tags: renewal-processes

Definition of renewal processes

A renewal process is a counting process where inter-arrival times, $\tau_j, j = 1, 2, \ldots$ between consecutive events are i.i.d. with cumulative distribution function $F_\tau(\cdot)$, $\mu := \mathbf{E}(\tau)$, $\tau > 0$ with probability 1.

State and prove three basic properties of renewal processes

Three properties are:

1. For any fixed finite t, $P(N_t = +\infty) = 0$.
2. $\lim_{t\to+\infty} N_t = +\infty$ with probability 1.
3. $\lim_{t\to+\infty} \mathbf{E}(N_t) = +\infty$.

State and prove the strong law of renewal processes

$\lim\limits_{t\to+\infty} \dfrac{N_t}{t} = \dfrac{1}{\mu}$ with probability 1.

Distribution of renewal process $\{N_t, t \geq 0\}$ with inter-arrival times τ_n

$P(N_t = n) = F_n(t) - F_{n+1}(t)$, where $F_n = F_\tau * F_\tau \cdots * F_\tau$

State central limit theorem for renewal processes

Assume that the inter-arrival times, τ_j for a renewal process, $\{N_t, t \geq 0\}$ have finite variance σ^2. Then $\xi_t = \frac{N_t - \frac{t}{\mu}}{\sigma\sqrt{t/\mu^3}} \to \mathcal{N}(0,1)$ in distribution when $t \to +\infty$.

State and prove the renewal equation

$m(t) = F_\tau(t) + \int_0^t m(t-x)F(dx)$

State and prove the elementary renewal theorem (τ is cont.)

$$\lim_{t\to+\infty} \frac{\mathbf{E}(N_t)}{t} = \frac{1}{\mathbf{E}(\tau)}$$

State and prove Blackwell's Theorem

If $\{N_t, t = 0, 1, \ldots\}$ has an inter-arrival distribution that is no-lattice, then $\forall \sigma > 0$,

$$\begin{aligned}
\lim_{t\to+\infty} [m(t+\sigma) - m(t)] &= \frac{\sigma}{\mu} \\
\lim_{t\to+\infty} P(N_{t+\sigma} - N_t = 1) &= \frac{\sigma}{\mu} \\
\lim_{t\to+\infty} P(N_{t+\sigma} - N_t = 0) &= 1 - \frac{\sigma}{\mu} + o(\sigma) \\
\lim_{t\to+\infty} P(N_{t+\sigma} - N_t = 2) &= o(\sigma)
\end{aligned}$$

If $\{N_t, t = 0, 1, \ldots\}$ has an inter-arrival distribution that is lattice with span d, then:

$$\lim_{t\to+\infty} [m(nd) - m((n-1)d)] = \frac{nd}{\mu}$$

Definition of renewal reward processes and fundamental assumption

Let $\{N_t\}, t \geq 0$ be a renewal process, $\{R(t), t \geq 0\}$ be a reward process associated with $\{N_t\}$. We assume R_t at a give time t depends only on the inter-renewal interval containing t, i.e.,

$$R_t = R(Z_t, \tau_t)$$

where $Z_t := t - T_{N_t}, \tau_t := T_{N_t+1} - T_{N_t}$.

Determine $\displaystyle\lim_{t\to+\infty} \frac{1}{t} \int_0^t R_s ds$

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t R_s ds = \frac{\mathbf{E}(R_n)}{\mu}$$

where $\mu := \mathbf{E}(\tau), R_n = \displaystyle\int_{T_{n-1}}^{T_n} R(Z_t, \tau_n) d\tau = \int_0^{\tau} R(s, \tau_n) ds$

Prove the key renewal reward theorem when $\{N_t\}$ is non-arithmetic

Let $R(\tau, Z)$ be a renewal reward function associated with a non-arithmetic renewal process $\{N_t, t \geq 0\}$; let $r(z) = \int_{\tau=z}^{+\infty} R(\tau, z) df(\tau)$ be a directly Riemann integrable function and let $m(t) := \mathbf{E}(N_t)$. Assume $\tau > 0$ with probability 1. We have the following:

$$\lim_{t \to +\infty} \mathbf{E}(R_t) = \lim_{t \to +\infty} \int_0^t r(z) d(m(t-z)) = \frac{1}{\mu} \int_0^{+\infty} r(z) dz$$

Determine $\lim\limits_{t \to +\infty} \mathbf{E}(R_t)$

By the key renewal reward theorem when $\{N_t\}$ is non-arithmetic, we have the following:

$$\lim_{t \to +\infty} \mathbf{E}(R_t) = \frac{\mathbf{E}(R_n)}{\mu}$$

where $\mu := \mathbf{E}(\tau)$.