economic interpretation needs to be dropped, see Remark 18.

We now point out the exact places in the last two papers in which the proofs were incomplete. In Lemma 1, [DJK13] state "Since $Y_i < y^c$, j is matched to i"; note that in this paper, goods are indexed by i and buyers by j. At this place, the proof needs to argue that there is no other good i', which is available to j and satisfies $Y_{i'} < Y_i < y^c$. Next, in Observation 1 in the proof of Lemma 2.3, [EFFS21] state, "This follows since buyer l_i derives higher utility from item r_j than from its match in M_{-j} ." Once again, the proof needs to show that in the run with all goods, an even better match than r_j does not show up for buyer l_i .

An important generalization of OBM is online b-matching. This problem is a special case of GENERAL in which the budget of each advertiser is \$b\$ and the bids are 0/1. [KP00] gave a simple optimal online algorithm, called BALANCE, for this problem. BALANCE awards the next query to the interested bidder who has been matched least number of times so far. [KP00] showed that as b tends to infinity, the competitive ratio of BALANCE tends to $(1 - \frac{1}{e})$.

The importance of online b-matching arises from the fact that it is a special case of SMALL, if b is large. Indeed, the first online algorithm [MSVV07] for SMALL was obtained by extending BALANCE as follows: [MSVV07] first gave a simpler proof of the competitive ratio of BALANCE using the notion of a *factor-revealing LP* [JMM $^+$ 03]. Then they gave the notion of a *tradeoff-revealing LP*, which yielded an algorithm achieving a competitive ratio of $(1-\frac{1}{e})$. [MSVV07] also proved that this is optimal for b-matching, and hence SMALL, by proving that no randomized algorithm can achieve a better ratio for online b-matching; previously, [KP00] had shown a similar result for deterministic algorithms. Following [MSVV07], a second optimal online algorithm for SMALL was given in [BJN07], using a primal-dual approach.

Another relevant generalization of OBM is online vertex weighted matching, in which the offline vertices have weights and the objective is to maximize the weight of the matched vertices. [AGKM11] extended RANKING to obtain an optimal online algorithm for this problem. Clearly, SINGLE-VALUED is intermediate between GENERAL and online vertex weighted matching; moreover, it can be reduced to the latter by creating k_j copies of each advertiser j. As observed by [AGKM11], via this reduction, their algorithm for online vertex weighted matching yields an optimal online algorithm for SINGLE-VALUED, see Section 1.3 for additional comments on this.

In the decade following the conference version (FOCS 2005) of [MSVV07], search engine companies generously invested in research on models derived from OBM and adwords. Their motivation was two-fold: the substantial impact of [MSVV07] and the emergence of a rich collection of digital ad tools. It will be impossible to do justice to this substantial body of work, involving

³For a succinct proof of optimality of the underlying function, e^{x-1} , see Section 2.1.1 in [HT22].

⁴Even though there is no overt use of LP-duality in the proof of [EFFS21], it is unclear if this proof could have been obtained directly, without going the LP-duality-route.

both algorithmic and game-theoretic ideas; for a start, see the surveys [Meh13, HT22].

1.3 Technical Ideas

Our proof of RANKING involves new combinatorial facts, given in Lemma 7 and Corollary 8, which help prove the no-surpassing property. We also introduce a new random variable, u_e , called *threshold*, corresponding to each edge $e = (i,j) \in E$ in the underlying graph, see Definition 9. Besides leading to a slight simplification of the analysis of RANKING, it sets the stage for extension to SINGLE-VALUED via the truncated threshold random variable described below. The key fact needed in the analysis of RANKING is that for any edge (i,j), its expected contribution is at least (1-1/e), and our proof of this fact crucially uses the threshold random variable for edge (i,j).

As noted in Section 1.2, RANKING has been extended all the way to SINGLE-VALUED by [AGKM11]. Our goal is to extend it to GENERAL, and thereby address SMALL. However, GENERAL is very different from SINGLE-VALUED in the following sense. Whereas the latter can be reduced to online vertex weighted matching, the former cannot. The reason is that the manner in which budget B_j of bidder j gets partitioned into bids is not predictable in the former; it depends on the queries, their order of arrival and the randomization executed in a run of the algorithm. Therefore, in order to solve GENERAL, we will first need to solve SINGLE-VALUED without reducing it to online vertex weighted matching. An immediate advantage is that such an algorithm for SINGLE-VALUED will require fewer random bits — only one random rank for each bidder j, as opposed one rank for each of the k_j copies of j.

This is done in Algorithm 20. Almost all of our new ideas, on the probabilistic front, were obtained in the process analyzing this algorithm. First, since vertex j is not split into k_j copies, we cannot talk about the contribution of edges anymore. Even worse, we don't have individual vertices for keeping track of the revenue accrued from each match, as per the scheme of [EFFS21]. Our algorithm gets around this difficulty by accumulating revenue in the same "account" each time bidder j gets matched. The corresponding random variable, r_j , is called the *total revenue* of bidder j, for want of a better name, see Remark 18. Lower bounding $\mathbb{E}[r_j]$ is much more tricky than lower bounding the revenue of a good in OBM, since it involves "teasing apart" the k_j accumulations made into this account; this is done in Lemma 28.

A replacement is also needed for the key lemma in the analysis of RANKING, namely Lemma 14, which lower bounds the contribution of each edge. For this purpose, we give the notion of a *j-star*, denoted X_j , which consists of bidder j together with edges to k_j of its neighbors in G, see Definition 24. The contribution of j-star X_j , is denoted by $\mathbb{E}[X_j]$, which is also defined in Definition 24. Finally, using the lower bound on $\mathbb{E}[r_j]$, Lemma 28 gives a lower $\mathbb{E}[X_j]$ for every j-star, X_j . This lemma crucially uses a new random variable, called *truncated threshold*, see Definition 23.

Next, we explain the reason for truncation in the definition of this random variable. Consider bidder j and a query i_l that is desired by j. Observe that in run \mathcal{R}_j , query i_l can get a bid as large as $B \cdot (1 - \frac{1}{e})$, where $B = \max_{k \in A} \{b_k\}$, whereas the largest bid that j can make to i_l is $b_j \cdot (1 - \frac{1}{e})$; in general, b_j may be smaller than B. Now, i_l contributes revenue to r_j only if i_l is matched to j in run \mathcal{R} , an event which will definitely not happen if $u_{e_l} > b_j \cdot (1 - \frac{1}{e})$. Therefore, whenever

 $u_{e_l} \in [b_j \cdot (1-\frac{1}{e}), \ B \cdot (1-\frac{1}{e})]$, the contribution to r_j is zero. By truncating u_{e_l} to $b_j \cdot (1-\frac{1}{e})$, we have effectively changed the probability density function of u_{e_l} so that the probability of the event $u_{e_l} \in [b_j \cdot (1-\frac{1}{e}), B \cdot (1-\frac{1}{e})]$ is now concentrated at the event $u_{e_l} = b_j \cdot (1-\frac{1}{e})$. From the viewpoint of lower bounding the revenue accrued in r_j , the two probability density functions are equivalent since the revenue accrued is zero under both these events. On the other hand, the truncated random variable enables us to apply the law of total expectation, in the proof of Lemma 28, in the same way as it was done in the proof of lemma 12, without introducing more difficulties.

The combinatorial facts given for RANKING, in Lemma 7 and Corollary 8, also need to be extended for SINGLE-VALUED; this is done in Lemma 21 and Corollary 22. The distinction is that the former facts are based on sets and the latter on multisets. In both cases, these facts help establish the no-surpassing property.

Finally, Algorithm 35 for GENERAL needs to get around the structural difficulties mentioned in Section 5.1. The idea of "fake" money helps get around this problem for the special case of SMALL, since for SMALL, the fake money can be upper-bounded in the worst case.

2 Preliminaries

Online Bipartite Matching (OBM): Let B be a set of n buyers and S a set of n goods. A bipartite graph G = (B, S, E) is specified on vertex sets B and S, and edge set E, where for $i \in B$, $j \in S$, $(i, j) \in E$ if and only if buyer i likes good j. G is assumed to have a perfect matching and therefore each buyer can be given a unique good she likes. Graph G is revealed in the following manner. The n goods are known up-front. On the other hand, the buyers arrive one at a time, and when buyer i arrives, the edges incident at i are revealed.

We are required to design an online algorithm \mathcal{A} in the following sense. At the moment a buyer i arrives, the algorithm needs to match i to one of its unmatched neighbors, if any; if all of i's neighbors are matched, i remains unmatched. The difficulty is that the algorithm does not "know" the edges incident at buyers which will arrive in the future and yet the size of the matching produced by the algorithm will be compared to the best *off-line matching*; the latter of course is a perfect matching. The formal measure for the algorithm is defined in Section 2.1.

General Adwords Problem (GENERAL): Let A be a set of m advertisers, also called bidders, and Q be a set of n queries. A bipartite graph G = (Q, A, E) is specified on vertex sets Q and A, and edge set E, where for $i \in Q$ and $j \in A$, $(i,j) \in E$ if and only if bidder j is interested in query i. Each query i needs to be matched⁵ to at most one bidder who is interested in it. For each edge (i,j), bidder j knows his bid for i, denoted by $bid(i,j) \in \mathbb{Z}_+$. Each bidder also has a budget $B_j \in \mathbb{Z}_+$ which satisfies $B_j \geq bid(i,j)$, for each edge (i,j) incident at j.

Graph G is revealed in the following manner. The m bidders are known up-front and the queries arrive one at a time. When query i arrives, the edges incident at i are revealed, together with the bids associated with these edges. If i gets matched to j, then the matched edge (i,j) is assigned a weight of bid(i,j). The constraint on j is that the total weight of matched edges incident at it be at most B_j . The objective is to maximize the total weight of all matched edges at all bidders.

⁵Clearly, this is not a matching in the usual sense, since a bidder may be matched to several queries.

Adwords under Single-Valued Bidders (SINGLE-VALUED): SINGLE-VALUED is a special case of GENERAL in which each bidder j will make bids of a single value, $b_j \in \mathbb{Z}_+$, for the queries he is interested in. If i accepts j's bid, then i will be matched to j and the weight of this matched edge will be b_j . Corresponding to each bidder j, we are also given $k_j \in \mathbb{Z}_+$, the maximum number of times j can be matched to queries. The objective is to maximize the total weight of matched edges. Observe that the matching M found in G is a b-matching with the b-value of each query i being 1 and of advertiser j being k_j .

Adwords under Small Bids (SMALL): SMALL is a special case of GENERAL in which for each bidder j, each bid of j is small compared to its budget. Formally, we will capture this condition by imposing the following constraint. For a valid instance I of SMALL, define

$$\mu(I) = \max_{j \in A} \left\{ \frac{\max_{(i,j) \in E} \left\{ \operatorname{bid}(i,j) - 1 \right\}}{B_j} \right\}.$$

Then we require that

$$\lim_{n(I)\to\infty} \mu(I) = 0,$$

where n(I) denotes the number of queries in instance I.

2.1 The competitive ratio of online algorithms

We will define the notion of competitive ratio of a randomized online algorithm in the context of OBM.

Definition 2. Let G = (B, S, E) be a bipartite graph as specified above. The competitive ratio of a randomized algorithm \mathcal{A} for OBM is defined to be:

$$c(A) = \min_{G = (B, S, E)} \min_{\rho(B)} \frac{\mathbb{E}[A(G, \rho(B))]}{n},$$

where $\mathbb{E}[\mathcal{A}(G, \rho(B))]$ is the expected size of matching produced by \mathcal{A} ; the expectation is over the random bits used by \mathcal{A} . We may assume that the worst case graph and the order of arrival of buyers, given by $\rho(B)$, are chosen by an adversary who knows the algorithm. It is important to note that the algorithm is provided random bits *after* the adversary makes its choices.

Remark 3. For each problem studied in this paper, we will assume that the offline matching is complete. It is easy to extend the arguments, without changing the competitive ratio, in case the offline matching is not complete. As an example, this is done for OBM in Remark 17.

3 Online Bipartite Matching: RANKING

Algorithm 4 presents an optimal algorithm for OBM. Note that this algorithm picks a random permutation of goods only once. Its competitive ratio is $(1 - \frac{1}{e})$, as shown in Theorem 16. Furthermore, as shown in [KVV90], it is an optimal online bipartite matching algorithm: no randomized algorithm can do better, up to an o(1) term.

Algorithm 4. (Algorithm RANKING)

- 1. **Initialization:** Pick a random permutation, π , of the goods in S.
- 2. **Online buyer arrival:** When a buyer, say i, arrives, match her to the first unmatched good she likes in the order π ; if none, leave i unmatched.

Output the matching, *M*, found.

We will analyze Algorithm 6 which is equivalent to Algorithm 4 and operates as follows. Before the execution of Step (1), the adversary determines the order in which buyers will arrive, say $\rho(B)$. In Step (1), each good j is assigned a *price* $p_j = e^{w_j-1}$, where w_j , called the *rank* of j, is picked at random from [0,1]; observe that $p_j \in [\frac{1}{e},1]$. In Step (2), buyers will arrive in the order $\rho(B)$, picked by the adversary, and will be matched to the cheapest available good. With probability 1 all n prices are distinct and sorting the goods by increasing prices results in a random permutation. Furthermore, since Algorithm 6 uses this sorted order only and is oblivious of the actual prices, it is equivalent to Algorithm 4. As we will see, the random variables representing actual prices are crucially important as well – in the analysis. We remark that for the generalizations of OBM studied in this paper, the prices are used not only in the analysis, but also by the algorithms.

3.1 Analysis of RANKING

We will use an *economic setting* for analyzing Algorithm 6 as follows. Each buyer i has *unit-demand* and 0/1 valuations over the goods she likes, i.e., she accrues unit utility from each good she likes, and she wishes to get at most one of them. The latter set is precisely the set of neighbors of i in G. If on arrival of i there are several of these which are still unmatched, i will pick one having the smallest price 6 . Therefore the buyers will maximize their utility as defined below.

For analyzing this algorithm, we will define two sets of random variables, u_i for $i \in B$ and r_j , for $j \in S$. These will be called utility of buyer i and revenue of good j, respectively. Each run of RANKING defines these random variables as follows. If RANKING matches buyer i to good j, then define $u_i = 1 - p_j$ and $r_j = p_j$, where p_j is the price of good j in this run of RANKING. Clearly, p_j is also a random variable, which is defined by Step (1) of the algorithm. If i remains unmatched, define $u_i = 0$, and if j remains unmatched, define $r_j = 0$. Observe that for each good j, $p_j \in \left[\frac{1}{e}, 1\right]$ and for each buyer i, $u_i \in [0, 1 - \frac{1}{e}]$. Let M be the matching produced by RANKING and let random variable |M| denote its size.

Lemma 5 pulls apart the contribution of each matched edge (i, j) into u_i and r_j . Next, we established in Lemma 14 that for each edge (i, j) in the graph, the total expected contribution of u_i and r_j is at least $1 - \frac{1}{e}$. Then, linearity of expectation allows us to reassemble the 2n terms in the right hand side of Lemma 5 so they are aligned with a perfect matching in G, and this yields Theorem 16.

⁶As stated above, with probability 1 there are no ties.

Algorithm 6. (Algorithm RANKING: Economic Viewpoint)

- 1. **Initialization:** $\forall j \in S$: Pick w_j independently and uniformly from [0,1]. Set price $p_j \leftarrow e^{w_j-1}$.
- 2. **Online buyer arrival:** When a buyer, say *i*, arrives, match her to the cheapest unmatched good she likes; if none, leave *i* unmatched.

Output the matching, *M*, found.

Lemma 5.

$$\mathbb{E}[|M|] = \sum_{i}^{n} \mathbb{E}[u_i] + \sum_{i}^{n} \mathbb{E}[r_j].$$

Proof. By definition of the random variables,

$$\mathbb{E}[|M|] = \mathbb{E}\left[\sum_{i=1}^n u_i + \sum_{j=1}^n r_j\right] = \sum_{i=1}^n \mathbb{E}[u_i] + \sum_{j=1}^n \mathbb{E}[r_j],$$

where the first equality follows from the fact that if $(i,j) \in M$ then $u_i + r_j = 1$ and the second follows from linearity of expectation.

While running Algorithm 6, assume that the adversary has picked the order of arrival of buyers, say $\rho(B)$, and Step (1) has been executed. We next define several ways of executing Step (2). Let \mathcal{R} denote the run of Step (2) on the entire graph G. Corresponding to each good f, let G_f denote graph G with vertex f removed. Define f to be the run of Step (2) on graph G_f .

Lemma 7 and Corollary 8 establish a relationship between the sets of available goods for a buyer i in the two runs \mathcal{R} and \mathcal{R}_j ; the latter is crucially used in the proof of Lemma 12. For ease of notation in proving these two facts, let us renumber the buyers so their order of arrival under $\rho(B)$ is $1, 2, \ldots n$. Let T(i) and $T_j(i)$ denote the sets of unmatched goods at the time of arrival of buyer i (i.e., just before the buyer i gets matched) in the graphs G and G_j , in runs \mathcal{R} and \mathcal{R}_j , respectively. Similarly, let S(i) and $S_j(i)$ denote the set of unmatched goods that buyer i is incident to in G and G_j , in runs \mathcal{R} and \mathcal{R}_j , respectively.

We have assumed that Step (1) of Algorithm 6 has already been executed and a price p_k has been assigned to each good k. With probability 1, the prices are all distinct. Let F_1 and F_2 be subsets of S containing goods k such that $p_k < p_j$ and $p_k > p_j$, respectively.

Lemma 7. For each i, $1 \le i \le n$, the following hold:

1.
$$(T_i(i) \cap F_1) = (T(i) \cap F_1)$$
.

2.
$$(T_i(i) \cap F_2) \subseteq (T(i) \cap F_2)$$
.

Proof. Clearly, in both runs, \mathcal{R} and \mathcal{R}_j , any buyer having an available good in F_1 will match to the most profitable one of these, without even considering the rest of the goods. Since $j \notin F_1$, the two runs behave in an identical manner on the set F_1 , thereby proving the first statement.

The proof of the second statement is by induction on i. The base case is trivially true since $j \notin F_2$. Assume that the statement is true for i = k and let us prove it for i = k + 1. By the first statement, we need to consider only the case that there are no available goods for the k^{th} buyer in F_1 in the runs \mathcal{R} and \mathcal{R}_j . Assume that in run \mathcal{R}_j , this buyer gets matched to good l; if she remains unmatched, we will take l to be null. Clearly, l is the most profitable good she is incident to in $T_j(k)$. Therefore, the most profitable good she is incident to in run \mathcal{R} is the best of l, the most profitable good in $T(k) - T_j(k)$, and j, in case it is available. In each of these cases, the induction step holds.

In the corollary below, the first two statements follow from Lemma 7 and the third statement follows from the first two.

Corollary 8. *For each* i, $1 \le i \le n$, the following hold:

- 1. $(S_i(i) \cap F_1) = (S(i) \cap F_1)$.
- 2. $(S_i(i) \cap F_2) \subseteq (S(i) \cap F_2)$.
- 3. $S_i(i) \subseteq S(i)$.

Next we define a new random variable, u_e , for each edge $e = (i, j) \in E$. This is called the *threshold* for edge e and is given in Definition 9. It is critically used in the proofs of Lemmas 12 and 14.

Definition 9. Let $e = (i, j) \in E$ be an arbitrary edge in G. Define random variable, u_e , called the *threshold* for edge e, to be the utility of buyer i in run \mathcal{R}_j . Clearly, $u_e \in [0, 1 - \frac{1}{e}]$.

Property 10. (No-Surpassing for OBM) Let p_j be such that the bid of j, namely $1 - p_j$, is better than the best bid that buyer i gets in run \mathcal{R}_j . Then, in run \mathcal{R} , no bid to i will surpass $1 - p_j$.

Lemma 11. The No-Surpassing Property holds for OBM.

Proof. Suppose the bid of j, namely $1 - p_j$, is better than the best bid that buyer i gets in run \mathcal{R}_j . If so, i gets no bid from F_1 in \mathcal{R}_j ; observe that they are all higher than $1 - p_j$. Now, by the first part of Corollary 8, i gets no bid from F_1 in run \mathcal{R} as well, i.e., in run \mathcal{R} , no bid to i will surpass $1 - p_j$.

Lemma 12. Corresponding to each edge $(i, j) \in E$, the following hold.

1. $u_i \geq u_e$, where u_i and u_e are the utilities of buyer i in runs \mathcal{R} and \mathcal{R}_i , respectively.

2. Let $z \in [0, 1 - \frac{1}{e}]$. Conditioned on $u_e = z$, if $p_j < 1 - z$, then j will definitely be matched in run \mathcal{R} .

Proof. **1).** By the third statement of Corollary 8, i has more options in run \mathcal{R} as compared to run \mathcal{R}_i , and therefore $u_i \geq u_e$.

2). In run \mathcal{R} , if j is already matched when i arrives, there is nothing to prove. Next assume that j is not matched when i arrives. The crux of the matter is that by Lemma 11, the No-Surpassing Property holds. Therefore, in run \mathcal{R} , i will not have any option that is better than j and will therefore get matched to j. Since $1 - p_j > z$, $S_j(i) \cap F_1 = \emptyset$. Therefore by the first statement of Corollary 8, $S(i) \cap F_1 = \emptyset$. Since i will get no bid better than j in \mathcal{R} , the no-surpassing property indeed holds and i must get matched to j.

Remark 13. The random variable u_e is called *threshold* because of the second statement of Lemma 12. It defines a value such that whenever p_j is smaller than this value, j is definitely matched in run \mathcal{R} .

The intuitive reason for the next, and most crucial, lemma is the following. The smaller u_e is, the larger is the range of values for p_j , namely $[0, 1 - u_e)$, over which (i, j) will be matched and j will accrue revenue of p_j . Integrating p_j over this range, and adding $\mathbb{E}[u_i]$ to it, gives the desired bound. Crucial to this argument is the fact that p_j is independent of u_e . This follows from the fact that u_e is determined by run \mathcal{R}_j on graph G_j , which does not contain vertex j.

Lemma 14. Corresponding to each edge $(i, j) \in E$,

$$\mathbb{E}[u_i + r_j] \ge 1 - \frac{1}{e}.$$

Proof. By the first part of Lemma 12, $\mathbb{E}[u_i] \geq \mathbb{E}[u_e]$.

Next, we will lower bound $\mathbb{E}[r_j]$. Let $z \in [0, 1 - \frac{1}{e}]$ and let us condition on the event $u_e = z$. The critical observation is that u_e is determined by the run \mathcal{R}_j . This is conducted on graph G_j , which does not contain vertex j. Therefore u_e is independent of p_j .

By the second part of Lemma 12, $r_j = p_j$ whenever $p_j < 1 - z$. We will ignore the contribution to $\mathbb{E}[r_j]$ when $p_j \ge 1 - z$. Let w be s.t. $e^{w-1} = 1 - z$.

Now p_j is obtained by picking x uniformly at random from the interval [0,1] and outputting e^{x-1} . In particular, when $x \in [0,w)$, $p_j < 1-z$. If so, by the second part of Lemma 12, j is matched and revenue is accrued in r_j , see Figure 2. Therefore,

$$\mathbb{E}[r_j \mid u_e = z] \ge \int_0^w e^{x-1} dx = e^{w-1} - \frac{1}{e} = 1 - \frac{1}{e} - z.$$

Let $f_{u_e}(z)$ be the probability density function of u_e ; clearly, $f_{u_e}(z) = 0$ for $z \notin [0, 1 - \frac{1}{e}]$. Therefore,

$$\mathbb{E}[r_j] = \mathbb{E}[\mathbb{E}[r_j \mid u_e]] = \int_{z=0}^{1-1/e} \mathbb{E}[r_j \mid u_e = z] \cdot f_{u_e}(z) dz$$

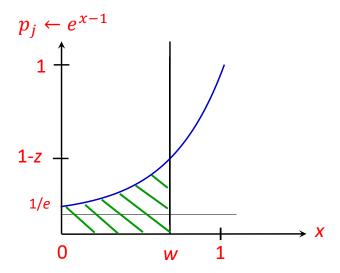


Figure 2: The shaded area is a lower bound on $\mathbb{E}[r_i \mid u_e = z]$.

$$\geq \int_{z=0}^{1-1/e} \left(1-\frac{1}{e}-z\right) \cdot f_{u_e}(z) dz = 1-\frac{1}{e} - \mathbb{E}[u_e],$$

where the first equality follows from the law of total expectation and the inequality follows from fact that we have ignored the contribution to $\mathbb{E}[r_j \mid u_e]$ when $p_j \geq 1-z$. Hence we get

$$\mathbb{E}[u_i + r_j] = \mathbb{E}[u_i] + \mathbb{E}[r_j] \ge 1 - \frac{1}{e}.$$

 $\mathbb{E}[u_i + r_j] = \mathbb{E}[u_i] + \mathbb{E}[r_j] \ge 1 - \frac{1}{e}.$

Remark 15. Observe that Lemma 14 is not a statement about i and j getting matched to each other, but about the utility accrued by i and the revenue accrued by j by being matched to various goods and buyers, respectively, over the randomization executed in Step (1) of Algorithm 6.

Theorem 16. The competitive ratio of RANKING is at least $1 - \frac{1}{e}$.

Proof. Let *P* denote a perfect matching in *G*. The expected size of matching produced by RANK-ING is

$$\mathbb{E}[|M|] = \sum_{i}^{n} \mathbb{E}[u_{i}] + \sum_{j}^{n} \mathbb{E}[r_{j}] = \sum_{(i,j) \in P} \mathbb{E}[u_{i} + r_{j}] \geq n \left(1 - \frac{1}{e}\right),$$

where the first equality uses Lemma 5, the second follows from linearity of expectation and the inequality follows from Lemma 14 and the fact that |P| = n. The theorem follows.

Remark 17. In case G does not have a perfect matching, let P denote a maximum matching in G, of size k, say. Then summing $\mathbb{E}[u_i]$ and $\mathbb{E}[r_j]$ over the the vertices i and j matched by P, we get that the expected size of matching produced by RANKING is at least k $(1-\frac{1}{\epsilon})$.

4 Algorithm for SINGLE-VALUED

Algorithm 20, which will be denoted by A_2 , is an online algorithm for SINGLE-VALUED. Before execution of Step (1) of A_2 , the order of arrival of queries, say $\rho(B)$, is fixed by the adversary. We will define several random variables whose purpose will be quite similar to that in RANKING and they will be given similar names as well; however, their function is not as closely tied to these economics-motivated names as in RANKING, see also Remark 18. Three of these random variables are the *price* p_j and *total revenue* r_j of each bidder $j \in A$, and the *utility* u_i of each query $i \in Q$.

We now describe how values are assigned to these random variables in a run of Algorithm 20. In Step (1), for each bidder j, A_2 picks a price $p_j \in [\frac{1}{e}, 1]$ via the specified randomized process. Furthermore, the revenue r_j and $degree\ d_j$ of bidder j are both initialized to zero, the latter represents the number of times j has been matched. During the run of A_2 , j will get matched to at most k_j queries; each match will add b_j to the total revenue generated by the algorithm. b_j is broken into a revenue and a utility component, with the former being added to r_j and the latter forming u_i . At the end of A_2 , r_j will contain all the revenue accrued by j.

In Step (2), on the arrival of query i, we will say that bidder j is available if $(i,j) \in E$ and $d_j < k_j$. At this point, for each available bidder j, the effective bid of j for i is defined to be $\operatorname{ebid}(j) = b_j \cdot (1 - p_j)$; clearly, $\operatorname{ebid}(j) \in [0, b_j \cdot (1 - \frac{1}{e})]$. Query i accepts the bidder whose effective bid is the largest. If there are no bids, matching M remains unchanged. If i accepts j's bid, then edge (i, j) is added to matching M and the weight of this edge is set to b_j . Furthermore, the utility of i, u_i , is defined to be $\operatorname{ebid}(j)$ and the revenue r_j of j is incremented by $b_j \cdot p_j$. Once all queries are processed, matching M and its weight W are output.

Remark 18. The economics-based names of random variables used in our proof of RANKING came from [EFFS21]. Although we have used the same names for similar random variables in Sections 4 and 5.2, for SINGLE-VALUED and GENERAL, the reader should not attribute an economic interpretation to these the names as was done in RANKING ⁷.

4.1 Analysis of Algorithm 20

For the analysis of Algorithm A_2 , we will use the random variables W, p_j , r_j and u_i defined above; their values are fixed during the execution of A_2 . In addition, corresponding to each edge $e = (i, j) \in E$, in Definition 23, we will introduce a new random variable, u_e , which will play a central role.

⁷We failed to come up with more meaningful names for these random variables and therefore have stuck to the old names.

Algorithm 20. (A_2 : Algorithm for SINGLE-VALUED)

1. Initialization: $M \leftarrow \emptyset$.

 $\forall j \in A$, do:

- (a) Pick w_i uniformly from [0,1] and set price $p_i \leftarrow e^{w_i-1}$.
- (b) $r_i \leftarrow 0$.
- (c) $d_i \leftarrow 0$.
- 2. **Query arrival:** When query *i* arrives, **do**:
 - (a) $\forall j \in A \text{ s.t. } (i, j) \in E \text{ and } d_i < k_i \text{ do:}$
 - i. $\operatorname{ebid}(j) \leftarrow b_i \cdot (1 p_i)$.
 - ii. Offer effective bid of ebid(j) to i.
 - (b) Query *i* accepts the bidder whose effective bid is the largest.

(If there are no bids, matching *M* remains unchanged.)

If i accepts j's bid, then **do**:

- i. Set utility: $u_i \leftarrow b_j \cdot (1 p_j)$.
- ii. Update revenue: $r_i \leftarrow r_i + b_i \cdot p_i$.
- iii. Update degree: $d_j \leftarrow d_j + 1$.
- iv. Update matching: $M \leftarrow M \cup (i, j)$. Define the weight of (i, j) to be b_i .
- (c) **Output:** Output matching *M* and its total weight *W*.

Lemma 19.

$$\mathbb{E}[W] = \sum_{i}^{n} \mathbb{E}[u_{i}] + \sum_{j}^{m} \mathbb{E}[r_{j}].$$

Proof. For each edge $(i, j) \in M$, its contribution to W is b_j . Furthermore, the sum of u_i and the contribution of (i, j) to r_j is also b_j . This gives the first equality below. The second equality follows from linearity of expectation.

$$\mathbb{E}[W] = \mathbb{E}\left[\sum_{i=1}^n u_i + \sum_{j=1}^m r_j\right] = \sum_{i=1}^n \mathbb{E}[u_i] + \sum_{j=1}^m \mathbb{E}[r_j],$$

As in the case of RANKING, we will define several runs of Algorithm 20. In these runs, we will assume Step (1) is executed once. We next define several ways of executing Step (2). Let \mathcal{R} denote the run of Step (2) on the entire graph G. Corresponding to each bidder $j \in A$, let G_j denote graph G with bidder j removed. Define \mathcal{R}_j to be the run of Step (2) on graph G_j .

Analogous to Lemma 7 and Corollary 8 proved for RANKING, we will prove Lemma 21 and Corollary 22, which establish a relationship between the available bidders for a query i in the two

runs \mathcal{R} and \mathcal{R}_j . One difference is that now bidders are available in multiplicity and therefore we will have to use the notion of a multiset rather than a set; this is established next.

A *multiset* contains elements with multiplicity. Let A and B be two multisets over n elements $\{1,2,\ldots n\}$, and let $a_i \geq 0$ and $b_i \geq 0$ denote the multiplicities of element i in A and B, respectively. We will say that $A \subseteq B$ if for each i, $a_i \leq b_i$, and A = B if for each i, $a_i = b_i$. We will say that $i \in A$ if $a_i \geq 1$. We will define $A \cap B$ to be the multiset containing each element i exactly $\min\{a_i,b_i\}$ times, and A - B to be the multiset containing each element i exactly $\max\{a_i - b_i,0\}$ times.

As before, let us renumber the queries so their order of arrival under $\rho(B)$ is $1, 2, \ldots n$. Let T(i) and $T_j(i)$ denote the multisets of available bidders at the time of arrival of query i (i.e., just before the query i gets matched) in runs \mathcal{R} and \mathcal{R}_j , respectively. In particular, T(1) will contain k_l copies of l for each bidder l and $T_j(1)$ will contain k_l copies of l for each bidder l, other than j. Similarly, let S(i) and $S_j(i)$ denote the projections of T(i) and $T_j(i)$ on the neighbors of i in G and G_j , respectively.

We have assumed that Step (1) of Algorithm 6 has already been executed and a price p_k has been assigned to each bidder k. The effective bid of bidder k is $\operatorname{ebid}(k) = b_k \cdot (1 - p_k)$. With probability 1, the effective bids of all bidders are distinct. Let F_1 be the multiset containing k_l copies of l for each $l \in A$ such that $b_l \cdot (1 - p_l) > b_j \cdot (1 - p_j)$. Similarly, let F_2 be the multiset containing k_l copies of l for each $l \in A$ such that and $b_l \cdot (1 - p_l) < b_j \cdot (1 - p_j)$. Observe that j is not contained in either multiset.

Lemma 21. For each i, $1 \le i \le n$, the following hold:

- 1. $(T_i(i) \cap F_1) = (T(i) \cap F_1)$.
- 2. $(T_i(i) \cap F_2) \subseteq (T(i) \cap F_2)$.

Proof. **1).** Clearly, in both runs, \mathcal{R} and \mathcal{R}_j , any query having an available bidder in F_1 will match to the most profitable one of these, without even considering the rest of the bidders. Since $j \notin F_1$, the two runs behave in an identical manner on the set F_1 , thereby proving the first statement.

2). The proof is by induction on i. The base case is trivially true because $(T_j(1) \cap F_2) = (T(1) \cap F_2)$, since $j \notin F_2$. Assume that the statement is true for i = k and let us prove it for i = k + 1. By the first statement, we need to consider only the case that there are no available bidders for the k^{th} query in F_1 in the runs \mathcal{R} and \mathcal{R}_j . Assume that in run \mathcal{R}_j , this query gets matched to bidder l; if it remains unmatched, we will take l to be null. Clearly, l is the most profitable bidder it is incident to in $T_j(k)$. Therefore, the most profitable bidder it is incident to in run \mathcal{R} is the best of l, the most profitable bidder in $T(k) - T_j(k)$, and j, in case it is available. In each of these cases, the induction step holds.

In the corollary below, the first two statements follow from Lemma 21 and the third statement follows from the first two statements.

Corollary 22. For each i, $1 \le i \le n$, the following hold:

- 1. $(S_i(i) \cap F_1) = (S(i) \cap F_1)$.
- 2. $(S_i(i) \cap F_2) \subseteq (S(i) \cap F_2)$.
- 3. $S_i(i) \subseteq S(i)$.

Next we define a new random variable, u_e , for each edge $e = (i, j) \in E$. This is called the *truncated threshold* for edge e and is given in Definition 23. It is critically used in the proofs of Lemmas 27 and 28.

Definition 23. Let $e = (i, j) \in E$ be an arbitrary edge in G. Define random variable, u_e , called the *truncated threshold* for edge e, to be $u_e = \min\{ut_i, b_j \cdot (1 - \frac{1}{e})\}$, where ut_i is the utility of query i in run \mathcal{R}_j .

Definition 24. Let $j \in A$. Henceforth, we will denote k_j by k in order to avoid triple subscripts. Let i_1, \ldots, i_k be queries such that for $1 \le l \le k$, $(i_l, j) \in E$. Then $(j; i_1, \ldots, i_k)$ is called a j-star. Let X_j denote this j-star. The contribution of X_j to $\mathbb{E}[W]$ is $\mathbb{E}[r_j] + \sum_{l=1}^k \mathbb{E}[u_{i_l}]$, and it will be denote by $\mathbb{E}[X_j]$.

Corresponding to *j*-star $X_j = (j; i_1, ..., i_k)$, denote by e_l the edge $(i_l, j) \in E$, for $1 \le l \le k$. Furthermore, let u_{e_l} denote the truncated threshold random variable corresponding to e_l .

Property 25. (No-Surpassing for SINGLE-VALUED) Let p_j be such that the effective bid of j to i_l , namely $\operatorname{ebid}(j) = b_j \cdot (1 - p_j)$, is better than the best bid that query i_l gets in run \mathcal{R}_j . Then, in run \mathcal{R} , no bid to i_l will surpass $b_j \cdot (1 - p_j)$.

Lemma 26. The No-Surpassing Property holds for SINGLE-VALUED.

Proof. Suppose the bid of j, namely $b_j \cdot (1 - p_j)$, is better than the best bid that buyer i gets in run \mathcal{R}_j . If so, i gets no bid from F_1 in \mathcal{R}_j ; observe that they are all higher than $b_j \cdot (1 - p_j)$. Now, by the first part of Corollary 22, i gets no bid from F_1 in run \mathcal{R} as well, i.e., in run \mathcal{R} , no bid to i will surpass $b_j \cdot (1 - p_j)$.

Lemma 27. Corresponding to j-star $X_i = (j; i_1, ..., i_k)$, the following hold.

• For $1 \le l \le k$, $u_{i_l} \ge u_{e_l}$.

Proof. By the third statement of Corollary 22, i_l has more options in run \mathcal{R} as compared to run \mathcal{R}_j . Furthermore, the truncation of the random variable only aids the inequality needed and therefore $u_{i_l} \geq u_{e_l}$.

Our next goal is to lower bound the contribution of an arbitrary j-star, $\mathbb{E}[X_j]$, which in turn involves lower bounding $\mathbb{E}[r_j]$. The latter crucially uses the fact that p_j is independent of u_{e_l} . This follows from the fact that u_{e_l} is determined by run \mathcal{R}_j on graph G_j , which does not contain vertex j.

Lemma 28. Let $j \in A$ and let $X_j = (j; i_1, ..., i_k)$ be a j-star. Then

$$\mathbb{E}[X_j] \geq k \cdot b_j \cdot \left(1 - \frac{1}{e}\right).$$

Proof. We will first lower bound $\mathbb{E}[r_j]$. Let $f_U(b_j \cdot z_1, \dots b_j \cdot z_k)$ be the joint probability density function of $(u_{e_1}, \dots u_{e_k})$; clearly, $f_U(b_j \cdot z_1, \dots b_j \cdot z_k)$ can be non-zero only if $z_l \in [0, 1 - \frac{1}{e}]$, for $1 \le l \le k$. By the law of total expectation,

$$\mathbb{E}[r_j] = \int_{(z_1,\ldots,z_k)} \mathbb{E}[r_j \mid u_{e_1} = b_j \cdot z_1,\ldots,u_{e_k} = b_j \cdot z_k] \cdot f_U(b_j \cdot z_1,\ldots b_j \cdot z_k) dz_1 \ldots dz_k,$$

where the integral is over $z_l \in [0, (1 - \frac{1}{e})]$, for $1 \le l \le k$.

For lower-bounding the conditional expectation in this integral, let $w_l \in [0,1]$ be s.t. $e^{w_l-1} = 1 - z_l$, for $1 \le l \le k$. For $x \in [0,1]$, define the set $S(x) = \{l \mid 1 \le l \le k \text{ and } x < w_l\}$.

Claim 29. Conditioned on $(u_{e_1} = b_j \cdot z_1, \dots, u_{e_k} = b_j \cdot z_k)$, if $p_j = e^{x-1}$, then the degree of j at the end of Algorithm A_2 is at least |S(x)|, i.e., the contribution to r_j in this run was $\geq b_j \cdot p_j \cdot |S(x)|$.

Proof. Suppose $l \in S(x)$, then $x < w_l$. In run \mathcal{R}_j , the maximum effective bid that i_l received has value $b_j \cdot z_l$. In run \mathcal{R} , if at the arrival of query i_l , j is already fully matched, the contribution to r_j in this run was $k \cdot b_j \cdot p_j$ and the claim is obviously true. If not, then since $x < w_l$, $b_j \cdot (1 - p_j) > b_j \cdot z_l$. The crux of the matter is that by Lemma 26, the No-Surpassing Property holds. Therefore, query i_l will receive its largest effective bid from j, i_l will get matched to it, and r_j will be incremented by $b_j \cdot p_j$. The claim follows.

For $1 \le l \le k$, define indicator functions $I_l : [0,1] \to \{0,1\}$ as follows.

$$I_l(x) = \begin{cases} 1 & \text{if } x < w_l, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $|S(x)| = \sum_{l=1}^{k} I_j(x)$. By Claim 29,

$$\mathbb{E}[r_j \mid u_{e_1} = b_j \cdot z_1, \dots, u_{e_k} = b_j \cdot z_k] \geq b_j \cdot \int_0^1 |S(x)| \cdot e^{x-1} dx$$

$$= b_j \cdot \int_0^1 \sum_{l=1}^k I_l(x) \cdot e^{x-1} dx = b_j \cdot \sum_{l=1}^k \int_0^1 I_l(x) \cdot e^{x-1} dx = b_j \cdot \sum_{l=1}^k \int_0^{w_l} e^{x-1} dx$$

$$= b_j \cdot \sum_{l=1}^k \left(e^{w_l - 1} - \frac{1}{e} \right) = b_j \cdot \sum_{l=1}^k \left(1 - \frac{1}{e} - z_l \right).$$

Since $I_l(x) = 0$ for $x \in [w_l, 1]$, we get that $\int_0^1 I_l(x) \cdot e^{x-1} dx = \int_0^{w_l} e^{x-1} dx$; this fact has been used above. Therefore,

$$\mathbb{E}[r_j] = \int_{(z_1, \dots, z_k)} \mathbb{E}[r_j \mid u_{e_1} = b_j \cdot z_1, \dots, u_{e_k} = b_j \cdot z_k] \cdot f_U(b_j \cdot z_1, \dots, b_j \cdot z_k) \ dz_1 \dots dz_k$$

$$\geq b_j \cdot \int_{(z_1,\ldots,z_k)} \sum_{l=1}^k \left(1 - \frac{1}{e} - z_l\right) \cdot f_U(b_j \cdot z_1,\ldots b_j \cdot z_k) \ dz_1 \ldots dz_k$$

$$= k \cdot b_j \cdot \left(1 - \frac{1}{e}\right) - \sum_{l=1}^k \mathbb{E}[u_{e_l}],$$

where both integrals are over $z_l \in [0, (1 - \frac{1}{e})]$, for $1 \le l \le k$.

By Lemma 27, $\mathbb{E}[u_{i_l}] \geq \mathbb{E}[u_{e_l}]$, for $1 \leq l \leq k$. Hence we get

$$\mathbb{E}[X_j] = \mathbb{E}[r_j] + \sum_{l=1}^k \mathbb{E}[u_{i_l}] \geq k \cdot b_j \cdot \left(1 - \frac{1}{e}\right),$$

Theorem 30. The competitive ratio of Algorithm A_2 is at least $1 - \frac{1}{e}$. Furthermore, it is budget-oblivious.

Proof. Let *P* denote a maximum weight *b*-matching in *G*, computed in an offline manner. By the assumption made in Remark 3, its weight is

$$w(P) = \sum_{j=1}^{m} k_j \cdot b_j.$$

Let T_j denote the j-star, under P, corresponding to each $j \in A$. The expected weight of matching produced by A_2 is

$$\mathbb{E}[W] = \sum_{i=1}^{n} \mathbb{E}[u_i] + \sum_{j=1}^{m} \mathbb{E}[r_j] = \sum_{j=1}^{m} \mathbb{E}[T_j] \ge \sum_{j=1}^{m} b_j \cdot k_j \left(1 - \frac{1}{e}\right) = \left(1 - \frac{1}{e}\right) \cdot w(P),$$

where the first equality uses Lemma 5, the second follows from linearity of expectation and the inequality follows from Lemma 28.

Finally, Algorithm A_2 is budget-oblivious because it does not need to know k_j for bidders j; it only needs to know during a run whether the k_j bids available to bidder j have been exhausted. The theorem follows.

5 Algorithm for SMALL, After Assuming No-Surpassing Property

Two new difficulties arise for the problem GENERAL The first is the inherent structural difficulty described in Section 5.1. Second, since bidders can have different bids for different queries, the no-surpassing property does not hold anymore, see Example 31.

Example 31. Assume that in the given instance for GENERAL, j,j' are two of the bidders, and $1, \ldots, k$ are k of the queries, where k is a large number. Assume $\operatorname{bid}(l,j) = \alpha$ for $1 \le l \le k$ and $\operatorname{bid}(l,j') = \alpha - 1$ for $1 \le l \le k - 1$. Further, assume that $\operatorname{bid}(k,j') = (\alpha - 1) \cdot (k - 1)$. Let the budgets be $B_j = \alpha \cdot k$ and $B_{j'} = (\alpha - 1) \cdot (k - 1)$.

Now consider a run in which $p_j = p_{j'} = p$. Assume that in run \mathcal{R}_j , the best effective bid to $1, \ldots, k-1$ comes from j', and in run \mathcal{R}_j , the best effective bid to $1, \ldots, k-1$ comes from j. In run \mathcal{R}_j , the budget of j' is exhausted when k arrives and assume that k does not get any bids, making $u_e = 0$ for e = (k, j). Now in run \mathcal{R} , $\operatorname{ebid}(k, j) = \alpha(1 - p)$ and $\operatorname{ebid}(k, j') = (\alpha - 1) \cdot (k - 1) \cdot (1 - p)$. Thus, even though $\operatorname{ebid}(k, j) > u_e$, k will be matched to j' and not j. Clearly, this phenomenon will hold for all runs in which $p_{j'}$ is not too much larger than p_j .

For the rest of this section, we will make this assumption:

Assumption of No-Surpassing for GENERAL: Let p_j be such that the effective bid of j to i_l , namely $\operatorname{ebid}(i_l, j) = \operatorname{bid}(i_l, j) \cdot (1 - p_j)$, is better than the best bid that query i_l gets in run \mathcal{R}_j . Then, in run \mathcal{R} , no bid to i_l will surpass $\operatorname{bid}(i_l, j) \cdot (1 - p_j)$.

Section 5.2 presents an algorithm for GENERAL, using fake money; the above-stated assumption is used in its analysis, in particular in the proof of Claim 38. Section 5.4 shows that by upper bounding the fake money used in the worst case, we get an optimal algorithm for SMALL, again based on the above-stated assumption.

5.1 Structural Difficulties in GENERAL

To describe the structural difficulties in GENERAL, we provide three instances in Example 32. In order to obtain a completely unconditional result, we would need to adopt the following convention: assume bidder j has L_j money leftover and impression i just arrived. Assume that j's bid for i is bid(i,j). If $bid(i,j) > L_j$, then j should not be allowed to bid for i, since j has insufficient money.

Under this convention, it is easy to see that even a randomized algorithm will accrue only \$W expected revenue on at least one of the instances given in Example 32, provided it is greedy, i.e., if a match is possible, it does not rescind this possibility; the latter condition is a simple way of ensuring that the algorithm is "fine tuned" for a particular type of example. Note that the optimal for each instance is \$2W.

Example 32. Let $W \in \mathbb{Z}_+$ be a large number. We define three instances of GENERAL, each having two bidders, b_1 and b_2 , with budgets of \$W each. Instances I_1 and I_2 have W + 1 queries, where for the first W queries, both bidders bid \$1 each. For the last query, under I_1 , b_1 bids \$W and b_2 is not interested. Under I_2 , b_2 bids \$W and b_1 is not interested. Instance I_3 has 2W queries and both bidders bid \$1 for each of them.

Therefore, to obtain a non-trivial competitive ratio, bidder j must be allowed to bid for i even if $L_j < \text{bid}(i,j)$. This amounts to the use of free disposal, since j will be allowed to obtain query i for less money than its value for i. Next, let's consider a second convention: if $L_j < \text{bid}(i,j)$, then j will bid L_j for i. As stated in Remark 40, this convention is not supported by our proof technique, since Claim 38 fails to hold, breaking the proof of Lemma 37 and hence Lemma 39.

This led us to a third convention: if $L_j < \text{bid}(i, j)$, then j will bid L_j real money and $\text{bid}(i, j) - L_j$ "fake" money for i. As a result, the total revenue of the algorithm consists of real money as well as fake money; in Algorithm 35, these are denoted by W and W_f , respectively. The problem now

is that Lemma 39, which compares the total revenue of the algorithm, namely $W + W_f$, with the optimal offline revenue, does not yield the competitive ratio of Algorithm 35. Remark 40 explains why our proof technique does not allow us to dispense with the use of fake money.

We note that when Algorithm 35 is run on instances of OBM, it reduces to RANKING. Therefore, it is indeed a (simple) extension of RANKING to GENERAL.

5.2 Algorithm for GENERAL

Algorithm 35, which will be denoted by A_3 , is an attempt at online algorithm for GENERAL. As stated in Section 5.1, because of the use of fake money, we will not be able to give a competitive ratio for it, instead, in Lemma 39, we will compare the sum of real and fake money spent by the algorithm with the real money spent by an optimal offline algorithm.

In algorithm A_3 , $L_j \in \mathbb{Z}_+$ will denote bidder j's leftover budget; it is initialized to B_j . At the arrival of query i, bidder j will bid for i if $(i,j) \in E$ and $L_j > 0$. In general, i will receive a number of bids. The exact procedure used by i to accept one of these bids is given in algorithm A_3 ; its steps are self-explanatory. If i accepts j's bid then i is matched to j, the edge (i,j) is assigned a weight of bid(i,j) and L_j is decremented by min $\{L_j, \text{bid}(i,j)\}$.

Note that we do not require that there is sufficient left-over money, i.e., $L_j \ge \operatorname{bid}(i,j)$, for j to bid for i. In case $L_j < \operatorname{bid}(i,j)$ and i accepts j's bid, then $\operatorname{bid}(i,j) - L_j$ of the money paid by j for i is fake money; this will be accounted for by incrementing W_f by $\operatorname{bid}(i,j) - L_j$. The rest, namely L_j , is real money and is added to W. If $\operatorname{bid}(i,j) \ge L_j$ and i accepts j's bid, then L_j becomes zero and j does not bid for any future queries. At the end of the algorithm, random variable W denotes the total real money spent and W_f denotes the total fake money spent.

The *offline optimal solution* to this problem is defined to be a matching of queries to advertisers that maximizes the weight of the matching; this is done with full knowledge of graph G. As stated in Remark 3, we will assume that under such a matching, P, the budget B_j of each bidder j is fully spent, i.e., $w(P) = \sum_{j=1}^m B_j$.

5.3 Analysis of Algorithm 35

Lemma 33.

$$\mathbb{E}[W + W_f] = \sum_{i=1}^{n} \mathbb{E}[u_i] + \sum_{i=1}^{m} \mathbb{E}[r_i].$$

Proof. For each edge $(i, j) \in M$, its contribution to $W + W_f$ is bid(i, j). Furthermore, the sum of u_i and the contribution of (i, j) to r_j is also bid(i, j). This gives the first equality below. The second equality follows from linearity of expectation.

$$\mathbb{E}[W+W_f] = \mathbb{E}\left[\sum_{i=1}^n u_i + \sum_{j=1}^m r_j\right] = \sum_i^n \mathbb{E}[u_i] + \sum_j^m \mathbb{E}[r_j],$$

Recall that for SINGLE-VALUED, we gave Lemma 21 and Corollary 22, which established a relationship between the available bidders for a query i in the two runs \mathcal{R} and \mathcal{R}_j . These facts dealt with multisets rather than sets; the latter sufficed for Lemma 7 and Corollary 8, which were used in the analysis of RANKING. In Section 4, we also defined operations on multisets.

We will need Lemma 21 and Corollary 22 for analyzing Algorithm 35 as well, though the definitions of the multisets will be guided by the following: If bidder $k \in A$ has leftover money of L_k , as determined by Algorithm 35, then we will say that i has L_k copies of k available to it. Furthermore, if i's bid for k is bid(i,k) and this bid is successful, then L_k will be decremented by min $\{L_k, \text{bid}(i,k)\}$, as stated in Step 2(b)(v) of the algorithm, and the available copies of k for the next bidder will decrease accordingly.

As before, let us renumber the queries so their order of arrival under $\rho(B)$ is 1, 2, ... n. Let T(i) and $T_j(i)$ denote the multisets of available copies of each bidder at the time of arrival of query i (i.e., just before the query i gets matched), in runs \mathcal{R} and \mathcal{R}_j , respectively. Similarly, let S(i) and $S_j(i)$ denote the multisets obtained by restricting T(i) and $T_j(i)$ to the bidders that have edges to query i in graphs G and G_i , respectively.

We have assumed that Step (1) of Algorithm 6 has already been executed and a price p_k has been assigned to each good k. With probability 1, the prices are all distinct. Let F_1 be the multiset containing B_l copies of l for each $l \in A$ such that $p_l < p_j$. Similarly, let F_2 be the multiset containing B_l copies of l for each $l \in A$ such that and $p_l > p_j$.

Under the definitions and operations stated above, it is easy to check that Lemma 21 and Corollary 22 hold for Algorithm 35 as well. Therefore, Lemma 27 also carries over. Definition 23 needs to be modified to the following.

Definition 34. Let $e = (i, j) \in E$ be an arbitrary edge in G. Define random variable, u_e , called the *truncated threshold* for edge e, to be $u_e = \min\{u_i, \operatorname{bid}(i, j) \cdot (1 - \frac{1}{e})\}$, where u_i is the utility of query i in run \mathcal{R}_j .

Definition 24 needs to be changed to the following.

Definition 36. Let $j \in A$. Let i_1, \ldots, i_k be queries such that for $1 \le l \le k$, $(i_l, j) \in E$ and $\sum_{l=1}^k \operatorname{bid}(i_l, j) = B_i$. Then $(j; i_1, \ldots, i_k)$ is called a B_j -star. Let X_j denote this B_j -star. The contribution of X_j to $\mathbb{E}[W]$ is $\mathbb{E}[r_j] + \sum_{l=1}^k \mathbb{E}[u_{i_l}]$, and it will be denote by $\mathbb{E}[X_j]$.

Corresponding to B_j -star $X_j = (j; i_1, ..., i_k)$, denote by e_l the edge $(i_l, j) \in E$, for $1 \le l \le k$. Furthermore, let u_{e_l} denote the truncated threshold random variable corresponding to e_l . The next lemma crucially uses the fact that p_j is independent of u_{e_l} ; the reason for this fact is the same as in SINGLE-VALUED.

Lemma 37. Let $j \in A$ and let $X_j = (j; i_1, ..., i_k)$ be a B_j -star. Then

$$\mathbb{E}[X_j] \geq B_j \cdot \left(1 - \frac{1}{e}\right).$$

Algorithm 35. (A_3 : Algorithm for GENERAL)

1. **Initialization:** $M \leftarrow \emptyset$, $W \leftarrow 0$ and $W_f \leftarrow 0$

 $\forall j \in A$, do:

- (a) Pick w_i uniformly from [0,1] and set price $p_i \leftarrow e^{w_i-1}$.
- (b) $r_i \leftarrow 0$.
- (c) $L_j \leftarrow B_j$.
- 2. **Query arrival:** When query *i* arrives, **do**:
 - (a) $\forall j \in A \text{ s.t. } (i, j) \in E \text{ and } L_i > 0 \text{ do: }$
 - i. $\operatorname{ebid}(i, j) \leftarrow \operatorname{bid}(i, j) \cdot (1 p_i)$.
 - ii. Offer effective bid of ebid(i, j) to i.
 - (b) Query *i* accepts the bidder whose effective bid is the largest.

(If there are no bids, matching M remains unchanged.)

If i accepts j's bid, then **do**:

- i. Set utility: $u_i \leftarrow \text{bid}(i, j) \cdot (1 p_i)$.
- ii. Update revenue: $r_i \leftarrow r_i + \text{bid}(i, j) \cdot p_i$.
- iii. Update matching: $M \leftarrow M \cup (i, j)$.
- iv. Update weight: $W \leftarrow \min\{L_i, \text{bid}(i, j)\}\$ and $W_f \leftarrow \max\{0, \text{bid}(i, j) L_i\}.$
- v. Update L_i : $L_i \leftarrow L_i \min\{L_i, \operatorname{bid}(i, j)\}$.
- 3. **Output:** Output matching M, real money spent W, and fake money spent W_f .

Proof. We will first lower bound $\mathbb{E}[r_j]$. Let $f_U(\text{bid}(i_1,j) \cdot z_1, \ldots, \text{bid}(i_k,j) \cdot z_k)$ be the joint probability density function of $(u_{e_1}, \ldots u_{e_k})$; clearly, $f_U(\text{bid}(i_1,j) \cdot z_1, \ldots, \text{bid}(i_k,j) \cdot z_k)$ can be non-zero only if $z_l \in [0,1-\frac{1}{e}]$, for $1 \leq l \leq k$.

By the law of total expectation, $\mathbb{E}[r_j] =$

$$\int_{(z_1,\ldots,z_k)} \mathbb{E}[r_j \mid u_{e_1} = \operatorname{bid}(i_1,j) \cdot z_1,\ldots,u_{e_k} = \operatorname{bid}(i_k,j) \cdot z_k] \cdot f_U(\operatorname{bid}(i_1,j) \cdot z_1,\ldots\operatorname{bid}(i_k,j) \cdot z_k) \, dz_1 \ldots dz_k,$$

where the integral is over $z_l \in [0, (1 - \frac{1}{e})]$, for $1 \le l \le k$.

For lower-bounding the conditional expectation in this integral, let $w_l \in [0,1]$ be s.t. $e^{w_l-1} = 1 - z_l$, for $1 \le l \le k$. Let $x \in [0,1]$. For $1 \le l \le k$, define indicator functions $I_l : [0,1] \to \{0,1\}$ as follows.

$$I_l(x) = \begin{cases} 1 & \text{if } x < w_l, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, define

$$V(x) = \sum_{l=1}^{k} I_l(x) \cdot \operatorname{bid}(i_l, j).$$

Claim 38. Conditioned on $(u_{e_1} = \text{bid}(i_1, j) \cdot z_1, \dots, u_{e_k} = \text{bid}(i_k, j) \cdot z_k)$, if $p_j = e^{x-1}$, where $x \in [0, 1]$, then the contribution to r_j in this run of algorithm A_3 was $\geq p_j \cdot V(x)$.

Proof. Suppose $I_l(x) = 1$, then $x < w_l$. In run \mathcal{R}_j , the maximum effective bid that i_l received has value $\operatorname{bid}(i_l, j) \cdot z_l$. In run \mathcal{R} , if on the arrival of query i_l , $L_j = 0$, i.e., j is already fully matched, then the contribution to r_j in this run was $B_j \cdot p_j$ and the claim is obviously true. If $L_j > 0$, then since $x < w_l$, $1 - p_j > z_l$. Therefore, by Corollary 22, query i_l will receive its largest effective bid from j. Hence, i_l will get matched to j and r_j will be incremented by $\operatorname{bid}(i_l, j) \cdot p_j$. The claim follows.

By Claim 38,

$$\mathbb{E}[r_j \mid u_{e_1} = \text{bid}(i_1, j) \cdot z_1, \dots, u_{e_k} = \text{bid}(i_k, j) \cdot z_k] \ge \int_0^1 V(x) \cdot e^{x-1} dx$$

$$= \sum_{l=1}^k \text{bid}(i_l, j) \cdot \int_0^1 I_l(x) \cdot e^{x-1} dx = \sum_{l=1}^k \text{bid}(i_l, j) \cdot \int_0^{w_l} e^{x-1} dx$$

$$= B_j \cdot \sum_{l=1}^k \left(e^{w_l - 1} - \frac{1}{e} \right) = B_j \cdot \sum_{l=1}^k \left(1 - \frac{1}{e} - z_l \right).$$

Therefore, $\mathbb{E}[r_i] =$

$$\int_{(z_1,\ldots,z_k)} \mathbb{E}[r_j \mid u_{e_1} = \operatorname{bid}(i_1,j) \cdot z_1,\ldots,u_{e_k} = \operatorname{bid}(i_k,j) \cdot z_k] \cdot f_U(\operatorname{bid}(i_1,j) \cdot z_1,\ldots\operatorname{bid}(i_k,j) \cdot z_k) \, dz_1 \ldots dz_k,$$

$$\geq B_j \cdot \int_{(z_1,\dots,z_k)} \sum_{l=1}^k \left(1 - \frac{1}{e} - z_l\right) \cdot f_U(\operatorname{bid}(i_1,j) \cdot z_1, \dots \operatorname{bid}(i_k,j) \cdot z_k) \ dz_1 \dots dz_k$$

$$= B_j \cdot \left(1 - \frac{1}{e}\right) - \sum_{l=1}^k \mathbb{E}[u_{e_l}].$$

By Lemma 27, $\mathbb{E}[u_{i_l}] \geq \mathbb{E}[u_{e_l}]$, for $1 \leq l \leq k$. Hence we get

$$\mathbb{E}[X_j] = \mathbb{E}[r_j] + \sum_{l=1}^k \mathbb{E}[u_{i_l}] \geq B_j \cdot \left(1 - \frac{1}{e}\right),$$

Lemma 39. Algorithm A_3 satisfies

$$\mathbb{E}\left[W+W_f\right] \ \geq \ \left(1-\frac{1}{e}\right)\cdot w(P).$$

Furthermore, it is budget-oblivious.

Proof. Let *P* denote a maximum weight *b*-matching in *G*. By the assumption made in Remark 3, its weight is

$$w(P) = \sum_{j=1}^{m} B_j.$$

Let T_j denote the j-star, under P, corresponding to each $j \in A$. The expected weight of matching produced by A_3 is

$$\mathbb{E}\left[W + W_f\right] = \sum_{i=1}^n \mathbb{E}\left[u_i\right] + \sum_{j=1}^m \mathbb{E}[r_j] = \sum_{j=1}^m \mathbb{E}[T_j] \geq \sum_{j=1}^m B_j \cdot \left(1 - \frac{1}{e}\right) = \left(1 - \frac{1}{e}\right) \cdot w(P),$$

where the first equality uses Lemma 5, the second follows from linearity of expectation and the inequality follows by using Lemma 37.

Finally, Algorithm A_3 is budget-oblivious because it does not need to know the budgets B_j for bidders j; it only needs to know during a run whether B_j has been exhausted. The lemma follows.

Remark 40. Let us consider the following two avenues for dispensing with the use of fake money altogether; we will show places where our proof technique breaks down for each one. Assume $L_i < \text{bid}(i, j)$.

- 1. Why not modify Step 2 of Algorithm 35 so that j's bid for i is taken to be L_j instead of bid(i,j)?
- 2. Why not modify Step 2(b)(i) so it sets u_i to $L_j \cdot (1 p_j)$ rather than $B_j \cdot (1 p_j)$

Under the first avenue, we cannot ensure $u_i \ge u_e$, since it may happen that $u_e > L_j \cdot (1 - p_j) = u_i$. The condition $u_i \ge u_e$ is used for deriving $\mathbb{E}[u_i] \ge \mathbb{E}[u_e]$, which is essential in the proof of Lemma 37.

To make the second avenue work, the proof of Claim 38 would need to be changed as follows: the last case, $L_j > 0$, will need to be split into the two cases given above. However, under Case 2, which applies if $L_j < \operatorname{bid}(i,j)$, even though $p_j < p$, the largest effective bid that query i_l receives may not be the one from j, since the effective bid of j has value $L_j \cdot (1 - p_j) < \operatorname{bid}(i_l, j) \cdot (1 - p_j)$. Therefore, i_l may not get matched to j, thereby invalidating Claim 38.

5.4 Algorithm for SMALL

We will use Lemma 39 to show that Algorithm 35 yields algorithms for SMALL by upper bounding the fake money used in the worst case. Their budget-obliviousness follows from that of Algorithm 35.

Conditional Theorem 41. Algorithm A_3 is an optimal online algorithm for SMALL; furthermore, it is budget-oblivious.

Proof. Let *I* be an instance of SMALL.

$$W_f \le \sum_{j \in A} \max_{(i,j) \in E} \left\{ \operatorname{bid}(i,j) - 1 \right\}$$

Therefore,

$$\mu(I) = \max_{j \in A} \left\{ \frac{\max_{(i,j) \in E} \left\{ \operatorname{bid}(i,j) - 1 \right\}}{B_j} \right\} \ge \frac{\sum_{j \in A} \max_{(i,j) \in E} \left\{ \operatorname{bid}(i,j) - 1 \right\}}{\sum_{j \in A} B_j} \ge \frac{W_f}{w(P)},$$

where $\mu(I)$ is defined in Section 2. Now, by definition of SMALL,

$$\lim_{n(I)\to\infty} \mu(I) = 0,$$

where n(I) denotes the number of queries in instance I.

Therefore

$$\lim_{n(I)\to\infty} \frac{W_f}{w(P)} = 0.$$

The theorem follows from Lemma 39.

6 Discussion

The open question mentioned in the Introduction, of removing the assumption of no-surpassing property from our proof of Algorithm 35 for SMALL, deserves special attention because of its potential impact in the ad auctions marketplace.

The main open question of the area is to make a substantial dent on GENERAL, or even better give an optimal online algorithm for it, without making any assumptions on bids vs budgets.

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