tags: lp-simplex

Standard form linear programming problem

The standard form linear programming problem has the following form:

$$
\begin{aligned}
\min \ & c^\top x \\
\text{s.t. } & Ax = b \\
& x \succeq 0
\end{aligned}
$$

where $A \in \mathbf{R}^{m \times n}$ and $\operatorname{rank}(A) = m$; let P denotes the corresponding feasible set.

Definition of feasible direction and how does simplex determine its feasible direction

Let $x \in P$, a vector $d \in \mathbf{R}^n$ is said to be a feasible direction at x if there exists a positive scalar θ for which $x + \theta d \in P$.

Simplex algorithm finds the optimal solution of a LP by searching for the best extreme points in P. Given a starting point, i.e. a basic feasible solution, we have m linearly independent basic constraints and $n - m$ nonbasic constraints associated with it. The jth feasible direction is determined by releasing the jth nonbasic constraints among the $n - m$ constraints in total: the jth feasible solution is the 1d null space of m basic constraints and $n - m - 1$ nonbasic constraints.

Derive primal simplex the jth feasible direction and corresponding reduce cost

Given a current basic feasible solution (BFS), $\begin{bmatrix} x_B \\ x_N \end{bmatrix}$, simplex does the following:

1. Current BFS satisfies:

$$
\tilde{A}x := \begin{bmatrix} A_B & A_N \\ 0 & I_{n-m} \end{bmatrix} \begin{bmatrix} x_B \\ x_N \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}
$$

Release the jth constraints, $j \in \{m+1, \dots, n\}$, solve for the null space of the left $(n-1) \times n$ matrix, $\tilde{A}^j$ using equality $\tilde{A}^j d = 0$, we have:

$$\tilde{A}^j d := \begin{bmatrix} A_B & A_N \\ 0 & I_{n-m}^j \end{bmatrix} \begin{bmatrix} d_B \\ d_N \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

where I_{n-m}^j denotes the identity matrix without the jth row. Evidently, the solution of d should satisfy:

$$\begin{aligned} A_B d_B + A_{N_j} d_{N_j} &= 0 \\ d_B &= -A_B^{-1} A_{N_j} d_{N_j} \\ d_N^j &= 0 \end{aligned}$$

where A_{N_j} denotes the jth column of A_N; d_{N_j} denotes the jth entry of d_N; d_N^j denotes d_N with the jth entry excluded.

2. Compute the reduction in objective value along the jth feasible direction with with step $d_{N_j} = 1$:

$$\begin{aligned} \delta &= c^\top \begin{bmatrix} d_B \\ d_N \end{bmatrix} \\ &= c_B^\top d_B + c_{N_j} \\ &= c_{N_j} - c_B^\top A_B^{-1} A_{N_j} d_{N_j} \end{aligned}$$

If all possible feasible direction leads to an increase in the objective value, we can say that current BFS is optimal

State and prove the relationship between reduce costs and optimality of basic feasible solution

Consider a basic feasible solution x associated with a basis matrix B, and let $\bar{c}$ be the corresponding vector of reduced costs.

1. If $\bar{c} \succeq 0$, then x is optimal.
2. If x is optimal and nondegernate, then $\bar{c} \succeq 0$.

Describe the primal simplex method

Simplex method searches for an optimal solution by moving from one basic feasible solution to another, along the edges of the feasible set, always in a cost reducing direction.