

EECS 592 Homework 3

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Problem 1: Evaluating Logical Statements

- a) $M := (A \wedge B) \models S := (A \Leftrightarrow B)$ is **true**:
1. The assignments for which $(A \wedge B)$ is true are: $A = 1, B = 1$.
 2. When $A = 1, B = 1$, $(A \Leftrightarrow B)$ is true. i.e., $A = 1, B = 1, S = (A \Leftrightarrow B)$ is true.
 3. $M \models S$ is true because for all models that evaluate M to be true S is also true.
- b) $M := (A \Leftrightarrow B) \models S := (A \vee B)$ is **false**:
1. Consider $A = 0, B = 0$ which evaluates $M := (A \Leftrightarrow B)$ to be true.
 2. However, $A = 0 \vee B = 0$ is evaluated to be false.
 3. Therefore $M \models S$ is false because there exists an assignment of A and B which makes M to be true and S is false.
- c) $M := (A \wedge B) \Rightarrow C \models S := (A \Rightarrow C) \vee (B \Rightarrow C)$ is **true**:
1. $S = (\neg A \vee C) \vee (\neg B \vee C) = \neg A \vee \neg B \vee C$.
 2. $M = \neg(A \wedge B) \vee C = (\neg A \vee \neg B) \vee C = S$.
- d) $M := (A \vee B) \wedge (\neg C \vee \neg D \vee E) \models S := (A \vee B) \wedge (\neg D \vee E)$ is **false**.
1. Consider $A = 1, B = 1, C = 0, D = 1, E = 0$ which evaluates M to be true.
 2. However, S is evaluated to be false.
 3. Therefore $M \models S$ is false because there exists an assignment of M which makes M to be true but S to be false.
- e) $M := (C \vee (\neg A \wedge \neg B)) \equiv S := ((A \Rightarrow C) \wedge (B \Rightarrow C))$ is **true**.
1. $S \equiv (\neg A \vee C) \wedge (\neg B \vee C)$.
 2. $M \equiv (C \vee \neg A) \wedge (C \vee \neg B)$.
 3. Therefore $M \equiv S$.
- f) The statement, $\alpha \models \beta \Leftrightarrow \alpha \Rightarrow \beta$ is valid, is **true**.
1. Denote $M(\alpha) = \{x \mid \alpha(x) = 1\}$ to be the set of all assignment x such that α is true; $M(\beta)$ is similarly defined.
 2. If $\alpha \models \beta$, then $M(\alpha) \subseteq M(\beta)$. Consider an arbitrary assignment x_0 . If $x_0 \in M(\alpha)$ then $x_0 \in M(\beta)$, i.e. if $\alpha(x_0) = 1, \beta(x_0) = 1, \alpha \Rightarrow \beta$ is true, $\forall x_0 \in M(\alpha)$. If $x_0 \notin M(\alpha)$, then $\alpha(x_0) = 0, \forall x_0 \notin M(\alpha)$, i.e. $\alpha(x_0) \Rightarrow \beta(x_0)$ is true, $\forall x_0 \notin M(\alpha)$. Therefore $\alpha(x_0) \Rightarrow \beta(x_0), \forall x_0$.
 3. If $\alpha(x_0) \Rightarrow \beta(x_0), \forall x_0$ then $\alpha(x_0) = 1, \beta(x_0) = 1, \forall x_0 \in M(\alpha)$. This implies that $x_0 \in M(\beta)$. Therefore $M(\alpha) \subseteq M(\beta)$, i.e. $\alpha \models \beta$.
- g) The statement, $\alpha \equiv \beta \Leftrightarrow (\neg(\alpha \Leftrightarrow \beta))$ is unsatisfiable) is **true**.
1. $\alpha \equiv \beta \Leftrightarrow (\alpha \models \beta) \wedge (\beta \models \alpha)$. By the conclusion above, $\alpha \equiv \beta \Leftrightarrow (\alpha \Rightarrow \beta \text{ is valid}) \wedge (\beta \Rightarrow \alpha \text{ is valid})$.
 2. $\alpha \equiv \beta \Leftrightarrow (\alpha(x) \Leftrightarrow \beta(x), \forall x)$. Thus $\alpha \equiv \beta \Leftrightarrow (\neg(\alpha \Leftrightarrow \beta))$ is unsatisfiable).
- h) $(\neg B \Rightarrow A) \Rightarrow (\neg A \Rightarrow \neg B)$ is **false**.
1. $(\neg B \Rightarrow A) = \neg(\neg B) \vee A = B \vee A$; $\neg A \Rightarrow \neg B = A \vee \neg B$.
 2. Let $A = 0, B = 1$, the left hand side is $B \vee A = 1$, the right hand side is $A \vee \neg B = 0$. Therefore

the above implication is false.

i) The claim that $(A \wedge B) \wedge \neg(A \Rightarrow B)$ is satisfiable is **false**.

$$1. \neg(A \Rightarrow B) = \neg(\neg A \vee B) = A \wedge \neg B$$

$$2. (A \wedge B) \wedge \neg(A \Rightarrow B) = (A \wedge B) \wedge (A \wedge \neg B) = A \wedge B \wedge A \wedge \neg B = A \wedge B \wedge \neg B. \text{ This is unsatisfiable.}$$

j) The claim that $(A \Leftrightarrow B) \wedge (\neg A \vee B)$ is unsatisfiable is **false**.

$$1. (A \Leftrightarrow B) \wedge (\neg A \vee B) = (A \Leftrightarrow B) \wedge (A \Rightarrow B). \text{ This is valid for all models.}$$

Conjunctive Normal Form and Resolution

A. Convert the following sentences into CNF

a)

$$D \wedge \neg B \Rightarrow E$$

$$\neg(D \wedge \neg B) \vee E$$

$$\neg D \vee B \vee E$$

b)

$$A \wedge C \wedge D \Rightarrow F$$

$$\neg(A \wedge C \wedge D) \vee F$$

$$\neg A \vee \neg C \vee \neg D \vee F$$

c)

$$A \wedge (D \vee F) \Rightarrow C$$

$$\neg(A \wedge (D \vee F)) \vee C$$

$$(\neg A \vee \neg(D \vee F)) \vee C$$

$$(\neg A \vee (\neg D \wedge \neg F)) \vee C$$

$$\neg A \vee C \vee (\neg D \wedge \neg F)$$

$$(\neg A \vee C \vee \neg D) \wedge (\neg A \vee C \vee \neg F)$$

d)

$$F \Leftrightarrow \neg B$$

$$(F \Rightarrow \neg B) \wedge (\neg B \Rightarrow F)$$

$$(\neg F \vee \neg B) \wedge (\neg(\neg B) \vee F)$$

$$(\neg F \vee \neg B) \wedge (B \vee F)$$

B. Prove that E is true using resolution.

$$1. \frac{(\neg A \vee C \vee \neg D) \wedge A \wedge D}{C}. \text{ } C \text{ is added to KB.}$$

$$2. \frac{(\neg A \vee \neg C \vee \neg D \vee \neg F) \wedge A \wedge D \wedge C}{F}. \text{ } F \text{ is added to KB.}$$

$$3. \frac{(\neg F \vee \neg B) \wedge F}{\neg B}. \text{ } \neg B \text{ is added to KB.}$$

$$4. \frac{(\neg D \vee B \vee E) \wedge \neg B \wedge D}{E}. \text{ } E \text{ is true.}$$

First-Order Logic Quantifiers

Translate sentences into FOL:

- a) $\forall y, Crust(y) \Rightarrow GoesWith(Peperoni, y).$
- b) $\exists x, Topping(x) \wedge GoesWith(x, Stuffed).$
- c) $\forall y \exists x Topping(x) \Rightarrow Crust(y) \wedge GoesWith(x, y).$

Translate FOL into English:

- d) As long as the pizza covered with mushrooms crust, Ann does not like it.
- e) Everyone likes thin pizza with some topping.
- f) There are some people love thick pizza no matter what topping comes with it.

First-Order Logic Semantics

- a) $WorthMore(quarter, penny) \wedge WorthMore(quarter, nickel) \wedge WorthMore(quarter, dime).$
- b) $WorthMore(quarter, penny) \wedge WorthMore(quarter, nickel) \wedge WorthMore(quarter, dime) \wedge penny \neq nickel \wedge nickel \neq dime \wedge penny \neq dime \wedge (\forall x WorthMore(quarter, x) \Rightarrow (x = penny \vee x = nickel \vee x = dime)).$

Unification

- a) $\{x/A, y/B, z/B\}.$
- b) Fail. To unify this two expressions, x, y need to satisfy:
 - $y = G(x, x)$ and $y = G(A, B).$ Since there does not exist x such that $G(x, x) = G(A, B)$ therefore no x, y can satisfy the above condition.
- c) $\{x/y, y/A\}.$
- d) $\{x/A, y/A, z/G(B)\}.$

Resolution Refutation

- 1) All wolves howl.
- 2) Anyone who has cats as pets will not have mice.
- 3) Anyone who is a light sleeper can't live near anything that howls.
- 4) Frank either has cats or lives near wolves.
- 5) If Frank is a light sleeper, Frank has mice.

Write all the sentences above in FOL

- 1) $\forall w Wolf(w) \Rightarrow Howl(w).$
- 2) $\forall x(\exists y Cat(y) \wedge Have(x, y) \Rightarrow \sim (\exists z Mouse(z) \wedge Have(x, z))).$
- 3) $\forall x LS(x) \Rightarrow \sim (\exists y Howl(y) \wedge Near(x, y)).$
- 4) $\exists x Cat(x) \wedge Have(Frank, x) \vee \exists w Wolf(w) \wedge Near(Frank, w).$
- 5) $LS(Frank) \Rightarrow \exists m Mouse(m) \wedge Have(Frank, m).$

Convert all the FOL to CNF

- 1) $\sim \text{Wolf}(w) \vee \text{Howl}(w)$.
- 2) $\sim \text{Cat}(y) \vee \sim \text{Have}(x, y) \vee \sim \text{Mouse}(z) \vee \sim \text{Have}(x, z)$.
- 3) $\sim \text{LS}(x) \vee \sim \text{Howl}(y) \vee \sim \text{Near}(x, y)$.
- 4) $(\text{Cat}(X_1) \vee \text{Wolf}(W_0)) \wedge (\text{Cat}(X_1) \vee \text{Near}(\text{Frank}, W_0)) \wedge (\text{Have}(\text{Frank}, X_1) \vee \text{Wolf}(W_0)) \wedge (\text{Have}(\text{Frank}, X_1) \vee \text{Near}(\text{Frank}, W_0))$.
- 5) $(\sim \text{LS}(\text{Frank}) \vee \text{Mouse}(M)) \wedge (\sim \text{LS}(\text{Frank}) \vee \text{Have}(\text{Frank}, M))$.

Prove “Frank is not a light sleeper” using Resolution Refutation.

- 1) Convert “Frank is not a light sleeper” to CNF: $\sim \text{LS}(\text{Frank})$.
- 2) Add $\text{LS}(\text{Frank})$ to KB and try to find controversy.
- 3)
$$\frac{(\sim \text{LS}(\text{Frank}) \vee \text{Mouse}(M)) \wedge \text{LS}(\text{Frank})}{\text{Mouse}(M)}$$
. Add $\text{Mouse}(M)$ to KB.
$$\frac{(\sim \text{LS}(F) \vee \text{Have}(\text{Frank}, M)) \wedge \text{LS}(\text{Frank})}{\text{Have}(\text{Frank}, M)}$$
. Add $\text{Have}(\text{Frank}, M)$ to KB.
- 4)
$$\frac{(\sim \text{Cat}(y) \vee \sim \text{Have}(x, y) \vee \sim \text{Mouse}(z) \vee \sim \text{Have}(x, z)) \wedge \text{Mouse}(M) \wedge \text{Have}(\text{Frank}, M)}{\sim \text{Cat}(y) \vee \sim \text{Have}(\text{Frank}, y)}$$
. Add $\sim \text{Cat}(y) \vee \sim \text{Have}(\text{Frank}, y)$ to KB.
- 5)
$$\frac{(\text{Cat}(X_1) \vee \text{Wolf}(W_0)) \wedge (\sim \text{Cat}(y))}{\text{Wolf}(W_0)}$$
. Add $\text{Wolf}(W_0)$ to KB. $\{y/X_1\}$.
- 6)
$$\frac{(\text{Have}(\text{Frank}, X_1) \vee \text{Near}(\text{Frank}, W_0)) \wedge (\text{Have}(\text{Frank}, X_1))}{\text{Near}(\text{Frank}, W_0)}$$
. Add $\text{Near}(\text{Frank}, W_0)$.
- 7)
$$\frac{(\sim \text{Wolf}(w) \vee \text{Howl}(w)) \wedge \text{Wolf}(W_0)}{\text{Howl}(W_0)}$$
. Add $\text{Howl}(W_0)$ to KB.
- 8)
$$\frac{(\sim \text{LS}(x) \vee \sim \text{Howl}(y) \vee \sim \text{Near}(x, y)) \wedge \text{LS}(\text{Frank}) \wedge \text{Howl}(W_0) \wedge \text{Have}(\text{Frank}, W_0)}{\{\}}$$
. Result in controversy. Therefore Frank is not a light sleeper.