tags: poisson-processes

Definition of Poisson Process

A counting process $\{N_t, t \geq\}$ is called a Poisson Process with parameter λ if it satisfies the following conditions:

1. $N_0 = 0$;
2. For any $0 \leq s < t < +\infty$, the increment $N_t - N_s$ is a Poisson random variable with parameter $\lambda(t-s)$;
3. $\{N_t, t \geq 0\}$ is an independent increment process.

Six properties of Poisson Processes

1. $\forall t > 0, N_t$ is a Poisson random varialbe with parameter λt;
2. $\mathbf{E}(N_t) = \lambda t, \mathbf{Var}(N_t) = \lambda t, \mathbf{E}(N_t^2) = \mathbf{Var}(N_t) + \mathbf{E}(N_t)^2 = \lambda t + (\lambda t)^2$
3. Consider $N_{t+\delta} - N_t$ where $\delta > 0$ is very small.

$$
\begin{aligned}
&P(N_{t+\delta} - N_t = 0) = e^{-\lambda\delta} = 1 - \lambda\delta + o(\delta) \\
&P(N_{t+\delta} - N_t = 1) = \lambda\delta e^{-\lambda\delta} = \lambda\delta + o(\delta) \\
&P(N_{t+\delta} - N_t \geq 2) = 1 - P(N_{t+\delta} - N_t = 0) - P(N_{t+\delta} - N_t = 1) = o(\delta)
\end{aligned}
$$

4. Let $T_n := \min\{t > 0 : N_t = 0\}$, then $\{T_n > t\} = \{N_t < n\}$. (Applies to all counting processes)
5. Let $S_n := T_n - T_{n-1}$, $S_1, S_2, \ldots, S_n$ are i.i.d. and exponentially distributed with parameter λ. Also note that they obey memoryless property.
6. The inter-arrival of a Poisson process are i.i.d exponential random variables with parameter λ and have momoryless property.

State and prove that the countable infinite sum of independent Poisson Processes is a Poisson Process

Sum of independent Poisson Processes:

Let $\{N_i, i = 1, 2, \ldots\}$ be a family of independent Poisson Processes with respective parameters $\lambda_i \geq 0, i = 1, 2, \ldots$, Then

1. If $\lambda := \sum_i^{+\infty} \lambda_i < +\infty$, then $\{N_t = \sum_1^{+\infty} N_t^i, t \geq 0\}$ is a Poisson Process with parameter λ.
2. Any two distinct Poisson Process from this family have no points in common.

Refs: For proof, please refer to notes on EECS 502

State and prove Competition Theorem for Poisson Processes

Competition Theorem:

Let $\{N_i, i = 1, 2, \ldots\}$ be a family of independent Poisson Processes with parameters $\lambda_i, i = 1, 2, \ldots$ respectively. Assume $\lambda := \sum_1^{+\infty} \lambda_i < +\infty$, define $N := \sum_1^{+\infty} N_i$ and let τ be the first event time of Poisson process N and Z be the index of the Poisson process responsible for it, that is τ is the first event time of N_Z, then

$$P(Z = i, \tau \geq t) = P(Z = i)P(\tau \geq t) = \frac{\lambda_i}{\lambda} e^{-\lambda t}$$

Refs: For proof, please refer to notes on EECS 502