

**tags:**    renewal-processes

Definition of renewal processes

A renewal process is a counting process where inter-arrival times,  $\tau_j, j = 1, 2, \dots$  between consecutive events are i.i.d. with cumulative distribution function  $F_\tau(\cdot)$ ,  $\mu := \mathbf{E}(\tau)$ ,  $\tau > 0$  with probability 1.

State and prove three basic properties of renewal processes

1. For any fixed finite  $t$ ,  $P(N_t = +\infty) = 0$ .
2.  $\lim_{t \rightarrow +\infty} N_t = +\infty$  with probability 1.
3.  $\lim_{t \rightarrow +\infty} \mathbf{E}(N_t) = +\infty$ .

State and prove the strong law of renewal processes

$\lim_{t \rightarrow +\infty} \frac{N_t}{t} = \frac{1}{\mu}$  with probability 1.

Solve for the distribution of renewal process with  $\{N_t, t \geq 0\}$  with inter-arrival times  $\tau_n, n = 1, 2, \dots$

$P(N_t = n) = F_n(t) - F_{n+1}(t)$ , where  $F_n = F_\tau * F_\tau \cdots * F_\tau$

State and prove central limit theorem for renewal processes

Assume that the inter-arrival times,  $\tau_j$  for a renewal process,  $\{N_t, t \geq 0\}$  have finite variance  $\sigma^2$ . Then  $\xi_t = \frac{N_t - \frac{t}{\mu}}{\sigma \sqrt{t/\mu^3}} \rightarrow \mathcal{N}(0, 1)$  in distribution when  $t \rightarrow +\infty$ .

State and prove the renewal equation

$$m(t) = F_\tau(t) + \int_0^t m(t-x)F(dx)$$

State and prove the elementary renewal theorem ( $\tau$  is cont.)

$$\lim_{t \rightarrow +\infty} \frac{\mathbf{E}(N_t)}{t} = \frac{1}{\mathbf{E}(\tau)}$$

State and prove Blackwell's Theorem

If  $\{N_t, t = 0, 1, \dots\}$  has an inter-arrival distribution that is no-lattice, then  $\forall \sigma > 0$ ,

$$\begin{aligned} \lim_{t \rightarrow +\infty} [m(t + \sigma) - m(t)] &= \frac{\sigma}{\mu} \\ \lim_{t \rightarrow +\infty} P(N_{t+\sigma} - N_t = 1) &= \frac{\sigma}{\mu} \\ \lim_{t \rightarrow +\infty} P(N_{t+\sigma} - N_t = 0) &= 1 - \frac{\sigma}{\mu} + o(\sigma) \\ \lim_{t \rightarrow +\infty} P(N_{t+\sigma} - N_t = 2) &= o(\sigma) \end{aligned}$$

If  $\{N_t, t = 0, 1, \dots\}$  has an inter-arrival distribution that is lattice with span  $d$ , then:

$$\lim_{t \rightarrow +\infty} [m(nd) - m((n-1)d)] = \frac{nd}{\mu}$$