tags: lp-geometry

Definition of extreme points and vertices of a polyhedron

Let $P \subset \mathbf{R}^n$ be a polyhedron. A vector $x \in P$ is an extreme point of P if we cannot find two vectors $y, z \in P$ both different from x, and a scalar $\lambda \in [0, 1]$, such that $x = \lambda y + (1 - \lambda)z$.

A vector $x \in P$ is a vertex of P if there exists some $c \in \mathbf{R}^n$ such that $c^\top x < c^\top y$ for all $y \in P, y \neq x$.

Defintion of basic (feasible) solutions

Consider a polyhedron $P \in \mathbf{R}^n$ defined by linear equality and inequality constraints, and let $\bar{x} \in \mathbf{R}^n$. $\bar{x}$ is a basic solution if (1) All equality constraints are active (i.e., satisfied); (2) Among the constraints that are active at $\bar{x}$, there exist n that are linearly independent.

If $\bar{x}$ is a basic solution that satisfies all of the constraints, we say it is a basic feasible solution.

Also note that if the number of constraints used to define $P \in \mathbf{R}^n$, m is less than n, then there is no basic solutions or basic feasible solutions.

State and prove the relation between extreme points, vertices and basic solutions

Let P be a nonempty polyhedron and let $x \in P$. Then, x being a vertex is equivalent to x being an extreme point and to being a basic feasible solution. Corollary: Given a finite number of linear inequality constraints, there can only be a finite number of basic or basic feasible solutions.

Definition of subspaces

We call $\varnothing \neq S \subseteq \mathbf{R}^n$ a subspace of $\mathbf{R}^n$ if $\forall x, y \in S, \alpha, \beta \in \mathbf{R}$, $\alpha x + \beta y \in S$. (Must contain 0!)