EECS 592 Homework 3

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Problem 1: Evaluating Logical Statements

a) $M := (A \wedge B) \models S := (A \Leftrightarrow B)$ is **true**:
   1. The assignments for which $(A \wedge B)$ is true are: $A = 1, B = 1$.
   2. When $A = 1, B = 1$, $(A \Leftrightarrow B)$ is true. i.e.,$A = 1, B = 1$, $S = (A \Leftrightarrow B)$ is true.
   3. $M \models S$ is true because for all models that evaluate M to be true S is also true.

b) $M := (A \Leftrightarrow B) \models S := (A \vee B)$ is **false**:
   1. Consider $A = 0, B = 0$ which evaluates $M := (A \Leftrightarrow B)$ to be true.
   2. However, $A = 0 \vee B = 0$ is evaluated to be false.
   3. Therefore $M \models S$ is false because there exists an assignment of A and B which makes M to be true and S is false.

c) $M := (A \wedge B) \Rightarrow C \models S := (A \Rightarrow C) \vee (B \Rightarrow C)$ is **true**:
   1. $S = (\neg A \vee C) \vee (\neg B \vee C) = \neg A \vee \neg B \vee C$.
   2. $M = \neg(A \wedge B) \vee C = (\neg A \vee \neg B) \vee C = S$.

d) $M := (A \vee B) \wedge (\neg C \vee \neg D \vee E) \models S := (A \vee B) \wedge (\neg D \vee E)$ is **false**.
   1. Consider $A = 1, B = 1, C = 0, D = 1, E = 0$ which evaluates M to be true.
   2. However, S is evaluated to be false.
   3. Therefore $M \models S$ is false because there exists an assignment of M which makes M to be true but S to be false.

e) $M := (C \vee (\neg A \wedge \neg B)) \equiv S := ((A \Rightarrow C) \wedge (B \Rightarrow C))$ is **true**.
   1. $S \equiv (\neg A \vee C) \wedge (\neg B \vee C)$.
   2. $M \equiv (C \vee \neg A) \wedge (C \vee \neg B)$.
   3. Therefore $M \equiv S$.

f) The statement, $\alpha \models \beta \Leftrightarrow \alpha \Rightarrow \beta$ is valid, is **true**.
   1. Denote $M(\alpha) = \{x \mid \alpha(x) = 1\}$ to be the set of all assignment x such that α is true; $M(\beta)$ is similarly defined.
   2. If $\alpha \models \beta$, then $M(\alpha) \subseteq M(\beta)$. Consider an arbitrary assignment x_0. If $x_0 \in M(\alpha)$ then $x_0 \in M(\beta)$, i.e. if $\alpha(x_0) = 1, \beta(x_0) = 1, \alpha \Rightarrow \beta$ is true, $\forall x_0 \in M(\alpha)$. If $x_0 \notin M(\alpha)$, then $\alpha(x_0) = 0, \forall x_0 \notin M(\alpha)$, i.e.$\alpha(x_0) \Rightarrow \beta(x_0)$ is true, $\forall x_0 \notin M(\alpha)$. Therefore $\alpha(x_0) \Rightarrow \beta(x_0)$, $\forall x_0$.
   3. If $\alpha(x_0) \Rightarrow \beta(x_0)$, $\forall x_0$ then $\alpha(x_0) = 1, \beta(x_0) = 1$, $\forall x_0 \in M(\alpha)$. This implies that $x_0 \in M(\beta)$. Therefore $M(\alpha) \subseteq M(\beta)$, i.e.$\alpha \models \beta$.

g) The statement, $\alpha \equiv \beta \Leftrightarrow (\neg(\alpha \Leftrightarrow \beta)$ is unsatisfiable) is **true**.
   1. $\alpha \equiv \beta \Leftrightarrow (\alpha \models \beta) \wedge (\beta \models \alpha)$. By the conclusion above, $\alpha \equiv \beta \Leftrightarrow (\alpha \Rightarrow \beta$ is valid) $\wedge$ ($\beta \Rightarrow \alpha$ is valid).
   2. $\alpha \equiv \beta \Leftrightarrow (\alpha(x) \Leftrightarrow \beta(x), \forall x)$. Thus $\alpha \equiv \beta \Leftrightarrow (\neg(\alpha \Leftrightarrow \beta)$ is unsatisfiable).

h) $(\neg B \Rightarrow A) \Rightarrow (\neg A \Rightarrow B)$ is **true**.
   1. $(\neg B \Rightarrow A) = \neg(\neg B) \vee A = B \vee A$; $\neg A \Rightarrow B = A \vee B$.
   2. Since for all models that make $(\neg B \Rightarrow A)$ true, $(\neg A \Rightarrow B)$ is also true, the above statement is true.

i) The claim that $(A \wedge B) \wedge \neg(A \Rightarrow B)$ is satisfiable is **false**.
   1. $\neg(A \Rightarrow B) = \neg(\neg A \vee B) = A \wedge \neg B$
   2. $(A \wedge B) \wedge \neg(A \Rightarrow B) = (A \wedge B) \wedge (A \wedge \neg B) = A \wedge B \wedge A \wedge \neg B = A \wedge B \wedge \neg B$. This is unsatisfiable.

j) The claim that $(A \Leftrightarrow B) \wedge (\neg A \vee B)$ is unsatisfiable is **false**.
   1. $(A \Leftrightarrow B) \wedge (\neg A \vee B) = (A \Leftrightarrow B) \wedge (A \Rightarrow B)$. This is valid for all models.

Conjunctive Normal Form and Resolution

A. Convert the following sentences into CNF

a)

$$D \wedge \neg B \Rightarrow E$$
$$\neg(D \wedge \neg B) \vee E$$
$$\neg D \vee B \vee E$$

b)

$$A \wedge C \wedge D \Rightarrow F$$
$$\neg(A \wedge C \wedge D) \vee F$$
$$\neg A \vee \neg C \vee \neg D \vee F$$

c)

$$A \wedge (D \vee F) \Rightarrow C$$
$$\neg(A \wedge (D \vee F)) \vee C$$
$$(\neg A \vee \neg(D \vee F)) \vee C$$
$$(\neg A \vee (\neg D \wedge \neg F)) \vee C$$
$$\neg A \vee C \vee (\neg D \wedge \neg F)$$
$$(\neg A \vee C \vee \neg D) \wedge (\neg A \vee C \vee \neg F)$$

d)

$$F \Leftrightarrow \neg B$$
$$(F \Rightarrow \neg B) \wedge (\neg B \Rightarrow F)$$
$$(\neg F \vee \neg B) \wedge (\neg(\neg B) \vee F)$$
$$(\neg F \vee \neg B) \wedge (B \vee F)$$

B. Prove that E is true using resolution.

1. $\frac{(\neg A \vee C \vee \neg D) \wedge A \wedge D}{C}$. C is added to KB.
2. $\frac{(\neg A \vee \neg C \vee \neg D \vee \neg F) \wedge A \wedge D \wedge C}{F}$. F is added to KB.
3. $\frac{(\neg F \vee \neg B) \wedge F}{\neg B}$. $\neg B$ is added to KB.
4. $\frac{(\neg D \vee B \vee E) \wedge \neg B \wedge D}{E}$. E is true.

First-Order Logic Quantifiers

Translate sentences into FOL:

a) $\forall y, Crust(y) \wedge GoesWith(Peperoni, y)$.
b) $\exists x, Topping(x) \wedge GoesWith(x, Stuffed)$.
c) $\forall y \exists\, x\ Topping(x) \wedge Crust(y) \wedge GoesWith(x, y)$.

Translate FOL into English:

d) Ann does not like curst mushrooms no matter what topping comes with it.
e) Everyone likes thin pizza with some topping.
f) There are some people love thick pizza no matter what topping comes with it.

First-Order Logic Semantics

a) $WorthMore(quarter, penny) \wedge WorthMore(quarter, Nickel) \wedge WorthMore(quarter, dime)$.
b) $WorthMore(quarter, penny) \wedge WorthMore(quarter, nickel) \wedge WorthMore(quarter, dime) \wedge penny \neq nickel \wedge nickel \neq dime \wedge penny \neq dime \wedge \forall\, x\; UScoins(x) \wedge x \neq quarter \wedge x \neq penny \wedge x \neq nickel \wedge x \neq dime \wedge WorthMore(x, quarter)$.

Unification

a) $x = A, y = B, z = B$.
b) Fail. To unify this two expressions, x, y need to satisfy:
   - $y = G(x, x)$ and $y = G(A, B)$. Since there does not exist x such that $G(x, x) = G(A, B)$ therefore no x, y can satisfy the above condition.

c) $x = A, y = A$.
d) $x = A, y = A, z = G(B)$.

Resolution Refutation

1) All wolves howl.
2) Anyone who has cats as pets will not have mice.
3) Anyone who is a light sleeper can't live near anything that howls.
4) Frank either has cats or lives near wolves.
5) If Frank is a light sleeper, Frank has mice.

Write all the sentences above in FOL

1) $\forall\, w\; Wolf(w) \Rightarrow Howl(w)$.
2) $\forall\, x (\exists y\; Cat(y) \wedge Have(x, y) \Rightarrow \sim (\exists\, z\; Mouse(z) \wedge Have(x, z)))$.
3) $\forall\, x\; LS(x) \Rightarrow \sim (\exists\, y\; Howl(y) \wedge Near(x, y))$.
4) $\exists\, x\; Cat(x) \wedge Have(Frank, x) \vee \exists\, w\; Wolf(w) \wedge Near(Frank, w)$.
5) $LS(Frank) \Rightarrow \exists\, m\; Mouse(m) \wedge Have(Frank, m)$.

Convert all the FOL to CNF

1) $\sim Wolf(w) \vee Howl(w)$.
2) $\sim Cat(y) \vee \sim Have(x, y) \vee \sim Mouse(z) \vee \sim Have(x, z)$.
3) $\sim LS(x) \vee \sim Howl(y) \vee \sim Near(x, y)$.
4) $(Cat(X_1) \vee Wolf(W_0)) \wedge (Cat(X_1) \vee Near(Frank, W_0)) \wedge (Have(Frank, X_1) \vee Wolf(W_0)) \wedge (Have(Frank, X_1) \vee Near(Frank, W_0))$.
5) $(\sim LS(Frank) \vee Mouse(M)) \wedge (\sim LS(Frank) \vee Have(Frank, M))$.

Prove "Frank is not a light sleeper" using Resolution Refutation.

1) Convert "Frank is not a light sleeper" to CNF: $\sim LS(Frank)$.

2) Add $LS(Frank)$ to KB and try to find controversy.

3) $\dfrac{(\sim LS(F) \vee Mouse(M)) \wedge LS(Frank)}{Mouse(M)}$. Add $Mouse(M)$ to KB. $\dfrac{(\sim LS(F) \vee Have(Frank, M)) \wedge LS(Frank)}{Have(Frank, M)}$. Add $Have(Frank, M)$ to KB.

4) $\dfrac{(\sim Cat(y) \vee \sim Have(x, y) \vee \sim Mouse(z) \vee \sim Have(x, z)) \wedge Mouse(M) \wedge Have(Frank, M)}{\sim Cat(y) \vee \sim Have(Frank, y)}$. Add $\sim Cat(y) \vee \sim Have(Frank, y)$ to KB.

5) $\dfrac{(Cat(X_1) \vee Wolf(W_0)) \wedge (\sim Cat(y) \vee \sim Have(Frank, y))}{Wolf(W_0) \vee \sim Have(Frank, X_1)}$. Add $Wolf(W_0) \vee \sim Have(Frank, X_1)$ to KB.

6) $\dfrac{(Have(Frank, X_1) \vee Near(Frank, W_0)) \wedge (Wolf(W_0) \vee \sim Have(Frank, X_1))}{Near(Frank, W_0) \vee Wolf(W_0)}$. Add $Near(Frank, W_0) \vee Wolf(W_0)$ to KB.

7) $\dfrac{(\sim Wolf(w) \vee Howl(w)) \wedge (Near(Frank, W_0) \vee Wolf(W_0))}{Near(Frank, W_0) \vee Howl(w)}$. Add $Near(Frank, W_0) \vee Howl(wW_0$ to KB.

8) $\dfrac{(\sim LS(x) \vee \sim Howl(y) \vee \sim Near(x, y)) \wedge (Near(Frank, W_0) \vee Howl(W_0))}{\sim LS(Frank)}$. $\sim LS(Frank)$ contradict with our assumption $LS(Frank)$. Therefore Frank is not a light sleeper.