tags: lp-geometry

Definition of extreme points and vertices of a polyhedron

Let $P \subset \mathbf{R}^n$ be a polyhedron. A vector $x \in P$ is an extreme point of P if we cannot find two vectors $y, z \in P$ both different from x, and a scalar $\lambda \in [0, 1]$, such that $x = \lambda y + (1 - \lambda)z$.

A vector $x \in P$ is a vertex of P if there exists some $c \in \mathbf{R}^n$ such that $c^\top x < c^\top y$ for all $y \in P, y \neq x$.

Three equivalent statement w.r.t. the set of indices of constraints active at $x^* \in \mathbf{R}^n$

Let x^* be an element of $\mathbf{R}^n$ and let $I = \{i \mid a_i^\top x^* = b_i\}$ be the set of indices of constraints that are active at x^*. Then the following are equivalent:

1. There exist n vectors in the set $\{a_i \mid i \in I\}$, which are linearly independent.
2. The span of the vectors $a_i, i \in I$ is all of $\mathbf{R}^n$, i.e., every elements of $\mathbf{R}^n$ can be expressed as a linear combination of the vectors $a_i, i \in I$.
3. The system of equations $a_i^\top x = b_i, i \in I$ has a unique solution.

Defintion of basic (feasible) solutions

Consider a polyhedron $P \in \mathbf{R}^n$ defined by linear equality and inequality constraints, and let $\bar{x} \in \mathbf{R}^n$. $\bar{x}$ is a basic solution if (1) All equality constraints are active (i.e., satisfied); (2) Among the constraints that are active at $\bar{x}$, there exist n that are linearly independent.

If $\bar{x}$ is a basic solution that satisfies all of the constraints, we say it is a basic feasible solution.

Also note that if the number of constraints used to define $P \in \mathbf{R}^n$, m is less than n, then there is no basic solutions or basic feasible solutions.

State and prove the relation between extreme points, vertices and basic solutions

Let P be a nonempty polyhedron and let $x \in P$. Then, x being a vertex is equivalent to x being an extreme point and to being a basic feasible solution. Corollary: Given a finite number of linear inequality constraints, there can only be a finite number of basic or basic feasible solutions.

Given $\{x \in \mathbf{R}^n \mid Ax = b, x \succeq 0\}$, state and prove how would you find basic solutions.

Consider the constraints $Ax = b$ and $x \succeq 0$ and assume that $A \in \mathbf{R}^{m \times n}$ has linear independent rows. A vector $x \in \mathbf{R}^n$ is a basic solution iff we have $Ax = b$ and there exists indices $B(1), \ldots, B(m)$ such that:

1. The columns $A_{B(1)}, \ldots, A_{B(m)}$ are linearly independent;
2. If $i \neq B(1), \ldots, B(m)$ then $x_i = 0$.

Note that there are actually $m + n$ constraints on x in total. Method above gives a way to find a way to identify corners. Also it is easy to judge whether the basic solution is feasible or not since we only need to check whether $x_B = A_B^{-1} b \succeq 0$ or not.

Definition of adjacent basic solutions

Two distinct basic solutions are said to be adjacent if there are $n-1$ linearly independent constraints that are active at both of them. For standard form problems, we also say two bases are adjacent if they share all but one basic column.

Why do we always assume A has full row rank?

Let $P = \{x \mid Ax = b, x \succeq 0\}$ be a nonempty polyhedron, where A is a matrix of dimensions $m \times n$, with rows $a_1^\top, \ldots, a_m^\top$. Suppose $\operatorname{rank}(A) = k < m$ and that the rows $a_{i_1}^\top, \ldots, a_{i_k}^\top$ are linearly independent. Consider the polyhedron

$$Q = \{x \mid a_{i_1}^\top x = b_{i_1}, \ldots, a_{i_k}^\top x = b_{i_k}, x \succeq 0\}$$

Then $Q = P$.

Definition of degeneracy and degeneracy in standard form polyhedra

A basic solution $x \in \mathbf{R}^n$ is said to be degenerate if more than n of the constraints are active at x.

Consider the standard form polyhedron $P = \{x \in \mathbf{R}^n \mid Ax = b, x \succeq 0\}$ and let x be a basic solution. Let m be the number of rows of A. The vectors x is degenerate basic solution if more than $n - m$ of the components of are zero.

Three equivalent statements on the existence of extreme point and its corollary

Suppose that the polyhedron $P = \{x \in \mathbf{R}^n \mid a_i^\top x \geq b_i, i = 1, 2, \ldots, m\}$ is nonempty. Then the following are equivalent:

1. The polyhedron P has at least one extreme point.
2. The polyhedron P does not contain a line.
3. There exist n vetors out of the family $a_1, \ldots, a_m$ which are linearly independent.

Corollary: Every nonempty bounded polyhedron and every nonempty polyhedron in standard form has at least one basic feasible solution.

State the optimality of extreme points

Consider the linear programming problem of minimizing $c^\top x$ over a polyhedron P. Suppose that P has at least one extreme point. Then either the optimal cost is $-\infty$ or there exists an extreme point which is optimal.

Definition of subspaces

We call $\varnothing \neq S \subseteq \mathbf{R}^n$ a subspace of $\mathbf{R}^n$ if $\forall x, y \in S, \alpha, \beta \in \mathbf{R}$, $\alpha x + \beta y \in S$. (Must contain 0!)