tags: analysis-number-systems

Definition of order and ordered sets

Let S be a set. An order on S is a relation, denoted by $<$, with the following two properties:

1. If $x \in S, y \in S$ then one and only one of the statements: $x < y,\ x = y,\ y < x$ is true.
2. If $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

An ordered set is a set S in which an order is defined.

Definition of upper (lower) bound and supremum (infimum)

Suppose S is an ordered set, and $E \subset S$. If $\exists \beta \in S$, s.t. $x \leq \beta$ for every $x \in E$, we say that E is bounded above, and call β and upper bound E. (lower bound is similarly defined)

Suppose S is an ordered set, $E \subset S$, and E is bounded above. Suppose there exists an $\alpha \in S$ with the following properties:

1. α is an upper bound of E.
2. If $\gamma < \alpha$ then γ is not an upper bound of E.

Then, α is called the least upper bound of E or the supremum of E and we denote it as $\alpha := \sup E$.

The least upper bound property of an ordered set

An ordered set S is said to have the least upper bound property if the following is true:

If $E \subset S$, E is not empty and E is bounded above, then $\sup E$ exists in S.

Prove the existance of the infimum of any subsets of a ordered set with least upper bounded property

Suppose S is an ordered set with the least upper bound property, $B \subset S$, B is not empty and is bounded below. Let L be the set of all lower bounds of B. Then $\alpha = \sup L$ exists in S and $\alpha = \inf B$

Definition of field

A field is a set F with two operations, called addition and multiplication, which satisfy the following field axioms:

- Axioms for addition
- Axioms for multiplication
- The distributive law

Definition of an ordered field

An ordered field is a field F which is also an ordered set, such that

1. $x + y < x + z$ if $x, y, z \in F$ and $y < z$,
2. $xy > 0$ if $x \in F, y \in F, x > 0$, and $y > 0$.

Existence of real fields

There exists an ordered field R which has the least upper bound property. Moreover, R contains $\mathbf{Q}$ as its subfield.

State and prove the archimedean property of $\mathbf{R}$

If $x \in R, y \in R$ and $x > 0$, then there is a positive integer n such that $nx > y$.

State and prove the relation between $\mathbf{Q}$ and $\mathbf{R}$

Rational numbers are dense in real numbers: If $x \in R, y \in R$ and $x < y$, then there exists a $p \in \mathbf{Q}$ such that $x < p < y$.

State and prove the Cauchy Inequality

If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are complex numbers, then

$$\| \sum_{j=1}^{n} a_j \bar{b}_j \|^2 \leq \sum_{j=1}^{n} \|a_j\|^2 \sum_{j=1}^{n} \|b_j\|^2.$$