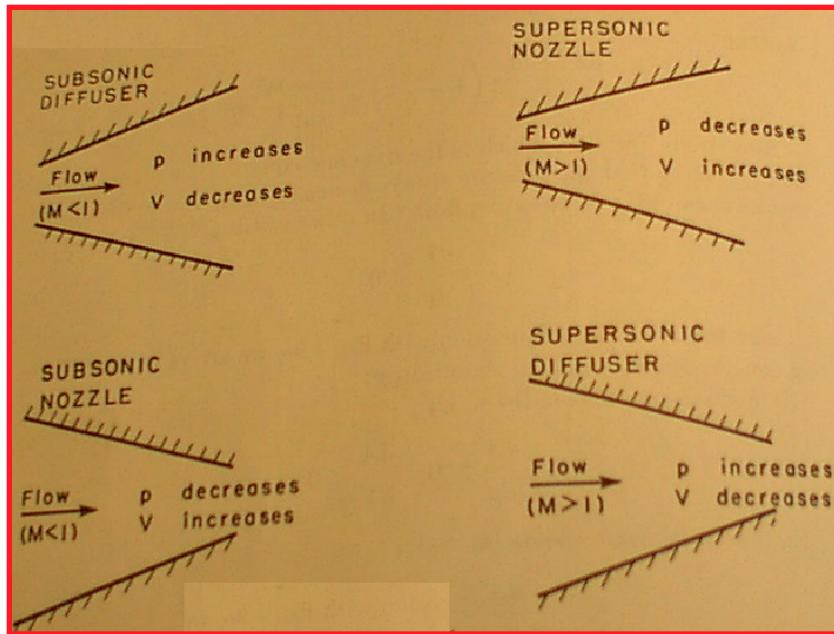


Section 5, Lecture 1:

Thermodynamics and Compressible Flow Review



Summary of Fundamental Properties and
Relationships

Sutton and Biblarz, Chapter 3

Equation of State for a Perfect Gas

- Relationship Between pressure, temperature, and density derived empirically in Modern form by John Dalton
- Theoretically derived by Ludwig Boltzmann using statistical Thermodynamics
- In perfect gas ... intermolecular (van der Waals) forces are neglected



John Dalton

$$p \ v = n \ R_u \ T$$

- p - pressure acting on gas
- v - volume of gas in system
- n - Number of moles of gas in system
- R_u - Universal gas constant
- T - Temperature of gas

p
 v
 T
 n

1-mole $\rightarrow 6.02 \times 10^{23}$

Avagadro's number

Equation of State for a Perfect Gas (cont'd)

- Re organizing the equation of state

$$p = \frac{M}{V} \frac{n}{M} R_u T = \rho \frac{R_u}{M/n} T = \rho \frac{R_u}{M_w} T = \rho R_g T$$

- p - pressure acting on gas
- v - volume of gas in system
- n - Number of moles of gas in system
- R_u - Universal gas constant
- T - Temperature of gas
- M_w - Molecular weight of gas
- R_g - Gas Specific Constant
- M - Mass of gas contained in volume

- Useful working form for Gas Dynamics

$$p = \rho \frac{R_u}{M_w} T = \rho R_g T \rightarrow R_g = \frac{R_u}{M_w}$$

Equation of State for a Perfect Gas (cont'd)

- Numerical Values for Universal Gas Constant

$$R_u = 1545.40 \text{ ft-lbf/}^{\circ}\text{R-(lbm-mole)}$$

$$R_u = 49722.01 \text{ ft-lbf/}^{\circ}\text{R-(slug-mole)}$$

$$R_u = 8314.4126 \text{ J/}^{\circ}\text{K-(kg-mole)}$$

- Gas Specific constant is Universal constant divided by the average molecular weight of the gas

- Molecular weight of various gases

Hydrogen	2.0
Helium	4.0
Methane	16.0
Ammonia	17.0
Water	18.0
Neon	20.2
Nitrogen	28.0
Air	28.9
Oxygen	32.0
Argon	39.9
CO ₂	44.0

Equation of State for a Perfect Gas (concluded)

- Molecular weight of Air

Average molecular weight of the gases in the atmosphere.

Air on earth at sea level is a mixture of approximately 78% nitrogen, 21% oxygen, with the remaining one percent a mix of argon, carbon dioxide, neon, helium and other rare gases,

~ 28.96443 kg/kg-mole

- Numerical Values for Air Specific Gas Constant

$$R_g = 53.355 \quad \text{ft-lbf/}^{\circ}\text{R-(lbm)}$$

$$R_g = 1716.658 \quad \text{ft-lbf/}^{\circ}\text{R-(slug)}$$

$$R_g = 287.056 \quad \text{J/}^{\circ}\text{K-(kg)}$$

Specific Heats, Internal Energy, and Enthalpy

(cont'd)

- e is the internal energy per unit mass

The total kinetic and potential energy associated with the motions and relative positions of the molecules a gas, solid, or liquid. An increase in internal energy results in a rise in temperature or a change in phase

- h is the enthalpy per unit mass

The measure of the heat content of a chemical or physical system; equivalent to the sum of the internal energy of the system plus the product of its volume multiplied by the pressure exerted on it by its Surroundings ... i.e.

$$h = e + p v$$

h is the specific enthalpy, (SI - joules/kg)

e is the specific internal energy, (SI - joules.kg)

P is the pressure of the system, (SI - pascals)

v is the specific volume, (SI cubic metres/kg) = $1/\rho$

Specific Heats, Internal Energy, and Enthalpy

- Specific heat definitions: (cont'd)

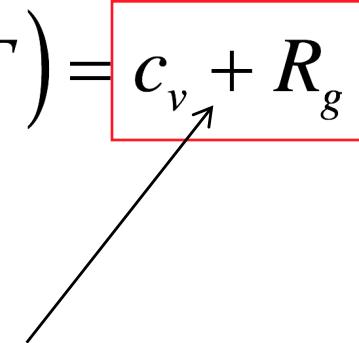
- Constant Volume:

$$c_v = \left(\frac{de}{dT} \right)_v$$

$$\boxed{c_p} = \left(\frac{dh}{dt} \right)_p = \frac{d}{dT}(e + pv) =$$

- Constant Pressure:

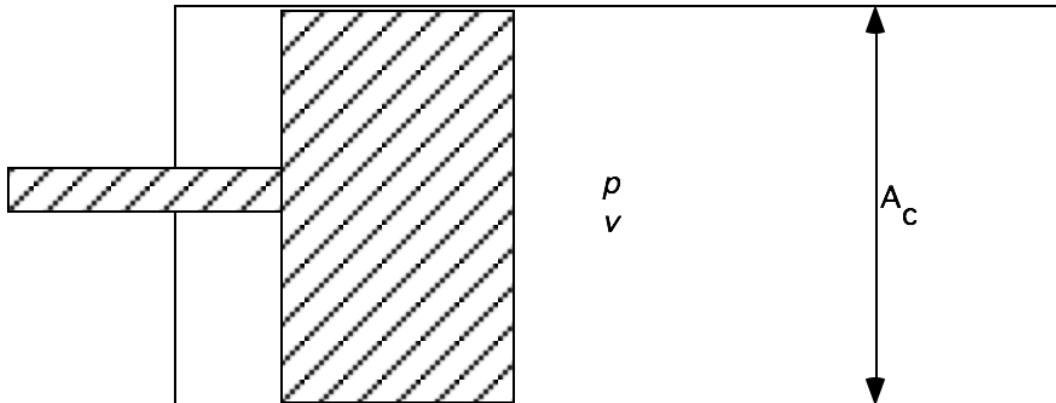
$$c_p = \left(\frac{dh}{dT} \right)_p$$

$$\frac{de}{dT} + \frac{d}{dT}(R_g T) = \boxed{c_v + R_g}$$


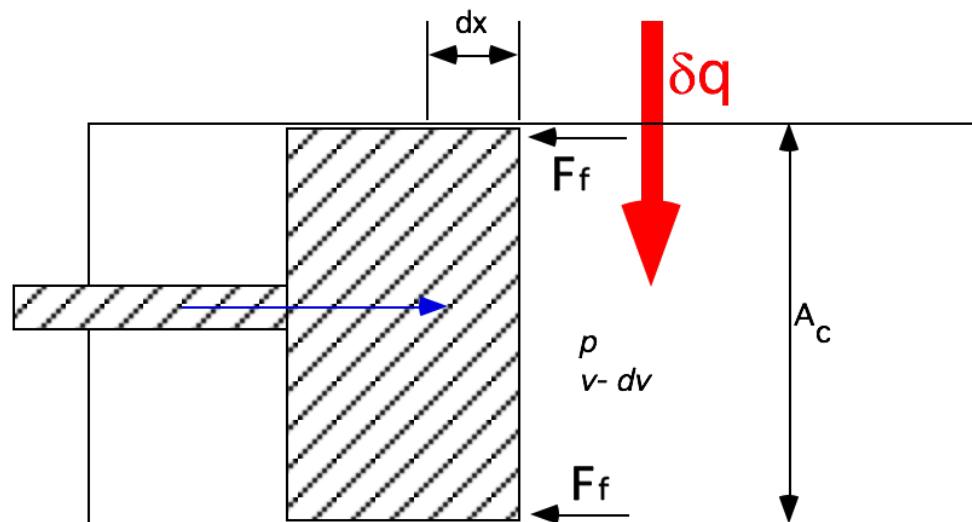
Universal relationship applies to all perfect gases

First Law of Thermodynamics

work performed by piston on system
with heat addition



- Change in internal Energy = heat added + work performed+ Energy dissipated



$$de = dq + dw_{rev} - dw_{irrev}$$

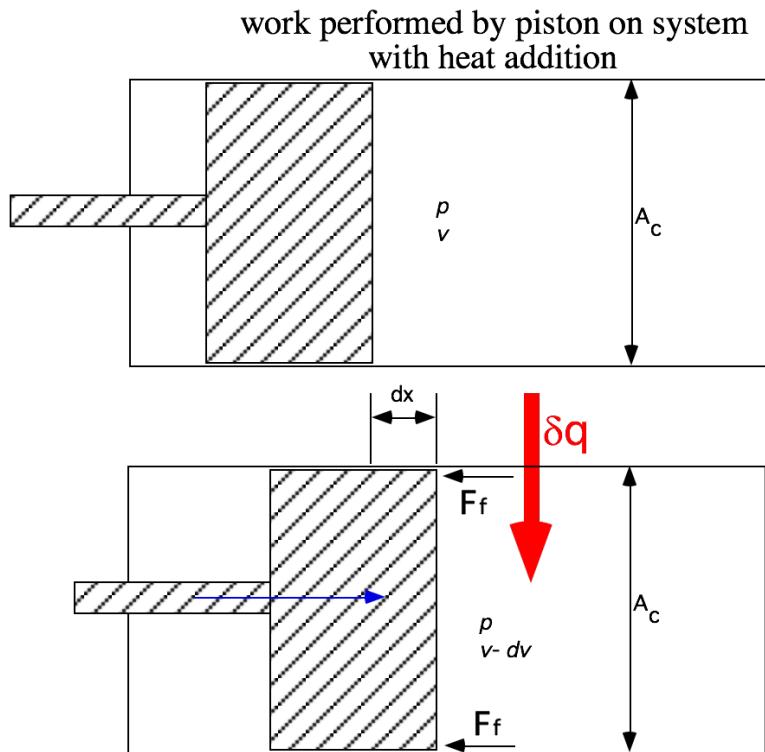
$$dw_{rev} = Fdx = pA_c dx = -pdv$$

$$dW_{irrev} = F_f dx$$

First Law of Thermodynamics (continued)

- Recall from earlier lecture

- 1) Adiabatic Process... no heat loss or addition
- 2) Reversible Process ... no dissipative phenomena occur
- 3) Reversible adiabatic process is referred to as *isentropic*



$$dq = 0 \text{ (adiabatic process)}$$

$$de = -(pdv + F_f dx)$$

$$F_f = 0 \text{ (isentropic process)}$$

$$de = -pdv$$

First Law of Thermodynamics (concluded)

- In terms of Enthalpy

- reversible process

$$de = dq - pdv$$

$$h = e + pv \rightarrow dh = de + vdp + pdv$$

$$dh = dq - pdv + vdp + pdv = dq + vdp$$

- isentropic process (reversible, adiabatic)

$$dh = vdp$$

Second Law of Thermodynamics (cont'd)

- Thermodynamic entropy, s , is a measure of the amount of energy in a physical system that cannot be used to do work.
- It is a relative measure of the disorder present in a system.
(SI unit of entropy is J/°K)

$$ds = \frac{dq}{T} + ds_{irrev}$$

Reversible heat addition Effect of dissipative processes

- Calculate change in entropy:

$$\text{First Law: } de = dq - pdv - dw_{irrev}$$

$$\text{Second Law: } Tds = dq + ds_{irrev}$$

Second Law of Thermodynamics (cont'd)

For a reversible process

$$de = dq - pdv$$

$$Tds = de + pdv$$

$$Tds = \delta q$$

Writing e in terms of enthalpy ($h = e + p v$)

$$dh = de + pdv + vdp$$

and

$$Tds = de + pdv$$

$$dh = de + pdv + vdp$$



$$dh = Tds - pdv + pdv + vdp =$$

$$dh = Tds + vdp$$

Second Law of Thermodynamics (cont'd)

Solving for ds and integrating between state 1 and state 2

$$s_2 - s_1 = \int_1^2 \left[\frac{dh}{T} - \frac{vdp}{T} \right]$$

But from definition $c_p = \left(\frac{dh}{dT} \right)_p \longrightarrow s_2 - s_1 = \int_1^2 \left[\frac{c_p dT}{T} - \frac{vdp}{T} \right]$

From the equation of state

$$p = \rho R_g T = \frac{1}{\nu} R_g T$$

$$\nu = \frac{R_g T}{p}$$

Second Law of Thermodynamics (concluded)

- Substituting equation of state into entropy integral

$$s_2 - s_1 = \int_1^2 \left[\frac{c_p dT}{T} - \frac{vdp}{T} \right] = \int_1^2 \left[\frac{c_p dT}{T} - \frac{\frac{R_g T}{p} dp}{T} \right] =$$

$$\int_1^2 \left[\frac{c_p dT}{T} - R_g \frac{dp}{p} \right] \quad \text{Valid for thermally perfect gas}$$

- For a thermally perfect gas, $c_p \sim \text{constant}$

$$s_2 - s_1 = c_p \ln_2 \left[\frac{T_2}{T_1} \right] - R_g \ln \left[\frac{p_2}{p_1} \right] \quad \text{Valid for adiabatic process}$$

Isentropic Process Relationships

- For an isentropic process, (adiabatic, reversible) $dq = 0$

$$\cancel{ds = \frac{dq}{T} + ds_{irrev}} = 0$$

and $c_p \ln_2 \left[\frac{T_2}{T_1} \right] - R_g \ln \left[\frac{p_2}{p_1} \right] \longrightarrow \left[\frac{T_2}{T_1} \right]^{c_p} = \left[\frac{p_2}{p_1} \right]^{R_g}$

Solving for p_2 in terms of T_2

$$\frac{p_2}{p_1} = \left[\frac{T_2}{T_1} \right]^{\frac{c_p}{R_g}} = \left[\frac{T_2}{T_1} \right]^{\frac{c_p}{c_p - c_v}}$$

Isentropic Process Relationships (cont'd)

- Defining a *ratio of specific heats*, $\gamma = c_p/c_v$

$$\frac{p_2}{p_1} = \left[\frac{T_2}{T_1} \right]^{\frac{\gamma}{\gamma-1}}$$

- Applying the equation of state

$$\frac{T_2}{T_1} = \frac{p_2}{p_1} \frac{\rho_1}{\rho_2}$$

- and

$$\left[\frac{p_2}{p_1} \right]^{1-\frac{\gamma}{\gamma-1}} = \left[\frac{p_2}{p_1} \right]^{\frac{-1}{\gamma-1}} = \left[\frac{\rho_2}{\rho_1} \right]^{\frac{-\gamma}{\gamma-1}} \rightarrow \boxed{\left[\frac{p_2}{p_1} \right] = \left[\frac{\rho_2}{\rho_1} \right]^\gamma}$$

Ratio of Specific Heats

- $\gamma = c_p/c_v$ is a critical parameter for compressible flow analysis

Approximate Specific Heat Ratio for Various Gases, at moderate temperatures

Gas	Ratio of Specific Heats
Carbon Dioxide	1.3
Helium	1.66
Hydrogen	1.41
Methane or Natural Gas	1.31
Nitrogen	1.4
Oxygen	1.4
Standard Air	1.4

Ratio of Specific Heats (cont'd)

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$$R_g = c_p - c_v = c_p \left(1 - \frac{1}{\gamma} \right) \rightarrow c_p = \frac{\gamma}{\gamma - 1} R_g$$

$$c_v = \frac{1}{\gamma} c_p = \frac{1}{\gamma - 1} R_g$$

- Air at Room Temperature -->

$$c_p = 1004.696 \text{ J/}^\circ\text{K-(kg-mole)}$$

$$c_v = c_p - R_g = 1004.696 - 287.056 = 717.64 \text{ J/}^\circ\text{K-(kg-mole)}$$

$$\gamma = c_p/c_v = 1007.696/717.64 = 1.400$$

Thermodynamics Summary

- Equation of State:

$$p = \rho R_g T \rightarrow R_g = \frac{R_u}{M_w}$$

- $R_u = 8314.4126 \text{ J/}^\circ\text{K-(kg-mole)}$
- $R_g \text{ (air)} = 287.056 \text{ J/}^\circ\text{K-(kg-mole)}$

- Relationship of R_g to specific heats, γ



$$C_p = C_v + R_g$$

- Internal Energy and Enthalpy

$$h = e + Pv$$

$$C_v = \left(\frac{de}{dT} \right)_v$$

$$C_p = \left(\frac{dh}{dT} \right)_p$$

Thermodynamics Summary (cont'd)

- First Law of Thermodynamics, *reversible process*

$$de = dq - pdv$$

$$dh = dq + vdp$$

- First Law of Thermodynamics, *isentropic process*
(adiabatic, reversible)

$$de = -pdv$$

$$dh = vdp$$

Thermodynamics Summary (cont'd)

- Second Law of Thermodynamics, *reversible process*

$$S_2 - S_1 = c_p \ln_2 \left[\frac{T_2}{T_1} \right] - R_g \ln \left[\frac{p_2}{p_1} \right]$$

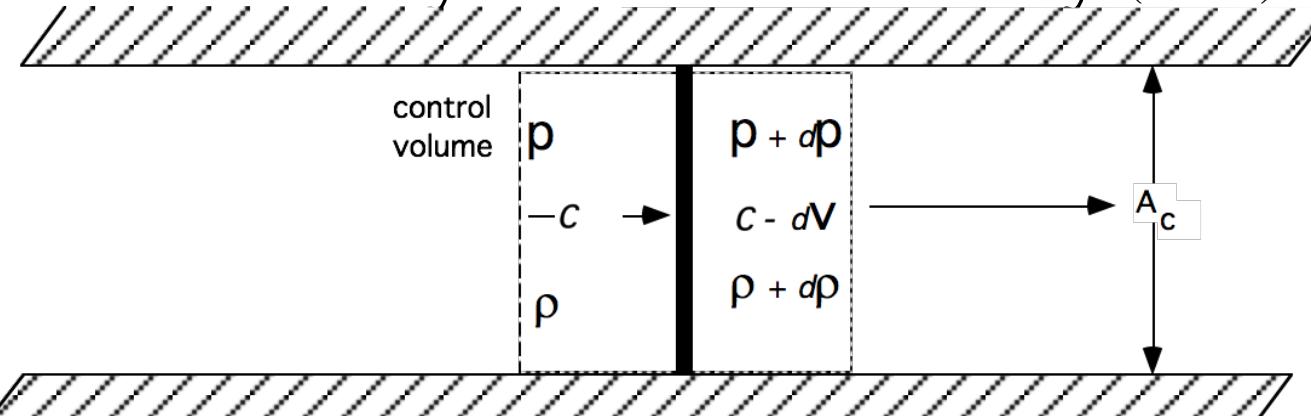
$$Tds = dh - vdp$$

- Second Law of Thermodynamics, *isentropic process*
(adiabatic, reversible) -----> $S_2 - S_1 = 0$

$$\frac{p_2}{p_1} = \left[\frac{T_2}{T_1} \right]^{\frac{\gamma}{\gamma-1}}$$

$$\left[\frac{p_2}{p_1} \right] = \left[\frac{\rho_2}{\rho_1} \right]^{\gamma}$$

Thermodynamics Summary (cont'd)



- From conservation of mass and momentum ..

sonic velocity is

$$c = \sqrt{\left[\frac{\partial p}{\partial \rho} \right]_{\Delta s=0}}$$

$$\left[\frac{p_2}{p_1} \right] = \left[\frac{\rho_2}{\rho_1} \right]^\gamma \rightarrow p_2 = [p_1 \rho_1^\gamma] \rho_2^\gamma \Rightarrow p = k \rho^\gamma \Rightarrow \boxed{\frac{\partial p}{\partial \rho} = \gamma k \rho^{\gamma-1}}$$

$$\frac{\partial p}{\partial \rho} = \gamma \frac{1}{\rho} [k \rho^\gamma] = \gamma \frac{p}{\rho} = \gamma R_g T \Rightarrow \boxed{c = \sqrt{\gamma R_g T}}$$

Thermodynamics Summary (concluded)

- Speed of Sound for calorically Perfect gas

$$c = \sqrt{\gamma R_g T}$$

- Mathematic definition of Mach Number

$$M = \frac{V}{\sqrt{\gamma R_g T}}$$

Review, General Integral Form of Equations of Compressible Flow

- Continuity (conservation of mass)

$$-\iiint_{C.V.} \left(\rho \vec{V} \bullet \vec{ds} \right) = \frac{\partial}{\partial t} \left(\iiint_{c.v.} \rho dv \right)$$

- Newton's Second law (momentum)-- Time rate of change of momentum Equals integral of external forces

$$\iiint_{C.V.} \rho \vec{f}_b dv - \iint_{C.S.} (p) \vec{dS} + \iint_{C.S.} \left| \left(\vec{f} \times \vec{dS} \right) \right| \frac{\vec{dS}}{S_{C.S.}} =$$

$$\iint_{C.S.} \left(\rho \vec{V} \bullet \vec{ds} \right) \vec{V} + \iiint_{C.V.} \frac{\partial}{\partial t} \left(\rho \vec{V} \right) dv$$

Review (concluded)

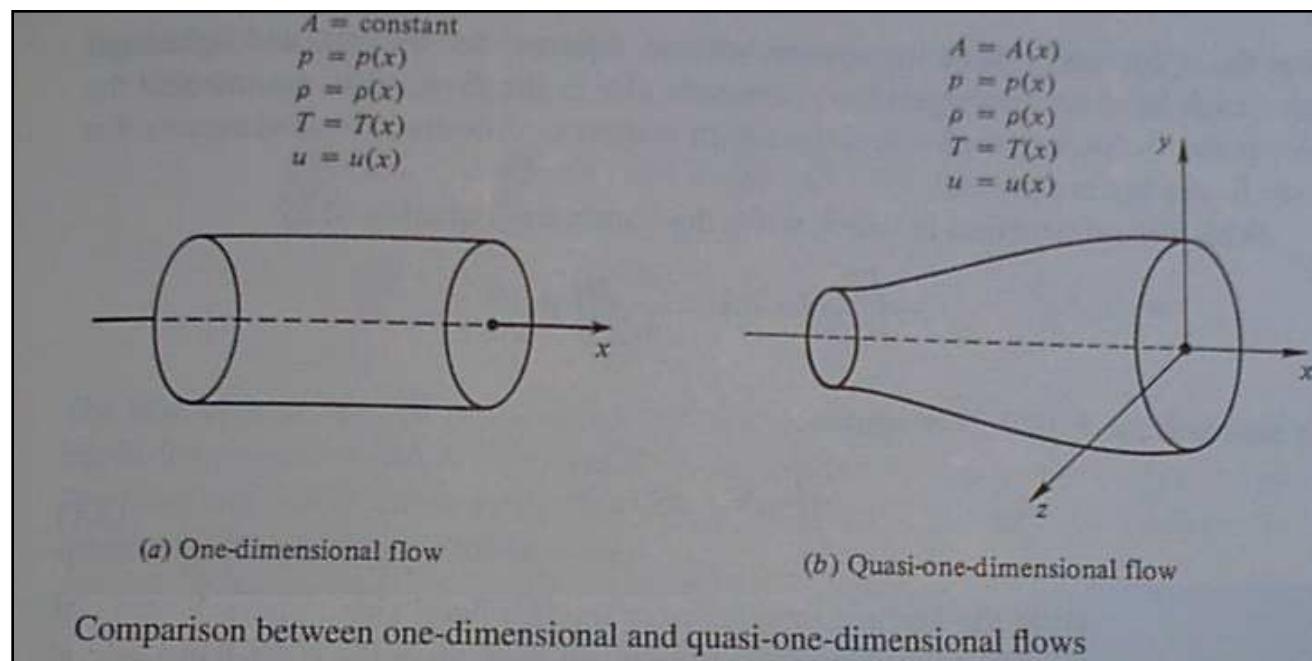
- Conservation of Energy--

$$\frac{\partial}{\partial t} \iiint_{C.V.} \left(\rho \left(e + \frac{\|V^2\|}{2} \right) dv \right) + \iint_{C.S.} \vec{\rho} \vec{V} \bullet d\vec{S} \left(e + \frac{\|V^2\|}{2} \right) =$$

$$\iiint_{C.V.} (\rho \vec{f} dv) \bullet \vec{V} - \iint_{C.S.} (pd\vec{S}) \bullet \vec{V} + \iiint_{C.V.} \left(\rho \left(\dot{q} \right) dv \right)$$

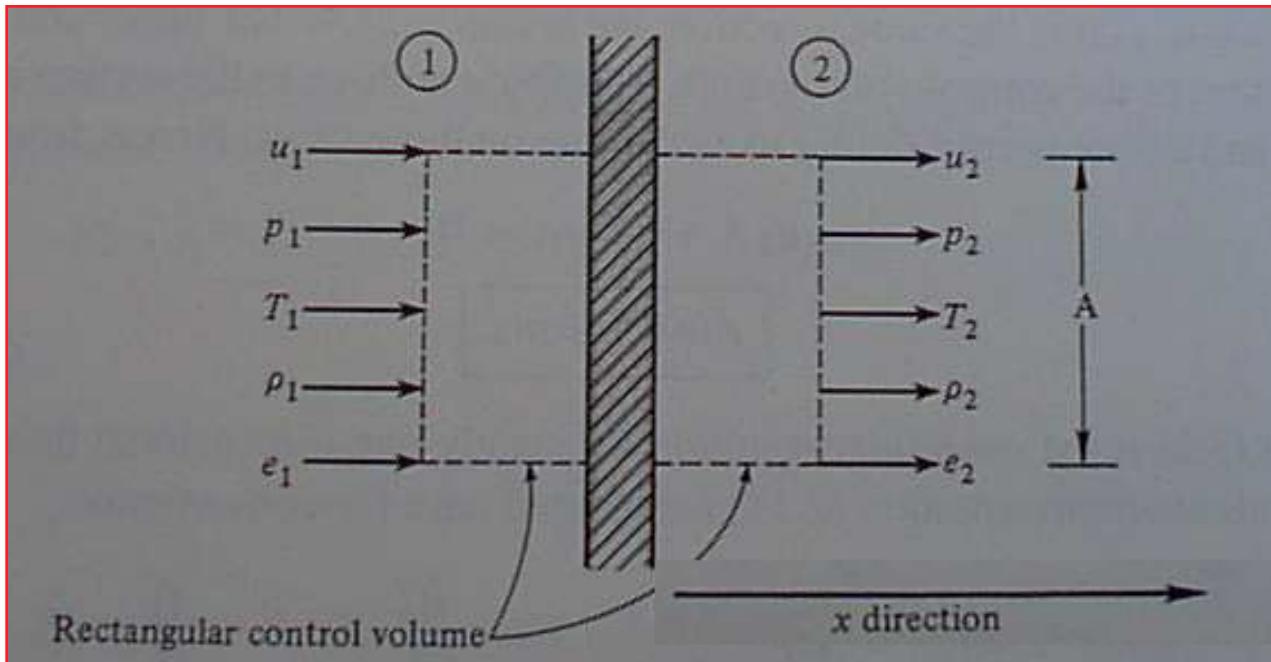
One Dimensional Flow Approximations

- Many Useful and practical Flow Situations can be Approximated by one-dimensional flow analyses



- “Air-goes-in, Air-goes-out, or both”

Control Volume for 1-D Flow



- Between C.V. entry (1) and exit (2), there could be
 - 1) Normal Shock wave (*supersonic engine duct*),
 - 2) Heat could be added or subtracted, (*heat exchanger*), or
 - 3) There could be work performed (*turbine element*)

Flow Characterized by motion only along longitudinal axis

1-D, Steady, Flow: Collected Equations

- Continuity

$$\rightarrow \rho_i V_i = \rho_e V_e$$

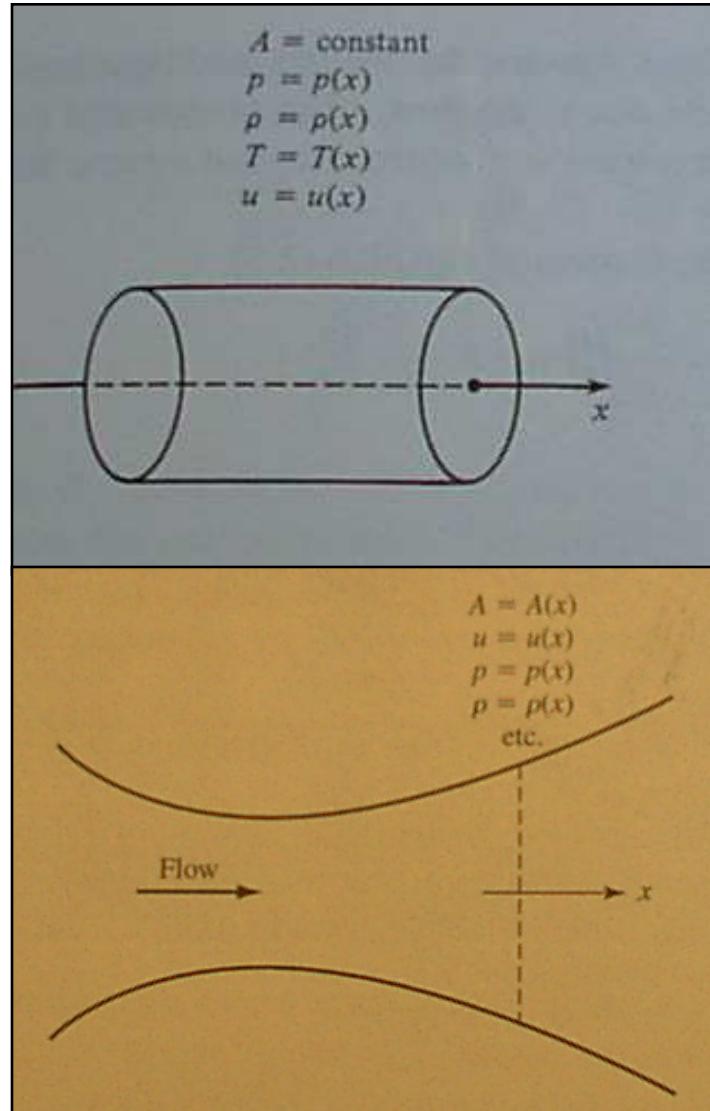
- Momentum

$$\rightarrow p_i + \rho_i V_i^2 = p_e + \rho_e V_e^2$$

- Energy

$$\rightarrow q + h_i + \frac{V_i^2}{2} = h_e + \frac{V_e^2}{2}$$

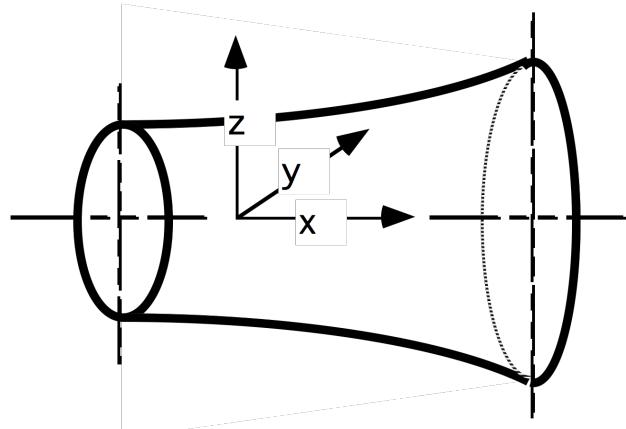
Distinction Between True 1-D Flow and Quasi 1-D Flow



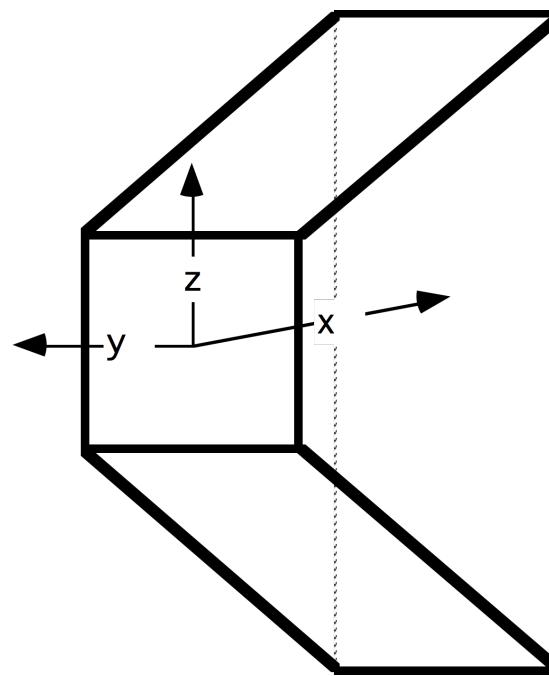
- In “true” 1-D flow Cross sectional area is strictly constant
- In quasi-1-D flow, cross section varies as a Function of the longitudinal coordinate, x
- Flow Properties are assumed constant across any cross-section
- Analytical simplification very useful for evaluating Flow properties in Nozzles, tubes, ducts, and diffusers Where the cross sectional area is large when compared to length

Distinction between True 1-D Flow and Quasi 1-D Flow (cont'd)

- One-D simplification requires axial symmetry in z and y directions
To insure that flow properties are constant across cross section



True Axi-Symmetric Duct
(Circular Cross-section)



Quasi-1-D Rectangular Duct

Quasi 1-D Flow: Governing Equations

- Inviscid, No Body Forces, Steady Flow

- Continuity

$$\iint_{C.S.} \left(\rho \vec{V} \bullet \vec{dS} \right) = 0$$

- Momentum

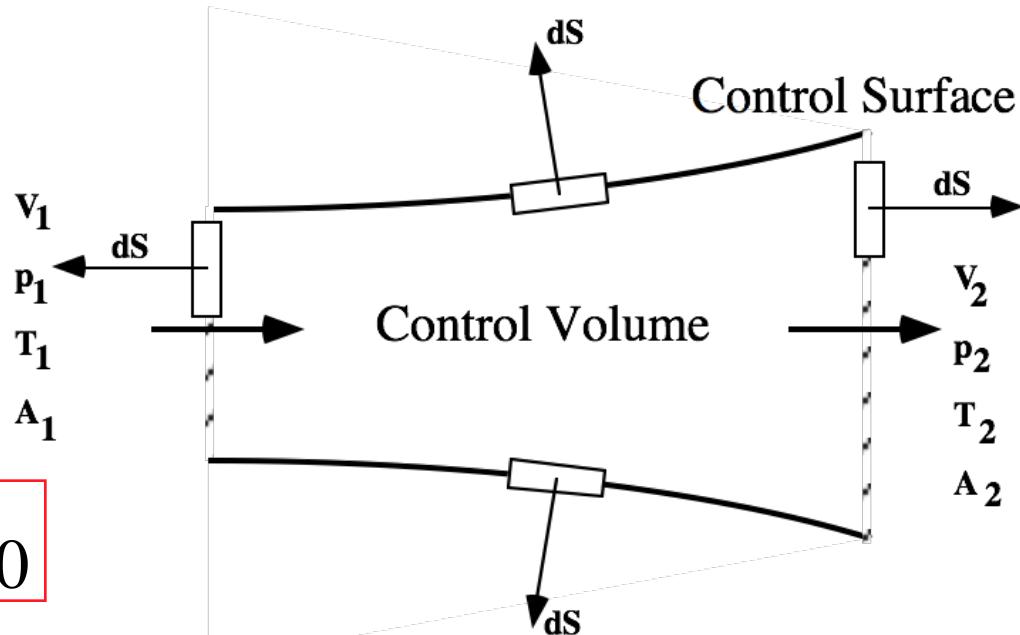
$$\iint_{C.S.} \left(\rho \vec{V} \bullet \vec{dS} \right) \vec{V} = - \iint_{C.S.} (p) \vec{dS}$$

- Energy

$$\Delta Q - \iint_{C.S.} (pd \vec{S}) \bullet \vec{V} = \iint_{C.S.} \rho \left(e + \frac{\|V^2\|}{2} \right) \vec{V} \bullet \vec{dS}$$

Review: Continuity Equation for Quasi 1-D Control Volume

$$\iint_{C.S.} \left(\rho \vec{V} \bullet \vec{ds} \right) = 0$$



- Upper and Lower Surfaces ...
no flow across boundary

$$\vec{V} \bullet \vec{ds} = 0$$

- Inlet (properties constant across Cross section) -->

$$\iint_1 \left(\rho \vec{V} \bullet \vec{ds} \right) = \rho_1 \vec{V}_1 \bullet \iint_1 \vec{ds} = \rho_1 V_1 \cos(180^\circ) \iint_1 ds = -\rho_1 V_1 A_1$$

- Inlet (properties constant across Cross section) -->

$$\iint_2 \left(\rho \vec{V} \bullet \vec{ds} \right) = \rho_2 \vec{V}_2 \bullet \iint_2 \vec{ds} = \rho_2 V_2 \cos(0^\circ) \iint_2 ds = \rho_2 V_2 A_2$$

$$\boxed{\rho_1 V_1 A_1 = \rho_2 V_2 A_2}$$

Review: Momentum Equation for Quasi 1-D Control Volume

- Similar Analysis for Momentum Equation Yields

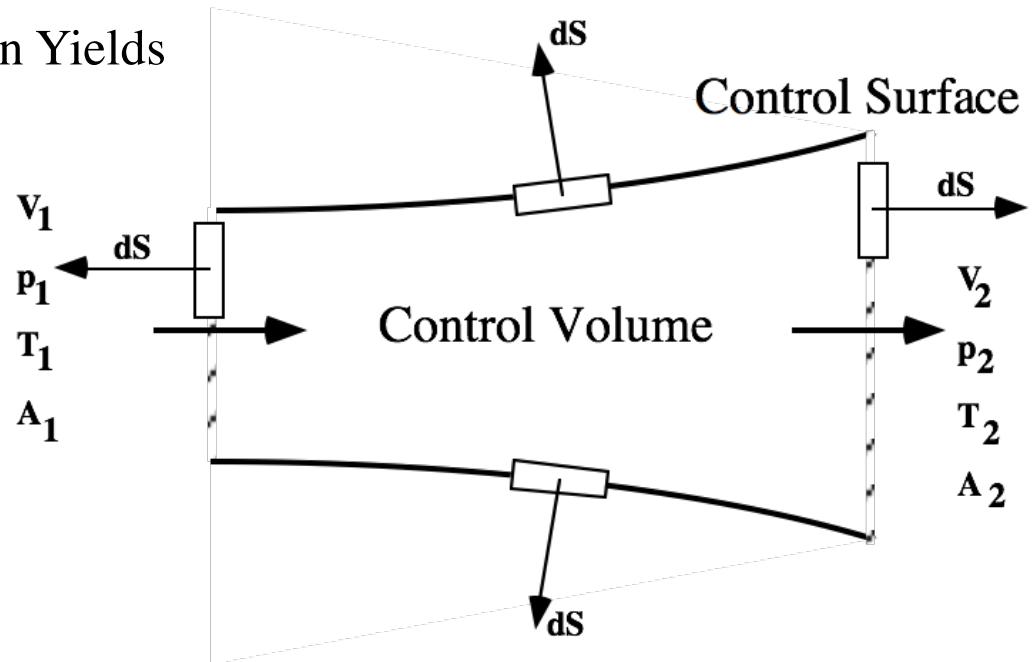
- Newton's Second law--

Time rate of change of momentum
Equals integral of external forces

$$\iint_{C.S.} \left(\rho \vec{V} \bullet \vec{ds} \right) \vec{V} = - \iint_{C.S.} (p) \vec{dS} \rightarrow$$

$$p_1 A_1 V_1 + \rho_1 V_1^2 A_1 + \int_1^2 p \vec{ds} \bullet \vec{i}_x = p_2 A_2 V_2 + \rho_2 V_2^2 A_2$$

- Because of duct symmetry the “Z-axis” Component of pressure integrated to zero



$$\vec{i}_x$$

“Unit vector” x-direction

Review: Momentum Equation for Quasi 1-D Control Volume

- Similar Analysis for Momentum Equation Yields

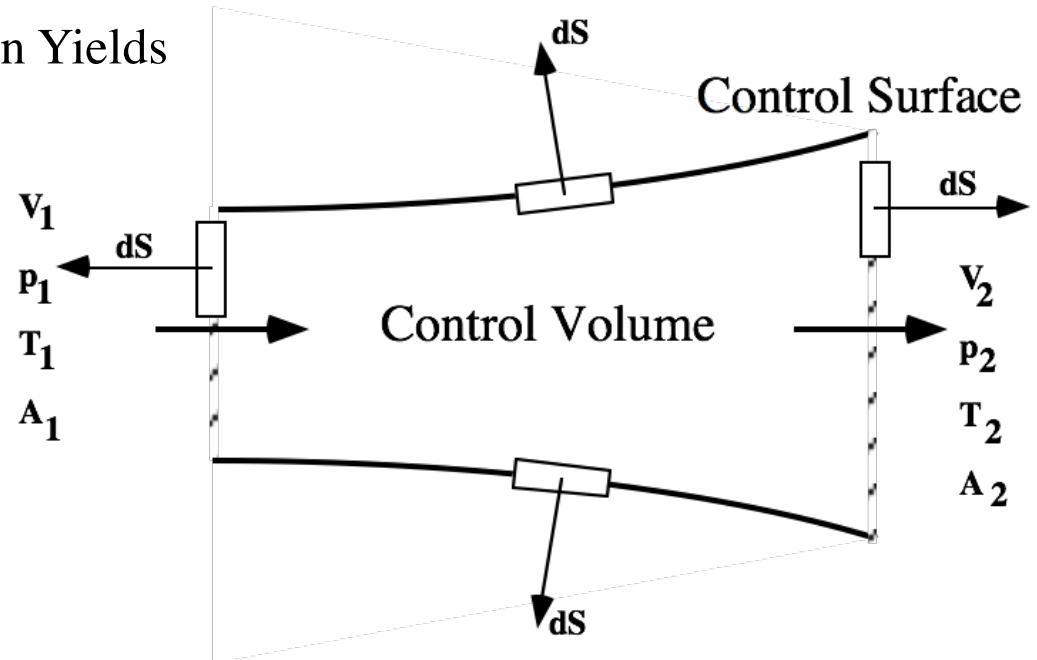
- Newton's Second law--

Time rate of change of momentum
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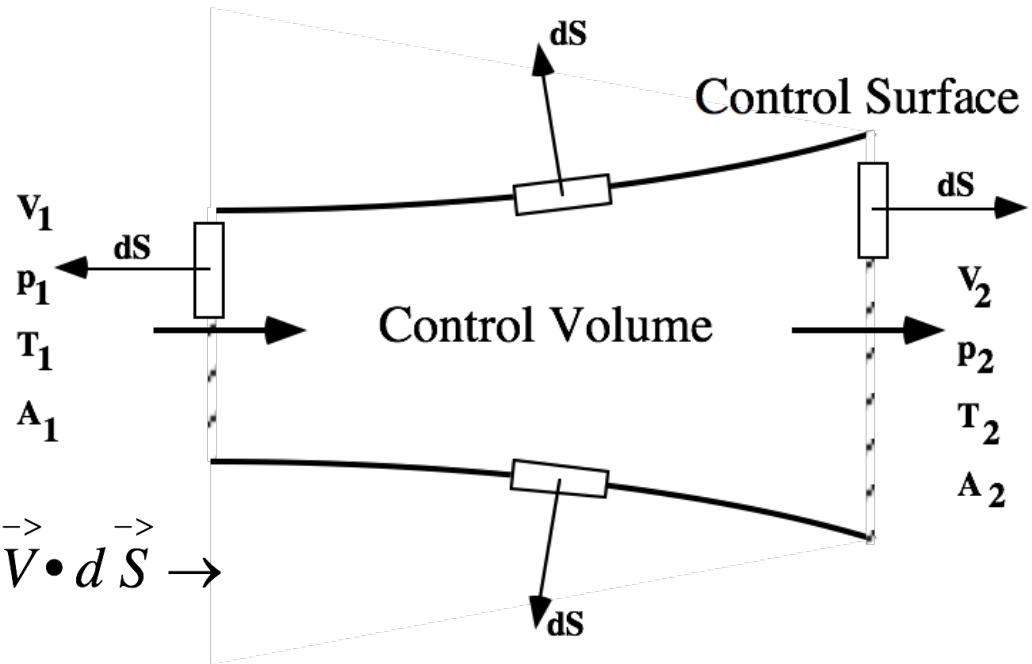
$$\iint_{C.S.} \left(\rho \vec{V} \bullet \vec{ds} \right) \vec{V} = - \iint_{C.S.} (p) \vec{dS} \rightarrow$$

$$p_1 A_1 V_1 + \rho_1 V_1^2 A_1 + \int_1^2 p \vec{ds} \bullet \vec{i}_x = p_2 A_2 V_2 + \rho_2 V_2^2 A_2$$

- Because of duct symmetry the “Z-axis” Component of pressure integrated to zero



Review: Energy Equation for Quasi 1-D Control Volume

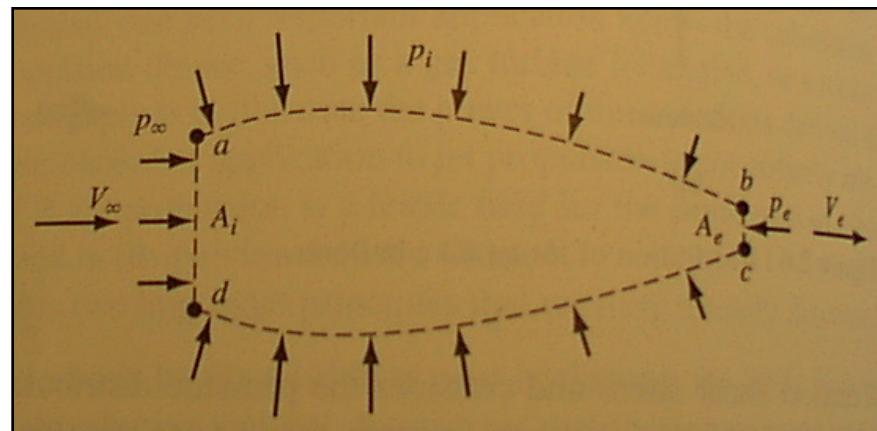
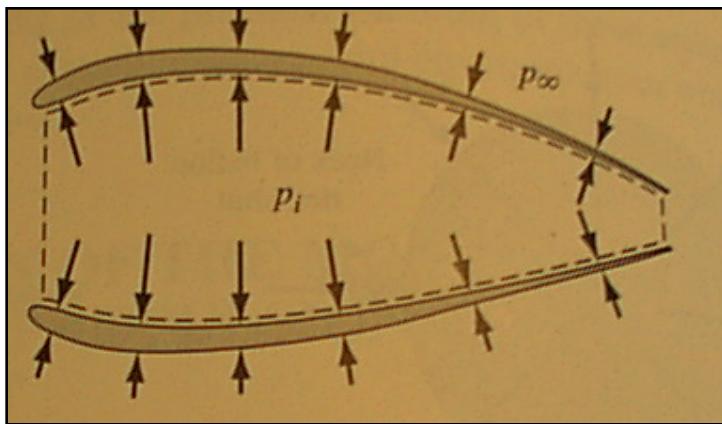


$$\dot{Q} - \iint_{C.S.} (pd\vec{S}) \cdot \vec{V} = \iint_{C.S.} \rho \left(e + \frac{\|V^2\|}{2} \right) \vec{V} \cdot d\vec{S}$$

$$\Delta q + h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2}$$

Example Usage: Engine Thrust Model

- Steady, Inviscid, One-Dimensional Flow Through Ramjet



$$-\iint_{C.S.} (\bar{p}) \vec{dS} = \iint_{C.S.} \left(\rho \vec{V} \bullet \vec{ds} \right) \vec{V}$$

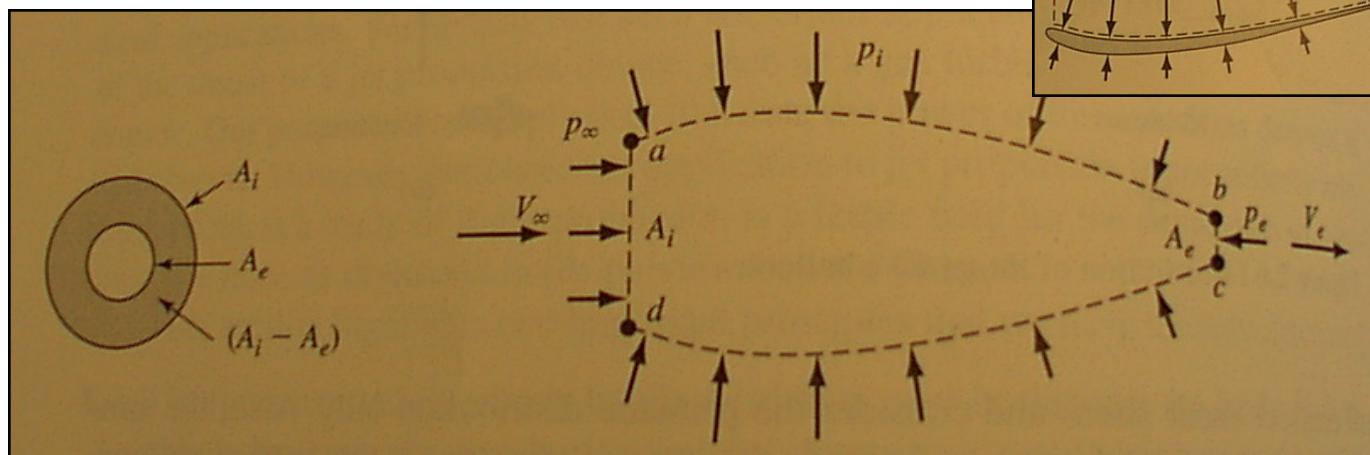
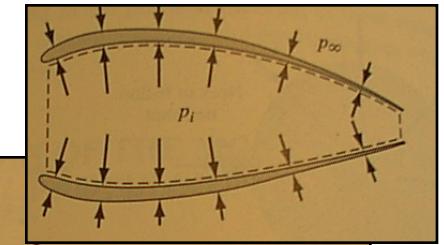
$$\int_{wall} P_{wall} dA_{wall} + p_{\infty} A_i - p_e A_e = \rho_e V_e^2 A_e - \rho_i V_i^2 A_i = \dot{m}_e V_e - \dot{m}_i V_i$$

Example Usage: Engine Thrust Model (cont'd)

- Adding $(p_e A_e - p_\infty A_e)$ to Both Sides, and collecting terms

$$\int_{\text{wall}} P_{\text{wall}} dA_{\text{wall}} + p_\infty A_i - p_e A_e + (p_e A_e - p_\infty A_e) = \dot{m}_e V_e - \dot{m}_i V_i + (p_e A_e - p_\infty A_e) \rightarrow$$

$$\int_{\text{wall}} \underline{P_{\text{wall}} dA_{\text{wall}}} + p_\infty (A_i - A_e) = \dot{m}_e V_e - \dot{m}_i V_i + (p_e A_e - p_\infty A_e)$$



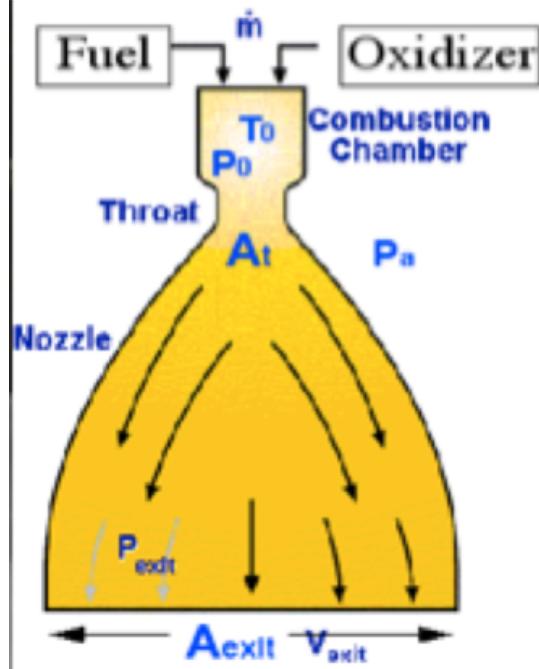
Integrated Pressure Forces Acting on External + Internal Surface of Engine Wall = *Thrust*

$$\text{Thrust} = \dot{m}_e V_e - \dot{m}_i V_i + (p_e A_e - p_\infty A_e)$$

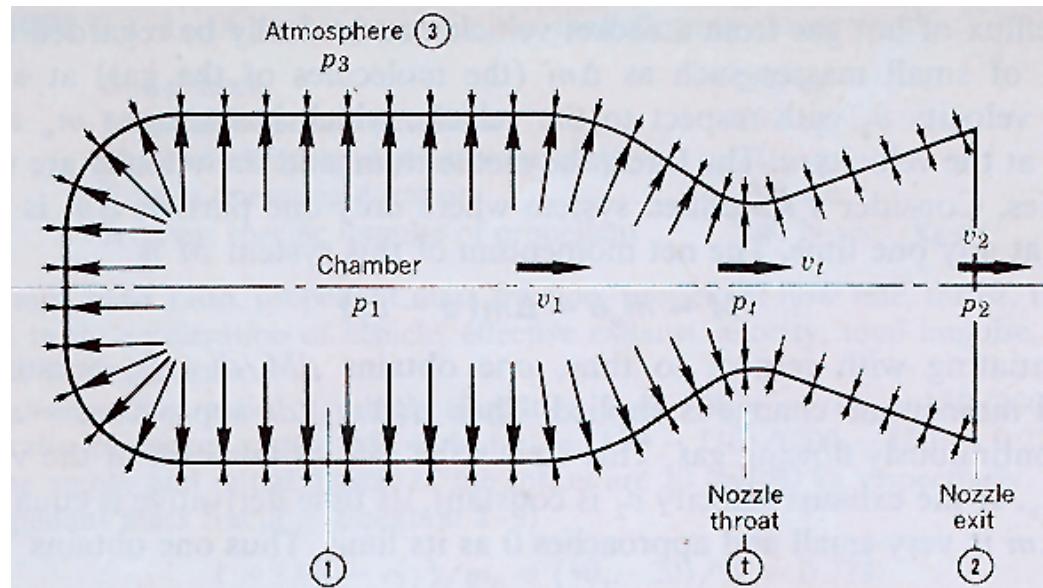
Rocket Thrust Equation

$$\bullet \quad Thrust = \dot{m}_e V_e + (p_e A_e - p_\infty A_e)$$

$$\bullet \quad m_i = 0$$



- Thrust + Oxidizer enters combustion Chamber at ~0 velocity, combustion Adds energy ... High Chamber pressure Accelerates flow through Nozzle
- Resultant pressure forces produce thrust*



Mach Number: Revisited

- From the definition of c_v

$$c_v = c_p - R_g \rightarrow$$

$$c_v = R_g \left(\frac{c_p}{R_g} - 1 \right) = R_g \left(\frac{c_p}{c_p - c_v} - 1 \right) =$$

$$R_g \left(\frac{c_p - c_p + c_v}{c_p - c_v} \right) = R_g \left(\frac{c_v}{c_p - c_v} \right) = \left(\frac{R_g}{\gamma - 1} \right)$$

Mach Number: Revisited (cont'd)

- Recall from the fundamental definition

$$M = \frac{V}{c} = \frac{V}{\sqrt{\gamma R_g T}}$$

- But if we take the ratio of kinetic to internal energy
Of a fluid *element*

$$\frac{V^2/2}{e} = \frac{V^2/2}{c_v T}$$

- calorically perfect gas

Mach Number: Revisited (cont'd)

- Then

$$\frac{V^2/2}{e} = \frac{V^2/2}{c_v T} = \frac{V^2/2}{\left(\frac{R_g}{\gamma - 1}\right) T} =$$

$$\frac{\gamma}{\gamma} \frac{V^2/2}{\left(\frac{R_g}{\gamma - 1}\right) T} = \frac{\gamma(\gamma - 1)}{2} \frac{V^2}{\gamma R_g T} = \frac{\gamma(\gamma - 1)}{2} M^2$$

i.e. Mach number is a measure of the ratio of the fluid Kinetic energy to the fluid internal energy (*direct motion To random thermal motion of gas molecules*)

Mach Number: Revisited (cont'd)

- Look at the Steady Flow Energy equation With no heat addition

$$h_i + \frac{V_i^2}{2} = h_e + \frac{V_e^2}{2} \rightarrow h + \frac{V^2}{2} = \text{const}$$

- Look at Differential form

$$d\left(h + \frac{V^2}{2}\right) = 0 \rightarrow dh + VdV = 0$$

Mach Number: Revisited (cont'd)

- For a reversible process

$$Tds = dh - vdp$$

- With no heat addition, $ds=0$

$$dh = vdp = \frac{dp}{\rho}$$

- *Subbing into previous*

$$dp = -\rho V dV \quad (\text{"euler's equation"})$$

Mach Number: Revisited (cont'd)

- Now look at Continuity equation

$$\rho_i V_i = \rho_e V_e \rightarrow \rho A V = \text{const}$$

$$\frac{d(\rho A V)}{\rho A V} = 0 = \frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V}$$

- Substituting in Euler's Equation $dp = -\rho V dV$

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{-dp}{\rho V} = 0$$

Mach Number: Revisited (cont'd)

- Solving for dA/A

$$\frac{dA}{A} = \frac{dp}{\rho} \frac{1}{V^2} - \frac{d\rho}{\rho} = \frac{dp}{\rho} \left(\frac{1}{V^2} - \frac{d\rho}{dp} \right)$$

- But since we are considering a process with $ds=0$

$$\frac{d\rho}{dp} = \frac{1}{c^2} \quad \text{and ...} \quad \frac{dA}{A} = \frac{dp}{\rho} \left(\frac{1}{V^2} - \frac{1}{c^2} \right) = \frac{(1 - M^2)}{\rho V^2} dp$$

- Substituting in Euler's Equation $dp = -\rho V dV$

$$\frac{dA}{A} = \frac{(1 - M^2)}{\rho V^2} (-\rho V dV) = (M^2 - 1) \left(\frac{dV}{V} \right)$$

Mach Number: Revisited (cont'd)

- Rearranging gives the TWO relationships

Whose ramifications are fundamental to this class

$$\left(\frac{dV}{dA} \right) = \frac{1}{(M^2 - 1)} \frac{V}{A}$$

$$\frac{dp}{dA} = \frac{-1}{(M^2 - 1)} \frac{\rho V^2}{A}$$

Mach Number: Revisited (cont'd)

$$\left(\frac{dV}{dA} \right) = \frac{1}{(M^2 - 1)} \frac{V}{A}$$

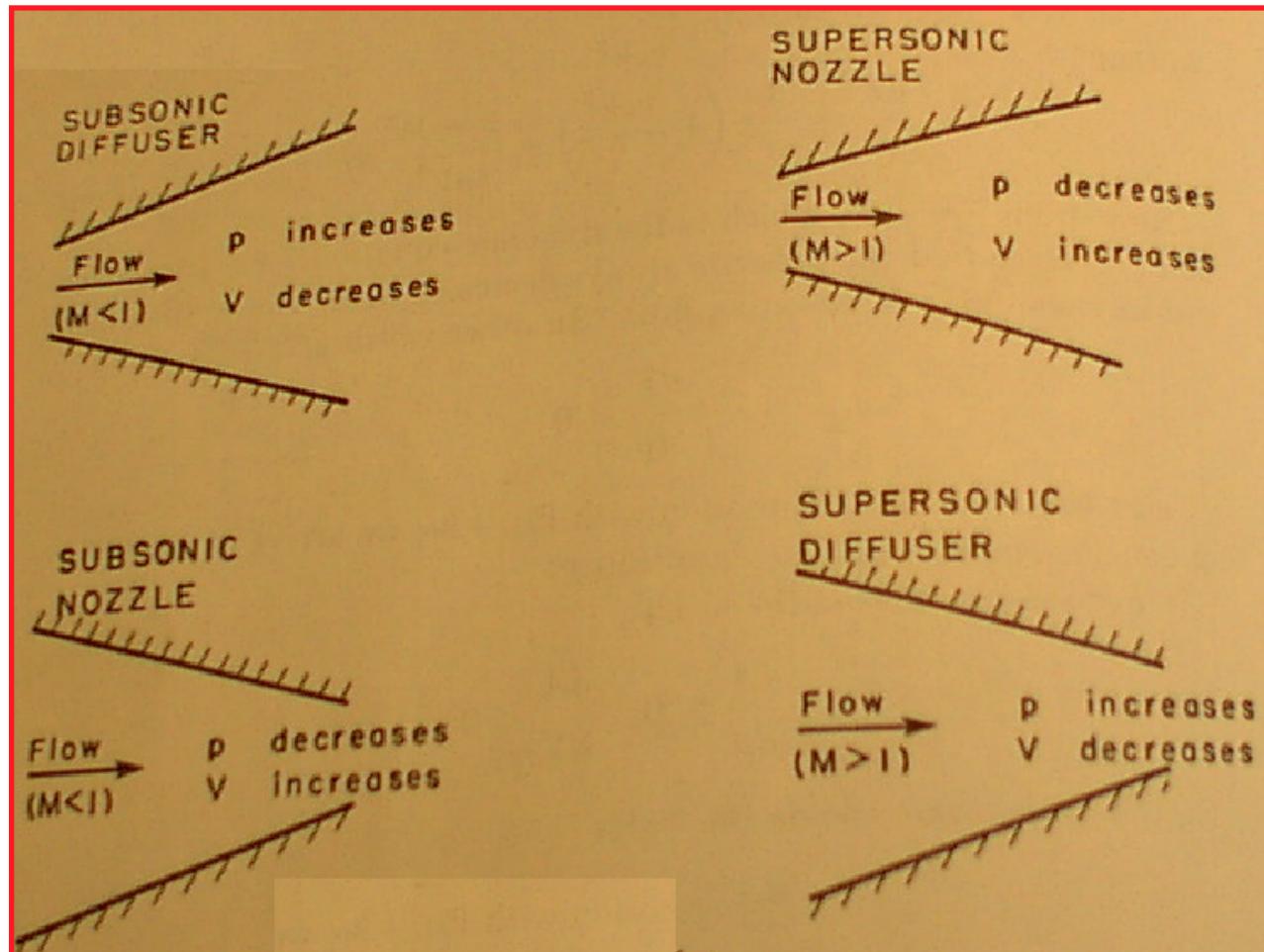
$$\frac{dp}{dA} = \frac{-1}{(M^2 - 1)} \frac{\rho V^2}{A}$$



$$M < 1 \rightarrow \left(\frac{dV}{dA} \right) < 0 \rightarrow \left(\frac{dp}{dA} \right) > 0$$

$$M > 1 \rightarrow \left(\frac{dV}{dA} \right) > 0 \rightarrow \left(\frac{dp}{dA} \right) < 0$$

Fundamental Properties of Supersonic and Supersonic Flow



... Hence the shape of the rocket Nozzle

