tags: markov-processes

Definition of stopping time

Let a random process, $\{X_t, t \in T\}$ defined on some probability space and taking values in the set of integers, D. Random variable τ is said to be a stopping time w.r.t. $\{X_t, t \in T\}$ if the event $\{\tau = m\}, \forall m \in D$ can be determined by $X_0, X_1, \ldots, X_m$.

Definition of strong Markov property

Let $\{X_t, t \in T\}$ be a Markov process and let τ be a stopping time w.r.t. $\{X_t, t \in T\}$. The process $\{X_t, t \in T\}$ satisfies the strong Markov property if $\forall k = 1, 2, \ldots,$

$$P(X_{\tau+k} \in A | X_\tau = x, X_{\tau-1} = x, \cdots, X_0 = x) = P(X_{\tau+k} \in A | X_\tau = x)$$

(note that when τ is a constant, it goes back to Markov property)

Definition of Markov Chain

A Markov chain is a Markov process $\{X_t, t \in T\}$ with finite or countably infinite state space. To characterize a Markov chain statistically, we need the following

$$\{\pi_s(x) : \forall x \in S, \pi_0(x) = p(X_0 = 0)\}$$
$$\{P_t(x, y) := p(X_{t+1} = y | X_t = x), x, y \in S, t \in T\}$$

where S is the state of space of the Markov chain.

Definition of Random Walk

Consider $\{\xi_1, \ldots, \xi_k, \ldots\}$, a collection of i.i.d. random variables taking values in a set of integers. Let X_t be a random variable that takes values in the set of integers and is independent of $\{\xi_1, \ldots, \xi_k, \ldots\}$, $\{X_t := X_0 + \xi_1 + \cdots + \xi_t\}$ is called a random walk.

Definition of hitting times

Considering a Markov Chain, $\{X_t, t \geq 0\}$ with state space S and transition probability, $P(x,y) = P(X_{t+1} = y|X_t = x), \forall x, y \in S, \forall t$. Let $A \subset S$, a stopping time T_A w.r.t. $\{X_t, t = 0, 1, 2, \ldots\}$ is defined as

$$T_A := \min\{t > 0 : X_t \in A\}.$$

Properties of Markov Chain

1. $P^n(x,y) = \sum_{m=1}^{n} P(T_y = m|X_n = x)P^{n-m}(y,y)$;
2. If a is an absorbing state, then

$$P(X_n = a|X_0 = x) = P(T_a \leq n|X_0 = x)$$

Definition of transient and recurrent states

Let $\rho_{xy} := P(T_y < +\infty|X_0 = x), \rho_{yy} = P(T_y < +\infty|X_0 = y), x, y \in S$. A state $y \in S$ is said to be recurrent if $\rho_{yy} = 1$. A state $y \in S$ is transient if $\rho_{yy} < 1$.

State and prove the properties of a transient state

Let $y \in S$ be a transient state, then

$$P(N(y) < +\infty|X_0 = x) = 1$$
$$G(x,y) = \mathbf{E}(N(y)|X_0 = x) = \frac{\rho_{xy}}{1 - \rho_{yy}}, \forall x \in S$$

where $N(y) := \sum_{n=1}^{+\infty} \mathbf{1}_{\{y\}}(X_n)$

State and prove properties of a recurrent state

Let $y \in S$ be a recurrent state, then

$$
\begin{aligned}
P(N(y) = +\infty | X_0 = y) &= 1 \\
G(y, y) = \mathbf{E}(N(y) | X_0 = y) &= +\infty \\
P(N(y) < +\infty | X_0 = x) &= \rho_{xy}, \forall x \in S
\end{aligned}
$$

where $N(y) := \sum_{n=1}^{+\infty} \mathbf{1}_{\{y\}}(X_n)$.

If $\rho_{xy} = 0$, then $G(x, y) = 0$; if $\rho_{xy} > 0$, then $G(x, y) = +\infty$.