tags: renewal-processes

Definition of renewal processes

A renewal process is a counting process where inter-arrival times, $\tau_j, j = 1, 2, \ldots$ between consecutive events are i.i.d. with cumulative distribution function $F_\tau(\cdot)$, $\mu := \mathbf{E}(\tau)$, $\tau > 0$ with probability 1.

State and prove three basic properties of renewal processes

Three properties are:

1. For any fixed finite t, $P(N_t = +\infty) = 0$.

2. $\lim_{t \to +\infty} N_t = +\infty$ with probability 1.

3. $\lim_{t \to +\infty} \mathbf{E}(N_t) = +\infty$.

State and prove the strong law of renewal processes

$$\lim_{t \to +\infty} \frac{N_t}{t} = \frac{1}{\mu} \text{ with probability 1.}$$

Distribution of renewal process $\{N_t, t \geq 0\}$ with inter-arrival times τ_n

---

$P(N_t = n) = F_n(t) - F_{n+1}(t)$, where $F_n = F_\tau * F_\tau \cdots * F_\tau$

State central limit theorem for renewal processes

Assume that the inter-arrival times, τ_j for a renewal process, $\{N_t, t \geq 0\}$ have finite variance σ^2. Then $\xi_t = \frac{N_t - \frac{t}{\mu}}{\sigma\sqrt{t/\mu^3}} \to \mathcal{N}(0, 1)$ in distribution when $t \to +\infty$.

State and prove the renewal equation

$m(t) = F_\tau(t) + \int_0^t m(t - x) F(dx)$

> State and prove the elementary renewal theorem (τ is cont.)

$$\lim_{t \to +\infty} \frac{\mathbf{E}(N_t)}{t} = \frac{1}{\mathbf{E}(\tau)}$$

> State and prove Blackwell's Theorem

If $\{N_t, t = 0, 1, \ldots\}$ has an inter-arrival distribution that is no-lattice, then $\forall \sigma > 0$,

$$\lim_{t \to +\infty} [m(t + \sigma) - m(t)] = \frac{\sigma}{\mu}$$

$$\lim_{t \to +\infty} P(N_{t+\sigma} - N_t = 1) = \frac{\sigma}{\mu}$$

$$\lim_{t \to +\infty} P(N_{t+\sigma} - N_t = 0) = 1 - \frac{\sigma}{\mu} + o(\sigma)$$

$$\lim_{t \to +\infty} P(N_{t+\sigma} - N_t = 2) = o(\sigma)$$

If $\{N_t, t = 0, 1, \ldots\}$ has an inter-arrival distribution that is lattice with span d, then:

$$\lim_{t \to +\infty} [m(nd) - m((n-1)d)] = \frac{nd}{\mu}$$