EECS 592 Homework 3

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Problem 1: Evaluating Logical Statements

a) $M := (A \wedge B) \models S := (A \Leftrightarrow B)$ is **true**:
   1. The assignments for which $(A \wedge B)$ is true are: $A = 1, B = 1$.
   2. When $A = 1, B = 1$, $(A \Leftrightarrow B)$ is true. i.e.,$A = 1, B = 1$, $S = (A \Leftrightarrow B)$ is true.
   3. $M \models S$ is true because for all models that evaluate M to be true S is also true.

b) $M := (A \Leftrightarrow B) \models S := (A \vee B)$ is **false**:
   1. Consider $A = 0, B = 0$ which evaluates $M := (A \Leftrightarrow B)$ to be true.
   2. However, $A = 0 \vee B = 0$ is evaluated to be false.
   3. Therefore $M \models S$ is false because there exists an assignment of A and B which makes M to be true and S is false.

c) $M := (A \wedge B) \Rightarrow C \models S := (A \Rightarrow C) \vee (B \Rightarrow C)$ is **true**:
   1. $S = (\neg A \vee C) \vee (\neg B \vee C) = \neg A \vee \neg B \vee C$.
   2. $M = \neg(A \wedge B) \vee C = (\neg A \vee \neg B) \vee C = S$.

d) $M := (A \vee B) \wedge (\neg C \vee \neg D \vee E) \models S := (A \vee B) \wedge (\neg D \vee E)$ is **false**.
   1. Consider $A = 1, B = 1, C = 0, D = 1, E = 0$ which evaluates M to be true.
   2. However, S is evaluated to be false.
   3. Therefore $M \models S$ is false because there exists an assignment of M which makes M to be true but S to be false.

e) $M := (C \vee (\neg A \wedge \neg B)) \equiv S := ((A \Rightarrow C) \wedge (B \Rightarrow C))$ is **true**.
   1. $S \equiv (\neg A \vee C) \wedge (\neg B \vee C)$.
   2. $M \equiv (C \vee \neg A) \wedge (C \vee \neg B)$.
   3. Therefore $M \equiv S$.

f) The statement, $\alpha \models \beta \Leftrightarrow \alpha \Rightarrow \beta$ is valid, is **true**.
   1. Denote $M(\alpha) = \{x \mid \alpha(x) = 1\}$ to be the set of all assignment x such that α is true; $M(\beta)$ is similarly defined.
   2. If $\alpha \models \beta$, then $M(\alpha) \subseteq M(\beta)$. Consider an arbitrary assignment x_0. If $x_0 \in M(\alpha)$ then $x_0 \in M(\beta)$, i.e. if $\alpha(x_0) = 1, \beta(x_0) = 1, \alpha \Rightarrow \beta$ is true, $\forall x_0 \in M(\alpha)$. If $x_0 \notin M(\alpha)$, then $\alpha(x_0) = 0, \forall x_0 \notin M(\alpha)$, i.e.$\alpha(x_0) \Rightarrow \beta(x_0)$ is true, $\forall x_0 \notin M(\alpha)$. Therefore $\alpha(x_0) \Rightarrow \beta(x_0)$, $\forall x_0$.
   3. If $\alpha(x_0) \Rightarrow \beta(x_0)$, $\forall x_0$ then $\alpha(x_0) = 1, \beta(x_0) = 1$, $\forall x_0 \in M(\alpha)$. This implies that $x_0 \in M(\beta)$. Therefore $M(\alpha) \subseteq M(\beta)$, i.e.$\alpha \models \beta$.

g) The statement, $\alpha \equiv \beta \Leftrightarrow (\neg(\alpha \Leftrightarrow \beta)$ is unsatisfiable) is **true**.
   1. $\alpha \equiv \beta \Leftrightarrow (\alpha \models \beta) \wedge (\beta \models \alpha)$. By the conclusion above, $\alpha \equiv \beta \Leftrightarrow (\alpha \Rightarrow \beta$ is valid) $\wedge$ ($\beta \Rightarrow \alpha$ is valid).
   2. $\alpha \equiv \beta \Leftrightarrow (\alpha(x) \Leftrightarrow \beta(x), \forall x)$. Thus $\alpha \equiv \beta \Leftrightarrow (\neg(\alpha \Leftrightarrow \beta)$ is unsatisfiable).

h) $(\neg B \Rightarrow A) \Rightarrow (\neg A \Rightarrow \neg B)$ is **false**.
   1. $(\neg B \Rightarrow A) = \neg(\neg B) \vee A = B \vee A$; $\neg A \Rightarrow \neg B = A \vee \neg B$.
   2. Let $A = 0$, $B = 1$, the left hand side is $B \vee A = 1$, the right hand side is $A \vee \neg B = 0$. Therefore

the above implication is false.

i) The claim that $(A \wedge B) \wedge \neg(A \Rightarrow B)$ is satisfiable is **false**.

   1. $\neg(A \Rightarrow B) = \neg(\neg A \vee B) = A \wedge \neg B$
   2. $(A \wedge B) \wedge \neg(A \Rightarrow B) = (A \wedge B) \wedge (A \wedge \neg B) = A \wedge B \wedge A \wedge \neg B = A \wedge B \wedge \neg B$. This is unsatisfiable.

j) The claim that $(A \Leftrightarrow B) \wedge (\neg A \vee B)$ is unsatisfiable is **false**.

   1. $(A \Leftrightarrow B) \wedge (\neg A \vee B) = (A \Leftrightarrow B) \wedge (A \Rightarrow B)$. This is valid for all models.

Conjunctive Normal Form and Resolution

A. Convert the following sentences into CNF

a)

$$\begin{aligned} D \wedge \neg B &\Rightarrow E \\ \neg(D \wedge \neg B) &\vee E \\ \neg D \vee B &\vee E \end{aligned}$$

b)

$$\begin{aligned} A \wedge C \wedge D &\Rightarrow F \\ \neg(A \wedge C \wedge D) &\vee F \\ \neg A \vee \neg C \vee \neg D &\vee F \end{aligned}$$

c)

$$\begin{aligned} A \wedge (D \vee F) &\Rightarrow C \\ \neg(A \wedge (D \vee F)) &\vee C \\ (\neg A \vee \neg(D \vee F)) &\vee C \\ (\neg A \vee (\neg D \wedge \neg F)) &\vee C \\ \neg A \vee C \vee (\neg D \wedge \neg F) & \\ (\neg A \vee C \vee \neg D) \wedge (\neg A \vee C \vee \neg F) & \end{aligned}$$

d)

$$\begin{aligned} F \Leftrightarrow \neg B & \\ (F \Rightarrow \neg B) \wedge (\neg B \Rightarrow F) & \\ (\neg F \vee \neg B) \wedge (\neg(\neg B) \vee F) & \\ (\neg F \vee \neg B) \wedge (B \vee F) & \end{aligned}$$

B. Prove that E is true using resolution.

1. $\dfrac{(\neg A \vee C \vee \neg D) \wedge A \wedge D}{C}$. C is added to KB.
2. $\dfrac{(\neg A \vee \neg C \vee \neg D \vee \neg F) \wedge A \wedge D \wedge C}{F}$. F is added to KB.
3. $\dfrac{(\neg F \vee \neg B) \wedge F}{\neg B}$. $\neg B$ is added to KB.
4. $\dfrac{(\neg D \vee B \vee E) \wedge \neg B \wedge D}{E}$. E is true.

First-Order Logic Quantifiers

Translate sentences into FOL:

a) $\forall y, Crust(y) \Rightarrow GoesWith(Peperoni, y)$.
b) $\exists x, Topping(x) \wedge GoesWith(x, Stuffed)$.
c) $\forall y \; \exists \; x \; Topping(x) \Rightarrow Crust(y) \wedge GoesWith(x, y)$.

Translate FOL into English:

d) As long as the pizza covered with mushrooms crust, Ann does not like it.
e) Everyone likes thin pizza with some topping.
f) There are some people love thick pizza no matter what topping comes with it.

First-Order Logic Semantics

a) $WorthMore(quarter, penny) \wedge WorthMore(quarter, nickel) \wedge WorthMore(quarter, dime)$.
b) $WorthMore(quarter, penny) \wedge WorthMore(quarter, nickel) \wedge WorthMore(quarter, dime) \wedge penny \neq nickel \wedge nickel \neq dime \wedge penny \neq dime \wedge (\forall \; x \; WorthMore(quarter, x) \Rightarrow (x = penny \vee x = nickel \vee x = dime))$.

Unification

a) $\{x/A, y/B, z/B\}$.
b) Fail. To unify this two expressions, x, y need to satisfy:
   - $y = G(x, x)$ and $y = G(A, B)$. Since there does not exist x such that $G(x, x) = G(A, B)$ therefore no x, y can satisfy the above condition.

c) $\{x/y, y/A\}$.
d) $\{x/A, y/A, z/G(B)\}$.

Resolution Refutation

1) All wolves howl.
2) Anyone who has cats as pets will not have mice.
3) Anyone who is a light sleeper can't live near anything that howls.
4) Frank either has cats or lives near wolves.
5) If Frank is a light sleeper, Frank has mice.

Write all the sentences above in FOL

1) $\forall \; w$ Wolf(w) $\Rightarrow$ Howl(w).
2) $\forall \; x(\exists y$ Cat(y) $\wedge$ Have(x, y) $\Rightarrow \sim$ $(\exists \; z$ Mouse(z) $\wedge$ Have(x, z))).
3) $\forall \; x$ LS(x) $\Rightarrow \sim$ $(\exists \; y$ Howl(y) $\wedge$ Near(x, y)).
4) $\exists \; x$ Cat(x) $\wedge$ Have(Frank, x) $\vee \exists \; w$ Wolf(w) $\wedge$ Near(Frank, w).
5) LS(Frank) $\Rightarrow \exists \; m$ Mouse(m) $\wedge$ Have(Frank, m).

Convert all the FOL to CNF

1) $\sim \text{Wolf}(w) \vee \text{Howl}(w)$.
2) $\sim \text{Cat}(y) \vee \sim \text{Have}(x, y) \vee \sim \text{Mouse}(z) \vee \sim \text{Have}(x, z)$.
3) $\sim \text{LS}(x) \vee \sim \text{Howl}(y) \vee \sim \text{Near}(x, y)$.
4) $(\text{Cat}(X_1) \vee \text{Wolf}(W_0)) \wedge (\text{Cat}(X_1) \vee \text{Near}(\text{Frank}, W_0)) \wedge (\text{Have}(\text{Frank}, X_1) \vee \text{Wolf}(W_0)) \wedge (\text{Have}(\text{Frank}, X_1) \vee \text{Near}(\text{Frank}, W_0))$.
5) $(\sim \text{LS}(\text{Frank}) \vee \text{Mouse}(M)) \wedge (\sim \text{LS}(\text{Frank}) \vee \text{Have}(\text{Frank}, M))$.

Prove "Frank is not a light sleeper" using Resolution Refutation.

1) Convert "Frank is not a light sleeper" to CNF: $\sim \text{LS}(\text{Frank})$.
2) Add LS(Frank) to KB and try to find controversy.
3) $\frac{(\sim \text{LS}(Frank) \vee \text{Mouse}(M)) \wedge \text{LS}(\text{Frank})}{\text{Mouse}(M)}$. Add $\text{Mouse}(M)$ to KB. $\frac{(\sim \text{LS}(F) \vee \text{Have}(\text{Frank}, M)) \wedge \text{LS}(\text{Frank})}{\text{Have}(\text{Frank}, M)}$. Add $\text{Have}(\text{Frank}, M)$ to KB.
4) $\frac{(\sim \text{Cat}(y) \vee \sim \text{Have}(x, y) \vee \sim \text{Mouse}(z) \vee \sim \text{Have}(x, z)) \wedge \text{Mouse}(M) \wedge \text{Have}(\text{Frank}, M)}{\sim \text{Cat}(y) \vee \sim \text{Have}(\text{Frank}, y)}$. Add $\sim \text{Cat}(y) \vee \sim \text{Have}(\text{Frank}, y)$ to KB.
5) $\frac{(\text{Cat}(X_1) \vee \text{Wolf}(W_0)) \wedge (\sim \text{Cat}(y) \vee \sim \text{Have}(\text{Frank}, y))}{\text{Wolf}(W_0) \vee \sim \text{Have}(\text{Frank}, X_1)}$. Add $\text{Wolf}(W_0) \vee \sim \text{Have}(\text{Frank}, X_1)$ to KB.
6) $\frac{(\text{Have}(\text{Frank}, X_1) \vee \text{Near}(\text{Frank}, W_0)) \wedge (\text{Wolf}(W_0) \vee \sim \text{Have}(\text{Frank}, X_1))}{\text{Near}(\text{Frank}, W_0) \vee \text{Wolf}(W_0)}$. Add $\text{Near}(\text{Frank}, W_0) \vee \text{Wolf}(W_0)$ to KB.
7) $\frac{(\sim \text{Wolf}(w) \vee \text{Howl}(w)) \wedge (\text{Near}(\text{Frank}, W_0) \vee \text{Wolf}(W_0))}{\text{Near}(\text{Frank}, W_0) \vee \text{Howl}(w)}$. Add $\text{Near}(\text{Frank}, W_0) \vee \text{Howl}(w W_0$ to KB.
8) $\frac{(\sim \text{LS}(x) \vee \sim \text{Howl}(y) \vee \sim \text{Near}(x, y)) \wedge (\text{Near}(\text{Frank}, W_0) \vee \text{Howl}(W_0))}{\sim \text{LS}(\text{Frank})}$. Add $\sim \text{LS}(\text{Frank})$ to KB.
9) $\frac{\sim \text{LS}(\text{Frank}) \wedge \text{LS}(\text{Frank})}{\{\}}$. Therefore Frank is not a light sleeper.